# 电动力学-第九次作业

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Problem 10.16

Answer: The potential of the particle is equal to:

$$V(\vec{r},t)=\frac{1}{4\pi\epsilon_0}\frac{qc}{\sqrt{\left(c^2t-\vec{r}\cdot\vec{v}\right)^2+\left(c^2-v^2\right)\left(r^2-c^2t^2\right)}}$$

We need to somehow "pull out" the factor of $Rc$ from the denominator. The expression under the root is:

$$\begin{aligned}
&\left(c^2t-\vec{r}\cdot\vec{v}\right)^2+\left(c^2-v^2\right)\left(r^2-c^2t^2\right)=\\
=&c^4t^2-2c^2t\vec{r}\cdot\vec{v}+(\vec{r}\cdot\vec{v})^2+c^2r^2-c^4t^2-v^2r^2+v^2c^2t^2\\
=&-2c^2t\vec{r}\cdot\vec{v}+(\vec{r}\cdot\vec{v})^2+c^2r^2-v^2r^2+v^2c^2t^2
\end{aligned}$$

$$\begin{aligned}
&\vec{R}=\vec{r}-\vec{v}t\Longrightarrow R^2=r^2+v^2t^2-2t\vec{v}\cdot\vec{r}\\
&\left(c^2t-\vec{r}\cdot\vec{v}\right)^2+\left(c^2-v^2\right)\left(r^2-c^2t^2\right)=\\
=&c^2\left(r^2+v^2t^2-2t\vec{v}\cdot\vec{r}\right)+(\vec{r}\cdot\vec{v})^2-v^2r^2\\
=&c^2R^2+(\vec{r}\cdot\vec{v})^2-v^2r^2
\end{aligned}$$

Now, the two other two terms are equal to:

$$\begin{aligned}
(\vec{r}\cdot\vec{v})^2-v^2r^2&=((\vec{R}+\vec{v}t)\cdot\vec{v})^2-v^2\left(R^2+2t\vec{r}\cdot\vec{v}-v^2t^2\right)\\
&=v^2R^2\cos^2\theta+2(\vec{R}\cdot\vec{v})v^2t+v^4t^2-v^2R^2+2v^2t\vec{r}\cdot\vec{v}+v^4t^2\\
&=v^2R^2\left(\cos^2\theta-1\right)+2((\vec{r}-\vec{v}t)\cdot\vec{v})v^2t+2v^2t\vec{r}\cdot\vec{v}+2v^4t^2\\
&=-v^2R^2\sin^2\theta+2v^2t\vec{r}\cdot\vec{v}-2v^4t^2-2v^2t\vec{r}\cdot\vec{v}+2v^4t^2\\
&=-v^2R^2\sin^2\theta
\end{aligned}$$

where $\theta$ is the angle the velocity makes with the vector $\vec{R}$. The expression under the root is then:

$$\left(c^2t - \vec{r}\cdot\vec{v}\right)^2 + \left(c^2 - v^2\right)\left(r^2 - c^2t^2\right) = c^2R^2 - v^2R^2\sin^2\theta$$
$$= c^2R^2\left(1 - \beta^2\sin^2\theta\right)$$

So,the potential is:

$$V = \frac{1}{4\pi\epsilon_0}\frac{qc}{\sqrt{\left(c^2t - \vec{r}\cdot\vec{v}\right)^2 + \left(c^2 - v^2\right)\left(r^2 - c^2t^2\right)}}$$
$$= \frac{1}{4\pi\epsilon_0}\frac{qc}{Rc\sqrt{1 - \beta^2\sin^2\theta}} = \sqrt{\frac{1}{4\pi\epsilon_0}\frac{q}{R\sqrt{1 - \beta^2\sin^2\theta}}}$$

Q.E.D

Problem 10.19

Answer:

Due to the limitations of this site I will be using letter $l$ as the distance from the charge to the point of interest.

The Lienard-Wiechert vector potential of the charge in motion is:

$$\vec{A} = \frac{\mu_0}{4\pi}\frac{qc\vec{v}}{l_c - \vec{l}\cdot\vec{v}} = \frac{\mu_0}{4\pi}\frac{qc\vec{v}}{\vec{l}\cdot\vec{u}}$$

all quantities evaluated at retarded time. We want the partial time derivative of this, but first:

$$l = c\left(t - t_r\right)$$
$$l^2 = c^2\left(t - t_r\right)^2$$
$$2\vec{l}\cdot\frac{\partial\vec{l}}{\partial t} = 2c^2\left(t - t_r\right)\left(1 - \frac{\partial t_r}{\partial t}\right) \Rightarrow \frac{\partial t_r}{\partial t} = 1 - \frac{\hat{l}}{c}\cdot\frac{\partial\vec{l}}{\partial t}$$

But:

$$\vec{l} = \vec{r} - \vec{w}\left(t_r\right)$$
$$\frac{\partial\vec{l}}{\partial t} = -\frac{\partial\vec{w}}{\partial t_r}\frac{\partial t_r}{\partial t} = -\vec{v}\left(t_r\right)\frac{\partial t_r}{\partial t}$$
$$\frac{\partial t_r}{\partial t} = 1 + \frac{\hat{l}}{c}\cdot\vec{v}\left(t_r\right)\frac{\partial t_r}{\partial t}$$

So, as we need to prove first:

$$\frac{\partial t_r}{\partial t} = \frac{cl}{\vec{l}\cdot(c\hat{l}-\vec{v})} = \frac{cl}{\vec{l}\cdot\vec{u}}$$

With this we can get the partial time derivative of the vector potential:

$$\frac{\partial \vec{A}}{\partial t} = \frac{\mu_0 qc}{4\pi}\left[\frac{1}{\vec{l}\cdot\vec{u}}\frac{\partial \vec{v}}{\partial t} - \frac{\vec{v}}{(\vec{l}\cdot\vec{u})^2}\frac{\partial}{\partial t}(\vec{l}\cdot\vec{u})\right]$$

$$\frac{\partial \vec{v}}{\partial t} = \frac{\partial \vec{v}}{\partial t_r}\frac{\partial t_r}{\partial t} = \vec{a}\frac{cl}{\vec{l}\cdot\vec{u}}$$

$$\frac{\partial}{\partial t}(\vec{l}\cdot\vec{u}) = c\frac{\partial l}{\partial t} - \frac{\partial \vec{l}}{\partial t}\cdot\vec{v} - \frac{\partial \vec{v}}{\partial t}\cdot\vec{l} = c\frac{\partial l}{\partial t} + v^2\frac{cl}{\vec{l}\cdot\vec{u}} - (\vec{a}\cdot\vec{l})\frac{cl}{\vec{l}\cdot\vec{u}}$$

Hence:

$$\begin{aligned}\frac{\partial \vec{A}}{\partial t} &= \frac{\mu_0 qc}{4\pi}\frac{1}{(\vec{l}\cdot\vec{u})^3}\left[lc\vec{a}(\vec{l}\cdot\vec{u}) + \vec{v}\left(cl\left(c^2 - v^2 + \vec{l}\cdot\vec{a}\right) + c^2(\vec{l}\cdot\vec{u})\right)\right] \\ &= \frac{qc}{4\pi\epsilon_0}\left[(\vec{l}\cdot\vec{u})\left(\frac{l}{c}\vec{a} - \vec{v}\right) + \frac{l}{c}\vec{v}\left(c^2 - v^2 + \vec{r}\cdot\vec{a}\right)\right]\end{aligned}$$

Q.E.D

Problem 10.22

Answer:

The electric field due to the leght of the wire of lenght $dx$ is the field of the uniformly moving point charge:

$$d\vec{E} = \frac{\lambda dx}{4\pi\epsilon_0}\frac{1-\beta^2}{\left(1-\beta^2\sin^2\theta\right)^{3/2}}\frac{\hat{R}}{R^2}$$

Only the vertical component of the field will survive the integration, so:

$$\begin{aligned}dE =& \frac{\lambda dx}{4\pi\epsilon_0}\frac{1-\beta^2}{\left(1-\beta^2\sin^2\theta\right)^{3/2}}\frac{\sin\theta}{R^2} \\ & R = \frac{d}{\sin\theta} \quad -x = R\cos\theta = d\cot\theta \quad dx = d\frac{d\theta}{\sin^2\theta} \\ =& \frac{\lambda d}{4\pi\epsilon_0}\frac{1-\beta^2}{\left(1-\beta^2\sin^2\theta\right)^{3/2}}\frac{\sin^3\theta}{d^2}\frac{d\theta}{\sin^2\theta} \\ =& \frac{\lambda}{4\pi\epsilon_0 d}\left(1-\beta^2\right)\frac{\sin\theta d\theta}{\left(1-\beta^2\sin^2\theta\right)^{3/2}}\end{aligned}$$

The total electric field is then the integral:

$$E = \frac{\lambda}{4\pi\epsilon_0 d}\left(1-\beta^2\right)\int_0^{\pi}\frac{\sin\theta d\theta}{\left(1-\beta^2\sin^2\theta\right)^{3/2}}$$
$$= \frac{\lambda}{2\pi\epsilon_0 d}\left(1-\beta^2\right)\int_0^{\pi/2}\frac{\sin\theta d\theta}{\left(1-\beta^2\sin^2\theta\right)^{3/2}}$$

since the integrand is symmetrical about $\theta = \pi/2$. Make the substitution:

$$u = \cos\theta \quad du = -\sin\theta d\theta$$
$$E = \tfrac{\lambda}{2\pi\epsilon_0 d}\left(1-\beta^2\right)\textstyle\int_0^1 \tfrac{du}{(1-\beta^2(1-u^2))^{3/2}}$$

The integral is of the table variety:

$$\int_0^1 \frac{du}{\left(1-\beta^2\left(1-u^2\right)\right)^{3/2}} = \frac{1}{\beta^3}\int_0^1\frac{du}{\left(\beta^{-2}-1+u^2\right)\right)^{3/2}}$$
$$= \frac{1}{\beta^3}\int_0^1\frac{du}{\left(A^2+u^2\right)^{3/2}}$$

$$A^2 = \beta^{-2}-1$$

$$\frac{1}{\beta^3}\int_0^1\frac{du}{\left(A^2+u^2\right)\right)^{3/2}} = \frac{1}{\beta^3}\frac{u}{A^2\sqrt{A^2+u^2}}\bigg|_0^1$$
$$= \frac{1}{\beta^3}\frac{1}{A^2\sqrt{A^2+1}} = \frac{1}{\beta^3}\frac{1}{\left(\beta^{-2}-1\right)\beta-1} = \frac{1}{1-\beta^2}$$

The electric field is thus:

$$E = \frac{\lambda}{2\pi\epsilon_n d}\left(1-\beta^2\right)\frac{1}{1-\beta^2} = \frac{\lambda}{2\pi\epsilon_0 d}$$

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0 d}\hat{s}$$