Coping with NP-completeness: Special Cases

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Advanced Algorithms and Complexity
Data Structures and Algorithms

The fact that a problem is **NP**-complete does not exclude an efficient algorithm for special cases of the problem.

Outline

1. 2-Satisfiability
2. Independent Sets in Trees

This part

- Striking connection between strongly connected components of a graph and formulas in 2-CNF
- A linear time algorithm for 2-SAT

2-Satisfiability (2-SAT)

Input: A set of clauses, each containing at most two literals (that is, a 2-CNF formula).

Output: Find a satisfying assignment (if exists).

Example

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- $(x \vee y)(x \vee \overline{y})(\overline{x} \vee y)(\overline{x} \vee \overline{y})$ is unsatisfiable

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- Essentially, it says that ℓ_1 and ℓ_2 cannot be both equal to 0
- In other words, if $\ell_1 = 0$, then $\ell_2 = 1$ and if $\ell_2 = 0$, then $\ell_1 = 1$

Definition

Implication is a binary logical operation denoted by $\Rightarrow$ and defined by the following truth table:

x	y	$x \Rightarrow y$
0	0	1
0	1	1
1	0	0
1	1	1

Definition

For a 2-CNF formula, its implication graph is constructed as follows:

- for each variable x, introduce two vertices labeled by x and $\overline{x}$;
- for each 2-clause $(\ell_1 \vee \ell_2)$, introduce two directed edges $\overline{\ell}_1 \to \ell_2$ and $\overline{\ell}_2 \to \ell_1$
- for each 1-clause (ℓ), introduce an edge $\overline{\ell} \to \ell$

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Encodes all implications imposed by the formula.

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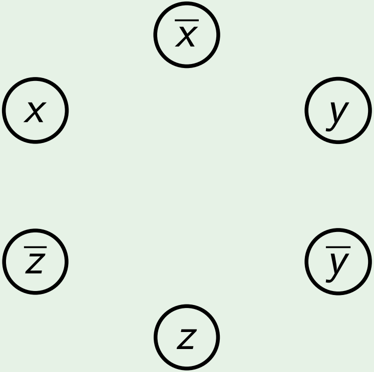

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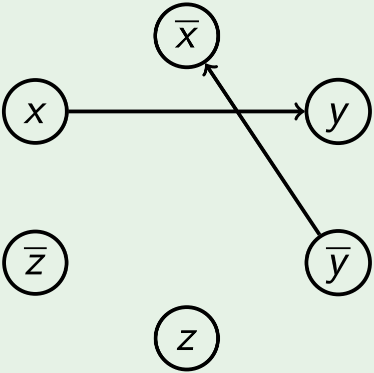

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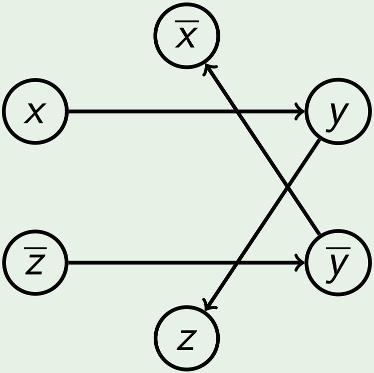

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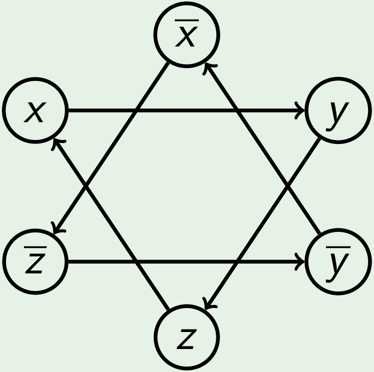
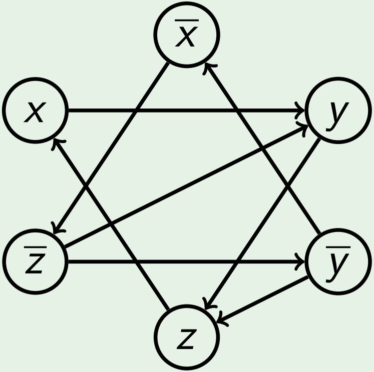


$$(\overline{x} \lor y)(\overline{y} \lor z)(x \lor \overline{z})(z \lor y)$$



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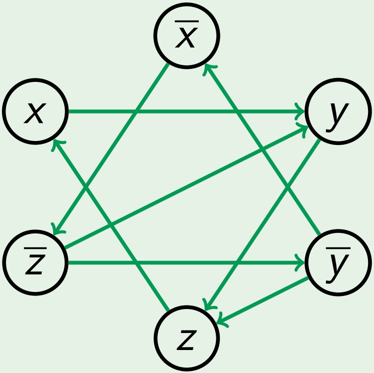

$x = 1, y = 1, z = 1$

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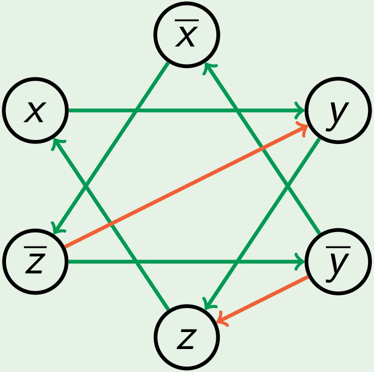

$$x = 0, y = 0, z = 0$$

Thus, our goal is to assign truth values to the variables so that each edge in the implication graph is "satisfied", that is, there is no edge from 1 to 0.

Skew-Symmetry

- The graph is skew-symmetric: if there is an edge $\ell_1 \rightarrow \ell_2$, then there is an edge $\overline{\ell}_2 \rightarrow \overline{\ell}_1$

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- This generalizes to paths: if there is a path from ℓ_1 to ℓ_2, then there is a path from $\overline{\ell}_2$ to $\overline{\ell}_1$

Transitivity

Lemma

If all the edges are satisfied by an assignment and there is a path from ℓ_1 to ℓ_2, then it cannot be the case that $\ell_1 = 1$ and $\ell_2 = 0$.

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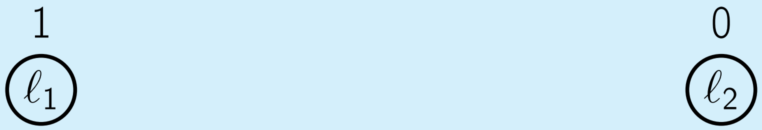

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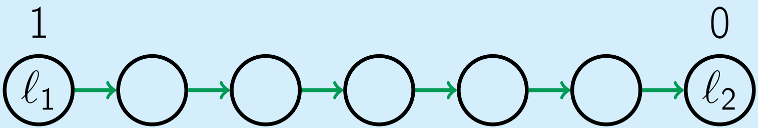

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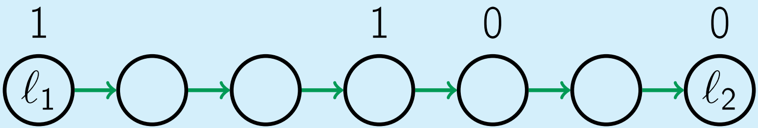

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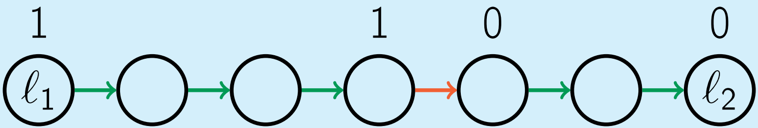

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- In particular, if a SCC contains a variable together with its negation, then the formula is unsatisfiable
- It turns out that otherwise the formula is satisfiable!

2SAT(2-CNF F)

```
construct the implication graph G
find SCC's of G
for all variables x:
  if x and x̄ lie in the same SCC of G:
    return "unsatisfiable"
find a topological ordering of SCC's
for all SCC's C in reverse order:
  if literals of C are not assigned yet:
    set all of them to 1
    set their negations to 0
return the satisfying assignment
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Running time: $O(|F|)$

Lemma

The algorithm `2SAT` is correct.

Proof

- When a literal is set to 1, all the literals that are reachable from it have already been set to 1 (since we process SCC's in reverse topological order).

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Proof

- When a literal is set to 1, all the literals that are reachable from it have already been set to 1 (since we process SCC's in reverse topological order).
- When a literal is set to 0, all the literals it is reachable from have already been set to 0 (by skew-symmetry). □

Outline

1. 2-Satisfiability

2. Independent Sets in Trees

Planning a company party

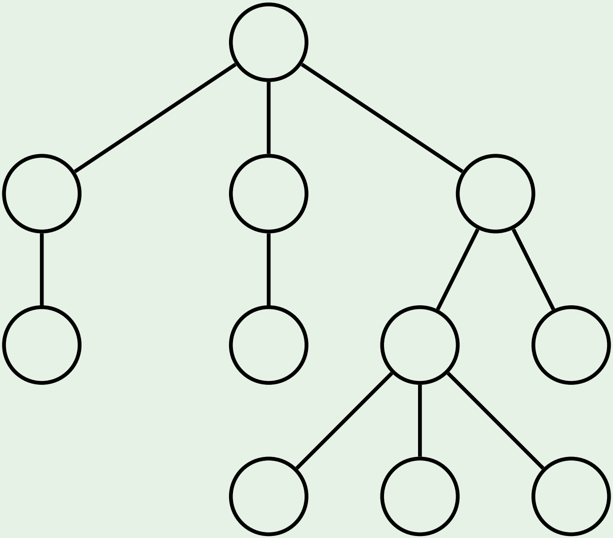
You are organizing a company party. You would like to invite as many people as possible with a single constraint: no person should attend a party with his or her direct boss.

Maximum independent set in a tree

Input: A tree.

Output: An independent set (i.e., a subset of vertices no two of which are adjacent) of maximum size.

Example



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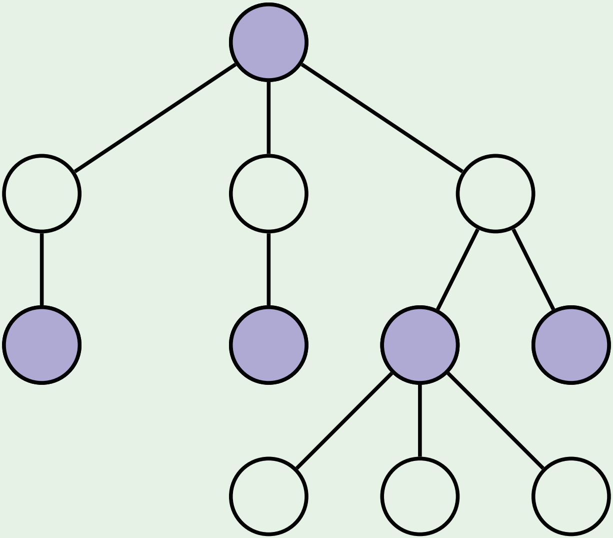

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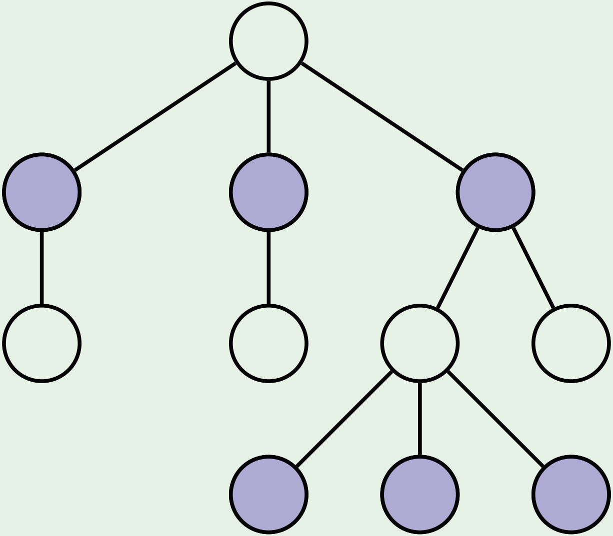

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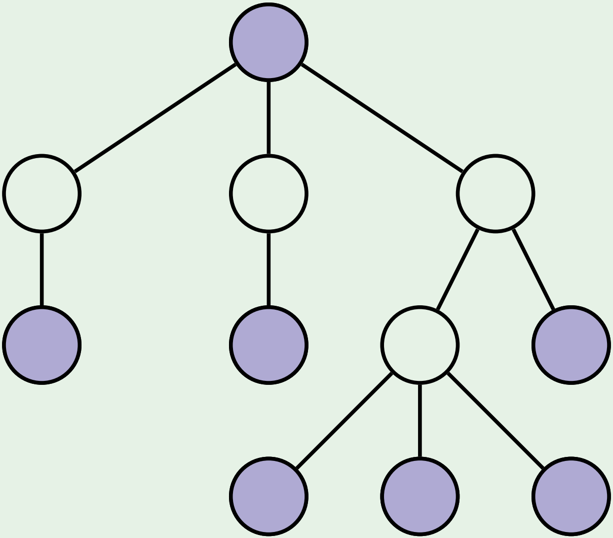
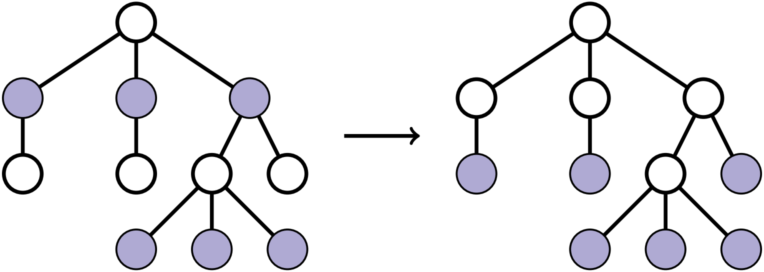


Safe move

For any leaf, there exists an optimal solution including this leaf.



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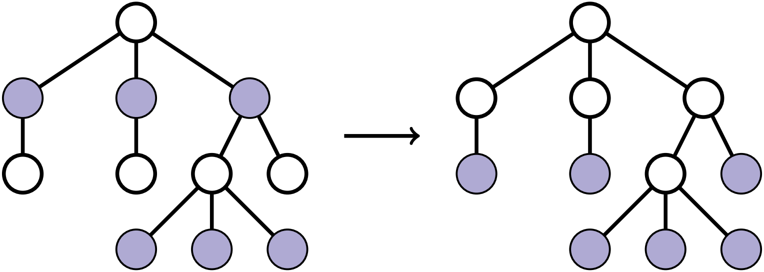

It is safe to take all the leaves.

PartyGreedy(T)

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while T is not empty:
  take all the leaves to the solution
  remove them and their parents from T
return the constructed solution
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Running time: $O(|T|)$ (for each vertex, maintain the number of its children).

Planning a company party

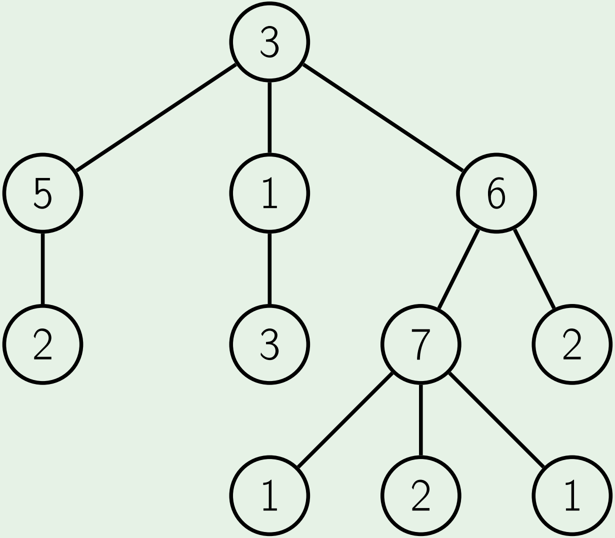
You are organizing a company party again. However this time, instead of maximizing the number of attendees, you would like to maximize the total fun factor.

Maximum weighted independent set in trees

Input: A tree T with weights on vertices.

Output: An independent set (i.e., a subset of vertices no two of which are adjacent) of maximum total weight.

Example



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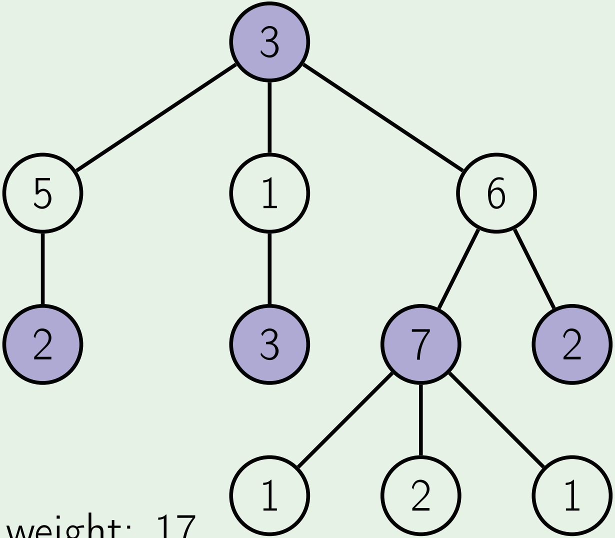
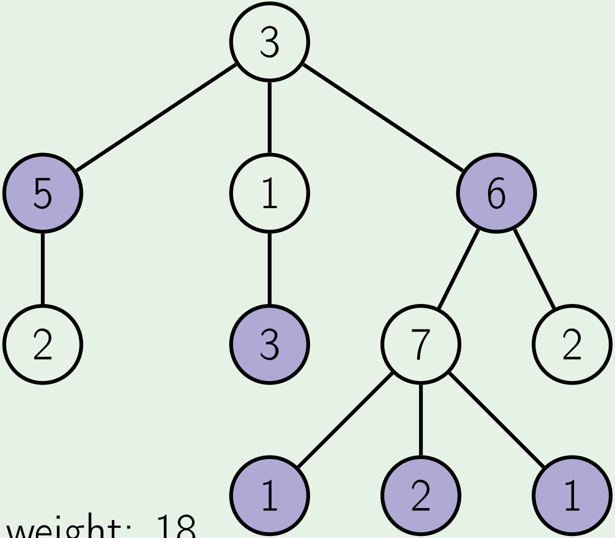


total weight: 17

Example



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Subproblems

- $D(v)$ is the maximum weight of an independent set in a subtree rooted at v

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- $D(v)$ is the maximum weight of an independent set in a subtree rooted at v
- Recurrence relation: $D(v)$ is

$$\max \left\{ w(v) + \sum_{\substack{\text{grandchildren} \\ w \text{ of } v}} D(w), \sum_{\substack{\text{children} \\ w \text{ of } v}} D(w) \right\}$$

Function FunParty(v)

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if D(v) = ∞:
  if v has no children:
    D(v) ← w(v)
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if D(v) = ∞:
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  else:
    m_1 ← w(v)
    for all children u of v:
      for all children w of u:
        m_1 ← m_1 + FunParty(w)
```

Function FunParty(v)

```
if D(v) = ∞:
  if v has no children:
    D(v) ← w(v)
  else:
    m₁ ← w(v)
    for all children u of v:
      for all children w of u:
        m₁ ← m₁ + FunParty(w)
    m₀ ← 0
    for all children u of v:
      m₀ ← m₀ + FunParty(u)
```

Function FunParty(v)

```
if D(v) = ∞:
  if v has no children:
    D(v) ← w(v)
  else:
    m1 ← w(v)
    for all children u of v:
      for all children w of u:
        m1 ← m1 + FunParty(w)
    m0 ← 0
    for all children u of v:
      m0 ← m0 + FunParty(u)
  D(v) ← max(m1, m0)
return D(v)
```

Example