# 1 Mathematical Preliminaries

*The enormous usefulness of mathematics in the natural sciences is something bordering on the mysterious.*
Eugene Wigner (1960)

## 1.1 Introduction

This chapter presents a collection of mathematical notation, definitions, identities, theorems, and transformations which play an important role in the study of electromagnetism. A brief discussion accompanies some of the less familiar topics and only a few proofs are given in detail. For more details and complete proofs, the reader should consult the books and papers listed in Sources, References, and Additional Reading at the end of the chapter. Appendix C at the end of the book summarizes the properties of Legendre polynomials, spherical harmonics, and Bessel functions.

## 1.2 Vectors

A vector is a geometrical object characterized by a magnitude and direction.[1] Although not necessary, it is convenient to discuss an arbitrary vector using its components defined with respect to a given coordinate system. An example is the right-handed coordinate system with orthogonal unit basis vectors $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$ shown in Figure 1.1, where

$$\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_1 = 1 \qquad \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_2 = 1 \qquad \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_3 = 1 \tag{1.1}$$

$$\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_2 = 0 \qquad \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_3 = 0 \qquad \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_1 = 0 \tag{1.2}$$

$$\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_3 \qquad \hat{\mathbf{e}}_2 \times \hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_1 \qquad \hat{\mathbf{e}}_3 \times \hat{\mathbf{e}}_1 = \hat{\mathbf{e}}_2. \tag{1.3}$$

We express an arbitrary vector $\mathbf{V}$ in this basis using components $V_k = \hat{\mathbf{e}}_k \cdot \mathbf{V}$,

$$\mathbf{V} = V_1\hat{\mathbf{e}}_1 + V_2\hat{\mathbf{e}}_2 + V_3\hat{\mathbf{e}}_3. \tag{1.4}$$

A vector can be decomposed in any coordinate system we please, so

$$\sum_{k=1}^{3} V_k\hat{\mathbf{e}}_k = \sum_{k=1}^{3} V'_k\hat{\mathbf{e}}'_k. \tag{1.5}$$

[1] A more precise definition of a vector is given in Section 1.8.

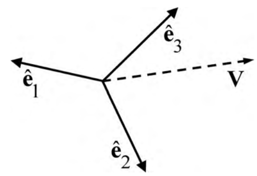



**Figure 1.1:** An orthonormal set of unit vectors $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, $\hat{\mathbf{e}}_3$. $\mathbf{V}$ is an arbitrary vector.

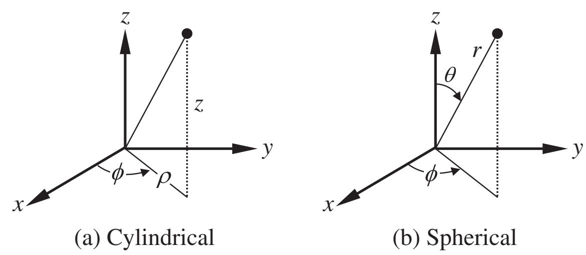



**Figure 1.2:** Two curvilinear coordinate systems.

## 1.2.1 Cartesian Coordinates

Our notation for Cartesian components and unit vectors is

$$\mathbf{V} = V_x\hat{\mathbf{x}} + V_y\hat{\mathbf{y}} + V_z\hat{\mathbf{z}}. \tag{1.6}$$

In particular, $r_k$ always denotes the Cartesian components of the position vector,

$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}. \tag{1.7}$$

It is not obvious geometrically (see Example 1.7 in Section 1.8), but the gradient operator is a vector with the Cartesian representation

$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}. \tag{1.8}$$

The divergence, curl, and Laplacian operations are, respectively,

$$\nabla \cdot \mathbf{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \tag{1.9}$$

$$\nabla \times \mathbf{V} = \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\hat{\mathbf{z}} \tag{1.10}$$

$$\nabla^2 A = \frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} + \frac{\partial^2 A}{\partial z^2}. \tag{1.11}$$

## 1.2.2 Cylindrical Coordinates

Figure 1.2(a) defines cylindrical coordinates $(\rho, \phi, z)$. Our notation for the components and unit vectors in this system is

$$\mathbf{V} = V_\rho\hat{\boldsymbol{\rho}} + V_\phi\hat{\boldsymbol{\phi}} + V_z\hat{\mathbf{z}}. \tag{1.12}$$

The transformation to Cartesian coordinates is

$$x = \rho\cos\phi \qquad y = \rho\sin\phi \qquad z = z. \tag{1.13}$$

The volume element in cylindrical coordinates is $d^3r = \rho d\rho d\phi dz$. The unit vectors $(\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\phi}}, \hat{\mathbf{z}})$ form a right-handed orthogonal triad. $\hat{\mathbf{z}}$ is the same as in Cartesian coordinates. Otherwise,

$$\hat{\boldsymbol{\rho}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi \qquad \hat{\mathbf{x}} = \hat{\boldsymbol{\rho}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi \tag{1.14}$$

$$\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi \qquad \hat{\mathbf{y}} = \hat{\boldsymbol{\rho}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi. \tag{1.15}$$

The gradient operator in cylindrical coordinates is

$$\nabla = \hat{\boldsymbol{\rho}}\frac{\partial}{\partial\rho} + \frac{\hat{\boldsymbol{\phi}}}{\rho}\frac{\partial}{\partial\phi} + \hat{\mathbf{z}}\frac{\partial}{\partial z}. \tag{1.16}$$

The divergence, curl, and Laplacian operations are, respectively,

$$\nabla\cdot\mathbf{V} = \frac{1}{\rho}\frac{\partial(\rho V_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial V_\phi}{\partial\phi} + \frac{\partial V_z}{\partial z} \tag{1.17}$$

$$\nabla\times\mathbf{V} = \left[\frac{1}{\rho}\frac{\partial V_z}{\partial\phi} - \frac{\partial V_\phi}{\partial z}\right]\hat{\boldsymbol{\rho}} + \left[\frac{\partial V_\rho}{\partial z} - \frac{\partial V_z}{\partial\rho}\right]\hat{\boldsymbol{\phi}} + \frac{1}{\rho}\left[\frac{\partial(\rho V_\phi)}{\partial\rho} - \frac{\partial V_\rho}{\partial\phi}\right]\hat{\mathbf{z}} \tag{1.18}$$

$$\nabla^2 A = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial A}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 A}{\partial\phi^2} + \frac{\partial^2 A}{\partial z^2}. \tag{1.19}$$

### 1.2.3 Spherical Coordinates

Figure 1.2(b) defines spherical coordinates $(r, \theta, \phi)$. Our notation for the components and unit vectors in this system is

$$\mathbf{V} = V_r\hat{\mathbf{r}} + V_\theta\hat{\boldsymbol{\theta}} + V_\phi\hat{\boldsymbol{\phi}}. \tag{1.20}$$

The transformation to Cartesian coordinates is

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta. \tag{1.21}$$

The volume element in spherical coordinates is $d^3r = r^2\sin\theta dr d\theta d\phi$. The unit vectors are related by

$$\hat{\mathbf{r}} = \hat{\mathbf{x}}\sin\theta\cos\phi + \hat{\mathbf{y}}\sin\theta\sin\phi + \hat{\mathbf{z}}\cos\theta \qquad \hat{\mathbf{x}} = \hat{\mathbf{r}}\sin\theta\cos\phi + \hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi \tag{1.22}$$

$$\hat{\boldsymbol{\theta}} = \hat{\mathbf{x}}\cos\theta\cos\phi + \hat{\mathbf{y}}\cos\theta\sin\phi - \hat{\mathbf{z}}\sin\theta \qquad \hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi \tag{1.23}$$

$$\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi \qquad \hat{\mathbf{z}} = \hat{\mathbf{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta. \tag{1.24}$$

The gradient operator in spherical coordinates is

$$\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial}{\partial\theta} + \frac{\hat{\boldsymbol{\phi}}}{r\sin\theta}\frac{\partial}{\partial\phi}. \tag{1.25}$$

The divergence, curl, and Laplacian operations are, respectively,

$$\nabla\cdot\mathbf{V} = \frac{1}{r^2}\frac{\partial(r^2V_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\,V_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial\phi} \tag{1.26}$$

$$\begin{aligned}\nabla\times\mathbf{V} &= \frac{1}{r\sin\theta}\left[\frac{\partial(\sin\theta\,V_\phi)}{\partial\theta} - \frac{\partial V_\theta}{\partial\phi}\right]\hat{\mathbf{r}} \\ &\quad + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial V_r}{\partial\phi} - \frac{\partial(rV_\phi)}{\partial r}\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial(rV_\theta)}{\partial r} - \frac{\partial V_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}\end{aligned} \tag{1.27}$$

$$\nabla^2 A = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial A}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 A}{\partial\phi^2}. \tag{1.28}$$

## 1.2.4 The Einstein Summation Convention

Einstein (1916) introduced the following convention. An index which appears exactly twice in a mathematical expression is implicitly summed over all possible values for that index. The range of this *dummy index* must be clear from context and the index cannot be used elsewhere in the same expression for another purpose. In this book, the range for a roman index like $i$ is from 1 to 3, indicating a sum over the Cartesian indices $x$, $y$, and $z$. Thus, $\mathbf{V}$ in (1.6) and its dot product with another vector $\mathbf{F}$ are written

$$\mathbf{V} = \sum_{k=1}^{3} V_k \hat{\mathbf{e}}_k \equiv V_k \hat{\mathbf{e}}_k \qquad \mathbf{V} \cdot \mathbf{F} = \sum_{k=1}^{3} V_k F_k \equiv V_k F_k. \tag{1.29}$$

In a Cartesian basis, the gradient of a scalar $\varphi$ and the divergence of a vector $\mathbf{D}$ can be variously written

$$\nabla \varphi = \hat{\mathbf{e}}_k \nabla_k \varphi = \hat{\mathbf{e}}_k \partial_k \varphi = \hat{\mathbf{e}}_k \frac{\partial \varphi}{\partial r_k} \tag{1.30}$$

$$\nabla \cdot \mathbf{D} = \nabla_k D_k = \partial_k D_k = \frac{\partial D_k}{\partial r_k}. \tag{1.31}$$

If an $N \times N$ matrix $\mathbf{C}$ is the product of an $N \times M$ matrix $\mathbf{A}$ and an $M \times N$ matrix $\mathbf{B}$,

$$C_{ik} = \sum_{j=1}^{M} A_{ij} B_{jk} = A_{ij} B_{jk}. \tag{1.32}$$

## 1.2.5 The Kronecker and Levi-Cività Symbols

The Kronecker delta symbol $\delta_{ij}$ and Levi-Cività permutation symbol $\epsilon_{ijk}$ have roman indices $i$, $j$, and $k$ which take on the Cartesian coordinate values $x$, $y$, and $z$. They are defined by

$$\delta_{ij} = \begin{cases} 1 & i = j, \\ 0 & i \neq j, \end{cases} \tag{1.33}$$

and

$$\epsilon_{ijk} = \begin{cases} 1 & ijk = \;\; xyz \quad yzx \quad zxy, \\ -1 & ijk = \;\; xzy \quad yxz \quad zyx, \\ 0 & \text{otherwise.} \end{cases} \tag{1.34}$$

Some useful Kronecker delta and Levi-Cività symbol identities are

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij} \qquad \delta_{kk} = 3 \tag{1.35}$$

$$\partial_k r_j = \delta_{jk} \qquad V_k \delta_{kj} = V_j \tag{1.36}$$

$$[\mathbf{V} \times \mathbf{F}]_i = \epsilon_{ijk} V_j F_k \qquad [\nabla \times \mathbf{A}]_i = \epsilon_{ijk} \partial_j A_k \tag{1.37}$$

$$\delta_{ij} \epsilon_{ijk} = 0 \qquad \epsilon_{ijk} \epsilon_{ijk} = 6. \tag{1.38}$$

A particularly useful identity involves a single sum over the repeated index $i$:

$$\epsilon_{ijk} \epsilon_{ist} = \delta_{js} \delta_{kt} - \delta_{jt} \delta_{ks}. \tag{1.39}$$

A generalization of (1.39) when there are no repeated indices to sum over is the determinant

$$\epsilon_{ki\ell} \epsilon_{mpq} = \begin{vmatrix} \delta_{km} & \delta_{im} & \delta_{\ell m} \\ \delta_{kp} & \delta_{ip} & \delta_{\ell p} \\ \delta_{kq} & \delta_{iq} & \delta_{\ell q} \end{vmatrix}. \tag{1.40}$$

Finally, let $\mathbf{C}$ be a $3 \times 3$ matrix with matrix elements $C_{11}$, $C_{12}$, etc. The determinant of $\mathbf{C}$ can be written using either an expansion by columns,

$$\det\mathbf{C} = \epsilon_{ijk} C_{i1} C_{j2} C_{k3}, \tag{1.41}$$

or an expansion by rows,

$$\det\mathbf{C} = \epsilon_{ijk} C_{1i} C_{2j} C_{3k}. \tag{1.42}$$

A closely related identity we will use in Section 1.8.1 is

$$\epsilon_{\ell mn} \det\mathbf{C} = \epsilon_{ijk} C_{\ell i} C_{mj} C_{nk}. \tag{1.43}$$

### 1.2.6 Vector Identities in Cartesian Components

The Kronecker and Levi-Cività symbols simplify the proof of vector identities. An example is

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}). \tag{1.44}$$

Using the left side of (1.37), the $i$th component of $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is

$$[\mathbf{a} \times (\mathbf{b} \times \mathbf{c})]_i = \epsilon_{ijk} a_j (\mathbf{b} \times \mathbf{c})_k = \epsilon_{ijk} a_j \epsilon_{k\ell m} b_l c_m. \tag{1.45}$$

The definition (1.34) tells us that $\epsilon_{ijk} = \epsilon_{kij}$. Therefore, the identity (1.39) gives

$$[\mathbf{a} \times (\mathbf{b} \times \mathbf{c})]_i = \epsilon_{kij}\epsilon_{k\ell m} a_j b_\ell c_m = (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}) a_j b_\ell c_m = a_j b_i c_j - a_j b_j c_i. \tag{1.46}$$

The final result, $b_i(\mathbf{a} \cdot \mathbf{c}) - c_i(\mathbf{a} \cdot \mathbf{b})$, is indeed the $i$th component of the right side of (1.44). The same method of proof applies to gradient-, divergence-, and curl-type vector identities because the components of the $\nabla$ operator transform like the components of a vector [see above (1.8)]. The next three examples illustrate this point.

**Example 1.1** Prove that $\nabla \cdot (\nabla \times \mathbf{g}) = 0$.

**Solution:** Begin with $\nabla \cdot (\nabla \times \mathbf{g}) = \partial_i \epsilon_{ijk} \partial_j g_k = \frac{1}{2}\partial_i \partial_j g_k \epsilon_{ijk} + \frac{1}{2}\partial_i \partial_j g_k \epsilon_{ijk}$. Exchanging the dummy indices $i$ and $j$ in the last term gives

$$\nabla \cdot \nabla \times \mathbf{g} = \tfrac{1}{2}\partial_i \partial_j g_k \epsilon_{ijk} + \tfrac{1}{2}\partial_j \partial_i g_k \epsilon_{jik} = \tfrac{1}{2}\{\epsilon_{ijk} + \epsilon_{jik}\}\partial_i \partial_j g_k = 0.$$

The final zero comes from $\epsilon_{ijk} = -\epsilon_{jik}$, which is a consequence of (1.34).

**Example 1.2** Prove that $\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}\nabla \cdot \mathbf{B} - (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} - \mathbf{B}\nabla \cdot \mathbf{A}$.

**Solution:** Focus on the $i$th Cartesian component and use the left side of (1.37) to write

$$[\nabla \times (\mathbf{A} \times \mathbf{B})]_i = \epsilon_{ijk}\partial_j (\mathbf{A} \times \mathbf{B})_k = \epsilon_{ijk}\epsilon_{kst}\partial_j (A_s B_t).$$

The cyclic properties of the Levi-Cività symbol and the identity (1.39) give

$$[\nabla \times (\mathbf{A} \times \mathbf{B})]_i = \epsilon_{kij}\epsilon_{kst}\partial_j (A_s B_t) = (\delta_{is}\delta_{jt} - \delta_{it}\delta_{js})(A_s \partial_j B_t + B_t \partial_j A_s).$$

Therefore,

$$[\nabla \times (\mathbf{A} \times \mathbf{B})]_i = A_i \partial_j B_j - A_j \partial_j B_i + B_j \partial_j A_i - B_i \partial_j A_j.$$

This proves the identity because the choice of $i$ is arbitrary.

**Example 1.3** Prove the "double-curl identity" $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$.

**Solution:** Consider the $i$th Cartesian component. The identity on the left side of (1.37) and the invariance of the Levi-Cività symbol with respect to cyclic permutations of its indices give

$$[\nabla \times (\nabla \times \mathbf{A})]_i = \epsilon_{ijk}\partial_j(\nabla \times \mathbf{A})_k = \epsilon_{ijk}\partial_j\epsilon_{kpq}\partial_p A_q = \epsilon_{kij}\epsilon_{kpq}\partial_j\partial_p A_q.$$

Now apply the identity (1.39) to get

$$[\nabla \times (\nabla \times \mathbf{A})]_i = (\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp})\partial_j\partial_p A_q = \partial_i\partial_j A_j - \partial_j\partial_j A_i = \nabla_i(\nabla \cdot \mathbf{A}) - \nabla^2 A_i.$$

The double-curl identity follows because

$$\nabla^2\mathbf{A} = \nabla^2(A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}) = \hat{\mathbf{x}}\nabla^2 A_x + \hat{\mathbf{y}}\nabla^2 A_y + \hat{\mathbf{z}}\nabla^2 A_z.$$

### 1.2.7 Vector Identities in Curvilinear Components

Care is needed to interpret the vector identities in Examples 1.2 and 1.3 when the vectors in question are decomposed into spherical or cylindrical components such as $\mathbf{A} = A_r\hat{\mathbf{r}} + A_\theta\hat{\boldsymbol{\theta}} + A_\phi\hat{\boldsymbol{\phi}}$. This can be seen from Example 1.3 where the final step is no longer valid because $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$ are not constant vectors. In other words,

$$\nabla^2\mathbf{A} \;=\; \nabla \cdot \nabla(A_r\hat{\mathbf{r}} + A_\theta\hat{\boldsymbol{\theta}} + A_\phi\hat{\boldsymbol{\phi}}) \neq \hat{\mathbf{r}}\,\nabla^2 A_r + \hat{\boldsymbol{\theta}}\,\nabla^2 A_\theta + \hat{\boldsymbol{\phi}}\,\nabla^2 A_\phi. \tag{1.47}$$

One way to proceed is to work out the components of $\nabla(A_r\hat{\mathbf{r}})$, $\nabla(A_\theta\hat{\boldsymbol{\theta}})$, and $\nabla(A_\phi\hat{\boldsymbol{\phi}})$. Alternatively, we may simply *define* the meaning of the operation $\nabla^2\mathbf{A}$ when $\mathbf{A}$ is expressed using curvilinear components. For example,

$$[\nabla^2\mathbf{A}]_\phi \equiv \partial_\phi(\nabla \cdot \mathbf{A}) - [\nabla \times (\nabla \times \mathbf{A})]_\phi, \tag{1.48}$$

and similarly for $(\nabla^2\mathbf{A})_r$ and $(\nabla^2\mathbf{A})_\theta$.

Exactly the same issue arises when we examine the last step in Example 1.2, namely

$$[\nabla \times (\mathbf{A} \times \mathbf{B})]_i = A_i\nabla \cdot \mathbf{B} - (\mathbf{A} \cdot \nabla)B_i + (\mathbf{B} \cdot \nabla)A_i - B_i\nabla \cdot \mathbf{A}. \tag{1.49}$$

By construction, this equation makes sense when $i$ stands for $x$, $y$, or $z$. It does *not* make sense if $i$ stands for, say, $r$, $\theta$, or $\phi$. On the other hand, the full vector version of the identity is correct as long as we retain the $r$, $\theta$, and $\phi$ variations of $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$. For example,

$$(\mathbf{A} \cdot \nabla)\mathbf{B} = \left[A_r\frac{\partial}{\partial r} + \frac{A_\theta}{r}\frac{\partial}{\partial\theta} + \frac{A_\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right](B_r\hat{\mathbf{r}} + B_\theta\hat{\boldsymbol{\theta}} + B_\phi\hat{\boldsymbol{\phi}}). \tag{1.50}$$

#### Application 1.1 Two Identities for $\nabla \times \mathbf{L}$

The $\hbar = 1$ version of the quantum mechanical angular momentum operator, $\mathbf{L} = -i\mathbf{r} \times \nabla$, plays a useful role in the analysis of classical spherical systems. In this application, we prove two operator identities which will appear later in the text:

(A) $\quad \nabla \times \mathbf{L} = -i\mathbf{r}\nabla^2 + i\nabla(1 + \mathbf{r} \cdot \nabla)$

(B) $\quad \nabla \times \mathbf{L} = (\hat{\mathbf{r}} \times \mathbf{L})\left(\dfrac{1}{r}\dfrac{\partial}{\partial r}r\right) + \hat{\mathbf{r}}\dfrac{i}{r}L^2.$

Proof of Identity (A):

We use (1.37), (1.39), and the cyclic property of the Levi-Cività symbol to evaluate the $k$th component of $\nabla \times \mathbf{L}$ acting on a scalar function $\phi$:

$$[\nabla \times \mathbf{L}]_k \phi = -i[\nabla \times (\mathbf{r} \times \nabla)]_k \phi = -i\epsilon_{mk\ell}\epsilon_{mst}\partial_\ell r_s \partial_t \phi = -i[\partial_\ell r_k \partial_\ell \phi - \partial_\ell r_\ell \partial_k \phi]. \quad (1.51)$$

Because $\partial_\ell r_\ell = 3$ and $\partial_\ell r_k = \delta_{\ell k}$,

$$\nabla \times \mathbf{L}\phi = [-i\mathbf{r}\nabla^2 + i2\nabla + i(\mathbf{r}\cdot\nabla)\nabla]\phi. \quad (1.52)$$

However,

$$\partial_k [r_\ell \partial_\ell \phi] = \partial_k \phi + r_\ell \partial_\ell \partial_k \phi, \quad (1.53)$$

which is the $k$th component of $\nabla(\mathbf{r}\cdot\nabla)\phi = \nabla\phi + (\mathbf{r}\cdot\nabla)\nabla\phi$. Substituting the latter into (1.53) gives Identity (A).

Proof of Identity (B):

We decompose the gradient operator into its radial and angular pieces:

$$\nabla = \hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\nabla) - \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \nabla) = \hat{\mathbf{r}}\frac{\partial}{\partial r} - \frac{i}{r}\hat{\mathbf{r}} \times \mathbf{L}. \quad (1.54)$$

Equation (1.54) and the Levi-Cività formalism produce the intermediate result

$$\nabla \times \mathbf{L} = (\hat{\mathbf{r}} \times \mathbf{L})\frac{\partial}{\partial r} - \frac{i}{r}(\hat{\mathbf{r}} \times \mathbf{L}) \times \mathbf{L} = (\hat{\mathbf{r}} \times \mathbf{L})\frac{\partial}{\partial r} - \frac{i}{r}\left[\hat{r}_k \mathbf{L} L_k - \hat{\mathbf{r}} L^2\right]. \quad (1.55)$$

However, the angular momentum operator obeys commutation relations which can be summarized as $\mathbf{L} \times \mathbf{L} = i\mathbf{L}$. Therefore,

$$\hat{\mathbf{r}} \times (\mathbf{L} \times \mathbf{L}) = i\hat{\mathbf{r}} \times \mathbf{L} \quad \Rightarrow \quad \hat{r}_k \mathbf{L} L_k - \hat{r}_k L_k \mathbf{L} = i\hat{\mathbf{r}} \times \mathbf{L}. \quad (1.56)$$

On the other hand, $r_k L_k = 0$ because $\mathbf{L}$ is perpendicular to both $\mathbf{r}$ and $\nabla$. Therefore, $\hat{r}_k \mathbf{L} L_k = i\hat{\mathbf{r}} \times \mathbf{L}$, which we can substitute into (1.55). The result is identity (B) because, for any scalar function $\phi$,

$$\frac{1}{r}\left[\frac{\partial}{\partial r}(r\phi)\right] = \frac{\partial \phi}{\partial r} + \frac{\phi}{r}. \quad (1.57)$$

■

# 1.3 Derivatives

## 1.3.1 Functions of r and |r|

The position vector is $\mathbf{r} = r\hat{\mathbf{r}}$ with $r = \sqrt{x^2 + y^2 + z^2}$. If $f(r)$ is a scalar function and $f'(r) = df/dr$,

$$\nabla r = \hat{\mathbf{r}} \qquad \nabla \times \mathbf{r} = 0 \quad (1.58)$$

$$\nabla f = f'\hat{\mathbf{r}} \qquad \nabla^2 f = \frac{(r^2 f')'}{r^2} \quad (1.59)$$

$$\nabla \cdot (f\mathbf{r}) = \frac{(r^3 f)'}{r^2} \qquad \nabla \times (f\mathbf{r}) = 0. \quad (1.60)$$

Similarly, if $\mathbf{g}(r)$ is a vector function and $\mathbf{c}$ is a constant vector,

$$\nabla \cdot \mathbf{g} = \mathbf{g}' \cdot \hat{\mathbf{r}} \qquad \nabla \times \mathbf{g} = \hat{\mathbf{r}} \times \mathbf{g}' \quad (1.61)$$

$$(\mathbf{g}\cdot\nabla)\mathbf{r} = \mathbf{g} \qquad (\mathbf{r}\cdot\nabla)\mathbf{g} = r\mathbf{g}' \quad (1.62)$$

$$\nabla(\mathbf{r}\cdot\mathbf{g}) = \mathbf{g} + \frac{(\mathbf{r}\cdot\mathbf{g}')\mathbf{r}}{r} \qquad \nabla\cdot(\mathbf{g}\times\mathbf{r}) = 0 \tag{1.63}$$

$$\nabla\times(\mathbf{g}\times\mathbf{r}) = 2\mathbf{g} + r\mathbf{g}' - \frac{(\mathbf{r}\cdot\mathbf{g}')\mathbf{r}}{r} \qquad \nabla(\mathbf{c}\cdot\mathbf{r}) = \mathbf{c}. \tag{1.64}$$

### 1.3.2 Functions of $\mathbf{r}-\mathbf{r}'$

Let $\mathbf{R} = \mathbf{r}-\mathbf{r}' = (x-x')\hat{\mathbf{x}} + (y-y')\hat{\mathbf{y}} + (z-z')\hat{\mathbf{z}}$. Then,

$$\nabla f(R) = f'(R)\hat{\mathbf{R}} \qquad \nabla\cdot\mathbf{g}(R) = \mathbf{g}'(R)\cdot\hat{\mathbf{R}} \qquad \nabla\times\mathbf{g}(R) = \hat{\mathbf{R}}\times\mathbf{g}'(R). \tag{1.65}$$

Moreover, because

$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z} \qquad \text{and} \qquad \nabla' = \hat{\mathbf{x}}\frac{\partial}{\partial x'} + \hat{\mathbf{y}}\frac{\partial}{\partial y'} + \hat{\mathbf{z}}\frac{\partial}{\partial z'}, \tag{1.66}$$

it it straightforward to confirm that

$$\nabla' f(R) = -\nabla f(R). \tag{1.67}$$

### 1.3.3 The Convective Derivative

Let $\phi(\mathbf{r},t)$ be a scalar function of space and time. An observer who repeatedly samples the value of $\phi$ at a fixed point in space, $\mathbf{r}$, records the time rate of change of $\phi$ as the partial derivative $\partial\phi/\partial t$. However, the same observer who repeatedly samples $\phi$ along a trajectory in space $\mathbf{r}(t)$ that moves with velocity $\boldsymbol{v}(t) = \dot{\mathbf{r}}(t)$ records the time rate of change of $\phi$ as the *convective derivative*,

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial t} + \frac{dx}{dt}\frac{\partial\phi}{\partial x} + \frac{dy}{dt}\frac{\partial\phi}{\partial y} + \frac{dz}{dt}\frac{\partial\phi}{\partial z} = \frac{\partial\phi}{\partial t} + (\boldsymbol{v}\cdot\nabla)\phi. \tag{1.68}$$

For a vector function $\mathbf{g}(\mathbf{r},t)$, the corresponding convective derivative is

$$\frac{d\mathbf{g}}{dt} = \frac{\partial\mathbf{g}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\mathbf{g}. \tag{1.69}$$

### 1.3.4 Taylor's Theorem

Taylor's theorem in one dimension is

$$f(x) = f(a) + (x-a)\left.\frac{df}{dx}\right|_{x=a} + \frac{1}{2!}(x-a)^2\left.\frac{d^2f}{dx^2}\right|_{x=a} + \cdots. \tag{1.70}$$

An alternative form follows from (1.70) if $x \to x+\epsilon$ and $a \to x$:

$$f(x+\epsilon) = f(x) + \epsilon\frac{df}{dx} + \frac{1}{2!}\epsilon^2\frac{d^2f}{dx^2} + \cdots. \tag{1.71}$$

Equivalently,

$$f(x+\epsilon) = \left[1 + \epsilon\frac{d}{dx} + \frac{1}{2!}\left(\epsilon\frac{d}{dx}\right)^2 + \cdots\right]f(x) = \exp\left(\epsilon\frac{d}{dx}\right)f(x). \tag{1.72}$$

This generalizes for a function of three variables to

$$f(x+\epsilon_x, y+\epsilon_y, z+\epsilon_z) = \exp\left(\epsilon_x\frac{\partial}{\partial x}\right)\exp\left(\epsilon_y\frac{\partial}{\partial y}\right)\exp\left(\epsilon_z\frac{\partial}{\partial z}\right)f(x,y,z), \tag{1.73}$$

or

$$f(\mathbf{r}+\boldsymbol{\epsilon}) = \exp\left(\boldsymbol{\epsilon}\cdot\nabla\right)f(\mathbf{r}) = \left[1 + \boldsymbol{\epsilon}\cdot\nabla + \frac{1}{2!}(\boldsymbol{\epsilon}\cdot\nabla)^2 + \cdots\right]f(\mathbf{r}). \tag{1.74}$$

## 1.4 Integrals

### 1.4.1 Jacobian Determinant

The determinant of the Jacobian matrix $\mathbf{J}$ relates volume elements when changing variables in an integral. For example, suppose $\mathbf{x}$ and $\mathbf{y}$ are $N$-dimensional space vectors in two different coordinates systems, e.g., Cartesian and spherical. The volume elements $d^N x$ and $d^N y$ are related by

$$d^N x = |\mathbf{J}(\mathbf{x}, \mathbf{y})| \, d^N y = \begin{vmatrix} \dfrac{\partial x_1}{\partial y_1} & \dfrac{\partial x_1}{\partial y_2} & \cdots & \dfrac{\partial x_1}{\partial y_N} \\ \vdots & \vdots & \vdots & \vdots \\ \dfrac{\partial x_N}{\partial y_1} & \dfrac{\partial x_N}{\partial y_2} & \cdots & \dfrac{\partial x_N}{\partial y_N} \end{vmatrix} d^N y. \tag{1.75}$$

### 1.4.2 The Divergence Theorem

Let $\mathbf{F}(\mathbf{r})$ be a vector function defined in a volume $V$ enclosed by a surface $S$ with an outward normal $\hat{\mathbf{n}}$. If $d\mathbf{S} = dS\hat{\mathbf{n}}$, the divergence theorem is

$$\int_V d^3r \, \nabla \cdot \mathbf{F} = \int_S d\mathbf{S} \cdot \mathbf{F}. \tag{1.76}$$

Special choices for the vector function $\mathbf{F}(\mathbf{r})$ produce various integral identities based on (1.76). For example, if $\mathbf{c}$ is an arbitrary constant vector, the reader can confirm that the choices $\mathbf{F}(\mathbf{r}) = \mathbf{c}\psi(\mathbf{r})$ and $\mathbf{F}(\mathbf{r}) = \mathbf{A}(\mathbf{r}) \times \mathbf{c}$ substituted into (1.76) respectively yield

$$\int_V d^3r \, \nabla\psi = \int_S d\mathbf{S}\psi \tag{1.77}$$

$$\int_V d^3r \, \nabla \times \mathbf{A} = \int_S d\mathbf{S} \times \mathbf{A}. \tag{1.78}$$

### 1.4.3 Green's Identities

The choice $\mathbf{F}(\mathbf{r}) = \phi(\mathbf{r})\nabla\psi(\mathbf{r})$ in (1.76) leads to *Green's first identity*,

$$\int_V d^3r \, [\phi\nabla^2\psi + \nabla\phi \cdot \nabla\psi] = \int_S d\mathbf{S} \cdot \phi\nabla\psi. \tag{1.79}$$

Writing (1.79) with the roles of $\phi$ and $\psi$ exchanged and subtracting that equation from (1.79) itself gives *Green's second identity*,

$$\int_V d^3r \, [\phi\nabla^2\psi - \psi\nabla^2\phi] = \int_S d\mathbf{S} \cdot [\phi\nabla\psi - \psi\nabla\phi]. \tag{1.80}$$

The choice $\mathbf{F} = \mathbf{P} \times \nabla \times \mathbf{Q}$ in (1.76) and the identity $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B}$ produces a vector analog of Green's first identity:

$$\int_V d^3r \, [\nabla \times \mathbf{P} \cdot \nabla \times \mathbf{Q} - \mathbf{P} \cdot \nabla \times \nabla \times \mathbf{Q}] = \int_S d\mathbf{S} \cdot (\mathbf{P} \times \nabla \times \mathbf{Q}). \tag{1.81}$$

Writing (1.81) with **P** and **Q** interchanged and subtracting that equation from (1.81) gives a vector analog of Green's second identity:

$$\int_V d^3r\,[\mathbf{Q}\cdot\nabla\times\nabla\times\mathbf{P}-\mathbf{P}\cdot\nabla\times\nabla\times\mathbf{Q}]=\int_S d\mathbf{S}\cdot[\mathbf{P}\times\nabla\times\mathbf{Q}-\mathbf{Q}\times\nabla\times\mathbf{P}]. \tag{1.82}$$

## 1.4.4 Stokes' Theorem

Stokes' theorem applies to a vector function $\mathbf{F}(\mathbf{r})$ defined on an open surface $S$ bounded by a closed curve $C$. If $d\boldsymbol{\ell}$ is a line element of $C$,

$$\int_S d\mathbf{S}\cdot\nabla\times\mathbf{F}=\oint_C d\boldsymbol{\ell}\cdot\mathbf{F}. \tag{1.83}$$

The curve $C$ in (1.83) is traversed in the direction given by the right-hand rule when the thumb points in the direction of $d\mathbf{S}$. As with the divergence theorem, variations of (1.83) follow from the choices $\mathbf{F}=\mathbf{c}\psi$ and $\mathbf{F}=\mathbf{A}\times\mathbf{c}$:

$$\int_S d\mathbf{S}\times\nabla\psi=\oint_C d\boldsymbol{\ell}\,\psi \tag{1.84}$$

$$\oint_C d\boldsymbol{\ell}\times\mathbf{A}=\int_S dS_k\nabla A_k-\int_S d\mathbf{S}(\nabla\cdot\mathbf{A}). \tag{1.85}$$

## 1.4.5 The Time Derivative of a Flux Integral

Leibniz' Rule for the time derivative of a one-dimensional integral is

$$\frac{d}{dt}\int_{x_1(t)}^{x_2(t)} dx\, b(x,t)=b(x_2,t)\frac{dx_2}{dt}-b(x_1,t)\frac{dx_1}{dt}+\int_{x_1(t)}^{x_2(t)} dx\,\frac{\partial b}{\partial t}. \tag{1.86}$$

This formula generalizes to integrals over circuits, surfaces, and volumes which move through space. Our treatment of Faraday's law makes use of the time derivative of a surface integral where the surface $S(t)$ moves because its individual area elements move with velocity $\boldsymbol{v}(\mathbf{r},t)$. In that case,

$$\frac{d}{dt}\int_{S(t)} d\mathbf{S}\cdot\mathbf{B}=\int_{S(t)} d\mathbf{S}\cdot\left[\boldsymbol{v}(\nabla\cdot\mathbf{B})-\nabla\times(\boldsymbol{v}\times\mathbf{B})+\frac{\partial\mathbf{B}}{\partial t}\right]. \tag{1.87}$$

*Proof:* We calculate the change in flux from

$$\delta\left[\int\mathbf{B}\cdot d\mathbf{S}\right]=\int\delta\mathbf{B}\cdot d\mathbf{S}+\int\mathbf{B}\cdot\delta(\hat{\mathbf{n}}dS). \tag{1.88}$$

The first term on the right comes from time variations of **B**. The second term comes from time variations of the surface. Multiplication of every term in (1.88) by $1/\delta t$ gives

$$\frac{d}{dt}\int\mathbf{B}\cdot d\mathbf{S}=\int\frac{\partial\mathbf{B}}{\partial t}\cdot d\mathbf{S}+\frac{1}{\delta t}\int\mathbf{B}\cdot\delta(\hat{\mathbf{n}}dS). \tag{1.89}$$

We can focus on the second term on the right-hand side of (1.89) because the first term appears already as the last term in (1.87). Figure 1.3 shows an open surface $S(t)$ with local normal $\hat{\mathbf{n}}(t)$ which moves and/or distorts to the surface $S(t+\delta t)$ with local normal $\hat{\mathbf{n}}(t+\delta t)$ in time $\delta t$.

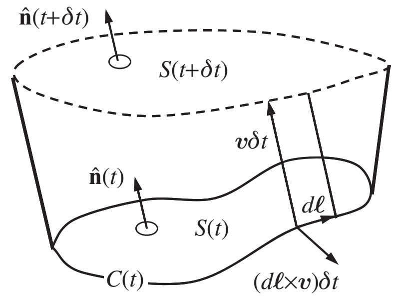



**Figure 1.3:** A surface $S(t)$ [bounded by the solid curve labeled $C(t)$] changes to the surface $S(t+\delta t)$ (dashed curve) because each element of surface moves by an amount $\boldsymbol{v}\delta t$.

Our strategy is to integrate $\nabla\cdot\mathbf{B}$ over the volume $V$ bounded by $S(t)$, $S(t+\delta t)$, and the ribbon-like surface of infinitesimal width that connects the two. Figure 1.3 shows that an area element of the latter is $d\boldsymbol{\ell}\times\boldsymbol{v}\delta t$. Therefore, using the divergence theorem,

$$\int_V d^3r\,\nabla\cdot\mathbf{B} = \int_{S(t+\delta t)} \mathbf{B}\cdot\hat{\mathbf{n}}dS - \int_{S(t)} \mathbf{B}\cdot\hat{\mathbf{n}}dS + \oint_{C(t)} \mathbf{B}\cdot(d\boldsymbol{\ell}\times\boldsymbol{v}\delta t). \tag{1.90}$$

The minus sign appears in (1.90) because the divergence theorem involves the *outward* normal to the surface bounded by $V$.

The volume integral on the left side of (1.90) can be rewritten as a surface integral over $S(t)$ because $d^3r = \hat{\mathbf{n}}\,dS\cdot\boldsymbol{v}\delta t$. In the circuit integral over $C(t)$, we use the fact that $\mathbf{B}\cdot(d\boldsymbol{\ell}\times\boldsymbol{v}) = d\boldsymbol{\ell}\cdot(\boldsymbol{v}\times\mathbf{B})$. Finally, the two surface integrals on the right side of (1.90) can be combined into one. Putting all this together transforms (1.90) to

$$\delta t\int_{S(t)} d\mathbf{S}\cdot\boldsymbol{v}(\nabla\cdot\mathbf{B}) = \int_{S(t)} \mathbf{B}\cdot\delta(\hat{\mathbf{n}}dS) + \delta t\oint_{C(t)} d\boldsymbol{\ell}\cdot(\boldsymbol{v}\times\mathbf{B}). \tag{1.91}$$

Finally, we use Stokes' theorem to write the circuit integral in (1.91) as a surface integral. This gives

$$\int_{S(t)} \mathbf{B}\cdot\delta(\hat{\mathbf{n}}dS) = \delta t\int_{S(t)} d\mathbf{S}\cdot\boldsymbol{v}(\nabla\cdot\mathbf{B}) - \delta t\int_{S(t)} d\mathbf{S}\cdot\nabla\times(\boldsymbol{v}\times\mathbf{B}). \tag{1.92}$$

Substitution of (1.92) into (1.89) produces (1.87).

## 1.5 Generalized Functions

### 1.5.1 The Delta Function in One Dimension

The one-dimensional generalized function $\delta(x)$ is defined by its "filtering" action on a smooth but otherwise arbitrary test function $f(x)$:

$$\int_{-\infty}^{\infty} dx f(x)\delta(x-x') = f(x'). \tag{1.93}$$

An informal definition consistent with (1.93) is

$$\delta(x) = 0 \quad \text{for} \quad x \neq 0 \qquad \text{but} \qquad \int_{-\infty}^{\infty} dx\, \delta(x) = 1. \tag{1.94}$$

If the variable $x$ has dimensions of length, the integrals in these equations make sense only if $\delta(x)$ has dimensions of inverse length. Note also that the integration ranges in (1.93) and (1.94) need only be large enough to include the point where the argument of the delta function vanishes.

The delta function can be understood as the limit of a sequence of functions which become more and more highly peaked at the point where its argument vanishes. Some examples are

$$\delta(x) = \lim_{m\to\infty} \frac{\sin mx}{\pi x} \tag{1.95}$$

$$\delta(x) = \lim_{m\to\infty} \frac{m}{\sqrt{\pi}} \exp(-m^2x^2) \tag{1.96}$$

$$\delta(x) = \lim_{\epsilon\to 0} \frac{\epsilon/\pi}{x^2+\epsilon^2}. \tag{1.97}$$

We prove the correctness of any of these proposed representations by showing that it possesses the filtering property (1.93). The same method is used to prove delta function identities like

$$\delta(ax) = \frac{1}{|a|}\delta(x), \qquad a \neq 0 \tag{1.98}$$

$$\int_{-\infty}^{\infty} dx f(x) \frac{d}{dx}\delta(x-x') = -\left.\frac{df}{dx}\right|_{x=x'} \tag{1.99}$$

$$\delta[g(x)] = \sum_n \frac{1}{|g'(x_n)|}\delta(x-x_n) \qquad \text{where} \qquad g(x_n)=0,\ \ g'(x_n)\neq 0 \tag{1.100}$$

$$\delta(x-x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ik(x-x')}. \tag{1.101}$$

Formula (1.101) may be read as a statement of the completeness of plane waves labeled with the continuous index $k$:

$$\psi_k(x) = \frac{1}{\sqrt{2\pi}} e^{-ikx}. \tag{1.102}$$

The general result for a complete set of normalized basis functions $\psi_n(x)$ labeled with the discrete index $n$ is[2]

$$\delta(x-x') = \sum_{n=1}^{\infty} \psi_n^*(x)\psi_n(x'). \tag{1.103}$$

**Example 1.4** Prove the identity (1.101) in the form

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx}.$$

[2] $\psi_n^*(x)$ is the complex conjugate of $\psi_n(x)$.

**Solution:** By direct calculation,

$$\int_{-\infty}^{\infty} dk\, e^{ikx} = \int_{0}^{\infty} dk\, e^{ikx} + \int_{0}^{\infty} dk\, e^{-ikx} = \lim_{\epsilon\to 0}\left[\int_{0}^{\infty} dk\, e^{ik(x+i\epsilon)} + \int_{0}^{\infty} dk\, e^{-ik(x-i\epsilon)}\right].$$

The convergence factors make the integrands zero at the upper limit, so

$$\int_{-\infty}^{\infty} dk\, e^{ikx} = \lim_{\epsilon\to 0}\left[\frac{i}{x+i\epsilon} - \frac{i}{x-i\epsilon}\right] = \lim_{\epsilon\to 0}\frac{2\epsilon}{x^2+\epsilon^2} = 2\pi\delta(x).$$

The final equality on the far right follows from (1.97).

## 1.5.2 The Principal Value Integral and Plemelj Formula

The Cauchy principal value is a generalized function defined by its action under an integral with an arbitrary function $f(x)$, namely,

$$\mathcal{P}\int_{-\infty}^{\infty} dx\,\frac{f(x)}{x-x_0} = \lim_{\epsilon\to 0}\left[\int_{-\infty}^{x_0-\epsilon} dx\,\frac{f(x)}{x-x_0} + \int_{x_0+\epsilon}^{\infty} dx\,\frac{f(x)}{x-x_0}\right]. \tag{1.104}$$

An important application where the principal value plays a role is the *Plemelj formula*:

$$\lim_{\epsilon\to 0}\frac{1}{x-x_0\pm i\epsilon} = \mathcal{P}\frac{1}{x-x_0} \mp i\pi\delta(x-x_0). \tag{1.105}$$

This expression is symbolic in the sense that it gains meaning when we multiply every term by an arbitrary function $f(x)$ and integrate over $x$ from $-\infty$ to $\infty$.

The correctness of (1.105) can be appreciated from Figure 1.4 and the identity

$$\frac{1}{x-x_0\pm i\epsilon} = \frac{x-x_0}{(x-x_0)^2+\epsilon^2} \mp i\frac{\epsilon}{(x-x_0)^2+\epsilon^2}. \tag{1.106}$$

The real part of (1.106) generates the principal value in (1.105) because it is a symmetrically cut-off version of $1/(x-x_0)$. The imaginary part of (1.106) generates the delta function in (1.105) by virtue of (1.97).

## 1.5.3 The Step Function and Sign Function

The Heaviside step function $\Theta(x)$ is defined by

$$\Theta(x) = \begin{cases} 0 & x<0, \\ 1 & x>0. \end{cases} \tag{1.107}$$

The delta function is the derivative of the theta function,

$$\frac{d\Theta(x)}{dx} = \delta(x). \tag{1.108}$$

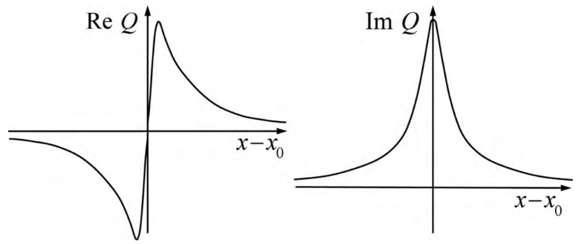



**Figure 1.4:** The real and imaginary parts of $Q(x) = 1/(x - x_0 - i\epsilon)$.

A useful representation is

$$\Theta(x) = \lim_{\epsilon \to 0} \frac{i}{2\pi} \int_{-\infty}^{\infty} dk \frac{1}{k + i\epsilon} e^{-ikx}. \tag{1.109}$$

The sign function $\mathrm{sgn}(x)$ is defined by

$$\mathrm{sgn}(x) = \frac{d}{dx}|x| = \begin{cases} -1 & x < 0, \\ 1 & x > 0. \end{cases} \tag{1.110}$$

A convenient representation is

$$\mathrm{sgn}(x) = -1 + 2 \int_{-\infty}^{x} dy\, \delta(y). \tag{1.111}$$

## 1.5.4 The Delta Function in Three Dimensions

The definition (1.93) leads us to define a three-dimensional delta function using an integral over a volume $V$ and a smooth but otherwise arbitrary "test" function $f(\mathbf{r})$:

$$\int_V d^3r\, f(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}') = \begin{cases} f(\mathbf{r}') & \mathbf{r}' \in V, \\ 0 & \mathbf{r}' \notin V. \end{cases} \tag{1.112}$$

A less formal definition consistent with (1.112) is

$$\delta(\mathbf{r}) = 0 \quad \text{for} \quad \mathbf{r} \neq 0 \qquad \text{but} \qquad \int_V d^3r\, \delta(\mathbf{r}) = \begin{cases} 1 & \mathbf{r} = 0 \in V, \\ 0 & \mathbf{r} = 0 \notin V. \end{cases} \tag{1.113}$$

These definitions tell us that $\delta(\mathbf{r})$ has dimensions of inverse volume. In Cartesian coordinates,

$$\delta(\mathbf{r}) = \delta(x)\delta(y)\delta(z). \tag{1.114}$$

In curvilinear coordinates, the constraint on the right side of (1.113) and the form of the volume elements for cylindrical and spherical coordinates imply that

$$\delta(\mathbf{r} - \mathbf{r}') = \frac{\delta(\rho - \rho')\delta(\phi - \phi')\delta(z - z')}{\rho} = \frac{\delta(r - r')\delta(\theta - \theta')\delta(\phi - \phi')}{r^2 \sin\theta}. \tag{1.115}$$

The special case $\mathbf{r}' = 0$ requires that we *define* the one-dimensional radial delta function so

$$\int_0^{\infty} dr\, \delta(r) = 1. \tag{1.116}$$

More generally, if $\mathbf{x}_0$ and $\mathbf{y}_0$ represent the same point in two different $N$-dimensional coordinate systems, we can use

$$\int d^N x\, \delta(\mathbf{x}-\mathbf{x}_0) = \int d^N y\, \delta(\mathbf{y}-\mathbf{y}_0) \tag{1.117}$$

and the Jacobian determinant result (1.75) to deduce that

$$\delta(\mathbf{x}-\mathbf{x}_0) = \frac{1}{|\mathbf{J}(\mathbf{x},\mathbf{y})|}\delta(\mathbf{y}-\mathbf{y}_0). \tag{1.118}$$

### 1.5.5 Some Useful Delta Function Identities

$$\delta(\mathbf{r}-\mathbf{r}') = \frac{1}{(2\pi)^3}\int d^3k\, e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \tag{1.119}$$

$$\int d^3r\, f(\mathbf{r})\delta[g(\mathbf{r})] = \int_S dS \frac{f(\mathbf{r}_S)}{|\nabla g(\mathbf{r}_S)|} \qquad \text{where } g(\mathbf{r}_S)=0 \text{ defines } S \tag{1.120}$$

$$\nabla^2 \frac{1}{|\mathbf{r}-\mathbf{r}'|} = -4\pi\delta(\mathbf{r}-\mathbf{r}') \tag{1.121}$$

$$\frac{\partial}{\partial r_k}\frac{\partial}{\partial r_m}\frac{1}{r} = \frac{3r_k r_m - r^2\delta_{km}}{r^5} - \frac{4\pi}{3}\delta_{ij}\delta(\mathbf{r}). \tag{1.122}$$

**Example 1.5** Use the divergence theorem to prove (1.121) in the form

$$\nabla^2\frac{1}{r} = -4\pi\delta(\mathbf{r}).$$

**Solution:** In spherical coordinates,

$$\nabla^2\frac{1}{r} = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\frac{1}{r} = 0 \quad \text{when} \quad r \neq 0.$$

To learn the behavior at $r=0$, we integrate $\nabla^2(1/r)$ over a tiny spherical volume $V$ centered at the origin. Since $d\mathbf{S} = r^2\sin\theta d\theta d\phi\hat{\mathbf{r}}$ and $\nabla(1/r) = -\hat{\mathbf{r}}/r^2$, the divergence theorem gives

$$\int_V d^3r\,\nabla^2\frac{1}{r} = \int_V d^3r\,\nabla\cdot\nabla\frac{1}{r} = \int_S d\mathbf{S}\cdot\left(-\frac{\hat{\mathbf{r}}}{r^2}\right) = -\int_0^{2\pi} d\phi\int_0^{\pi} d\theta\,\sin\theta = -4\pi.$$

In light of (1.113), these two facts taken together establish the identity.

## 1.6 Fourier Analysis

Every periodic function $f(x+L) = f(x)$ has a Fourier series representation

$$f(x) = \sum_{m=-\infty}^{\infty}\hat{f}_m e^{i2\pi mx/L}. \tag{1.123}$$

The Fourier expansion coefficients in (1.123) are given by

$$\hat{f}_m = \frac{1}{L}\int_0^L dx f(x)\, e^{-i2\pi mx/L}. \tag{1.124}$$

For non-periodic functions, the sum over integers in (1.123) becomes an integral over the real line. When the integral converges, we find the *Fourier transform* pair:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, \hat{f}(k)\, e^{ikx} \tag{1.125}$$

$$\hat{f}(k) = \int_{-\infty}^{\infty} dx\, f(x)\, e^{-ikx}. \tag{1.126}$$

If $f(x)$ happens to be a real function, it follows from these definitions that

$$f(x) = f^*(x) \;\Rightarrow\; \hat{f}(k) = \hat{f}(-k) \tag{1.127}$$

In the time domain, it is conventional to write

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \hat{g}(\omega)\, e^{-i\omega t} \qquad\qquad \hat{g}(\omega) = \int_{-\infty}^{\infty} dt\, g(t)\, e^{i\omega t}. \tag{1.128}$$

Thus, our convention for the Fourier transform and inverse Fourier transform of a function $f(\mathbf{r}, t)$ of time and all three spatial variables is

$$f(\mathbf{r}, t) = \frac{1}{(2\pi)^4}\int d^3k \int_{-\infty}^{\infty} d\omega\, \hat{f}(\mathbf{k}|\omega)\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \tag{1.129}$$

$$\hat{f}(\mathbf{k}|\omega) = \int d^3r \int_{-\infty}^{\infty} dt\, f(\mathbf{r}, t)\, e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}. \tag{1.130}$$

### 1.6.1 Parseval's Theorem

$$\int_{-\infty}^{\infty} dt g_1^*(t)\, g_2(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \hat{g}_1^*(\omega)\, \hat{g}_2(\omega). \tag{1.131}$$

The proof follows by substituting the left member of (1.128) into (1.131) for $g_1^*(t)$ and $g_1(t)$ and using the representation (1.101) of the delta function.

### 1.6.2 The Convolution Theorem

A function $h(t)$ is called the *convolution* of $f(t)$ and $g(t)$ if

$$h(t) = \int_{-\infty}^{\infty} dt' f(t-t')\, g(t'). \tag{1.132}$$

The convolution theorem states that the Fourier transforms $\hat{h}(\omega)$, $\hat{f}(\omega)$, and $\hat{g}(\omega)$ are related by

$$\hat{h}(\omega) = \hat{f}(\omega)\hat{g}(\omega). \tag{1.133}$$

We prove this assertion by using the left side of (1.128) to rewrite (1.132) as

$$h(t) = \int_{-\infty}^{\infty} dt' \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \hat{f}(\omega)\, e^{-i\omega(t-t')} \right] \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega' \hat{g}(\omega')\, e^{-i\omega' t'} \right]. \tag{1.134}$$

Rearranging terms gives

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\, \hat{f}(\omega) \int_{-\infty}^{\infty} d\omega'\, \hat{g}(\omega') \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} dt'\, e^{-i(\omega'-\omega)t'} \right]. \tag{1.135}$$

The identity (1.101) identifies the quantity in square brackets as the delta function $\delta(\omega - \omega')$. Therefore,

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\, \hat{f}(\omega)\, \hat{g}(\omega). \tag{1.136}$$

Comparing (1.136) to the definition on the left side of (1.128) completes the proof.

### 1.6.3 A Time-Averaging Theorem

Let $A(\mathbf{r}, t) = a(\mathbf{r}) \exp(-i\omega t)$ and $B(\mathbf{r}, t) = b(\mathbf{r}) \exp(-i\omega t)$, where $a(\mathbf{r})$ and $b(\mathbf{r})$ are complex-valued functions. If $T = 2\pi/\omega$, it is useful to know that

$$\langle \mathrm{Re}\,[A(\mathbf{r}, t)]\, \mathrm{Re}\,[B(\mathbf{r}, t)] \rangle = \frac{1}{T} \int_0^T dt\, \mathrm{Re}\,[A(\mathbf{r}, t)]\, \mathrm{Re}\,[B(\mathbf{r}, t)] = \tfrac{1}{2} \mathrm{Re}\,[a(\mathbf{r}) b^*(\mathbf{r})]\,. \tag{1.137}$$

We prove (1.137) by writing $\mathrm{Re}\,[A] = \frac{1}{2}(A + A^*)$ and $\mathrm{Re}\,[B] = \frac{1}{2}(B + B^*)$ so

$$\langle \mathrm{Re}\,[A]\, \mathrm{Re}\,[B] \rangle = \frac{1}{4T} \int_0^T dt \left\{ ab\, e^{-2i\omega t} + a^* b^*\, e^{2i\omega t} + ab^* + ba^* \right\}. \tag{1.138}$$

The time-dependent terms in the integrand of (1.138) integrate to zero over one full period. Therefore,

$$\langle \mathrm{Re}\,[A]\, \mathrm{Re}\,[B] \rangle = \frac{1}{4} \left[ ab^* + a^* b \right] = \frac{1}{2} \mathrm{Re} \left[ ab^* \right] = \frac{1}{2} \mathrm{Re} \left[ a^* b \right]. \tag{1.139}$$

**Example 1.6** Find the Fourier series which represents the periodic function

$$f(x) = \sum_{p=-\infty}^{\infty} \delta(x - 2\pi p).$$

**Solution:** This function is periodic in the interval $-\pi \leq x < \pi$. Therefore, using (1.124),

$$\hat{f}_m = \frac{1}{2\pi} \sum_{p=-\infty}^{\infty} \int_{-\pi}^{\pi} dx\, \delta(x - 2\pi p) e^{-imx} = \frac{1}{2\pi}.$$

Substituting $\hat{f}_m$ into (1.123) gives

$$f(x) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{imx} = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{m=1}^{\infty} \cos mx.$$

## 1.7 Orthogonal Transformations

Let $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$ and $(\hat{\mathbf{e}}'_1, \hat{\mathbf{e}}'_2, \hat{\mathbf{e}}'_3)$ be two sets of orthogonal Cartesian unit vectors. Each is a complete basis for vectors in three dimensions, so

$$\hat{\mathbf{e}}'_i = A_{ij} \hat{\mathbf{e}}_j. \tag{1.140}$$

The set of scalars $A_{ij}$ are called *direction cosines*. Using the unit vector properties from Section 1.2,

$$\delta_{ij} = \hat{\mathbf{e}}'_i \cdot \hat{\mathbf{e}}'_j = A_{ik} A_{jk}. \tag{1.141}$$

Equation (1.141) says that the transpose of the matrix $\mathbf{A}$, called $\mathbf{A}^{\mathsf{T}}$, is identical to the inverse of the matrix $\mathbf{A}$, called $\mathbf{A}^{-1}$. This is the definition of a matrix that describes an *orthogonal* transformation,

$$\mathbf{A}\mathbf{A}^{\mathsf{T}} = \mathbf{A}\mathbf{A}^{-1} = 1. \tag{1.142}$$

There are two classes of orthogonal coordinate transformations. These follow from the determinant of (1.142):

$$\det\,[\mathbf{A}\mathbf{A}^{\mathsf{T}}] = \det\mathbf{A}\, \det\mathbf{A}^{\mathsf{T}} = (\det\mathbf{A})^2 = 1. \tag{1.143}$$

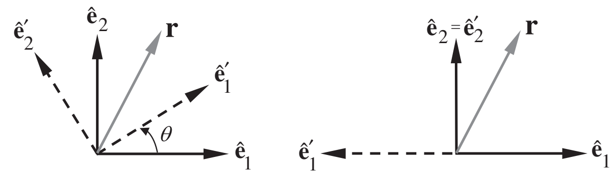
A **rotation** has $\det \mathbf{A} = 1$. Figure 1.5(a) shows an example where

$$\mathbf{A} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.144}$$

A **reflection** has $\det \mathbf{A} = -1$. Figure 1.5(b) shows an example where

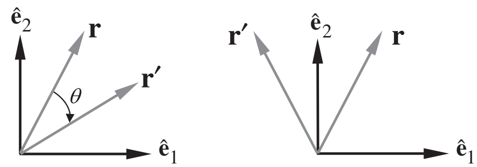
$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.145}$$

The **inversion** transformation is represented by $A_{ij} = -\delta_{ij}$ so $\det \mathbf{A} = -1$ like a reflection. However, a sequence of reflections can have $\det \mathbf{A} = 1$ like a rotation.




(a) Rotation (b) Reflection

**Figure 1.5:** Orthogonal transformations from the *passive* point of view. The Cartesian coordinate system transforms. The position vector $\mathbf{r}$ is fixed. (a) a rotation where $\det \mathbf{A} = 1$; (b) a reflection where $\det \mathbf{A} = -1$.




(a) Rotation (b) Reflection

**Figure 1.6:** Orthogonal transformations from the *active* point of view. The vector $\mathbf{r}$ transforms to the vector $\mathbf{r}'$. The Cartesian coordinate system is fixed. (a) a rotation where $\det \mathbf{A} = 1$; (b) a reflection where $\det \mathbf{A} = -1$.

## 1.7.1 Passive Point of View

Consider the position vector $\mathbf{r}$ drawn in Figure 1.5. This object can be decomposed using either the $\{\hat{\mathbf{e}}\}$ basis or the $\{\hat{\mathbf{e}}'\}$ basis:

$$\mathbf{r} = r_i \hat{\mathbf{e}}_i = r_j' \hat{\mathbf{e}}_j' = (\mathbf{r})'. \tag{1.146}$$

The notation on the far right side of (1.146) indicates that the vector $\mathbf{r}$ is represented in the primed coordinate system. Substitution of (1.140) in (1.146) shows that the components of $\mathbf{r}$ transform like the unit vectors in (1.140):

$$r_i' = A_{ij} r_j \qquad\qquad r_j = A_{kj} r_k'. \tag{1.147}$$

This description is called the *passive* point of view. The position vector $\mathbf{r}$ is a spectator fixed in space while the coordinate system transforms. The matrix $\mathbf{A}$ is regarded as an operator that transforms the components of $\mathbf{r}$ in the $\{\hat{\mathbf{e}}\}$ basis to the components of $\mathbf{r}$ in the $\{\hat{\mathbf{e}}'\}$ basis. The matrix form of the transformation connects components of the *same* vector in *different* coordinate systems:

$$(\mathbf{r})' = \mathbf{A}\mathbf{r}. \tag{1.148}$$

## 1.7.2 Active Point of View

The *active* point of view is an alternative (and equivalent) way to think about an orthogonal transformation. Here, the matrix $\mathbf{A}$ is regarded as an operator which transforms $\mathbf{r}$ to a new vector $\mathbf{r}'$ with no change in the underlying coordinate system. The matrix form of the transformation connects the components of *different* vectors in the *same* coordinate system:

$$\mathbf{r}' = \mathbf{A}\mathbf{r}. \tag{1.149}$$

Direct calculation using (1.149) confirms the active point of view illustrated in Figure 1.6 for transformations represented by the rotation matrix (1.144) and the reflection matrix (1.145). Figure 1.6(a) shows that $\mathbf{r}'$ is rotated in the *clockwise* direction with respect to $\mathbf{r}$ as compared to the *counterclockwise* rotation of the set $\{\hat{\mathbf{e}}'\}$ with respect to the set $\{\hat{\mathbf{e}}\}$ in Figure 1.5. Figure 1.6(b) shows that $\mathbf{r}'$ is the image of $\mathbf{r}$ reflected in a mirror at $x = 0$.

## 1.8 Cartesian Tensors

Tensors are mathematical objects defined by their behavior under orthogonal coordinate transformations. Physical quantities are classified as *rotational tensors* of various ranks depending on how they transform under rotations. In this section, we adopt the passive point of view (Section 1.7.1) where the Cartesian coordinate system alone transforms.

A *tensor of rank 0* is a one-component quantity where the result of a rotational transformation from the passive point of view is

$$f'(\mathbf{r}) = f(\mathbf{r}). \tag{1.150}$$

An ordinary **scalar** is a tensor of rank 0. A *tensor of rank 1* is an object whose three components transform under rotation like the three components of $\mathbf{r}$ in (1.147):

$$V_i'(\mathbf{r}) = A_{ij} V_j(\mathbf{r}). \tag{1.151}$$

A geometrical **vector** is a tensor of rank 1 because (1.151) together with (1.140) and (1.141) guarantees that [*cf.* Equation (1.5)]

$$V_i \hat{\mathbf{e}}_i = V_j' \hat{\mathbf{e}}_j'. \tag{1.152}$$

Many authors use the transformation rule (1.151) to *define* the components of a vector. In light of (1.141), a vector is characterized by the preservation of its length under a change of coordinates:

$$V_i' V_i' = A_{ij} V_j A_{ik} V_k = V_k V_k. \tag{1.153}$$

A *tensor of rank 2* is a nine-component quantity whose components transform under rotation by the rule

$$T_{ij}'(\mathbf{r}) = A_{ik} A_{jm} T_{km}(\mathbf{r}). \tag{1.154}$$

A **dyadic** is a tensor of rank 2 composed of a linear combination of pairs of juxtaposed (not multiplied) vectors. The examples we will encounter in this book all have the form[3]

$$\mathbf{T} \equiv \hat{\mathbf{e}}_i T_{ij} \hat{\mathbf{e}}_j. \tag{1.155}$$

The structure of (1.155) implies that the scalar product (or vector product) of a dyadic with a vector that stands to its left (or to its right) is also a vector. For example, the *unit dyadic* $\mathbf{I}$ has $I_{ij} = \delta_{ij}$ so $\mathbf{I} = \hat{\mathbf{e}}_i \hat{\mathbf{e}}_i$. A brief calculation confirms that the scalar product of a test vector with $\mathbf{I}$ returns the test vector:

$$\mathbf{v} \cdot \mathbf{I} = \mathbf{I} \cdot \mathbf{v} = \mathbf{v}. \tag{1.156}$$

[3] We use boldface type to denote both vectors and dyadics. Context should be sufficient to avoid confusion.

**Example 1.7** Show that the gradient operator (1.8) transforms like a vector under an orthogonal coordinate transformation.

**Solution:** The proof follows from the chain rule and the equation on the right side of (1.147). The latter shows that $A_{kj} = \partial r_j / \partial r'_k$. Therefore, if $\varphi(\mathbf{r})$ is a scalar function,

$$\left(\frac{\partial \varphi}{\partial r_k}\right)' = \frac{\partial \varphi}{\partial r'_k} = \frac{\partial \varphi}{\partial r_j}\frac{\partial r_j}{\partial r'_k} = A_{kj}\frac{\partial \varphi}{\partial r_j}.$$

This is the transformation rule (1.151) for a vector.

## 1.8.1 Reflection, Inversion and Pseudotensors

We study here the transformation properties of rotational tensors (Section 1.8) under a general orthogonal transformation $\mathbf{A}$. An instructive example is the cross product $\mathbf{m} = \mathbf{p} \times \mathbf{w}$. In component form,

$$m_i = \epsilon_{ijk} p_j w_k. \tag{1.157}$$

By direct calculation using (1.151) and the definition (1.33) of the Kronecker delta,

$$m'_i = \epsilon_{ijk} p'_j w'_k = \epsilon_{ijk} A_{js} p_s A_{k\ell} w_\ell = \epsilon_{pjk} \delta_{ip} A_{js} A_{k\ell} p_s w_\ell. \tag{1.158}$$

Using (1.141) to eliminate $\delta_{ip}$ in (1.158) gives

$$m'_i = \epsilon_{pjk} A_{iq} A_{pq} A_{js} A_{k\ell} p_s w_\ell = (\epsilon_{pjk} A_{pq} A_{js} A_{k\ell}) A_{iq} p_s w_\ell. \tag{1.159}$$

In light of (1.43), (1.159) is exactly

$$m'_i = (\det\mathbf{A})\, A_{iq} \epsilon_{qs\ell} p_s w_\ell. \tag{1.160}$$

Therefore, using (1.157),

$$m'_i = (\det \mathbf{A}) A_{iq} m_q. \tag{1.161}$$

Equation (1.161) shows that $\mathbf{m}$ transforms like an ordinary vector under rotations when $\det \mathbf{A} = 1$. An extra minus sign occurs in (1.161) when $\mathbf{A}$ corresponds to a reflection or an inversion where $\det \mathbf{A} = -1$.

More generally, any rotational vector where an explicit determinant factor appears in its transformation rule is called a "pseudovector" or an *axial vector* to distinguish it from an "ordinary" or *polar vector* where no such determinant factor appears in the transformation rule. The nature of the vector produced by the cross product of two other vectors is summarized by

$$\text{axial vector} \times \text{polar vector} = \text{polar vector} \tag{1.162}$$

$$\text{polar vector} \times \text{polar vector} = \text{axial vector} \tag{1.163}$$

$$\text{axial vector} \times \text{axial vector} = \text{axial vector}. \tag{1.164}$$

### Application 1.2 Inversion and Reflection

The position vector $\mathbf{r}$ is an ordinary polar vector because the transformation law (1.147) does not include the determinant factor in (1.161). Therefore, if the orthogonal transformation $\mathbf{A}$ corresponds to inversion, the active point of view (Section 1.7.2) tells us that the effect of inversion is

$$\mathbf{r} \to \mathbf{r}' = -\mathbf{r}. \tag{1.165}$$

By definition, any other polar vector $\mathbf{P}$ behaves the same way under inversion:

$$\mathbf{P} \to \mathbf{P}' = -\mathbf{P} \qquad \text{(inversion of a polar vector).} \tag{1.166}$$

Also by definition, an axial vector $\mathbf{Q}$ feels the effect of the determinant in (1.161). This introduces a minus sign for the case of inversion (see Section 1.7), so

$$\mathbf{Q} \to \mathbf{Q}' = \mathbf{Q} \qquad \text{(inversion of an axial vector).} \tag{1.167}$$

This idea generalizes immediately to the case of a *pseudoscalar* or a *pseudotensor* of any rank. For example, if $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are polar vectors, the triple product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is a pseudoscalar. The Levi-Cività symbol (1.34) is a third-rank pseudotensor.

The operation of mirror reflection through the $x$-$y$ plane inverts the $z$-component of the position vector so

$$(x, y, z) \to (x', y', z') = (x, y, -z). \tag{1.168}$$

A polar vector $\mathbf{P}$ behaves the same way under inversion:

$$(P_x, P_y, P_z) \to (P'_x, P'_y, P'_z) = (P_x, P_y, -P_z) \qquad \text{(reflection of a polar vector).} \tag{1.169}$$

The transformation matrix for reflection satisfies $\det \mathbf{A} = -1$ (see Section 1.7). Therefore, (1.161) dictates that the transformation law for an axial vector $\mathbf{Q}$ includes an overall factor of $-1$ (compared to a polar vector). In other words,

$$(Q_x, Q_y, Q_z) \to (Q'_x, Q'_y, Q'_z) = (-Q_x, -Q_y, Q_z) \qquad \text{(reflection of an axial vector).} \tag{1.170}$$

■

**Example 1.8** Show that the magnetic field $\mathbf{B}$ is a pseudovector.

**Solution:** By definition, the position vector $\mathbf{r}$ is a polar vector. Therefore, so are the velocity $\mathbf{v} = d\mathbf{r}/dt$ and the current density $\mathbf{j} = \rho\mathbf{v}$. The gradient operator $\nabla = \partial/\partial\mathbf{r}$ transforms like $\mathbf{r}$ under inversion or reflection, so it is also a polar vector. Finally, Ampère's law tells us that $\nabla \times \mathbf{B} = \mu_0\mathbf{j}$. Since $\nabla$ and $\mathbf{j}$ are polar vectors, we conclude from (1.162) that $\mathbf{B}$ is an axial vector.

## 1.9 The Helmholtz Theorem

**Statement:**
An arbitrary vector field $\mathbf{C}(\mathbf{r})$ can always be decomposed into the sum of two vector fields: one with zero divergence and one with zero curl. Specifically,

$$\mathbf{C} = \mathbf{C}_\perp + \mathbf{C}_\parallel, \tag{1.171}$$

where

$$\nabla \cdot \mathbf{C}_\perp = 0 \qquad \text{and} \qquad \nabla \times \mathbf{C}_\parallel = 0. \tag{1.172}$$

An explicit representation of special interest is

$$\mathbf{C}(\mathbf{r}) = \nabla \times \mathbf{F}(\mathbf{r}) - \nabla\Omega(\mathbf{r}). \tag{1.173}$$

When the following integrals over all space converge, $\Omega(\mathbf{r})$ and $\mathbf{F}(\mathbf{r})$ are uniquely given by

$$\Omega(\mathbf{r}) = \frac{1}{4\pi}\int d^3r' \, \frac{\nabla' \cdot \mathbf{C}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \tag{1.174}$$

and

$$\mathbf{F}(\mathbf{r}) = \frac{1}{4\pi}\int d^3r' \frac{\nabla' \times \mathbf{C}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \tag{1.175}$$

This result is valid for both static and time-dependent vector fields.

**Existence:**
The delta function properties (1.112) and (1.121) imply that

$$\mathbf{C}(\mathbf{r}) = \int d^3r' \mathbf{C}(\mathbf{r}')\,\delta(\mathbf{r}-\mathbf{r}') = -\frac{1}{4\pi}\int d^3r'\,\mathbf{C}(\mathbf{r}')\,\nabla^2 \frac{1}{|\mathbf{r}-\mathbf{r}'|}. \tag{1.176}$$

Exchanging $\mathbf{C}(\mathbf{r}')$ and $\nabla^2$ in the last term and using the double-curl identity in Example 1.3 gives

$$\mathbf{C}(\mathbf{r}) = -\frac{1}{4\pi}\nabla\int d^3r'\,\nabla\cdot\left[\frac{\mathbf{C}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right] + \frac{1}{4\pi}\nabla\times\int d^3r'\,\nabla\times\left[\frac{\mathbf{C}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right]. \tag{1.177}$$

Now, since $\nabla f(|\mathbf{r}-\mathbf{r}'|) = -\nabla' f(|\mathbf{r}-\mathbf{r}'|)$, we deduce that

$$\nabla'\cdot\left[\frac{\mathbf{C}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right] = \frac{\nabla'\cdot\mathbf{C}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} - \mathbf{C}(\mathbf{r}')\cdot\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}. \tag{1.178}$$

Moving $\mathbf{C}(\mathbf{r}')$ to the right of $\nabla$ in the last term and rearranging gives

$$\nabla\cdot\left[\frac{\mathbf{C}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right] = \frac{\nabla'\cdot\mathbf{C}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} - \nabla'\cdot\frac{\mathbf{C}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \tag{1.179}$$

In exactly the same way,

$$\nabla\times\left[\frac{\mathbf{C}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right] = \frac{\nabla'\times\mathbf{C}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} - \nabla'\times\frac{\mathbf{C}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \tag{1.180}$$

Inserting (1.179) and (1.180) into (1.177) generates four terms:

$$\begin{aligned}\mathbf{C}(\mathbf{r}) = &-\frac{1}{4\pi}\nabla\int d^3r'\,\frac{\nabla'\cdot\mathbf{C}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} + \frac{1}{4\pi}\nabla\times\int d^3r'\,\frac{\nabla'\times\mathbf{C}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\\ &+\frac{1}{4\pi}\nabla\int d^3r'\,\nabla'\cdot\left[\frac{\mathbf{C}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right] - \frac{1}{4\pi}\nabla\times\int d^3r'\,\nabla'\times\left[\frac{\mathbf{C}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right].\end{aligned} \tag{1.181}$$

The last two integrals in (1.181) are *zero* when the first two integrals converge. This follows from (1.76) and (1.78), which show that the divergence theorem transforms the volume integrals in the last two terms in (1.181) into the surface integrals

$$\int d\mathbf{S}'\cdot\frac{\mathbf{C}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \qquad \text{and} \qquad \int d\mathbf{S}'\times\frac{\mathbf{C}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \tag{1.182}$$

The surface of integration for both integrals in (1.182) lies at infinity. Therefore, both integrals vanish if $\mathbf{C}(\mathbf{r})$ goes to zero faster than $1/r$ as $r \to \infty$. The same condition guarantees that the integrals in the first two terms in (1.181) converge. The final result is exactly the representation of $\mathbf{C}(\mathbf{r})$ given in the statement of the theorem.

**Uniqueness:**
Suppose that $\nabla\cdot\mathbf{C}_1 = \nabla\cdot\mathbf{C}_2$ and $\nabla\times\mathbf{C}_1 = \nabla\times\mathbf{C}_2$. Then, if $\mathbf{W} = \mathbf{C}_1 - \mathbf{C}_2$, we have $\nabla\cdot\mathbf{W} = 0$, $\nabla\times\mathbf{W} = 0$. The double-curl identity in Example 1.3 tells us that $\nabla^2\mathbf{W} = 0$ also. With this information, Green's first identity (1.79) with $\phi = \psi = W$ ($W$ is any Cartesian component of $\mathbf{W}$) takes the form

$$\int_V d^3r\,|\nabla W|^2 = \int_S d\mathbf{S}\cdot W\nabla W. \tag{1.183}$$

The surface integral on the right side of (1.183) goes to zero when $\mathbf{C}(\mathbf{r})$ behaves at infinity as indicated above. Therefore, $\nabla W = 0$ or $W = const$. But $W \to 0$ at infinity so $W = 0$ or $\mathbf{C}_1 = \mathbf{C}_2$ as required.

## 1.10 Lagrange Multipliers

Suppose we wish to minimize (or maximize) a function of two variables $f(x, y)$. The rules of calculus tell us to set the total differential equal to zero:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0. \tag{1.184}$$

But $dx$ and $dy$ are arbitrary, so (1.184) implies that

$$\frac{\partial f}{\partial x} = 0 \qquad \text{and} \qquad \frac{\partial f}{\partial y} = 0. \tag{1.185}$$

Equation (1.185) is the correct requirement for an extremum if $x$ and $y$ are independent variables. However, suppose the two variables are constrained by the equation

$$g(x, y) = const. \tag{1.186}$$

Equation (1.186) implies that

$$dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy = 0. \tag{1.187}$$

Therefore, (1.184) and (1.187) together tell us that

$$\frac{\partial f/\partial x}{\partial g/\partial x} = \frac{\partial f/\partial y}{\partial g/\partial y} = \lambda. \tag{1.188}$$

The constant $\lambda$ appears because the two ratios in (1.188) cannot otherwise be equal for all values of $x$ and $y$. In other words,

$$\frac{\partial f}{\partial x} - \lambda\frac{\partial g}{\partial x} = 0 \qquad \text{and} \qquad \frac{\partial f}{\partial y} - \lambda\frac{\partial g}{\partial y} = 0. \tag{1.189}$$

These are the equations we would have gotten in the first place by trying to extremize, *without constraint*, the function

$$F(x, y) = f(x, y) - \lambda g(x, y). \tag{1.190}$$

The *Lagrange constant* $\lambda$ is not determined (nor does it need to be) by this procedure. Its value can be adjusted to fix the constant in (1.186) if desired.

## Sources, References, and Additional Reading

The quotation at the beginning of the chapter is from

E.P. Wigner, "The unreasonable effectiveness of mathematics in the natural sciences", *Communications on Pure and Applied Mathematics* **13**, 1 (1960).

**Section 1.1** Most of the material in this chapter is discussed in general treatments of mathematical physics. Three textbooks with rather different styles are

J. Mathews and R.L. Walker, *Mathematical Methods of Physics* (Benjamin/Cummings, Menlo Park, CA, 1970).

G. Arfken, *Mathematical Methods for Physicists*, 3rd edition (Academic, San Diego, 1985).

M. Stone and P. Goldbart, *Mathematics for Physics* (University Press, Cambridge, 2009).

Four textbooks of electromagnetism with particularly complete treatments of the mathematical preliminaries are

B. Podolsky and K.S. Kunz, *Fundamentals of Electrodynamics* (Marcel Dekker, New York, 1969).

W. Hauser, *Introduction to the Principles of Electromagnetism* (Addison-Wesley, Reading, MA, 1971).

R.H. Good, Jr. and T.J. Nelson, *Classical Theory of Electric and Magnetic Fields* (Academic, New York, 1971).

A.M. Portis, *Electromagnetic Fields: Sources and Media* (Wiley, New York, 1978).

**Section 1.2** Two readable textbooks of vector analysis and vector calculus are

D.E. Bourne and P.C. Kendall, *Vector Analysis* (Oldbourne, London, 1967).

H.F. Davis and A. D. Snider, *Introduction to Vector Analysis*, 7th edition (William C. Brown, Dubuque, IA, 1995).

**Section 1.4** Our proof of the flux theorem in Section 1.4.5 is adapted from

M. Abraham and R. Becker, *The Classical Theory of Electricity and Magnetism* (Blackie, London, 1932), pp. 39-40.

**Section 1.5** *Lighthill* and *Barton* discuss the delta function and other generalized functions with clarity and precision.

M.J. Lighthill, *Fourier Analysis and Generalized Functions* (University Press, Cambridge, 1964).

G. Barton, *Elements of Green's Functions and Propagation* (Clarendon, Oxford, 1989).

*Frayn* and *Hnizdo* discuss delta function identities with less and more rigor, respectively, in

C.P. Frayn, "Some novel delta function identities", *American Journal of Physics* **51**, 826 (1983).

V. Hnizdo, "Generalized second-order partial derivatives of $1/r$", *European Journal of Physics* **32**, 287 (2011).

**Section 1.6** A good, short introduction to Fourier transforms and their applications to physics is

T. Schücker, *Distributions, Fourier Transforms and Some of Their Applications to Physics* (World Scientific, Singapore, 1991).

**Section 1.7** The distinction between the passive and active points of view for an orthogonal transformation is made in all editions of

H. Goldstein, *Classical Mechanics* (Addison-Wesley, Cambridge, MA, 1950).

**Section 1.8** *Hauser* (see Section 1.1 above) is particularly good on Cartesian tensors. See also

J. Rosen, "Transformation properties of electromagnetic quantities under space inversion, time reversal, and charge conjugation", *American Journal of Physics* **41**, 586 (1973).

**Section 1.9** For more on the Helmholtz theorem, see

R.B. McQuistan, *Scalar and Vector Fields: A Physical Interpretation* (Wiley, New York, 1965)

J. Van Bladel, "A discussion of Helmholtz' theorem", *Electromagnetics* **13**, 95 (1993).

*Van Bladel* discusses a generalization of the Helmholtz decomposition theorem to the case when $\mathbf{C}(\mathbf{r}) \to 0$ does not go to zero more rapidly than $1/r$ as $r \to \infty$.

# Problems

## 1.1 Levi-Cività Practice I

(a) Let $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_2)$ be unit vectors of a right-handed, orthogonal coordinate system. Show that the Levi-Cività symbol satisfies

$$\epsilon_{ijk} = \hat{\mathbf{e}}_i \cdot (\hat{\mathbf{e}}_j \times \hat{\mathbf{e}}_k).$$

(b) Prove that

$$\mathbf{a} \times \mathbf{b} = \det \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_2 \end{vmatrix} = \epsilon_{ijk}\hat{\mathbf{e}}_i a_j b_k.$$

(c) Prove that $\epsilon_{ijk}\epsilon_{ist} = \delta_{js}\delta_{kt} - \delta_{jt}\delta_{ks}$.

(d) In quantum mechanics, the Cartesian components of the angular momentum operator $\hat{\mathbf{L}}$ obey the commutation relation $[\hat{L}_i, \hat{L}_j] = i\hbar\epsilon_{ijk}\hat{L}_k$. Let **a** and **b** be constant vectors and prove the commutator identity

$$[\hat{\mathbf{L}} \cdot \mathbf{a}, \hat{\mathbf{L}} \cdot \mathbf{b}] = i\hbar\hat{\mathbf{L}} \cdot (\mathbf{a} \times \mathbf{b}).$$

**1.2 Levi-Cività Practice II** Evaluate the following expressions which exploit the Einstein summation convention.

(a) $\delta_{ii}$
(b) $\delta_{ij}\epsilon_{ijk}$
(c) $\epsilon_{ijk}\epsilon_{\ell jk}$
(d) $\epsilon_{ijk}\epsilon_{ijk}$

**1.3 Vector Identities** Use the Levi-Cività symbol to prove that

(a) $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C})$.
(b) $\nabla \cdot (\mathbf{f} \times \mathbf{g}) = \mathbf{g} \cdot (\nabla \times \mathbf{f}) - \mathbf{f} \cdot (\nabla \times \mathbf{g})$.
(c) $(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C} \times \mathbf{D})\mathbf{B} - (\mathbf{B} \cdot \mathbf{C} \times \mathbf{D})\mathbf{A}$.
(d) The $2 \times 2$ Pauli matrices $\sigma_x$, $\sigma_y$, and $\sigma_z$ used in quantum mechanics satisfy $\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$. If **a** and **b** are ordinary vectors, prove that $(\boldsymbol{\sigma} \cdot \mathbf{a})(\boldsymbol{\sigma} \cdot \mathbf{b}) = \mathbf{a} \cdot \mathbf{b} + i\boldsymbol{\sigma} \cdot (\mathbf{a} \times \mathbf{b})$.

**1.4 Vector Derivative Identities** Use the Levi-Cività symbol to prove that

(a) $\nabla \cdot (f\mathbf{g}) = f\nabla \cdot \mathbf{g} + \mathbf{g} \cdot \nabla f$.
(b) $\nabla \times (f\mathbf{g}) = f\nabla \times \mathbf{g} - \mathbf{g} \times \nabla f$.
(c) $\nabla \times (\mathbf{g} \times \mathbf{r}) = 2\mathbf{g} + r\dfrac{\partial \mathbf{g}}{\partial r} - \mathbf{r}(\nabla \cdot \mathbf{g})$.

**1.5 Delta Function Identities** A test function as part of the integrand is required to prove any delta function identity. With this in mind,

(a) Prove that $\delta(ax) = \dfrac{1}{|a|}\delta(x), \quad a \neq 0$.
(b) Use the identity in part (a) to prove that

$$\delta[g(x)] = \sum_m \frac{1}{|g'(x_m)|}\delta(x - x_m), \qquad \text{where} \quad g(x_m) = 0 \text{ and } g'(x_m) \neq 0.$$

(c) Confirm that

$$I = \int_0^\infty dx\, \delta(\cos x) \exp(-x) = \frac{1}{2\sinh(\pi/2)}.$$

**1.6 Radial Delta Functions**

(a) Show that $\delta(r)/r = -\delta'(r)$ when it appears as part of the integrand of a three-dimensional integral in spherical coordinates. Convince yourself that the test function $f(r)$ does not provide any information. Then try $f(r)/r$.
(b) Show that $\nabla \cdot [\delta(r - a)\hat{\mathbf{r}}] = (a^2/r^2)\delta'(r - a)$ when it appears as part of the integrand of a three-dimensional integral in spherical coordinates.

**1.7 A Representation of the Delta Function** Show that $D(x) = \lim\limits_{m\to\infty} \dfrac{\sin mx}{\pi x}$ is a representation of $\delta(x)$ by showing that $\int_{-\infty}^{\infty} dx f(x)D(x) = f(0)$.

**1.8 An Application of Stokes' Theorem** Without using vector identities,

(a) use Stokes' Theorem $\int d\mathbf{S}\cdot(\nabla\times\mathbf{A}) = \oint d\mathbf{s}\cdot\mathbf{A}$ with $\mathbf{A}=\mathbf{c}\times\mathbf{F}$ where $\mathbf{c}$ is an arbitrary constant vector to establish the equality on the left side of

$$\oint_C d\mathbf{s}\times\mathbf{F} = \int_S dS\,\{\hat{n}_i\nabla F_i - \hat{\mathbf{n}}(\nabla\cdot\mathbf{F})\} = \int_S dS\,(\hat{\mathbf{n}}\times\nabla)\times\mathbf{F}.$$

(b) confirm the equality on the right side of this expression.
(c) show that $\oint\limits_C \mathbf{r}\times d\mathbf{s} = 2\int\limits_S d\mathbf{S}$.

**1.9 Three Derivative Identities** Without using vector identities, prove that

(a) $\nabla f(\mathbf{r}-\mathbf{r}') = -\nabla' f(\mathbf{r}-\mathbf{r}')$.
(b) $\nabla\cdot[\mathbf{A}(r)\times\mathbf{r}] = 0$.
(c) $d\mathbf{Q} = (d\mathbf{s}\cdot\nabla)\mathbf{Q}$ where $d\mathbf{Q}$ is a differential change in $\mathbf{Q}$ and $d\mathbf{s}$ is an element of arc length.

**1.10 Derivatives of** $\exp(i\mathbf{k}\cdot\mathbf{r})$ Let $\mathbf{A}(\mathbf{r}) = \mathbf{c}\exp(i\mathbf{k}\cdot\mathbf{r})$ where $\mathbf{c}$ is constant. Show that, in every case, the replacement $\nabla\to i\mathbf{k}$ produces the correct answer for $\nabla\cdot\mathbf{A}$, $\nabla\times\mathbf{A}$, $\nabla\times(\nabla\times\mathbf{A})$, $\nabla(\nabla\cdot\mathbf{A})$, and $\nabla^2\mathbf{A}$.

**1.11 Some Integral Identities** Assume that $\varphi(\mathbf{r})$ and $|\mathbf{G}(\mathbf{r})|$ both go to zero faster than $1/r$ as $r\to\infty$.

(a) Let $\mathbf{F}=\nabla\varphi$ and $\nabla\cdot\mathbf{G}=0$. Show that $\int d^3r\,\mathbf{F}\cdot\mathbf{G}=0$.
(b) Let $\mathbf{F}=\nabla\varphi$ and $\nabla\times\mathbf{G}=0$. Show that $\int d^3r\,\mathbf{F}\times\mathbf{G}=0$.
(c) Begin with the vector with components $\partial_j(P_j\mathbf{G})$ and prove that

$$\int\limits_V d^3r\,\mathbf{P} = -\int\limits_V d^3r\,\mathbf{r}(\nabla\cdot\mathbf{P}) + \int\limits_S dS(\hat{\mathbf{n}}\cdot\mathbf{P})\mathbf{r}.$$

**1.12 Unit Vector Practice** Express the following in terms of $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$:

$$\frac{\partial\hat{\mathbf{r}}}{\partial\theta}\qquad\frac{\partial\hat{\mathbf{r}}}{\partial\phi}\qquad\frac{\partial\hat{\boldsymbol{\theta}}}{\partial\theta}\qquad\frac{\partial\hat{\boldsymbol{\theta}}}{\partial\phi}\qquad\frac{\partial\hat{\boldsymbol{\phi}}}{\partial\theta}\qquad\frac{\partial\hat{\boldsymbol{\phi}}}{\partial\phi}$$

**1.13 Compute the Normal Vector** Compute the unit normal vector $\hat{\mathbf{n}}$ to the ellipsoidal surfaces defined by constant values of

$$\Phi(x,y,z) = V\left[\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right].$$

Check that you get the expected answer when $a=b=c$.

**1.14 A Variant of the Helmholtz Theorem I** Mimic the proof of Helmholtz' theorem in the text and prove that

$$\varphi(\mathbf{r}) = -\nabla\cdot\frac{1}{4\pi}\int\limits_V d^3r'\,\frac{\nabla'\varphi(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} + \nabla\cdot\frac{1}{4\pi}\int\limits_S d\mathbf{S}'\,\frac{\varphi(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}.$$

**1.15 A Variant of the Helmholtz Theorem II** A vector function $\mathbf{Z}(\mathbf{r})$ satisfies $\nabla\cdot\mathbf{Z}=0$ and $\nabla\times\mathbf{Z}=0$ everywhere in a simply-connected volume $V$ bounded by a surface $S$. Modify the proof of the Helmholtz theorem in the text and show that $\mathbf{Z}(\mathbf{r})$ can be found everywhere in $V$ if its value is specified at every point on $S$.

**1.16 Densities of States** Let $E$ be a positive real number. Evaluate

(a) $g_1(E) = \int\limits_{-\infty}^{\infty} dk_x\, \delta(E - k_x^2)$.

(b) $g_2(E) = \int\limits_{-\infty}^{\infty} dk_x \int\limits_{-\infty}^{\infty} dk_y\, \delta(E - k_x^2 - k_y^2)$.

(c) $g_3(E) = \int\limits_{-\infty}^{\infty} dk_x \int\limits_{-\infty}^{\infty} dk_y \int\limits_{-\infty}^{\infty} dk_z\, \delta(E - k_x^2 - k_y^2 - k_z^2)$.

**1.17 Dot and Cross Products** Let $\mathbf{b}$ be a vector and $\hat{\mathbf{n}}$ a unit vector.

(a) Use the Levi-Cività symbol to prove that $\mathbf{b} = (\mathbf{b} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} + \hat{\mathbf{n}} \times (\mathbf{b} \times \hat{\mathbf{n}})$.
(b) Interpret the decomposition in part (a) geometrically.
(c) Let $\omega = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ where $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are any three non-coplanar vectors. Now let

$$\mathbf{A} = \frac{\mathbf{b} \times \mathbf{c}}{\omega} \qquad \mathbf{B} = \frac{\mathbf{c} \times \mathbf{a}}{\omega} \qquad \mathbf{C} = \frac{\mathbf{a} \times \mathbf{b}}{\omega}.$$

Express $\Omega = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ entirely in terms of $\omega$.

**1.18 $S_{ij}$ and $T_{ij}$**

(a) What property must $S_{ij}$ have if $\epsilon_{ijk} S_{ij} = 0$?
(b) Let $\mathbf{b}$ and $\mathbf{y}$ be vectors. The components of the latter are defined by $y_i = b_k T_{ki}$ where $T_{ij} = -T_{ji}$ is an anti-symmetric object. Find a vector $\boldsymbol{\omega}$ such that $\mathbf{y} = \mathbf{b} \times \boldsymbol{\omega}$. Why does it makes sense that $\mathbf{T}$ and $\boldsymbol{\omega}$ could have the same information content?

**1.19 Two Surface Integrals** Let $S$ be the surface that bounds a volume $V$. Show that (a) $\int\limits_S d\mathbf{S} = 0$; (b) $\frac{1}{3}\int\limits_S d\mathbf{S} \cdot \mathbf{r} = V$.

**1.20 Electrostatic Dot and Cross Products** If $\mathbf{a}$ and $\mathbf{b}$ are constant vectors, $\varphi(\mathbf{r}) = (\mathbf{a} \times \mathbf{r}) \cdot (\mathbf{b} \times \mathbf{r})$ is the electrostatic potential in some region of space. Find the electric field $\mathbf{E} = -\nabla\varphi$ and then the charge density $\rho = \epsilon_0 \nabla \cdot \mathbf{E}$ associated with this potential.

**1.21 A Decomposition Identity** Let $\mathbf{A}$ and $\mathbf{B}$ be vectors. Show that

$$A_i B_j = \frac{1}{2}\epsilon_{ijk}(\mathbf{A} \times \mathbf{B})_k + \frac{1}{2}(A_i B_j + A_j B_i).$$