Chapter 9: Part B
Quantum statistics of ideal gases

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Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics
9.1 Identical particles and symmetry requirements

Quantum state description:

$$\{s_1, s_2, \ldots, s_N\}$$

Wave function:

$$\Psi = \Psi_{\{s_1, \ldots, s_N\}}(Q_1, Q_2, \ldots, Q_N)$$

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics
9.1 Identical particles and symmetry requirements

"Classical limit":

1, distinguishable;
2, any number of particles can be in the same state;
3, no symmetry requirement

Quantum mechanics:

1, exchange two particles does not obey a new state;
2, indistinguishable;
3, symmetry requirement← fundamental postulate of quantum mechanics

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics
9.1 Identical particles and symmetry requirements

A, Particles with integral spin (Bose-Einstein statistics)

$$\Psi(\cdots Q_j \cdots Q_i \cdots) = \Psi(\cdots Q_i \cdots Q_j \cdots)$$

1, exchange two particles does not obey a new state;
2, indistinguishable;
3, no restriction for number of particles in the same state

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics
9.1 Identical particles and symmetry requirements

B, Particles with half-integral spin (Fermi-Dirac statistics)

$$\Psi(\cdots Q_j \cdots Q_i \cdots) = -\Psi(\cdots Q_i \cdots Q_j \cdots)$$

1, exchange two particles does not obey a new state;
2, indistinguishable;
3, restriction for number of particles in the same state??

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.1 Identical particles and symmetry requirements

B, Particles with half-integral spin (Fermi-Dirac statistics)

$$\Psi(\cdots Q_j \cdots Q_i \cdots) = -\Psi(\cdots Q_i \cdots Q_j \cdots)$$

If two particles i and j are in the same state, then

$$\Psi(\cdots Q_j \cdots Q_i \cdots) = \Psi(\cdots Q_i \cdots Q_j \cdots)$$

$\Psi = 0$ when particles i and j are in the same state s

No state!!! **Pauli exclusion principle!!!**

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.1 Identical particles and symmetry requirements

Illustration:

Two particles A and B can be In three states: 1, 2 ,3

Maxwell-Boltzmann statistics

1	2	3
AB	...	...
...	AB	...
...	...	AB
A	B	...
B	A	...
A	...	B
B	...	A
...	A	B
...	B	A

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.1 Identical particles and symmetry requirements

Illustration:

Two particles A and B can be
In three states: 1, 2 ,3

Bose-Einstein statistics

1	2	3
AA	. . .	. . .
. . .	AA	. . .
. . .	. . .	AA
A	A	. . .
A	. . .	A
. . .	A	A

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.1 Identical particles and symmetry requirements

Illustration:

Two particles A and B can be

In three states: 1, 2 ,3

Fermi-Dirac statistics

1	2	3
A	A	. . .
A	. . .	A
. . .	A	A

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.2 Formulation of the statistical problem

Energy:

$$\boxed{E_R = n_1\epsilon_1 + n_2\epsilon_2 + n_3\epsilon_3 + \cdots = \sum_r n_r\epsilon_r}$$

restriction:

$$\boxed{\sum_r n_r = N}$$

Partition function:

$$\boxed{Z = \sum_R e^{-\beta E_R} = \sum_R e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + \cdots)}}$$

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics
9.2 Formulation of the statistical problem

Mean number in *s* state:

$$\bar{n}_s = \frac{\sum_R n_s e^{-\beta(n_1\epsilon_1+n_2\epsilon_2+\cdots)}}{\sum_R e^{-\beta(n_1\epsilon_1+n_2\epsilon_2+\cdots)}}$$

$$= \frac{1}{Z}\sum_R \left(-\frac{1}{\beta}\frac{\partial}{\partial \epsilon_s}\right) e^{-\beta(n_1\epsilon_1+n_2\epsilon_2+\cdots)}$$

$$= -\frac{1}{\beta Z}\frac{\partial Z}{\partial \epsilon_s}$$

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics
9.2 Formulation of the statistical problem

Maxwell-Boltzmann statistics

$n_r = 0, 1, 2, 3, \ldots \quad \text{for each } r$

$\sum_r n_r = N$

distinguishable

Bose-Einstein statistics

$n_r = 0, 1, 2, 3, \ldots \quad \text{for each } r$

$\sum_r n_r = N$

indistinguishable

Fermi-Dirac statistics

$n_r = 0, 1 \quad \text{for each } r$

$\sum_r n_r = N$

indistinguishable

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics
9.3 Quantum distribution functions

$$\bar{n}_s = \frac{\sum\limits_{n_1, n_2, \ldots} n_s \, e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + \cdots + n_s\epsilon_s + \cdots)}}{\sum\limits_{n_1, n_2, \ldots} e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + \cdots + n_s\epsilon_s + \cdots)}}$$

$$\sum_r n_r = N$$

$$\bar{n}_s = \frac{\sum\limits_{n_s} n_s \, e^{-\beta n_s \epsilon_s} \sum\limits_{n_1, n_2, \ldots}^{(s)} e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + \cdots)}}{\sum\limits_{n_s} e^{-\beta n_s \epsilon_s} \sum\limits_{n_1, n_2, \ldots}^{(s)} e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + \cdots)}}$$

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.3 Quantum distribution functions

Photon statistics: BE statistics without restricted *N*

$$\bar{n}_s = \frac{\sum_{n_s} n_s e^{-\beta n_s \epsilon_s}}{\sum_{n_s} e^{-\beta n_s \epsilon_s}}$$

$$\bar{n}_s = \frac{\sum_{n_s} n_s e^{-\beta n_s \epsilon_s} \sum_{n_1, n_2, \ldots}^{(s)} e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + \cdots)}}{\sum_{n_s} e^{-\beta n_s \epsilon_s} \sum_{n_1, n_2, \ldots}^{(s)} e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + \cdots)}}$$

$$\bar{n}_s = \frac{(-1/\beta)(\partial/\partial\epsilon_s)\Sigma\, e^{-\beta n_s \epsilon_s}}{\Sigma\, e^{-\beta n_s \epsilon_s}}$$

$$= -\frac{1}{\beta}\frac{\partial}{\partial\epsilon_s}\ln\left(\Sigma\, e^{-\beta n_s \epsilon_s}\right)$$

$$= \frac{1}{1 - e^{-\beta\epsilon_s}}$$

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.3 Quantum distribution functions

Photon statistics: BE statistics without restricted *N*

$$\bar{n}_s = \frac{1}{\beta}\frac{\partial}{\partial \epsilon_s}\ln\left(1 - e^{-\beta\epsilon_s}\right) = \frac{e^{-\beta\epsilon_s}}{1 - e^{-\beta\epsilon_s}}$$

$$\bar{n}_s = \frac{1}{e^{\beta\epsilon_s} - 1}$$

Plank distribution

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.3 Quantum distribution functions

$$\sum_r n_r = N$$

Fermi-Dirac statistics $n_r = 0 \text{ and } 1$

define

$$Z_s(N) = \sum_{n_1, n_2, \dots}^{(s)} e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + \cdots)}$$

$n_s=0$ 1

$$\boxed{\bar{n}_s = \frac{0 + e^{-\beta\epsilon_s} Z_s(N-1)}{Z_s(N) + e^{-\beta\epsilon_s} Z_s(N-1)}}$$

$$\bar{n}_s = \frac{1}{[Z_s(N)/Z_s(N-1)]\, e^{\beta\epsilon_s} + 1}$$

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.3 Quantum distribution functions

Fermi-Dirac statistics

$$\bar{n}_s = \frac{1}{[Z_s(N)/Z_s(N-1)]\, e^{\beta\epsilon_s} + 1}$$

define

$$\alpha = \frac{\partial \ln Z}{\partial N}$$

$$\boxed{Z(N)/Z(N-1) = e^{\alpha}}$$

$$\boxed{\bar{n}_s = \frac{1}{e^{\alpha+\beta\epsilon_s} + 1}}$$

Fermi-Dirac distribution

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.3 Quantum distribution functions

Fermi-Dirac statistics

define $$\alpha = \frac{\partial \ln Z}{\partial N}$$

$$\alpha = -\frac{1}{kT}\frac{\partial F}{\partial N} =$$

$$= -\frac{\mu}{kT} = -\beta\mu$$

$$\bar{n}_s = \frac{1}{e^{\alpha+\beta\epsilon_s}+1}$$

$$0 \leq \bar{n}_s \leq 1$$

$$\bar{n}_s \to 0$$

if ϵ_s becomes large enough

Chemical potential per particle

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.3 Quantum distribution functions

Bose-Einstein statistics $n_s = 0, 1, 2, 3, \ldots$

$n_s =$ 0 1 2

$$\bar{n}_s = \frac{0 + e^{-\beta\epsilon_s} Z_s(N-1) + 2e^{-2\beta\epsilon_s} Z_s(N-2) + \cdots}{Z_s(N) + e^{-\beta\epsilon_s} Z_s(N-1) + e^{-2\beta\epsilon_s} Z_s(N-2) + \cdots}$$

$$Z(N)/Z(N-1) = e^{\alpha}$$

$$\bar{n}_s = \frac{Z_s(N)[0 + e^{-\beta\epsilon_s} e^{-\alpha} + 2e^{-2\beta\epsilon_s} e^{-2\alpha} + \cdots]}{Z_s(N)[1 + e^{-\beta\epsilon_s} e^{-\alpha} + e^{-2\beta\epsilon_s} e^{-2\alpha} + \cdots]}$$

$$\bar{n}_s = \frac{\sum_s n_s e^{-n_s(\alpha+\beta\epsilon_s)}}{\sum_s e^{-n_s(\alpha+\beta\epsilon_s)}}$$

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.3 Quantum distribution functions

Bose-Einstein statistics

$$\bar{n}_s = \frac{\sum_s n_s e^{-n_s(\alpha+\beta\varepsilon_s)}}{\sum_s e^{-n_s(\alpha+\beta\varepsilon_s)}}$$

$$n_s = -\frac{1}{\beta}\frac{\partial}{\partial\varepsilon_s}\ln\left(\sum_{n_s} e^{-n_s(\alpha+\beta\varepsilon_s)}\right)$$

$$\frac{1}{1-e^{-(\alpha+\beta\varepsilon_s)}}$$

$$n_s = \frac{e^{-(\alpha+\beta\varepsilon_s)}}{1-e^{-(\alpha+\beta\varepsilon_s)}}$$

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.3 Quantum distribution functions

Bose-Einstein statistics

$$\bar{n}_s = \frac{\sum_s n_s e^{-n_s(\alpha+\beta\varepsilon_s)}}{\sum_s e^{-n_s(\alpha+\beta\varepsilon_s)}}$$

$$\bar{n}_s = \frac{1}{e^{\alpha+\beta\varepsilon_s} - 1}$$

α can be determined by

$$\sum_r \frac{1}{e^{\alpha+\beta\varepsilon_r} - 1} = N$$

$$\alpha = -\beta\mu$$

$$\bar{n}_s = \frac{1}{e^{\beta(\varepsilon_s-\mu)} - 1}$$

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.4 Maxwell-Boltzmann statistics: comparison

Partition function:

$$Z = \sum_{R} e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + \cdots)}$$

for *given* values of $\{n_1, n_2, \ldots\}$

Possible way:

$$\frac{N!}{n_1!n_2!\cdots}$$

$$Z = \sum_{n_1, n_2, \ldots} \frac{N!}{n_1!n_2!\cdots} e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + \cdots)}$$

$$\sum_{r} n_r = N$$

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.4 Maxwell-Boltzmann statistics

Partition function:

$$Z = \sum_{n_1,n_2,\ldots} \frac{N!}{n_1!n_2!\cdots} e^{-\beta(n_1\epsilon_1+n_2\epsilon_2+\cdots)}$$

$$Z = \sum_{n_1,n_2,\ldots} \frac{N!}{n_1!n_2!\cdots} (e^{-\beta\epsilon_1})^{n_1}(e^{-\beta\epsilon_2})^{n_2}\cdots$$

$$Z = (e^{-\beta\epsilon_1} + e^{-\beta\epsilon_2} + \cdots)^N$$

$$\ln Z = N \ln\left(\sum_r e^{-\beta\epsilon_r}\right)$$

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.4 Maxwell-Boltzmann statistics

Partition function:

$$\boxed{\ln Z = N \ln \left(\sum_r e^{-\beta \epsilon_r} \right)}$$

$$\bar{n}_s = -\frac{1}{\beta}\frac{\partial \ln Z}{\partial \epsilon_s} = -\frac{1}{\beta} N \frac{-\beta\, e^{-\beta \epsilon_s}}{\sum_r e^{-\beta \epsilon_r}}$$

$$\boxed{\bar{n}_s = N \frac{e^{-\beta \epsilon_s}}{\sum_r e^{-\beta \epsilon_r}}}$$

Maxwell-Boltzmann distribution

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.5 Photon statistics

Partition function:

$$Z = \sum_R e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + \cdots)}$$

$$Z = \sum_{n_1, n_2, \ldots} e^{-\beta n_1 \epsilon_1} e^{-\beta n_2 \epsilon_2} e^{-\beta n_3 \epsilon_3} \cdots$$

$$Z = \left(\sum_{n_1=0}^{\infty} e^{-\beta n_1 \epsilon_1} \right) \left(\sum_{n_2=0}^{\infty} e^{-\beta n_2 \epsilon_2} \right) \left(\sum_{n_3=0}^{\infty} e^{-\beta n_3 \epsilon_3} \right) \cdots$$

$$Z = \left(\frac{1}{1 - e^{-\beta\epsilon_1}} \right) \left(\frac{1}{1 - e^{-\beta\epsilon_2}} \right) \left(\frac{1}{1 - e^{-\beta\epsilon_3}} \right) \cdots$$

$$\boxed{\ln Z = - \sum_r \ln\left(1 - e^{-\beta\epsilon_r}\right)}$$

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics
9.5 Photon statistics

Partition function:

$$\ln Z = -\sum_r \ln\left(1 - e^{-\beta\epsilon_r}\right)$$

$$\bar{n}_s = -\frac{1}{\beta}\frac{\partial \ln Z}{\partial \epsilon_s} = \frac{e^{-\beta\epsilon_s}}{1 - e^{-\beta\epsilon_s}}$$

$$\bar{n}_s = \frac{1}{e^{\beta\epsilon_s} - 1}$$

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.6 Bose-Einstein statistics

Partition function:

$$Z = \sum_R e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + \cdots)}$$

$$n_r = 0, 1, 2, \ldots \qquad \sum_r n_r = N$$

Considering Z(N').

Z(N') increases rapidly with N', but we are only interested in Z at N'=N .

Multiply $e^{-\alpha N'}$ to produce a function $Z(N')e^{-\alpha N'}$ with maximum at N'=N by a proper choice of *α*.

A sum of all N' must select only terms of interest near N

$$\sum_{N'} Z(N')\, e^{-\alpha N'} = Z(N)\, e^{-\alpha N}\, \Delta^* N'$$

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics
9.6 Bose-Einstein statistics

Define Grand partition function

$$\sum_{N'} Z(N')\, e^{-\alpha N'} = Z(N)\, e^{-\alpha N} \Delta^* N'$$

$$\mathbf{Z} \equiv \sum_{N'} Z(N') e^{-\alpha N'}$$

$$\ln Z(N) = \alpha N + \ln \mathbf{Z}$$

?

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics
9.6 Bose-Einstein statistics

Grand partition function $Z = \sum_R e^{-\beta(n_1\epsilon_1+n_2\epsilon_2+\cdots)}\, e^{-\alpha(n_1+n_2+\cdots)}$

$$Z = \sum_{n_1,n_2,\ldots} e^{-(\alpha+\beta\epsilon_1)n_1-(\alpha+\beta\epsilon_2)n_2-\cdots}$$

$$= \left(\sum_{n_1=0}^{\infty} e^{-(\alpha+\beta\epsilon_1)n_1}\right)\left(\sum_{n_2=0}^{\infty} e^{-(\alpha+\beta\epsilon_2)n_2}\right)\cdots$$

$$Z = \left(\frac{1}{1-e^{-(\alpha+\beta\epsilon_1)}}\right)\left(\frac{1}{1-e^{-(\alpha+\beta\epsilon_2)}}\right)\cdots$$

$$\ln Z = -\sum_r \ln\left(1-e^{-\alpha-\beta\epsilon_r}\right)$$

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.6 Bose-Einstein statistics

$$\ln Z(N) = \alpha N + \ln Z$$

Grand partition function $\ln Z = -\sum_r \ln\left(1 - e^{-\alpha-\beta\epsilon_r}\right)$

$$\ln Z = \alpha N - \sum_r \ln\left(1 - e^{-\alpha-\beta\epsilon_r}\right)$$

Keep N’=N by a proper choice of α

$$Z(N')e^{-\alpha N'}$$

$$\frac{\partial}{\partial N'}\left[\ln Z(N') - \alpha N'\right] = \frac{\partial \ln Z(N)}{\partial N} - \alpha = 0$$

$$\left[\alpha + \left(N + \frac{\partial \ln Z}{\partial \alpha}\right)\frac{\partial \alpha}{\partial N}\right] - \alpha = 0$$

$$N + \frac{\partial \ln Z}{\partial \alpha} = \frac{\partial \ln Z}{\partial \alpha} = 0$$

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.6 Bose-Einstein statistics

$$\ln Z = \alpha N - \sum_r \ln\left(1 - e^{-\alpha-\beta\epsilon_r}\right)$$

$$\ln \mathrm{Z} = -\sum_r \ln\left(1 - e^{-\alpha-\beta\epsilon_r}\right)$$

$$N + \frac{\partial \ln \mathrm{Z}}{\partial \alpha} = \frac{\partial \ln Z}{\partial \alpha} = 0$$

$$N - \sum_r \frac{e^{-\alpha-\beta\epsilon_r}}{1 - e^{-\alpha-\beta\epsilon_r}} = 0$$

$$\sum_r \frac{1}{e^{\alpha+\beta\epsilon_r} - 1} = N$$

0

$$\bar{n}_s = -\frac{1}{\beta}\frac{\partial \ln Z}{\partial \epsilon_s} = -\frac{1}{\beta}\left[-\frac{\beta e^{-\alpha-\beta\epsilon_s}}{1 - e^{-\alpha-\beta\epsilon_s}} + \frac{\partial \ln Z}{\partial \alpha}\frac{\partial \alpha}{\partial \epsilon_s}\right]$$

$$\boxed{\bar{n}_s = \frac{1}{e^{\alpha+\beta\epsilon_s} - 1}}$$

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.6 Bose-Einstein statistics

$$\boxed{\bar{n}_s = \frac{1}{e^{\alpha+\beta\epsilon_s} - 1}}$$

$$\boxed{\frac{\partial \ln Z(N)}{\partial N} - \alpha = 0}$$

$$\Rightarrow \quad \sum_r \bar{n}_r = N$$

$$\mu = \frac{\partial F}{\partial N} = -kT\frac{\partial \ln Z}{\partial N} = -kT\alpha$$

$$\boxed{\alpha = -\beta\mu}$$

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.7 Fermi-Dirac statistics

$$n_r = 0 \text{ and } 1 \qquad \text{for each } r$$

Similar to the treatment in BE statistics

$$Z = \sum_{n_1, n_2, n_3} e^{-\beta(n_1\varepsilon_1 + n_2\varepsilon_2 + \cdots) - \alpha(n_1 + n_2 + \cdots)}$$

$$= \left(\sum_{n_1=0}^{1} e^{-(\alpha+\beta\varepsilon_1)n_1} \right) \left(\sum_{n_2=0}^{1} e^{-(\alpha+\beta\varepsilon_2)n_2} \right) \cdots$$

$$Z = (1 + e^{-\alpha-\beta\varepsilon_1})(1 + e^{-\alpha-\beta\varepsilon_2}) \cdots$$

$$\ln Z = \sum_r \ln(1 + e^{-\alpha-\beta\varepsilon_r})$$

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.7 Fermi-Dirac statistics

$$\ln Z = \sum_r \ln\left(1 + e^{-\alpha-\beta\epsilon_r}\right)$$

$$\boxed{\ln Z = \alpha N + \sum_r \ln\left(1 + e^{-\alpha-\beta\epsilon_r}\right)}$$

α is also determined by the condition

$$\frac{\partial \ln Z}{\partial \alpha} = N - \sum_r \frac{e^{-\alpha-\beta\epsilon_r}}{1 + e^{-\alpha-\beta\epsilon_r}} = 0$$

$$\boxed{\sum_r \frac{1}{e^{\alpha+\beta\epsilon_r} + 1} = N}$$

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.7 Fermi-Dirac statistics

$$\bar{n}_s = -\frac{1}{\beta}\frac{\partial \ln Z}{\partial \epsilon_s} = \frac{1}{\beta}\frac{\beta e^{-\alpha-\beta\epsilon_s}}{1+e^{-\alpha-\beta\epsilon_s}}$$

$$\Rightarrow \quad \boxed{\bar{n}_s = \frac{1}{e^{\alpha+\beta\epsilon_s}+1}}$$

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics
9.8 Quantum statistics in the classic limit

BE and FD distributions: $\bar{n}_r = \dfrac{1}{e^{\alpha+\beta\epsilon_r} \pm 1}$

Total particles: $\sum_r \bar{n}_r = \sum_r \dfrac{1}{e^{\alpha+\beta\epsilon_r} \pm 1} = N$

Partition function: $\ln Z = \alpha N \pm \sum_r \ln\left(1 \pm e^{-\alpha-\beta\epsilon_r}\right)$

Limiting cases:

very low concentration $\bar{n}_r \ll 1$ $\quad \exp(\alpha + \beta\epsilon_r) \gg 1$

Very high T $\beta \longrightarrow 0$ $\quad \beta\epsilon_r \ll \alpha$

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.8 Quantum statistics in the classic limit

Limiting cases:

very low concentration $\bar{n}_r \ll 1$ $\exp(\alpha + \beta\epsilon_r) \gg 1$

Very high T $\beta \longrightarrow 0$ $\beta\epsilon_r \ll \alpha$

$$\boxed{\sum_r \bar{n}_r = \sum_r \frac{1}{e^{\alpha+\beta\epsilon_r} \pm 1} = N}$$

Number of terms contribute substantially to summation increases
Requires α must be large enough
To keep sum ==N

$\Rightarrow$ $\exp(\alpha + \beta\epsilon_r) \gg 1$

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.8 Quantum statistics in the classic limit

Limiting cases:

very low concentration、very high T

$$e^{\alpha+\beta\epsilon_r} \gg 1 \Rightarrow \bar{n}_r \ll 1$$

$$\bar{n}_r = \frac{1}{e^{\alpha+\beta\epsilon_r} \pm 1} \Rightarrow \bar{n}_r = e^{-\alpha-\beta\epsilon_r}$$

$$\sum_r e^{-\alpha-\beta\epsilon_r} = e^{-\alpha} \sum_r e^{-\beta\epsilon_r} = N$$

$$e^{-\alpha} = N \left(\sum_r e^{-\beta\epsilon_r}\right)^{-1}$$

Limiting cases:

low concentration

high T --→MB dis.

$$\bar{n}_r = N \frac{e^{-\beta\epsilon_r}}{\sum_r e^{-\beta\epsilon_r}}$$

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

$$\ln Z = \alpha N \pm \sum_r \ln\left(1 \pm e^{-\alpha-\beta\epsilon_r}\right)$$

9.8 Quantum statistics in the classic limit

Partition function:

$$\ln Z = \alpha N \pm \sum_r \left(\pm e^{-\alpha-\beta\epsilon_r}\right) = \alpha N + N$$

$$\alpha = -\ln N + \ln\left(\sum_r e^{-\beta\epsilon_r}\right)$$

$$\ln Z = -N \ln N + N + N \ln\left(\sum_r e^{-\beta\epsilon_r}\right)$$

While MB gives:

$$\ln Z = N \ln\left(\sum_r e^{-\beta\epsilon_r}\right)$$

???????????????

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.8 Quantum statistics in the classic limit

Partition function:

$$\ln Z = \ln Z_{MB} - (N \ln N - N)$$

$$\ln Z = \ln Z_{MB} - \ln N!$$

$$Z = \frac{Z_{MB}}{N!}$$

<<< distinguishable

Class-work

P 398   9.1

Homework

P 398   9.2-9.3

Homework

configuration			no. of states		
0	ϵ	3ϵ	MB	BE	FD
xx			1	1	--
	xx		1	1	--
		xx	1	1	--
x	x		2	1	1
x		x	2	1	1
	x	x	2	1	1

(a) $Z_{MB} = 1 + e^{-2\epsilon\beta} + e^{-6\epsilon\beta} + 2e^{-\epsilon\beta} + 2e^{-3\epsilon\beta} + 2e^{-4\epsilon\beta}$

(b) $Z_{BE} = 1 + e^{-2\epsilon\beta} + e^{-6\epsilon\beta} + e^{-\epsilon\beta} + e^{-3\epsilon\beta} + e^{-4\epsilon\beta}$

(c) $Z_{FD} = e^{-\epsilon\beta} + e^{-3\epsilon\beta} + e^{-4\epsilon\beta}$