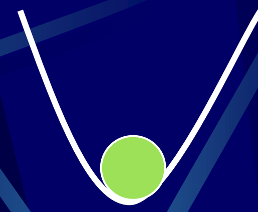
# 第五章　微振动(Small Oscillations)

振动分类

- 满足微分方程
  - 线性
  - 非线性
- 自由度
  - 单自由度
  - 多单自由度
- 能量是否守恒
  - 自由振动
  - 非自由振动

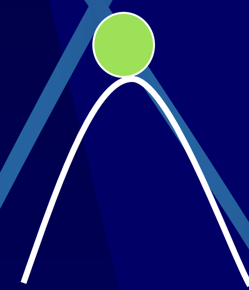
## 一．平衡的种类

稳定平衡



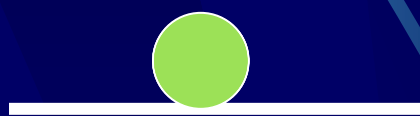
$V(x_0)=V_{min}$

非稳定平衡



$V(x_0)=V_{max}$

随遇平衡



$V(x)=const.$

$$F(x_0)=-\left.\frac{dV}{dx}\right|_{x_0}=0$$

$$F(x_0+\varepsilon)=-\left.\frac{dV}{dx}\right|_{x_0}-\left(\frac{d^2V}{dx^2}\right)_{x_0}\varepsilon+\ldots\approx-\left(\frac{d^2V}{dx^2}\right)_{x_0}\varepsilon$$

## 二. 平衡的判定

单自由度 $q \Rightarrow$ 广义坐标

$q_0 \Rightarrow$ 平衡位形

$$V(q) = V(q_0) + \left(\frac{dV}{dq}\right)_{q_0}(q - q_0) + \frac{1}{2}\left(\frac{d^2V}{dq^2}\right)_{q_0}(q - q_0)^2 + \ldots$$

参考值

=0 势能极值条件

$$F = -\frac{dV}{dq} = -\left(\frac{d^2V}{dq^2}\right)_{q_0}\varepsilon \qquad \varepsilon \equiv q - q_0$$

# Discussion:

$$F = -\frac{dV}{dq} = -\left(\frac{d^2V}{dq^2}\right)_{q_0} \varepsilon$$

- 如果 $\left(\frac{d^2V}{dq^2}\right)_{q_0} > 0 \begin{cases} \varepsilon > 0 \quad F < 0 \\ \varepsilon < 0 \quad F > 0 \end{cases}$ ⇨ 稳定平衡
- 如果 $\left(\frac{d^2V}{dq^2}\right)_{q_0} < 0 \begin{cases} \varepsilon > 0 \quad F > 0 \\ \varepsilon < 0 \quad F < 0 \end{cases}$ ⇨ 非稳定平衡
- 如果 $\left(\frac{d^2V}{dq^2}\right)_{q_0} = 0 \quad F = 0$ ⇨ 随遇平衡?

$$V(q_1\ q_2\ q_3 \ldots\ldots q_s) = V(0\,0\,0 \ldots 0) + \left(\frac{\partial V}{\partial q_\alpha}\right)_0 q_\alpha + \frac{1}{2}\left(\frac{\partial^2 V}{\partial q_\alpha \partial q_\beta}\right)_{00} q_\alpha q_\beta + \ldots\ldots$$

$$(\alpha\ \beta = 1,2,3,\ldots\ldots S)$$

多自由度系统

如 $\left(\frac{\partial^2 V}{\partial q_\alpha \partial q_\beta}\right)_0 > 0$ ⇒ 稳定平衡

如 $\left(\frac{\partial^2 V}{\partial q_\alpha \partial q_\beta}\right)_0 < 0$ ⇒ 非稳定平衡

如 $\left(\frac{\partial^2 V}{\partial q_\alpha \partial q_\beta}\right)_0 = 0$ ⇒ 随遇平衡?

## 三. 一维简谐振动

取平衡位形$q_0 = 0$

$$V(q) = V(0) + \left(\frac{dV}{dq}\right)_0 q + \frac{1}{2}\left(\frac{d^2V}{dq^2}\right)_0 q^2 = \frac{1}{2}kq^2 \Longleftarrow \boxed{k \equiv \left(\frac{d^2V}{dq^2}\right)_0}$$

$$T = \frac{1}{2}a(q)\dot{q}^2 \qquad T = \frac{1}{2}m\dot{q}^2$$

$$\approx \frac{1}{2}a(0)\dot{q}^2 \qquad m \equiv a(0)$$

$$L = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2$$

$$\boxed{\ddot{q} + \omega^2 q = 0} \qquad \omega = \sqrt{\frac{k}{m}} \qquad q = A\cos(\omega t + \theta)$$

# 四. 多自由度系统谐振动

设系统自由度为S　取平衡位形为(0000......)

$$V(q_1.q_2.q_3....q_s)=V(0.0.0....0)+(\frac{\partial V}{\partial q_\alpha})_{0...0}q_\alpha$$

$$+.....................+\frac{1}{2}(\frac{\partial^2 V}{\partial q_\alpha \partial q_\beta})_{0.......0}q_\alpha q_\beta$$

$$(\alpha\ \beta=1.2.3...S)$$

等效弹性系数

令$k_{\alpha\beta}\equiv(\frac{\partial^2 V}{\partial q_\alpha \partial q_\beta})_{0......0}$

$$k_{\alpha\beta}=k_{\beta\alpha}$$

$$V(q_1 \ldots\ldots q_s) = \frac{1}{2} k_{\alpha\beta} q_\alpha q_\beta$$

$$V(q_1 \ldots\ldots q_s) = \frac{1}{2}\left(q_1\, q_2 \ldots\ldots q_s\right) \begin{bmatrix} k_{11} & k_{12} & k_{13} \ldots\ldots k_{1s} \\ k_{21} & k_{22} & k_{23} \ldots\ldots k_{2s} \\ \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ k_{s1} & k_{s2} & k_{s3} \ldots\ldots k_{ss} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_s \end{bmatrix}$$

$$k_{\alpha\beta} = k_{\beta\alpha}$$

实对称矩阵

$$T=\frac{1}{2}a_{\alpha\beta}\dot{q}_{\alpha}\dot{q}_{\beta} \qquad (\alpha.\beta=1.2.3\ldots\ldots S)$$

$$\because a_{\alpha\beta}=m_i\frac{\partial\vec{r}_i}{\partial q_{\alpha}}\frac{\partial\vec{r}_i}{\partial q_{\beta}}$$

$$\boxed{\therefore a_{\alpha\beta}=a_{\beta\alpha}}$$

实对称矩阵

$$(\,i=1.2.3\ldots\ldots N\,)$$

$$a_{\alpha\beta}\approx a_{\alpha\beta}(0.0.0\ldots.0)+\left(\frac{\partial a}{\partial q_{\alpha}}\right)_0 q_{\alpha}+\ldots..$$

$$\boxed{\text{令}\, a_{\alpha\beta}(0.0.0\ldots.0)=m_{\alpha\beta}}$$

⇐ 准惯性系数矩阵

$$T=\frac{1}{2}a_{\alpha\beta}\dot{q}_{\alpha}\dot{q}_{\beta}$$

$$T=\frac{1}{2}m_{\alpha\beta}\dot{q}_{\alpha}\dot{q}_{\beta}$$

$$m_{\alpha\beta}=m_{\beta\alpha}$$

$$T=\frac{1}{2}(\dot{q}_1\ \dot{q}_2\ldots\ldots\dot{q}_s)\begin{pmatrix} m_{11} & m_{12} & m_{13}\ldots\ldots m_{1s} \\ m_{21} & m_{22} & m_{23}\ldots\ldots m_{2s} \\ \ldots & \ldots & \ldots\ldots\ldots\ldots \\ m_{s1} & m_{s2} & m_{s3}\ldots\ldots m_{ss} \end{pmatrix}\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_s \end{pmatrix}$$

$$L=\frac{1}{2}m_{\alpha\beta}\dot{q}_{\alpha}\dot{q}_{\beta}-\frac{1}{2}k_{\alpha\beta}q_{\alpha}q_{\beta}$$

$$(\alpha\ \beta=1.2.3\ldots\ldots s)$$

$$L=\frac{1}{2}m_{\alpha\beta}\dot{q}_{\alpha}\dot{q}_{\beta}-\frac{1}{2}k_{\alpha\beta}q_{\alpha}q_{\beta}$$

$$\frac{\partial L}{\partial \dot{q}_{\sigma}}=\quad m_{11}\dot{q}_{1}\dot{q}_{1}+m_{12}\dot{q}_{1}\dot{q}_{2}+\ldots\ldots m_{1s}\dot{q}_{1}\dot{q}_{s}$$

$$+m_{21}\dot{q}_{2}\dot{q}_{1}+m_{22}\dot{q}_{2}\dot{q}_{2}+\ldots\ldots m_{2s}\dot{q}_{2}\dot{q}_{s}$$

$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{\partial L}{\partial \dot{q}_{\sigma}}=+m_{\sigma 1}\dot{q}_{\sigma}\dot{q}_{1}+m_{\sigma 2}\dot{q}_{\sigma}\dot{q}_{2}+\ldots\ldots m_{\sigma s}\dot{q}_{\sigma}\dot{q}_{s}$$

$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{\partial L}{\partial \dot{q}_{\sigma}}=+m_{s1}\dot{q}_{s}\dot{q}_{1}+m_{s2}\dot{q}_{s}\dot{q}_{2}+\ldots\ldots m_{ss}\dot{q}_{s}\dot{q}_{s}$$

$$L=\frac{1}{2}m_{\alpha\beta}\dot{q}_{\alpha}\dot{q}_{\beta}-\frac{1}{2}k_{\alpha\beta}q_{\alpha}q_{\beta}$$

同理可得

$$\frac{\partial L}{\partial q_{\sigma}}=-k_{\sigma\beta}q_{\beta}\quad(\beta=1\,2\,3......s)$$

$$\frac{\partial L}{\partial \dot{q}_{\sigma}}=m_{\sigma\beta}\dot{q}_{\beta}$$

$q_{\beta}$满足拉格朗日方程

$$m_{\sigma\beta}\ddot{q}_{\beta}+k_{\sigma\beta}q_{\beta}=0\quad(\beta=1\,2\,3......s)$$

求和与哑指标无关

$$m_{\alpha\beta}\ddot{q}_{\beta}+k_{\alpha\beta}q_{\beta}=0\quad......(1)$$

$$m_{\alpha\beta}\ddot{q}_{\beta}+k_{\alpha\beta}q_{\beta}=0 \qquad \text{......(1)}$$

关于S个q的线性齐次二阶微分方程组

$$\begin{cases} m_{11}\ddot{q}_1+m_{12}\ddot{q}_2\ldots+m_{1s}\ddot{q}_s+k_{11}q_1+k_{12}q_2+\ldots k_{1s}q_s=0 \\ m_{21}\ddot{q}_1+m_{22}\ddot{q}_2\ldots+m_{1s}\ddot{q}_s+k_{21}q_1+k_{22}q_2+\ldots k_{2s}q_s=0 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ m_{s1}\ddot{q}_1+m_{s2}\ddot{q}_2\ldots+m_{ss}\ddot{q}_s+k_{s1}q_1+k_{s2}q_2+\ldots k_{ss}q_s=0 \end{cases}$$

$$m_{\alpha\beta}\ddot{q}_{\beta}+k_{\alpha\beta}q_{\beta}=0 \quad \ldots\ldots(1)$$

$$(\beta=1\ 2\ 3\ldots s)$$

设它的解形式为

$$q_{\beta}=e^{i\omega t}B_{\beta} \quad \ldots\ldots(2)$$

$\omega, B_{\beta}$为待定常数。

本征矢方程

$$B_{\beta}(k_{\alpha\beta}-\omega^{2}m_{\alpha\beta})=0 \quad \ldots\ldots(3)$$

本征矢　本征值

$$(\beta=1\ 2\ 3\ldots\ldots s)$$

$$B_\beta(k_{\alpha\beta}-\omega^2 m_{\alpha\beta})=0 \qquad ......(3)$$

$$q_\beta = e^{i\omega t} B_\beta \qquad ......(2)$$

$$\begin{cases} (k_{11}-\omega^2 m_{11})B_1+(k_{12}-\omega^2 m_{12})B_2+.....+(k_{1s}-\omega^2 m_{1s})B_s=0 \\ (k_{21}-\omega^2 m_{21})B_1+(k_{22}-\omega^2 m_{22})B_2+.....+(k_{2s}-\omega^2 m_{2s})B_s=0 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ (k_{s1}-\omega^2 m_{s1})B_1+(k_{s2}-\omega^2 m_{s2})B_2+.....+(k_{ss}-\omega^2 m_{ss})B_s=0 \end{cases}$$

$$B_\beta(k_{\alpha\beta}-\omega^2 m_{\alpha\beta})=0 \qquad \ldots\ldots(3)$$

欲使上式有非零解 系数行列式为零

$$\left\| k_{\alpha\beta}-\omega^2 m_{\alpha\beta} \right\|=0 \qquad \ldots\ldots(4)$$

**关于S个本征值ω²的本征值方程**

$$\begin{vmatrix} (k_{11}-\omega^2 m_{11}) & (k_{12}-\omega^2 m_{12}) & \ldots\ldots & (k_{1s}-\omega^2 m_{1s}) \\ (k_{21}-\omega^2 m_{21}) & (k_{22}-\omega^2 m_{22}) & \ldots\ldots & (k_{2s}-\omega^2 m_{2s}) \\ \cdots & \cdots & \cdots & \cdots \\ (k_{s1}-\omega^2 m_{s1}) & (k_{s2}-\omega^2 m_{s2}) & \ldots\ldots & (k_{ss}-\omega^2 m_{ss}) \end{vmatrix}=0$$

对于每一根$\omega_{\mathrm{i}}$，

可以得到一组常数$B_\beta^i$为待定常数。

$$\Downarrow \quad \left\| k_{\alpha\beta} - \omega^2 m_{\alpha\beta} \right\| = 0$$

$$(k_{11} - \omega_i^{\ 2} m_{11}) B_1^i + (k_{12} - \omega_i^{\ 2} m_{12}) B_2^i + \ldots\ldots + (k_{1s} - \omega_i^{\ 2} m_{1s}) B_s^i = 0$$

$$(k_{21} - \omega_i^{\ 2} m_{21}) B_1^i + (k_{22} - \omega_i^{\ 2} m_{22}) B_2^i + \ldots\ldots + (k_{2s} - \omega_i^{\ 2} m_{2s}) B_s^i = 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(k_{s1} - \omega_i^{\ 2} m_{11}) B_1^i + (k_{12} - \omega_i^{\ 2} m_{s2}) B_2^i + \ldots\ldots + (k_{ss} - \omega_i^{\ 2} m_{ss}) B_s^i = 0$$

$$(k_{11}-\omega_i^2 m_{11})B_1+(k_{12}-\omega_i^2 m_{12})B_2+\ldots\ldots+(k_{1s}-\omega_i^2 m_{1s})B_s=0$$

$$(k_{21}-\omega_i^2 m_{21})B_1+(k_{22}-\omega_i^2 m_{22})B_2+\ldots\ldots+(k_{2s}-\omega_i^2 m_{2s})B_s=0$$

......................................................................

$$(k_{s1}-\omega_i^2 m_{11})B_1+(k_{12}-\omega_i^2 m_{s2})B_2+\ldots\ldots+(k_{ss}-\omega_i^2 m_{ss})B_s=0$$

假定不出现重根

$$B^i_\beta(k_{\alpha\beta}-\omega_i^2 m_{\alpha\beta})=0 \qquad \ldots\ldots(3)$$

$(B_1^i\ \ B_2^i\ \ B_3^i\ldots\ldots B_s^i)$，s-1个独立，因系数矩阵的秩小于s。

在s个$B^i_\beta$中有一个任意常数 假定 $B_s^i$

$$B_{\beta}^{i}(k_{\alpha\beta}-\omega_{i}^{2}m_{\alpha\beta})=0 \qquad \text{......(3)}$$

**挑出第S项**

$$B_{\beta}^{i'}(k_{\alpha\beta}-\omega_{i}^{2}m_{\alpha\beta})=\omega_{i}^{2}m_{\alpha s}-k_{\alpha s} \text{......(5)}$$

(5)为s -1个非齐次线性代数方程。

$$B_{\beta}^{i'}=\frac{B_{\beta}^{i}}{B_{s}^{i}} \qquad (\alpha\ \beta=1\ 2\ 3......s-1)$$

**假定ω²不出现重根,**

$$B_{\beta}^{i'}=\frac{\Delta_{\beta}(\omega_{i}^{2})}{\Delta_{s-1}(\omega_{i}^{2})}$$

$\Delta_{s-1}(\omega_i^2)$的$\beta$列顺次替换为$(m_{\alpha s}\omega_i^2-k_{\alpha s})$所得行列式。

方程组(5)的系数行列式。

$$B_\beta^{i'} = \frac{\Delta_\beta(\omega_i^2)}{\Delta_{s-1}(\omega_i^2)} \quad + \quad B_\beta^{i'} = \frac{B_\beta^i}{B_s^i}$$

$$\Downarrow$$

$$q_\beta = e^{i\omega t} B_\beta \leftarrow B_\beta^i = B_s^i B_\beta^{i'} = B_s^i \frac{\Delta_\beta(\omega_i^2)}{\Delta_{s-1}(\omega_i^2)}$$

$$\boxed{q_\beta = c(\omega_i^2)\Delta_\beta(\omega_i^2)e^{i\omega t}}$$

$$\downarrow$$

$$c(\omega_i^2) = \frac{B_s^i}{\Delta_{s-1}(\omega_i^2)}$$

本征值ω必为实数:

$$B_\beta(k_{\alpha\beta}-\omega^2 m_{\alpha\beta})=0 \qquad (\beta=123.....s)$$

$$B_\beta(k_{\alpha\beta}-\omega^2 m_{\alpha\beta})B_\alpha^*=0 \qquad (\alpha\ \beta=123......s)$$

$$\omega^2=\frac{(k_{\alpha\beta}B_\alpha^* B_\beta)}{m_{\alpha\beta}B_\alpha^* B_\beta}$$

$$\because (k_{\alpha\beta}B_\alpha^* B_\beta)^*=k_{\alpha\beta}B_\alpha B_\beta^* \Leftarrow \because k_{\beta\alpha}=k_{\alpha\beta}$$

$$=k_{\beta\alpha}B_\beta B_\alpha^*=k_{\alpha\beta}B_\alpha^* B_\beta$$

$$(k_{\alpha\beta}B_\beta B_\alpha^*)^* = k_{\alpha\beta}B_\beta B_\alpha^*$$

$(k_{\alpha\beta}B_\beta B_\alpha^*)$必为实数 $\quad \because B_\beta$是振幅$>0$

$$\omega^2 = \frac{(k_{\alpha\beta}B_\beta B_\alpha^*)}{m_{\alpha\beta}B_\alpha^* B_\beta}\text{必为实数}$$

$\pm\omega$均为本征值方程的解

$$q_\beta = c(\omega_i^2)\Delta_\beta(\omega_i^2)e^{i\omega t}$$

$$q_\beta = C_i e^{i\omega_i t} + C_i^{'} e^{-i\omega_i t} \quad (i = 1\,2\,3\ldots\ldots s)$$

$$q_\beta = (C_i e^{i\omega_i t} + C_i' e^{-i\omega_i t})\Delta_\beta(\omega_i^2) \quad (i = 1\,2\,3\ldots\ldots s)$$

取实部

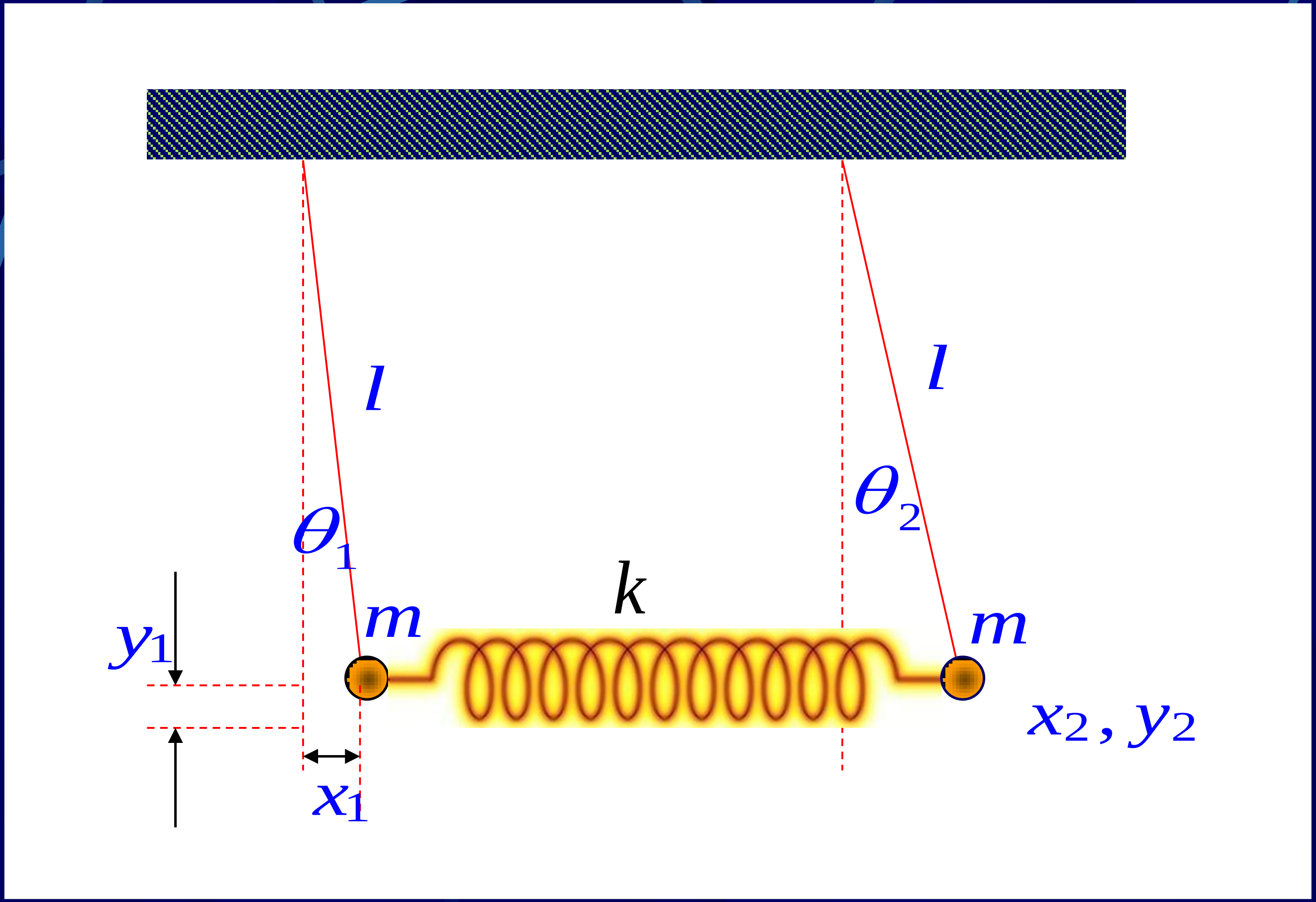
$$q_\beta = D_i \Delta_\beta(\omega_i^2)\cos(\omega_i t + \varphi_i) \quad (i = 123\ldots\ldots s)$$

$q_\beta$随时间的变化是复杂振动:

相同的频谱$\omega_{1,}\ldots\ \ \omega_s$,但合成的权重不同。$\omega_i$:简正频率。

例题（P.178） **假设ω不出现重根**

# 例题：耦合摆的微振动。



解：❶体系自由度 S=2 $q_1=\theta_1 \quad q_2=\theta_2$

$$T=\frac{1}{2}ml^2(\dot{\theta}_1^{\ 2}+\dot{\theta}_2^{\ 2})$$

$$V=\frac{1}{2}mgl(\theta_1^{\ 2}+\theta_2^{\ 2})+\frac{1}{2}kl^2(\theta_1-\theta_2)^2$$

$$1-\cos\theta=\frac{\theta^2}{2}+O(4)\approx\frac{\theta^2}{2}$$

$$\Delta x=x_1-x_2=l(\sin\theta_1-\sin\theta_2)$$

$$\approx l(\theta_1-\theta_2)$$

L函数

$$L=\frac{1}{2}ml^2(\dot{\theta}_1^{\ 2}+\dot{\theta}_2^{\ 2})-\frac{1}{2}mgl(\theta_1^{\ 2}+\theta_2^{\ 2})-\frac{1}{2}kl^2(\theta_1-\theta_2)^2$$

代入拉氏方程

$$\begin{cases}\ddot{\theta}_1+\dfrac{g}{l}\theta_1+\dfrac{k}{m}(\theta_1-\theta_2)=0\\[2ex]\ddot{\theta}_2+\dfrac{g}{l}\theta_2-\dfrac{k}{m}(\theta_1-\theta_2)=0\end{cases}$$

$$k_{\alpha\beta}=?$$

$$m_{\alpha\beta}=?$$

设方程的解 $\begin{cases}\theta_1 = A_1\cos(\omega_1 t+\varphi_1)\\ \theta_2 = A_2\cos(\omega_2 t+\varphi_2)\end{cases}$

代入微分方程 ⇨ 代数方程

$$\begin{cases}(-\omega_1^2+\dfrac{g}{l}+\dfrac{k}{m})A_1\cos(\omega_1 t+\varphi_1)-\dfrac{k}{m}A_2\cos(\omega_2 t+\varphi_2)=0\\ (-\omega_2^2+\dfrac{g}{l}+\dfrac{k}{m})A_2\cos(\omega_2 t+\varphi_2)-\dfrac{k}{m}A_1\cos(\omega_1 t+\varphi_1)=0\end{cases}$$

任何时刻成立，必有 $\omega_1=\omega_2=\omega,\varphi_1=\varphi_2=\varphi$

否则只能 $A_1=A_2=0$ 无意义

解方程

$\cos(\omega t+\varphi)$ 前的系数必须＝0

$$\begin{cases}(-\omega^2+\dfrac{g}{l}+\dfrac{k}{m})A_1-\dfrac{k}{m}A_2=0\\ -\dfrac{k}{m}A_1+(-\omega^2+\dfrac{g}{l}+\dfrac{k}{m})A_2=0\end{cases}$$

*

$A_1, A_2$ 有非零解的充要条件：系数行列式＝0

$$\begin{vmatrix}-\omega^2+\dfrac{g}{l}+\dfrac{k}{m} & -\dfrac{k}{m}\\ -\dfrac{k}{m} & -\omega^2+\dfrac{g}{l}+\dfrac{k}{m}\end{vmatrix}=0$$

$$T=\frac{1}{2}ml^2(\dot{\theta}_1^2+\dot{\theta}_2^2)$$

$$V=\frac{1}{2}mgl(\theta_1^2+\theta_2^2)+\frac{1}{2}kl^2(\theta_1-\theta_2)^2$$

$$T=\frac{1}{2}\begin{pmatrix}\dot{\theta}_1 & \dot{\theta}_2\end{pmatrix}\begin{pmatrix}ml^2 & 0\\ 0 & ml^2\end{pmatrix}\begin{pmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{pmatrix}$$

$$V=\frac{1}{2}\begin{pmatrix}\theta_1 & \theta_2\end{pmatrix}\begin{pmatrix}mgl+kl^2 & -kl^2\\ -kl^2 & mgl+kl^2\end{pmatrix}\begin{pmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{pmatrix}$$

$$k_{\alpha\beta}=\begin{pmatrix}mgl+kl^2 & -kl^2\\ -kl^2 & mgl+kl^2\end{pmatrix}$$

$$m_{\alpha\beta}=\begin{pmatrix}ml^2 & 0\\ 0 & ml^2\end{pmatrix}$$

$$\left\|k_{\alpha\beta}-\omega^2 m_{\alpha\beta}\right\|=0$$

解方程

$$\omega_\alpha = \sqrt{\frac{g}{l}} \qquad \omega_\beta = \sqrt{\frac{g}{l} + \frac{2k}{m}} \qquad \text{本征频率}$$

方程的特解

$$\omega_\alpha = \sqrt{\frac{g}{l}} \longrightarrow \begin{cases} \theta_1 = A_1^{(\alpha)} \cos(\omega_\alpha t + \varphi_\alpha) \\ \theta_2 = A_2^{(\alpha)} \cos(\omega_\alpha t + \varphi_\alpha) \end{cases}$$

$$\omega_\beta = \sqrt{\frac{g}{l} + \frac{2k}{m}} \longrightarrow \begin{cases} \theta_1 = A_1^{(\beta)} \cos(\omega_\beta t + \varphi_\beta) \\ \theta_2 = A_2^{(\beta)} \cos(\omega_\beta t + \varphi_\beta) \end{cases}$$

方程的通解

$$\begin{cases}\theta_1 = C_\alpha A_1^{(\alpha)}\cos(\omega_\alpha t+\varphi_\alpha)+C_\beta A_1^{(\beta)}\cos(\omega_\beta t+\varphi_\beta)\\ \theta_2 = C_\alpha A_2^{(\alpha)}\cos(\omega_\alpha t+\varphi_\alpha)+C_\beta A_2^{(\beta)}\cos(\omega_\beta t+\varphi_\beta)\end{cases}$$

$$\begin{cases}(-\omega^2+\dfrac{g}{l}+\dfrac{k}{m})A_1-\dfrac{k}{m}A_2=0\\ -\dfrac{k}{m}A_1+(-\omega^2+\dfrac{g}{l}+\dfrac{k}{m})A_2=0\end{cases}$$

代数方程只有一个是独立的 $\begin{cases}A_1^{(\alpha)}/A_2^{(\alpha)}=1\\ A_1^{(\beta)}/A_2^{(\beta)}=-1\end{cases}$

$$\begin{cases}\theta_1 = A_1^{(\alpha)}\cos(\omega_\alpha t+\varphi_\alpha)+A_1^{(\beta)}\cos(\omega_\beta t+\varphi_\beta)\\ \theta_2 = A_1^{(\alpha)}\cos(\omega_\alpha t+\varphi_\alpha)-A_1^{(\beta)}\cos(\omega_\beta t+\varphi_\beta)\end{cases}$$

由初始条件定 $A_1^{(\alpha)}, A_1^{(\beta)}, \varphi_\alpha, \varphi_\beta$

若 $t=0,\theta_1=\theta_0,\theta_2=0,\dot{\theta}_1=0,\dot{\theta}_2=0$

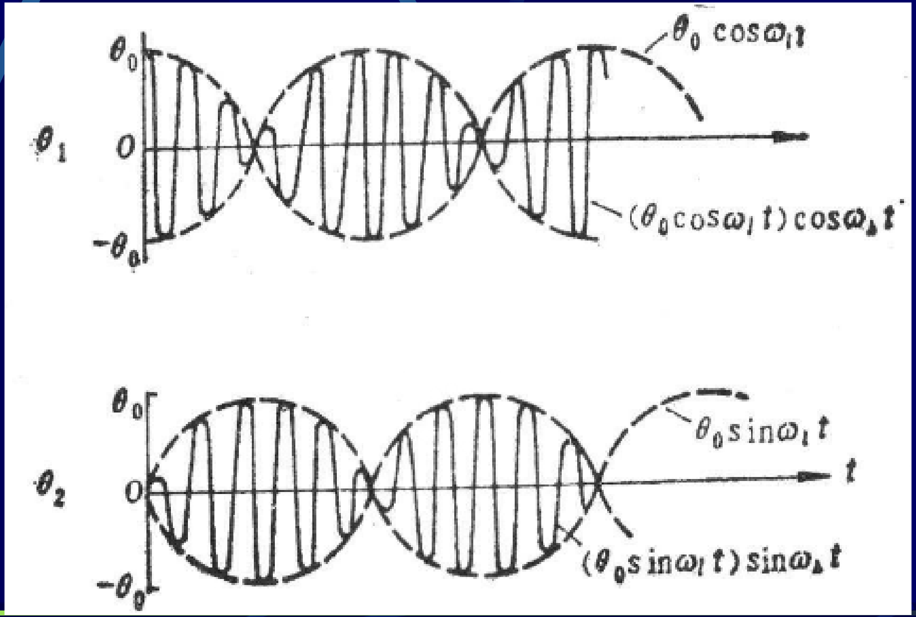
$$A_1^{(\alpha)}=A_1^{(\beta)}=\frac{1}{2}\theta_0,\varphi_\alpha=\varphi_\beta=0$$

运动方程的解
$$\begin{cases}\theta_1=\dfrac{1}{2}\theta_0(\cos\omega_\alpha t+\cos\omega_\beta t)\\[2ex]\theta_2=\dfrac{1}{2}\theta_0(\cos\omega_\alpha t-\cos\omega_\beta t)\end{cases}$$

变换解的形式

令 $\omega_l=\dfrac{\omega_\beta-\omega_\alpha}{2}\quad\omega_h=\dfrac{\omega_\beta+\omega_\alpha}{2}$

$$\begin{cases}\theta_1=(\theta_0\cos\omega_l t)\cos\omega_h t\\\theta_2=(\theta_0\sin\omega_l t)\sin\omega_h t\end{cases}$$



# 五. 自然坐标(Natural Coordinates)

对$\omega_i$ $$(k_{\alpha\beta} - \omega_i^2 m_{\alpha\beta})B_\beta^i = 0 \qquad \ldots\ldots(1)$$

$$(\beta = 1\,2\,3\ldots\ldots s)$$

此处i不求和

(1)×$q_\alpha$并对α求和

$$q_\alpha k_{\alpha\beta} B_\beta^i - \omega_i^2 q_\alpha m_{\alpha\beta} B_\beta^i = 0 \qquad \ldots\ldots(2)$$

$$(\alpha\ \beta = 1\,2\,3\ldots\ldots s)$$

$$q_\alpha k_{\alpha\beta} B_\beta^i - \omega_i^2 q_\alpha m_{\alpha\beta} B_\beta^i = 0 \qquad \ldots\ldots(2)$$

$$\because m_{\alpha\beta}\ddot{q}_\beta + k_{\alpha\beta} q_\beta = 0 \qquad (\beta = 1\,2\,3\ldots\ldots s)$$

$$k_{\alpha\beta} q_\beta = -m_{\alpha\beta}\ddot{q}_\beta$$

$$\because \quad k_{\alpha\beta} = k_{\beta\alpha} \quad m_{\alpha\beta} = m_{\beta\alpha} \qquad \therefore k_{\alpha\beta} q_\beta = k_{\beta\alpha} q_\alpha = k_{\alpha\beta} q_\alpha$$

此处i不求和!!!

$$\therefore k_{\alpha\beta} q_\alpha = -m_{\alpha\beta}\ddot{q}_\alpha$$

$$\ddot{q}_\alpha m_{\alpha\beta} B_\beta^i + \omega_i^2 q_\alpha m_{\alpha\beta} B_\beta^i = 0 \qquad \ldots\ldots(3)$$

$$(\alpha\ \beta = 1\,2\,3\ldots s)$$

$$\ddot{q}_\alpha m_{\alpha\beta} B_\beta^i + \omega_i^2 q_\alpha m_{\alpha\beta} B_\beta^i = 0 \qquad \ldots\ldots(3)$$

$$(\alpha\ \beta = 1\ 2\ 3 \ldots\ldots s)$$

此处i不求和

$$令\ Q_i \equiv q_\alpha m_{\alpha\beta} B_\beta^i \qquad \ldots\ldots(4)$$

⇒ 自然坐标

$$\ddot{Q}_i + \omega_i^2 Q_i = 0 \qquad (5)$$

此处i不求和

$$Q_i = (q_1\ q_2 \ldots\ldots\ q_s)\begin{pmatrix} m_{11} & m_{12} & m_{13} \ldots\ldots m_{1s} \\ m_{21} & m_{22} & m_{23} \ldots\ldots m_{2s} \\ \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ m_{s1} & m_{s2} & m_{s3} \ldots\ldots m_{ss} \end{pmatrix}\begin{pmatrix} B_1^i \\ B_2^i \\ \vdots \\ B_s^i \end{pmatrix}$$

# S个自然坐标

$$\left[ Q_1 \; Q_2 \; Q_3 \ldots\ldots Q_s \right]$$

$$= (q_1 \, q_2 \, q_3 \ldots . q_s) \begin{bmatrix} m_{11} & m_{12} & m_{13} \ldots\ldots m_{1s} \\ m_{21} & m_{22} & m_{23} \ldots\ldots m_{2s} \\ \cdots & \cdots & \cdots \\ m_{s1} & m_{s2} & m_{s3} \ldots\ldots m_{ss} \end{bmatrix} \begin{bmatrix} B_1^1 & B_1^2 & B_1^3 \ldots\ldots B_1^s \\ B_2^1 & B_2^2 & B_2^3 \ldots\ldots B_2^s \\ \cdots & \cdots & \cdots \\ B_s^1 & B_s^2 & B_s^3 \ldots\ldots B_s^s \end{bmatrix}$$

$$Q = qMB \qquad q = QB^{-1}M^{-1}$$

# 六. 简正坐标(Normal Coordinates)

对频率$\omega_i$:

$$(k_{\alpha\beta}-\omega_i^2 m_{\alpha\beta})B_\beta^i=0\ldots\ldots(1)$$

$$(\beta=1\ 2\ 3\ldots\ldots s)$$

对频率$\omega_j$:

$$(k_{\alpha\beta}-\omega_j^2 m_{\alpha\beta})B_\alpha^j=0\ldots\ldots(2)$$

$$(\alpha=1\ 2\ 3\ldots\ldots s)$$

$(1)\times B_\alpha^j$并对$\alpha$求和

$$B_\alpha^j(k_{\alpha\beta}-\omega_i^2 m_{\alpha\beta})B_\beta^i=0\ldots\ldots(3)$$

$(2)\times B_\beta^i$并对$\beta$求和

$$\begin{cases} B_\beta^i(k_{\alpha\beta}-\omega_j^{\;2}m_{\alpha\beta})B_\alpha^j=0\ldots\ldots(4) \\ B_\alpha^j(k_{\alpha\beta}-\omega_i^{\;2}m_{\alpha\beta})B_\beta^i=0\ldots\ldots(3) \end{cases}$$

$$(\alpha\ \beta=1\ 2\ 3\ldots\ldots s)$$

$(i\ j$不求和$)$

$(3)-(4)$

$$(\omega_j^2-\omega_i^2)m_{\alpha\beta}B_\beta^iB_\alpha^j=0\ldots\ldots(5)$$

(此处i j 不求和, α β = 1 2 3 ...s)

$$(\omega_j^2-\omega_i^2)m_{\alpha\beta}B_\beta^iB_\alpha^j=0\ldots\ldots(5)$$

$\because\ \omega_i\neq\omega_j$ $\qquad(\alpha\ \beta=1\ 2\ 3\ldots s)$

$$\therefore\quad m_{\alpha\beta}B_\beta^iB_\alpha^j=0\ldots\ldots(6)$$

两个不同频率本征矢带权重因子正交　但不一定归一化

定义：

$$\hat{B}_k^i=\frac{B_k^i}{[m_{\alpha\beta}B_\alpha^iB_\beta^i]^{\frac{1}{2}}}\ldots\ldots(7)$$

$$\hat{B}_k^i = \frac{B_k^i}{[m_{\alpha\beta} B_\alpha^i B_\beta^i]^{\frac{1}{2}}} \ldots\ldots(7)$$

($i$不求和!!!)

$$m_{\alpha\beta} B_\beta^i B_\alpha^j = 0 \ldots\ldots(6)(i \neq j)$$

$$m_{\alpha\beta} \hat{B}_\alpha^i \hat{B}_\beta^j = \frac{m_{\alpha\beta} B_\alpha^i B_\beta^j}{[m_{\alpha\beta} B_\alpha^i B_\beta^i]^{\frac{1}{2}} [m_{\alpha\beta} B_\alpha^j B_\beta^j]^{\frac{1}{2}}}$$

此处i，j 均不求和

$$m_{\alpha\beta} \hat{B}_\alpha^i \hat{B}_\beta^j = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \ldots\ldots(8)$$

$$m_{\alpha\beta}\hat{B}^{i}_{\alpha}\hat{B}^{j}_{\beta}=\delta_{ij}=\begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}\quad\ldots\ldots(8)$$

(8)$\times\hat{B}^{j}_{\gamma}$并对 j 求和

$$\hat{B}^{i}_{\alpha}\underbrace{m_{\alpha\beta}\hat{B}^{j}_{\beta}\hat{B}^{j}_{\gamma}}_{?}=\delta_{ij}\hat{B}^{j}_{\gamma}$$

$$(\alpha\ \beta\ j=1\,2\,3\ldots\ldots s)$$

$$\hat{B}^{i}_{\gamma}$$

$$m_{\alpha\beta}\hat{B}^{j}_{\beta}\hat{B}^{j}_{\gamma}=\delta_{\alpha\gamma}=\begin{cases}1 & \alpha=\gamma\\ 0 & \alpha\neq\gamma\end{cases}\quad\ldots\ldots(9)$$

自然坐标 $\Rightarrow$ $Q_i \equiv q_\alpha m_{\alpha\beta} B^i_\beta$

$\hat{q}_i \equiv q_\alpha m_{\alpha\beta} \hat{B}^i_\beta$......(10) $\Leftarrow$ 简正坐标

$\hat{q}_i = c_i \cos(\omega_i t + \varphi_i) \quad (i = 123......s)$

$$(\hat{q}_1\,\hat{q}_2\,\hat{q}_3\,....\hat{q}_s\,)$$

$$=(q_1\,q_2\,q_3\,....q_s)\begin{pmatrix} m_{11} & m_{12} & m_{13}......m_{1s} \\ m_{21} & m_{22} & m_{23}......m_{2s} \\ \cdots & \cdots & \cdots \\ m_{s1} & m_{s2} & m_{s3}......m_{ss} \end{pmatrix}\begin{pmatrix} \hat{B}^1_1 & \hat{B}^2_1 & \hat{B}^3_1......\hat{B}^s_1 \\ \hat{B}^1_2 & \hat{B}^2_2 & \hat{B}^3_2......\hat{B}^s_2 \\ \cdots & \cdots & \cdots \\ \hat{B}^1_s & \hat{B}^2_s & \hat{B}^3_s......\hat{B}^s_s \end{pmatrix}$$

$$\hat{q}_i \equiv q_\alpha m_{\alpha\beta} \hat{B}^i_\beta \ldots\ldots(10)$$

$(10)\times \hat{B}^i_\gamma$并对所有i求和

$$\hat{B}^i_\gamma \hat{q}_i = \hat{B}^i_\gamma q_\alpha m_{\alpha\beta} \hat{B}^i_\beta$$

$$\{\alpha\ \beta\ i = 1\,2\,3\ldots\ldots S\}$$

$$= q_\alpha \underbrace{m_{\alpha\beta} \hat{B}^i_\gamma \hat{B}^i_\beta}_{\delta_{\alpha\gamma}}$$

$$m_{\alpha\beta} \hat{B}^j_\beta \hat{B}^j_\gamma = \delta_{\alpha\gamma}$$

$$= q_\alpha \delta_{\alpha\gamma} = q_\gamma$$

$$\hat{B}^i_\gamma \hat{q}_i = q_\gamma \quad (i = 1\,2\,3\ldots\ldots s)\ldots\ldots(11)$$

$$m_{\alpha\beta}\hat{B}^{i}_{\alpha}\hat{B}^{j}_{\beta}=\delta_{ij}=\begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}\ldots\ldots(8)$$

$$m_{\alpha\beta}\hat{B}^{j}_{\beta}\hat{B}^{j}_{\gamma}=\delta_{\alpha\gamma}=\begin{cases}1 & \alpha=\gamma\\ 0 & \alpha\neq\gamma\end{cases}\ldots\ldots(9)$$

$$\hat{B}^{i}_{\gamma}\hat{q}_{i}=q_{\gamma}\quad(i=1\,2\,3\ldots\ldots s)\ldots\ldots(11)$$

$$\hat{q}_{i}\equiv q_{\alpha}m_{\alpha\beta}\hat{B}^{i}_{\beta}\ldots\ldots(10)$$

多自由度体系的复杂性 $\Longrightarrow$ 交叉项出现!!!

$$V(q_1 \ldots\ldots q_s) = \frac{1}{2}\left(q_1\, q_2 \ldots\ldots q_s\right) \begin{pmatrix} k_{11} & k_{12} & k_{13} \cdots\cdots K_{1s} \\ k_{21} & k_{22} & k_{23} \cdots\cdots k_{2s} \\ \cdots & \cdots & \cdots\cdots\cdots \\ k_{s1} & k_{s2} & k_{s3} \cdots\cdots k_{ss} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_s \end{pmatrix}$$

$$T = \frac{1}{2}(\dot{q}_1\, \dot{q}_2 \ldots\ldots \dot{q}_s) \begin{pmatrix} m_{11} & m_{12} & m_{13} \cdots\cdots m_{1s} \\ m_{21} & m_{22} & m_{23} \cdots\cdots m_{2s} \\ \cdots & \cdots & \cdots\cdots\cdots \\ m_{s1} & m_{s2} & m_{s3} \cdots\cdots m_{ss} \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_s \end{pmatrix}$$

如果能把T和V同时对角化 ⟹ 处理将得到简化

$$\hat{B}_\gamma^i \hat{q}_i = q_\gamma \quad (i = 1\,2\,3\ldots\ldots s)\ldots\ldots(11)$$

$$\hat{q}_i \equiv q_\alpha m_{\alpha\beta} \hat{B}_\beta^i \ldots\ldots(10)$$

$$T = \frac{1}{2} m_{\alpha\beta} \dot{q}_\alpha \dot{q}_\beta \quad (\alpha\ \beta = 1\,2\,3\ldots\ldots s)$$

$$T = \frac{1}{2} m_{\alpha\beta} (\dot{\hat{q}}_i \hat{B}_\alpha^i)(\dot{\hat{q}}_j \hat{B}_\beta^j)$$

$$(i\ j\ \alpha\ \beta = 1\,2\,3\ldots\ldots s)$$

$$T = \frac{1}{2} \dot{\hat{q}}_i \dot{\hat{q}}_j (m_{\alpha\beta} \hat{B}_\alpha^i \hat{B}_\beta^j) \longrightarrow \delta_{ij}$$

$$T = \frac{1}{2} \dot{\hat{q}}_i \dot{\hat{q}}_j \delta_{ij} = \frac{1}{2} \dot{\hat{q}}_i \dot{\hat{q}}_i \quad (i = 1\,2\,3\ldots\ldots s)$$

$$\dot{q}_\gamma = \dot{\hat{q}}_i \hat{B}^i_\gamma$$

$$\hat{q}_i \equiv q_\alpha m_{\alpha\beta} \hat{B}^i_\beta \ldots\ldots(10)$$

$$= \dot{q}_\alpha m_{\alpha\beta} \hat{B}^i_\beta \hat{B}^i_\gamma$$

$$= \dot{q}_\alpha \delta_{\alpha\gamma}$$

$$= \dot{q}_\gamma \longrightarrow \dot{q}_\alpha$$

$$\boxed{\hat{B}_{\gamma}^{i}\hat{q}_{i} = q_{\gamma} \quad (i = 1\,2\,3......s)}$$

$$\Downarrow$$

$$V(q) = \frac{1}{2}k_{\alpha\beta}q_{\alpha}q_{\beta} \quad (\alpha\ \beta = 123......s)$$

$$V(q) = \frac{1}{2}k_{\alpha\beta}(\hat{q}_{i}\hat{B}_{\alpha}^{i})(\hat{q}_{j}\hat{B}_{\beta}^{j}) \quad (i\ j\ \alpha\ \beta = 123......s)$$

$$= \frac{1}{2}\hat{q}_{i}\hat{q}_{j}\boxed{(k_{\alpha\beta}\hat{B}_{\alpha}^{i}\hat{B}_{\beta}^{j})}$$

$$\because (k_{\alpha\beta} - \omega_{i}^{2}m_{\alpha\beta})B_{\beta}^{i} = 0 \ \therefore k_{\alpha\beta}B_{\beta}^{i} = \omega_{i}^{2}m_{\alpha\beta}B_{\beta}^{i}$$

$$\therefore k_{\alpha\beta}\hat{B}_{\beta}^{i} = \omega_{i}^{2}m_{\alpha\beta}\hat{B}_{\beta}^{i}$$

$$V(q)=\frac{1}{2}\hat{q}_i\hat{q}_j \quad \boxed{(k_{\alpha\beta}\hat{B}^i_\alpha \hat{B}^j_\beta)}$$

$$\because k_{\alpha\beta}\hat{B}^i_\beta = \omega_i{}^2 m_{\alpha\beta}\hat{B}^i_\beta$$

$$\Downarrow$$

$$V(q)=\frac{1}{2}\hat{q}_i\hat{q}_j\omega_j^2 \boxed{(m_{\alpha\beta}\hat{B}^j_\beta \hat{B}^i_\alpha)}$$

(i,j,α,β=123…s)

$$m_{\alpha\beta}\hat{B}^i_\alpha \hat{B}^j_\beta = \delta_{ij}$$

$$\Downarrow$$

$$V(q)=\frac{1}{2}\hat{q}_i\hat{q}_j\omega_j^2\delta_{ij}$$

$$\boxed{V(q)=\frac{1}{2}\omega_i^2\hat{q}_i\hat{q}_i \quad (i=123.....s)}$$

$$L = T - V$$

$$T = \frac{1}{2}\dot{\hat{q}}_i\dot{\hat{q}}_i$$

$$V(q) = \frac{1}{2}\omega_i^2\hat{q}_i\hat{q}_i$$

$$L = \frac{1}{2}\dot{\hat{q}}_i\dot{\hat{q}}_i - \frac{1}{2}\omega_i^2\hat{q}_i\hat{q}_i$$

$$\ddot{\hat{q}}_i + \omega_i^2\hat{q}_i = 0 \quad \text{(此处i不求和)}$$

关键

$$\hat{q}_i \equiv q_\alpha m_{\alpha\beta}\hat{B}_\beta^i$$

# 耦合摆简正坐标:

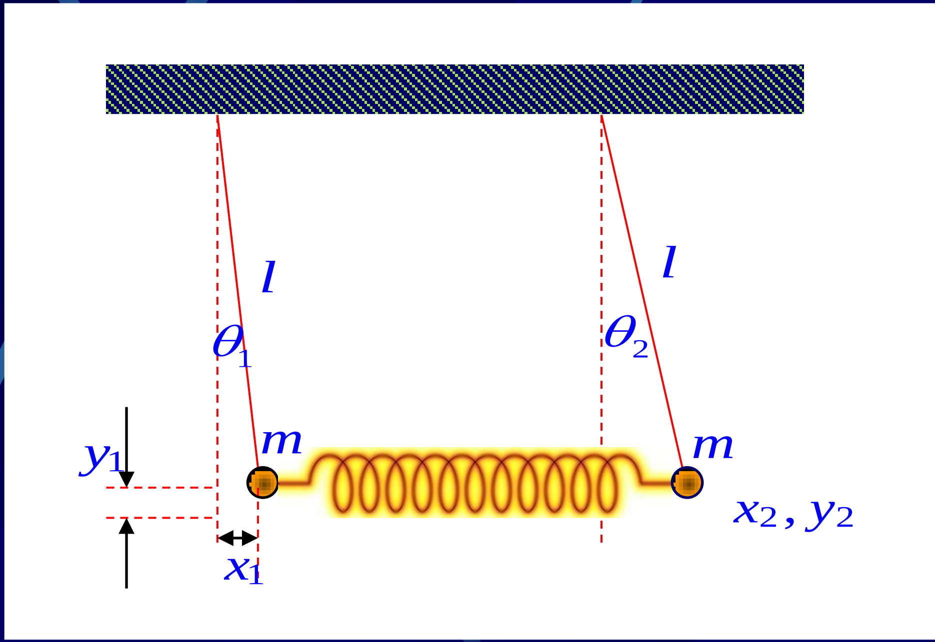
原先广义坐标

$$q_1 = \theta_1 \qquad q_2 = \theta_2$$

新的广义坐标

$$q_1 = \varphi_1 = \frac{1}{\sqrt{2}}(\theta_1 + \theta_2)$$

$$q_2 = \varphi_2 = \frac{1}{\sqrt{2}}(\theta_1 - \theta_2)$$




$q_1$,2个$m$的同向振动;

$q_2$ 2个$m$的相向振动;

体系的动能和势能

$$T=\frac{1}{2}ml^2(\dot{\varphi}_1{}^2+\dot{\varphi}_2{}^2)$$

$$V=\frac{1}{2}mgl(\varphi_1{}^2+\varphi_2{}^2)+\frac{1}{2}kl^2\varphi_2{}^2$$

体系的L函数

$$L=\frac{1}{2}ml^2(\dot{\varphi}_1{}^2+\dot{\varphi}_2{}^2)-\frac{1}{2}mgl(\varphi_1{}^2+\varphi_2{}^2)-\frac{1}{2}kl^2\varphi_2{}^2$$

代入拉氏方程 $\begin{cases}\ddot{\varphi}_1+\dfrac{g}{l}\varphi_1=0\\ \ddot{\varphi}_2+(\dfrac{g}{l}+\dfrac{2k}{m})\varphi_2=0\end{cases}$

$$\omega_1=\sqrt{\frac{g}{l}}\qquad \omega_2=\sqrt{\frac{g}{l}+\frac{2k}{m}}$$

方程的解 $\begin{cases}\varphi_1=C_1\cos(\omega_1 t+\alpha_1)\\ \varphi_2=C_2\cos(\omega_2 t+\alpha_2)\end{cases}$

简正坐标 ！

- 系统中以相同频率作简谐振动的方式称为简正模式;
- 每个模式对应的振动频率称为简正频率;
- n自由度的振动系统有n个简正模式，n个简正频率;
- 一般而言，系统中每个振子的振动状态将是系统的各简正模式按一定的权重叠加的结果;

# 多自由度微振动求解：

1.确定广义坐标；

2、写出广义动能和势能（或Lagrangian）；

3、得出等效惯性系数和等效弹性系数；

T,V在平衡位形处泰勒展开，并取一级近似为：

$$L=\frac{1}{2}m_{\alpha\beta}\dot{q}_\alpha\dot{q}_\beta-\frac{1}{2}k_{\alpha\beta}q_\alpha q_\beta$$

4、写出微分方程及通解；$m_{\alpha\beta}\ddot{q}_\beta+k_{\alpha\beta}q_\beta=0$

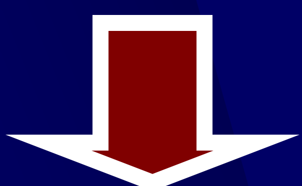
5、得出本征矢方程。 $\left\|k_{\alpha\beta}-\omega^2 m_{\alpha\beta}\right\|=0$

6、自然坐标中的表示,简正坐标….

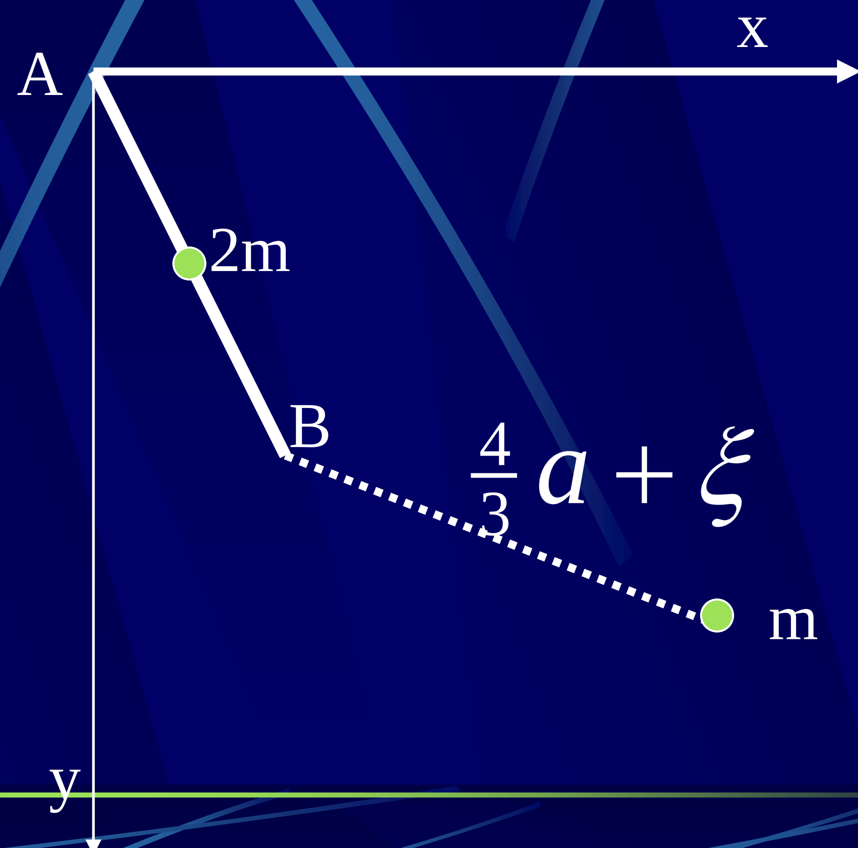
$$Q_i\equiv q_\alpha m_{\alpha\beta}B_\beta^i \qquad \hat{B}_k^i=\frac{B_k^i}{[m_{\alpha\beta}B_\alpha^i B_\beta^i]^{\frac{1}{2}}}. \qquad \hat{q}_i\equiv q_\alpha m_{\alpha\beta}\hat{B}_\beta^i$$

一质量为2m，长为2a 的匀质棒AB可绕过A点的水平轴自由转动。轻弹性绳一端系在B点，另一端系一质量为m的质点，当系统处于平衡位置时弹性绳伸长量为ε，弹性绳原长度为4/3a。假定系统在竖直面内作微振动，试求相应等值单摆长和自由坐标。

自由度？？

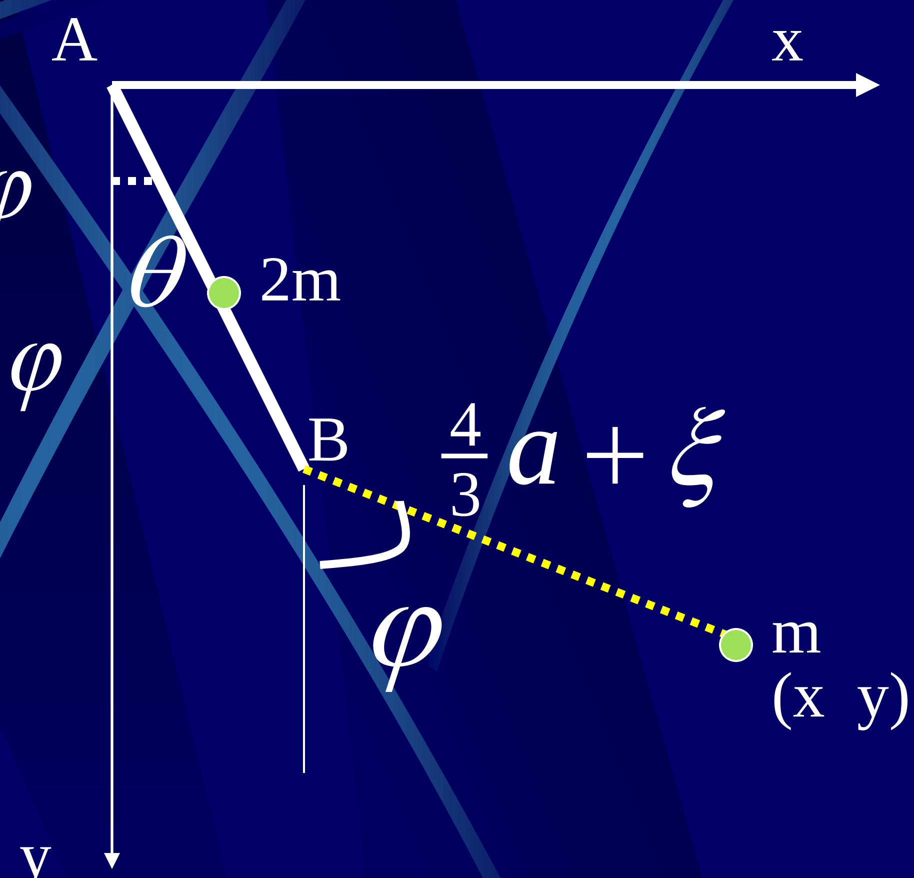


**3**



广义坐标：$\theta \quad \varphi \quad \xi$

$$\begin{cases} x = 2a\sin\theta + (\xi + \frac{4}{3}a)\sin\varphi \\ y = 2a\cos\theta + (\xi + \frac{4}{3}a)\cos\varphi \end{cases}$$




质点的动能$T_1$

$$T_1 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$\approx 2ma^2\dot{\theta}^2 + \frac{1}{2}m\dot{\xi}^2 + \frac{8}{9}ma^2\dot{\theta}\dot{\varphi} + \frac{8}{9}ma^2\dot{\varphi}^2$$

$$\sin x \approx x$$

$$\cos x \approx 1$$

略去三次方小量

棒的动能$T_2$

$$T_2=\frac{1}{2}\frac{1}{3}(2m)(2a)^2\dot{\theta}^2=\frac{4}{3}ma^2\dot{\theta}^2$$

$$T=T_1+T_2$$

$$\boxed{T=\frac{10}{3}ma^2\dot{\theta}^2+\frac{1}{2}m\dot{\xi}^2+\frac{8}{9}ma^2\dot{\varphi}^2+\frac{8}{3}ma^2\dot{\theta}\dot{\varphi}}$$

**重力势能$V_1$**

$$V_1=-2mga\cos\theta-mg[2a\cos\theta+(\xi+\frac{4}{3}a)\cos\varphi]$$

$$V_1 = -2mga\cos\theta - mg[2a\cos\theta + (\xi + \tfrac{4}{3}a)\cos\varphi]$$

$$\cos x \approx 1 - \tfrac{1}{2}x^2$$

$$\boxed{V_1 = 2mga\theta^2 + \tfrac{2}{3}mga\varphi^2 - mg\xi + const}$$

$$|const| = \left|-\frac{16}{3}mga\right| >> mg\xi$$

弹性势能$V_2$

$$V_2 = \tfrac{1}{2}k\xi^2$$

平衡时

$$mg = k\varepsilon$$

$$k = ? = \frac{mg}{\varepsilon}$$

$$V_2 = \frac{mg}{2\varepsilon}\xi^2$$

$$V = V_1 + V_2$$

$$V = 2mga\theta^2 + \tfrac{2}{3}mga\varphi^2 + \frac{mg}{2\varepsilon}\xi^2 + const$$

$$L = T - V$$

$$L = \frac{10}{3}ma^2\dot{\theta}^2 + \frac{1}{2}m\dot{\xi}^2 + \frac{8}{9}ma^2\dot{\varphi}^2 + \frac{8}{3}ma^2\dot{\theta}\dot{\varphi}$$

$$- 2mga\theta^2 - \frac{mg}{2\varepsilon}\xi^2 - \frac{2}{3}mga\varphi^2 + const$$

拉格朗日方程

$$\begin{cases} \ddot{\xi} + \dfrac{g}{\varepsilon}\xi = 0 & (1) \\ \dfrac{20}{3}a\ddot{\theta} + \dfrac{8}{3}a\ddot{\varphi} + 4g\theta = 0 & (2) \\ \dfrac{8}{3}a\ddot{\theta} + \dfrac{16}{9}a\ddot{\varphi} + \dfrac{4}{3}g\varphi = 0 & (3) \end{cases}$$

（1）式即标准谐振动方程

$\xi$ 即自然坐标

ε即等值单摆长

$$\frac{20}{3}a\ddot{\theta}+\frac{8}{3}a\ddot{\varphi}+4g\theta=0 \qquad (2)$$

$$\frac{8}{3}a\ddot{\theta}+\frac{16}{9}a\ddot{\varphi}+\frac{4}{3}g\varphi=0 \qquad (3)$$

令

$$\begin{cases}\theta=\theta_0\sin(\omega t+\delta) & (4)\\ \varphi=\varphi_0\sin(\omega t+\delta) & (5)\end{cases}$$

本征矢方程

$$\begin{cases}(5a\omega^2-3g)\theta_0+2a\omega^2\varphi_0=0 & (6)\\ 6a\omega^2\theta_0+(4a\omega^2-3g)\varphi_0=0 & (7)\end{cases}$$

欲使$\theta_0$和$\varphi_0$有非零解

$$\begin{vmatrix} (5a\omega^2-3g) & 2a\omega^2 \\ 6a\omega^2 & (4a\omega^2-3g) \end{vmatrix} = 0$$

本征值方程

$$\begin{cases} \omega_1^2 = \dfrac{3g}{a} = \dfrac{g}{\frac{a}{3}} \\ \omega_2^2 = \dfrac{3g}{8a} = \dfrac{g}{\frac{8a}{3}} \end{cases}$$

等值单摆长

$\dfrac{a}{3}$

$\dfrac{8a}{3}$

欲求自然坐标 $\Rightarrow$ 本征矢

$$(5a\omega^2-3g)\theta_0+2a\omega^2\varphi_0=0 \qquad (6)$$

$$\frac{\theta_0}{\varphi_0}=\frac{-2a\omega^2}{5a\omega^2-3g}\quad \begin{cases} \omega_1^2=\dfrac{3g}{a} \Rightarrow \dfrac{\theta_0}{\varphi_0}=-\dfrac{1}{2} \\ \omega_2^2=\dfrac{3g}{8a} \Rightarrow \dfrac{\theta_0}{\varphi_0}=\dfrac{2}{3} \end{cases}$$

$\because \dfrac{\theta_0}{\varphi_0}=-\dfrac{1}{2}$ 如果取 $\theta_0=c_1$ 则 $\varphi_0=-2c_1$

$\because \dfrac{\theta_0}{\varphi_0}=\dfrac{2}{3}$ 如果取 $\theta_0=2c_2$ 则 $\varphi_0=3c_2$

$$\begin{cases} \theta = \theta_0 \sin(\omega t + \delta) & (4) \\ \varphi = \varphi_0 \sin(\omega t + \delta) & (5) \end{cases}$$

$$\omega_1 \begin{cases} \theta_0 = c_1 \\ \varphi_0 = -2c_1 \end{cases} \qquad \omega_2 \begin{cases} \theta_0 = 2c_2 \\ \varphi_0 = 3c_2 \end{cases}$$

$$\begin{cases} \theta = c_1 \sin(\omega_1 t + \delta_1) + 2c_2 \sin(\omega_2 t + \delta_2) & (8) \\ \varphi = -2c_1 \sin(\omega_1 t + \delta_1) + 3c_2 \sin(\omega_2 t + \delta_2) & (9) \end{cases}$$

$$\begin{cases} \theta = c_1 \sin(\omega_1 t + \delta_1) + 2c_2 \sin(\omega_2 t + \delta_2) & (8) \\ \varphi = -2c_1 \sin(\omega_1 t + \delta_1) + 3c_2 \sin(\omega_2 t + \delta_2) & (9) \end{cases}$$

自然坐标

$$2\theta + \varphi = 7c_2 \sin(\omega_2 t + \delta_2)$$

$$2\varphi - 3\theta = -7c_1 \sin(\omega_1 t + \delta_1)$$

质量为m，半径为R的圆环上的o点用铰链固定可在竖直面内摆动，另有一质量为m的质点可在弯管内无摩擦滑动。试求其作微振动频率和振幅比。

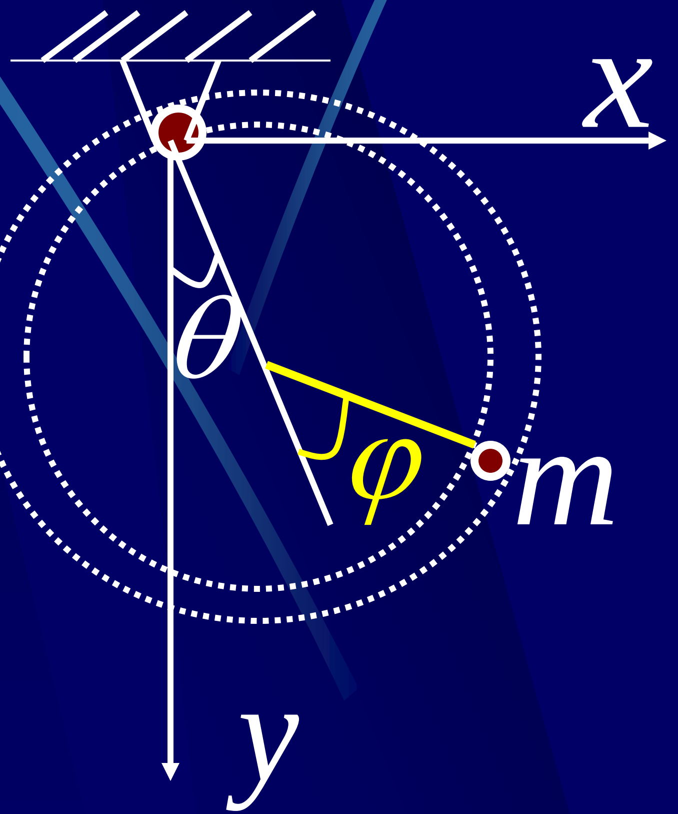
解：

坐标数 ⟹ 3

约束数 ⟹ 1

自由度 ⟹ 2

取θ和$\varphi$为广义坐标



环的动能

$$T_1=\frac{1}{2}I\dot{\theta}^2=\frac{1}{2}(mr^2+mr^2)\dot{\theta}^2=mr^2\dot{\theta}^2$$

质点动能

$$x=r\sin\theta+r\sin(\theta+\varphi)$$

$$y=r\cos\theta+r\cos(\theta+\varphi)$$

$$T_2=\frac{1}{2}m(\dot{x}^2+\dot{y}^2)$$

$$T=T_1+T_2$$

$$=\frac{3}{2}mr^2\dot{\theta}^2+\frac{1}{2}mr^2(\dot{\theta}+\dot{\varphi})^2+mr^2\dot{\theta}\cos\varphi(\dot{\theta}+\dot{\varphi})$$

$$V=-2mgr\cos\theta-mgr\cos(\theta+\varphi)$$

$$T=\frac{3}{2}mr^2\dot{\theta}^2+\frac{1}{2}mr^2(\dot{\theta}+\dot{\varphi})^2+mr^2\dot{\theta}\cos\varphi(\dot{\theta}+\dot{\varphi})$$

$$L=T-V$$

$$L=\frac{3}{2}mr^2\dot{\theta}^2+\frac{1}{2}mr^2(\dot{\theta}+\dot{\varphi})^2+mr^2\cos\varphi\dot{\theta}(\dot{\theta}+\dot{\varphi})$$

$$+2mgr\cos\theta+mgr\cos(\theta+\varphi)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta}=0 \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right)-\frac{\partial L}{\partial \varphi}=0$$

$$mr^2[2\ddot{\theta}(2+\cos\varphi)+\ddot{\varphi}(1+\cos\varphi)-$$

$$\dot{\varphi}(2\dot{\theta}+\dot{\varphi})\sin\varphi]+mgr[2\sin\theta+\sin(\theta+\varphi)]=0 \quad (1)$$

$$mr^2[\ddot{\varphi}+\ddot{\theta}(1+\cos\varphi)+\dot{\theta}^2\sin\varphi]$$

$$+mgr\sin(\theta+\varphi)=0 \quad (2)$$

对微振动 $\cos\varphi=\cos\theta\approx 1 \quad \sin\varphi\approx\varphi$

$$\sin(\theta+\varphi)\approx\theta+\varphi$$

$$\begin{cases} 6mr^2\ddot{\theta}+2mr^2\ddot{\varphi}+2mgr\theta+mgr(\theta+\varphi)=0 \\ 2mr^2\ddot{\theta}+mr^2\ddot{\varphi}+mgr(\theta+\varphi)=0 \end{cases}$$

$$\begin{cases} 6r\ddot{\theta}+2r\ddot{\varphi}+3g\theta+g\varphi=0 & (3) \\ 2r\ddot{\theta}+r\ddot{\varphi}+g\theta+g\varphi=0 & (4) \end{cases}$$

设试探解为

$$\begin{cases} \theta=A_1\cos\omega t & (5) \\ \varphi=A_2\cos\omega t & (6) \end{cases}$$

本征矢方程

$$\begin{cases}(3g-6\omega^2 r)A_1+(g-2\omega^2 r)A_2=0 & (7)\\ (g-2\omega^2 r)A_1+(g-\omega^2 r)A_2=0 & (8)\end{cases}$$

欲得到非零振幅解

$$\begin{vmatrix}(3g-6\omega^2 r) & (g-2\omega^2 r)\\ (g-2\omega^2 r) & (g-\omega^2 r)\end{vmatrix}=0 \quad (9)$$

$$\omega_1=\sqrt{\frac{2g}{r}} \qquad \omega_2=\sqrt{\frac{g}{2r}}$$

$$\omega_1^2 = \frac{2g}{r}$$

$$(3g - 6\omega^2 r)A_1 + (g - 2\omega^2 r)A_2 = 0 \quad (7)$$

$$\frac{A_1}{A_2} = \frac{2\omega^2 r - g}{3g - 6\omega^2 r} = \frac{3}{-9} = -\frac{1}{3}$$

$$\omega_2^2 = \frac{g}{2r}$$

$$(3g - 6\omega^2 r)A_1 + (g - 2\omega^2 r)A_2 = 0 \quad (7)$$

$$\frac{A_1}{A_2} = \frac{2\omega^2 r - g}{3g - 6\omega^2 r} = 0 \quad (8)$$

当$\dfrac{A_1}{A_2}=-\dfrac{1}{3}$ ⇒ 令 $\underbrace{A_1=C_1 \quad A_2=-3C_1}$

当$\dfrac{A_1}{A_2}=0$ ⇒ 令 $\underbrace{A_1=0 \quad A_2=c_2}$

$$\left.\begin{aligned}\theta &= A_1\cos\omega t \\ \varphi &= A_2\cos\omega t\end{aligned}\right\}\qquad \begin{aligned}&(5)\\&(6)\end{aligned}$$

$$\begin{cases}\theta = c_1\cos(\omega_1 t+\delta_1) \\ \varphi = -3C_1\cos(\omega_1 t+\delta_1)+C_2\cos(\omega_2 t+\delta_2)\end{cases}$$

或者简单写出 $k_{\alpha\beta}, m_{\alpha\beta}$

$$V = -2mgr\cos\theta - mgr\cos(\theta+\varphi)$$

$$= -2mgr\cos\theta - mgr(\cos\theta\cos\varphi - \sin\theta\sin\varphi)$$

$$\approx -3mgr + mgr\theta\varphi = \frac{1}{2}k_{\alpha\beta}q_\alpha q_\beta$$

$$\Rightarrow \quad \xcancel{k_{\alpha\beta} = \begin{bmatrix} 0 & mgr \\ mgr & 0 \end{bmatrix}} \quad ?$$

$$\cos x \approx 1 - \tfrac{1}{2} x^2$$

$$V = -2mgr\cos\theta - mgr\cos(\theta+\varphi)$$

$$= -2mgr\cos\theta - mgr(\cos\theta\cos\varphi - \sin\theta\sin\varphi)$$

$$\approx -2mgr(1-\frac{1}{2}\theta^2) - mgr(1-\frac{1}{2}\theta^2)(1-\frac{1}{2}\varphi^2) + mgr\theta\varphi$$

$$\approx -2mgr + \frac{3}{2}mgr\theta^2 + mgr\theta\varphi + \frac{1}{2}mgr\varphi^2$$

$$= C + \frac{1}{2}k_{\alpha\beta}q_\alpha q_\beta$$

$$k_{\alpha\beta} = \begin{bmatrix} 3mgr & mgr \\ mgr & mgr \end{bmatrix}$$

$$T=\frac{3}{2}mr^2\dot{\theta}^2+\frac{1}{2}mr^2(\dot{\theta}+\dot{\varphi})^2+mr^2\dot{\theta}\cos\varphi(\dot{\theta}+\dot{\varphi})$$

$$\approx\frac{3}{2}mr^2\dot{\theta}^2+\frac{3}{2}mr^2\dot{\theta}^2+\frac{1}{2}mr^2\dot{\varphi}^2+2mr^2\dot{\theta}\dot{\varphi}$$

$$=\frac{1}{2}(6mr^2\dot{\theta}^2+2mr^2\dot{\theta}\dot{\varphi}$$

$$+2mr^2\dot{\theta}\dot{\varphi}+mr^2\dot{\varphi}^2)$$

$$\cos x\approx 1$$

$$\cos x\approx 1-\tfrac{1}{2}x^2\,?$$

$$m_{\alpha\beta}=\begin{bmatrix}6mr^2 & 2mr^2\\ 2mr^2 & mr^2\end{bmatrix}$$

级数展开,

二次型的系数近似!

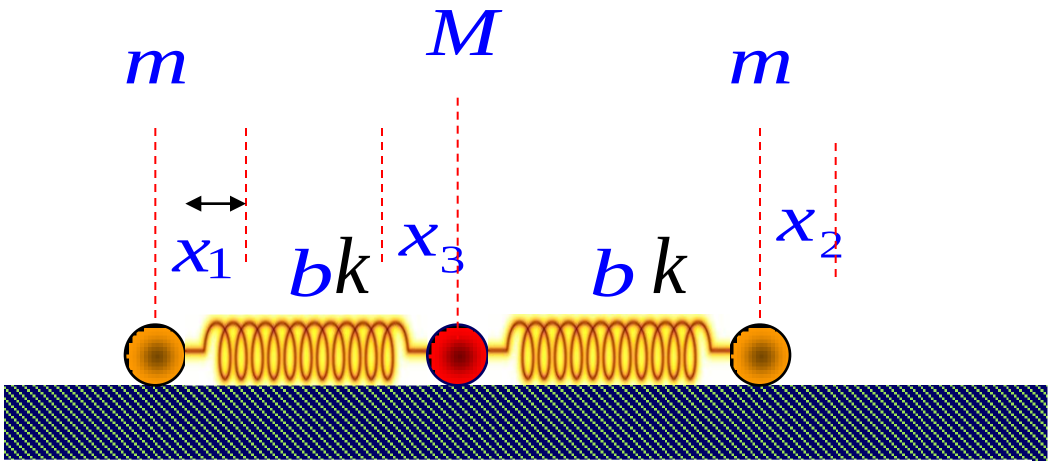
$$\left\| k_{\alpha\beta} - \omega^2 m_{\alpha\beta} \right\| = 0$$

We have,

$$\begin{vmatrix} (3g - 6\omega^2 r) & (g - 2\omega^2 r) \\ (g - 2\omega^2 r) & (g - \omega^2 r) \end{vmatrix} = 0 \qquad \ldots\ldots$$

注意：利用拉格朗日方程不易出错！

# 例：三原子纵向微振动。



解：三个原子，纵向位移$x_1,x_2,x_3$,有一个约束：

$$m(x_1+x_2)+Mx_3=0$$

取$x_1,x_2$为广义坐标：

$$T=\frac{1}{2}m\dot{x}_1^2+\frac{1}{2}m\dot{x}_2^2+\frac{1}{2}M\dot{x}_3^2$$

$$=\frac{1}{2}m\frac{m+M}{M}(\dot{x}_1^2+\dot{x}_2^2)+\frac{1}{2}\frac{m^2}{M}(2\dot{x}_1\dot{x}_2)$$

$$\{m_{\alpha\beta}\} = \begin{vmatrix} \dfrac{m(m+M)}{M} & \dfrac{m^2}{M} \\ \dfrac{m^2}{M} & \dfrac{m(m+M)}{M} \end{vmatrix}$$

势能为:

$$V = \frac{1}{2}k（x_1 - x_3)^2 + \frac{1}{2}k（x_2 - x_3)^2$$

$$= \frac{1}{2}k\frac{（m+M)^2 + m^2}{M^2}(x_1^2 + x_2^2)^2$$

$$+ \frac{1}{2}k\frac{4m（m+M)}{M^2}x_1 x_2$$

$$\{k_{\alpha\beta}\} = \begin{vmatrix} k\dfrac{m^2+(m+M)^2}{M^2} & 2k\dfrac{m(m+M)}{M^2} \\ 2k\dfrac{m(m+M)}{M^2} & k\dfrac{m^2+(m+M)^2}{M^2} \end{vmatrix}$$

$$\left\| k_{\alpha\beta} - \omega^2 m_{\alpha\beta} \right\| = 0 \qquad \ldots\ldots$$

Then, $\omega_1^2 = \dfrac{k}{m}$, and $\omega_2^2 = \dfrac{k}{m}\dfrac{2m+M}{M}$, Since $(k_{\alpha\beta} - \omega_i^2 m_{\alpha\beta})\begin{pmatrix} A_1^i \\ A_2^i \end{pmatrix} = 0$

With $\omega_1^2 = \dfrac{k}{m}$, we can have,

$$A_1^1 = A_2^1 = c_1, const.$$

and with $\omega_2^2 = \dfrac{k}{m}\dfrac{2m+M}{M}$, one can obtain,

$$A_1^2 = -A_2^2 = c_2, const.$$

Finally， we obtained,

$$x_1 = c_1 \cos(\omega_1 t + \delta_1) + c_2 \cos(\omega_2 t + \delta_2),$$

$$x_2 = c_1 \cos(\omega_1 t + \delta_1) - c_2 \cos(\omega_2 t + \delta_2).$$

And the natural cordinates are,

$$q_1 = x_1 + x_2 = 2c_1 \cos(\omega_1 t + \delta_1)$$

$$q_2 = x_1 - x_2 = 2c_2 \cos(\omega_2 t + \delta_2).$$

$$\left(m_{\alpha\beta} A_{\alpha}^{1} A_{\beta}^{1}\right)^{1/2}$$

$$= \left(m_{11} A_1^1 A_1^1 + m_{12} A_1^1 A_2^1 + m_{21} A_2^1 A_1^1 + m_{22} A_2^1 A_2^1\right)^{1/2}$$

$$= A_1^1 \sqrt{\frac{2m(2m+M)}{M}}$$

$$\left(m_{\alpha\beta} A_{\alpha}^{2} A_{\beta}^{2}\right)^{1/2}$$

$$= \left(m_{11} A_1^2 A_1^2 + m_{12} A_1^2 A_2^2 + m_{21} A_2^2 A_1^2 + m_{22} A_2^2 A_2^2\right)^{1/2}$$

$$= A_1^2 \sqrt{2m}$$

$$\hat{A}_1^1 = \frac{A_1^1}{\left(m_{\alpha\beta} A_\alpha^1 A_\beta^1\right)^{1/2}} = \sqrt{\frac{M}{2m(2m+M)}} = \hat{A}_2^1$$

$$\hat{A}_1^2 = \frac{A_1^2}{\left(m_{\alpha\beta} A_\alpha^1 A_\beta^1\right)^{1/2}} = \sqrt{\frac{1}{2m}} = -\hat{A}_2^2$$

The normal cordinates are

$$\hat{q}_i = x_\alpha m_{\alpha\beta} \hat{A}_\beta^i$$

$$\hat{q}_i = x_\alpha m_{\alpha\beta} \hat{A}^i_\beta$$

$$\hat{q}_1 = x_\alpha m_{\alpha\beta} \hat{A}^1_\beta$$

$$= x_1 m_{11} \hat{A}^1_1 + x_1 m_{12} \hat{A}^1_2 + x_2 m_{21} \hat{A}^1_1 + x_2 m_{22} \hat{A}^1_2$$

$$= \sqrt{\frac{m(2m+M)}{2M}}(x_1 + x_2)$$

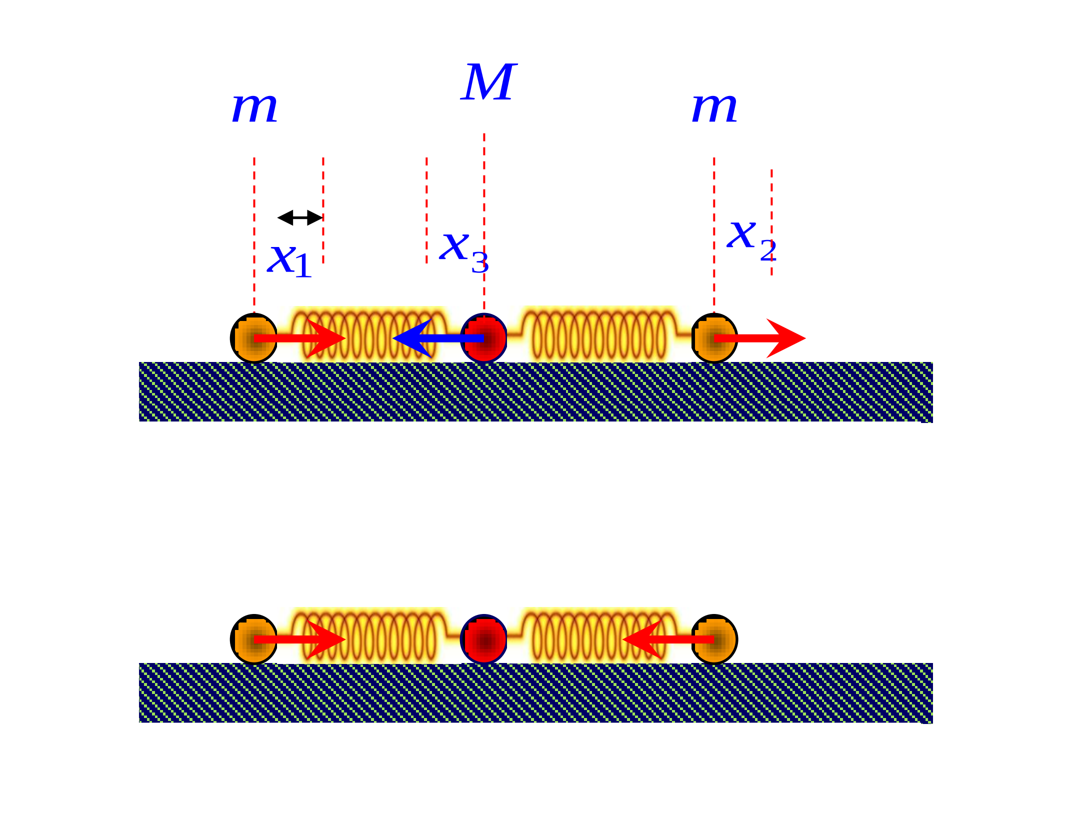
$$\hat{q}_2 = \sqrt{\frac{m}{2}}(x_1 - x_2).$$

*Natural Cordinates :*

$$q_1 = x_1 + x_2$$

$$q_2 = x_1 - x_2.$$

自由振动，质心不动：$m(x_1+x_2)+Mx_3=0$




$$\hat{q}_1=\sqrt{\frac{m(2m+M)}{2M}}(x_1+x_2),$$

2个$m$的同向振动；

$$\hat{q}_2=\sqrt{\frac{m}{2}}(x_1-x_2).$$

2个$m$的相向振动；

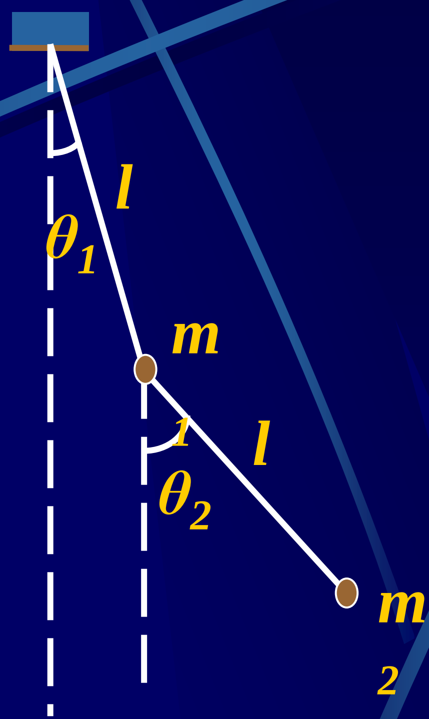
$$\hat{q}_i=c_i\cos(\omega_i t+\varphi_i)\quad(i=1,2)$$

$$T=\frac{1}{2}\dot{\hat{q}}_i\dot{\hat{q}}_i \qquad V(q)=\frac{1}{2}\omega_i^2\hat{q}_i\hat{q}_i$$

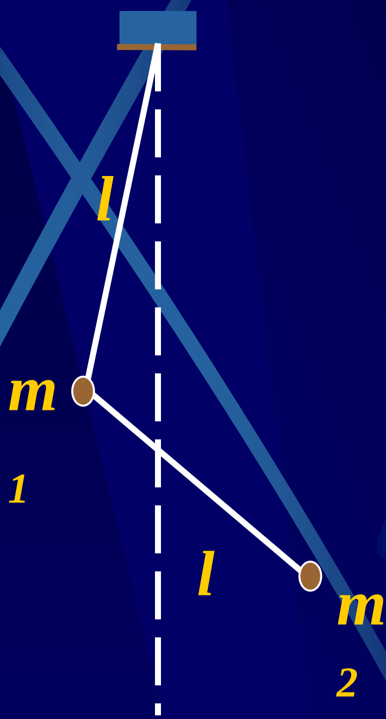
$x_1=\sqrt{\frac{M}{2m(2m+M)}}\hat{q}_1+\sqrt{\frac{1}{2m}}\hat{q}_2,\ x_2=\sqrt{\frac{M}{2m(2m+M)}}\hat{q}_1-\sqrt{\frac{1}{2m}}\hat{q}_2$ 2个$m$的振动；

$x_3=-\sqrt{\frac{m}{2M(2m+M)}}\hat{q}_1$，$M$的振动.

例：




简振模式1




简振模式2