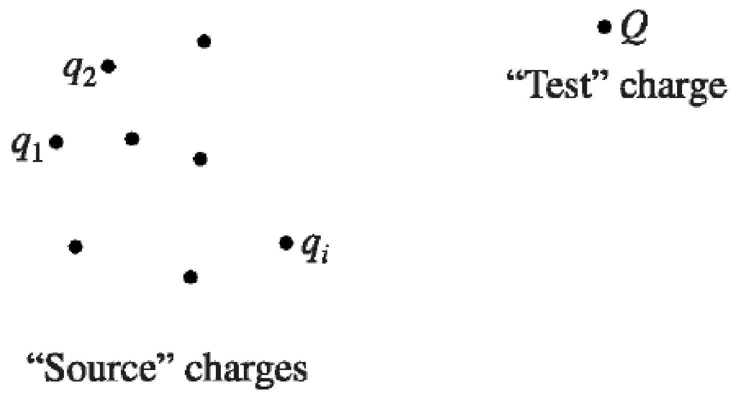
Chapter 2

Electrostatics

- 2.1 The Electronic Field
- 2.2 Divergence and Curl of Electronic Fields
- 2.3 Electric Potential
- 2.4 Work and Energy In Electronics

The Electronic Field

Coulomb's Law



What is the force on a test charge Q due to a single point charge q which is at *rest* a distance $\mathscr{r}$ away? The answer (based on experiments) is given by **Coulomb's law**:

$$\boxed{\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}.} \tag{2.1}$$

The constant ϵ_0 is called the **permitivity of free space.** In SI units, where force is in Newtons (N), distance in meters (m), and charge in coulombs (C),

$$\epsilon_0 = 8.85 \times 10^{-12} \frac{\mathrm{C}^2}{\mathrm{N} \cdot \mathrm{m}^2}.$$

The Electric Field

If we have *several* point charges $q_1, q_2, \ldots, q_n$, at distances $\mathscr{r}_1, \mathscr{r}_2, \ldots, \mathscr{r}_n$ from Q, the total force on Q is evidently

$$\begin{aligned}\mathbf{F} &= \mathbf{F}_1 + \mathbf{F}_2 + \ldots = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 Q}{\mathscr{r}_1^2}\hat{\boldsymbol{\mathscr{r}}}_1 + \frac{q_2 Q}{\mathscr{r}_2^2}\hat{\boldsymbol{\mathscr{r}}}_2 + \ldots\right) \\ &= \frac{Q}{4\pi\epsilon_0}\left(\frac{q_1\hat{\boldsymbol{\mathscr{r}}}_1}{\mathscr{r}_1^2} + \frac{q_2\hat{\boldsymbol{\mathscr{r}}}_2}{\mathscr{r}_2^2} + \frac{q_3\hat{\boldsymbol{\mathscr{r}}}_3}{\mathscr{r}_3^2} + \ldots\right),\end{aligned}$$

or

$$\boxed{\mathbf{F} = Q\mathbf{E},} \tag{2.3}$$

where

$$\mathbf{E}(\mathbf{r}) \equiv \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i^2}\hat{\boldsymbol{\mathscr{r}}}_i. \tag{2.4}$$

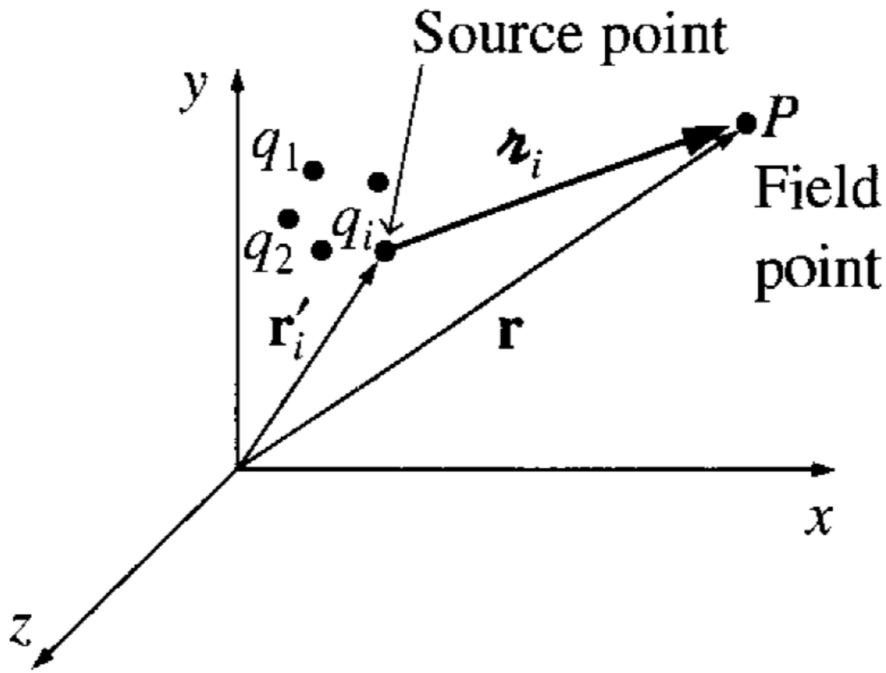

Source point
y
q_1
q_2
q_i
$\boldsymbol{\mathscr{r}}_i$
P
Field
point
$\mathbf{r}'_i$
$\mathbf{r}$
x
z

Figure 2.3

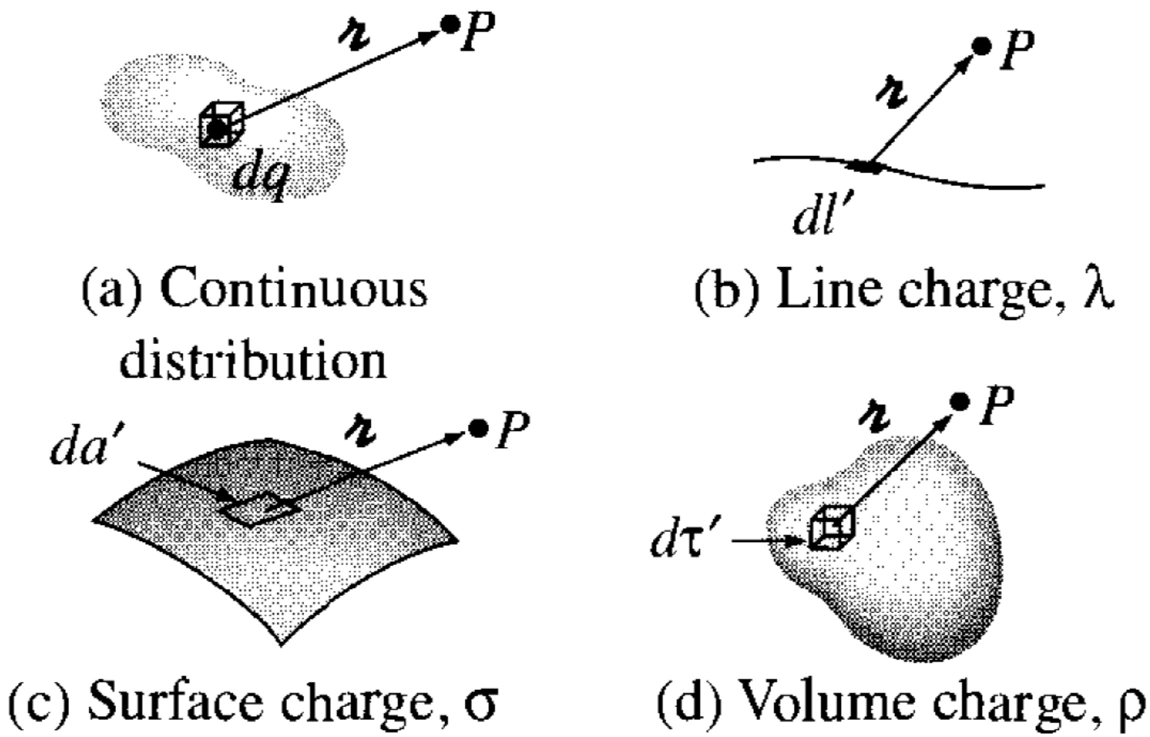

P
dq
(a) Continuous distribution
P
dl′
(b) Line charge, λ
da′
P
(c) Surface charge, σ
P
dτ′
(d) Volume charge, ρ

Figure 2.5

$$dq \to \lambda\, dl' \sim \sigma\, da' \sim \rho\, d\tau'.$$

Thus the electric field of a line charge is

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{P}} \frac{\lambda(\mathbf{r}')}{\mathscr{r}^2} \hat{\boldsymbol{\mathscr{r}}}\, dl'; \tag{2.6}$$

for a surface charge,

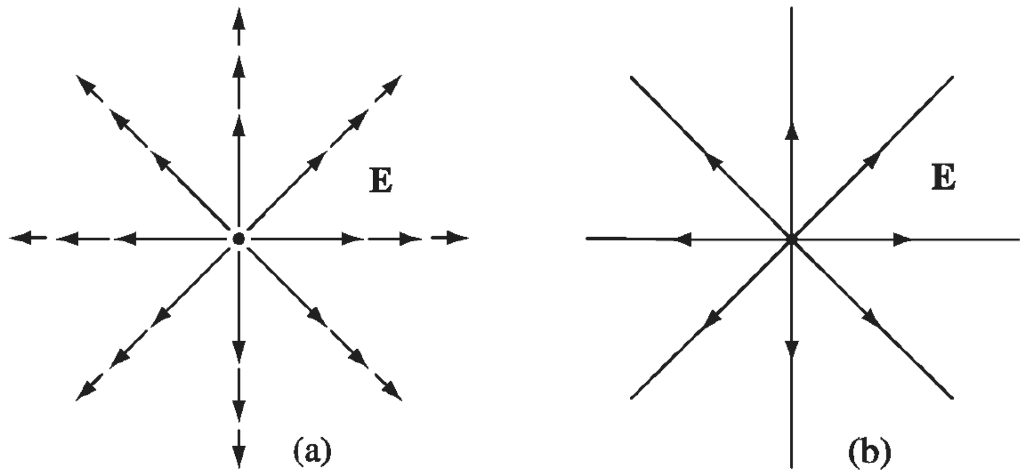
$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{S}} \frac{\sigma(\mathbf{r}')}{\mathscr{r}^2} \hat{\boldsymbol{\mathscr{r}}}\, da'; \tag{2.7}$$

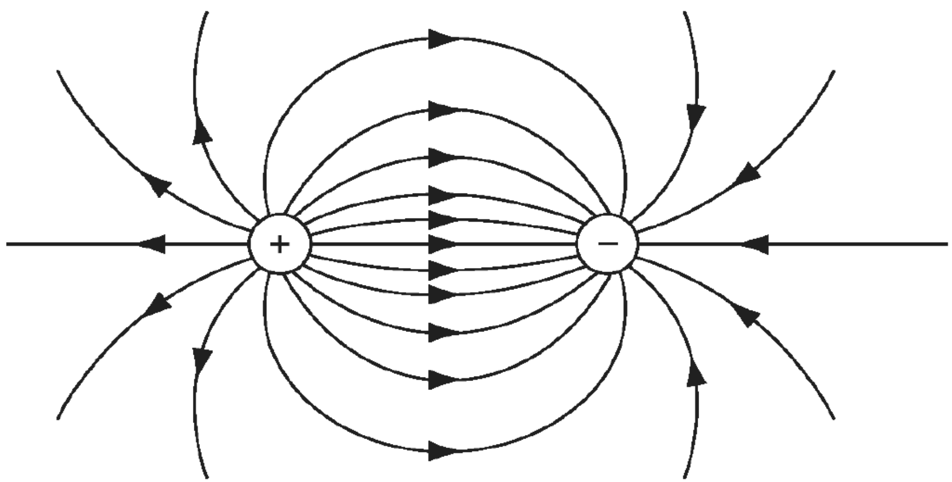
and for a volume charge,

$$\boxed{\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \frac{\rho(\mathbf{r}')}{\mathscr{r}^2} \hat{\boldsymbol{\mathscr{r}}}\, d\tau'.} \tag{2.8}$$

Electric field lines

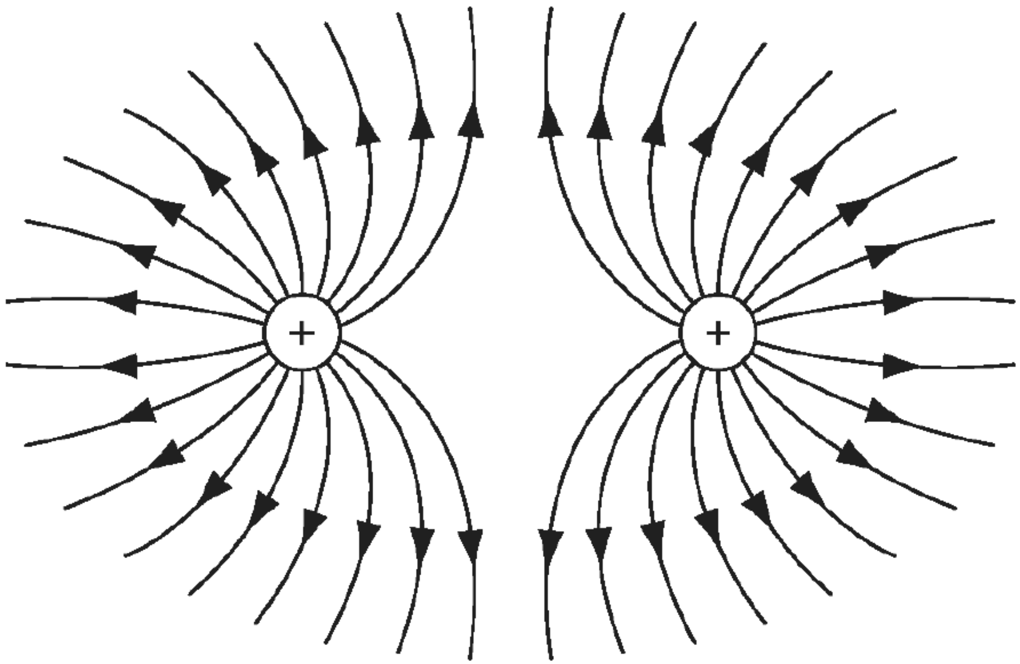
$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$

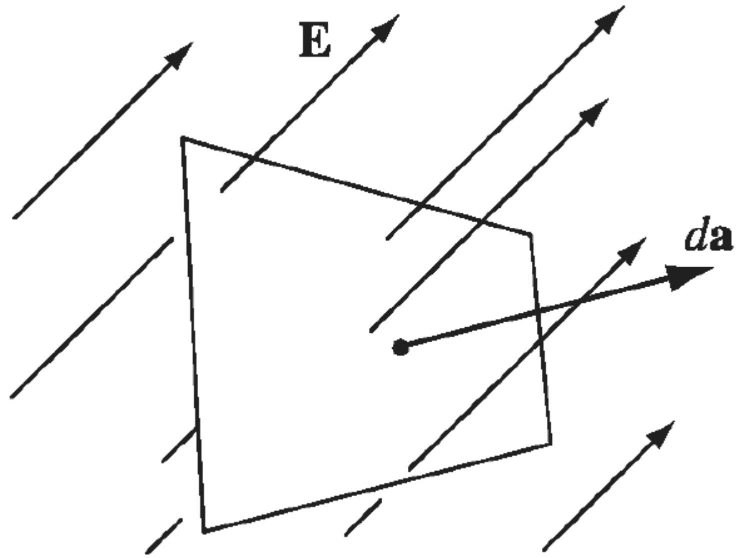




Opposite charges

Electric field lines





the *flux* of $\mathbf{E}$ through a surface $\mathcal{S}$,

$$\Phi_E \equiv \int_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a},$$

In the case of a point charge q at the origin, the flux of **E** through a sphere of radius r is

$$\oint \mathbf{E} \cdot d\mathbf{a} = \int \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r^2}\hat{\mathbf{r}}\right) \cdot (r^2 \sin\theta \, d\theta \, d\phi \, \hat{\mathbf{r}}) = \frac{1}{\epsilon_0} q. \qquad (2.12)$$

$$\mathbf{E} = \sum_{i=1}^{n} \mathbf{E}_i .$$

The flux through a surface that encloses them all, then, is

$$\oint \mathbf{E} \cdot d\mathbf{a} = \sum_{i=1}^{n} \left(\oint \mathbf{E}_i \cdot d\mathbf{a} \right) = \sum_{i=1}^{n} \left(\frac{1}{\epsilon_0} q_i \right)$$

For any closed surface, then,

$$\oint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}}$$

$$\oint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} = \int_{\mathcal{V}} (\nabla \cdot \mathbf{E})\, d\tau.$$

Rewriting Q_{enc} in terms of the charge density ρ, we have

$$Q_{\text{enc}} = \int_{\mathcal{V}} \rho \, d\tau.$$

So Gauss's law becomes

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{E})\, d\tau = \int_{\mathcal{V}} \left(\frac{\rho}{\epsilon_0}\right) d\tau.$$

And since this holds for *any* volume, the integrands must be equal:

$$\boxed{\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho.}$$

The Divergence of E

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int\limits_{\text{all space}} \frac{\hat{\boldsymbol{\imath}}}{\imath^2} \rho(\mathbf{r}')\, d\tau'$$

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{1}{4\pi\epsilon_0} \int \boldsymbol{\nabla} \cdot \left(\frac{\hat{\boldsymbol{\imath}}}{\imath^2} \right) \rho(\mathbf{r}')\, d\tau'.$$

$$\boldsymbol{\nabla} \cdot \left(\frac{\hat{\boldsymbol{\imath}}}{\imath^2} \right) = 4\pi \delta^3(\boldsymbol{\imath})$$

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{1}{4\pi\epsilon_0} \int 4\pi \delta^3(\mathbf{r} - \mathbf{r}')\rho(\mathbf{r}')\, d\tau' = \frac{1}{\epsilon_0}\rho(\mathbf{r})$$

$$\int\limits_{\mathcal{V}} \boldsymbol{\nabla} \cdot \mathbf{E}\, d\tau = \oint\limits_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} \int\limits_{\mathcal{V}} \rho\, d\tau = \frac{1}{\epsilon_0} Q_{\text{enc}}.$$

Applications of Gauss's Law

Example :

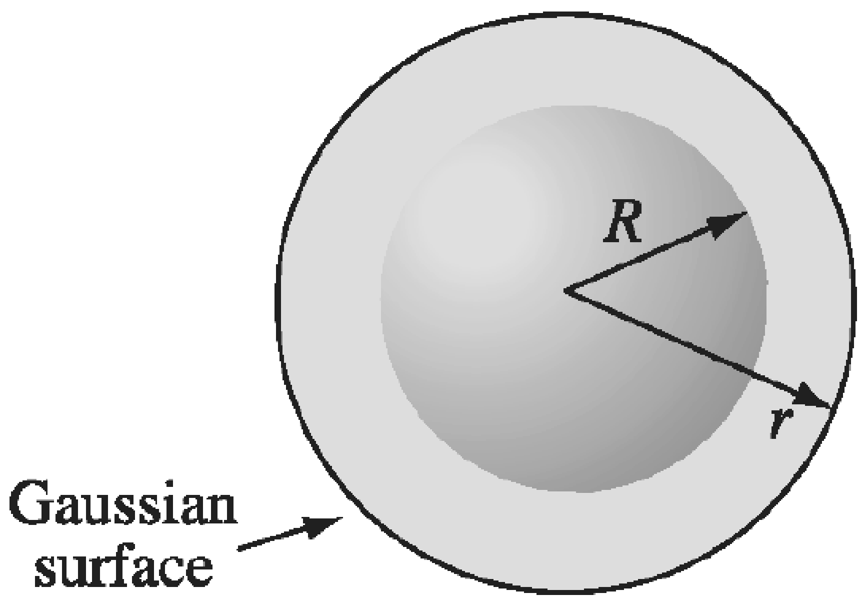
Find the field outside a uniformly charged solid sphere of radius R and total charge q.

$$\int_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} = \int_{\mathcal{S}} |\mathbf{E}| \, da,$$

$$\int_{\mathcal{S}} |\mathbf{E}| \, da = |\mathbf{E}| \int_{\mathcal{S}} da = |\mathbf{E}| \, 4\pi r^2.$$

$$|\mathbf{E}| \, 4\pi r^2 = \frac{1}{\epsilon_0} q,$$

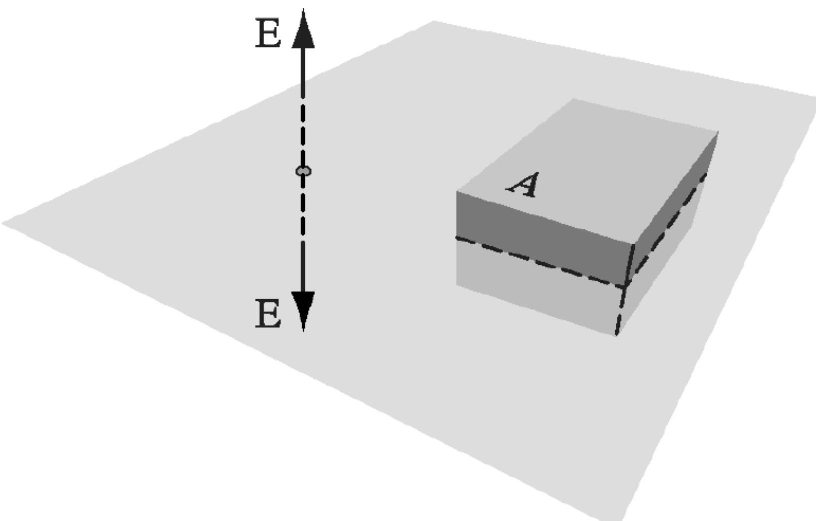
$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}.$$



Example:

An infinite plane carries a uniform surface charge σ. Find its electric field.

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}}$$



$$\int \mathbf{E} \cdot d\mathbf{a} = 2A|\mathbf{E}|$$

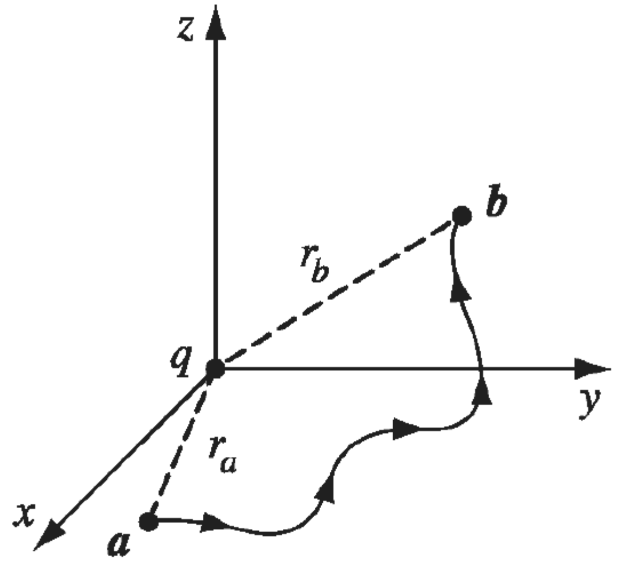
Thus,

$$2A\,|\mathbf{E}| = \frac{1}{\epsilon_0}\sigma A$$

or

$$\mathbf{E} = \frac{\sigma}{2\epsilon_0}\hat{\mathbf{n}},$$

The Curl of E



$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}.$$

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}.$$

In spherical coordinates, $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$, so

$$\mathbf{E} \cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}dr.$$

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0} \int_{\mathbf{a}}^{\mathbf{b}} \frac{q}{r^2} dr = \frac{-1}{4\pi\epsilon_0} \frac{q}{r} \Big|_{r_a}^{r_b} = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{r_a} - \frac{q}{r_b} \right),$$

$$\boxed{\oint \mathbf{E} \cdot d\mathbf{l} = 0,}$$

Stokes' theorem:

$$\int_S d\mathbf{S} \cdot \nabla \times \mathbf{F} = \oint_C d\ell \cdot \mathbf{F}$$

$$\boxed{\boldsymbol{\nabla} \times \mathbf{E} = \mathbf{0}.}$$

If there are many charges, by using the principle of superposition, the total **E**:

$$\boldsymbol{\nabla} \times \mathbf{E} = \boldsymbol{\nabla} \times (\mathbf{E}_1 + \mathbf{E}_2 + \ldots) = (\boldsymbol{\nabla} \times \mathbf{E}_1) + (\boldsymbol{\nabla} \times \mathbf{E}_2) + \ldots = \mathbf{0}.$$

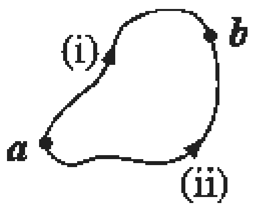
Electric Potential

$$\boxed{V(\mathbf{r}) \equiv -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l}.}$$

O: reference point,
which could be arbitrary

$$V(\mathbf{b}) - V(\mathbf{a}) = -\int_{\mathcal{O}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} + \int_{\mathcal{O}}^{\mathbf{a}} \mathbf{E} \cdot d\mathbf{l}$$

$$= -\int_{\mathcal{O}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} - \int_{\mathbf{a}}^{\mathcal{O}} \mathbf{E} \cdot d\mathbf{l} = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}.$$



$$V(\mathbf{b}) - V(\mathbf{a}) = \int_{\mathbf{a}}^{\mathbf{b}} (\nabla V) \cdot d\mathbf{l},$$

$$\boxed{\mathbf{E} = -\nabla V.}$$

E Vector
V Scalar

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla V) \cdot d\mathbf{l} = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}.$$

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{\mathscr{r}^2} \hat{\mathscr{r}}\, d\tau' \qquad V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{\mathscr{r}}\, d\tau'$$

Poisson's Equation and Laplace's Equation

$$\mathbf{E} = -\nabla V$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

Poisson's equation:

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}.$$

If $\rho=0$, then Poisson's equation becomes Laplace's equation:

$$\nabla^2 V = 0$$

All formula in electrostatics follow from two experimental observations:
(1) The principle of superposition of the electromagnetic forces.
(2) Coulomb's law

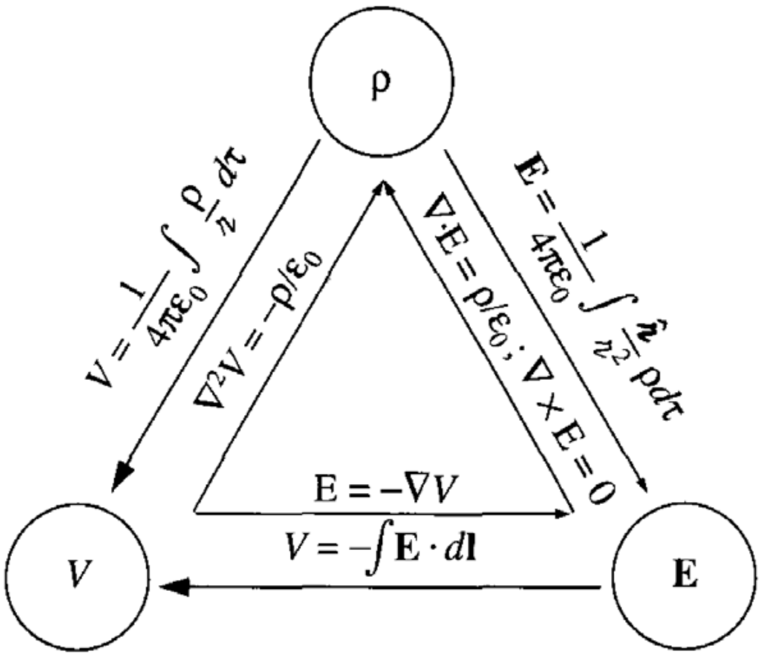
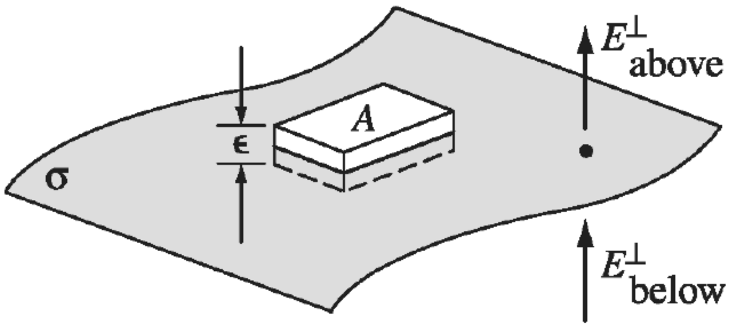
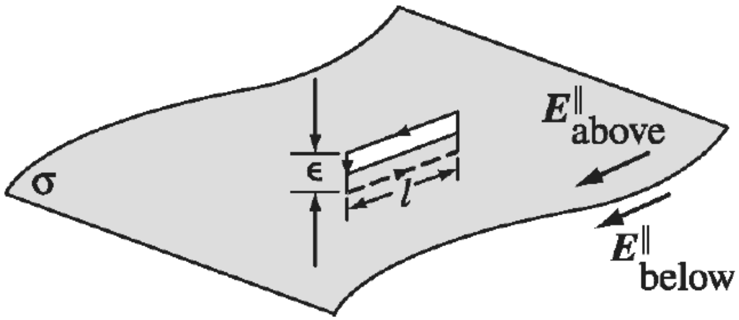


Figure 2.35

Boundary Conditions





$$\oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}} = \frac{1}{\epsilon_0} \sigma A$$

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0$$

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{1}{\epsilon_0} \sigma$$

$$\mathbf{E}^{\parallel}_{\text{above}} = \mathbf{E}^{\parallel}_{\text{below}}$$

$$\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\epsilon_0} \hat{\mathbf{n}}$$

Derive them by yourself

Boundary Conditions

The potential is continuous:

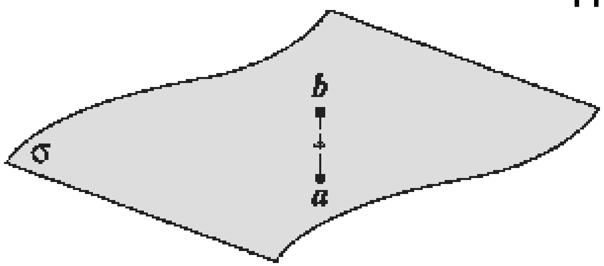


FIGURE 2.38

$$V_{\text{above}} - V_{\text{below}} = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}$$

d**L** becomes 0 $\Rightarrow$ $V_{\text{above}} = V_{\text{below}}$

However, for the gradient of the potential:

$$\mathbf{E} = -\nabla V \quad \Rightarrow \quad \nabla V_{\text{above}} - \nabla V_{\text{below}} = -\frac{1}{\epsilon_0}\sigma\,\hat{\mathbf{n}}$$

Or, more conveniently, $\dfrac{\partial V_{\text{above}}}{\partial n} - \dfrac{\partial V_{\text{below}}}{\partial n} = -\dfrac{1}{\epsilon_0}\sigma$

Where $\dfrac{\partial V}{\partial n} = \nabla V \cdot \hat{\mathbf{n}}$

Work and energy in electrostatics

The work to move a charge Q

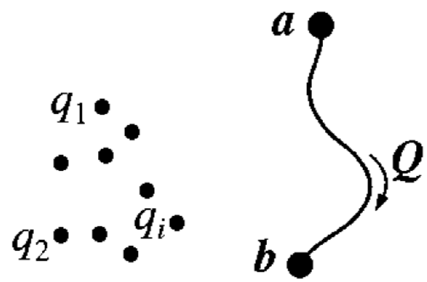


Figure 2.39

$$W = \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l} = -Q \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} = Q[V(\mathbf{b}) - V(\mathbf{a})]$$

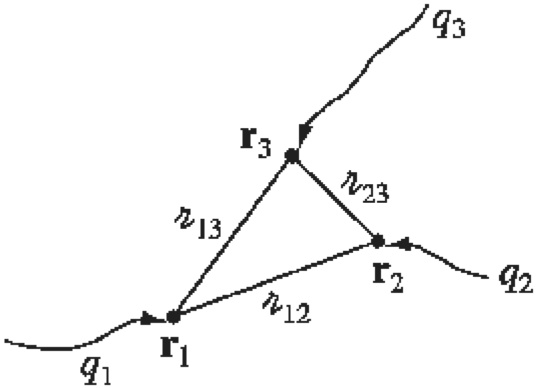
$$V(\mathbf{b}) - V(\mathbf{a}) = \frac{W}{Q}$$

$$W = Q[V(\mathbf{r}) - V(\infty)]$$

If you have set the reference point at infinity ,

$$W = QV(\mathbf{r})$$

The energy of a point charge distribution



$$W_2 = \frac{1}{4\pi\epsilon_0} q_2 \left(\frac{q_1}{\mathscr{r}_{12}}\right)$$

$$W_3 = \frac{1}{4\pi\epsilon_0} q_3 \left(\frac{q_1}{\mathscr{r}_{13}} + \frac{q_2}{\mathscr{r}_{23}}\right)$$

$$W_4 = \frac{1}{4\pi\epsilon_0} q_4 \left(\frac{q_1}{\mathscr{r}_{14}} + \frac{q_2}{\mathscr{r}_{24}} + \frac{q_3}{\mathscr{r}_{34}}\right)$$

$$W = \frac{1}{4\pi\epsilon_0} \left(\frac{q_1 q_2}{\mathscr{r}_{12}} + \frac{q_1 q_3}{\mathscr{r}_{13}} + \frac{q_1 q_4}{\mathscr{r}_{14}} + \frac{q_2 q_3}{\mathscr{r}_{23}} + \frac{q_2 q_4}{\mathscr{r}_{24}} + \frac{q_3 q_4}{\mathscr{r}_{34}}\right)$$

$$W = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j>i}}^{n} \frac{q_i q_j}{\mathscr{r}_{ij}}. \tag{2.40}$$

Generally, $$W = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j>i}}^{n} \frac{q_i q_j}{\mathscr{r}_{ij}}$$

A nicer way : $$W = \frac{1}{8\pi\epsilon_0} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{q_i q_j}{\mathscr{r}_{ij}}$$

Finally, pull out all the factor q_i: $$W = \frac{1}{2} \sum_{i=1}^{n} q_i \left(\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{4\pi\epsilon_0} \frac{q_j}{\mathscr{r}_{ij}} \right)$$

$$W = \frac{1}{2} \sum_{i=1}^{n} q_i V(\mathbf{r}_i)$$

The energy of a continuous charge distribution

$$W = \frac{1}{2} \int \rho V \, d\tau. \tag{2.43}$$

$$\rho = \epsilon_0 \nabla \cdot \mathbf{E} \qquad W = \frac{\epsilon_0}{2} \int (\nabla \cdot \mathbf{E}) V \, d\tau$$

But $\nabla V = -E$,so

$$W = \frac{\epsilon_0}{2} \left(\int_{\mathcal{V}} E^2 \, d\tau + \oint_{\mathcal{S}} V \mathbf{E} \cdot d\mathbf{a} \right)$$

$$W = \frac{\epsilon_0}{2} \int_{\text{all space}} E^2 \, d\tau$$