Chapter 11

Radiation

- 11.1 Dipole Radiation
- 11.2 Point Charges

Dipole Radiation

What is Radiation?

In Chapter 9 we discussed the propagation of plane electromagnetic waves through various media, but how the waves got started in the first place? It takes accelerating charges, and changing currents, as we shall see how they radiate.

Assume the source is localized near the origin. Imagine a gigantic spherical shell, out at radius r ; the total power passing out through this surface is the integral of the Poynting vector:

$$P(r,t) = \oint \mathbf{S} \cdot d\mathbf{a} = \frac{1}{\mu_0} \oint (\mathbf{E} \times \mathbf{B}) \cdot d\mathbf{a}.$$

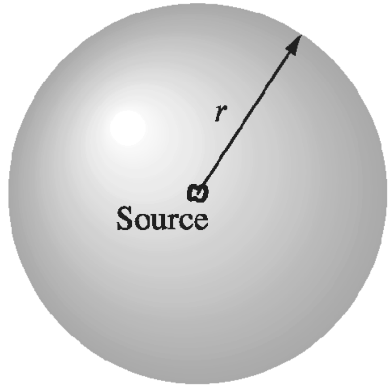


FIGURE 11.1

The power radiated is the limit of this quantity as r goes to infinity:

$$P_{\text{rad}}(t_0) = \lim_{r\to\infty} P\left(r, t_0 + \frac{r}{c}\right)$$

This is the energy (per unit time) that is transported out to infinity, and never comes back.

Now, the area of the sphere is $4\pi r^2$, so for radiation to occur the Poynting vector must decrease (at large r) no faster than $\frac{1}{r^2}$ (if it went like $\frac{1}{r^3}$, for example, then $\mathbf{P}(r)$ would go like $\frac{1}{r}$, and P_{rad} would be zero).

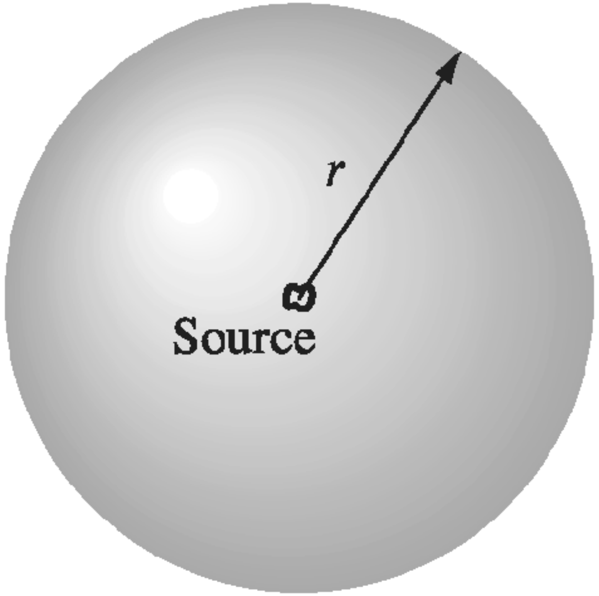


FIGURE 11.1

Electric Dipole Radiation

Picture two tiny metal spheres separated by a distance d and connected by a fine wire $(Fig. 11.2)$; at time t the charge on the upper sphere is $q(t)$, and the charge on the lower sphere is $-q(t)$. Suppose that we drive the charge back and forth through the wire, from one end to the other, at an angular frequency ω:

$$q(t) = q_0 \cos(\omega t)$$

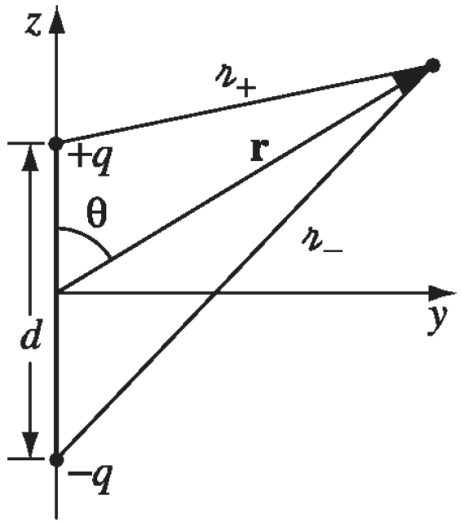


FIGURE 11.2

The result is an oscillating electric dipole

$$\mathbf{p}(t) = p_0 \cos(\omega t)\, \hat{\mathbf{z}}.$$

where

$$p_0 \equiv q_0 d$$

is the maximum value of the dipole moment.

The retarded potential is

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q_0\cos[\omega(t - \mathscr{r}_+/c)]}{\mathscr{r}_+} - \frac{q_0\cos[\omega(t - \mathscr{r}_-/c)]}{\mathscr{r}_-}\right\} \qquad (11.5)$$

where, by the law of cosines

$$\mathscr{r}_\pm = \sqrt{r^2 \mp rd\cos\theta + (d/2)^2}.$$

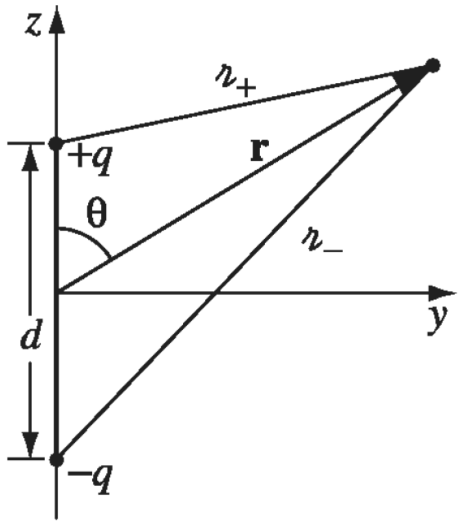


FIGURE 11.2

Now, to make this physical dipole into a perfect dipole, we want the separation distance to be extremely small:

$$\textbf{approximation 1}: \quad d \ll r.$$

what we want is an expansion carried to first order in d. Thus

$$\mathscr{r}_\pm \cong r\left(1 \mp \frac{d}{2r}\cos\theta\right)$$

It follows that
$$\frac{1}{\mathscr{r}_\pm} \cong \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right)$$

and
$$\cos[\omega(t - \mathscr{r}_\pm/c)] \cong \cos\left[\omega(t - r/c) \pm \frac{\omega d}{2c}\cos\theta\right]$$
$$= \cos[\omega(t - r/c)]\cos\left(\frac{\omega d}{2c}\cos\theta\right)$$
$$\mp \sin[\omega(t - r/c)]\sin\left(\frac{\omega d}{2c}\cos\theta\right)$$

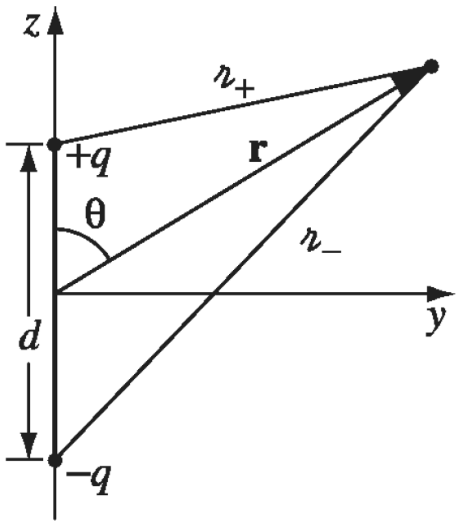


FIGURE 11.2

In the perfect dipole limit we have, further

$$\textbf{approximation 2}: \quad d \ll \frac{c}{\omega}$$

Since waves of frequency ω have a wavelength) $\lambda = {}^{2\pi c}/_{\omega}$, thus $d \ll \lambda$
Under these conditions,

$$\cos[\omega(t - \mathscr{r}_{\pm}/c)] \cong \cos[\omega(t - r/c)] \mp \frac{\omega d}{2c} \cos\theta \sin[\omega(t - r/c)]$$

Therefore

$$V(r, \theta, t) = \frac{p_0 \cos\theta}{4\pi\epsilon_0 r} \left\{ -\frac{\omega}{c} \sin[\omega(t - r/c)] + \frac{1}{r} \cos[\omega(t - r/c)] \right\}$$

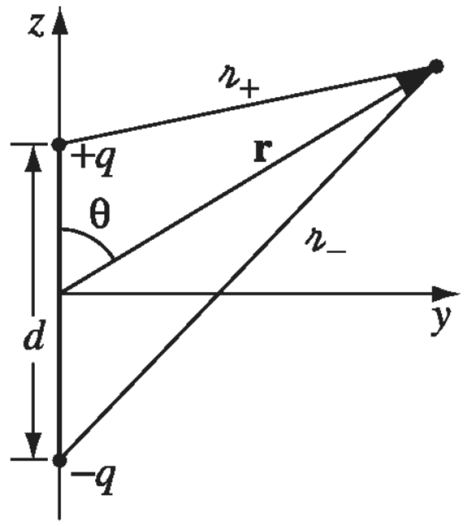


FIGURE 11.2

In the static limit ($\omega \longrightarrow 0$) the second term reproduces the old formula for the potential of a stationary dipole :

$$V = \frac{p_0 \cos\theta}{4\pi\epsilon_0 r^2}$$

However, we are interested in the fields that survive at *large distances from the source*, in the so-called radiation zone:

$$\textbf{approximation 3}: \quad r \gg \frac{c}{\omega}$$

(or, in terms of the wavelength, $r \gg \lambda$). In this region the potential reduces to

$$\boxed{V(r, \theta, t) = -\frac{p_0\omega}{4\pi\epsilon_0 c}\left(\frac{\cos\theta}{r}\right)\sin[\omega(t - r/c)].} \qquad (11.14)$$

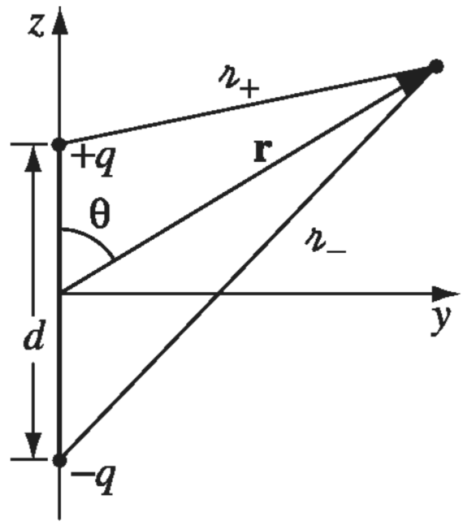


FIGURE 11.2

Meanwhile, the vector potential is determined by the current flowing in the wire:

$$\mathbf{I}(t) = \frac{dq}{dt}\,\hat{\mathbf{z}} = -q_0\omega\sin(\omega t)\,\hat{\mathbf{z}}$$

Referring to $Fig. 11.3$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int_{-d/2}^{d/2} \frac{-q_0 \omega \sin[\omega(t - \mathscr{r}/c)]\,\hat{\mathbf{z}}}{\mathscr{r}}\, dz.$$

$$\boxed{\mathbf{A}(r, \theta, t) = -\frac{\mu_0 p_0 \omega}{4\pi r} \sin[\omega(t - r/c)]\,\hat{\mathbf{z}}.}$$

From the potentials, it is a straightforward matter to compute the fields

$$\nabla V = \frac{\partial V}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\boldsymbol{\theta}}$$

$$= -\frac{p_0 \omega}{4\pi\epsilon_0 c}\left\{\cos\theta\left(-\frac{1}{r^2}\sin[\omega(t - r/c)] - \frac{\omega}{rc}\cos[\omega(t - r/c)]\right)\hat{\mathbf{r}} - \frac{\sin\theta}{r^2}\sin[\omega(t - r/c)]\,\hat{\boldsymbol{\theta}}\right\}$$

$$\cong \frac{p_0 \omega^2}{4\pi\epsilon_0 c^2}\left(\frac{\cos\theta}{r}\right)\cos[\omega(t - r/c)]\,\hat{\mathbf{r}}.$$

Here we dropped the first and last terms, in accordance with approximation 3

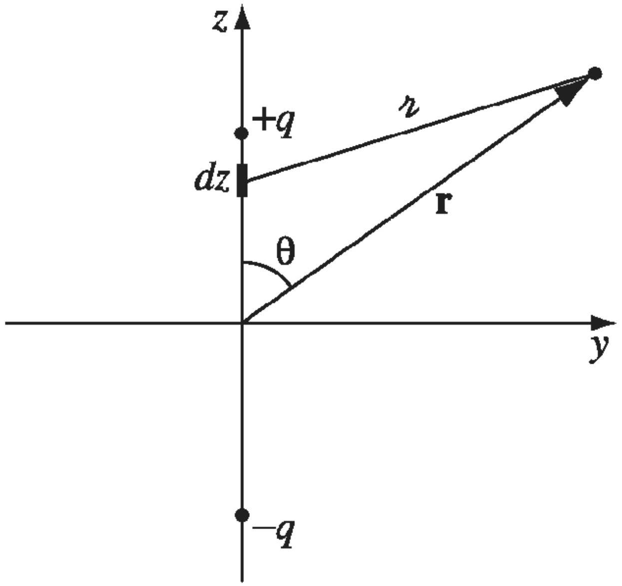


FIGURE 11.3

Likewise

$$\frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 p_0 \omega^2}{4\pi r} \cos[\omega(t - r/c)](\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}})$$

and therefore

$$\boxed{\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 p_0 \omega^2}{4\pi} \left(\frac{\sin\theta}{r}\right) \cos[\omega(t - r/c)]\,\hat{\boldsymbol{\theta}}.} \quad (11.18)$$

Meanwhile

$$\nabla \times \mathbf{A} = \frac{1}{r}\left[\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial A_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

$$= -\frac{\mu_0 p_0 \omega}{4\pi r}\left\{\frac{\omega}{c}\sin\theta\cos[\omega(t - r/c)] + \frac{\sin\theta}{r}\sin[\omega(t - r/c)]\right\}\hat{\boldsymbol{\phi}}$$

The second term is again eliminated by approximation 3, so

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A} = -\frac{\mu_0 p_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\,\hat{\boldsymbol{\phi}}.} \tag{11.19}$$

Equations 11.18 and 11.19 represent monochromatic waves of frequency ω traveling in the radial direction at the speed of light. The fields are in phase mutually perpendicular, and transverse; the ratio of their amplitudes is $E/B = c$. All of which is precisely what we expect for electromagnetic waves in free space. (These are actually *spherical* waves, not plane waves, and their amplitude decreases like $^1/_r$ as they progress. But for large *r*, they are approximately plane over small regions.

The energy radiated is determined by the Poynting vector:

$$\mathbf{S}(\mathbf{r}, t) = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}) = \frac{\mu_0}{c}\left\{\frac{p_0\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\right\}^2 \hat{\mathbf{r}} \tag{11.20}$$

The intensity is obtained by averaging (in time) over a complete cycle:

$$\langle \mathbf{S} \rangle = \left(\frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c} \right) \frac{\sin^2\theta}{r^2} \hat{\mathbf{r}}$$

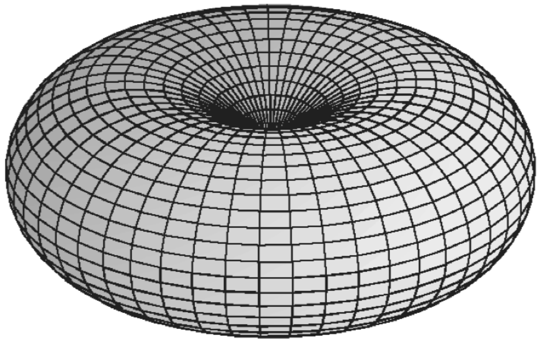


FIGURE 11.4

Notice that there is no radiation along the axis of the dipole (here $\sin\theta = 0$); the intensity profile takes the form of a donut, The total power radiated is found by integrating $< S >$ over a sphere of radius r:

$$\langle P \rangle = \int \langle \mathbf{S} \rangle \cdot d\mathbf{a} = \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c} \int \frac{\sin^2\theta}{r^2} r^2 \sin\theta \, d\theta \, d\phi = \frac{\mu_0 p_0^2 \omega^4}{12\pi c} \tag{11.22}$$

It is independent of the radius of the sphere, as one would expect from conversation of energy.

Example 11.1

The strong frequency dependence of the power formula is what accounts for the blueness of the sky. Sunlight passing through the atmosphere stimulates atoms to oscillate as tiny dipoles. The incident solar radiation covers a broad range of frequencies (white light), but the energy absorbed and reradiated by the atmospheric dipoles is stronger at the higher frequencies because of the ω^4 in $Eq. 11.22$. It is more intense in the blue than in the red. It is this reradiated light that you see when you look up in the sky-unless, of course, you're staring directly at the sun.

Because electromagnetic waves are transverse,
the dipoles oscillate in a plane orthogonal to the sun's rays.

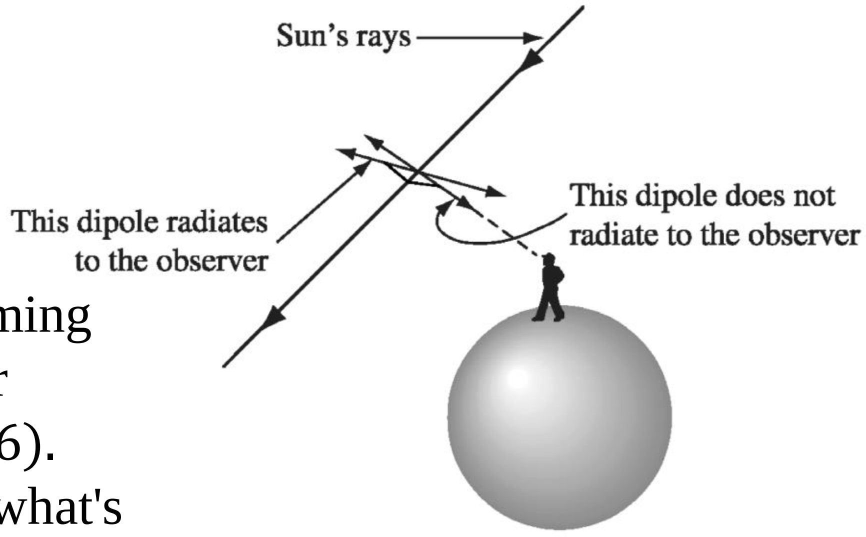


FIGURE 11.5

The redness of sunset is the other side of the same coin: Sunlight coming in at a tangent to the earth's surface must pass through a much longer stretch of atmosphere than sunlight coming from overhead ($Fig. 11.6$). Accordingly, much of the blue has been removed by scattering, and what's left is red.

Magnetic Dipole Radiation

Suppose now that we have a wire loop of radius b ($Fig. 11.8$), around which we drive an alternating current:

$$I(t) = I_0 \cos(\omega t)$$

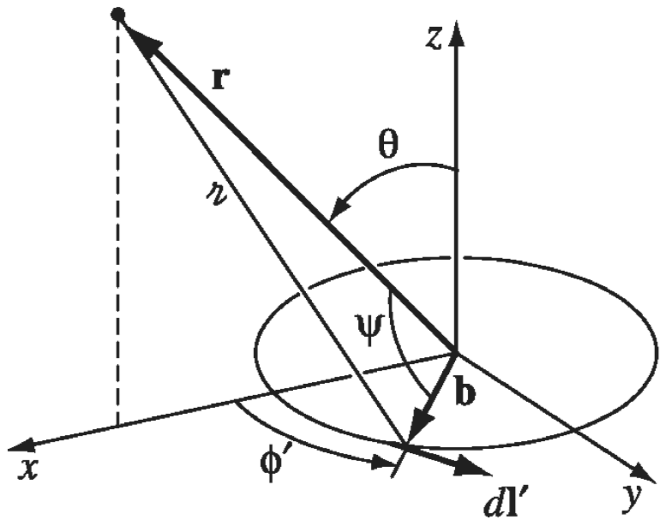


FIGURE 11.8

This is a model for an oscillating magnetic dipole

$$\mathbf{m}(t) = \pi b^2 I(t)\,\hat{\mathbf{z}} = m_0 \cos(\omega t)\,\hat{\mathbf{z}},$$

where $$m_0 \equiv \pi b^2 I_0$$

is the maximum value of the magnetic dipole moment.

The loop is uncharged, so the scalar potential is zero. The retarded vector potential is

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{I_0 \cos[\omega(t - \mathscr{r}/c)]}{\mathscr{r}}\, d\mathbf{l}'$$

For a point r directly above the x axis ($Fig.$ 11.8), **A** must aim in the y direction, since the x components from symmetrically placed points on either side of the x axis will cancel. Thus

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0 I_0 b}{4\pi} \hat{\mathbf{y}} \int_0^{2\pi} \frac{\cos[\omega(t - \imath/c)]}{\imath} \cos\phi' \, d\phi' \qquad (11.27)$$

$\cos\Phi'$serves to pick out the component of dl'. By the law of cosines,

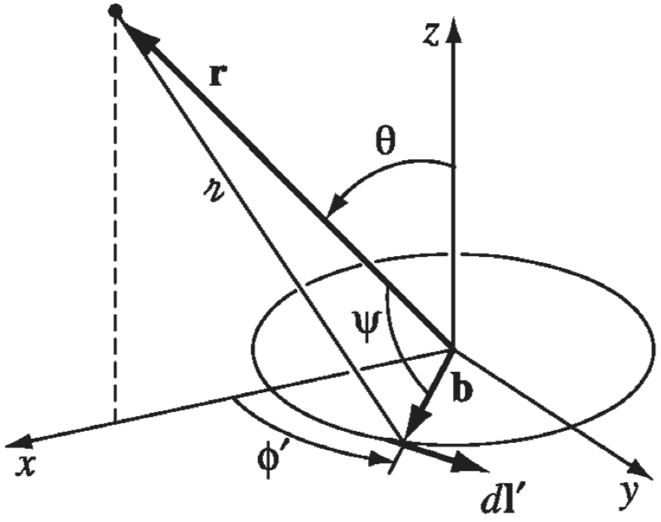


FIGURE 11.8

$$\imath = \sqrt{r^2 + b^2 - 2rb\cos\psi}$$

where ψ is the angle between the vectors r and b:

$$\mathbf{r} = r\sin\theta\,\hat{\mathbf{x}} + r\cos\theta\,\hat{\mathbf{z}}, \quad \mathbf{b} = b\cos\phi'\,\hat{\mathbf{x}} + b\sin\phi'\,\hat{\mathbf{y}}$$

So $rb\cos\psi = \mathbf{r}\cdot\mathbf{b} = rb\sin\theta\cos\phi'$, and therefore

$$\imath = \sqrt{r^2 + b^2 - 2rb\sin\theta\cos\phi'}$$

For a "perfect" dipole, we want the loop to be extremely small:

$$\textbf{approximation 1}: \quad b \ll r$$

To first order in b:

$$\mathscr{r} \cong r\left(1 - \frac{b}{r}\sin\theta\cos\phi'\right)$$

so $$\frac{1}{\mathscr{r}} \cong \frac{1}{r}\left(1 + \frac{b}{r}\sin\theta\cos\phi'\right)$$

and
$$\begin{aligned}\cos[\omega(t-\mathscr{r}/c)] &\cong \cos\left[\omega(t-r/c) + \frac{\omega b}{c}\sin\theta\cos\phi'\right] \\ &= \cos[\omega(t-r/c)]\cos\left(\frac{\omega b}{c}\sin\theta\cos\phi'\right) \\ &\quad - \sin[\omega(t-r/c)]\sin\left(\frac{\omega b}{c}\sin\theta\cos\phi'\right)\end{aligned}$$

As before, we also assume the size of the dipole is small compared to the wavelength radiated:

$$\textbf{approximation 2}: \quad b \ll \frac{c}{\omega}$$

In that case,

$$\cos[\omega(t - \mathscr{r}/c)] \cong \cos[\omega(t - r/c)] - \frac{\omega b}{c} \sin\theta \cos\phi' \sin[\omega(t - r/c)]. \qquad (11.32)$$

Inserting $Eqs.$ 11.30 and 11.32 into $Eq.$ 11.27, and dropping the second-order term:

$$\mathbf{A}(\mathbf{r}, t) \cong \frac{\mu_0 I_0 b}{4\pi r} \hat{\mathbf{y}} \int_0^{2\pi} \left\{ \cos[\omega(t - r/c)] + b \sin\theta \cos\phi' \right.$$
$$\left. \times \left(\frac{1}{r} \cos[\omega(t - r/c)] - \frac{\omega}{c} \sin[\omega(t - r/c)] \right) \right\} \cos\phi' \, d\phi'$$

$$\mathbf{A}(\mathbf{r}, t) \cong \frac{\mu_0 I_0 b}{4\pi r} \hat{\mathbf{y}} \int_0^{2\pi} \left\{ \cos[\omega(t - r/c)] + b \sin\theta \cos\phi' \times \left(\frac{1}{r} \cos[\omega(t - r/c)] - \frac{\omega}{c} \sin[\omega(t - r/c)] \right) \right\} \cos\phi' \, d\phi'$$

The first term integrates to zero

$$\int_0^{2\pi} \cos\phi' \, d\phi' = 0$$

The second term involves the integral of cosine squared:

$$\int_0^{2\pi} \cos^2\phi' \, d\phi' = \pi$$

The vector potential of an oscillating perfect magnetic dipole is

$$\mathbf{A}(r, \theta, t) = \frac{\mu_0 m_0}{4\pi} \left(\frac{\sin\theta}{r} \right) \left\{ \frac{1}{r} \cos[\omega(t - r/c)] - \frac{\omega}{c} \sin[\omega(t - r/c)] \right\} \hat{\boldsymbol{\phi}}. \quad (11.33)$$

In the static limit ($\omega = 0$) we recover the familiar formula for the potential of a magnetic dipole (Eq. 5.87)

$$\mathbf{A}(r, \theta) = \frac{\mu_0}{4\pi} \frac{m_0 \sin\theta}{r^2} \hat{\boldsymbol{\phi}}$$

In the radiation zone, **approximation 3 :** $r \gg \dfrac{c}{\omega}$

the first term in $\boldsymbol{A}$ is negligible, so

$$\boxed{\mathbf{A}(r,\theta,t) = -\frac{\mu_0 m_0 \omega}{4\pi c}\left(\frac{\sin\theta}{r}\right)\sin[\omega(t - r/c)]\,\hat{\boldsymbol{\phi}}.} \tag{11.35}$$

From A we obtain the fields at large r

$$\boxed{\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} = \frac{\mu_0 m_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\,\hat{\boldsymbol{\phi}},} \tag{11.36}$$

and

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A} = -\frac{\mu_0 m_0 \omega^2}{4\pi c^2}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\,\hat{\boldsymbol{\theta}}.} \qquad (11.37)$$

These fields are in phase, mutually perpendicular, and transverse to the direction of propagation ($\mathbf{r}$). The ratio of their amplitudes is $E/B = c$, all of which is as expected for electromagnetic waves. They are, in fact, remarkably similar in structure to the fields of an oscillating electric dipole

The energy flux for magnetic dipole radiation is

$$\mathbf{S}(\mathbf{r}, t) = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}) = \frac{\mu_0}{c}\left\{\frac{m_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\right\}^2 \hat{\mathbf{r}}$$

the intensity is

$$\langle \mathbf{S} \rangle = \left(\frac{\mu_0 m_0^2 \omega^4}{32\pi^2 c^3}\right)\frac{\sin^2\theta}{r^2}\,\hat{\mathbf{r}}$$

and the total radiated power is

$$\langle P \rangle = \frac{\mu_0 m_0^2 \omega^4}{12\pi c^3}$$

Once again, the intensity profile has the shape of a donut $(Fig. 11.4)$, and the power radiated goes like ω^4.

For configurations with comparable dimensions, the power radiated electrically is enormously greater. Comparing $Eqs. 11.22$ and 11.40

$$\frac{P_{\text{magnetic}}}{P_{\text{electric}}} = \left(\frac{m_0}{p_0 c}\right)^2$$

Where $m_0 = \pi b^2 I_0,$ and $p_0 = q_0 d$ Setting $d = \pi b$, for the sake of comparison

$$\frac{P_{\text{magnetic}}}{P_{\text{electric}}} = \left(\frac{\omega b}{c}\right)^2$$

Electric dipole

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 p_0 \omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\,\hat{\boldsymbol{\theta}}.$$

$$\mathbf{B} = \nabla \times \mathbf{A} = -\frac{\mu_0 p_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\,\hat{\boldsymbol{\phi}}.$$

Magnetic dipole

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} = \frac{\mu_0 m_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\,\hat{\boldsymbol{\phi}},$$

$$\mathbf{B} = \nabla \times \mathbf{A} = -\frac{\mu_0 m_0 \omega^2}{4\pi c^2}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\,\hat{\boldsymbol{\theta}}.$$

B that points in the $\widehat{\theta}$ direction and ***E*** in the $\widehat{\Phi}$ direction, whereas for electric dipoles it's the other way around.

Radiation from an Arbitrary Source

A configuration of charge and current that is entirely arbitrary, except that it is localized within some finite volume near the origin ($Fig.$ 11.9). The retarded scalar potential is

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t - \mathscr{r}/c)}{\mathscr{r}} d\tau'$$

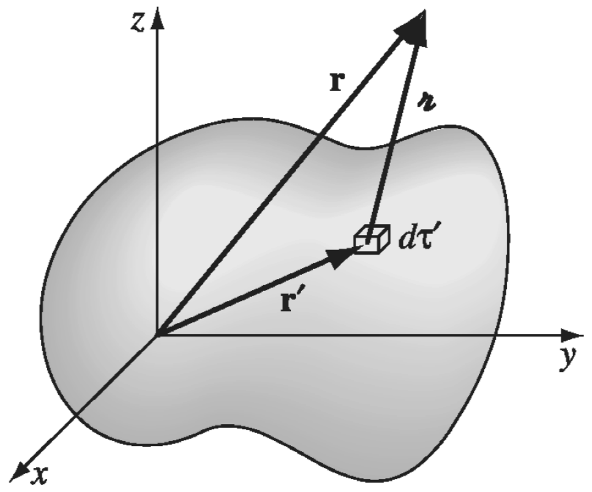


FIGURE 11.9

where $\mathscr{r} = \sqrt{r^2 + r'^2 - 2\mathbf{r}\cdot\mathbf{r}'}$.

we shall assume that the field point r is far away, in comparison to the dimensions of the source:

$$\textbf{approximation 1}: \quad r' \ll r$$

On this assumption $\mathscr{r} \cong r\left(1 - \frac{\mathbf{r}\cdot\mathbf{r}'}{r^2}\right)$

so $$\frac{1}{\mathscr{r}} \cong \frac{1}{r}\left(1 + \frac{\mathbf{r}\cdot\mathbf{r}'}{r^2}\right)$$

and $$\rho(\mathbf{r}', t - \mathscr{r}/c) \cong \rho\left(\mathbf{r}', t - \frac{r}{c} + \frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}\right)$$

Expanding ρ as a Taylor series in t about the retarded time at the origin,

$$t_0 \equiv t - \frac{r}{c} \qquad (11.48)$$

we have $$\rho(\mathbf{r}', t - \mathscr{r}/c) \cong \rho(\mathbf{r}', t_0) + \dot{\rho}(\mathbf{r}', t_0)\left(\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}\right) + \ldots \qquad (11.49)$$

where the dot signifies differentiation with respect to time. The next terms in the series would be

$$\frac{1}{2}\ddot{\rho}\left(\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}\right)^2, \quad \frac{1}{3!}\dddot{\rho}\left(\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}\right)^3, \ldots .$$

We can afford to drop them, provided

$$\textbf{approximation 2}: \quad r' \ll \frac{c}{|\ddot{\rho}/\dot{\rho}|}, \ \frac{c}{|\dddot{\rho}/\dot{\rho}|^{1/2}}, \ \frac{c}{|\ddddot{\rho}/\dot{\rho}|^{1/3}}, \ \ldots \tag{11.50}$$

For an oscillating system, each of these ratios is $^{c}/_{\omega}$, and we recover the old approximation 2. The general approximations 1 and 2 amount to keep only the first order terms in r'

Putting $Eqs.$ 11.47 and 11.49 into the formula for $\boldsymbol{V}$ ($Eq.$ 11.43), and again discarding the second-order term:

$$V(\mathbf{r}, t) \cong \frac{1}{4\pi\epsilon_0 r}\left[\int \rho(\mathbf{r}', t_0)\, d\tau' + \frac{\hat{\mathbf{r}}}{r} \cdot \int \mathbf{r}' \rho(\mathbf{r}', t_0)\, d\tau' + \frac{\hat{\mathbf{r}}}{c} \cdot \frac{d}{dt} \int \mathbf{r}' \rho(\mathbf{r}', t_0)\, d\tau'\right]$$

The first integral is simply the total charge, Q, at time t_0. Because charge is conserved, Q is independent of time. The other two integrals represent the electric dipole moment at time t_0. Thus

$$V(\mathbf{r}, t) \cong \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r} + \frac{\hat{\mathbf{r}} \cdot \mathbf{p}(t_0)}{r^2} + \frac{\hat{\mathbf{r}} \cdot \dot{\mathbf{p}}(t_0)}{rc}\right] \qquad (11.51)$$

In the static case, the first two terms are the monopole and dipole contributions to the multipole expansion for $\boldsymbol{V}$. Meanwhile, the vector potential is :

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}', t - \imath/c)}{\imath}\, d\tau'$$

To the first order in r' it suffices to replace $\imath$ by r in the integrand:

$$\mathbf{A}(\mathbf{r}, t) \cong \frac{\mu_0}{4\pi r}\int \mathbf{J}(\mathbf{r}', t_0)\, d\tau'$$

According to Prob. 5.7, the integral of $\boldsymbol{J}$ is the time derivative of the dipole moment, so

$$\mathbf{A}(\mathbf{r}, t) \cong \frac{\mu_0}{4\pi}\frac{\dot{\mathbf{p}}(t_0)}{r}$$

Next we must calculate the fields. Once again, we are interested in the radiation zone so we keep only those terms that go like $^{1}/_{r}$:

$$\textbf{approximation 3}: \quad \text{discard } 1/r^2 \text{ terms in } \mathbf{E} \text{ and } \mathbf{B}$$

For instance, the Coulomb field,

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}}$$

coming from the first term in $Eq.$ 11.51, does not contribute to the electromagnetic radiation. From $Eq.$ 11.48

$$t_0 \equiv t - \frac{r}{c} \qquad \nabla t_0 = -\frac{1}{c}\nabla r = -\frac{1}{c}\hat{\mathbf{r}}$$

and hence

$$\nabla V \cong \nabla\left[\frac{1}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}\cdot\dot{\mathbf{p}}(t_0)}{rc}\right] \cong \frac{1}{4\pi\epsilon_0}\left[\frac{\hat{\mathbf{r}}\cdot\ddot{\mathbf{p}}(t_0)}{rc}\right]\nabla t_0 = -\frac{1}{4\pi\epsilon_0 c^2}\frac{[\hat{\mathbf{r}}\cdot\ddot{\mathbf{p}}(t_0)]}{r}\hat{\mathbf{r}}$$

Similarly

$$\nabla \times \mathbf{A} \cong \frac{\mu_0}{4\pi r}[\nabla \times \dot{\mathbf{p}}(t_0)] = \frac{\mu_0}{4\pi r}[(\nabla t_0) \times \ddot{\mathbf{p}}(t_0)] = -\frac{\mu_0}{4\pi r c}[\hat{\mathbf{r}} \times \ddot{\mathbf{p}}(t_0)]$$

while $$\frac{\partial \mathbf{A}}{\partial t} \cong \frac{\mu_0}{4\pi} \frac{\ddot{\mathbf{p}}(t_0)}{r}$$

So $$\boxed{\mathbf{E}(\mathbf{r}, t) \cong \frac{\mu_0}{4\pi r}[(\hat{\mathbf{r}} \cdot \ddot{\mathbf{p}})\hat{\mathbf{r}} - \ddot{\mathbf{p}}] = \frac{\mu_0}{4\pi r}[\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \ddot{\mathbf{p}})],}$$

where $\ddot{p}$ is evaluated at time $t_0 = t - {}^{r}/_{c}$, and

$$\boxed{\mathbf{B}(\mathbf{r}, t) \cong -\frac{\mu_0}{4\pi r c}[\hat{\mathbf{r}} \times \ddot{\mathbf{p}}].}$$

In particular, if we use spherical polar coordinates, with the z axis in the direction of $\ddot{P}(t_0)$, then

$$\left.\begin{aligned} \mathbf{E}(r,\theta,t) &\cong \frac{\mu_0 \ddot{p}(t_0)}{4\pi}\left(\frac{\sin\theta}{r}\right)\hat{\boldsymbol{\theta}}, \\ \mathbf{B}(r,\theta,t) &\cong \frac{\mu_0 \ddot{p}(t_0)}{4\pi c}\left(\frac{\sin\theta}{r}\right)\hat{\boldsymbol{\phi}}. \end{aligned}\right\}$$

The Poynting vector is

$$\mathbf{S}(\mathbf{r},t) \cong \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \frac{\mu_0}{16\pi^2 c}[\ddot{p}(t_0)]^2\left(\frac{\sin^2\theta}{r^2}\right)\hat{\mathbf{r}}$$

and the total radiated power $(Eq. 11.2)$ is

$$P_{\text{rad}}(t_0) \cong \frac{\mu_0}{6\pi c}\left[\ddot{p}(t_0)\right]^2$$

Because charge is conserved, an electric monopole does not radiate.
If charge were not conserved, the first term in $Eq.$ 11.51 would read

$$V_{\text{mono}} = \frac{1}{4\pi\epsilon_0}\frac{Q(t_0)}{r}$$

and we would get a monopole field proportional to $^1/_r$:

$$\nabla t_0 = -\frac{1}{c}\nabla r = -\frac{1}{c}\hat{\mathbf{r}}$$

$$\mathbf{E}_{\text{mono}} = \frac{1}{4\pi\epsilon_0 c}\frac{\dot{Q}(t_0)}{r}\,\hat{\mathbf{r}}$$

$$V(\mathbf{r}, t) \cong \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r} + \frac{\hat{\mathbf{r}}\cdot\mathbf{p}(t_0)}{r^2} + \frac{\hat{\mathbf{r}}\cdot\dot{\mathbf{p}}(t_0)}{rc}\right] \qquad (11.51)$$

Point Charges

In Chapter 10 we derived the fields of a point charge q in arbitrary motion ($Eqs.$ 10.72 and 10.73):

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}} \cdot \mathbf{u})^3} \left[\left(c^2 - v^2 \right) \mathbf{u} + \boldsymbol{\mathscr{r}} \times (\mathbf{u} \times \mathbf{a}) \right] \qquad (11.62)$$

where $\mathbf{u} = c\hat{\boldsymbol{\mathscr{r}}} - \mathbf{v}$, and

$$\mathbf{B}(\mathbf{r}, t) = \frac{1}{c} \hat{\boldsymbol{\mathscr{r}}} \times \mathbf{E}(\mathbf{r}, t)$$

The first term in $Eq.$ 11.62 is the velocity field, and the second one is the acceleration field.

The Poynting vector is

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}) = \frac{1}{\mu_0 c}[\mathbf{E} \times (\hat{\boldsymbol{\mathscr{r}}} \times \mathbf{E})] = \frac{1}{\mu_0 c}[E^2 \hat{\boldsymbol{\mathscr{r}}} - (\hat{\boldsymbol{\mathscr{r}}} \cdot \mathbf{E})\mathbf{E}] \qquad (11.64)$$

The radiated energy is the stuff that, in effect, detaches itself from the charge and propagates off to infinity. To calculate the total power *radiated* by the particle at time t_r , we draw a huge sphere of radius $\mathscr{r}$ $(Fig. 11.10)$, centered at the position of the particle (at time t_r), wait the appropriate interval

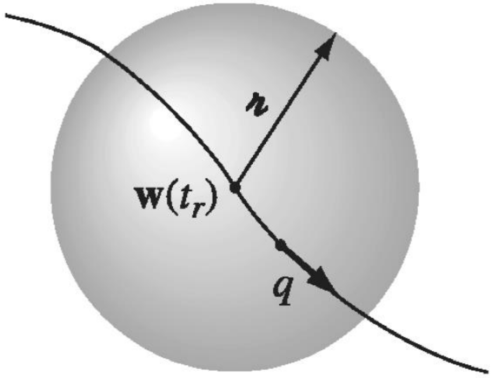


FIGURE 11.10

$$t - t_r = \frac{\mathscr{r}}{c}$$

for the radiation to reach the sphere, and at that moment integrate the Poynting vector over the surface.

Now, the area of the sphere is proportional to $\mathscr{r}^2$, so any term in $\boldsymbol{S}$ that goes like $1/\mathscr{r}^2$ will yield a finite answer, but terms like $1/\mathscr{r}^3$ or $1/\mathscr{r}^4$ will contribute nothing in the limit $\mathscr{r} \to \infty$. For this reason, only the *acceleration* fields represent true radiation (hence their other name, **radiation fields):**

$$\mathbf{E}_{\text{rad}} = \frac{q}{4\pi\epsilon_0} \frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}} \cdot \mathbf{u})^3} [\boldsymbol{\mathscr{r}} \times (\mathbf{u} \times \mathbf{a})]$$

Now $\boldsymbol{E}_{rad}$ is perpendicular to $\hat{\mathscr{r}}$, so the second term in $Eq. 11.64$ vanishes:

$$\mathbf{S}_{\text{rad}} = \frac{1}{\mu_0 c} E_{\text{rad}}^2 \, \hat{\mathscr{r}}$$

If the charge is instantaneously at rest (at time t_r), then $\mathbf{u} = c\hat{\mathscr{r}}$, and

$$\mathbf{E}_{\text{rad}} = \frac{q}{4\pi\epsilon_0 c^2 \mathscr{r}}[\hat{\mathscr{r}} \times (\hat{\mathscr{r}} \times \mathbf{a})] = \frac{\mu_0 q}{4\pi \mathscr{r}}[(\hat{\mathscr{r}} \cdot \mathbf{a})\,\hat{\mathscr{r}} - \mathbf{a}]$$

$$\mathbf{S}_{\text{rad}} = \frac{1}{\mu_0 c}\left(\frac{\mu_0 q}{4\pi \mathscr{r}}\right)^2 \left[a^2 - (\hat{\mathscr{r}} \cdot \mathbf{a})^2\right] \hat{\mathscr{r}} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\left(\frac{\sin^2\theta}{\mathscr{r}^2}\right) \hat{\mathscr{r}}, \qquad (11.69)$$

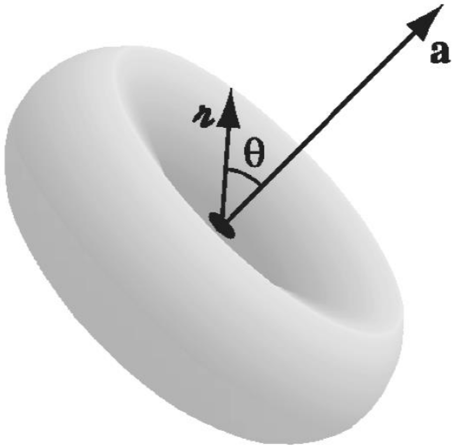


FIGURE 11.11

where θ is the angle between $\hat{\mathscr{r}}$ and **a.** No power is radiated in the forward or backward direction. It is emitted in a donut about the direction of instantaneous acceleration ($Fig. 11.11$).

The total power radiated is :

$$P = \oint \mathbf{S}_{\text{rad}} \cdot d\mathbf{a} = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \int \frac{\sin^2\theta}{r^2} r^2 \sin\theta \, d\theta \, d\phi, \qquad (11.70)$$

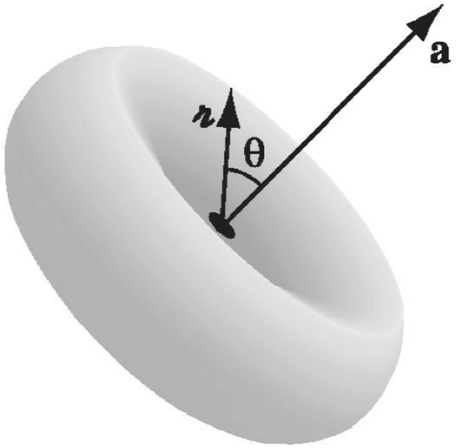


FIGURE 11.11

or $$\boxed{P = \frac{\mu_0 q^2 a^2}{6\pi c}.}$$

This is the Larmor formula.

Eqs. 11.69 and 11.70 actually hold to good approximation as long as $v \ll c$. An exact treatment of the case $v \neq 0$ is harder, both for the obvious reason that $\boldsymbol{E}_{rad}$ is more complicated, and also $\boldsymbol{S}_{rad}$, The rate at which energy passes through the sphere, is not the same as the rate at which energy left the particle. Suppose someone is firing a stream of bullets out the window of a moving car ($Fig. 11.12$). **The rate N_t at which the bullets strike a stationary target is not the same as the rate N_g at which they left the gun, because of the motion of the car**. One can prove that

$$N_g = \left(1 - \frac{\hat{\mathscr{r}} \cdot \mathbf{v}}{c}\right) N_t$$

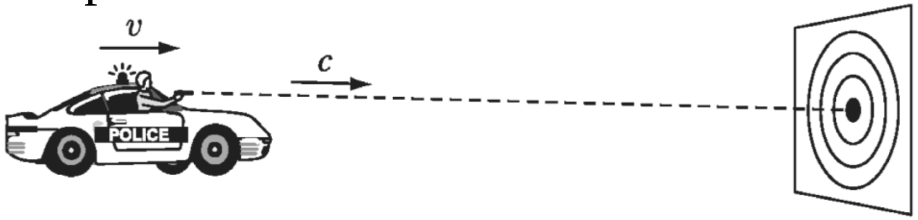


FIGURE 11.12

for arbitrary directions (here **v** is the velocity of the car, c is that of the bullets relative to the ground, and $\mathscr{r}$ is a unit vector from car to target). In our case, if $^{dW}/_{dt}$ is the rate at which energy passes through the sphere at radius, then the rate at which energy left the charge was

$$\frac{dW}{dt_r} = \frac{dW/dt}{\partial t_r/\partial t} = \left(\frac{\mathscr{r} \cdot \mathbf{u}}{\mathscr{r} c}\right) \frac{dW}{dt}$$

$$\boxed{\frac{\partial t_r}{\partial t}} = \frac{\mathscr{r} c}{\mathscr{r} \cdot \mathbf{u}}$$

The volume counted during time dt depends on **v** and $\mathscr{r}$

Here we use $Eq. 10.78$ (Prob. 10.19) to express $^{\partial t_r}/_{\partial t}$. But

$$\mathbf{u} = c\hat{\mathscr{r}} - \mathbf{v}, \qquad \frac{\mathscr{r} \cdot \mathbf{u}}{\mathscr{r} c} = 1 - \frac{\hat{\mathscr{r}} \cdot \mathbf{v}}{c}$$

which is precisely the ratio of N_g to N_t; it's a purely geometrical factor (the same as in the Doppler effect).

The power into a patch of area $\mathscr{r}^2 \sin\theta \, d\theta \, d\phi = \mathscr{r}^2 \, d\Omega$ on the sphere is therefore given by

$$\mathbf{S}_{\text{rad}} = \frac{1}{\mu_0 c} E_{\text{rad}}^2 \hat{\boldsymbol{\mathscr{r}}} \Longrightarrow \frac{dP}{d\Omega} = \left(\frac{\boldsymbol{\mathscr{r}} \cdot \mathbf{u}}{\mathscr{r} c}\right) \frac{1}{\mu_0 c} E_{\text{rad}}^2 \mathscr{r}^2 = \frac{q^2}{16\pi^2 \epsilon_0} \frac{|\hat{\boldsymbol{\mathscr{r}}} \times (\mathbf{u} \times \mathbf{a})|^2}{(\hat{\boldsymbol{\mathscr{r}}} \cdot \mathbf{u})^5}$$

where $d\Omega = \sin\theta \, d\theta \, d\phi$, is the **solid angle** into which this power is radiated. Integrating over θ and Φ to get total power radiated :

$$P = \frac{\mu_0 q^2 \gamma^6}{6\pi c} \left(a^2 - \left| \frac{\mathbf{v} \times \mathbf{a}}{c} \right|^2 \right)$$

$$\boxed{P = \frac{\mu_0 q^2 a^2}{6\pi c}.}$$

where $\gamma \equiv 1/\sqrt{1 - v^2/c^2}$. This is **Lienard's generalization** of the Larmor formula (to which it reduces when $v \ll c$). The factor γ^6 means that the radiated power increases enormously as the particle velocity approaches the speed of light.

Example 11.3

Suppose $\boldsymbol{v}$ and $\boldsymbol{a}$ are instantaneously collinear (at time t_r), as, for example, in straight-line motion. Find the angular distribution of the radiation ($Eq. 11.72$) and the total power emitted.

$$\frac{dP}{d\Omega} = \left(\frac{\boldsymbol{\mathscr{r}} \cdot \mathbf{u}}{\mathscr{r} c}\right) \frac{1}{\mu_0 c} E_{\text{rad}}^2 \mathscr{r}^2 = \frac{q^2}{16\pi^2\epsilon_0} \frac{|\hat{\boldsymbol{\mathscr{r}}} \times (\mathbf{u} \times \mathbf{a})|^2}{(\hat{\boldsymbol{\mathscr{r}}} \cdot \mathbf{u})^5}$$

Solution

$$\mathbf{u} = c\hat{\boldsymbol{\mathscr{r}}} - \mathbf{v}$$

In this case $(\mathbf{u} \times \mathbf{a}) = c(\hat{\boldsymbol{\mathscr{r}}} \times \mathbf{a})$, so

$$\frac{dP}{d\Omega} = \frac{q^2c^2}{16\pi^2\epsilon_0} \frac{|\hat{\boldsymbol{\mathscr{r}}} \times (\hat{\boldsymbol{\mathscr{r}}} \times \mathbf{a})|^2}{(c - \hat{\boldsymbol{\mathscr{r}}} \cdot \mathbf{v})^5}$$

Now $\hat{\boldsymbol{\mathscr{r}}} \times (\hat{\boldsymbol{\mathscr{r}}} \times \mathbf{a}) = (\hat{\boldsymbol{\mathscr{r}}} \cdot \mathbf{a})\hat{\boldsymbol{\mathscr{r}}} - \mathbf{a},$ so $|\hat{\boldsymbol{\mathscr{r}}} \times (\hat{\boldsymbol{\mathscr{r}}} \times \mathbf{a})|^2 = a^2 - (\hat{\boldsymbol{\mathscr{r}}} \cdot \mathbf{a})^2$

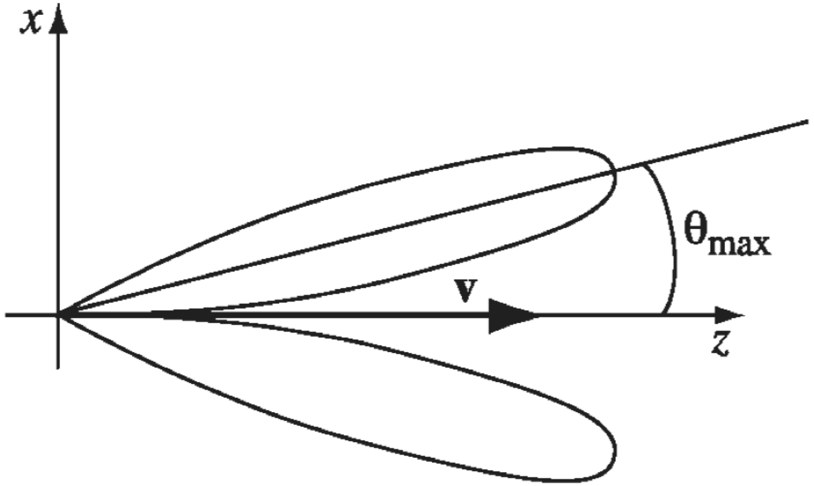
In particular, if we let the z axis point along v, then

$$\frac{dP}{d\Omega} = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \frac{\sin^2\theta}{(1 - \beta\cos\theta)^5}$$

$$\beta \equiv v/c$$

For very large v $(\beta \approx 1)$ the donut of radiation $(Fig. 11.11)$ is stretched out and pushed forward by the factor $(1-\beta\cos\theta)^{-5}$, as indicated in $Fig. 11.13$

The total power emitted is found by integrating $Eq. 11.74$ over all angles:



$$P = \int \frac{dP}{d\Omega} d\Omega = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \int \frac{\sin^2\theta}{(1-\beta\cos\theta)^5} \sin\theta \, d\theta \, d\phi$$

The ϕ integral is 2π; the θ integral is simplified by the substitution $x \equiv \cos\theta$:

FIGURE 11.13

$$P = \frac{\mu_0 q^2 a^2}{8\pi c} \int_{-1}^{+1} \frac{(1-x^2)}{(1-\beta x)^5} dx$$

$$P = \frac{\mu_0 q^2 a^2}{8\pi c} \int_{-1}^{+1} \frac{(1 - x^2)}{(1 - \beta x)^5} dx$$

Integration by parts yields $\frac{4}{3}(1 - \beta^2)^{-3}$, therefore we have

$$P = \frac{\mu_0 q^2 a^2 \gamma^6}{6\pi c}$$

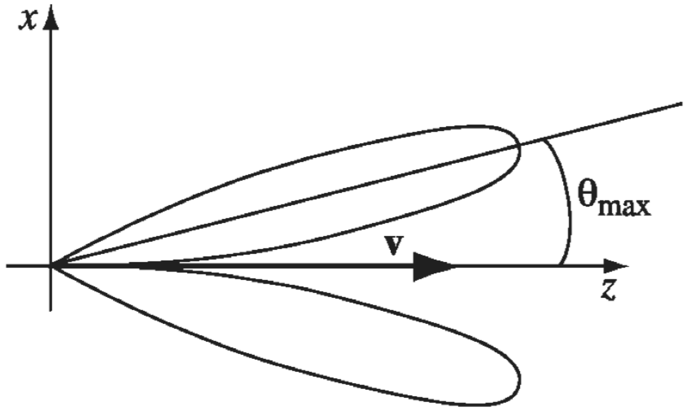


FIGURE 11.13

- This result is consistent with the Lienard formula ($Eq. 11.73$), for the case of collinear v and a.
- Notice that the angular distribution of the radiation is the same whether the particle is *accelerating* or *decelerating;* it only depends on the square of a and is concentrated in the forward direction (with respect to the velocity) in either case.
- When a high-speed electron hits a metal target it rapidly decelerates, giving off what is called **bremsstrahlung**, or "braking radiation."

Radiation Reaction

Radiation carries off energy, which comes, ultimately, at the expense of the particle's kinetic energy, In this section we'll derive the radiation reaction force from the conservation of energy.

For a nonrelativistic particle ($v \ll c$), the total power radiated is given by the Larmor formula ($Eq. 11.70$)

$$P = \frac{\mu_0 q^2 a^2}{6\pi c}$$

Conservation of energy suggests that this is also the rate at which the particle loses energy, under the influence of the radiation reaction force F_{rad}:

$$\mathbf{F}_{\text{rad}} \cdot \mathbf{v} = -\frac{\mu_0 q^2 a^2}{6\pi c}$$

This equation is actually wrong. For we calculated the radiated power by integrating the Poynting vector over a sphere of "infinite" radius; in this calculation the velocity fields played no part. But the velocity fields *do* carry energy-they just don't transport it out to infinity.

The energy lost by the particle in any given time interval, then, must equal the energy carried away by the radiation plus whatever extra energy has been pumped into the velocity fields. If we agree to consider only intervals over which the system returns to its initial state, then the energy in the velocity fields is the same at both ends, and the only *net* loss is in the form of radiation. Thus $Eq. 11.77,$ while incorrect instantaneously, is valid on the *average:*

$$\int_{t_1}^{t_2} \mathbf{F}_{\text{rad}} \cdot \mathbf{v}\, dt = -\frac{\mu_0 q^2}{6\pi c} \int_{t_1}^{t_2} a^2\, dt,$$

with the stipulation that the state of the system is identical at t_1 and t_2. In the case of periodic motion, for instance, we must integrate over an integral number of full cycles. Now, the right side of $Eq. 11.78$ can be integrated by parts:

$$\int_{t_1}^{t_2} a^2\, dt = \int_{t_1}^{t_2} \left(\frac{d\mathbf{v}}{dt}\right) \cdot \left(\frac{d\mathbf{v}}{dt}\right) dt = \left(\mathbf{v} \cdot \frac{d\mathbf{v}}{dt}\right)\Bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d^2\mathbf{v}}{dt^2} \cdot \mathbf{v}\, dt$$

The boundary term drops out, since the velocities and accelerations are identical at t_1 and t_2, so $Eq.$ 11.78 can be written equivalently as

$$\int_{t_1}^{t_2} \left(\mathbf{F}_{\text{rad}} - \frac{\mu_0 q^2}{6\pi c} \dot{\mathbf{a}} \right) \cdot \mathbf{v}\, dt = 0 \tag{11.79}$$

will certainly be satisfied if

$$\boxed{\mathbf{F}_{\text{rad}} = \frac{\mu_0 q^2}{6\pi c} \dot{\mathbf{a}}.} \tag{11.80}$$

This is the **Abraham-Lorentz formula** for the radiation reaction force.

$Eq.$ 11.79 doesn‘t prove $Eq.$ 11.80, it only tells you the *time average* of the parallel component. We'll see in the next section, there are other reasons for believing in the **Abraham-Lorentz formula**, but for now, It represents the *simplest* form the radiation reaction force could take, consistent with conservation of energy.

The Mechanism Responsible for the Radiation Reaction

- Let's avoid this problem by considering an *extended* charge distribution, for which the field is finite everywhere; at the end, we'll take the limit as the size of the charge goes to zero.
- In general, the electromagnetic force of one part (A) on another part (B) is *not* equal and opposite to the force of *B* on *A* (Fig. 11.16).
- If the distribution is divided up into infinitesimal chunks, and the imbalances are added up for all such pairs, the result is a *net force of the charge* on itself.
- It is this **self-force,** resulting from the breakdown of Newton's third law within the structure of the particle, that accounts for the radiation reaction.

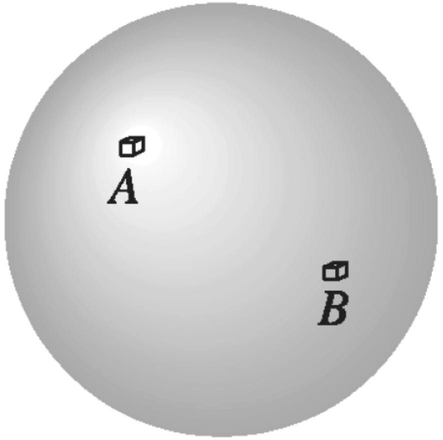


FIGURE 11.16

$$\mathbf{E}_{\text{rad}} = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}[\boldsymbol{\mathscr{r}}\times(\mathbf{u}\times\mathbf{a})]$$

Using a less realistic model: a "dumbbell" in which the total charge q is divided into two halves separated by a fixed distance d ($Fig.$ 11.17). This is the simplest possible arrangement of the charge that permits the essential mechanism (imbalance of internal electromagnetic forces) to function.

Let's assume the dumbbell moves in the x direction and is (instantaneously) at rest at the retarded time. The electric field at (1) due to (2) is

$$\mathbf{E}_1 = \frac{(q/2)}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}\left[(c^2+\boldsymbol{\mathscr{r}}\cdot\mathbf{a})\mathbf{u}-(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})\mathbf{a}\right]$$

where $\mathbf{u} = c\,\hat{\boldsymbol{\mathscr{r}}}$ and $\boldsymbol{\mathscr{r}} = l\,\hat{\mathbf{x}} + d\,\hat{\mathbf{y}}$

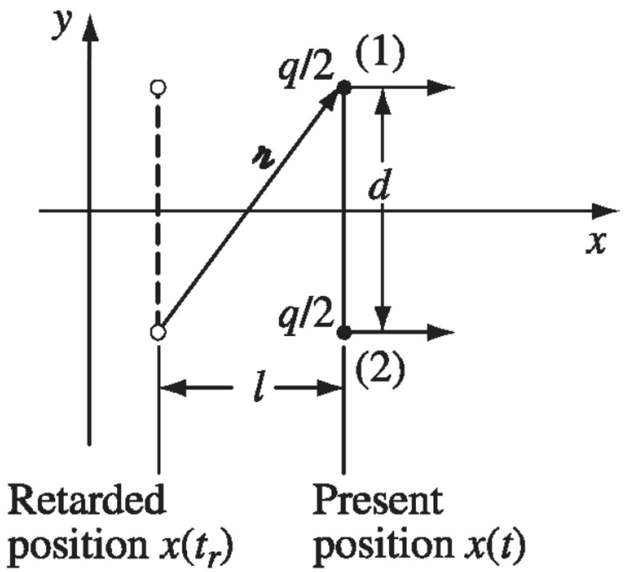


FIGURE 11.17

$$\mathbf{E}_1 = \frac{(q/2)}{4\pi\epsilon_0} \frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}} \cdot \mathbf{u})^3} \left[(c^2 + \boldsymbol{\mathscr{r}} \cdot \mathbf{a})\mathbf{u} - (\boldsymbol{\mathscr{r}} \cdot \mathbf{u})\mathbf{a} \right]$$

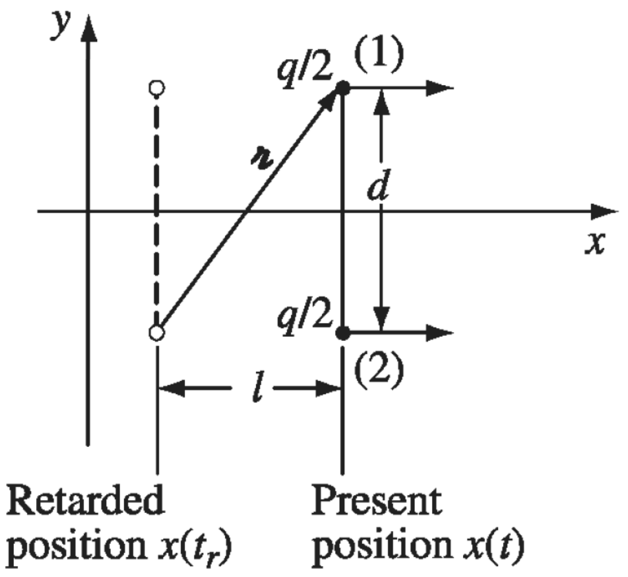


FIGURE 11.17

We're only interested in the x component of $E1$, since the y components will cancel when we add the forces on the two ends (for the same reason, we don't need to worry about the magnetic forces). Now

$$u_x = \frac{cl}{\mathscr{r}}, \quad \boldsymbol{\mathscr{r}} \cdot \mathbf{u} = c\mathscr{r}, \quad \boldsymbol{\mathscr{r}} \cdot \mathbf{a} = la, \quad \text{and} \quad \mathscr{r} = \sqrt{l^2 + d^2}.$$

and hence $$E_{1_x} = \frac{q}{8\pi\epsilon_0 c^2} \frac{(lc^2 - ad^2)}{(l^2 + d^2)^{3/2}}.$$

By symmetry, $E_{2_x} = E_{1_x}$, so the net force on the dumbbell is

$$\mathbf{F}_{\text{self}} = \frac{q}{2}(\mathbf{E}_1 + \mathbf{E}_2) = \frac{q^2}{8\pi\epsilon_0 c^2} \frac{(lc^2 - ad^2)}{(l^2 + d^2)^{3/2}} \hat{\mathbf{x}}.$$

The idea now is to expand in powers of d; when the size of the particle goes to zero, Using Taylor's theorem

$$x(t) = x(t_r) + \dot{x}(t_r)(t-t_r) + \frac{1}{2}\ddot{x}(t_r)(t-t_r)^2 + \frac{1}{3!}\dddot{x}(t_r)(t-t_r)^3 + \cdots$$

we have, $$l = x(t) - x(t_r) = \frac{1}{2}aT^2 + \frac{1}{6}\dot{a}T^3 + \cdots$$

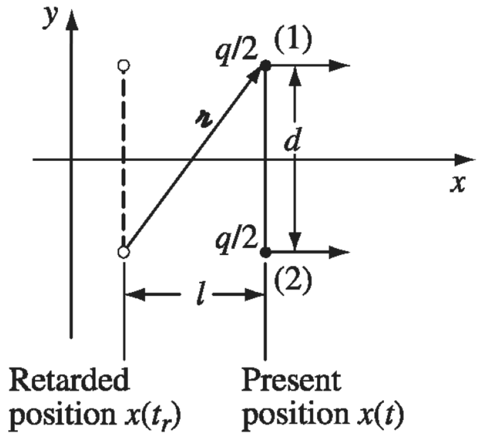


FIGURE 11.17

where $T \equiv t - t_r$. Now T is determined by the retarded time condition

$$(cT)^2 = l^2 + d^2$$

so $$d = \sqrt{(cT)^2 - l^2} = cT\sqrt{1 - \left(\frac{aT}{2c} + \frac{\dot{a}T^2}{6c} + \cdots\right)^2} = cT - \frac{a^2}{8c}T^3 + (\ )T^4 + \cdots$$

we need to "solve" it for T as a function of d. Ignoring all higher powers of T

$$d \cong cT \quad \Rightarrow \quad T \cong \frac{d}{c}$$

using this as an approximation for the cubic term,

$$d \cong cT - \frac{a^2}{8c}\frac{d^3}{c^3} \quad \Rightarrow \quad T \cong \frac{d}{c} + \frac{a^2 d^3}{8c^5}$$

and so on

$$T = \frac{1}{c}d + \frac{a^2}{8c^5}d^3 + (\ )d^4 + \cdots$$

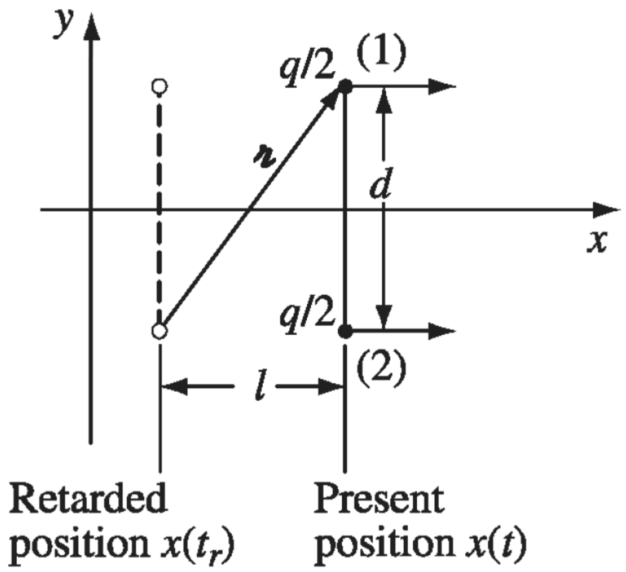


FIGURE 11.17

Returning to $Eq.$ 11.91, we construct the power series for l in terms of d:

$$l = x(t) - x(t_r) = \frac{1}{2}aT^2 + \frac{1}{6}\dot{a}T^3 + \cdots$$

$$l = \frac{a}{2c^2}d^2 + \frac{\dot{a}}{6c^3}d^3 + (\ )d^4 + \cdots$$

Putting this into Eq. 11.90

$$\mathbf{F}_{\text{self}} = \frac{q^2}{4\pi\epsilon_0}\left[-\frac{a}{4c^2 d} + \frac{\dot{a}}{12c^3} + (\ )d + \cdots\right]\hat{\mathbf{x}}.$$

Here a and its time derivative are evaluated at the retarded time (t_r), but it's easy to rewrite the result in terms of the present time t:

$$a(t_r) = a(t) + \dot{a}(t)(t_r - t) + \cdots = a(t) - \dot{a}(t)T + \cdots = a(t) - \dot{a}(t)\frac{d}{c} + \cdots$$

$$\mathbf{F}_{\text{self}} = \frac{q^2}{4\pi\epsilon_0}\left[-\frac{a(t)}{4c^2 d} + \frac{\dot{a}(t)}{3c^3} + (\ )d + \cdots\right]\hat{\mathbf{x}}$$

The first term on the right is proportional to the acceleration of the charge; if we pull it over to the other side of Newton's second law, it simply adds to the dumbbell's mass.

$$m = 2m_0 + \frac{1}{4\pi\epsilon_0}\frac{q^2}{4dc^2}$$

where m_0 is the mass of either end alone. In the context of special relativity, it is not surprising that the electrical repulsion of the charges should enhance the mass of the dumbbell. For the potential energy of this configuration (in the static case) is

$$\frac{1}{4\pi\epsilon_0}\frac{(q/2)^2}{d}$$

and according to Einstein's formula $E = mc^2$, this energy contributes to the inertia of the object.

The second term in Eq. 11.96 is the radiation reaction:

$$F_{\text{rad}}^{\text{int}} = \frac{\mu_0 q^2 \dot{a}}{12\pi c}.$$

It alone (apart from the mass correction) survives in the "point dumbbell" limit $d \to 0$.

It differs from the Abraham-Lorentz formula by a factor of 2. But then, this is only the self-force associated with the interaction between 1 and 2. There remains the force of each end on itself. When the latter is included (see $Prob.$ 11.20), the result is

$$F_{\text{rad}} = \frac{\mu_0 q^2 \dot{a}}{6\pi c}$$

reproducing the Abraham-Lorentz formula exactly. Conclusion: The radiation reaction is due to the force of the charge on itself.