Chapter 4

Electric Fields in Matter

- 4.1 Polarization
- 4.2 The Field of a Polarized Object
- 4.3 The Electric Displacement
- 4.4 Linear Dielectrics

Polarization

- In this chapter, we shall study electric fields in matter.
- Matter, of course, comes in many varieties. Most everyday objects belong (at least, in good approximation) to one of two large classes: conductors and insulators (or dielectrics).
- We have already talked about conductors that contain an "unlimited" supply of charges that are free to move about through the material .
- In dielectrics, all charges are attached to specific atoms or molecules. There are actually two principle mechanisms by which electric fields can distort the charge distribution of a dielectric atom or molecule: stretching and rotating.

Induced Dipoles

What happens to a neutral atom when it is placed in an electric field E?

Although the atom as a whole is electrically neutral, there is a positively charged core (the nucleus) and a negatively charged electron cloud surrounding it. These two regions of charge within the atom are influenced by the field: the nucleus is pushed in the direction of the field, and the electrons the opposite way.

The two opposing forces - E pulling the electrons and nucleus apart, their mutual attraction drawing them back together - reach a balance, leaving the atom polarized, with plus charge shifted slightly one way, and minus the other. The atom now has a tiny dipole moment p, which points in the same direction as E.

$$P = \alpha E$$

The constant of proportionality a is called atomic polarizability.

H	He	Li	Be	C	Ne	Na	Ar	K	Cs
0.667	0.205	24.3	5.60	1.67	0.396	24.1	1.64	43.4	59.4

TABLE 4.1 Atomic Polarizabilities ($\alpha/4\pi\epsilon_0$, in units of 10^{-30} m^3). *Data from: Handbook of Chemistry and Physics*, 91st ed. (Boca Raton: CRC Press, 2010).

Example

A primitive model for an atom consists of a point nucleus (+q) surrounded by a uniformly charged spherical cloud (-q) of radius a (Fig. 4.1). Calculate the atomic polarizability of such an atom

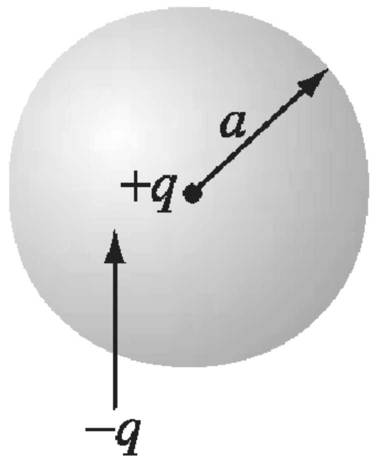


FIGURE 4.1

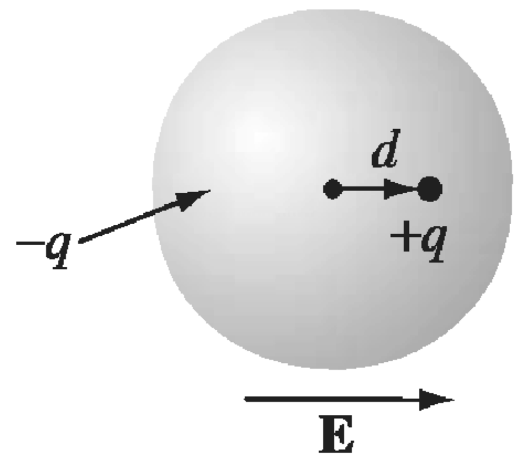


FIGURE 4.2

The field at a distance d from the center of a uniformly charged sphere is

$$E_e = \frac{1}{4\pi\epsilon_0}\frac{qd}{a^3}$$ (From Gaussian's Law)

$$E = \frac{1}{4\pi\epsilon_0}\frac{qd}{a^3}, \qquad \text{or} \quad p = qd = (4\pi\epsilon_0 a^3)E.$$

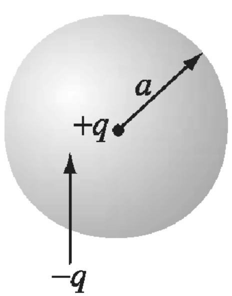


FIGURE 4.1

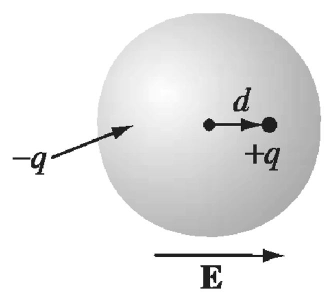


FIGURE 4.2

$$\alpha = 4\pi\epsilon_0 a^3 = 3\epsilon_0 v$$

v : volume of the atom

For molecules :

$$\mathbf{p} = \alpha_{\perp}\mathbf{E}_{\perp} + \alpha_{\parallel}\mathbf{E}_{\parallel}$$

$$\left.\begin{aligned} p_x &= \alpha_{xx}E_x + \alpha_{xy}E_y + \alpha_{xz}E_z \\ p_y &= \alpha_{yx}E_x + \alpha_{yy}E_y + \alpha_{yz}E_z \\ p_z &= \alpha_{zx}E_x + \alpha_{zy}E_y + \alpha_{zz}E_z \end{aligned}\right\}$$

Note:
The set of nine constants α_{ij} constitute the polarizability tensor for the molecule.

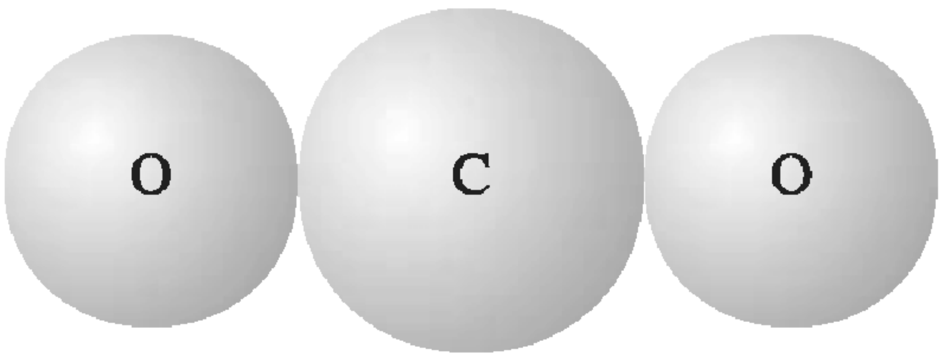


FIGURE 4.3

Alignment of Polar Molecules

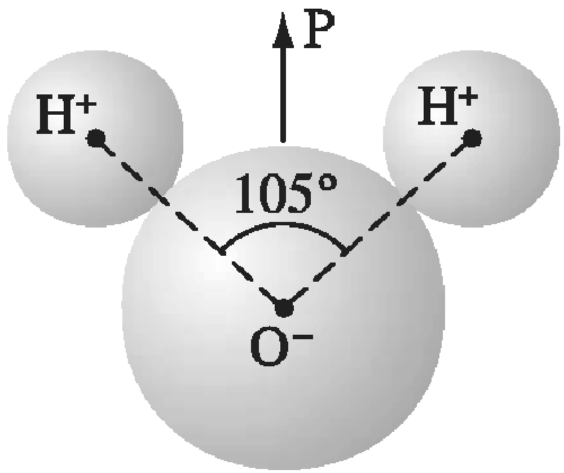


FIGURE 4.4

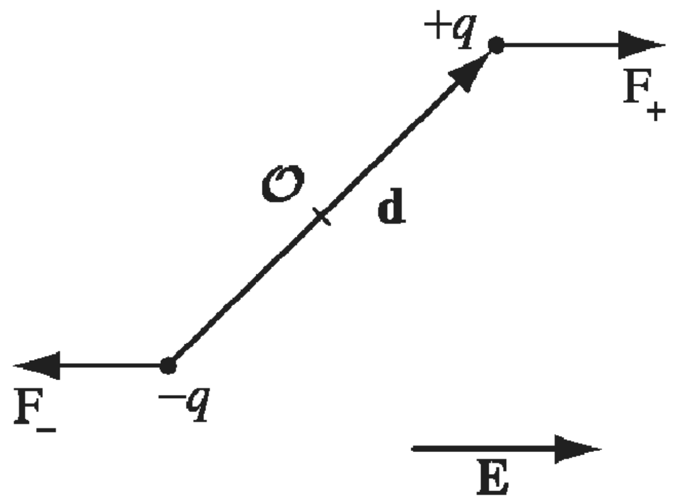


FIGURE 4.5

What happens when polar molecules are placed in an electric field?

If the field is uniform, the *force* on the positive end, $\mathbf{F}_+ = q\mathbf{E}$, exactly cancels the force on the negative end, $\mathbf{F}_- = -q\mathbf{E}$ (Fig. 4.5). However, there will be a *torque:*

$$\begin{aligned}\mathbf{N} &= (\mathbf{r}_+ \times \mathbf{F}_+) + (\mathbf{r}_- \times \mathbf{F}_-)\\ &= \left[(\mathbf{d}/2) \times (q\mathbf{E})\right] + \left[(-\mathbf{d}/2) \times (-q\mathbf{E})\right] = q\mathbf{d} \times \mathbf{E}.\end{aligned}$$

Thus a dipole $\mathbf{p} = q\mathbf{d}$ in a uniform field $\mathbf{E}$ experiences a torque

$$\mathbf{N} = \mathbf{p} \times \mathbf{E}$$

Notice that N is in such a direction as to line **p** up parallel to **E**: a polar molecule that is free to rotate will swing around until it points in the direction of the applied field .

Polarization

What happens to a piece of dielectric material when it is placed in an electric field?

If the material is made up of polar molecules, each permanent dipole will experience a torque, tending to line it up along the field direction. (Random thermal motions compete with this process, so the alignment is never complete, especially at higher temperatures, and disappears almost at once when the field is removed.)

A lot of little dipoles pointing along the direction of the field such that the material becomes polarized.

$$\boldsymbol{P} \equiv \text{dipole moment per unit volume}$$

$\boldsymbol{P}$ is called **polarization**

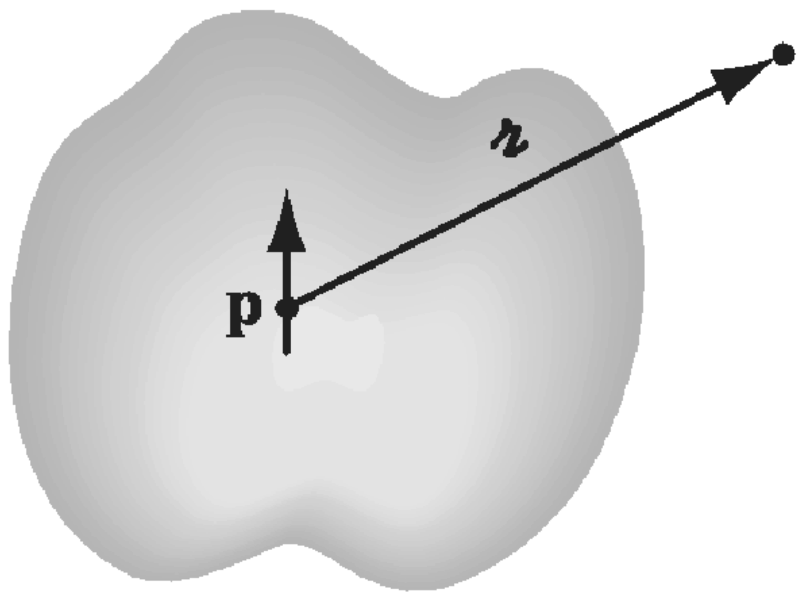
The Field of a Polarized Object

Bound Charges

The dipole moment per unit volume **P** is given .

Question :

What is the field produced by this object (not the field that may have caused the polarization, but the field the polarization itself causes)?



$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p}\cdot\hat{\boldsymbol{\imath}}}{\imath^2},$$

A dipole moment $p = Pd\tau'$ in each volume element $d\tau'$,so the potential is :

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \frac{\mathbf{P}(\mathbf{r}')\cdot\hat{\boldsymbol{\imath}}}{\imath^2} d\tau'$$

$$\nabla'\left(\frac{1}{\imath}\right) = \frac{\hat{\boldsymbol{\imath}}}{\imath^2}$$

$$V = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \mathbf{P}\cdot\nabla'\left(\frac{1}{\mathscr{r}}\right)\, d\tau'.$$

Integrating by parts, using product rule number 5 (in the front cover), gives

$$V = \frac{1}{4\pi\epsilon_0}\left[\int_{\mathcal{V}} \nabla'\cdot\left(\frac{\mathbf{P}}{\mathscr{r}}\right) d\tau' - \int_{\mathcal{V}} \frac{1}{\mathscr{r}}(\nabla'\cdot\mathbf{P})\, d\tau'\right],$$

or, invoking the divergence theorem,

$$V = \frac{1}{4\pi\epsilon_0}\oint_{\mathcal{S}} \frac{1}{\mathscr{r}}\mathbf{P}\cdot d\mathbf{a}' - \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{1}{\mathscr{r}}(\nabla'\cdot\mathbf{P})\, d\tau'.$$

The first term looks like the potential of a surface charge

$$\sigma_b \equiv \mathbf{P}\cdot\hat{\mathbf{n}}$$

(where $\hat{\mathbf{n}}$ is the normal unit vector), while the second term looks like the potential of a volume charge

$$\boxed{\rho_b \equiv -\nabla \cdot \mathbf{P}.} \tag{4.12}$$

$$\boxed{V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \oint_{\mathcal{S}} \frac{\sigma_b}{\mathscr{r}}\, da' + \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \frac{\rho_b}{\mathscr{r}}\, d\tau'}$$

$$\sigma_b \equiv \mathbf{P} \cdot \hat{\mathbf{n}}$$

Physical Interpretation of Bound Charges

The field of a polarized object is identical to the field that would be produced by a certain distribution of "bound charges"

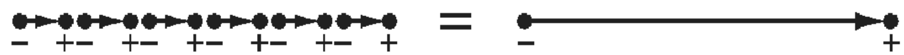


FIGURE 4.11

To calculate the actual amount of bound charge resulting from a given polarization, examine a "tube" of dielectric parallel to P. In terms of the charge (q) at the end, this same dipole moment can be written as

$$q = PA$$

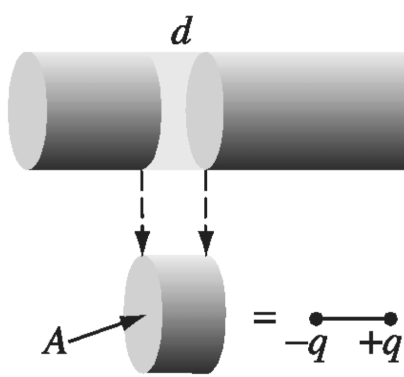


FIGURE 4.12

If the ends have been sliced off perpendicularly, the surface charge density is

$$\sigma_b = \frac{q}{A} = P$$

A more general expression:

$$\sigma_b = \frac{q}{A_{\text{end}}} = P\cos\theta = \mathbf{P}\cdot\hat{\mathbf{n}}.$$

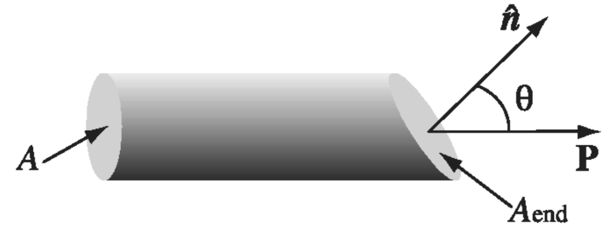


FIGURE 4.13

Indeed, the net bound charge $\int \rho_b d\tau$ in a given volume is equal and opposite to the amount that has been pushed out through the surface .

$$\int_{\mathcal{V}} \rho_b\, d\tau = -\oint_{\mathcal{S}} \mathbf{P}\cdot d\mathbf{a} = -\int_{\mathcal{V}} (\nabla\cdot\mathbf{P})\, d\tau$$

Since this is true for any volume :

$$\rho_b = -\nabla\cdot\mathbf{P}$$

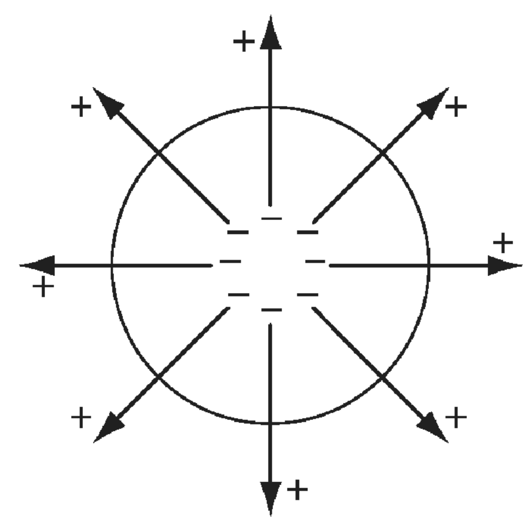
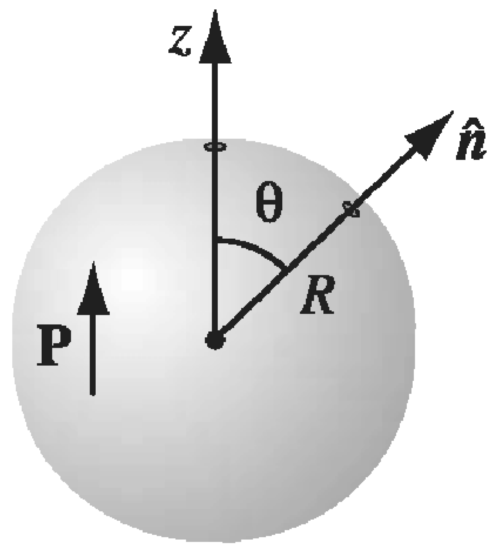


FIGURE 4.14

Example

Find the electric field produced by a uniformly polarized sphere of radius R .



Solution

We may as well choose the z axis to coincide with the direction of polarization (Fig. 4.9). The volume bound charge density ρ_b is zero, since $\mathbf{P}$ is uniform, but

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = P \cos\theta,$$

$$V(r,\theta) = \begin{cases} \dfrac{P}{3\epsilon_0} r\cos\theta, & \text{for} \quad r \le R, \\ \dfrac{P}{3\epsilon_0}\dfrac{R^3}{r^2}\cos\theta, & \text{for} \quad r \ge R. \end{cases}$$

See Example (3.9)

Since $r\cos\theta = z$, the *field* inside the sphere is *uniform*:

$$\mathbf{E} = -\nabla V = -\frac{P}{3\epsilon_0}\hat{\mathbf{z}} = -\frac{1}{3\epsilon_0}\mathbf{P}, \quad \text{for} \quad r < R. \tag{4.14}$$

This remarkable result will be very useful in what follows. Outside the sphere the potential is identical to that of a perfect dipole at the origin,

$$V = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}, \quad \text{for} \quad r \ge R, \tag{4.15}$$

$$\mathbf{p} = \frac{4}{3}\pi R^3 \mathbf{P}.$$

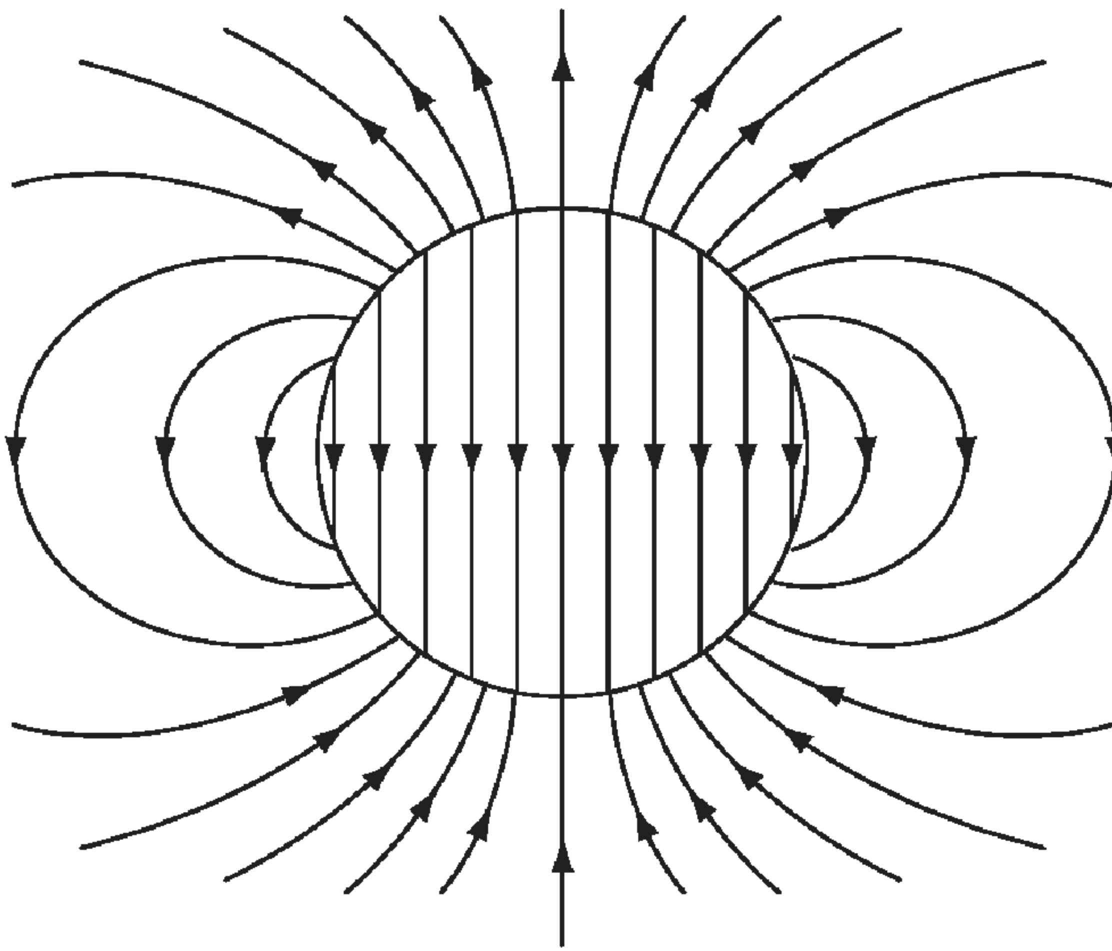


FIGURE 4.10

The Electric Displacement

Gauss's Law in the Presence of Dielectrics

Now we found that the effect of polarization is to produce accumulations of (bound) charge, $\rho_b = -\nabla \cdot \boldsymbol{P}$ within the dielectric and $\sigma_b = \mathbf{P} \cdot \boldsymbol{n}$ on the surface. The field due to polarization of the medium is just the field of this bound charge. We are now ready to put it all together: the field attributable to bound charge plus the field due to everything else (which, for want of a better term, we call free charge, ρ_{f}). The free charge might consist of electrons on a conductor or ions embedded in the dielectric material or whatever; any charge, in other words, that is not a result of polarization. Within the dielectric, the total charge density can be written:

$$\rho = \rho_b + \rho_f$$

Gauss's law reads

$$\epsilon_0 \nabla \cdot \mathbf{E} = \rho = \rho_b + \rho_f = -\nabla \cdot \mathbf{P} + \rho_f$$

$$\nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho_f$$

Electric Displacement $\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P}$

Gauss's law reads $\nabla \cdot \mathbf{D} = \rho_f$

in integral form $\oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}}$

Q_{fenc} denotes the total free charge enclosed in the volume, it makes reference only to free charges

A Deceptive Parallel

There is no ‘Coulomb’s law’ for **D**

$$\mathbf{D}(\mathbf{r}) \neq \frac{1}{4\pi} \int \frac{\hat{\boldsymbol{\imath}}}{\imath^2} \rho_f(\mathbf{r}')\, d\tau'$$

$$\nabla \times \mathbf{D} = \epsilon_0(\nabla \times \mathbf{E}) + (\nabla \times \mathbf{P}) = \nabla \times \mathbf{P}$$

There is no reason, in general, to suppose that the curl of **P** vanishes

Advice:

When you are asked to compute the electric displacement, first look for symmetry. If the problem exhibits spherical, cylindrical, or plane symmetry, then you can get D directly from Eq. 4.23 by the usual Gauss's law methods.

Boundary Conditions

$$D^{\perp}_{\text{above}} - D^{\perp}_{\text{below}} = \sigma_f$$

$$\mathbf{D}^{\parallel}_{\text{above}} - \mathbf{D}^{\parallel}_{\text{below}} = \mathbf{P}^{\parallel}_{\text{above}} - \mathbf{P}^{\parallel}_{\text{below}}$$

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{1}{\epsilon_0}\sigma$$

$$\mathbf{E}^{\parallel}_{\text{above}} - \mathbf{E}^{\parallel}_{\text{below}} = \mathbf{0}$$

Liner Dielectrics

Susceptibility, Permittivity, Dielectric Constant

For many substances, in fact, the polarization is proportional to the field, provided E is not too strong:

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$$

χ_e is called the electric susceptibility of the medium
(a factor of ϵ_0 has been extracted to make χ_e dimensionless)

We shall call materials that obey the equation above **linear dielectrics**

Note :
E in the equation above is the total field ; it may be due in part to free charges and in part to the polarization itself.
The simplest approach is to begin with the ***displacement*** , at least in those cases where **D** can be deduced directly from the charge distribution.

In linear media we have

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 \mathbf{E} + \epsilon_0 \chi_e \mathbf{E} = \epsilon_0 (1 + \chi_e) \mathbf{E}$$

$$\mathbf{D} = \epsilon \mathbf{E}$$

permittivity of the material $\epsilon \equiv \epsilon_0 (1 + \chi_e)$

relative permittivity,
or dielectric constant: $\epsilon_r \equiv 1 + \chi_e = \frac{\epsilon}{\epsilon_0}$

Example

A metal sphere of radius a carries charge Q (Fig. 4.20). It is surrounded, out to radius b, by linear dielectric material of permittivity ϵ. Find the potential at the center (relative to infinity).

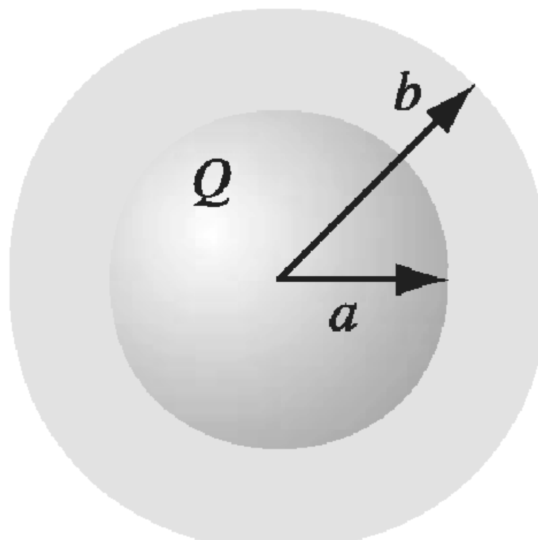
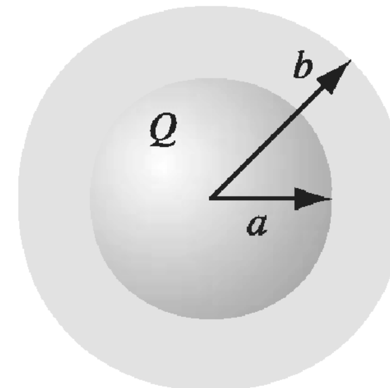


FIGURE 4.20

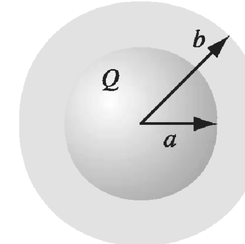
Solution



For all points $r > a$ $$\mathbf{D} = \frac{Q}{4\pi r^2}\hat{\mathbf{r}}.$$

Note: Inside the metal sphere, of course, **E** = **P** = **D** = 0

$$\mathbf{E} = \begin{cases} \dfrac{Q}{4\pi\epsilon r^2}\hat{\mathbf{r}}, & \text{for } a < r < b \\ \dfrac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}, & \text{for } r > b. \end{cases}$$



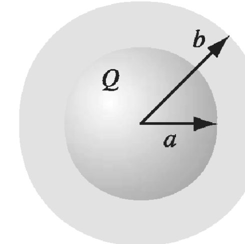
The potential at the center is therefore

$$V = -\int_{\infty}^{0} \mathbf{E} \cdot d\mathbf{l} = -\int_{\infty}^{b} \left(\frac{Q}{4\pi\epsilon_0 r^2} \right) dr - \int_{b}^{a} \left(\frac{Q}{4\pi\epsilon r^2} \right) dr - \int_{a}^{0} (0)\, dr$$

$$= \frac{Q}{4\pi} \left(\frac{1}{\epsilon_0 b} + \frac{1}{\epsilon a} - \frac{1}{\epsilon b} \right).$$

In the dielectric $\mathbf{P} = \epsilon_0 \chi_e \mathbf{E} = \frac{\epsilon_0 \chi_e Q}{4\pi\epsilon r^2} \hat{\mathbf{r}}$

$$\rho_b = -\nabla \cdot \mathbf{P} = 0,$$



$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = \begin{cases} \dfrac{\epsilon_0 \chi_e Q}{4\pi\epsilon b^2}, & \text{at the outer surface} \\[2ex] \dfrac{-\epsilon_0 \chi_e Q}{4\pi\epsilon a^2}, & \text{at the inner surface} \end{cases}$$

Note:
The surface bound charge at a is *negative* (**n** points outward *with respect to the dielectric,* which is $+\mathbf{r}$ at b but $-\mathbf{r}$ at a). This is natural, since the charge on the metal sphere attracts its opposite in all the dielectric molecules. It is this layer of negative charge that reduces the field, within the dielectric, from $\frac{Q}{4\pi\epsilon_0 r^2}\mathbf{r}$ to $\frac{Q}{4\pi\epsilon r^2}\mathbf{r}$. In this respect, a dielectric is rather like an imperfect conductor: on a *conducting* shell the induced surface charge would be such as to cancel the field of Q *completely* in the region $a < r < b;$ the dielectric does the best it can, but the cancellation is only partial.

Boundary Value Problems with Linear Dielectrics

In a (homogeneous isotropic) linear dielectric, the bound charge density (ρ_b) is proportional to the free charge density ρ_b:

$$\rho_b = -\nabla \cdot \mathbf{P} = -\nabla \cdot \left(\epsilon_0 \frac{\chi_e}{\epsilon} \mathbf{D}\right) = -\left(\frac{\chi_e}{1+\chi_e}\right) \rho_f$$

In particular, unless free charge is actually embedded in the material, $\rho = 0$, and any net charge must reside at the surface. Within such a dielectric, then, the potential obeys Laplace's equation .

$$\epsilon_{\text{above}} E_{\text{above}}^{\perp} - \epsilon_{\text{below}} E_{\text{below}}^{\perp} = \sigma_f$$

$$\epsilon_{\text{above}} \frac{\partial V_{\text{above}}}{\partial n} - \epsilon_{\text{below}} \frac{\partial V_{\text{below}}}{\partial n} = -\sigma_f$$

$$V_{\text{above}} = V_{\text{below}}$$

Example

A sphere of homogeneous linear dielectric material is placed in an otherwise uniform electric field $\mathbf{E}_0$ (Fig. 4.27). Find the electric field inside the sphere.

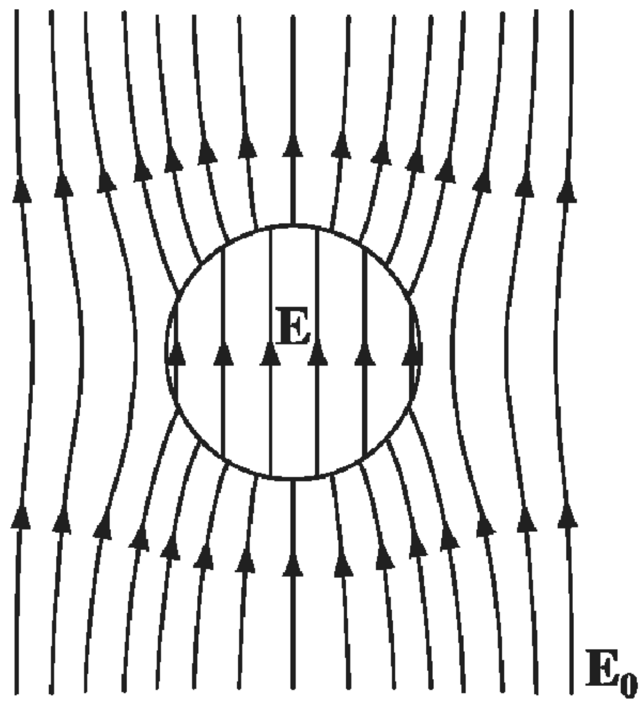


FIGURE 4.27

Solution

Our problem is to solve Laplace's equation:

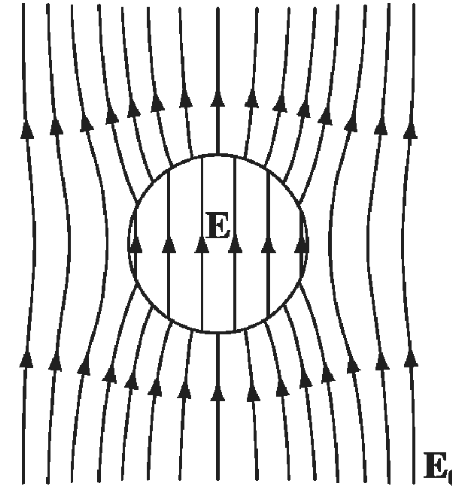


FIGURE 4.27

$$V_{\text{in}}(r,\theta) = \sum_{l=0}^{\infty} A_l\, r^l P_l(\cos\theta)$$

$$V_{\text{out}}(r,\theta) = -E_0 r\cos\theta + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

$$\left.\begin{array}{llll} \text{(i)} & V_{\text{in}} = V_{\text{out}}, & \text{at} & r = R, \\ \text{(ii)} & \epsilon\dfrac{\partial V_{\text{in}}}{\partial r} = \epsilon_0\dfrac{\partial V_{\text{out}}}{\partial r}, & \text{at} & r = R, \\ \text{(iii)} & V_{\text{out}} \to -E_0 r\cos\theta, & \text{for} & r \gg R. \end{array}\right\}$$

$$\left.\begin{array}{llll}\text{(i)} & V_{\text{in}} = V_{\text{out}}, & \text{at } r = R, \\ \text{(ii)} & \epsilon \dfrac{\partial V_{\text{in}}}{\partial r} = \epsilon_0 \dfrac{\partial V_{\text{out}}}{\partial r}, & \text{at } r = R, \\ \text{(iii)} & V_{\text{out}} \to -E_0 r \cos\theta, & \text{for } r \gg R.\end{array}\right\}$$

Boundary condition (i) requires that

$$\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = -E_0 R \cos\theta + \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta)$$

$$\left.\begin{array}{l} A_l R^l = \dfrac{B_l}{R^{l+1}}, \quad \text{for } l \neq 1, \\ A_1 R = -E_0 R + \dfrac{B_1}{R^2}. \end{array}\right\}$$

Meanwhile, condition (ii) yields

$$\epsilon_r \sum_{l=0}^{\infty} l A_l R^{l-1} P_l(\cos\theta) = -E_0 \cos\theta - \sum_{l=0}^{\infty} \frac{(l+1) B_l}{R^{l+2}} P_l(\cos\theta)$$

$$\left.\begin{array}{l} \epsilon_r l A_l R^{l-1} = -\dfrac{(l+1) B_l}{R^{l+2}}, \text{ for } l \neq 1, \\ \epsilon_r A_1 = -E_0 - \dfrac{2 B_1}{R^3}. \end{array}\right\}$$

$$\left.\begin{array}{llll} \text{(i)} & V_{\text{in}} = V_{\text{out}}, & \text{at } r = R, \\ \text{(ii)} & \epsilon \dfrac{\partial V_{\text{in}}}{\partial r} = \epsilon_0 \dfrac{\partial V_{\text{out}}}{\partial r}, & \text{at } r = R, \\ \text{(iii)} & V_{\text{out}} \to -E_0 r \cos\theta, & \text{for } r \gg R. \end{array}\right\}$$

It follows that

$$\left.\begin{array}{ll} A_l = B_l = 0, & \text{for } l \neq 1, \\ A_1 = -\dfrac{3}{\epsilon_r + 2} E_0 & B_1 = \dfrac{\epsilon_r - 1}{\epsilon_r + 2} R^3 E_0. \end{array}\right\}$$

$$V_{\text{in}}(r, \theta) = -\frac{3E_0}{\epsilon_r + 2} r \cos\theta = -\frac{3E_0}{\epsilon_r + 2} z,$$

The electric field inside the sphere

$$\mathbf{E} = \frac{3}{\epsilon_r + 2} \mathbf{E}_0.$$

Energy in Dielectric System

It takes work to charge up a capacitor

$$W = \frac{1}{2} C V^2$$

If the capacitor is filled with linear dielectric

$$C = \epsilon_r C_{vac}$$

A general formula for the energy stored in any electrostatic system

$$W = \frac{\epsilon_0}{2} \int \epsilon_r E^2 \, d\tau = \frac{1}{2} \int \mathbf{D} \cdot \mathbf{E} \, d\tau.$$