哈密顿正则方程：简单、但求解不简单

运动积分的寻找？

适当变换

哈密的函数出现循环坐标
（循环坐标尽量多）

哈密顿正则方程求解简化！

第六章 正则变换(Canonical Transformation)

一. 正则变换的定义

哈密顿函数

系统有s个自由度

变换前
$$\begin{cases} H(q\ \ p\ \ t) \\ (q_1\ q_2\ q_3\ \ldots\ldots q_s) \\ (p_1\ p_2\ p_3\ \ldots\ldots p_s) \end{cases}$$

$$\begin{cases} \dot{q}_\alpha = \dfrac{\partial H}{\partial p_\alpha} \\ \dot{p}_\alpha = -\dfrac{\partial H}{\partial q_\alpha} \end{cases} \quad \ldots\ldots(1)$$

变换后
$$\begin{cases} K(Q\ \ P\ \ t) \\ (Q_1\ Q_2\ Q_3\ \ldots\ldots Q_s) \\ (P_1\ P_2\ P_3\ \ldots\ldots P_s) \end{cases}$$

$$\begin{cases} \dot{Q}_\alpha = \dfrac{\partial K}{\partial P_\alpha} \\ \dot{P}_\alpha = -\dfrac{\partial K}{\partial Q_\alpha} \end{cases} \quad \ldots\ldots(2)$$

变换关系

$$\begin{cases} Q_\alpha = Q_\alpha(q_1 \ldots\ldots q_s \; p_1 \ldots\ldots p_s \; t) \\ P_\alpha = P_\alpha(q_1 \ldots\ldots q_s \; p_1 \ldots\ldots p_s \; t) \end{cases} \quad \ldots\ldots(3)$$

$$\begin{cases} \dot{Q}_\alpha = \dfrac{\partial K}{\partial P_\alpha} \\ \dot{P}_\alpha = -\dfrac{\partial K}{\partial Q_\alpha} \end{cases}$$

$$q_{,} \; p_{,} \; H \rightarrow Q_{,} \; P_{,} \; K$$

满足正则方程具有协变性的变换 ⇒ 正则变换

二. 正则变换判别

相空间

正则方程　**等 价**　哈密顿原理

$$\delta\int_{t_1}^{t_2}\left(-H+\dot{q}_\alpha p_\alpha+\frac{df_1}{dt}\right)dt=0 \qquad f_1=f_1(q\ t)$$

$$\delta\int_{t_1}^{t_2}\left(-K+\dot{Q}_\alpha P_\alpha+\frac{df_2}{dt}\right)dt=0 \qquad f_1=f_2(Q\ \ t)$$

$$-H+\dot{q}_\alpha p_\alpha=-K+P_\alpha\dot{Q}_\alpha+\frac{dF}{dt} \qquad \ldots\ldots(4)$$

$$(\alpha=1\ 2\ 3\ldots\ldots s) \qquad F=f_2-f_1=F(q\ Q\ \ p\ \ P)$$

$$\delta\int_{t_1}^{t_2}\left(-H+\dot{q}_\alpha p_\alpha+\frac{df_1}{dt}\right)dt=0$$

$$\delta\int_{t_1}^{t_2}\frac{df_1}{dt}dt=?$$

$$=\int_{t_1}^{t_2}\delta\frac{df_1}{dt}dt \quad =\int_{t_1}^{t_2}\frac{d(\delta f_1)}{dt}dt$$

$$=\delta f_1\Big|_{t_1}^{t_2} \quad =\frac{\partial f_1}{\partial q_\alpha}\delta q_\alpha\Bigg|_{t_1}^{t_2}$$

$$=0$$

$$-H+\dot{q}_{\alpha}p_{\alpha}=-K+P_{\alpha}\dot{Q}_{\alpha}+\frac{dF}{dt} \qquad ......(4)$$

$$(K-H)dt+(p_{\alpha}dq_{\alpha}-P_{\alpha}dQ_{\alpha})=dF \qquad ......(4)$$

Attention:

- F的函数形式决定 (q　p) $\Rightarrow$ (Q　P)

生成函数

- 如果 $K(Q\ P\ t)=-L(Q\ \dot{Q}\ t)+\dot{Q}_{\alpha}P_{\alpha}$

  正则方程不一定具有协变性，为了保证正则方程的形式不变，新的哈密顿函数必须由（4）式确定

$$(K-H)dt+(p_\alpha dq_\alpha - P_\alpha dQ_\alpha)=dF \quad \ldots\ldots(4)$$

- （4）式中有4S+1个变量，但独立变量只有2S+1个

$$\begin{cases} Q_\alpha = Q_\alpha(q_\alpha, p_\alpha, t) \\ P_\alpha = P_\alpha(q_\alpha, p_\alpha, t) \end{cases}$$

- 满足（4）式时正则方程的形式具有协变性，

如果（4）式右边能写成某一函数的全微分，

则相应的变换

$(q,p,t) \Longrightarrow (Q,P,t) \Longrightarrow$ 正则变换

三. 四类生成函数

第一类: $F_1 = F_1(Q\ q\ t)$

$$dF_1 = \frac{\partial F_1}{\partial q_\alpha} dq_\alpha + \frac{\partial F_1}{\partial Q_\alpha} dQ_\alpha + \frac{\partial F_1}{\partial t} dt \qquad \ldots\ldots(5)$$

$$(\alpha = 1\ 2\ 3 \ldots\ldots s)$$

$$\boxed{(K - H)dt} + (p_\alpha dq_\alpha - P_\alpha dQ_\alpha) = dF \qquad ......(4)$$

$$dF_1 = \frac{\partial F_1}{\partial q_\alpha} dq_\alpha + \frac{\partial F_1}{\partial Q_\alpha} dQ_\alpha + \boxed{\frac{\partial F_1}{\partial t} dt} \qquad ......(5)$$

比较(4)和(5)

$$\begin{cases} p_\alpha = \dfrac{\partial F_1}{\partial q_\alpha} = p_\alpha(q_\alpha \ Q_\alpha \ t) \\ P_\alpha = -\dfrac{\partial F_1}{\partial Q_\alpha} = P_\alpha(q_\alpha \ Q_\alpha \ t) \\ K = H + \dfrac{\partial F_1}{\partial t} \end{cases}$$

$$\Longrightarrow Q_\alpha(q_\alpha \ p_\alpha \ t)$$

$$\Downarrow$$

$$P_\alpha(q_\alpha \ Q_\alpha(q_\alpha \ p_\alpha) \ t)$$

第二类: $F_2 = F_2(P\ q\ t)$

$$(K-H)dt+(p_\alpha dq_\alpha - P_\alpha dQ_\alpha) = dF_1 \quad \ldots\ldots(4)$$

$$-P_\alpha dQ_\alpha = -d(P_\alpha Q_\alpha) + Q_\alpha dP_\alpha$$

$$(K-H)dt + p_\alpha dq_\alpha + Q_\alpha dP_\alpha = dF_2 \qquad (\alpha = 1\ 2\ 3 \ldots\ldots s)$$

$$\begin{cases} p_\alpha = \dfrac{\partial F_2}{\partial q_\alpha} \\ Q_\alpha = \dfrac{\partial F_2}{\partial P_\alpha} \\ K = H + \dfrac{\partial F_2}{\partial t} \end{cases}$$

$$F_2 = F_1 + P_\alpha Q_\alpha$$

$$F_2(\mathrm{q}, \mathrm{P}, \mathrm{t}) = F_1 + P_\alpha Q_\alpha$$

$$(K - H)dt + p_\alpha dq_\alpha + Q_\alpha dP_\alpha = dF_2$$

$$\begin{cases} p_\alpha = \dfrac{\partial F_2}{\partial q_\alpha} \\[2ex] Q_\alpha = \dfrac{\partial F_2}{\partial P_\alpha} \\[2ex] K = H + \dfrac{\partial F_2}{\partial t} \end{cases}$$

第三类: $F_3 = F_3(p\ Q\ t)$

$$(K - H)dt + (p_\alpha dq_\alpha - P_\alpha dQ_\alpha) = dF_1 \quad \text{......}(4)$$

$$p_\alpha dq_\alpha = d(p_\alpha q_\alpha) - q_\alpha dp_\alpha$$

$$-q_\alpha dp_\alpha - P_\alpha dQ_\alpha + (K - H)dt = dF_3 \quad (\alpha = 1\ 2\ 3\text{......}s)$$

$$\begin{cases} q_\alpha = -\dfrac{\partial F_3}{\partial p_\alpha} \\ P_\alpha = -\dfrac{\partial F_3}{\partial Q_\alpha} \\ K = H + \dfrac{\partial F_3}{\partial t} \end{cases}$$

$$F_3(p\ Q\ t) = F_1 - q_\alpha p_\alpha$$

第四类: $F_4 = F_4(p\ P\ t)$

$$\begin{cases} p_\alpha dq_\alpha = d(p_\alpha q_\alpha) - q_\alpha dp_\alpha \\ -P_\alpha dQ_\alpha = -d(P_\alpha Q_\alpha) + Q_\alpha dP_\alpha \end{cases}$$

$$(K - H)dt + (p_\alpha dq_\alpha - P_\alpha dQ_\alpha) = dF \quad ......(4)$$

$$(\alpha = 1\ 2\ 3......s)$$

$$Q_\alpha dP_\alpha - q_\alpha dp_\alpha + (K - H)dt = dF_4$$

$$\begin{cases} q_\alpha = -\dfrac{\partial F_4}{\partial p_\alpha} \\ Q_\alpha = \dfrac{\partial F_4}{\partial P_\alpha} \\ K = H + \dfrac{\partial F_4}{\partial t} \end{cases}$$

$$F_4(p\ P\ t) = F_1 + P_\alpha Q_\alpha - p_\alpha q_\alpha$$

- 如果生成函数F不显含时间t $\Rightarrow \dfrac{\partial F}{\partial t}=0 \Rightarrow K=H$

变换前后哈密顿函数相等

$$(K-H)dt+(p_\alpha dq_\alpha - P_\alpha dQ_\alpha)=dF \quad \ldots\ldots(4)$$

与时间无关的正则变换满足

H=K

$$p_\alpha dq_\alpha - P_\alpha dQ_\alpha = dU$$

$U \Rightarrow$ 生成函数

- 变换不显含时间，$\dfrac{\partial F}{\partial t}=0$，$K=H$
- 四类生成函数之间实质是勒让特变换

四类正则变换Pfaff方程

$$p_\alpha dq_\alpha - P_\alpha dQ_\alpha + (K - H)dt = dF_1(q, Q, t) \cdots\cdots\cdots (1)$$

$$p_\alpha dq_\alpha + Q_\alpha dP_\alpha + (K - H)dt = dF_2(q, P, t) \cdots\cdots\cdots (2)$$

$$-q_\alpha dp_\alpha - P_\alpha dQ_\alpha + (K - H)dt = dF_3(p, Q, t) \cdots\cdots\cdots (3)$$

$$-q_\alpha dp_\alpha + Q_\alpha dP_\alpha + (K - H)dt = dF_4(p, P, t) \cdots\cdots (4)$$

四类正则变换

$$F_1 = F_1(q, Q, t)$$

$$\begin{cases} p_\alpha = \dfrac{\partial F_1}{\partial q_\alpha} \\ P_\alpha = -\dfrac{\partial F_1}{\partial Q_\alpha} \\ K = H + \dfrac{\partial F_1}{\partial t} \end{cases}$$

$$F_2 = F_2(q, P, t) = F_1 + Q_\alpha P_\alpha$$

$$\begin{cases} p_\alpha = \dfrac{\partial F_2}{\partial q_\alpha} \\ Q_\alpha = \dfrac{\partial F_2}{\partial P_\alpha} \\ K = H + \dfrac{\partial F_2}{\partial t} \end{cases}$$

$$F_3 = F_3(p, Q, t) = F_1 - q_\alpha p_\alpha$$

$$\begin{cases} q_\alpha = -\dfrac{\partial F_3}{\partial p_\alpha} \\ P_\alpha = -\dfrac{\partial F_3}{\partial Q_\alpha} \\ K = H + \dfrac{\partial F_3}{\partial t} \end{cases}$$

$$F_4 = F_4(p, P, t) = F_1 + Q_\alpha P_\alpha - q_\alpha p_\alpha$$

$$\begin{cases} q_\alpha = -\dfrac{\partial F_4}{\partial p_\alpha}, \\ Q_\alpha = \dfrac{\partial F_4}{\partial P_\alpha} \\ K = H + \dfrac{\partial F_4}{\partial t} \end{cases}$$

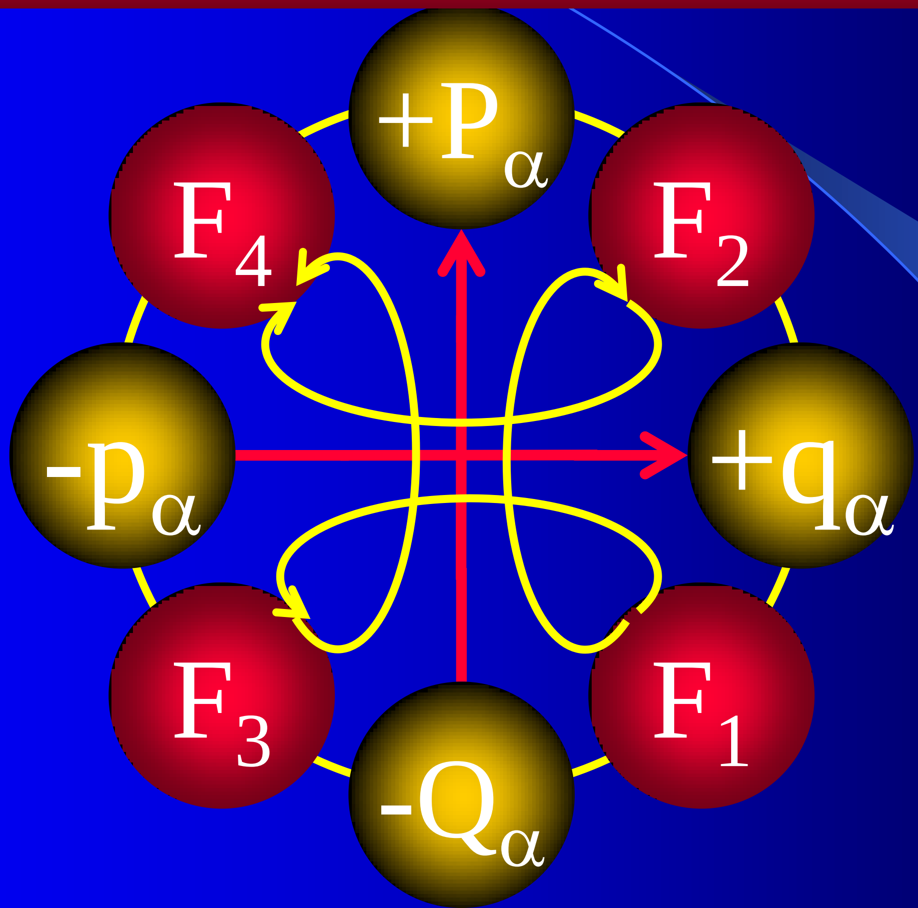
$$p_\alpha dq_\alpha + Q_\alpha dP_\alpha + (K - H)dt = dF_2(q, P, t)$$

$$-q_\alpha dp_\alpha + Q_\alpha dP_\alpha + (K - H)dt = dF_4(p, P, t)$$

$F_4 = F_4(p \ldots$ $\qquad = F_1 + Q_\alpha P_\alpha$

$$\left.\begin{aligned} q_\alpha &= -\frac{\partial F_4}{\partial p_\alpha} \\ Q_\alpha &= \frac{\partial F_4}{\partial P_\alpha} \\ K &= H + \frac{\partial F_4}{\partial t} \end{aligned}\right\}$$

$$\left\{\begin{aligned} \ldots &= \frac{\partial F_2}{\partial q_\alpha} \\ Q_\alpha &= \frac{\partial F_2}{\partial P_\alpha} \\ K &= H + \frac{\partial F_2}{\partial t} \end{aligned}\right.$$



$$\left.\begin{aligned} q_\alpha &= -\frac{\partial F_3}{\partial p_\alpha} \\ P_\alpha &= -\frac{\partial F_3}{\partial Q_\alpha} \\ K &= H + \frac{\partial F_3}{\partial t} \end{aligned}\right\}$$

$$\left\{\begin{aligned} p_\alpha &= \frac{\partial F_1}{\partial q_\alpha} \\ P_\alpha &= -\frac{\partial F_1}{\partial Q_\alpha} \\ K &= H + \frac{\partial F_1}{\partial t} \end{aligned}\right.$$

$$p_\alpha dq_\alpha - P_\alpha dQ_\alpha + (K - H)dt = dF_1(q, Q, t)$$

$$-q_\alpha dp_\alpha - P_\alpha dQ_\alpha + (K - H)dt = dF_3(p, Q, t)$$

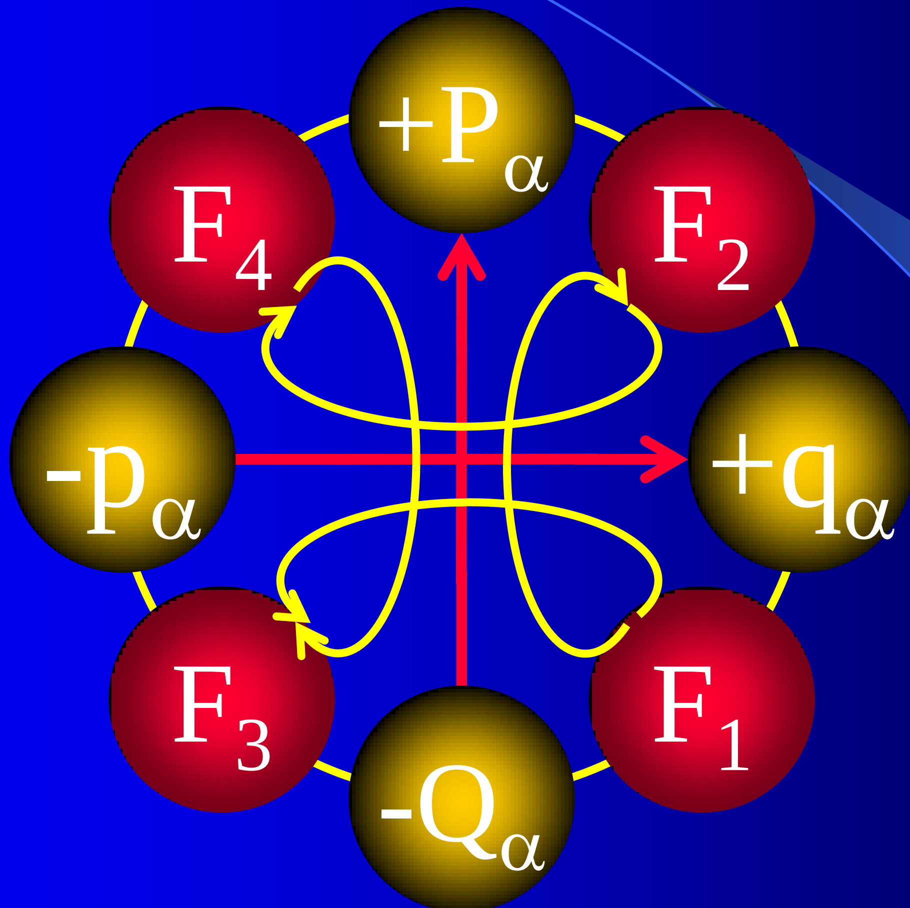
$F_3 = F_3(p \ldots$ $\qquad = F_1(q, Q, t)$

$$F_4 = F_4(p, P, t) = F_1 + Q_\alpha P_\alpha - q_\alpha p_\alpha$$

$$\left. \begin{aligned} q_\alpha &= -\frac{\partial F_4}{\partial p_\alpha}, \\ Q_\alpha &= \frac{\partial F_4}{\partial P_\alpha} \\ K &= H + \frac{\partial F_4}{\partial t} \end{aligned} \right\}$$

$$F_2 = F_2(q, P, t) = F_1 + Q_\alpha P_\alpha$$

$$\left\{ \begin{aligned} p_\alpha &= \frac{\partial F_2}{\partial q_\alpha} \\ Q_\alpha &= \frac{\partial F_2}{\partial P_\alpha} \\ K &= H + \frac{\partial F_2}{\partial t} \end{aligned} \right.$$



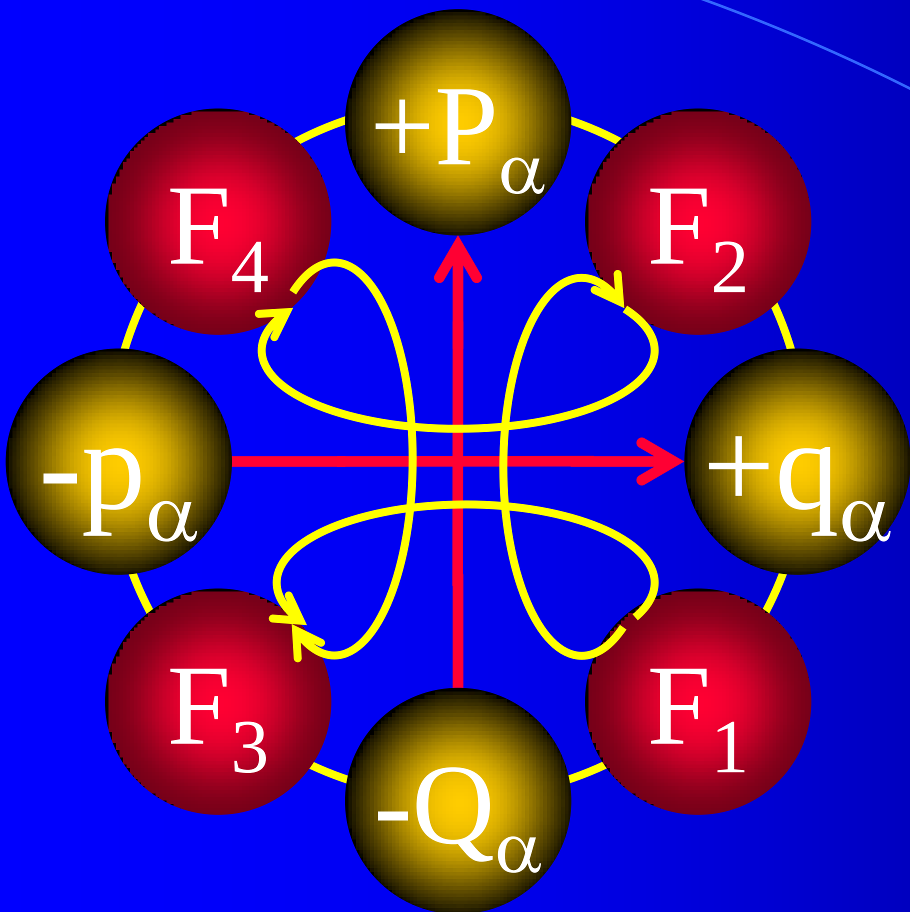
$$\left. \begin{aligned} q_\alpha &= -\frac{\partial F_3}{\partial p_\alpha} \\ P_\alpha &= -\frac{\partial F_3}{\partial Q_\alpha} \\ K &= H + \frac{\partial F_3}{\partial t} \end{aligned} \right\}$$

$$F_3 = F_3(p, Q, t) = F_1 - q_\alpha p_\alpha$$

$$\left\{ \begin{aligned} p_\alpha &= \frac{\partial F_1}{\partial q_\alpha} \\ P_\alpha &= -\frac{\partial F_1}{\partial Q_\alpha} \\ K &= H + \frac{\partial F_1}{\partial t} \end{aligned} \right.$$

$$F_1 = F_1(q, Q, t)$$

$$p_\alpha dq_\alpha - P_\alpha dQ_\alpha + (K-H)dt = dF_1(q,Q,t)$$



$$\begin{cases} p_\alpha = \dfrac{\partial F_1}{\partial q_\alpha} \\ P_\alpha = -\dfrac{\partial F_1}{\partial Q_\alpha} \\ K = H + \dfrac{\partial F_1}{\partial t} \end{cases}$$

两次偏导可以交换，即：

$$\frac{\partial^2 F_1}{\partial q_i \partial q_k} = \frac{\partial^2 F_1}{\partial q_k \partial q_i}, \qquad \frac{\partial p_i}{\partial q_k} = \frac{\partial p_k}{\partial q_i},$$

$$\frac{\partial^2 F_1}{\partial Q_i \partial Q_k} = \frac{\partial^2 F_1}{\partial Q_k \partial Q_i}, \qquad \frac{\partial(-P_i)}{\partial Q_k} = \frac{\partial(-P_k)}{\partial Q_i},$$

$$\frac{\partial^2 F_1}{\partial q_i \partial Q_k} = \frac{\partial^2 F_1}{\partial Q_k \partial q_i} \qquad \frac{\partial p_i}{\partial Q_k} = \frac{\partial(-P_k)}{\partial q_i}$$

矢量（$p_1,\dots \quad p_s, -P_1,\dots \quad -P_s$）在2s维空间（$q_1,\dots \quad q_s, Q_1,\dots \quad Q_s$）是无旋的?

以上也是$p_i dq_i - P_i dQ_i$可积分条件！

正则变换从等时变分角度可以表示为：$p_i \delta q_i - P_i \delta Q_i = \delta F_1$

四. 几种特殊的正则变换

- 点变换
- 交替变换
- 恒等变换
- 相空间中的平移变换
- 无穷小正则变换

❖点变换

$\Rightarrow$ 位形空间中的坐标变换

$$Q_\alpha = Q_\alpha(q_\alpha \; t)$$

$$\boxed{F_2 = F_1 + P_\alpha Q_\alpha} \quad F_1 = 0 \Rightarrow F_2 = Q_\alpha(q_\alpha t)P_\alpha$$

$$Q_\alpha = \frac{\partial F_2}{\partial P_\alpha} = Q_\alpha(q \; t)$$

例:有心力场

$$p_\beta = \frac{\partial F_2}{\partial q_\beta} = P_\alpha \frac{\partial Q_\alpha}{\partial q_\beta}, (\beta\text{不求和})$$

广义坐标 $\begin{cases} (r \; \theta) \\ (x \; y) \end{cases}$

$\Downarrow$

广义动量改变

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

❖交替变换

$$p_\alpha dq_\alpha - P_\alpha dQ_\alpha = dF_1 \quad ......(4)$$ 令 $F_1 = q_\alpha Q_\alpha$

$$dF_1 = q_\alpha dQ_\alpha + Q_\alpha dq_\alpha$$

$$\begin{cases} Q_\alpha = \dfrac{\partial F_1}{\partial q_\alpha} = p_\alpha \\ P_\alpha = -\dfrac{\partial F_1}{\partial Q_\alpha} = -q_\alpha \end{cases}$$

$$K = H$$

❖恒等变换

$$F_2 = q_\alpha P_\alpha \qquad dF_2 = q_\alpha dP_\alpha + P_\alpha dq_\alpha$$

$$(K - H)dt + p_\alpha dq_\alpha + Q_\alpha dP_\alpha = dF_2$$

$$\begin{cases} p_\alpha = \dfrac{\partial F_2}{\partial q_\alpha} = P_\alpha \\[2ex] Q_\alpha = \dfrac{\partial F_2}{\partial P_\alpha} = q_\alpha \\[2ex] K = H \end{cases}$$

❖相空间中的平移变换

$$Q_\alpha = q_\alpha + a_\alpha \qquad a_\alpha = a_\alpha(t)\text{亦可为常数}$$

$$P_\alpha = p_\alpha + b_\alpha \qquad b_\alpha = b_\alpha(t)\text{亦可为常数}$$

$$\boxed{F_2 = q_\alpha P_\alpha + a_\alpha P_\alpha - b_\alpha q_\alpha}$$

$$\because \quad (K - H)dt + p_\alpha dq_\alpha + Q_\alpha dP_\alpha = dF_2$$

$$dF_2 = (P_\alpha - b_\alpha)dq_\alpha + (q_\alpha + a_\alpha)dP_\alpha$$

$$\begin{cases} Q_\alpha = \dfrac{\partial F_2}{\partial P_\alpha} = q_\alpha + a_\alpha \\ p_\alpha = \dfrac{\partial F_2}{\partial q_\alpha} = P_\alpha - b_\alpha \end{cases}$$

❖无穷小正则变换

$$Q_\alpha = q_\alpha + dq_\alpha$$

$$P_\alpha = p_\alpha + dp_\alpha$$

$$F_2 = q_\alpha P_\alpha + \varepsilon G(q_\alpha P_\alpha \ t)$$

ε与$(q\ P)$无关的小参数

$G(q_\alpha P_\alpha \ t)$为无穷小正则变换生成函数

$$dF_2 = (q_\alpha + \varepsilon \frac{\partial G}{\partial P_\alpha})dP_\alpha + (P_\alpha + \varepsilon \frac{\partial G}{\partial q_\alpha})dq_\alpha + \varepsilon \frac{\partial G}{\partial t} dt$$

$$= p_\alpha dq_\alpha + Q_\alpha dP_\alpha + (K - H)dt$$

$$\because \quad (K - H)dt + p_\alpha dq_\alpha + Q_\alpha dP_\alpha = dF_2$$

$$dF_2 = (q_\alpha + \varepsilon \frac{\partial G}{\partial P_\alpha}) dP_\alpha + (P_\alpha + \varepsilon \frac{\partial G}{\partial q_\alpha}) dq_\alpha + \varepsilon \frac{\partial G}{\partial t} dt$$

$$F_2 = q_\alpha P_\alpha + \varepsilon G(q_\alpha P_\alpha \; t)$$

$$p_\alpha = \frac{\partial F_2}{\partial q_\alpha} = P_\alpha + \varepsilon \frac{\partial G}{\partial q_\alpha} \Rightarrow dp_\alpha = P_\alpha - p_\alpha = -\varepsilon \frac{\partial G}{\partial q_\alpha}$$

$$Q_\alpha = \frac{\partial F_2}{\partial P_\alpha} = q_\alpha + \varepsilon \frac{\partial G}{\partial P_\alpha} \Rightarrow dq_\alpha = Q_\alpha - q_\alpha = \varepsilon \frac{\partial G}{\partial P_\alpha}$$

$$K = H + \varepsilon \frac{\partial G}{\partial t} \quad \begin{cases} dp_\alpha = -\varepsilon \dfrac{\partial G(q_\alpha, P_\alpha, t)}{\partial q_\alpha} \approx -\varepsilon \dfrac{\partial G(q_\alpha, p_\alpha, t)}{\partial q_\alpha} \\ dq_\alpha = \varepsilon \dfrac{\partial G(q_\alpha, P_\alpha, t)}{\partial P_\alpha} \approx \varepsilon \dfrac{G(q_\alpha, p_\alpha, t)}{\partial p_\alpha} \end{cases}$$

$$F_2 = q_\alpha P_\alpha + \varepsilon G(q_\alpha P_\alpha \ t)$$

$G(q_\alpha P_\alpha \ t)$为无穷小正则变换生成函数

$$\begin{cases} dp_\alpha = -\varepsilon \dfrac{\partial G}{\partial q_\alpha} \\ dq_\alpha \approx \varepsilon \dfrac{\partial G}{\partial p_\alpha} \end{cases}$$

$+$

$$\begin{cases} \text{若取 } \varepsilon = dt \\ \text{若取 } G = H \end{cases}$$

$$\begin{cases} \dot{p}_\alpha = -\dfrac{\partial H}{\partial q_\alpha} \\ \dot{q}_\alpha = \dfrac{\partial H}{\partial p_\alpha} \end{cases}$$

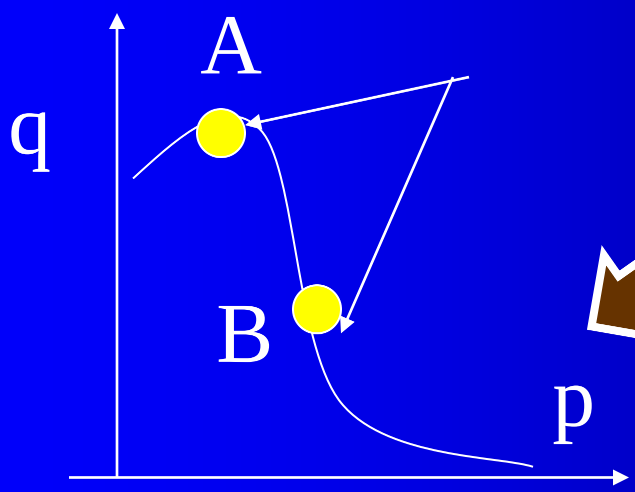
Conclusion:

H在每时每刻生成无穷小正则变换

连续的无穷小正则变换序列

描述了力学系统随时间的演化

两个正则变换的积仍是正则变换



某一正则变换相联系

无穷小正则变换和泊松括号有密切联系

如果f=f(q p),(不显含t)

$$q_\alpha \quad \Rightarrow \quad q_\alpha + dq_\alpha$$

$$p_\alpha \quad \Rightarrow \quad p_\alpha + dp_\alpha$$

相应地

$$df = \frac{\partial f}{\partial q_\alpha} dq_\alpha + \frac{\partial f}{\partial p_\alpha} dp_\alpha \qquad (\alpha = 1\,2\,3\ldots\ldots s)$$

$$dq_\alpha = \varepsilon \frac{\partial G}{\partial p_\alpha} \qquad dp_\alpha = -\varepsilon \frac{\partial G}{\partial q_\alpha}$$

$$df = \varepsilon\left[\frac{\partial f}{\partial q_\alpha}\frac{\partial G}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial G}{\partial q_\alpha}\right] \Rightarrow df = \varepsilon[f \quad G]$$

$$\delta f = \varepsilon[f \quad G]$$

如果取f = H $\Rightarrow$ $$\delta H = \varepsilon[H \quad G]$$

设G为运动积分且不显含时间 $\Rightarrow$ $$[H \quad G] = -[G \quad H] = 0$$

$\Downarrow$

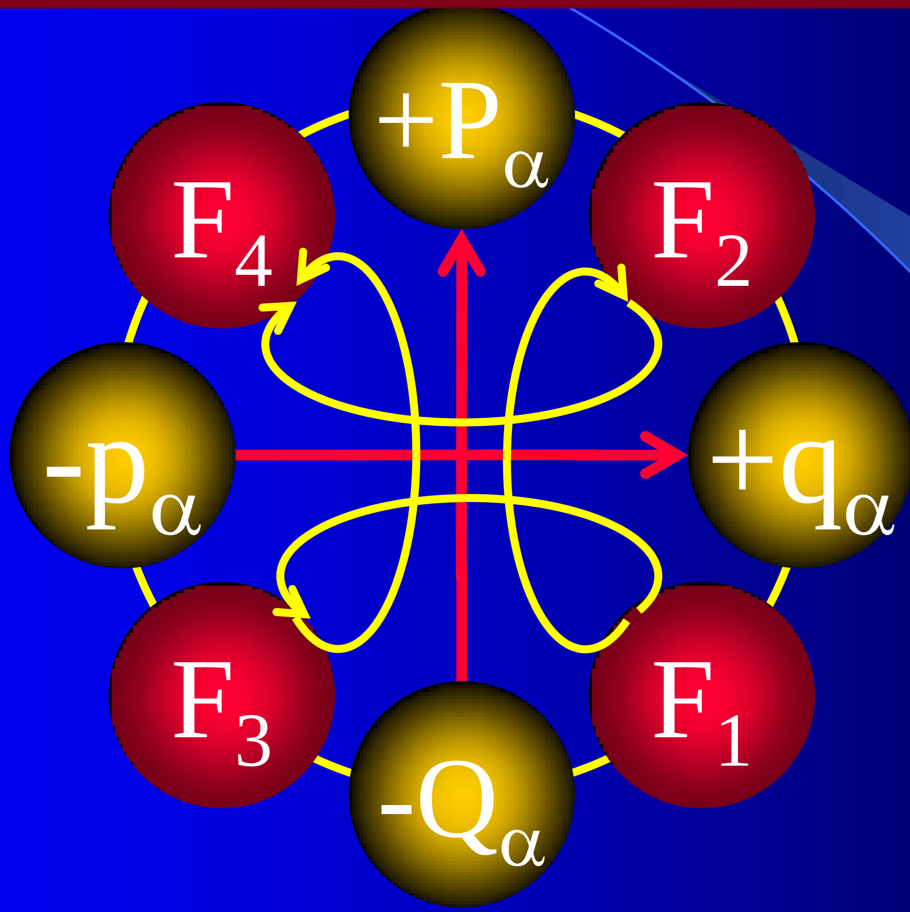
$$\delta H = 0$$

这表明:

以运动积分作为无穷小正则变换的生成函数时, 这样的正则变换不改变系统的哈密顿函数

$$p_\alpha dq_\alpha + Q_\alpha dP_\alpha + (K-H)dt = dF_2(q,P,t)$$

$$-q_\alpha dp_\alpha + Q_\alpha dP_\alpha + (K-H)dt = dF_4(p,P,t)$$



平衡态热力学中的正则变量：(T，S）(P，V）

F_1：U
F_2：H
F_3：F
F_4：G

参考沈惠川《力学季刊》第24卷，2003 p462，

$$p_\alpha dq_\alpha - P_\alpha dQ_\alpha + (K-H)dt = dF_1(q,Q,t)$$

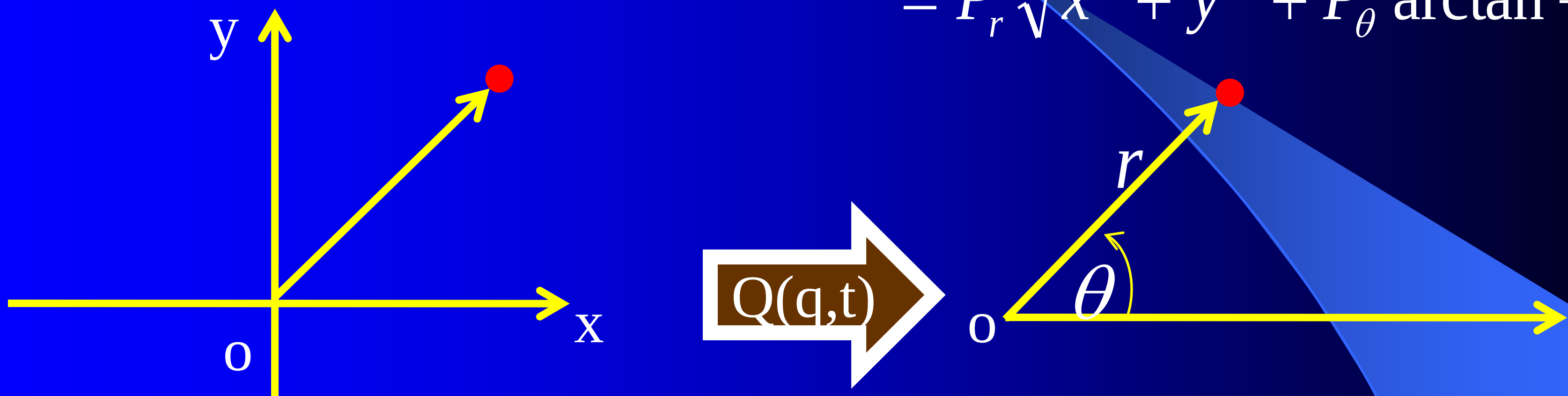
$$-q_\alpha dp_\alpha - P_\alpha dQ_\alpha + (K-H)dt = dF_3(p,Q,t)$$

正则变换让求解哈密顿正则方程简化实例

例：有心力场V中质点m。

$$F_2 = P_\alpha Q_\alpha(q) = P_r r + P_\theta \theta$$

$$= P_r\sqrt{x^2+y^2} + P_\theta \arctan\frac{y}{x}$$



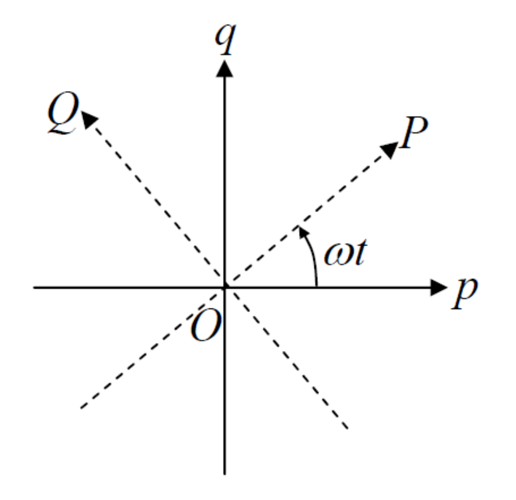
$$q_1 = x = r\cos\theta, \quad q_2 = y = r\sin\theta,$$

$$q_1 = x, \quad q_2 = y, \qquad \Rightarrow \qquad Q_1 = r, \quad Q_2 = \theta,$$

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + V(x,y) \qquad K = \frac{1}{2m}\left(P_r^2 + \frac{P_\theta^2}{r^2}\right) + V(r)$$

$$\dot{P}_\theta = -\frac{\partial K}{\partial \theta} = 0, \qquad \boxed{P_\theta = mr^2\dot{\theta} = J_0}$$

例:相空间中的等角速度旋转。



该变换的生成函数:

$$F_1 = \frac{q^2\cos\omega t - 2qQ + Q^2\cos\omega t}{2\sin\omega t}$$

$$p = \frac{\partial F_1}{\partial q} = \frac{q\cos\omega t - Q}{\sin\omega t}$$

$$P = -\frac{\partial F_1}{\partial Q} = -\frac{-q + Q\cos\omega t}{\sin\omega t}$$

$$\Rightarrow \quad Q = -p\sin\omega t + q\cos\omega t$$

$$\Rightarrow \quad q = P\sin\omega t + Q\cos\omega t$$

整理得到:

$$P = p\cos\omega t + q\sin\omega t$$
$$Q = -p\sin\omega t + q\cos\omega t$$

或

$$p = P\cos\omega t - Q\sin\omega t$$
$$q = P\sin\omega t + Q\cos\omega t$$

$$\frac{\partial F_1}{\partial t} = -\frac{\omega}{2\sin^2\omega t}\left(q^2 + Q^2 - 2qQ\cos\omega t\right)$$

$$= -\frac{\omega}{2\sin^2\omega t}\left[\left(q - Q\cos\omega t\right)^2 + Q^2\sin^2\omega t\right]$$

$$= -\frac{\omega}{2\sin^2\omega t}\left(P^2\sin^2\omega t + Q^2\sin^2\omega t\right)$$

$$= -\frac{\omega}{2}\left(Q^2 + P^2\right)$$

变换后新Hamiltonian：

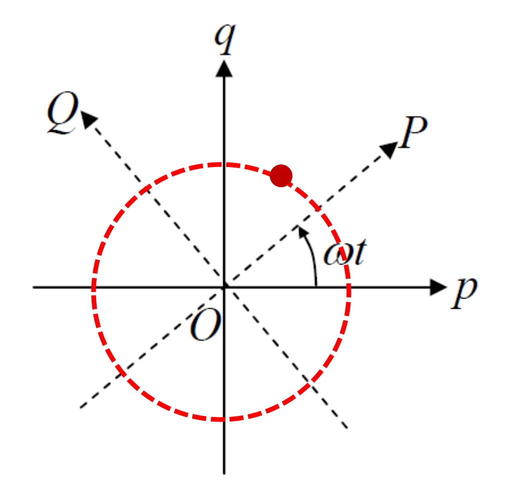
$$H^* = H\left(Q, P\right) - \frac{\omega}{2}\left(Q^2 + P^2\right)$$

$$H^* = H(Q,P) - \frac{\omega}{2}\left(Q^2 + P^2\right)$$

$$p = P\cos\omega t - Q\sin\omega t$$
$$q = P\sin\omega t + Q\cos\omega t$$

谐振子: $H = \frac{\omega}{2}\left(q^2 + p^2\right) = \frac{\omega}{2}\left(Q^2 + P^2\right)$

$$H^* = H + \frac{\partial F_1}{\partial t} = 0$$



$$\dot{Q} = \frac{\partial H^*}{\partial P} = 0, \qquad \dot{P} = -\frac{\partial H^*}{\partial Q} = 0$$

$Q = \text{const.}$，$P = \text{const.}$

谐振子在Q-P空间"静止"。

四. 正则变换的性质

1. $Q=Q(q,p),P=P(q,p)$ 不显含t. 正则变换 ⇒ 充要条件：

$$J=\left|\frac{\partial(QP)}{\partial(qp)}\right|=\begin{vmatrix}\frac{\partial Q_1}{\partial q_1} & \frac{\partial Q_1}{\partial q_2} & \cdots\cdots & \frac{\partial Q_1}{\partial p_s}\\ \frac{\partial Q_2}{\partial q_1} & \frac{\partial Q_2}{\partial q_2} & \cdots\cdots & \frac{\partial Q_2}{\partial p_s}\\ \cdots & \cdots & \cdots & \cdots\\ \frac{\partial P_s}{\partial q_1} & \frac{\partial P_s}{\partial q_2} & \cdots\cdots & \frac{\partial P_s}{\partial p_s}\end{vmatrix}=1$$

证明：略（参考《经典力学》沈惠川、李书民著）

2. $\because J=1 \quad \therefore$ 逆变换换存在

若 $(q\ p) \Rightarrow (Q\ P)$ ⇨ 正则变换

则 $(Q\ P) \Rightarrow (q\ p)$ ⇨ 正则变换

3.积运算存在

若 $(q\ p) \Rightarrow (q'\ p')$ ⇨ 正则变换

$(q'\ p') \Rightarrow (q''\ p'')$ ⇨ 正则变换

则 $(q\ p) \Rightarrow (q''\ p'')$ ⇨ 正则变换

例1、证明以下变换为正则变换，并利用它求谐振子的运动方程(谐振子H为)： $H(x,p)=\dfrac{p^2}{2m}+\dfrac{1}{2}m\omega^2q^2$

$$\begin{cases} p = m\omega q ctgQ \\ P=\dfrac{1}{2}m\omega q^2\csc^2 Q \end{cases}$$

Solution：

$$\begin{cases} p\mathrm{d}q = m\omega q ctgQ\mathrm{d}q=\dfrac{1}{2}m\omega ctgQ\mathrm{d}q^2 \\ Pd\mathrm{Q}=\dfrac{1}{2}m\omega q^2\csc^2 Qd\mathrm{Q}=-\dfrac{1}{2}m\omega q^2\mathrm{dctg}Q \end{cases}$$

$$\therefore \begin{cases} p\mathrm{d}q-Pd\mathrm{Q}=\mathrm{d}(\dfrac{1}{2}m\omega q^2 ctgQ)=dF \\ \mathrm{F}=\dfrac{1}{2}m\omega q^2 ctgQ \end{cases}$$

Thus,

$$(\mathrm{K}-\mathrm{H})\, dt + pdq - PdQ = dF, \quad and\ \mathrm{K} = \mathrm{H}$$

$$\begin{cases} p = m\omega q\, ctg Q \\ P = \dfrac{1}{2} m\omega q^2 \csc^2 Q \end{cases}$$

Above is a canonical transformation.

2) The Hamiltonian of a one-dimensional harmonic oscillator is,

$$H(x, p) = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2$$

From above canonical transformation, we have

$$\begin{cases} q = \sqrt{\dfrac{2P}{m\omega}} \sin Q \\ p = \sqrt{2m\omega P} \cos Q \end{cases}$$

$$F(q,Q) = \frac{m\omega}{2} q^2 \cot Q$$

F is not an explicit function of t , thus *K=H*

Hence, we have

$$H(x,p) = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2 = \omega P = K(Q,P) = E(const.)$$

$$\begin{cases} \dot{Q} = \dfrac{\partial K}{\partial P} = \dfrac{\partial H}{\partial P} = \omega \\ \dot{P} = -\dfrac{\partial K}{\partial Q} = 0 \end{cases} \Rightarrow \begin{array}{c} P = \dfrac{E}{\omega} \\ Q = wt + b \end{array} \Rightarrow q = \sqrt{\frac{2E}{m\omega^2}} \sin(wt + b)$$

例2：求下列变换时，（m,n）的取值；并给出该正则变换的第三类母函数：

$$\begin{cases} Q = q^n \cos(\mathrm{m}p) \\ P = q^n \sin(\mathrm{m}p) \end{cases},\qquad （m,n为常数）。$$

Solution： (1)First, we can easily have ,

$$\begin{cases} \dfrac{\partial Q}{\partial q} = nq^{n\text{-}1}\cos(\mathrm{m}p) & \dfrac{\partial P}{\partial q} = nq^{n\text{-}1}\sin(\mathrm{m}p) \\ \dfrac{\partial Q}{\partial p} = \text{-}mq^{n}\sin(\mathrm{m}p) & \dfrac{\partial P}{\partial p} = mq^{n}\cos(\mathrm{m}p) \end{cases}, \qquad （m,n为常数）。$$

From $J = \left|\dfrac{\partial(Q\,P)}{\partial(q\,p)}\right| = 1$, we have

$$J=\begin{vmatrix}\dfrac{\partial Q}{\partial q} & \dfrac{\partial Q}{\partial p}\\ \dfrac{\partial P}{\partial q} & \dfrac{\partial P}{\partial p}\end{vmatrix}=\begin{vmatrix} nq^{n-1}\cos(\mathrm{m}p) & mq^{n}\sin(\mathrm{m}p)\\ -nq^{n-1}\sin(\mathrm{m}p) & mq^{n}\cos(\mathrm{m}p)\end{vmatrix}=1,$$

Thus, the following results can be obtained,

$$nmq^{2n-1}=1$$

$$2n-1=0,\quad nm=1,$$

$$n=\frac{1}{2},\quad m=2,$$

(2) Then Canonical transformation can be written as follows,

$$\begin{cases} Q = \sqrt{q}\cos(2p) \\ P = \sqrt{q}\sin(2p) \end{cases},$$

From above we can have, $\begin{cases} q = Q^2\sec^2(2p) \\ P = Q\tan(2p) \end{cases},$

$(q, p) \Rightarrow (Q, P)$ is not an explicit function of t , $\frac{\partial F_3}{\partial t} = 0$, thus $K = H$, and

$$dF_3(p, Q) = -qdp - PdQ$$

$$= -Q^2\sec^2(2p)dp - Q\tan(2p)dQ$$

$$= -\frac{1}{2}Q^2 d\tan(2p) - \tan(2p)\frac{1}{2}dQ^2$$

$$= d\left[-\frac{Q^2\tan(2p)}{2}\right]$$

$$\therefore F_3(p, Q) = -\frac{1}{2}Q^2\tan(2p).$$

(2)Solution 2:

$$\because \begin{cases} q = Q^2 \sec^2(2p), \\ P = Q\tan(2p) \end{cases}$$

$(q,p) \Rightarrow (Q,P)$ is not an explicit function of t, $\dfrac{\partial F_3}{\partial t} = 0$, thus $K = H$, and

$$F_3(p,Q) = \int_{p_0,Q_0}^{p,Q} dF_3(p,Q) = \int_{p_0,Q_0}^{p,Q} (-qdp - PdQ)$$

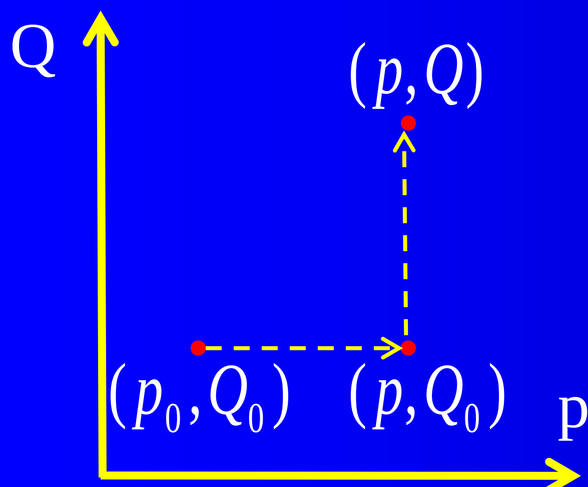
$$= \int_{p_0,Q_0}^{p,Q_0} \left(-Q^2 \sec^2(2p)dp - Q\tan(2p)dQ\right)$$

$$+ \int_{p,Q_0}^{p,Q} \left(-Q^2 \sec^2(2p)dp - Q\tan(2p)dQ\right)$$

$$= -Q_0^2 \int_{p_0}^{p} \sec^2(2p)dp + \tan(2p)\int_{Q_0}^{Q} QdQ$$

$$= -\frac{1}{2}\left(Q_0^2 \tan(2p)\Big|_{p_0}^{p} + Q^2 \tan(2p)\Big|_{Q_0}^{Q} \right)$$

$$= -\frac{1}{2}Q^2\tan(2p) + \frac{1}{2}Q_0^2\tan(2p_0)$$



where $\frac{1}{2}Q_0^2\tan(2p_0) = \text{const}$, we can assume it is 0. thus, $F_3 = -\frac{1}{2}Q^2\tan(2p)$.

例3：求证下列变换为正则变换，并求该正则变换的第三类母函数：

$$\begin{cases} Q = \ln(1+\sqrt{q}\cos p) \\ P = 2(1+\sqrt{q}\cos p)\sqrt{q}\sin p \end{cases}.$$

Proof： First, we can easily have ,

$$q = (e^Q - 1)^2 \sec^2 p$$

$$P = 2(1+\sqrt{q}\cos p)\sqrt{q}\sin p$$

$$= 2e^Q(e^Q - 1)\tan p$$

$$-qdp - PdQ = -(e^Q - 1)^2 \sec^2 p dp - 2e^Q(e^Q - 1)\tan p dQ$$

$$= -(e^Q - 1)^2 d\tan p - \tan p d(e^Q - 1)^2$$

$$= d\left[-(e^Q - 1)^2 \tan p\right] = dF_3(p, Q)$$

Finally we have,

$$F_3(p,Q) = -(e^Q - 1)^2 \tan p$$

$(q,p) \Rightarrow (Q,P)$, $F_3(p,Q)$ is not relvant to the time t, thus $K = H$, and

$$-q_\alpha dp_\alpha - P_\alpha dQ_\alpha + (K - H)dt = dF_3(p,Q)$$

$$\begin{cases} q = -\dfrac{\partial F_3}{\partial p} \\ P = -\dfrac{\partial F_3}{\partial Q} \\ K = H \end{cases}$$

We also can use the following formula to prove

$$J = \begin{vmatrix} \dfrac{\partial Q}{\partial q} & \dfrac{\partial Q}{\partial p} \\ \dfrac{\partial P}{\partial q} & \dfrac{\partial P}{\partial p} \end{vmatrix} = 1,$$

例4：已知正则变换母函数为：$F_1(q,Q)=mg(\frac{1}{6}gQ^3+qQ)$ 利用该正则变换，由正则方程求竖直上抛物体的运动规律。

解：F_1不显含时间t，故正则变换为：

$$\begin{cases} p=\dfrac{\partial F_1}{\partial q}=mgQ \\ P=-\dfrac{\partial F_1}{\partial Q}=-\dfrac{mg^2}{2}Q^2-mgq \\ \text{且}K=H \end{cases}$$

$$\Rightarrow q=-\frac{P}{mg}-\frac{g}{2}Q^2$$

$$H=\frac{p^2}{2m}+mgq$$

$$K=\frac{(mgQ)^2}{2m}+mg(-\frac{P}{mg}-\frac{g}{2}Q^2)=-P$$

$$\therefore\begin{cases} \dot{Q}=\dfrac{\partial K}{\partial P}=-1 \\ \dot{P}=-\dfrac{\partial K}{\partial Q}=0 \end{cases}$$

$$\therefore\begin{cases} Q=-t+\beta \\ P=\alpha \end{cases}(\alpha,\beta\text{ 为}const.)$$

$$\therefore q=-\frac{\alpha}{mg}-\frac{g}{2}(-t+\beta)^2=q_0+\dot{q}_0t-\frac{1}{2}gt^2,$$

其中$q_0=-\dfrac{\alpha}{mg}$-$\dfrac{\beta^2}{2g}$, $\dot{q}_0=g\beta$。

五.寻找合适生成函数

如果所选生成函数使得新的哈密顿函数等于零

$$K = H + \frac{\partial F}{\partial t} = 0$$

$$\begin{cases} \dot{Q}_\alpha = \dfrac{\partial K}{\partial P_\alpha} = 0 \\ \dot{P}_\alpha = \dfrac{\partial K}{\partial Q_\alpha} = 0 \end{cases}$$

$\Rightarrow$ $Q_\alpha = a_\alpha \,(\text{constant})$

$\Rightarrow$ $P_\alpha = b_\alpha \,(\text{constant})$

取第二类的生成函数F $\Rightarrow$ $F_2(q\ P\ t)$

$$H(q,p,t) + \frac{\partial F_2(q,P,t)}{\partial t} = 0$$

$$H(q,p,t)+\frac{\partial F_2(q,P,t)}{\partial t}=0$$

$$\begin{cases} p_\alpha = \dfrac{\partial F_2}{\partial q_\alpha} \\ Q_\alpha = \dfrac{\partial F_2}{\partial P_\alpha} \\ K = H + \dfrac{\partial F_2}{\partial t} \end{cases}$$

$$(K-H)dt + p_\alpha dq_\alpha + Q_\alpha dP_\alpha = dF_2$$

$$\boxed{P_\alpha = b_\alpha \qquad (cont.)}$$

$$S(q\ t) = F_2(q\ t) + A(cons\tan t)$$

$$H\left(q_1 q_2 \cdots\cdots q_f, \frac{\partial S}{\partial q_1}\frac{\partial S}{\partial q_2}\cdots\cdots\frac{\partial S}{\partial q_f}, t\right) + \frac{\partial S(q_1\cdots\cdots q_1, b_1\cdots\cdots b_f, t)}{\partial t} = 0$$

$$H(q_1 q_2 \cdots\cdots q_f, \frac{\partial S}{\partial q_1}, \frac{\partial S}{\partial q_2} \cdots\cdots \frac{\partial S}{\partial q_f}, t) + \frac{\partial S(q_1 \cdots\cdots q_1, b_1 \cdots\cdots b_f, t)}{\partial t} = 0$$

一阶偏微分方程 $\Rightarrow$ *Halmiton - Jacobi Equation*

$$P_\alpha = b_\alpha (\text{constant}), \text{i} = 1, .., \text{f}$$

$S = S(q\ t) \quad \Rightarrow$ 哈密顿主函数

求解正则方程 $\Rightarrow$

寻找H - J方程的生成函数（S（q,t））

$S = S(q\ t)$ 依赖n个独立的常数，

经S生成的$\text{P}_\alpha = b_\alpha (\text{constant})$.

$$H\left(q_1 q_2 \cdots\cdots q_f, \frac{\partial S}{\partial q_1}\frac{\partial S}{\partial q_2}\cdots\cdots\frac{\partial S}{\partial q_f}, t\right) + \frac{\partial S(q_1 \cdots\cdots q_1, b_1 \cdots\cdots b_f, t)}{\partial t} = 0$$

$$\because \frac{ds}{dt} = \frac{\partial S}{\partial q_\alpha}\dot{q}_\alpha + \frac{\partial S}{\partial t} \qquad (\alpha = 123 \ldots f)$$

$$\because K = 0, \ \frac{\partial S}{\partial q_\alpha} = p_\alpha, \quad \frac{\partial S}{\partial t} = -H$$

$$\begin{cases} p_\alpha = \dfrac{\partial S}{\partial q_\alpha} \\ Q_\alpha = \dfrac{\partial S}{\partial P_\alpha} \\ K = H + \dfrac{\partial S}{\partial t} \end{cases} \qquad (K - H)dt + p_\alpha dq_\alpha + Q_\alpha dP_\alpha = dS$$

$$P_\alpha = b_\alpha \qquad (cont.)$$

$$\therefore \frac{ds}{dt} = p_\alpha \dot{q}_\alpha - H = L \qquad \therefore S = \int L\,dt$$

S表示积分限不确定的哈密顿主函数

白居易《埇桥旧业》诗曰：
“改移新径路，变换旧村邻。”