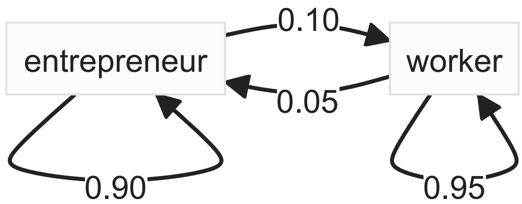


Markov chain

States $s = 1, 2, \dots K$.

Probability distribution across states:

probability *mass* function pmf,

$$\pi_{kt} := \Pr(x_t = k)$$

(as a special case, $\pi_{kt} = 1$ for some k)

KxK transition matrix $\mathbf{P}$. Start from π_0.

$$\Pr(x_1 = k | x_0 \sim \pi_0) = \pi_1$$

$$\mathbf{P} = \begin{bmatrix} 0.9 & 0.1 \\ 0.05 & 0.95 \end{bmatrix}$$

$$\Pr(x_1 = k | x_0 = 1) = (1 \quad 0)\mathbf{P}$$

$$\Pr(x_1 = k | x_0 = 2) = (0 \quad 1)\mathbf{P}$$

If we don't know where we started from, only π_0. Then with π_{01} we were entrepreneurs and

$$\pi_{01} \times (1 \quad 0)\mathbf{P},$$

with complementary

$$\pi_{02} \times (0 \quad 1)\mathbf{P}.$$

Both can happen, so the overall probability of ending up in state k in period 1

$$(\pi_{01} \quad \pi_{02})\mathbf{P} = \pi_0 \mathbf{P}$$

Suppose we only know that e had 50% prob

$$(0.5\, 0.5)\mathbf{P} = (0.475\, 0.525),$$

generically

$$\pi_t \mathbf{P} = \pi_{t+1}$$

Properties of P

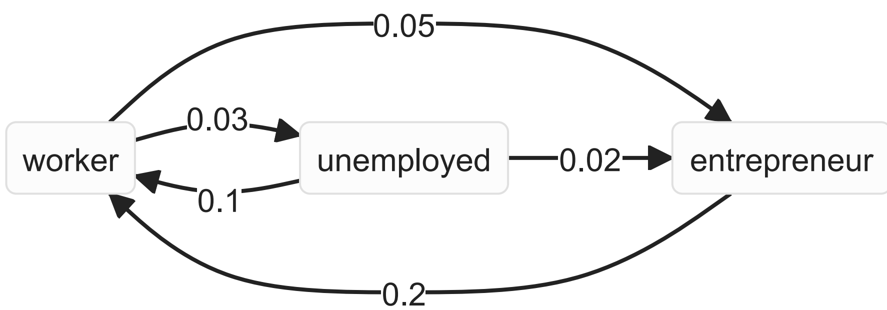
Remember, elements are conditional probabilities.

1. KxK square
2. $\mathbf{P}_{ij} \geq 0$
3. $\mathbf{P}_{ij} \leq 1$
4. $\sum_{j=1}^{K} \mathbf{P}_{ij} = 1$ for all $i = 1, \ldots, K$.

Matrices with these properties are called *stochastic*.

All stochastic matrices are valid transition matrices.

2023-11-22



Our goal today is to characterize how state changes over time and whether there is "stability."

1. Where will we be in T periods?
2. Will we have reached E by period 3?
3. Are there any states never reached? Are all states reached eventually?
4. As T grows large, "where will I be"? Is there an answer to this question?

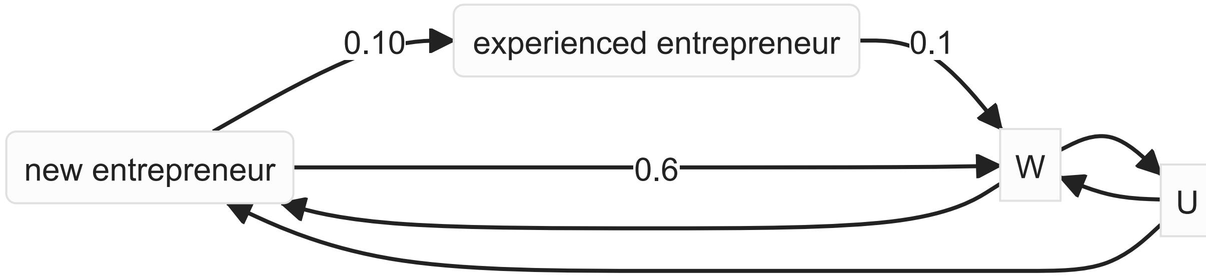
The transition matrix P has the edges of the graph, P_{ij} as well as the diagonal elements:

$$P_{ii} = 1 - \sum_{j \neq i} P_{ij}$$

For example above,

$$\begin{bmatrix} 0.92 & 0.03 & 0.05 \\ 0.10 & 0.88 & 0.02 \\ 0.20 & 0.00 & 0.80 \end{bmatrix}$$

> Can Es learn from experience? Yes, but we need more states



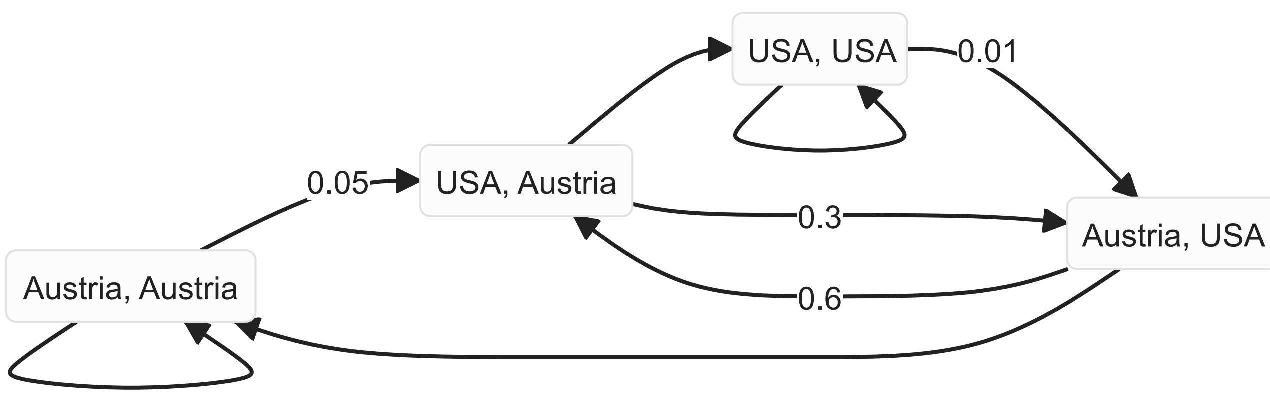
> Can dynamics be second- or higher order? Yes, for example migration.

First-order migration flow could be characterized as

The second move is much higher chance to be "back to" original country. Better characterized by 2nd-order Markov chain. "country of birth"

$$\Pr(x_t = k | x_{t-1}, x_{t-2})$$

I could record "current country" and "country last year"



All of this can be represented by a KxK transition matrix P.

An "ergodic set" of states???

Different definitions of "long run"

$$\lim_{t \to \infty} \pi_0 \mathbf{P}^t =? = \pi_\infty$$

Subquestions

1. Does π_∞ exist?
2. Is it independent of π_0?

A steady state

$$\pi_* = \pi_* \mathbf{P} = \pi_* \mathbf{P}^t \text{ for any } t \geq 1$$

Unlike the first, this **always exists**. May not be unique, though.

Let's transpose

$$\pi'_* = \mathbf{P}' \pi'_*$$

This is an eigenvalue equation,

$$\mathbf{x} = \mathbf{A}\mathbf{x}$$

when 1 is an eigenvalue.

> Question: Does $\mathbf{P}'$ have an eigenvalue of 1?

Proof:

$$\det(\mathbf{P}' - \mathbf{I}) = 0?$$

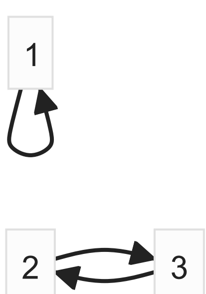
$$\det(\mathbf{P} - \mathbf{I}) = 0?$$

Yes, because each row of P sums to 1, so that each row of $P - I$ sums to zero.

$$\mathbf{P1} = \mathbf{1}$$

If so, π_* is the eigenvector corresponding to $\lambda = 1$ of $\mathbf{P}'$.

Counterexample on unicity:



> Theorem: if $[\mathbf{P}^t]_{ij} > 0$ for some finite t for all i and j, then
>
> 1. π_* is unique
> 2. π_∞ exists as a limit
> 3. independent of π_0
> 4. $\pi_\infty = \pi_*$

> Graph representation: each state can be reached from every other state in finite steps.