§ 7.4 Sinusoidal grating diffraction

Diffraction grating

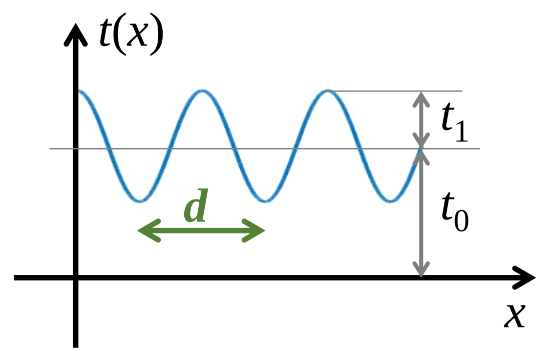
- A diffraction screen that produces a periodic change in the phase, amplitude, or both of the exiting lightwaves.

Grating classification (different point of view):

Transmission or reflection type;
Amplitude or phase type;
Rectangular, sinusoidal ...

- The screen function of a sinusoidal (cosinoidal) diffraction grating is

$$t(x) = t_0 + t_1 \cos 2\pi f x_0$$



Grating period d
spatial frequency $f = 1/d$

Incidence $\tilde{E}_{\mathrm{i}} = A$, transmission wave on the back surface of the grating is:

$$\tilde{E}(x_0) = \tilde{E}_{\mathrm{i}}(x_0)\tilde{t}(x_0) = A\left[t_0 + t_1 \cos 2\pi f x_0\right]$$

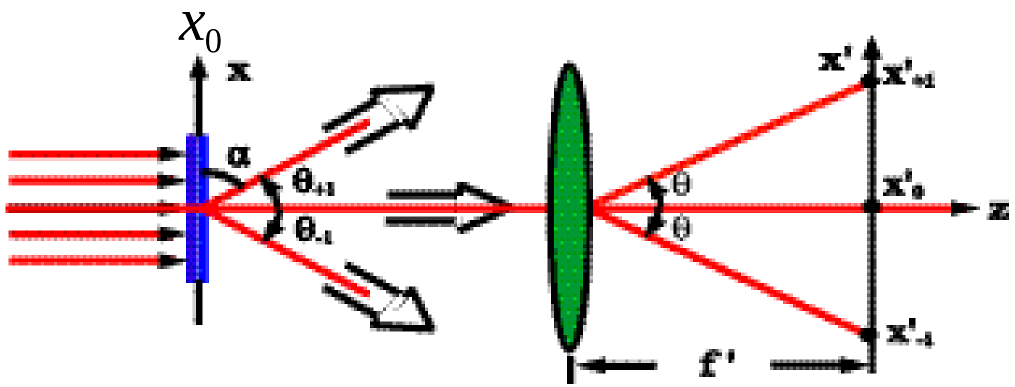
$$= \frac{A}{2} t_1 \left(e^{\mathrm{i}2\pi f x_0} + e^{-\mathrm{i}2\pi f x_0}\right) + A t_0$$

$$\tilde{E}(f_x, f_y) = \tilde{C} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{t}(x_0, y_0) \mathrm{e}^{-i2\pi(f_x x_0 + f_y y_0)} \mathrm{d}x_0 \mathrm{d}y_0$$

$$\tilde{E}(f_x, f_y) = \tilde{C} A \frac{t_1}{2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(e^{\mathrm{i}2\pi(f - f_x)x_0} + e^{-\mathrm{i}2\pi(f + f_x)x_0}\right) \mathrm{e}^{-i2\pi f_y y_0} \mathrm{d}x_0 \mathrm{d}y_0$$

$$+ \tilde{C} A t_0 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-i2\pi(f_x x_0 + f_y y_0)} \mathrm{d}x_0 \mathrm{d}y_0$$

$$\begin{cases} f_x = f \\ f_x = -f \\ f_x = 0 \end{cases}$$



three spot

$$\tilde{E}\left(x_0\right)=\frac{A}{2}t_1\left(e^{\mathrm{i}2\pi f x_0}+e^{-\mathrm{i}2\pi f x_0}\right)+At_0$$

The diffraction angles of the three columns of plane waves are:

$\sin\theta=0$ 0^{th} order

$\sin\theta_{+1}=\lambda f=\dfrac{\lambda}{d}$ $+1^{\mathrm{st}}$ order

$\sin\theta_{-1}=-\lambda f=-\dfrac{\lambda}{d}$ -1^{st} order

$$f_x=\frac{k_x}{2\pi}=\frac{k}{2\pi}\sin\theta=\frac{\sin\theta}{\lambda}$$

The information of the diffraction grating can be divided into three categories: $f \equiv 1/d$

$f \ll \dfrac{1}{\lambda}$ Low frequency

$f \leq \dfrac{1}{\lambda}$ High frequency

$f > \dfrac{1}{\lambda}$ Ultra high frequency

when $d \approx \lambda$, the scalar diffraction theory no longer applies. The near-field component become important.

$$\Rightarrow \quad \sin\theta = \frac{\lambda}{d} \approx 1 \qquad \theta \approx 90^\circ$$

When $d < \lambda$, there is a evanescent wave

$$k_z = k\sqrt{1-\sin^2\theta} = k\sqrt{1-\left(\lambda/d\right)^2} = \mathrm{i}\Lambda$$

The evanescent wave only exists in the near-field region and cannot propagate to the observation screen, that is, it cannot transport information with a spatial frequency greater than $1/\lambda$ to the far field.

The upper and lower limits of the integral equivalent to the Fourier transform are changed from $\pm\infty$ to $\pm 1/\lambda$.

This is the physical mechanisms of the **diffraction limit** (see the Rayleigh criterion).

⇨ Optical microscope resolution ~ 0.5 μm

Electron microscope resolution ~ 0.1 nm

§ 7.5 Fraunhofer diffraction: Single slit

- The intensity distribution in the diffracted field can be discussed by the integral method, the half-period zone, and the phasor.

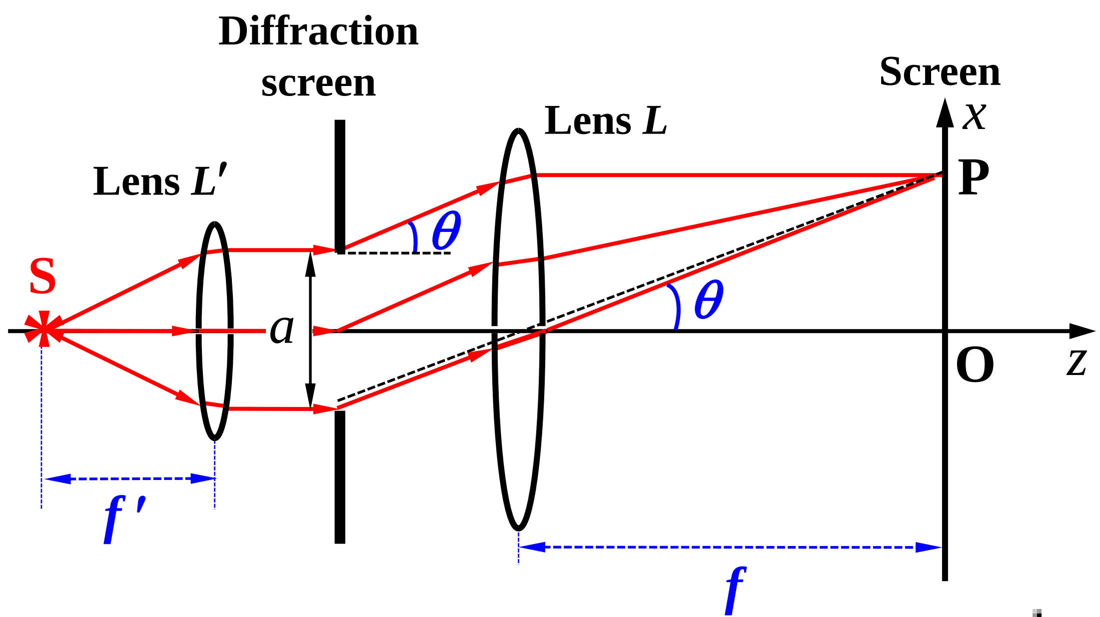
- The Fraunhofer diffraction pattern on the viewing screen visually reflects the spatial spectrum of the light field distribution at the diffraction screen.

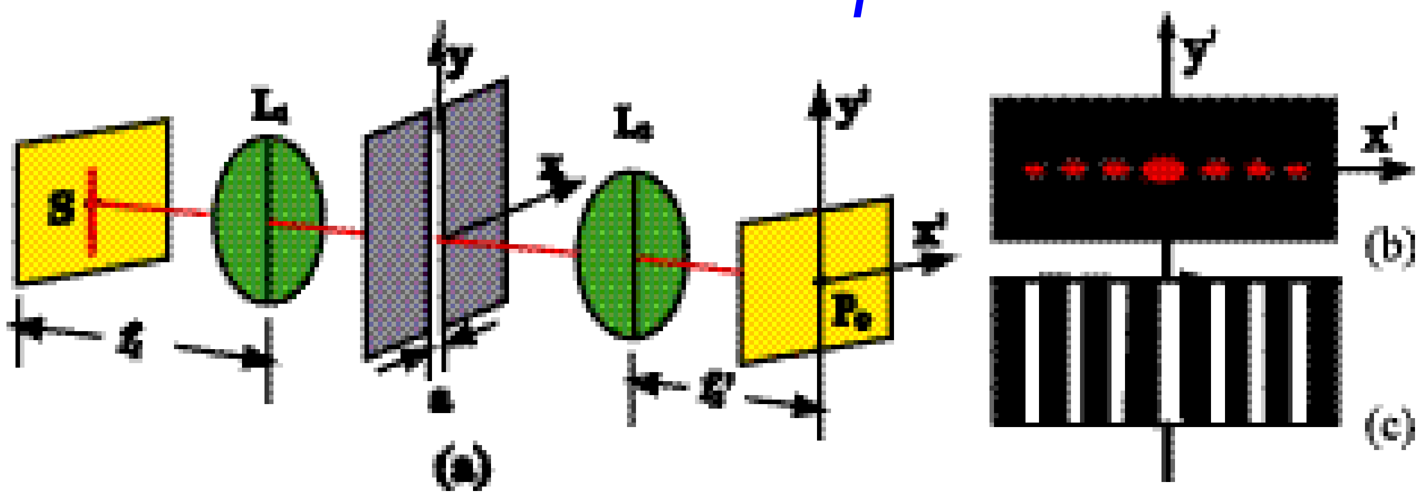
$$\underline{\tilde{E}\left(f_x, f_y\right)} = \tilde{C}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\underline{\tilde{t}\left(x_0, y_0\right)}\,\mathrm{e}^{-i2\pi\left(f_x x_0 + f_y y_0\right)}\mathrm{d}x_0\mathrm{d}y_0$$

is the spectral function of the screen function $\tilde{t}\left(x_0, y_0\right)$, reflecting the structure of the diffraction screen.

Demonstration
（single、double、many、grating、 Circular aperture）

Experiment setup





Intensity distribution

① Integral method

$$\tilde{E}(f_x, f_y) = \tilde{C} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{t}(x_0, y_0) \mathrm{e}^{-i2\pi(f_x x_0 + f_y y_0)} \mathrm{d}x_0 \mathrm{d}y_0$$

- slit： l (length of the slit)>> a (width of the slit), 2D->1D

$$\tilde{t}(x_0) = \begin{cases} 1, & |x_0| \le a/2 \\ 0, & \text{other} \end{cases}$$

$$f_x = \frac{x}{z\lambda} = \frac{k_x}{2\pi} = \frac{k}{2\pi}\sin\theta \approx \frac{k}{2\pi}\tan\theta$$

$$\Longrightarrow \quad \tilde{E}(x) = \tilde{C} \int_{-a/2}^{a/2} \mathrm{e}^{-ik\sin\theta x_0} \mathrm{d}x_0$$

$$-2i\sin\alpha = \left(\mathrm{e}^{-i\alpha} - \mathrm{e}^{i\alpha}\right)$$

- That is

$$\tilde{E}(\theta) = \frac{\tilde{C}}{-ik\sin\theta}\left(\mathrm{e}^{-i\underbrace{ka\sin\theta/2}_{\alpha}} - \mathrm{e}^{ika\sin\theta/2}\right)$$

$$= \tilde{C}a\frac{\sin\alpha}{\alpha}$$

Intensity distribution

$$\tilde{E}(\theta)=\tilde{C}a\frac{\sin\alpha}{\alpha} \Rightarrow \boxed{I(\theta)=\left|\tilde{A}_0\right|^2\left(\frac{\sin\alpha}{\alpha}\right)^2} \qquad \alpha=\frac{ka\sin\theta}{2}$$

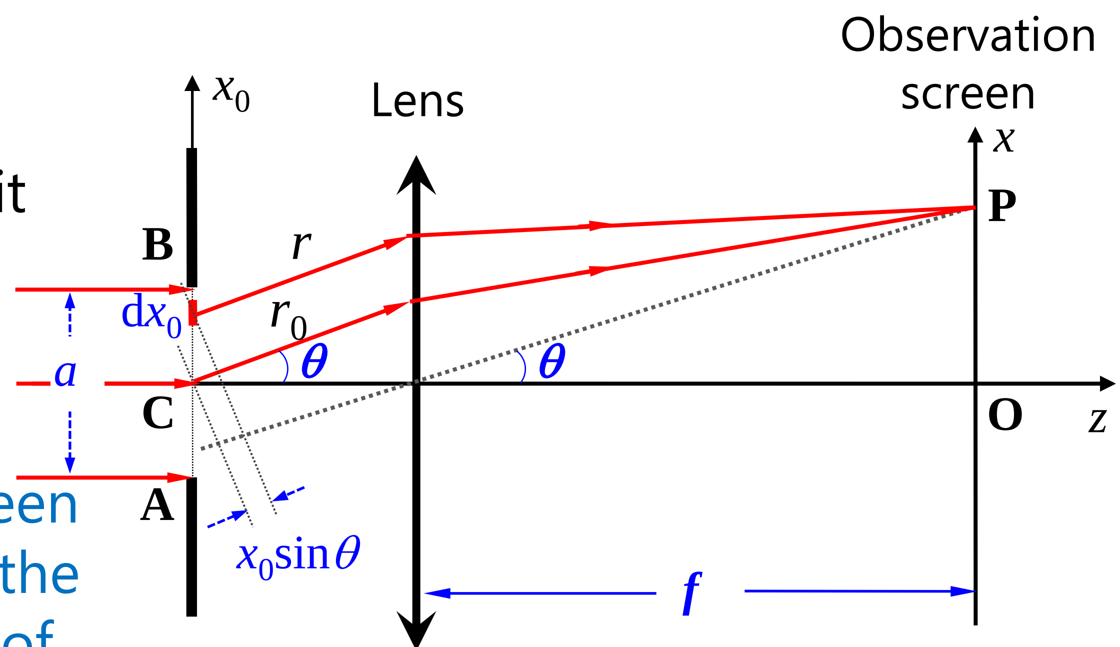
- Considering a slit unit with width $\mathrm{d}x_0$, the distance from point C to slit is x_0, the OPL difference between x_0-P and CP is

$$\Delta l = r_0 - r = x_0\sin\theta$$

At the edge of the slit

$$|x_0| = a/2$$

- α is the phase difference at P between the lightwaves from the edge and the cental of the slit.

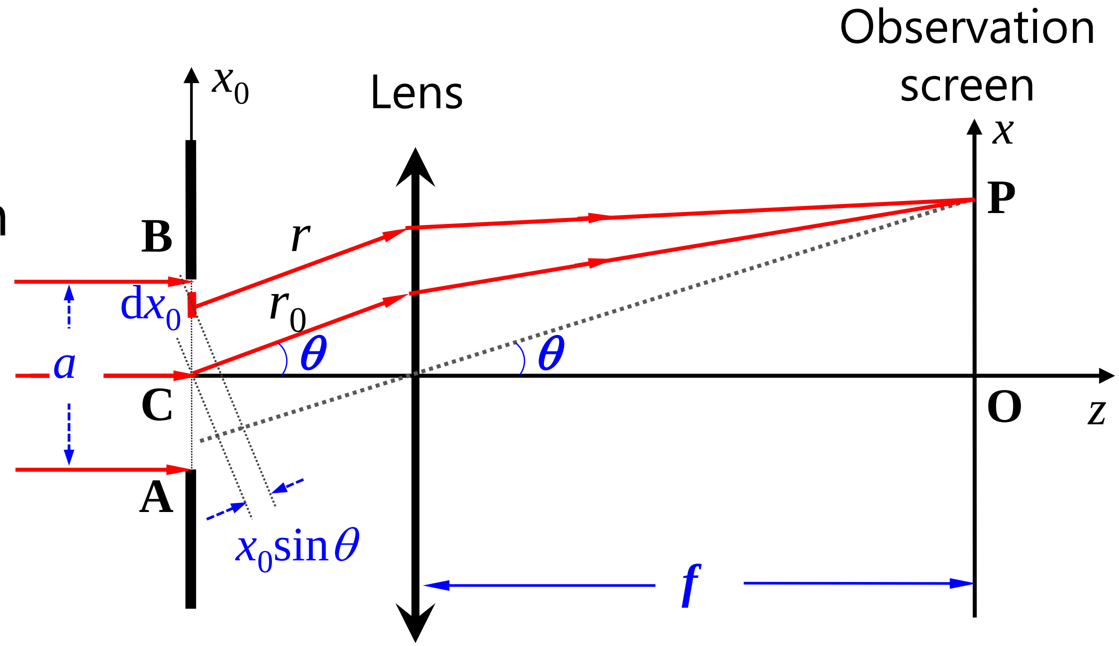


Intensity distribution

② Phasor addition

Divide the slit into N narrow subbands, each with a narrow width:

$$\Delta x = \frac{a}{N}$$



The wavelets of each narrow band are approximately equal in amplitude at point P, and are set to ΔE_p。

The phase difference between the wavelets coming from adjacent narrow bands to point P is:

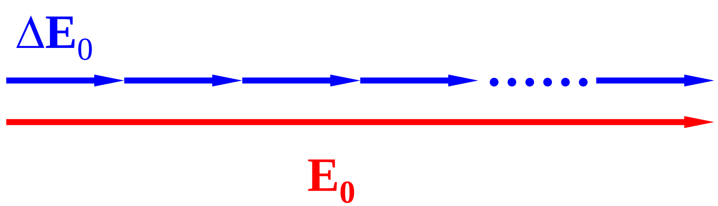
$$\Delta\varphi = k \cdot \Delta x \sin\theta = \frac{2\pi}{\lambda} \cdot \frac{a}{N} \sin\theta \qquad (N \text{ is large})$$

Intensity distribution

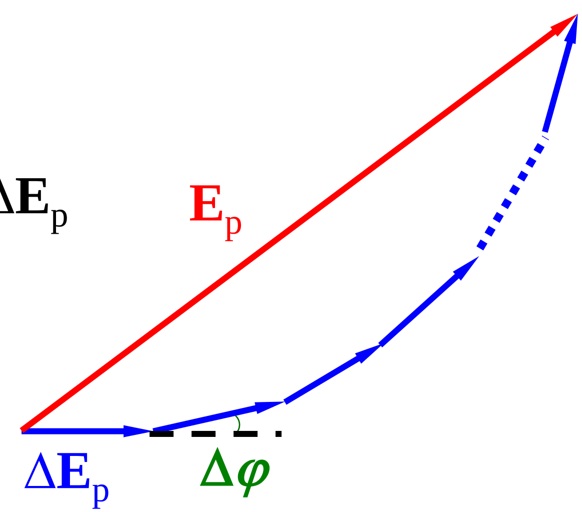
The combined amplitude E_p of point P is the modulus of the sum of the phasors of the respective wavelets.

At point P, the vibration is sum of a number of simple harmonic vibrations with the same direction, the same frequency, the same amplitude, but with an initial phase difference of $\Delta\varphi$.

For the center point, $\theta = 0$, $\Delta\varphi = 0$.



$$\mathbf{E}_0 = N\,\Delta\mathbf{E}_p$$



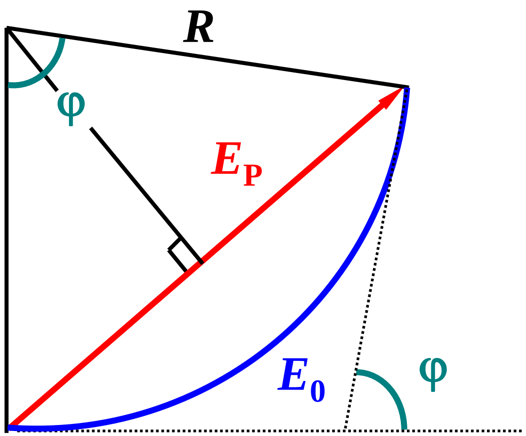
Other points, $\Delta\varphi \neq 0$: $\mathbf{E}_p < \mathbf{E}_0$

When $N \to \infty$, The N connected polylines will become an arc.

Intensity distribution

$$\Delta\varphi = \frac{2\pi}{\lambda} \cdot \frac{a}{N} \sin\theta \qquad \alpha = \frac{ka\sin\theta}{2}$$

$$\varphi = N\Delta\varphi = \frac{2\pi a}{\lambda} \cdot \sin\theta = 2\alpha$$



$$E_0 = N \cdot \Delta E_p = N \cdot R\Delta\varphi = R\varphi$$

$$E_p = 2R\sin\frac{\varphi}{2}$$

$$= 2\frac{E_0}{\varphi}\sin\frac{\varphi}{2}$$

$$= E_0\frac{\sin\alpha}{\alpha}$$

$$\boxed{I = I_0\left(\frac{\sin\alpha}{\alpha}\right)^2}$$

Diffraction pattern

Discussions: $I = I_0 \left(\frac{\sin\alpha}{\alpha} \right)^2$ $\alpha = \frac{ka\sin\theta}{2}$

(1) Central maximum (principal maximum) position:

$\theta = 0 \quad \alpha = 0 \quad \Rightarrow \quad \frac{\sin\alpha}{\alpha} = 1 \quad I = I_0 = I_{max}$

(2) Minimum position (dark line) :

$\alpha = \pm m\pi, \; m = 1, 2, 3\cdots \quad \sin\alpha = 0 \quad I = 0 \quad \alpha \neq 0$

$\Rightarrow \quad \frac{a}{2}\sin\theta = m\frac{\lambda}{2}$

The OPL difference from point A, point B (the edge of the slit) and center point C to the point P is integer times of half a wavelength. >>**destructive interference**

Diffraction pattern

Discussions: $$I = I_0\left(\frac{\sin\alpha}{\alpha}\right)^2 \qquad \alpha = \frac{ka\sin\theta}{2}$$

(3) Submaximal (high-order diffraction spot) position:

$$\frac{\mathrm{d}I}{\mathrm{d}\alpha} = I_0\frac{2\sin\alpha\left(\alpha\cos\alpha - \sin\alpha\right)}{\alpha^3} = 0$$

$\Rightarrow$ $\alpha = \tan\alpha$ Intersection of $y = x$ and $y = \tan x$

$$\alpha = \pm 1.43\pi,\ \pm 2.46\pi,\ \pm 3.47\pi\cdots$$

$$\sin\theta = \pm 1.43\frac{\lambda}{a},\ \pm 2.46\frac{\lambda}{a},\ \pm 3.47\frac{\lambda}{a}\cdots$$

The submaximal condition can be approximated as

$$a\sin\theta = \left(n + 1/2\right)\lambda$$

Diffraction pattern

Submaximal intensity

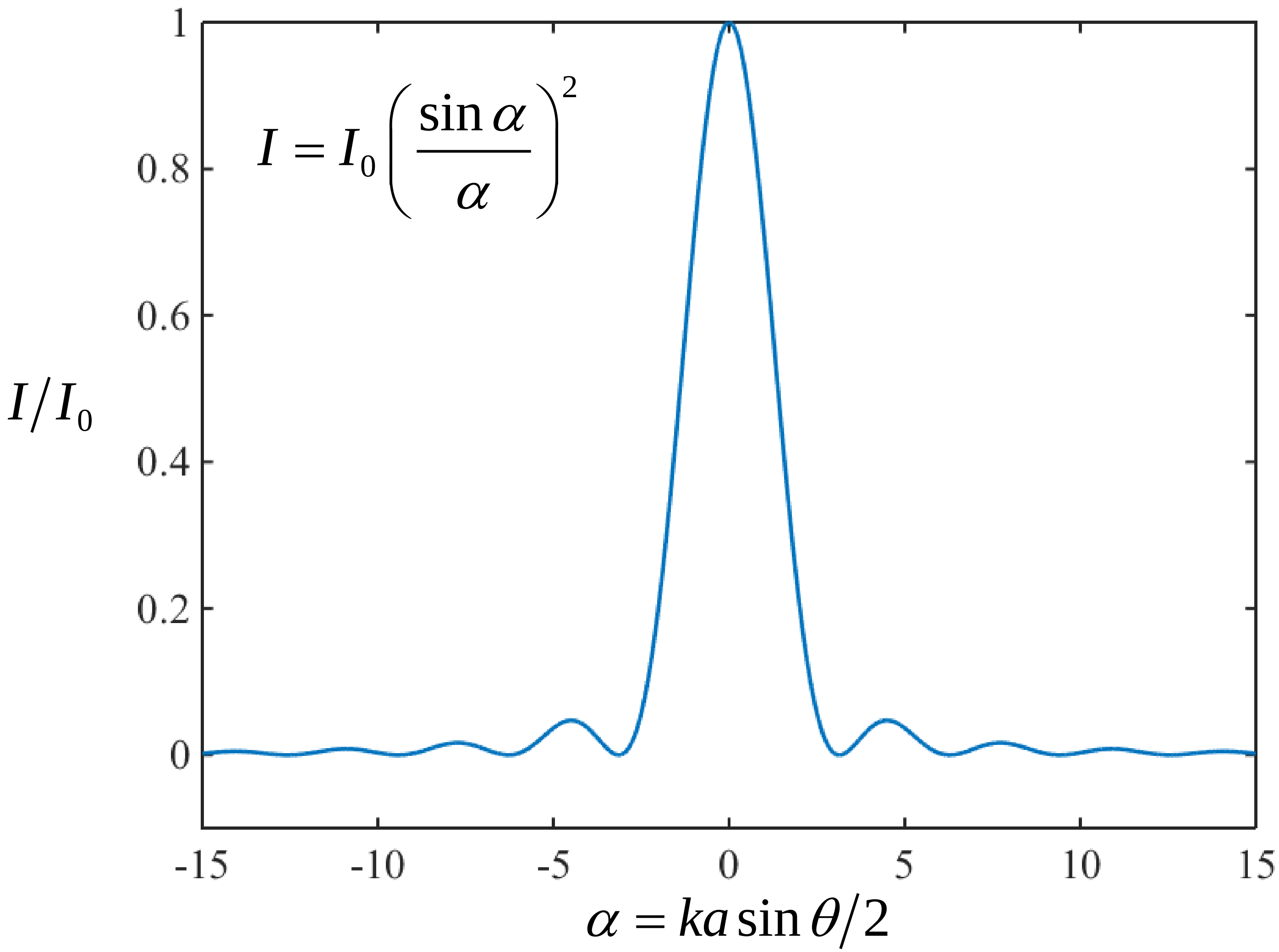
$$I_1 = I_0 \left(\frac{\sin 1.43\pi}{1.43\pi} \right)^2 \approx 4.72\% I_0$$

$$I_2 \approx 1.65\% I_0$$

$$I_3 \approx 0.83\% I_0$$

※ More than **80%** of the energy is concentrated in the principal maximum.

Diffraction pattern



Diffraction pattern

Discussions: $I = I_0\left(\frac{\sin\alpha}{\alpha}\right)^2$ $\alpha = \frac{ka\sin\theta}{2}$

（4） Angular width

Angular width: The angular distance between the dark lines on both sides of the bright line.

0^{th} order bright center： $\theta = 0$

1^{st} dark center： $\theta_1 \approx \sin\,\theta_1 = \lambda/a$ $\frac{a}{2}\sin\theta = m\frac{\lambda}{2}$

Half angular width of the central maximum: $\Delta\theta \approx \lambda/a$

Angular width of the 1^{st} bright fringe： $\Delta\theta_1 = 2\lambda/a - \lambda/a = \lambda/a$

※The 0^{th} bright-angle corner width is twice that of other levels of bright lines.

※ $\Delta\theta$ can be used as a sign of how strong is the diffraction.

Diffraction pattern

(5) Effect of slit width

$$\boxed{\Delta\theta = \frac{\lambda}{a}}$$

For a given λ, if $a \downarrow \rightarrow \Delta\theta \uparrow$, i.e., the larger the 0th diffraction spot >>the more obvious of the diffraction.

If $a \uparrow \rightarrow \Delta\theta \downarrow$, all fringes are close to the center

When $a >> \lambda$, $\Delta\theta \rightarrow 0$. The **diffraction fringes become indistinguishable**, showing only a single bright line (the shadow).

At this time, the light follows the law of **rectilinear propagation**.

Diffraction pattern

When the slit is extremely narrow ($a \approx \lambda$) $\sin\theta_1 \approx 1$, $\theta_1 \approx \pi/2$

Diffraction → Scattering

(break down of scalar diffraction)

The central maximum extend over a large range of direction.
The observation screen only receive the central part of it.
That's why it looks more uniform.

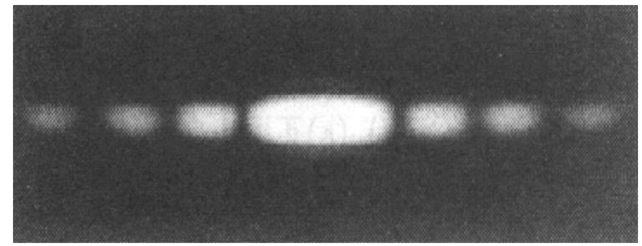
That is why we didn't consider the effect of diffraction of the thin slits in previous Young's experiment.

(6) The effect of wavelength

Different wavelengths → ribbons

§ 7.6 Fraunhofer diffraction: aperture

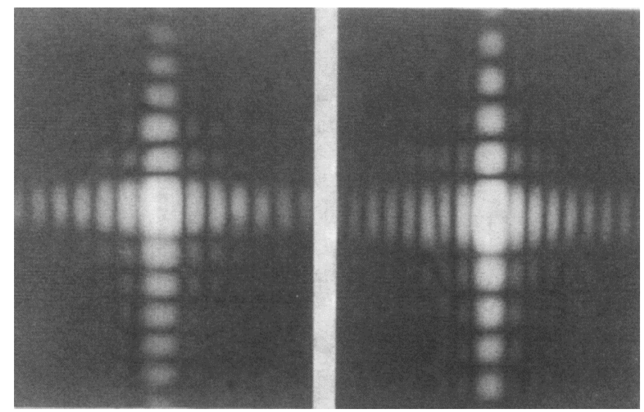
- If we shrink down the long side of the single slit, the diffraction fringes perpendicular to it appear. And the cross terms appears.



$$I = I_0\left(\frac{\sin\alpha}{\alpha}\right)^2\left(\frac{\sin\beta}{\beta}\right)^2$$

$$\alpha = \frac{ka\sin\theta}{2} \qquad \sin\theta = \frac{x}{z} = \frac{x}{f}$$

$$\beta = \frac{kb\sin\theta'}{2}$$



Fourier transform of rectangular holes

Circular aperture

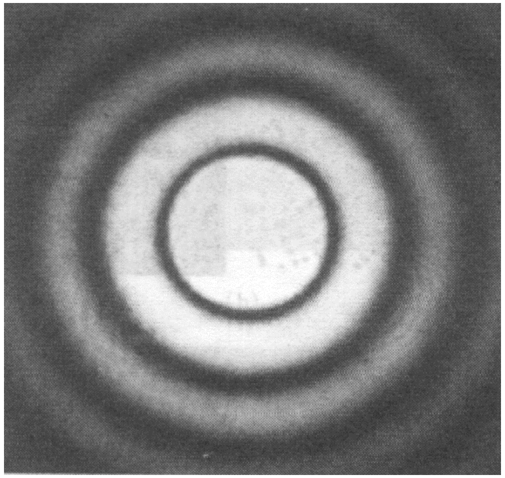
- Circular aperture diffraction intensity expression (1st Bessel function)

$$\alpha = \frac{ka\sin\theta}{2}$$

$$I = I_0\left(\frac{J_1(\alpha/2)}{\alpha/2}\right)^2$$

- The center of the diffraction pattern is called the **Airy disk**, and its angular radius

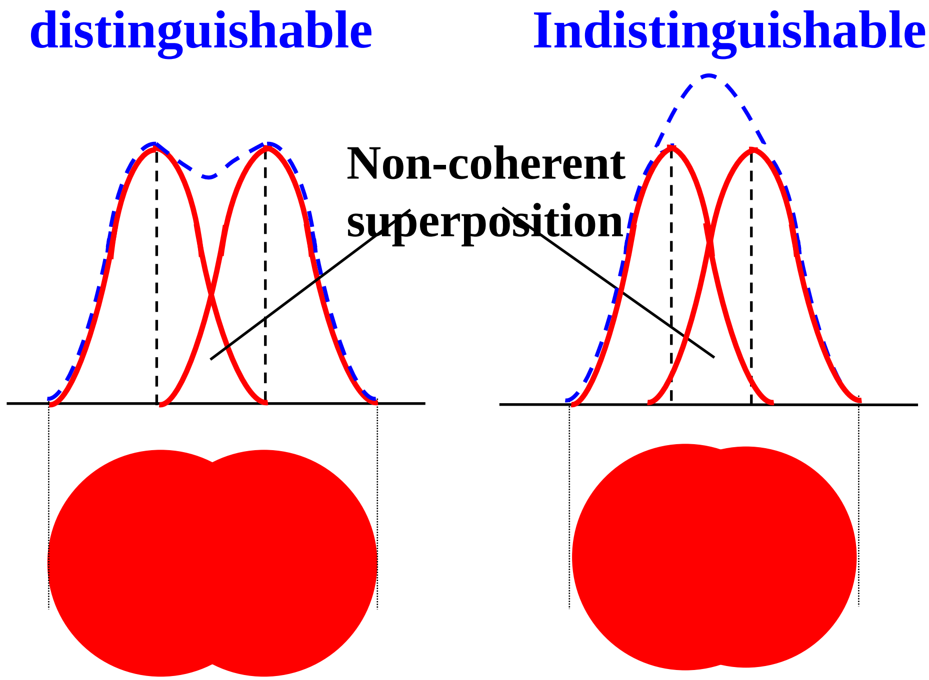
$$\Delta\theta \approx \sin\theta_1 = 0.61\frac{\lambda}{a} = 1.22\frac{\lambda}{D}$$



Circular aperture diffraction

- The Airy spot concentrates about 84% of the diffracted light energy.

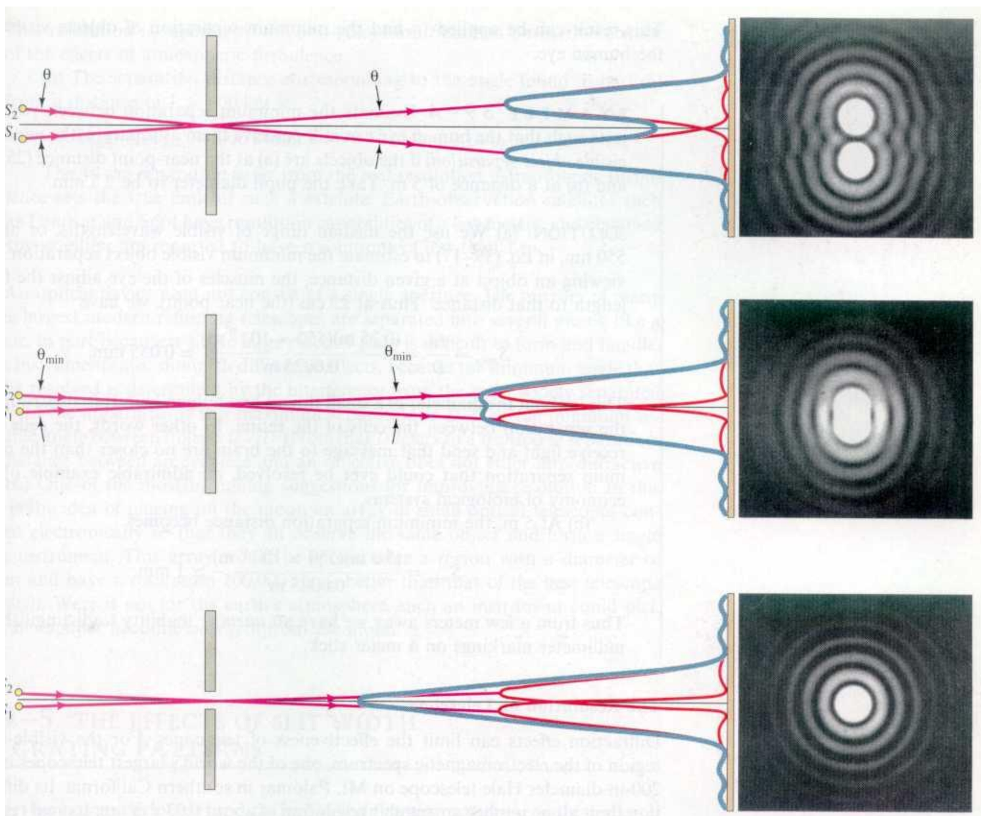
Rayleigh criterion



Rayleigh criterion: When the central principal maxima of one diffraction pattern coincides with the first minimum of the other diffraction pattern, the two object points are said to be distinguishable.

Rayleigh criterion

The resolution of the small hole (diameter D) to two distant point sources.



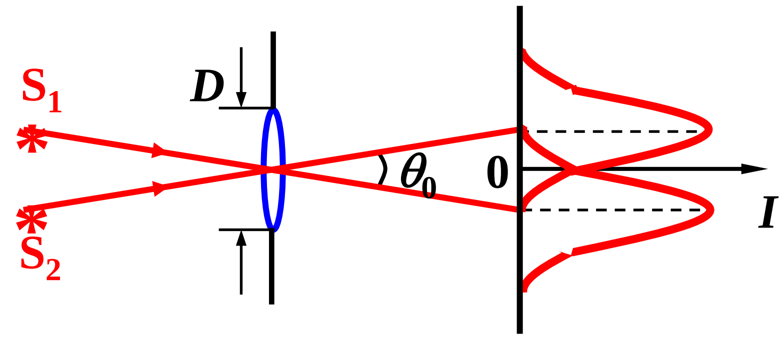
Point source distance is large
Distinguished

Rayleigh criterion

Point source distance is small
Undistinguished

Rayleigh criterion

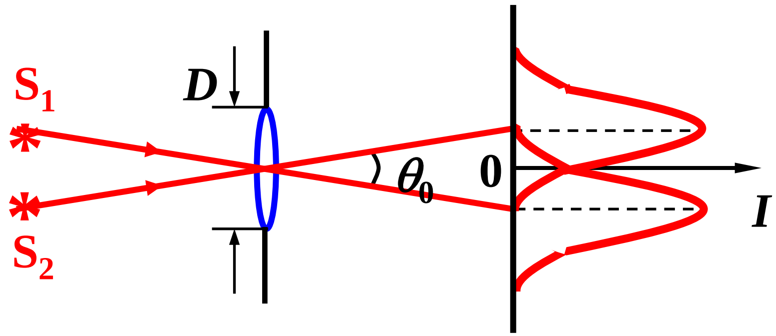
Minimum resolvable angle of the optical system θ_0 **:** When the diffraction spots of the two object points are exactly distinguishable, the angle of the two object points to the center of the diffraction hole.



- If the aperture is a circular hole, θ_0 is equal to the angular distance between the main maxima of the two diffraction spots, which is the half-width of the Airy disk $\Delta\theta$.

$$\boxed{\theta_0 = \Delta\theta = 1.22\frac{\lambda}{D}}$$

Rayleigh criterion



$$\theta_0 = \Delta\theta = 1.22\frac{\lambda}{D}$$

- If the clear aperture is a slit, the minimum resolvable angle θ_0 is the half-width angle $\Delta\theta$ of the central maximum of the single slit diffraction.

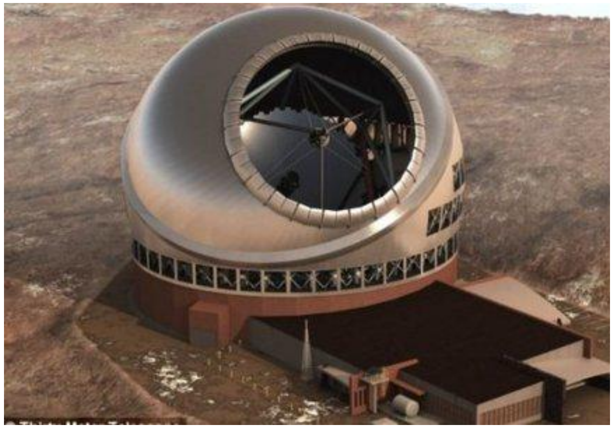
$$\theta_0 = \Delta\theta = \lambda/a \qquad \left.\begin{matrix} D\uparrow \\ \lambda\downarrow \end{matrix}\right\} \to R\uparrow$$

For an instrument $\theta_0 \downarrow \to$ resolution $\uparrow$

Rayleigh criterion

Telescope: λ can not be selected, but $D\uparrow \Rightarrow R\uparrow$

- Largest optical telescope (is building): D = 30 m, Mauna Kea, Hawaii.



- Largest single-antenna radio telescope: Five-hundred-meter Aperture Spherical radio Telescope (FAST), D = 500 m, Pingtang, Guizhou.

§ 7.7* Super-resolution imaging

Super-resolution microscope

Eric Betzig, Stefan W. Hell and William E. Moerner *are awarded the* ***Nobel Prize in Chemistry 2014*** *for having bypassed a presumed scientific limitation stipulating that an optical microscope can never yield a resolution better than 0.2 micrometres. Using the fluorescence of molecules, scientists can now monitor the interplay between individual molecules inside cells; they can observe disease-related proteins aggregate and they can track cell division at the nanolevel.*

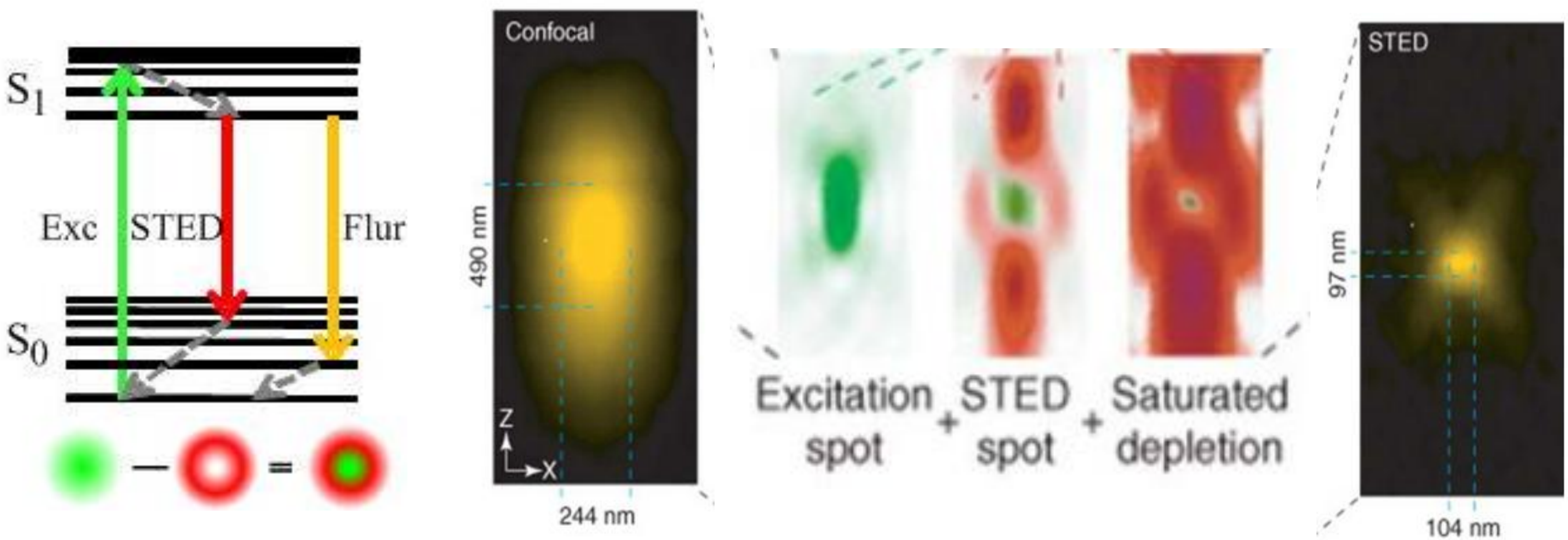
受激辐射耗尽 (STED)

Stimulated Emission Depletion

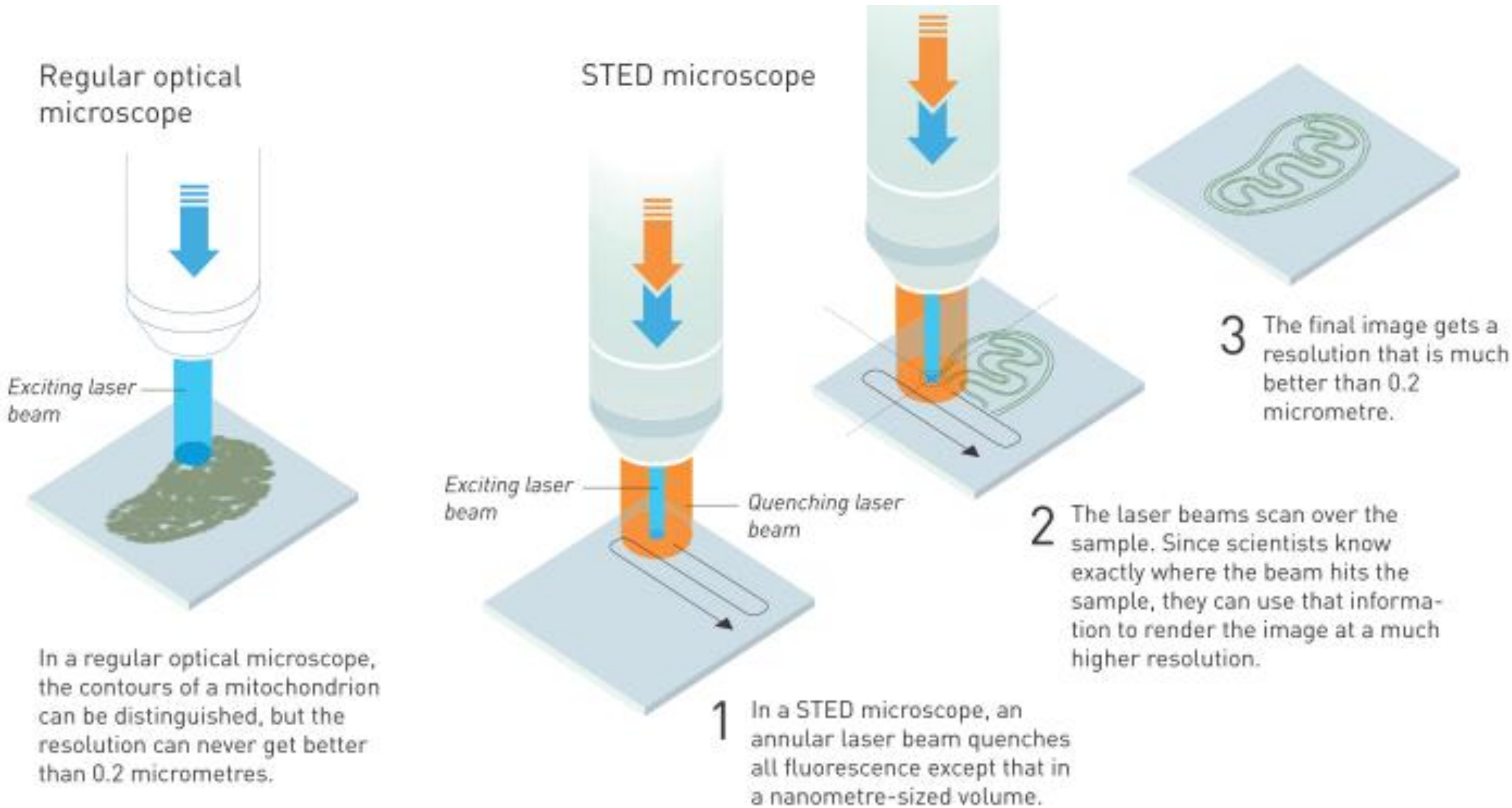
Hell SW and Kroug M (1995) Ground-state depletion fluorescence microscopy, a concept for breaking the diffraction resolution limit. Appl. Phys. B. 60:495-497.

Hell SW and Wichman J (1994) Breaking the diffraction resolution limit by stimulated emission: stimulated-emission-depletion-microscopy. Opt. Lett. 19:780-782.

Stefan W. Hell



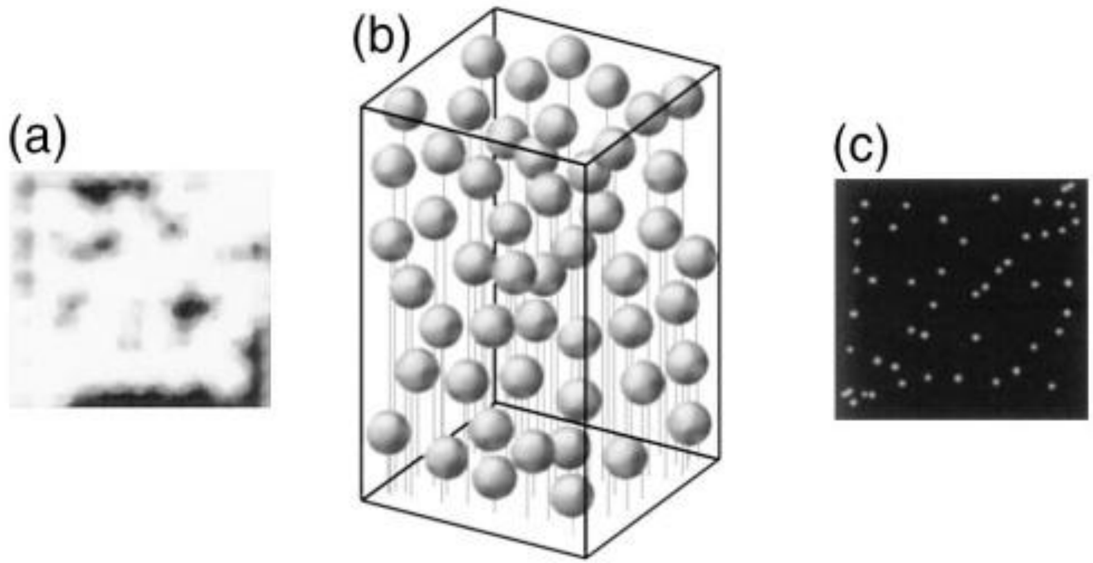
受激辐射耗尽 (STED)



分辨率 ~ 10 nm

超分辨荧光成像技术构想

需要发不同颜色荧光的分子，并且荧光峰比较窄



Eric Betzig

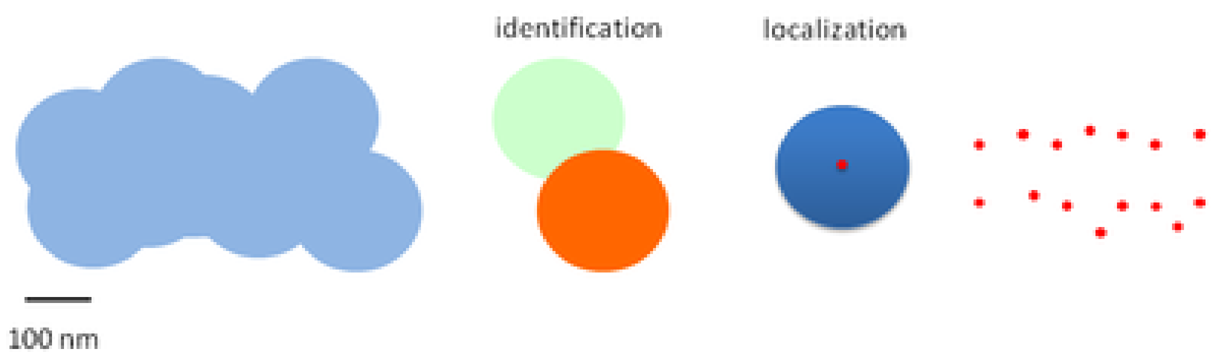
Betzig E. Proposed method for molecular optical imaging. Opt. Lett. 20:237-239 (1995).

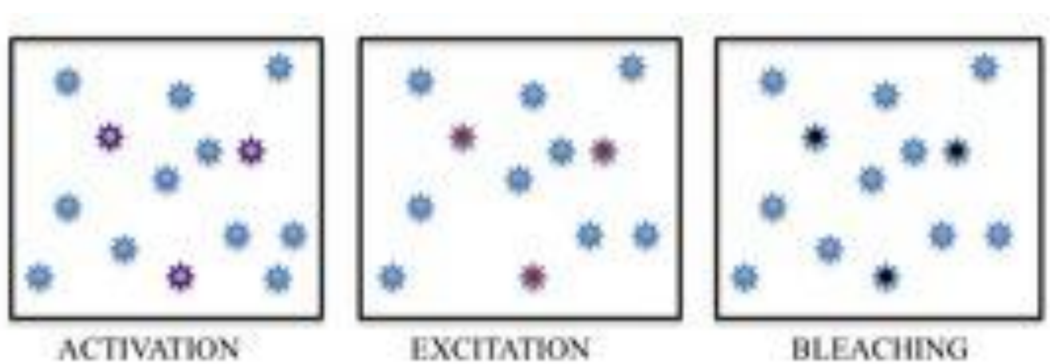
- **1991-1995年，新泽西Bell实验室SNOM；1995年提出超分辨分子成像的思路；同年，感觉SNOM无望，离开学术生涯；2005年发现荧光可以随意操控，想到了PALM的实现。**

超分辨荧光成像的物理实现

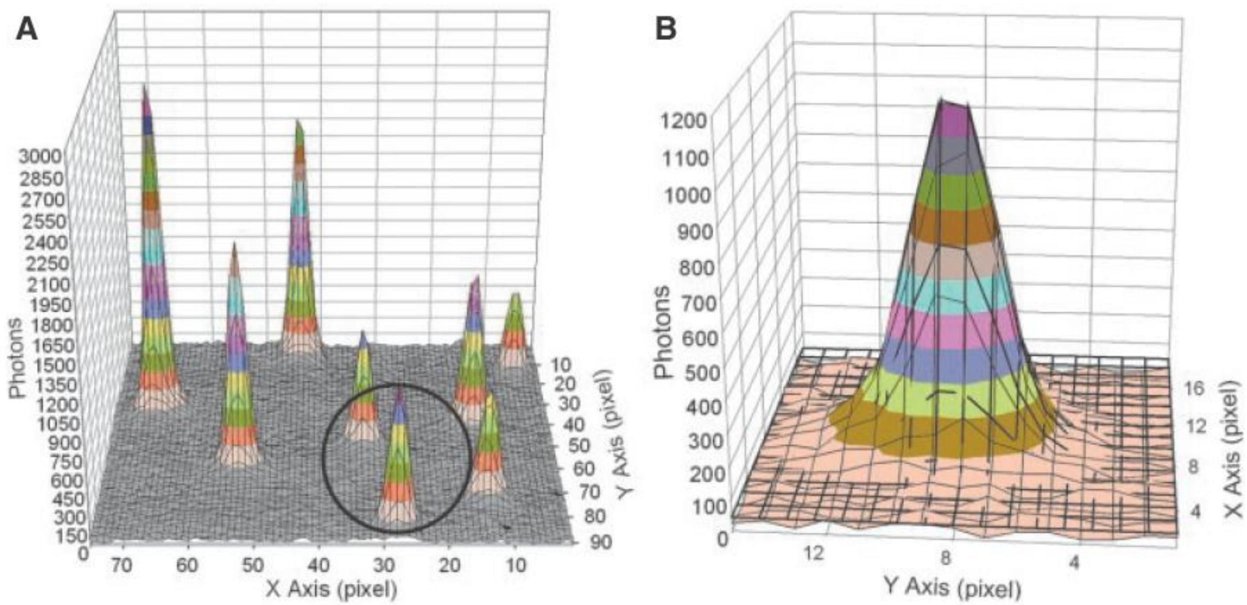
- 光敏定位显微镜 Photoactivated localization microscopy (PALM)

不同颜色的荧光 → 不同时间发出荧光





单分子成像定位



Paul R. Silven, Myosin V Walks Hand-Over-Hand: Single Fluorophore Imaging with 1.5-nm Localization, Science 2003

PLAM成像技术原理

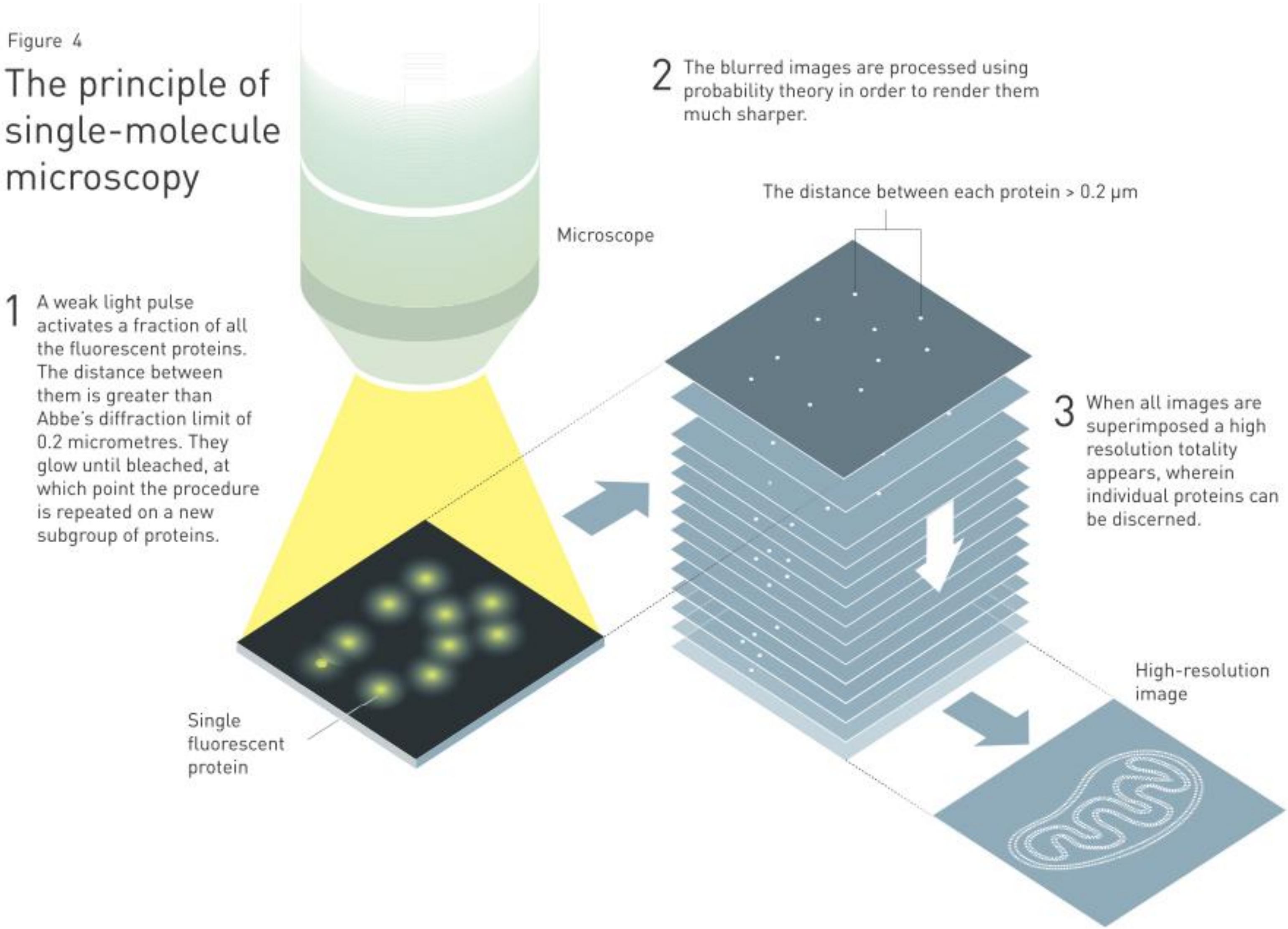
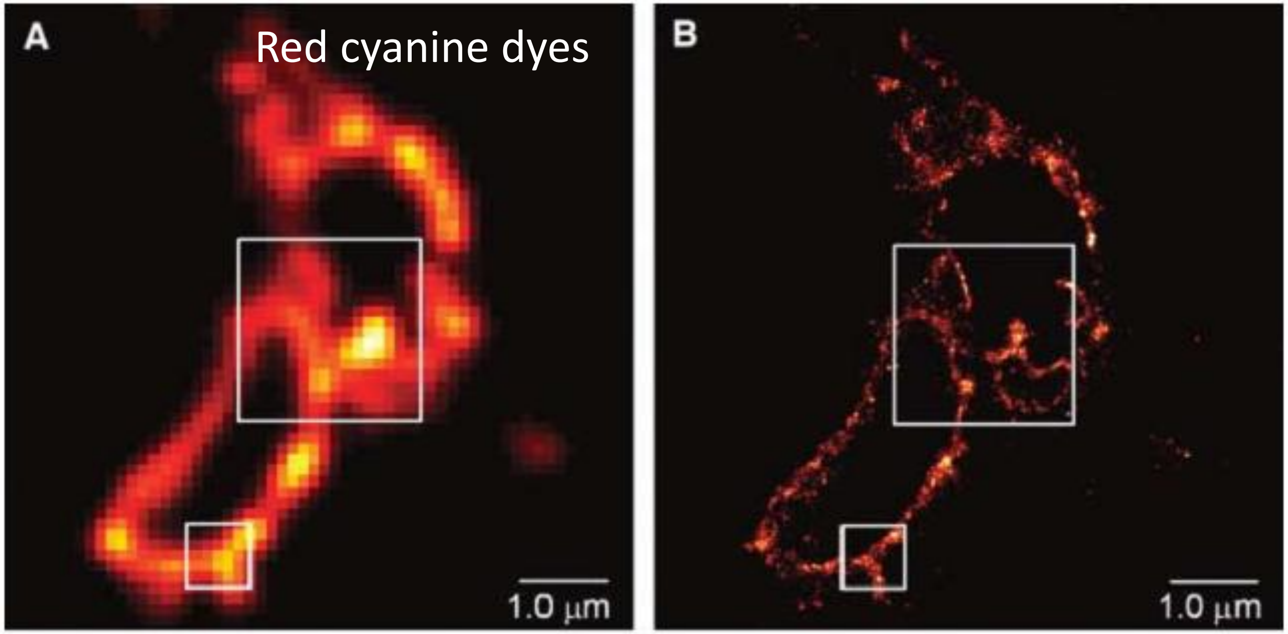


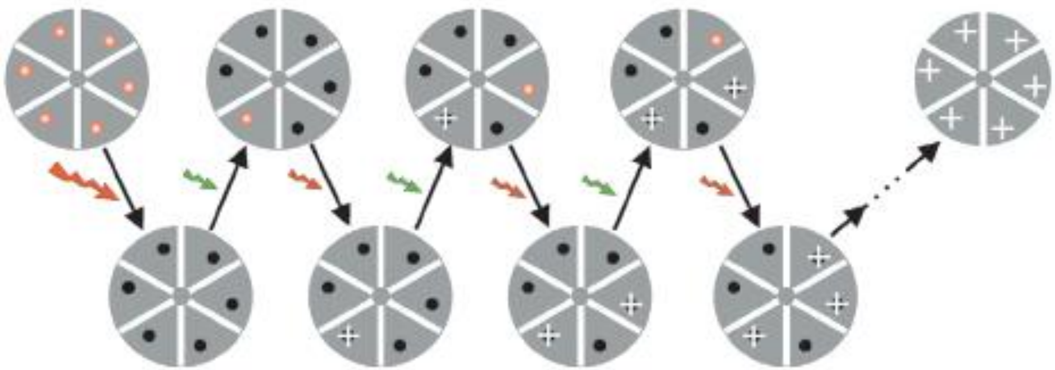
Figure 4
The principle of single-molecule microscopy
Microscope
1 A weak light pulse activates a fraction of all the fluorescent proteins. The distance between them is greater than Abbe's diffraction limit of 0.2 micrometres. They glow until bleached, at which point the procedure is repeated on a new subgroup of proteins.
2 The blurred images are processed using probability theory in order to render them much sharper.
The distance between each protein > 0.2 µm
3 When all images are superimposed a high resolution totality appears, wherein individual proteins can be discerned.
Single fluorescent protein
High-resolution image

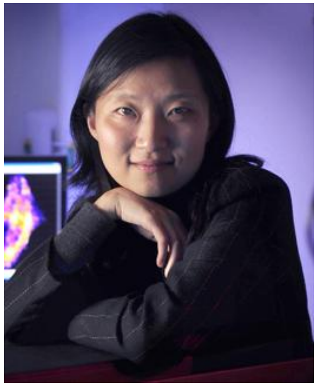
Example of PLAM

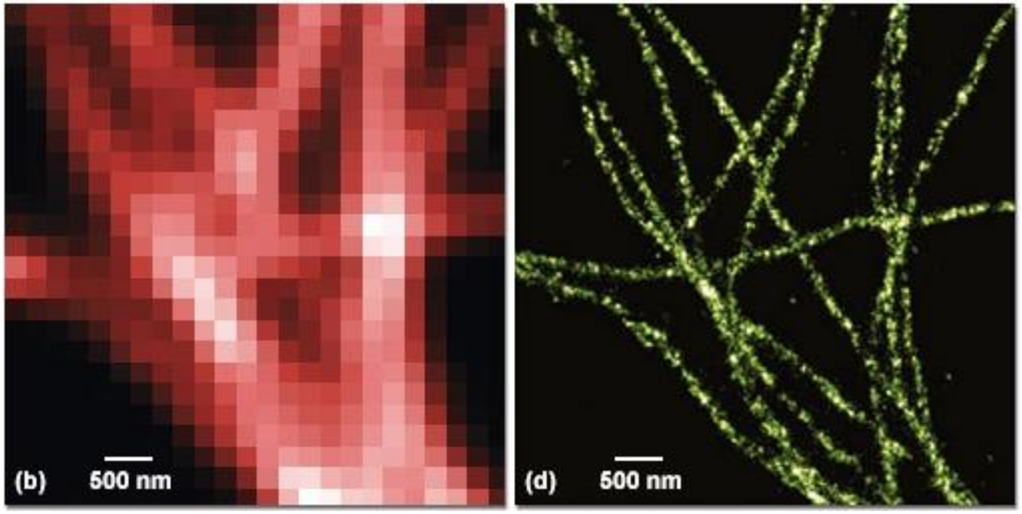


随机光学重构显微镜(STORM)

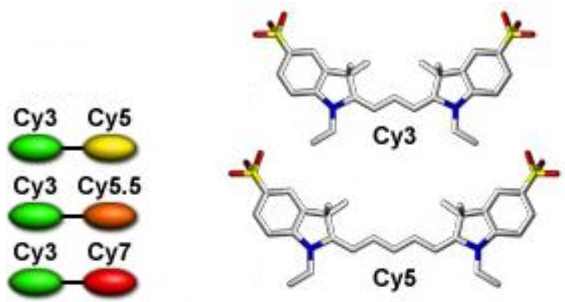
- 随机光学重构显微镜 Stochastic Optical Reconstruction Microscopy (STORM)







activator-reporter dye pairs



来源尼康公司

类似技术　　英雄所见略同 & 出名要趁早

PLAM

Eric Betzig, George H. Patterson, Rachid Sougrat, O. Wolf Lindwasser, Scott Olenych,Juan S. Bonifacino, Michael W. Davidson, Jennifer Lippincott-Schwartz, HaraldF. Hess, “Imaging Intracellular Fluorescent Proteins at Nanometer Resolution”, Science 2006 Vol. 313 no. 5793 pp. 1642-1645. (Received for publication **13 March** 2006. Accepted for publication 2 August 2006.）

STORM

Rust, Michael J., Mark Bates, and **Xiaowei Zhuang**. "Sub-diffraction-limit imaging by stochastic optical reconstruction microscopy (STORM)." *Nature methods* 3.10 (2006): 793-796. (Received **7 July**; accepted 31 July; published online 9 August 2006)

FPLAM

Samuel T. Hess, T. P. K. Girirajan, and M. D. Mason, “Ultra-High Resolution Imaging by Fluorescence Photoactivation Localization Microscopy,” Biophysical Journal, vol. 91, no. 11, pp. 4258-4272, 2006. Submitted **June 12**, 2006, and accepted for publication August 28, 2006. Published 1 December 2006

定位分辨率能有多高?

一般的PALM/STORM的分辨率在10 nm左右，没有理论极限

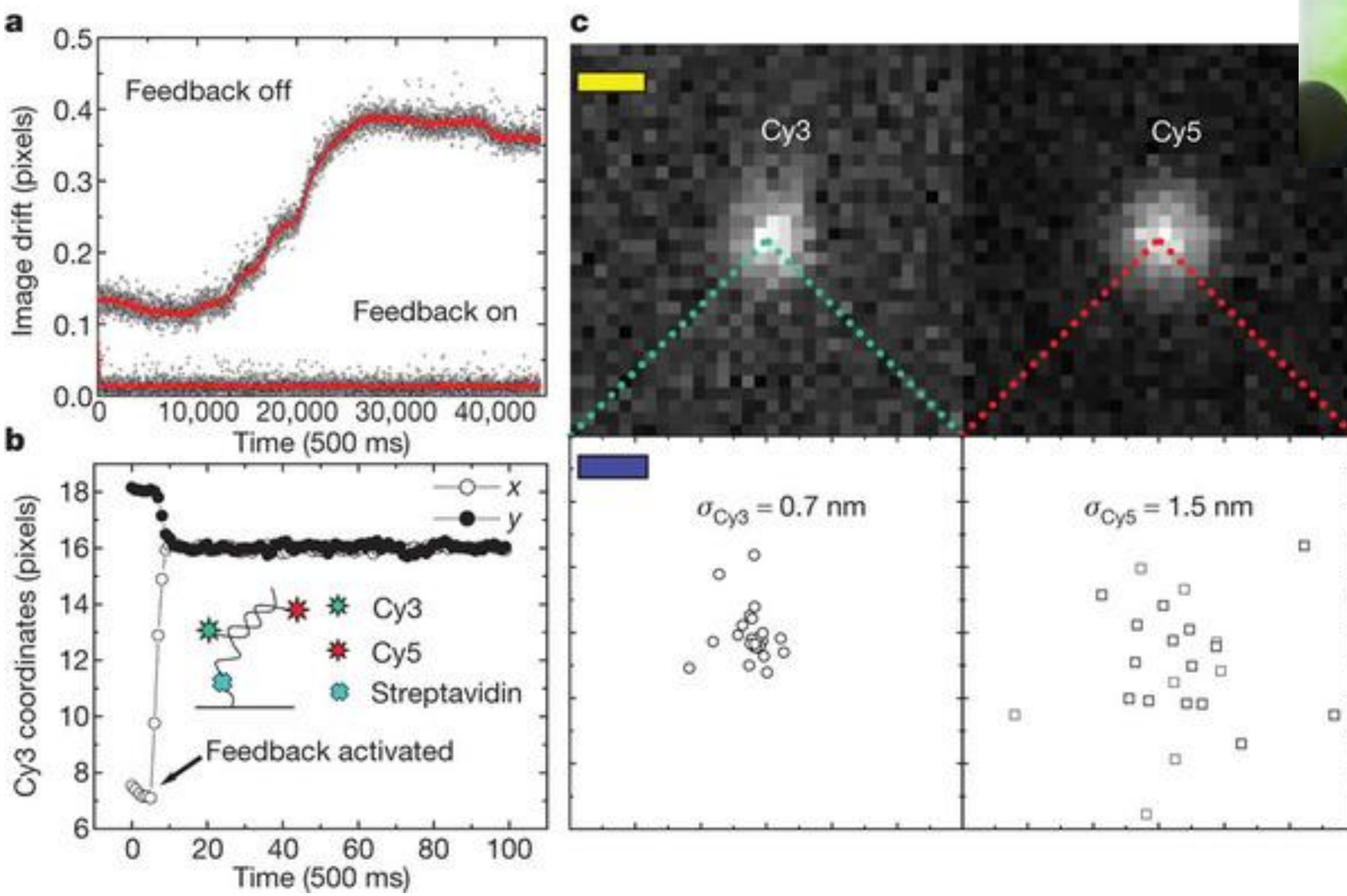
$$\sigma = \frac{s}{\sqrt{N}}$$

s是点扩散函数的标准差，*N*是探测到的光子数。

荧光量子产率、温度等因素

机械漂移、流体流动与布朗运动等等

定位分辨率能有多高?



Subnanometre single-molecule localization, registration and distance measurements， Nature Volume: 466, Pages: 647–651 Date published: (29 July 2010)

Homework

Prove the screen function of a convex lens.

Homework*

You are free. Again!

Next week

Many slits, Gratings, Fresnel Diffraction.

Sections 10.2.3、10.2.8、10.3