电动力学-第九次作业

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Problem 9.19

Answer:

(a)

For glass:

$$\epsilon = 4.7\epsilon_0 \quad \sigma = 1 \cdot 10^{-13} \tag{1.1}$$

With this the relaxation time τ is:

$$\tau = \frac{\epsilon}{\sigma} \approx 416.138\text{s} \tag{1.2}$$

(b)

The skin depth d is given by:

$$d = \frac{1}{\kappa} = \sqrt{\frac{2}{\epsilon\mu}}[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2} - 1]^{-1/2} \tag{1.3}$$

Using the values of the constants for the silver (again, in SI units), and the provided angular frequncy:

$$\epsilon \approx \epsilon_0 \quad \mu \approx \mu_0 \quad \sigma = 6.29 \cdot 10^7 \tag{1.4}$$

The skin depth has the value:

$$d \approx 6.35 \cdot 10^{-7}\text{m} = 0.64\mu\text{m} \tag{1.5}$$

(c)

The frequency is $1\text{MHz} = 10^6\text{Hz}$. To get the wave lenght we calculate the wave number k:

$$k = \sqrt{\frac{2}{\epsilon\mu}}[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2} + 1]^{1/2} \tag{1.6}$$

Simliarly with (b), we can get:

$$k \approx 15299.807\text{m}^{-1} \tag{1.7}$$

So:

$$\lambda = \frac{2\pi}{k} \approx 4.107 \cdot 10^{-4}\text{m} \tag{1.8}$$

$$v = \frac{\omega}{k} = f\lambda \approx 410.7\text{m/s} \tag{1.9}$$

Problem 9.31

Answer:

For the TM mode $B_z = 0$, so:

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2\right] E_z = 0 \tag{2.1}$$

with the boundary condition $\vec{E}_{\parallel} = 0$, meaning that $E_z = 0$ at the boundary of the waveguide. This translates into:

$$E_z(x, b) = E_z(x, 0) = E_z(0, y) = E_z(a, y) = 0 \tag{2.2}$$

Now, let us assume that $E_z = X(x)Y(y)$, so:

$$\frac{X''}{X} + \frac{Y''}{Y} = -(\omega/c)^2 + k^2 \tag{2.3}$$

$$-k_x^2 - k_y^2 = -(\omega/c)^2 + k^2 \tag{2.4}$$

Then:

$$X(x) = A\cos(k_x x) + B\sin(k_x x) \tag{2.5}$$

$$Y(y) = C\cos(k_y y) + D\sin(k_y y) \tag{2.6}$$

Apply the boundary conditions, which yield:

$$\begin{aligned} E_z(x, 0) = 0 &\rightarrow C = 0 \\ E_z(0, y) = 0 &\rightarrow A = 0 \\ E_z(x, b) = 0 &\rightarrow k_y = \frac{m\pi}{b} \\ E_z(a, y) = 0 &\rightarrow k_x = \frac{n\pi}{a} \end{aligned} \tag{2.7}$$

where m and n are integers larger than zero, otherwise the solution is trivial (zero).
Thus, the z -component of the field is:

$$E_z = XY = E_0 \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \tag{2.8}$$

The cutoff frequencies:

$$(\omega/c)^2 - k^2 - k_x^2 - k_y^2 = 0 \tag{2.9}$$

$$\omega = c\sqrt{k^2 + \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2} = c\sqrt{k^2 + \frac{\omega_{mn}^2}{c^2}} \tag{2.10}$$

So:

$$\omega_{mn} = c\pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}} \tag{2.11}$$

The lowest cutoff frequency is, as we mentioned, the (1,1) mode, so:

$$\omega_{11} = c\pi\sqrt{\frac{1}{a^2} + \frac{1}{b^2}} \tag{2.12}$$

The wave velocity and the group velocity are as follows:

$$v = \frac{\omega}{k} = \frac{\omega}{\frac{1}{c}\sqrt{\omega^2 - \omega_{mn}^2}} = \frac{c}{\sqrt{1 - (\omega_{mn}/\omega)^2}} \tag{2.13}$$

$$\begin{aligned} v_g &= \frac{d\omega}{dk} = \frac{d}{dk} c\sqrt{k^2 + (\omega_{mn}/c)^2} = \frac{ck}{\sqrt{k^2 + (\omega_{mn}/c)^2}} = \frac{ck}{\omega/c} \\ &= \frac{c^2}{v} = c\sqrt{1 - (\omega_{mn}/\omega)^2} \end{aligned} \tag{2.14}$$

Finally, the ratio of lowest TE cutoff frequency to the lowest TM cutoff frequency is:

$$\frac{\omega_{11}}{\omega_{10}} = \frac{c\pi\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}}{\frac{c\pi}{a}} = \sqrt{1 + (a/b)^2} \tag{2.15}$$

Problem 3.36

Answer:

Let the two planes be at $z = 0$ and $z = d$, let the electric field travel along the z -axis and be polarized along the x -axis, then the fields are, in material 1:

$$\vec{E}_I = E_{0I} e^{i(k_1 z - \omega t)} \hat{x} \tag{3.1}$$

$$\vec{B}_I = \frac{E_{0I}}{v_1} e^{i(k_1 z - \omega t)} \hat{z} \times \hat{x} = \frac{E_{0I}}{v_1} e^{i(k_1 z - \omega t)} \hat{y} \tag{3.2}$$

$$\vec{E}_R = E_{0R} e^{i(-k_{12} - \omega t)} \hat{x} \tag{3.3}$$

$$\vec{B}_R = \frac{E_{0R}}{v_1} e^{i(-k_1 z - \omega t)} (-\hat{z}) \times \hat{x} = -\frac{E_{0R}}{v_1} e^{i(-k_1 z - \omega t)} \hat{y} \tag{3.4}$$

For material 2:

$$\vec{E}_r = E_{0r} e^{i(k_2 z - \omega t)} \hat{x} \tag{3.5}$$

$$\vec{B}_r = \frac{E_{0r}}{v_2} e^{i(k_2 z - \omega t)} \hat{y} \tag{3.6}$$

$$\vec{E}_l = E_{0l} e^{i(-k_2 z - \omega t)} \hat{x} \tag{3.7}$$

$$\vec{B}_l = -\frac{E_{0l}}{v_2} e^{i(-k_{2z} - \omega t)} \hat{y} \tag{3.8}$$

where r stands for wave going to the right, and l for one going to the left. Material 3 only has the transmitted wave:

$$\vec{E}_T = E_{0T} e^{i(k_3 z - \omega t)} \hat{x} \tag{3.9}$$

$$\vec{B}_T = \frac{E_{0T}}{v_3} e^{i(k_3 z - \omega t)} \hat{y} \tag{3.10}$$

On both of the planes we impose boundary conditions. The first one, $\vec{E}_{\parallel,1} = \vec{E}_{\parallel,2}$, gives:

$$E_{0I} + E_{0R} = E_{0r} + E_{0l} \tag{3.11}$$

$$E_{0T} e^{ik_3 d} = E_{0r} e^{ik_2 d} + E_{0l} e^{-ik_2 d} \tag{3.12}$$

and the second one, $\vec{B}_{\parallel,1}/\mu_1 = \vec{B}_{\parallel,2}/\mu_2$, gives:

$$\frac{1}{\mu_1 v_1} (E_{0I} - E_{0R}) = \frac{1}{\mu_2 v_2} (E_{0r} - E_{0l}) \tag{3.13}$$

$$\frac{1}{\mu_3 v_3} E_{0T} e^{ik_3 d} = \frac{1}{\mu_2 v_2} \left(E_{0r} e^{ik_2 d} - E_{0l} e^{-ik_2 d} \right) \tag{3.14}$$

Combine (3.11) (3.12) (3.13) (3.14) to solve simultaneous equations.

So:

$$
\begin{aligned}
T &= \frac{n_3^2}{n_1^2}\frac{1}{\beta_{13}}\frac{4}{\left(1+\beta_{13}\right)^2+\sin^2\left(k_2 d\right)\left(\beta_{12}^2+\beta_{23}^2+2\beta_{12}\beta_{23}-1-2\beta_{13}-\beta_{13}^2\right)} \\
&= \frac{n_3}{n_1}\frac{4}{\left(1+n_3/n_1\right)^2+\sin^2 k_2 d\left(\left(n_3/n_2\right)^2+\left(n_2/n_1\right)^2-\left(n_3/n_1\right)^2-1\right)} \\
&= \frac{4n_1 n_3}{\left(n_1+n_3\right)^2+\sin^2 k_2 d\left(n_1^2\left(n_3/n_2\right)^2+n_2^2-n_3^2-n_1^2\right)} \\
&= \frac{4n_1 n_3}{\left(n_1+n_3\right)^2+\sin^2\left(\frac{\omega n_2 d}{c}\right)\frac{\left(n_1^2-n_2^2\right)\left(n_3^2-n_2^2\right)}{n_2^2}}
\end{aligned}
\tag{3.15}
$$