Module-4: Boundary value problem

Definition (Boundary value problem)

A boundary value problem (BVP) consists of an ODE and given boundary conditions referring to the two boundary points (endpoints) $x = a$ and $x = b$ of a given interval $a \leq x \leq b$.

Definition (Strum-Liouville Problem)

A second order homogeneous linear differential equation of the form $\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [q(x) + \lambda r(x)]y = 0$ on some interval $a \leq x \leq b$, satisfying conditions of the form

$$a_1 y(a) + a_2 y^{'}(a) = 0, \;\; b_1 y(b) + b_2 y^{'}(b) = 0 \text{ or}$$

$$y(a) = y(b), y^{'}(a) = y^{'}(b), p(a) = p(b)$$

where p, q and r are real-valued continuous functions on $[a, b]$ and λ is a real parameter is called a **Strum-Liouville Problem.**

Applications of Strum-Liouville Problem

Definition

If a and b are finite and $P(x) \neq 0, \forall x \in [a, b]$, then the BVP is called **regular BVP**. A BVP which is not regular is called a **singular BVP.**

Example Find nontrivial solutions of the Strum-Liouville Problem

$$\frac{d^2y}{dx^2} + \lambda y = 0, y(0) = 0, y^{'}(\pi) = 0.$$

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$$\frac{d^2y}{dx^2} + \lambda y = 0, y(0) = 0, y^{'}(\pi) = 0.$$

Answer $\lambda = \frac{(2n-1)^2}{4}$ and $y = c_n \sin \frac{2n-1}{2} x, \; n = 1, 2, \ldots .$

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Note The values of λ for which the BVP possesses nontrivial solutions are called **eigenvalues** and the corresponding solutions are called **eigenfunctions**.

Orthogonality of eigenfunctions

Example An elastic string of a violin is stretched a little and fixed at its ends $x = 0$ and $x = \pi$ and then allowed to vibrate. For this instance, the following Sturm-Liouville problem arises

$$\frac{d^2y}{dx^2} + \lambda y = 0, \; y(0) = 0, \; y(\pi) = 0$$

Find eigenvalues and eigenfunctions of this problem.

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Example An elastic string of a violin is stretched a little and fixed at its ends $x = 0$ and $x = L/2$ and then allowed to vibrate. For this instance, the following Sturm-Liouville problem arises

$$\frac{d^2y}{dx^2} + \lambda y = 0, \ \ 0 < x < \frac{L}{2}, \ \ y^{'}(0) = 0, y^{'}(\frac{L}{2}) = 0$$

Find eigenvalues and eigenfunctions of this problem.

Orthogonality of eigenfunctions

Definition (Orthogonal functions)

Two distinct continuous function $f(x)$ and $g(x)$ on $[a, b]$ are said to be **orthogonal** with respect to a continuous weight function $r(x)$ if

$$\int_a^b r(x)f(x)g(x) = 0.$$

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Theorem

If $y_m(x)$ and $y_n(x)$ are eigenfunctions of the Strum-Liouville problem corresponding to the distinct eigenvalues λ_m and λ_n respectively, then they are orthogonal with respect to the weight function $r(x)$.

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Note An infinite set of functions defined on $[a,b]$ is said to be an **orthogonal** system with respect to the weight function $r(x)$ on $[a,b]$ if every pair of distinct functions of the set are orthogonal w.r.t. $r(x)$.

Orthogonality of eigenfunctions

Example Show that eigenfunctions of the Sturm-Liouville problem

$$\frac{d^2y}{dx^2} + \lambda y = 0, \;\; y(0) = 0, \;\; y(\pi) = 0$$

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In Strum-Liouville problem (SLP) $\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [q(x) + \lambda r(x)]y = 0$, if $r(a) = r(b)$ with boundary conditions $y(a) = y(b)$ and $y^{'}(a) = y^{'}(b)$ is called periodic SLP.

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$$\frac{d^2y}{dx^2} + \lambda y = 0, y(-\pi) = y(\pi), y^{'}(-\pi) = y^{'}(\pi).$$

Expansion of a function in terms of eigen functions

The eigenfunctions of a Sturm-Liouville problem can be used to describe piecewise continuous functions.
Let $y_0, y_1, y_2, \cdots$ be orthogonal with respect to a weight function $r(x)$ on an interval $a \leqq x \leqq b$, and let $f(x)$ be a piecewise continuous function, then

$$f(x) = \sum_{m=0}^{\infty} a_m y_m(x) = a_0 y_0(x) + a_1 y_1(x) + \cdots$$

This is called an orthogonal series, orthogonal expansion, or generalized Fourier series. If the y_m are the eigenfunctions of a Sturm-Liouville problem, we call this as an eigenfunction expansion.

where

$$a_m = \frac{(f, y_m)}{\|y_m\|^2} = \frac{1}{\|y_m\|^2} \int_a^b r(x) f(x) y_m(x)\, dx \qquad (n = 0, 1, \cdots$$

Here

$$(y_m, y_n) = \int_a^b r(x) y_m(x) y_n(x) dx$$

(y_m, y_n) is a standard notation for this integral.
The norm $\|y_m\|$ of y_m is defined as

$$\|y_m\| = \sqrt{(y_m, y_m)} = \sqrt{\int_a^b r(x) y_m^2(x) dx}$$

Note: For Strum-Liouville Problem,

$$(y_m, y_n) = \int_a^b r(x) y_m(x) y_n(x) dx = \begin{cases} 0 & \text{if} \quad m \neq n \\ 1 & \text{if} \quad m = \boldsymbol{n}. \end{cases}$$

Example: Expand $f(x) = 1$ in terms of $\{\sqrt{2/L}\sin(n\pi x/L)\}_{n=1}^{\infty}$, which are eigenfunctions of the Sturm-Liouville problem

$$y'' + \lambda y = 0, \quad 0 < x < L, \quad y(0) = y(L) = 0.$$

We have

$$f(x) = 1 = \sqrt{\frac{2}{L}} \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L},$$

where

$$\begin{aligned} b_n &= \left\langle \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}, 1 \right\rangle = \sqrt{\frac{2}{L}} \int_0^L \sin \frac{n\pi x}{L} dx \\ &= -\sqrt{\frac{2}{L}} \frac{L}{n\pi} \cos \frac{n\pi x}{L} \Big|_0^L \\ &= -\frac{\sqrt{2L}}{n\pi} \left[(-1)^n - 1\right] \\ &= \frac{\sqrt{2L}}{n\pi} \left[1 - (-1)^n\right] = \begin{cases} 0 & n \text{ even} \\ \frac{2\sqrt{2L}}{n\pi} & n \text{ odd} \end{cases} \end{aligned}$$

This yields

$$f(x) = \sqrt{\frac{2}{L}} \sum_{k=1}^{\infty} \frac{2\sqrt{2L}}{(2k-1)\pi} \sin \frac{(2k-1)\pi x}{L} = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{2k-1} \sin \frac{(2k-1)\pi x}{L}.$$

Example: Expand $f(x) = x$ in terms of $\{\sqrt{1/L}\} \cup \{\sqrt{2/L}\cos(n\pi x/L)\}_{n=1}^{\infty}$, which are eigenfunctions of the Sturm-Liouville problem

$$y'' + \lambda y = 0, \quad 0 < x < L, \quad y'(0) = y'(L) = 0.$$

We have

$$f(x) = x = \sqrt{\frac{1}{L}} a_0 + \sqrt{\frac{2}{L}} \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

where

$$a_0 = \left\langle \sqrt{\frac{1}{L}}, x \right\rangle = \sqrt{\frac{1}{L}} \int_0^L x dx = \sqrt{\frac{1}{L}} \frac{L^2}{2} = \frac{L^{3/2}}{2}$$

For $n = 1, 2, \ldots,$

$$
\begin{aligned}
a_n &= \left\langle \sqrt{\frac{2}{L}} \cos \frac{n\pi x}{L}, x \right\rangle \\
&= \sqrt{\frac{2}{L}} \int_0^L x \cos \frac{n\pi x}{L} dx \\
&= \sqrt{\frac{2}{L}} \left[\frac{L}{n\pi} x \sin \frac{n\pi x}{L} \Big|_0^L - \frac{L}{n\pi} \int_0^L \sin \frac{n\pi x}{L} dx \right] \\
&= \sqrt{\frac{2}{L}} \frac{L^2}{(n\pi)^2} \cos \frac{n\pi x}{L} \Big|_0^L \\
&= \sqrt{\frac{2}{L}} \frac{L^2}{(n\pi)^2} \left[(-1)^n - 1 \right] \\
&= \begin{cases} 0 & n \text{ even} \\ -\frac{2L\sqrt{2L}}{(n\pi)^2} & n \text{ odd} \end{cases}
\end{aligned}
$$

We conclude that

$$f(x) = \sqrt{\frac{1}{L}}\frac{L^{3/2}}{2} - \sqrt{\frac{2}{L}}\sum_{k=1}^{\infty}\frac{2L\sqrt{2L}}{((2k-1)\pi)^2}\cos\frac{(2k-1)\pi x}{L}$$

$$= \frac{L}{2} - \frac{4L}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos\frac{(2k-1)\pi x}{L}.$$

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Definition (Power series)

A power series is a series of the form

$$\sum_{m=0}^{\infty} a_m (x - x_0)^m = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots$$

where x is a variable and a_i are constants, called the coefficients of the series. x_0 is a constant, called the center of the series.

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In particular, if $x_0 = 0$, $\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \cdots$.

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Some standard Maclaurin series

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Definition (Analytic function)

A function $f(x)$ which can be expanded in Taylor's series on interval containing the point x_0. The series converges to $f(x)$ for all x in the interval of convergence.

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Definition

A point x_0 is called an **Ordinary point** of the differential equation if both $P(x)$ and $Q(x)$ are analytic at x_0. If one (or both) of these functions is not analytic at x_0 then x_0 is called a **singular point** of the differential equation.

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$(x-2)\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} + (x^2-2x)y = 0$

Existence of Power series solutions

Operations of Power series

1. **Termwise Addition**

$\sum_{n=0}^{\infty} a_n(x-x_0)^n + \sum_{n=0}^{\infty} b_n(x-x_0)^n = \sum_{n=0}^{\infty}(a_n+b_n)(x-x_0)^n$

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2. **Termwise Multiplication**

$\sum_{n=0}^{\infty} a_n(x-x_0)^n \times \sum_{n=0}^{\infty} b_n(x-x_0)^n =$
$a_0b_0 + (a_0b_1 + a_1b_0)(x-x_0) + (a_0b_2 + a_1b_1 + a_2b_0)(x-x_0)^2 + \cdots$

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3. **Termwise Differentiation**

$y(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$, $y^{'}(x) = \sum_{n=1}^{\infty} na_n(x-x_0)^{n-1}$,
$y^{''}(x) = \sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2}$

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Theorem

Let x_0 be an ordinary point of the differential equation, then it has two non-trivial linearly independent solutions of the form $\sum_{n=0}^{\infty} a_n(x-x_0)^n$.

Example Solve $y' - y = 0$ by power series method.

Power series solutions

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Answer $y = a_0 \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) = a_0 e^x$

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Answer

Example Solve $y'' - 2xy' + y = 0, y(0) = 0, y'(0) = 1$ by power series method.

Power series

Definition

A power series is **convergent** at a specified value of x if its sequence of partial sum $S_N(x)$ converges, that is,
$\lim_{N\to\infty} S_N(x) = \lim_{N\to\infty} \sum_{n=0}^{N} c_n(x - x_0)^n$ exists.

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Definition

Every power series has an **interval of convergence.** The interval of convergence is the set of all real numbers x for which the series convergences. The center of the interval of the convergence is the center x_0 of the series.

Power series

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The radius R of the interval of convergence a power series is called its **radius of convergence**. If $R > 0$, then the power series convergences for $|x - x_0| < R$ and diverges for $|x - x_0| > R$.

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Ratio test: $R = \frac{1}{\lim\limits_{n\to\infty} |\frac{a_{n+1}}{a_n}|}$ **Root test:** $R = \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{|a_n|}}$

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The radius of convergence can be determined from the coefficients of the series

Ratio test: $R = \frac{1}{\lim\limits_{n\to\infty} |\frac{a_{n+1}}{a_n}|}$ **Root test:** $R = \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{|a_n|}}$

Example Find the radius of convergence and circle of convergence.

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♣ The power series may or may not converge at the endpoints $x_0 - R$ and $x_0 + R$ of the interval.

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Note: If n is an even integer ($n=2$), $y_1(x)$ terminates, but $y_2(x)$ is an infinite series.

$y_1(x) = c_0\left[1 - \frac{2.3}{2!}x^2\right] = c_0\left[1 - 3x^2\right]$

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Similarly, when n is an odd integer, the series for $y_2(x)$ terminates with x^n, we obtain an nth-degree polynomial solution of Legendre's equation.

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Definition (Legendre polynomial)

The resulting solution of Legendre's differential equation is called **Legendre polynomial** of degree n and are denoted by $P_n(x)$.

$$P_n(x) = \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!} x^{n-2m}$$
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where $M = n/2$ or $(n-1)/2$, whichever is an integer.

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$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = \frac{1}{2}(3x^2 - 1),$
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Question Plot the Legendre's polynomial $P_i(x), i = 1, 2, 3, 4.$

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(hint take $v = (x^2-1)^n$ and show that v is a solution of Legendre's equation, then use the condition $P_n(1) = 1$.)

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If P_n is the Legendre polynomial of degree n, then
(a) $\int_{-1}^{1} P_m(x)P_n(x)dx = 0$ *if* $m \neq n$.

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Legendre's equation

1. Find the solution of differential equation $(1-x^2)y^{''}-2xy^{'}+12y=0$.
2. Show that the differential equation

$$\sin\theta\frac{d^2y}{d\theta^2}+\cos\theta\frac{dy}{d\theta}+n(n+1)(\sin\theta)y=0$$

can be transformed into Legendre's equation by means of the substitution $x=\cos\theta$ and write the solution.

Fourier Legendre Series

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The Fourier-Legendre series is an eigen function expansion of given function $f(x)$ on the interval $-1 \leq x \leq 1$ w.r.t. the weight function $P(x) = 1$ and is given by

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Theorem

If x_0 is a regular singular point of the differential equation, then it has at least one non-trivial solution of the form $\sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$, where the number r is a constant to be determined. The series will converge at least on some interval $0 < x - x_0 < R$.

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$Case - 1$ If r_1 and r_2 are **distinct and** $r_1 - r_2$ **is not an integer**, then there exist two linearly independent solutions $y_1(x) = x^{r_1}(c_0 + c_1 x + c_2 x^2 + \cdots)$ and $y_2(x) = x^{r_2}(C_0 + C_1 x + C_2 x^2 + \cdots)$

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$Case - 2$ If r_1 and r_2 are **distinct and** $r_1 - r_2$ **is an integer**, then there exist two linearly independent solutions $y_1(x) = x^{r_1}(c_0 + c_1 x + c_2 x^2 + \cdots)$ and $y_2(x) = k y_1(x) \ln x + x^{r_2}(C_0 + C_1 x + C_2 x^2 + \cdots)$, where k is a constant that could be zero.

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Case -3 If r_1 and r_2 are **equal** $(r_1 = r_2 = r)$, then there exist two linearly independent solutions $y_1(x) = x^r(c_0 + c_1 x + c_2 x^2 + \cdots)$ and
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(iii) $x(x-1)y'' + (3x-1)y' + y = 0$

(iV) $9x(1-x)y'' - 12y' + 4y = 0$

Bessel's equation

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<u>**Bessel Functions $J_n(x)$ for integer $v = n$**</u>

$a_{2m} = \frac{(-1)^m a_0}{2^{2m} m!(n+1)(n+2)\cdots(n+m)}$ $m = 1, 2, \cdots$. Let $a_0 = \frac{1}{2^n n!}$,

$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!(n+m)!}$ $n \geq 0$.
$J_n(x)$ is called the **Bessel function of first kind** of order n.

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Case-3 If 2ν is not an odd positive integer, then J_{-v} and J_{-v} are linear dependent. So the general solution of Bessel's equation is $y(x) = C_1 J_v(x) + C_2 Y_v(x)$, where $Y_v(x) = \frac{J_v(x)\cos v\pi - J_{-v}(x)}{\sin v\pi}$.
$Y_v(x)$ is called the **Bessel function of the second kind** of order v.

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Question Prove that $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$ and $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$ (use $\Gamma(\frac{1}{2}) = \sqrt{\pi}$).

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Question Calculate the value of $J_{3/2}, J_{-3/2}, J_{5/2}, J_{-5/2}$ etc.

Application of Bessel's equation

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Fourier-Bessel Series

Definition

The Fourier-Bessel series is an orthogonal expansion of a given function $f(x)$ defined on the interval $0 \leq x \leq R$ w.r.t. the weight function $P(x) = x$ and in terms of orthogonal Bessel function $J_n(K_{1n}x), J_n(K_{2n}x), \ldots$ where n is fixed and RK_{mn} are the zeros of the Bessel functions. (i.e. $J_n(RK_{mn}) = 0$), it is given by

$$f(x) = \sum_{m=1}^{\infty} a_m J_n(K_{mn}x)$$

. Here $a_m = \frac{2}{R^2 J_{n+1}^2(K_{mn}R)} \int_0^R x.f(x).J_n(K_{mn}x)dx,\ m = 1, 2, 3\ldots$

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Example Find $f(x) = x^2$, in Fourier-Bessel series in terms of Bessel functions of order zero($n = 0$) and w.r.t. the weight function $P(x) = x$ over the interval $0 < x < R$.

Fourier-Bessel Series

Answer $f(x) = a_1 J_0(K_{10}x) + a_2 J_0(K_{20}x) + a_3 J_0(K_{30}x) + \cdots$

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Note If the fundamental period of the is T, then
fundamental frequency is $f_0 = \frac{1}{T}$,
fundamental angular frequency is $\omega_0 = 2\pi f_0$,

Practice set

1. Find the steady state current $I(t)$ in the RLC-circuit, where $R = 10\Omega, L = 1H, C = 10^{-1}F$ and $E(t) = \begin{cases} -2, -\pi < t < 0 \\ 2,\ 0 < t < \pi \end{cases}$.
   ($E(t)$ is a periodic function of period 2π).
2. Use the change of variables $u = \frac{2}{3}e^{-\frac{1}{2}(3t-2)}$ to show that the differential equation of the aging spring $y''(t) + e^{-3t+2}y(t) = 0$, becomes Bessel's equation and hence write the solution.
3. A spring for which stiffness $k = 12\text{N/m}$ hangs in a vertical position with its upper end fixed. A mass of 3kg is attached to the lower end. The standard form of mass spring system is given by the differential equation $m\frac{d^2x}{dt^2} + kx = 0$, find the power series solution of resulting motion of the mass by neglecting air resistance.