电动力学-第八次作业

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Problem 8.4

Answer:

(a)

$$(\overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a})_z = T_{zx}da_x + T_{zy}da_y + T_{zz}da_z \tag{1.1}$$

For x-y plane:

$$(\overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a})_z = \epsilon_0 \left(E_zE_z - \frac{1}{2}E^2\right)(-rdrd\phi) \tag{1.2}$$

Now:

$$\begin{cases} \mathbf{E} = \dfrac{1}{4\pi\epsilon_n} 2\dfrac{q}{r^2}\cos\theta\hat{\mathbf{r}} \\ \cos\theta = \dfrac{r}{s} \end{cases} \tag{1.3}$$

So:

$$E_z = 0 \tag{1.4}$$

$$E^2 = \left(\frac{q}{2\pi\epsilon_0}\right)^2 \frac{r^2}{\left(r^2+a^2\right)^3} \tag{1.5}$$

Then:

$$F_z = \frac{1}{2}\epsilon_0 \left(\frac{q}{2\pi\epsilon_0}\right)^2 2\pi \int_0^{\epsilon fty} \frac{r^3dr}{\left(r^2+a^2\right)^3} = \frac{q^2}{4\pi\epsilon_0}\frac{1}{(2a)^2} \tag{1.6}$$

(b)

In this case:

$$\begin{cases} \mathbf{E} = -\dfrac{1}{4\pi\epsilon_0} 2\dfrac{q}{s^2}\sin\theta\hat{\mathbf{z}} \\ \sin\theta = \dfrac{a}{s} \end{cases} \tag{1.7}$$

So:

$$E^2 = E_z^2 = \left(\frac{qa}{2\pi\epsilon_0}\right)^2 \frac{1}{\left(r^2+a^2\right)^3} \tag{1.8}$$

So:

$$F_z = -\frac{\epsilon_0}{2}\left(\frac{qa}{2\pi\epsilon_0}\right)^2 2\pi \int_0^\infty \frac{rdr}{\left(r^2+a^2\right)^3} = -\frac{q^2}{4\pi\epsilon_0}\frac{1}{(2a)^2} \tag{1.9}$$

Problem 8.7

Answer:

(a)

$E_x = E_y = 0, E_z = -\sigma/\epsilon_0$,Then:

$$\begin{cases} T_{xy} = T_{xz} = T_{yz} = \cdots = 0 \\ T_{xx} = T_{yy} = -\frac{\epsilon_0}{2}E^2 = -\frac{\sigma^2}{2\epsilon_0} \\ T_{zz} = \epsilon_0\left(E_z^2 - \frac{1}{2}E^2\right) = \frac{\epsilon_0}{2}E^2 = \frac{\sigma^2}{2\epsilon_0} \end{cases} \tag{2.1}$$

So:

$$\overleftrightarrow{\mathbf{T}} = \frac{\sigma^2}{2\epsilon_0}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & +1 \end{pmatrix} \tag{2.2}$$

(b)

$$\mathbf{F} = \oint \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a} \tag{2.3}$$

Do integrata over the xy plane;

$$F_z = \int T_{zz} da_z = -\frac{\sigma^2}{2\epsilon_0}A \tag{2.4}$$

So, the force per unit area is:

$$f = \frac{\mathbf{F}}{A} = -\frac{\sigma^2}{2\epsilon_0}\hat{\mathbf{z}} \tag{2.5}$$

(c)

$$-T_{zz} = \frac{\sigma^2}{2\epsilon_0} \tag{2.6}$$

(d)

The recoil force is the momentum delivered per unit time, so the force per unit area on the top plate is:

$$f = -\frac{\sigma^2}{2\epsilon_0}\hat{\mathbf{z}} \tag{2.7}$$

This result is same as (b)

Problem 8.9

Answer:

(a)

The angular momentum stored in the fields is:

$$\vec{l} = \vec{r} \times \epsilon_0 \vec{E} \times \vec{B} = \frac{QB_0}{4\pi r}\hat{r} \times (\hat{r} \times \hat{z}) \tag{3.1}$$

We have:

$$\hat{r} \times (\hat{r} \times \hat{z}) = \hat{r}(\hat{r} \cdot \hat{z}) - \hat{z}(\hat{r} \cdot \hat{r}) = \hat{r}\cos\theta - \hat{z} \tag{3.2}$$

Over the sphere, only the z-component of the angular momentum will survive the integration so:

$$(\hat{r} \times (\hat{r} \times \hat{z}))_z = (\hat{r}\cos\theta - \hat{z}) \cdot \hat{z} = \cos^2\theta - 1 = -\sin^2\theta \tag{3.3}$$

Then:

$$L_z = \int_V l_z dV = \int_a^b \int_0^\pi \int_0^{2\pi} \frac{QB_0}{4\pi r}\left(-\sin^2\theta\right) r^2 dr \sin\theta d\theta d\phi = -\frac{QB_0}{4\pi} 2\pi \int_a^b r dr \int_0^\pi \sin^3\theta d\theta \tag{3.4}$$

We have:

$$x = \cos\theta \quad dx = -\sin\theta d\theta \tag{3.5}$$

Then:

$$L_z = -\frac{\pi Q B_0}{2}\frac{1}{2}\left(b^2 - a^2\right)\int_1^{-1} \sin^3\theta \frac{dx}{-\sin\theta} = -\frac{\pi Q B_0}{3}\left(b^2 - a^2\right) \tag{3.6}$$

So:

$$\vec{L} = -\frac{\pi Q B_0}{3}\left(b^2 - a^2\right)\hat{z} \tag{3.7}$$

(b)

When the magnetic field is turning off an electric field is induced. By Faraday's law:

$$2E\pi s = -s^2 \pi \dot{B} \Longrightarrow \vec{E} = -\frac{s}{2}\dot{B}\hat{\phi} \tag{3.8}$$

Torque on the patch of surface on a sphere is with (on the sphere) $s = a\sin\theta$:

$$d\vec{N} = \vec{a}\times d\vec{F} = \vec{a}\times\sigma dS\vec{E} = -\vec{a}\times\sigma dS\frac{a\sin\theta}{2}\dot{B}\hat{\phi} == -\sigma a^2\sin^2\theta d\theta d\phi\vec{a}\times\frac{a}{2}\dot{B}\hat{\phi} \tag{3.9}$$

Now, again only the z-component will survive, so the projection of vector $\vec{a}\times\hat{\phi}$ on the z-axis is:

$$(\vec{a}\times\hat{\phi})_z = (\vec{a}\times\hat{\phi})\cdot\hat{z} = a\cos\left(\pi - \frac{\pi}{2} - \theta\right) = a\sin\theta \tag{3.10}$$

With this:

$$dN_z = -\sigma a^4\frac{1}{2}\dot{B}\sin^3\theta d\theta d\phi \tag{3.11}$$

$$N_z = -\frac{1}{2}\dot{B}\sigma a^4\int_0^{\pi}\sin^3\theta d\theta\int_0^{2\pi}d\phi = -\pi\dot{B}\sigma a^4\frac{4}{3} \tag{3.12}$$

where we solved the same integral $\left(\sin^3\theta\right)$ in the (a) part of the problem. Now this is valid for both spheres, so for the bigger sphere just replace a with b. Using $\sigma = Q/4\pi a^2$(small sphere) and $\sigma = -Q/4\pi b^2$ the torques are:

$$N_{z,a} = -\frac{\pi}{3}\dot{B}Qa^2 \quad N_{z,b} = \frac{\pi}{3}\dot{B}Qb^2 \tag{3.13}$$

The total angular momentum of the system is then:

$$\begin{aligned} L_z &= \int_0^{t_f}\left(N_{z,a} + N_{z,b}\right)dt = \frac{\pi}{3}Q\left(b^2 - a^2\right)\int_0^{t_f}\dot{B}dt = \frac{\pi}{3}Q\left(b^2 - a^2\right)\int_{B_0}^{0}dB \\ &= -\frac{\pi}{3}QB_0\left(b^2 - a^2\right) \end{aligned} \tag{3.14}$$

So:

$$\vec{L} = -\frac{\pi}{3}QB_0\left(b^2 - a^2\right)\hat{z} \tag{3.15}$$

Problem 8.14

Answer:

(a)

$$
\begin{aligned}
\frac{U}{l} &= \int_0^{2\pi}\int_a^b u s ds d\phi \\
&= \int_0^{2\pi}\int_a^b \left[\frac{\epsilon_0}{2}\left(\frac{\lambda}{2\pi\epsilon_0 s}\right)^2 + \frac{1}{2\mu_0}\left(\frac{\mu_0\lambda v}{2\pi s}\right)^2\right] s ds d\phi \\
&= 2\pi\frac{1}{2\epsilon_0}\left(\frac{\lambda}{2\pi}\right)^2 \int_a^b \left(\frac{1}{s^2} + \frac{1}{s^2}\frac{v^2}{c^2}\right) s ds \\
&= \frac{\lambda^2}{4\pi\epsilon_0}\left(1+\beta^2\right)\int_a^b \frac{ds}{s} = \frac{\lambda^2}{4\pi\epsilon_0}\left(1+\beta^2\right)\ln\frac{b}{a}
\end{aligned}
\tag{4.1}
$$

(b)

Now, for the momentum per unit lenght stored in the fields just integrate the momentum density over a unit lenght of the system:

$$
\begin{aligned}
\frac{\vec{p}}{l} &= \int_0^{2\pi}\int_a^b \vec{g} s ds d\phi = \int_0^{2\pi}\int_a^b \epsilon_0 \vec{E}\times\vec{B} s ds d\phi \\
&= \int_0^{2\pi}\int_a^b \epsilon_0 \frac{\lambda}{2\pi\epsilon_0 s}\hat{s}\times\frac{\mu_0\lambda v}{2\pi s}\hat{\phi} s ds d\phi \\
&= \frac{\mu_0\lambda^2 v}{4\pi^2}\int_a^b \frac{ds}{s}\int_0^{2\pi} d\phi \hat{s}\times\hat{\phi} = \frac{\mu_0\lambda^2 v}{2\pi}\ln\frac{b}{a}\hat{z}
\end{aligned}
\tag{4.2}
$$

(c)

For the power transported down the wires integrate the Poynting vector over the annular surface in between the cyllinders, with the positive orientation in the z-direction:

$$
\begin{aligned}
P = \int_a^b\int_0^{2\pi} \vec{S}\cdot d\vec{A} &= \frac{1}{\mu_0}\int_a^b\int_0^{2\pi} (\vec{E}\times\vec{B})\cdot s ds d\phi \hat{z} \\
&= \frac{1}{\mu_0}\int_a^b\int_0^{2\pi}\left(\frac{\lambda}{2\pi\epsilon_0 s}\hat{s}\times\frac{\mu_0\lambda v}{2\pi s}\hat{\phi}\right)\cdot s ds d\phi \\
&= \frac{\lambda^2 v}{4\pi^2\epsilon_0}\int_a^b\frac{ds}{s}\int_0^{2\pi} d\phi \\
&= \frac{\lambda^2 v}{2\pi\epsilon_0}\ln\frac{b}{a}
\end{aligned}
\tag{4.3}
$$