Chapter 10

Potentials and Fields

- 10.1 The Potential Formulation
- 10.2 Continuous Distributions
- 10.3 Point Charges

The Potential Formulation

Scalar and Vector Potentials

In this chapter we seek the general solution to Maxwell's equations

$$\left.\begin{array}{llll} \text{(i)} & \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho, & \text{(iii)} & \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \\ \text{(ii)} & \nabla \cdot \mathbf{B} = 0, & \text{(iv)} & \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0\epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \end{array}\right\} \qquad (10.1)$$

Given $\rho(\mathbf{r},t)$ and $\mathbf{J}(\mathbf{r},t)$, what are the fields $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$? In the static case, Coulomb's law and the Biot-Savart law provide the answer. What we're looking for, then, is the generalization of those laws to time-dependent configurations.

In electrostatics $\nabla \times \mathbf{E} = 0$ allows as to write $\mathbf{E}$ as $\mathbf{E} = -\nabla V$, and for $\mathbf{B}$

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A},} \qquad (10.2)$$

$$
\left.\begin{array}{llll}
\text{(i)} & \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho, & \text{(iii)} & \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \\
\text{(ii)} & \nabla \cdot \mathbf{B} = 0, & \text{(iv)} & \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}.
\end{array}\right\} \qquad (10.1)
$$

Putting this into Faraday's law (iii) yields

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{A})$$

or

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = \mathbf{0}.$$

Here curl does vanish therefore:

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla V.$$

Then

$$\boxed{\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}.} \qquad (10.3)$$

Putting $Eq.$ 10.3 into (i)

$$\boxed{\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{1}{\epsilon_0}\rho} \qquad (10.4)$$

this replaces Poisson's equation. Putting $Eqs.$ 10.2 and 10.3 into (iv) yields

$$\nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J} - \mu_0 \epsilon_0 \nabla \left(\frac{\partial V}{\partial t}\right) - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2}$$

using the vector identity $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$,

$$\boxed{\left(\nabla^2 \mathbf{A} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2}\right) - \nabla \left(\nabla \cdot \mathbf{A} + \mu_0 \epsilon_0 \frac{\partial V}{\partial t}\right) = -\mu_0 \mathbf{J}} \qquad (10.5)$$

Equations 10.4 and 10.5 contain all the information in Maxwell's equations.

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{1}{\epsilon_0}\rho$$

Gauge Transformations

$$\left(\nabla^2 \mathbf{A} - \mu_0\epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2}\right) - \nabla\left(\nabla \cdot \mathbf{A} + \mu_0\epsilon_0 \frac{\partial V}{\partial t}\right) = -\mu_0 \mathbf{J}$$

Equations 10.4 and 10.5 are ugly, however, we have succeeded in reducing six problems-finding E and B (three components each)-down to four: V (one component) and A (three more). Moreover, Eqs. 10.2 and 10.3 do not uniquely define the potentials; we are free to impose extra conditions on V and A, as long as nothing happens to E and B---gauge freedom

$$\mathbf{A}' = \mathbf{A} + \boldsymbol{\alpha} \quad \text{and} \quad V' = V + \beta$$

Since the two A's give the same B, their curls must be equal, and hence

$$\nabla \times \boldsymbol{\alpha} = \mathbf{0} \qquad\qquad \boldsymbol{\alpha} = \nabla\lambda$$

The two potentials also give the same **E**, using $\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla V$,

$$\nabla\beta + \frac{\partial \boldsymbol{\alpha}}{\partial t} = \mathbf{0}$$

or $$\nabla\left(\beta + \frac{\partial\lambda}{\partial t}\right) = \mathbf{0}$$

The term in parentheses is therefore independent of position (it could, however, depend on time); call it $k(t)$:

$$\beta = -\frac{\partial\lambda}{\partial t} + k(t)$$

defining a new λ by adding $\int_0^t k(t')dt'$ to the old one, then there is no k dependence that

$$\left.\begin{aligned} \mathbf{A}' &= \mathbf{A} + \nabla\lambda, \\ V' &= V - \frac{\partial\lambda}{\partial t}. \end{aligned}\right\} \quad (10.7)$$

Conclusion: For any old scalar function $\lambda(r,t)$, we can add $\nabla\lambda$ to **A**, provided simultaneously subtract $\partial\lambda/\partial t$ from V. This will not affect the physical quantities **E** and **B**. Such changes in V and **A** are called **gauge transformations**. In magnetostatics, it was best to choose $\nabla \cdot \boldsymbol{A} = 0$; in electrodynamics, the situation is not so clear, and the most convenient gauge depends to some extent on the problem at hand. There are many famous gauges in the literature; we will show two most popular ones in the following.

$$\boxed{\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{1}{\epsilon_0}\rho} \qquad (10.4)$$

Coulomb Gauge and Lorenz Gauge

$$\boxed{\left(\nabla^2 \mathbf{A} - \mu_0\epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2}\right) - \nabla\left(\nabla \cdot \mathbf{A} + \mu_0\epsilon_0 \frac{\partial V}{\partial t}\right) = -\mu_0 \mathbf{J}} \qquad (10.5)$$

The Coulomb Gauge. As in magnetostatics

$$\nabla \cdot \mathbf{A} = 0.$$

With this, $Eq.$ 10.4 becomes

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho$$

This is Poisson's equation, and we already know how to solve it: setting V = 0 at infinity,

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}', t)}{\mathscr{r}}\, d\tau'$$

Unlike electrostatics, V by itself doesn’t tell you **E**; you have to know **A** as well.

- The advantage of the Coulomb gauge is that the scalar potential is particularly simple to calculate;
- The disadvantage is that **A** is particularly difficult to calculate.

$$\boxed{\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{1}{\epsilon_0}\rho} \qquad (10.4)$$

$$\boxed{\left(\nabla^2 \mathbf{A} - \mu_0\epsilon_0\frac{\partial^2 \mathbf{A}}{\partial t^2}\right) - \nabla\left(\nabla \cdot \mathbf{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t}\right) = -\mu_0\mathbf{J}} \qquad (10.5)$$

The Lorenz gauge

$$\boxed{\nabla \cdot \mathbf{A} = -\mu_0\epsilon_0\frac{\partial V}{\partial t}.} \qquad (10.12)$$

Eq. 10.5 becomes

$$\nabla^2 \mathbf{A} - \mu_0\epsilon_0\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J}$$

Meanwhile, (Eq. 10.4), becomes

$$\nabla^2 V - \mu_0\epsilon_0\frac{\partial^2 V}{\partial t^2} = -\frac{1}{\epsilon_0}\rho$$

Here, the same differential operator

$$\boxed{\nabla^2 - \mu_0\epsilon_0\frac{\partial^2}{\partial t^2} \equiv \Box^2,} \qquad (10.15)$$

d' Alembertian operator

$$\boxed{\begin{aligned} &\text{(i)} \quad \Box^2 V = -\frac{1}{\epsilon_0}\rho, \\ &\text{(ii)} \quad \Box^2 \mathbf{A} = -\mu_0 \mathbf{J}. \end{aligned}} \qquad (10.16)$$

This democratic treatment of V and **A** is especially nice in the context of special relativity, where the d' Alembertian is the natural generalization of the Laplacian, and $Eqs.$ 10.16 can be regarded as four-dimensional versions of Poisson's equation. In the Lorenz gauge, V and **A** satisfy the inhomogeneous wave equation, with a "source" term. From now on, we will use the Lorenz gauge exclusively, and the whole of electrodynamics reduces to the problem of *solving the inhomogeneous wave equation for a specified source.*

Continuous Distributions

$$\text{(i)} \quad \Box^2 V = -\frac{1}{\epsilon_0}\rho,$$

$$\text{(ii)} \quad \Box^2 \mathbf{A} = -\mu_0 \mathbf{J}.$$

Retarded Potentials

In the static case, Eq. 10.16 reduces to (four copies of) Poisson's equation,

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho, \quad \nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$$

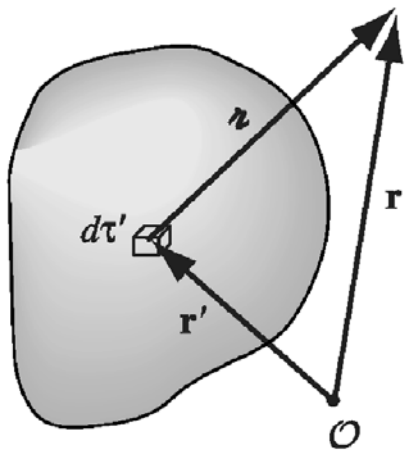

FIGURE 10.3

with the familiar solutions

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}}\, d\tau', \qquad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}}\, d\tau' \qquad (10.24)$$

Now, electromagnetic "news" travels at the speed of light. In the nonstatic case, therefore, it's not the status of the source right now that matters, but rather its condition at some earlier time t_r (retarded time) when the "message" left.

$$\boxed{t_r \equiv t - \frac{\mathscr{r}}{c}.}$$

The natural generalization of $Eq.$ 10.24 for nonstatic sources is therefore

$$\boxed{V(\mathbf{r},t)=\frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}',t_r)}{\mathscr{r}}\,d\tau', \qquad \mathbf{A}(\mathbf{r},t)=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}',t_r)}{\mathscr{r}}\,d\tau'.}$$

Because the integrands are evaluated at the retarded time, these are called retarded potentials.

To prove them, they satisfy the inhomogeneous wave equation ($Eq.$ 10.16) and meet the Lorenz condition ($Eq.$ 10.12). If you apply the same logic to the fields you'll get entirely the wrong answer:

$$\mathbf{E}(\mathbf{r},t)\neq\frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}',t_r)}{\mathscr{r}^2}\hat{\mathscr{r}}\,d\tau', \quad \mathbf{B}(\mathbf{r},t)\neq\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}',t_r)\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau'$$

In calculating the Laplacian of $V(\mathbf{r}, t)$, the crucial point to notice is that the integrand depends on $\mathbf{r}$ in two places: explicitly, in the denominator $(\mathscr{r}=|\mathbf{r}-\mathbf{r}'|)$ and implicitly, through $t_r=t-\mathscr{r}/c$ in the numerator. Thus

$$\nabla V=\frac{1}{4\pi\epsilon_0}\int\left[(\nabla\rho)\frac{1}{\mathscr{r}}+\rho\nabla\left(\frac{1}{\mathscr{r}}\right)\right]d\tau' \qquad (10.27)$$

and $$\nabla \rho = \dot{\rho}\, \nabla t_r = -\frac{1}{c}\dot{\rho}\, \nabla \mathscr{r} \qquad (10.28)$$

The dot denotes differentiation with respect to time $\nabla \mathscr{r} = \hat{\mathscr{r}}$ and $\nabla(1/\mathscr{r}) = -\hat{\mathscr{r}}/\mathscr{r}^2$

$$\nabla V = \frac{1}{4\pi\epsilon_0}\int \left[-\frac{\dot{\rho}}{c}\frac{\hat{\mathscr{r}}}{\mathscr{r}} - \rho\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right] d\tau' \qquad (10.29)$$

Taking the divergence

$$\nabla^2 V = \frac{1}{4\pi\epsilon_0}\int \left\{-\frac{1}{c}\left[\frac{\hat{\mathscr{r}}}{\mathscr{r}}\cdot(\nabla\dot{\rho}) + \dot{\rho}\,\nabla\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}}\right)\right] - \left[\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\cdot(\nabla\rho) + \rho\nabla\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\right]\right\} d\tau'$$

But $$\nabla\dot{\rho} = -\frac{1}{c}\ddot{\rho}\,\nabla\mathscr{r} = -\frac{1}{c}\ddot{\rho}\,\hat{\mathscr{r}},$$

and $$\nabla\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}}\right) = \frac{1}{\mathscr{r}^2}$$

whereas $$\nabla \cdot \left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) = 4\pi \delta^3(\mathscr{r})$$

So $$\nabla^2 V = \frac{1}{4\pi\epsilon_0} \int \left[\frac{1}{c^2} \frac{\ddot{\rho}}{\mathscr{r}} - 4\pi\rho\delta^3(\mathscr{r}) \right] d\tau' = \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} - \frac{1}{\epsilon_0} \rho(\mathbf{r}, t)$$

confirming that the retarded potential ($Eq.$ 10.26) satisfies the inhomogeneous wave equation ($Eq.$ 10.16).

Incidentally, this proof applies equally well to the advanced potentials

$$V_a(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t_a)}{\mathscr{r}} d\tau', \qquad \mathbf{A}_a(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t_a)}{\mathscr{r}} d\tau'$$

advanced time $$t_a \equiv t + \frac{\mathscr{r}}{c}$$

Note:
Because the d' Alembertian involves t^2 (as opposed to t), the theory itself is time-reversal invariant, and does not distinguish "past" from "future."

Although the advanced potentials are entirely consistent with Maxwell's equations, they violate the most sacred tenet in all of physics: the principle of causality. They suggest that the potentials now depend on what the charge and the current distribution will be at sometime in the future. Although the advanced potentials are of some theoretical interest, they have no direct physical significance.

Jefimenko's Equations

Given the retarded potentials

$$V(\mathbf{r},t)=\frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}',t_r)}{\mathscr{r}}d\tau',\qquad \mathbf{A}(\mathbf{r},t)=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}',t_r)}{\mathscr{r}}d\tau' \tag{10.33}$$

it is, a straightforward matter to determine the fields:

$$\mathbf{E}=-\nabla V-\frac{\partial\mathbf{A}}{\partial t},\qquad \mathbf{B}=\nabla\times\mathbf{A}$$

But the details are not entirely trivial $\mathscr{r}=|\mathbf{r}-\mathbf{r}'|\qquad t_r=t-\mathscr{r}/c$

The gradient of V is obtained in $Eq.$ 10.29; the time derivative of A is :

$$\frac{\partial\mathbf{A}}{\partial t}=\frac{\mu_0}{4\pi}\int\frac{\dot{\mathbf{J}}}{\mathscr{r}}d\tau'$$

Putting them together and using $c^2 = 1/\mu_0\epsilon_0$

$$\boxed{\mathbf{E}(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int\left[\frac{\rho(\mathbf{r}',t_r)}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}} + \frac{\dot{\rho}(\mathbf{r}',t_r)}{c\mathscr{r}}\hat{\boldsymbol{\mathscr{r}}} - \frac{\dot{\mathbf{J}}(\mathbf{r}',t_r)}{c^2\mathscr{r}}\right]d\tau'.}$$

This is the time-dependent generalization of Coulomb's law

As for **B**, $$\nabla\times\mathbf{A} = \frac{\mu_0}{4\pi}\int\left[\frac{1}{\mathscr{r}}(\nabla\times\mathbf{J}) - \mathbf{J}\times\nabla\left(\frac{1}{\mathscr{r}}\right)\right]d\tau'.$$

$$(\nabla\times\mathbf{J})_x = \frac{\partial J_z}{\partial y} - \frac{\partial J_y}{\partial z},$$

and $$\frac{\partial J_z}{\partial y} = \dot{J}_z\frac{\partial t_r}{\partial y} = -\frac{1}{c}\dot{J}_z\frac{\partial \mathscr{r}}{\partial y}.$$

so $$(\nabla\times\mathbf{J})_x = -\frac{1}{c}\left(\dot{J}_z\frac{\partial\mathscr{r}}{\partial y} - \dot{J}_y\frac{\partial\mathscr{r}}{\partial z}\right) = \frac{1}{c}\left[\dot{\mathbf{J}}\times(\nabla\mathscr{r})\right]_x$$

But $\nabla \eta = \hat{\boldsymbol{\eta}}$,so

$$\nabla \times \mathbf{J} = \frac{1}{c} \dot{\mathbf{J}} \times \hat{\boldsymbol{\eta}}.$$

Meanwhile $\nabla(1/\eta) = -\hat{\boldsymbol{\eta}}/\eta^2$ and hence

$$\boxed{\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \left[\frac{\mathbf{J}(\mathbf{r}', t_r)}{\eta^2} + \frac{\dot{\mathbf{J}}(\mathbf{r}', t_r)}{c\eta} \right] \times \hat{\boldsymbol{\eta}}\, d\tau'.} \qquad (10.38)$$

This is the time-dependent generalization of the Biot-Savart law

Equations 10.36 and 10.38 are the (causal) solutions to Maxwell's equations. they do not seem to have been published until quite recently by Oleg Jefimenko, in 1966. In practice **Jefimenko's equations** are of limited utility, since it is typically easier to calculate the retarded potentials and differentiate them, rather than going directly to the fields.

Point Charges

Lienard-Wiechert Potentials

Next is to calculate the (retarded) potentials, $V(r,t)$ and $A(r,t)$, of a point charge q that is moving on a specified trajectory

$$\mathbf{w}(t) \equiv \text{position of q at time t}$$

the retarded time is determined implicitly by the equation

$$|\mathbf{r} - \mathbf{w}(t_r)| = c(t - t_r)$$

The left side is the distance the "news" must travel, and $(t - t_r)$ is the time it takes to make the trip; $\boldsymbol{\mathscr{r}}$ is the vector from the retarded position to the field point r:

$$\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{w}(t_r)$$

It is important to note that at most one point on the trajectory is "in communication" at any particular time t. If there are two points:

$$\mathscr{r}_1 = c(t - t_1) \quad \text{and} \quad \mathscr{r}_2 = c(t - t_2) \quad \text{Then } \mathscr{r}_1 - \mathscr{r}_2 = c(t_2 - t_1)$$

The average speed of the charged particle along $\boldsymbol{\mathscr{r}}$ is c (even not consider the speed along other directions yet)!

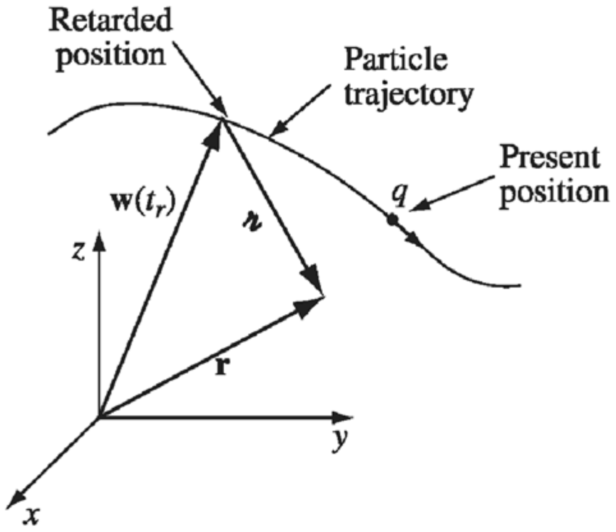

FIGURE 10.8

A naive reading of the formula

$$V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}',t_r)}{\mathscr{r}}\,d\tau'$$

might suggest to you that the potential is simply

$$\frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}}$$

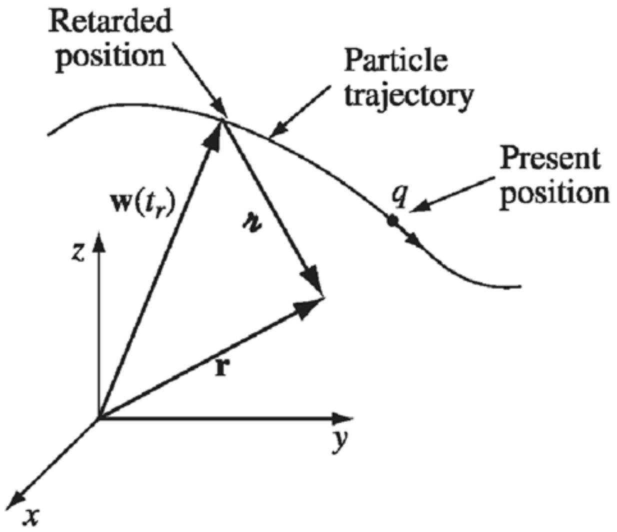

FIGURE 10.8

But this is wrong. It is true that for a point source the denominator r comes outside the integral, but what remains,

$$\int \rho(\mathbf{r}',t_r)\,d\tau' \qquad (10.41)$$

is not equal to the charge of the particle. To calculate the total charge of a configuration, you must integrate ρ over the entire distribution at one instant of time, but here the retardation $t_r = t - \mathscr{r}/c$, obliges us to evaluate ρ at different times for different parts of the configuration.

And for an extended particle

$$\int \rho(\mathbf{r}', t_r)\, d\tau' = \frac{q}{1 - \hat{\boldsymbol{\mathscr{r}}} \cdot \mathbf{v}/c}$$

Proof. This is a purely geometrical effect a train coming towards you looks a little longer than it really is, because the light you receive from the caboose left earlier than the light you receive simultaneously from the engine, and at that earlier time the train was farther away. In the interval it takes light from the caboose to travel the extra distance L' the train itself moves a distance $L' - L$:

$$\frac{L'}{c} = \frac{L' - L}{v}, \quad \text{or} \quad L' = \frac{L}{1 - v/c}$$

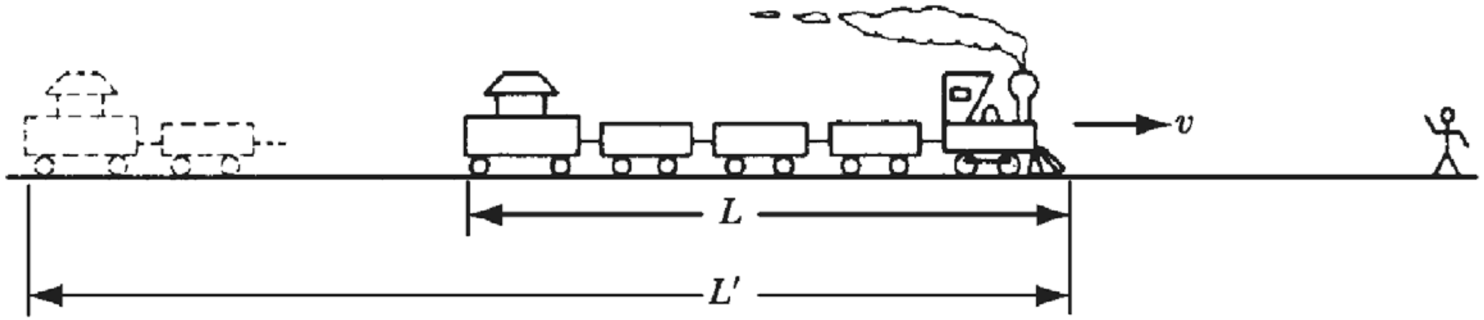

FIGURE 10.6

So approaching trains appear longer, by a factor $(1 - v/c)^{-1}$. By contrast, a train going away from you looks shorter, by a factor $(1 + v/c)^{-1}$. In general, if the train's velocity makes an angle with your line of sight, the extra distance light from the caboose must cover is $L' cos\theta$. In the time $L' \cos\theta/c$, the train moves a distance $(L' - L)$:

$$\frac{L' \cos\theta}{c} = \frac{L' - L}{v}, \quad \text{or} \quad L' = \frac{L}{1 - v\cos\theta/c}$$

Notice that this effect does not distort the dimensions perpendicular to the motion (the height and width of the train). since there's no motion in that direction, The apparent volume τ' of the train, then, is related to the actual volume τ by

$$\tau' = \frac{\tau}{1 - \hat{\boldsymbol{\imath}} \cdot \mathbf{v}/c}$$

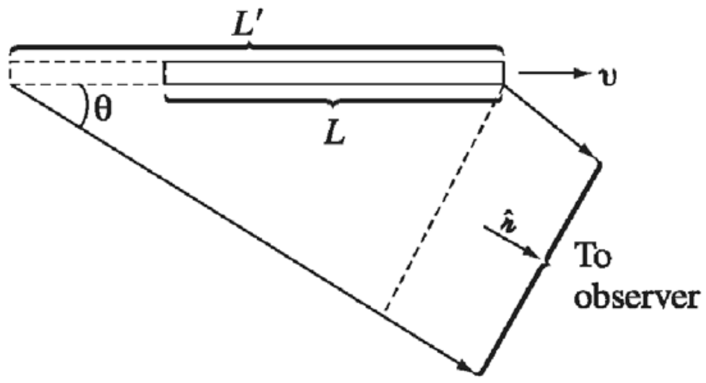

FIGURE 10.7

where $\hat{\boldsymbol{\imath}}$ is a unit vector from the train to the observer.

$$\int \rho(\mathbf{r}', t_r)\, d\tau'$$

Whenever you do an integral of the type in Eq. 10.41, in which the integrand is evaluated at the retarded time, the effective volume is modified by the factor in Eq. 10.43, just as the apparent volume of the train was.

$$\tau' = \frac{\tau}{1 - \hat{\boldsymbol{\mathscr{r}}} \cdot \mathbf{v}/c}$$

It follows, then, that

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t_r)}{\mathscr{r}}\, d\tau'$$

$$\boxed{V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{qc}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})},} \qquad (10.46)$$

where v is the velocity of the charge at the retarded time, and $\boldsymbol{\mathscr{r}}$ is the vector from the retarded position to the field point $\boldsymbol{r}$. Moreover, since the current density is ρv $(Eq. 5.26)$, the vector potential is

Lienard-Wiechert potentials for a moving point charge:

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\rho(\mathbf{r}', t_r)\mathbf{v}(t_r)}{\mathscr{r}}\, d\tau' = \frac{\mu_0}{4\pi} \frac{\mathbf{v}}{\mathscr{r}} \int \rho(\mathbf{r}', t_r)\, d\tau'.$$

or

$$\boxed{\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \frac{qc\mathbf{v}}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})} = \frac{\mathbf{v}}{c^2} V(\mathbf{r}, t).}$$

Example 10.3

Find the potentials of a point charge moving with constant velocity

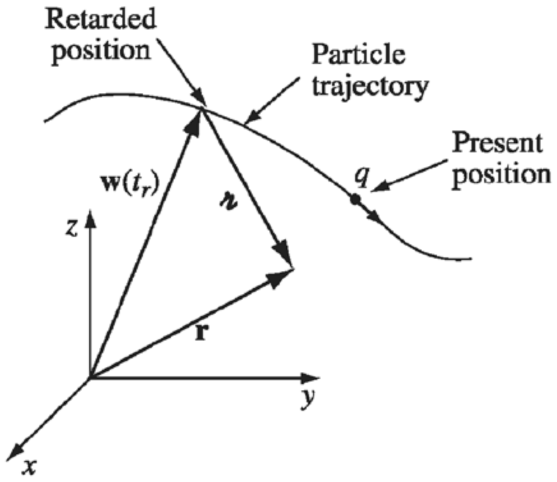

FIGURE 10.8

Solution

For convenience, let's say the particle passes through the origin at time t = 0, so that

$$\mathbf{w}(t) = \mathbf{v}t$$

We first compute the retarded time, using

$$|\mathbf{r} - \mathbf{v}t_r| = c(t - t_r)$$

squaring:

$$r^2 - 2\mathbf{r}\cdot\mathbf{v}t_r + v^2t_r^2 = c^2(t^2 - 2tt_r + t_r^2)$$

Solving for t_r by the quadratic formula,

$$t_r = \frac{(c^2 t - \mathbf{r}\cdot\mathbf{v}) \pm \sqrt{(c^2 t - \mathbf{r}\cdot\mathbf{v})^2 + (c^2 - v^2)(r^2 - c^2 t^2)}}{c^2 - v^2}$$

To fix the sign, consider the limit $v = 0$:

$$t_r = t \pm \frac{r}{c}$$

In this case the charge is at rest at the origin, and the retarded time should be $(t - r/c)$ evidently we want the minus sign.

Now, from Eqs. 10.44 and 10.45 $\mathscr{r} = c(t - t_r)$, and $\hat{\mathscr{r}} = \dfrac{\mathbf{r} - \mathbf{v}t_r}{c(t - t_r)}$

so $$\mathscr{r}(1-\hat{\mathscr{r}}\cdot\mathbf{v}/c)=c(t-t_r)\left[1-\frac{\mathbf{v}}{c}\cdot\frac{(\mathbf{r}-\mathbf{v}t_r)}{c(t-t_r)}\right]=c(t-t_r)-\frac{\mathbf{v}\cdot\mathbf{r}}{c}+\frac{v^2}{c}t_r$$

$$=\frac{1}{c}\left[(c^2t-\mathbf{r}\cdot\mathbf{v})-(c^2-v^2)t_r\right]$$

$$=\frac{1}{c}\sqrt{(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)}$$

Therefore, $$V(\mathbf{r},t)=\frac{1}{4\pi\epsilon_0}\frac{qc}{\sqrt{(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)}}$$

And,

$$\mathbf{A}(\mathbf{r},t)=\frac{\mu_0}{4\pi}\frac{qc\mathbf{v}}{\sqrt{(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)}}$$

The Fields of a Moving Point Charge

To calculate the electric and magnetic fields of a point charge in arbitrary motion, using the Lienard-Wiechert potentials:

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\frac{qc}{(\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v})}, \qquad \mathbf{A}(\mathbf{r}, t) = \frac{\mathbf{v}}{c^2}V(\mathbf{r}, t)$$

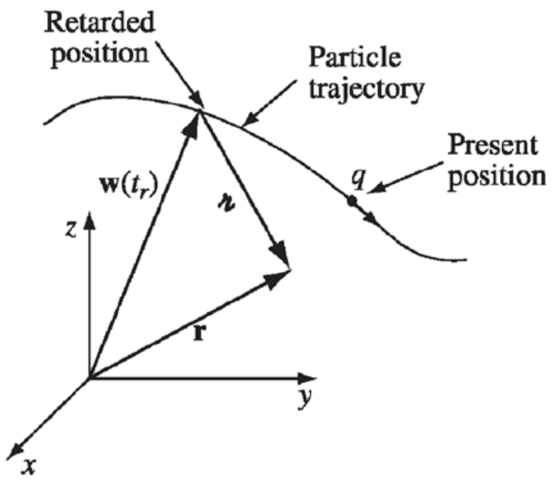

FIGURE 10.8

and the equations for ***E*** and ***B***

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

The differentiation is tricky, however, because

$$\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{w}(t_r) \quad \text{and} \quad \mathbf{v} = \dot{\mathbf{w}}(t_r) \qquad (10.54)$$

are both evaluated at the retarded time, and t_r defined implicitly by the equation

$$|\mathbf{r} - \mathbf{w}(t_r)| = c(t - t_r) \qquad (10.55)$$

is itself a function of r and t.

Let's begin with the gradient of V:

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{qc}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})}, \qquad \mathbf{A}(\mathbf{r}, t) = \frac{\mathbf{v}}{c^2} V(\mathbf{r}, t)$$

$$\nabla V = \frac{qc}{4\pi\epsilon_0} \frac{-1}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})^2} \nabla(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})$$

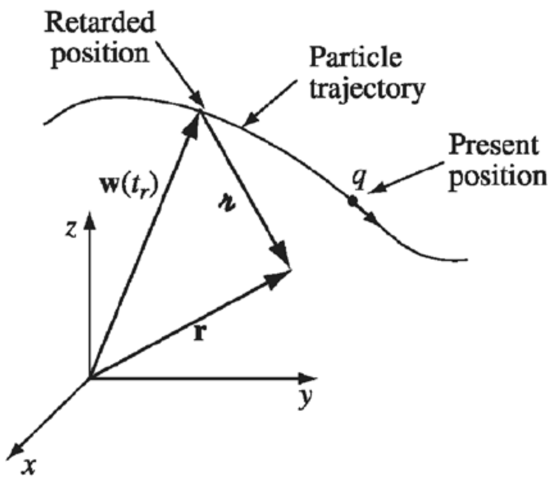

FIGURE 10.8

Since $\quad \mathscr{r} = c(t - t_r)$

$$\nabla \mathscr{r} = -c \nabla t_r \qquad (10.57)$$

As for the second term, product rule 4 gives

$$\nabla(\boldsymbol{\mathscr{r}} \cdot \mathbf{v}) = (\boldsymbol{\mathscr{r}} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\boldsymbol{\mathscr{r}} + \boldsymbol{\mathscr{r}} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \boldsymbol{\mathscr{r}}) \qquad (10.58)$$

$$\nabla(\boldsymbol{\mathscr{r}} \cdot \mathbf{v}) = (\boldsymbol{\mathscr{r}} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\boldsymbol{\mathscr{r}} + \boldsymbol{\mathscr{r}} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \boldsymbol{\mathscr{r}})$$

Evaluating these terms one at a time

$$(\boldsymbol{\mathscr{r}} \cdot \nabla)\mathbf{v} = \left(\mathscr{r}_x \frac{\partial}{\partial x} + \mathscr{r}_y \frac{\partial}{\partial y} + \mathscr{r}_z \frac{\partial}{\partial z}\right)\mathbf{v}(t_r)$$

$$= \mathscr{r}_x \frac{d\mathbf{v}}{dt_r}\frac{\partial t_r}{\partial x} + \mathscr{r}_y \frac{d\mathbf{v}}{dt_r}\frac{\partial t_r}{\partial y} + \mathscr{r}_z \frac{d\mathbf{v}}{dt_r}\frac{\partial t_r}{\partial z}$$

$$= \mathbf{a}(\boldsymbol{\mathscr{r}} \cdot \nabla t_r), \qquad (10.59)$$

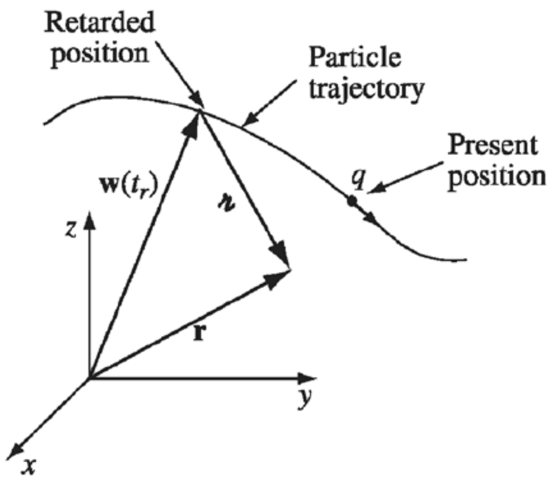

FIGURE 10.8

where $\mathbf{a} \equiv \dot{\mathbf{v}}$ is the acceleration of the particle at the retarded time. Next

$$(\mathbf{v} \cdot \nabla)\boldsymbol{\mathscr{r}} = (\mathbf{v} \cdot \nabla)\mathbf{r} - (\mathbf{v} \cdot \nabla)\mathbf{w} \qquad (10.60)$$

while $$(\mathbf{v} \cdot \nabla)\mathbf{r} = \left(v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z}\right)(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}})$$

$$= v_x\,\hat{\mathbf{x}} + v_y\,\hat{\mathbf{y}} + v_z\,\hat{\mathbf{z}} = \mathbf{v},$$

$$\nabla(\boldsymbol{\mathscr{r}} \cdot \mathbf{v}) = (\boldsymbol{\mathscr{r}} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\boldsymbol{\mathscr{r}} + \boldsymbol{\mathscr{r}} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \boldsymbol{\mathscr{r}})$$

and $$(\mathbf{v} \cdot \nabla)\mathbf{w} = \mathbf{v}(\mathbf{v} \cdot \nabla t_r)$$

(same reasoning as Eq. 10.59). Moving on to the third term in Eq. 10.58,

$$\begin{aligned}
\nabla \times \mathbf{v} &= \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}} \\
&= \left(\frac{dv_z}{dt_r}\frac{\partial t_r}{\partial y} - \frac{dv_y}{dt_r}\frac{\partial t_r}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{dv_x}{dt_r}\frac{\partial t_r}{\partial z} - \frac{dv_z}{dt_r}\frac{\partial t_r}{\partial x}\right)\hat{\mathbf{y}} \\
&\quad + \left(\frac{dv_y}{dt_r}\frac{\partial t_r}{\partial x} - \frac{dv_x}{dt_r}\frac{\partial t_r}{\partial y}\right)\hat{\mathbf{z}} \\
&= -\mathbf{a} \times \nabla t_r. \qquad (10.62)
\end{aligned}$$

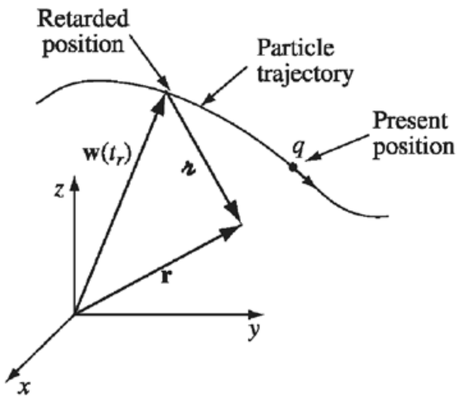

FIGURE 10.8

Finally, $$\nabla \times \boldsymbol{\mathscr{r}} = \nabla \times \mathbf{r} - \nabla \times \mathbf{w} \qquad (10.63)$$

$$\nabla(\boldsymbol{\mathscr{r}} \cdot \mathbf{v}) = (\boldsymbol{\mathscr{r}} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\boldsymbol{\mathscr{r}} + \boldsymbol{\mathscr{r}} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \boldsymbol{\mathscr{r}})$$

But $\nabla \times \mathbf{r} = \mathbf{0}$ while, by the same argument as $Eq.$ 10.62

$$\nabla \times \mathbf{w} = -\mathbf{v} \times \nabla t_r \qquad (10.64)$$

Putting all this back into Eq. 10.58, and using the "BAC-CAB" rule to reduce the triple cross products,

$$\begin{aligned} \nabla(\boldsymbol{\mathscr{r}} \cdot \mathbf{v}) &= \mathbf{a}(\boldsymbol{\mathscr{r}} \cdot \nabla t_r) + \mathbf{v} - \mathbf{v}(\mathbf{v} \cdot \nabla t_r) - \boldsymbol{\mathscr{r}} \times (\mathbf{a} \times \nabla t_r) + \mathbf{v} \times (\mathbf{v} \times \nabla t_r) \\ &= \mathbf{v} + (\boldsymbol{\mathscr{r}} \cdot \mathbf{a} - v^2)\nabla t_r . \end{aligned} \qquad (10.65)$$

Collecting $Eqs.$ 10.57 and 10.65, we have

$$\nabla V = \frac{qc}{4\pi\epsilon_0} \frac{1}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})^2} \left[\mathbf{v} + (c^2 - v^2 + \boldsymbol{\mathscr{r}} \cdot \mathbf{a})\nabla t_r \right]$$

To complete the calculation, we need to know ∇t_r

$$-c\nabla t_r = \nabla \mathscr{r} = \nabla\sqrt{\boldsymbol{\mathscr{r}}\cdot\boldsymbol{\mathscr{r}}} = \frac{1}{2\sqrt{\boldsymbol{\mathscr{r}}\cdot\boldsymbol{\mathscr{r}}}}\nabla(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{\mathscr{r}}) \qquad \mathscr{r} = c(t - t_r)$$

$$= \frac{1}{\mathscr{r}}\left[(\boldsymbol{\mathscr{r}}\cdot\nabla)\boldsymbol{\mathscr{r}} + \boldsymbol{\mathscr{r}}\times(\nabla\times\boldsymbol{\mathscr{r}})\right].$$

But $$(\boldsymbol{\mathscr{r}}\cdot\nabla)\boldsymbol{\mathscr{r}} = \boldsymbol{\mathscr{r}} - \mathbf{v}(\boldsymbol{\mathscr{r}}\cdot\nabla t_r)$$

(same idea as Eq. 10.60), while (from Eqs. 10.63 and 10.64)

$$\nabla\times\boldsymbol{\mathscr{r}} = (\mathbf{v}\times\nabla t_r)$$

Thus $$-c\nabla t_r = \frac{1}{\mathscr{r}}\left[\boldsymbol{\mathscr{r}} - \mathbf{v}(\boldsymbol{\mathscr{r}}\cdot\nabla t_r) + \boldsymbol{\mathscr{r}}\times(\mathbf{v}\times\nabla t_r)\right] = \frac{1}{\mathscr{r}}\left[\boldsymbol{\mathscr{r}} - (\boldsymbol{\mathscr{r}}\cdot\mathbf{v})\nabla t_r\right]$$

and hence $$\nabla t_r = \frac{-\boldsymbol{\mathscr{r}}}{\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v}}$$

Incorporating this result into $Eq. 10.66$

$$\nabla V = \frac{1}{4\pi\epsilon_0} \frac{qc}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})^3} \left[(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})\mathbf{v} - (c^2 - v^2 + \boldsymbol{\mathscr{r}} \cdot \mathbf{a})\boldsymbol{\mathscr{r}} \right]$$

A similar calculation yields

$$\frac{\partial \mathbf{A}}{\partial t} = \frac{1}{4\pi\epsilon_0} \frac{qc}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})^3} \left[(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})(-\mathbf{v} + \mathscr{r}\mathbf{a}/c) + \frac{\mathscr{r}}{c}(c^2 - v^2 + \boldsymbol{\mathscr{r}} \cdot \mathbf{a})\mathbf{v} \right].$$

Combining these results, and introducing the vector

$$\mathbf{u} \equiv c\hat{\boldsymbol{\mathscr{r}}} - \mathbf{v}$$

$$\boxed{\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}} \cdot \mathbf{u})^3} \left[(c^2 - v^2)\mathbf{u} + \boldsymbol{\mathscr{r}} \times (\mathbf{u} \times \mathbf{a}) \right].} \qquad (10.72)$$

Meanwhile, $$\nabla \times \mathbf{A} = \frac{1}{c^2}\nabla \times (V\mathbf{v}) = \frac{1}{c^2}[V(\nabla \times \mathbf{v}) - \mathbf{v} \times (\nabla V)]$$

$\nabla \times \mathbf{v}$ (Eq. 10.62) ∇V (Eq. 10.69). Putting these together,

$$\nabla \times \mathbf{A} = -\frac{1}{c}\frac{q}{4\pi\epsilon_0}\frac{1}{(\mathbf{u}\cdot\boldsymbol{\mathscr{r}})^3}\boldsymbol{\mathscr{r}} \times \left[(c^2 - v^2)\mathbf{v} + (\boldsymbol{\mathscr{r}}\cdot\mathbf{a})\mathbf{v} + (\boldsymbol{\mathscr{r}}\cdot\mathbf{u})\mathbf{a}\right]$$

The quantity in brackets is strikingly similar to the one in $Eq. 10.72$

$$\mathbf{B}(\mathbf{r}, t) = \frac{1}{c}\hat{\boldsymbol{\mathscr{r}}} \times \mathbf{E}(\mathbf{r}, t).$$

The first term in **E** falls off as the inverse square of the distance from the particle. If the velocity and acceleration are both zero,

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}.$$

$$\boxed{\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}\left[(c^2 - v^2)\mathbf{u} + \boldsymbol{\mathscr{r}}\times(\mathbf{u}\times\mathbf{a})\right].} \quad (10.72)$$

For this reason, the first term in **E** is sometimes called the generalized Coulomb field. Because it does not depend on the acceleration, it is also known as the velocity field. The second term falls off as the inverse first power of **r** - and is therefore dominant at large distances. It is this term that is responsible for electromagnetic radiation; it is called the radiation field - or, since it is proportional to **a**, the acceleration field. The same terminology applies to the magnetic field.

the Lorentz force law determines the resulting force:

$$\mathbf{F} = \frac{qQ}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}\Big\{[(c^2 - v^2)\mathbf{u} + \boldsymbol{\mathscr{r}}\times(\mathbf{u}\times\mathbf{a})] + \frac{\mathbf{V}}{c}\times\left[\hat{\boldsymbol{\mathscr{r}}}\times[(c^2 - v^2)\mathbf{u} + \boldsymbol{\mathscr{r}}\times(\mathbf{u}\times\mathbf{a})]\right]\Big\}, \quad (10.74)$$

where **V** is the velocity of Q, and $\boldsymbol{\mathscr{r}}$, **u**, **v**, and **a** are all evaluated at the retarded time.

Example 10.4.

Calculate the electric and magnetic fields of a point charge moving with constant velocity.

Solution

Putting $a = 0$ in $Eq. 10.72$,

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0} \frac{(c^2 - v^2)\mathscr{r}}{(\boldsymbol{\mathscr{r}} \cdot \mathbf{u})^3}$$

In this case, using $w = vt$, $\mathbf{u} \equiv c\hat{\boldsymbol{\mathscr{r}}} - \mathbf{v}$

$$\mathscr{r}\mathbf{u} = c\boldsymbol{\mathscr{r}} - \mathscr{r}\mathbf{v} = c(\mathbf{r} - \mathbf{v}t_r) - c(t - t_r)\mathbf{v} = c(\mathbf{r} - \mathbf{v}t)$$

In Ex. 10.3 we found that

$$\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v} = \boldsymbol{\mathscr{r}} \cdot \mathbf{u} = \sqrt{(c^2t - \mathbf{r} \cdot \mathbf{v})^2 + (c^2 - v^2)(r^2 - c^2t^2)}$$

In Prob. 10.16, this radical could be written as

$$Rc\sqrt{1 - v^2 \sin^2\theta / c^2}$$

where $\mathbf{R} \equiv \mathbf{r} - \mathbf{v}t$

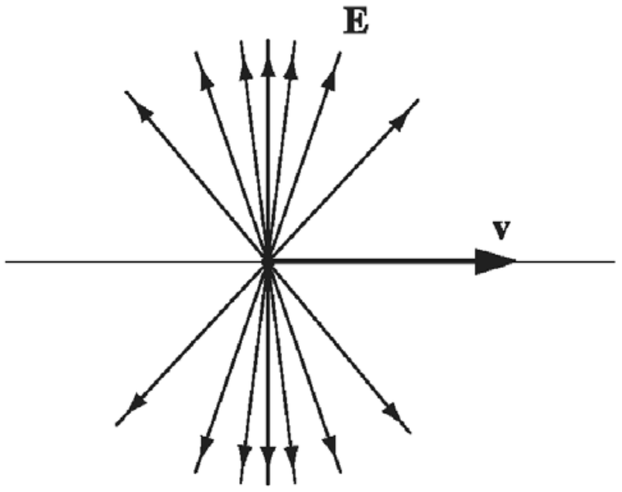

FIGURE 10.10

is the vector from the present location of the particle to r, and theta is the angle between R and v (Fig. 10.9). Thus

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1 - v^2/c^2}{\left(1 - v^2 \sin^2\theta/c^2\right)^{3/2}} \frac{\hat{\mathbf{R}}}{R^2}. \quad (10.75)$$

Notice that E points along the line from the present position of the particle. This is an extraordinary coincidence, since the “message" came from the retarded position. Because of the $sin^2\theta$ in the denominator, the field of a fast-moving charge is flattened out like a pancake in the direction perpendicular to the motion($Fig.$ 10.10). In the forward and backward directions E is reduced by a factor $(1 - v^2/c^2)$ elative to the field of a charge at rest; in the perpendicular direction, it is enhanced by a factor $1/\sqrt{1 - v^2/c^2}$

As for **B**, we have

$$\hat{\boldsymbol{\mathscr{r}}} = \frac{\mathbf{r} - \mathbf{v}t_r}{\mathscr{r}} = \frac{(\mathbf{r} - \mathbf{v}t) + (t - t_r)\mathbf{v}}{\mathscr{r}} = \frac{\mathbf{R}}{\mathscr{r}} + \frac{\mathbf{v}}{c}$$

and therefore

$$\mathbf{B} = \frac{1}{c}(\hat{\boldsymbol{\mathscr{r}}} \times \mathbf{E}) = \frac{1}{c^2}(\mathbf{v} \times \mathbf{E}) \qquad (10.76)$$

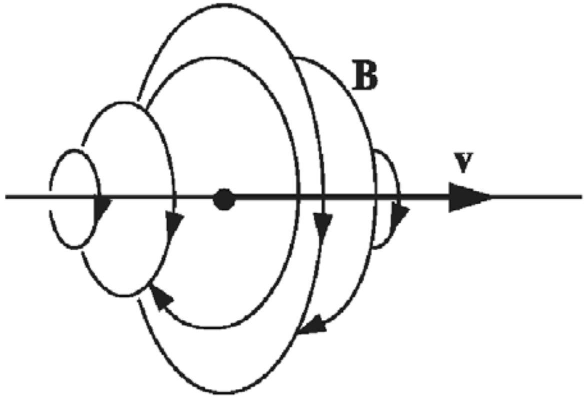

FIGURE 10.11

Lines of **B** circle around the charge, as shown in Fig. 10.11

The fields of a point charge moving at constant velocity (Eqs. 10.75 *and* 10.76) were first obtained by Oliver Heaviside in 1888. When $v^2 \ll c^2$ they reduce to

$$\mathbf{E}(\mathbf{r}, t) \approx \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}\hat{\mathbf{R}} \qquad \mathbf{B}(\mathbf{r}, t) \approx \frac{\mu_0}{4\pi}\frac{q}{R^2}(\mathbf{v} \times \hat{\mathbf{R}})$$

The first is essentially Coulomb's law, and the second is the "Biot-Savart law for a point charge"