Lecture 5

Magnetostasics

- 5.1 The Lorentz Force Law
- 5.2 The Biot-Savart Law
- 5.3 The Divergence and Curl of B
- 5.4 Magnetic Vector Potential

Magnetic Fields

Magnetic Field

Up to now,we have confined our attention to the simplest case, electrostatics, in which the source charge is at rest (though the test charge need not be)

FIGURE 5.1

Magnetic Forces

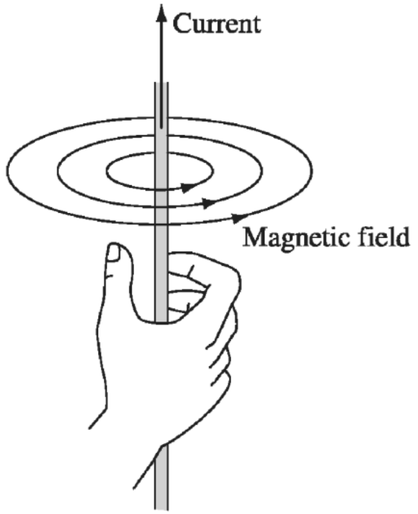


FIGURE 5.3

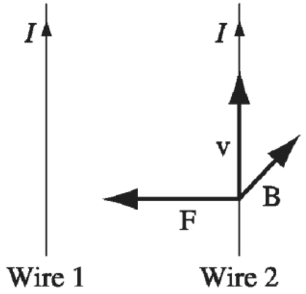


FIGURE 5.4

Lorentz force law

$$\mathbf{F}_{\text{mag}} = Q(\mathbf{v} \times \mathbf{B})$$

In the presence of both electric and magnetic fields

$$\mathbf{F} = Q[\mathbf{E} + (\mathbf{v} \times \mathbf{B})]$$

Example

Cyclotron motion

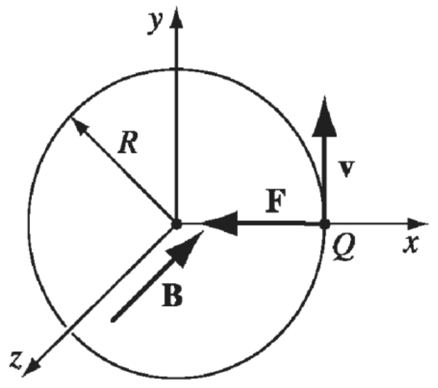


FIGURE 5.5

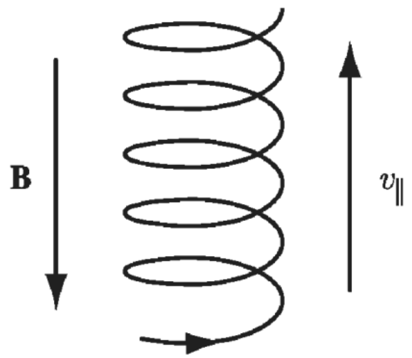


FIGURE 5.6

cyclotron formula: $$QvB = m\frac{v^2}{R}, \quad \text{or} \quad p = QBR$$

It also suggests a simple experimental technique for finding the momentum of a charged particle: send it through a region of known magnetic field, and measure the radius of its trajectory. This is in fact the standard means for determining the momenta of elementary particles

If it starts out with some additional speed $v||$ parallel to $\boldsymbol{B}$, this component of the motion is unaffected by the magnetic field

Example

Cycloid Motion

A more exotic trajectory occurs if we include a uniform electric field, at right angles to the magnetic one. Suppose, for instance, that $\boldsymbol{B}$ points in the x-direction, and $\boldsymbol{E}$ in the z-direction, as shown in Fig. 5.7. A positive charge is released from the origin; what path will it follow?

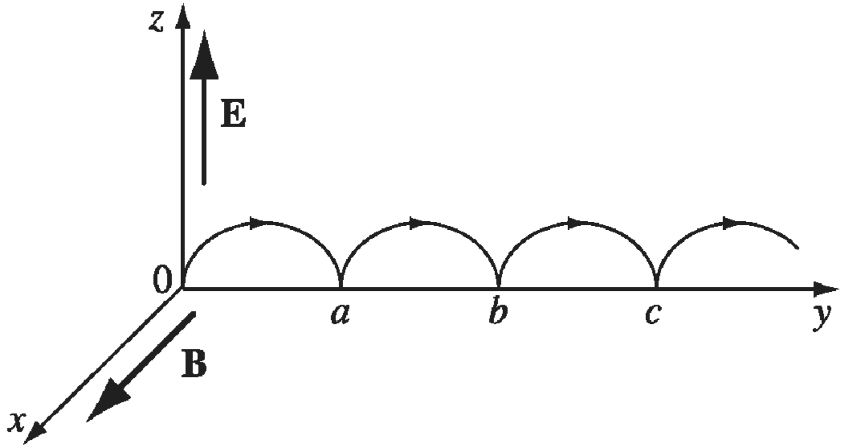


FIGURE 5.7

Let's think it through qualitatively:
Initially, the particle is at rest, so the magneticforce is zero, and the electric field accelerates the charge in the z-direction.
As it picks up speed, a magnetic force develops which, according to Eq. 5.1, pulls the charge around to the right.
The faster it goes, the stronger $Fmag$ becomes;
eventually, it curves the particle back around towards the y axis.
At this point the charge is moving against the electrical force, so it begins to slow down- the magnetic force then decreases, and the electrical force takes over, bringing the particle to rest at point a, in Fig. 5.7. There the entire process commences anew, carrying the particle over to point b, and so on

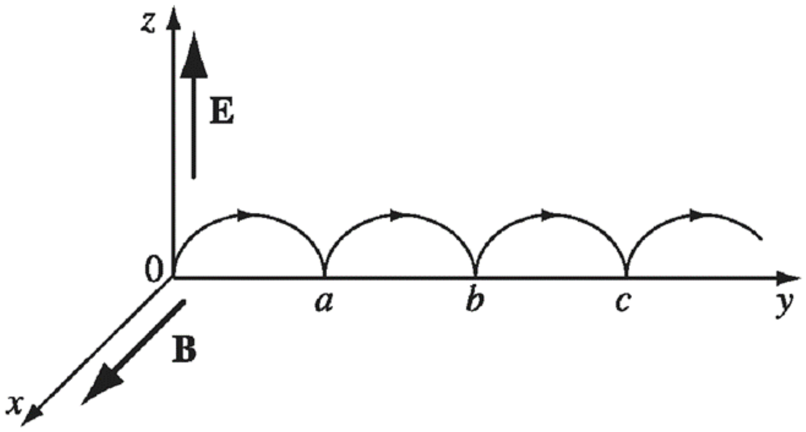


FIGURE 5.7

Note:
Magnetic forces may alter the direction in which a particle moves, but they cannot speed it up or slow it down.

Magnetic forces do not work

$$\mathbf{F}_{\text{mag}} = Q(\mathbf{v} \times \mathbf{B})$$

For if Q moves an amount $d\boldsymbol{l} = \boldsymbol{v}dt$, the work done is

$$dW_{\text{mag}} = \mathbf{F}_{\text{mag}} \cdot d\mathbf{l} = Q(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v}\,dt = 0$$

Magnetic forces may alter the direction in which a particle moves, but they cannot speed it up or slow it down

Currents

The **current** in a wire is the ***charge per unit time*** passing a given point.
It depends on the **product of charge and velocity**

Current is measured in coulombs-per-second, or amperes (A):

$$1\ \text{A} = 1\ \text{C/s}$$

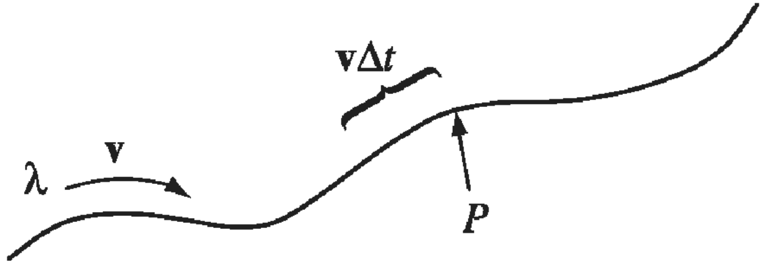


FIGURE 5.9

A line charge λ traveling down a wire at speed v

$$I = \lambda v$$

The magnetic force on a segment of current -carrying wire is

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B})\, dq = \int (\mathbf{v} \times \mathbf{B})\lambda\, dl = \int (\mathbf{I} \times \mathbf{B})\, dl$$

I and ***dl*** both point in the same direction

$$\mathbf{F}_{\text{mag}} = \int I\,(d\mathbf{l} \times \mathbf{B})$$

Typically, the current is constant (in magnitude) along the wire, and in that case *I* comes outside the integral:

$$\mathbf{F}_{\text{mag}} = I \int (d\mathbf{l} \times \mathbf{B})$$

When charge flows over a surface, we describe it by the surface current density, **K**, defined as follows: Consider a "ribbon" of infinitesimal width d⊥, running parallel to the flow (Fig. 5.13). If the current in this ribbon is dI, the surface current density is:

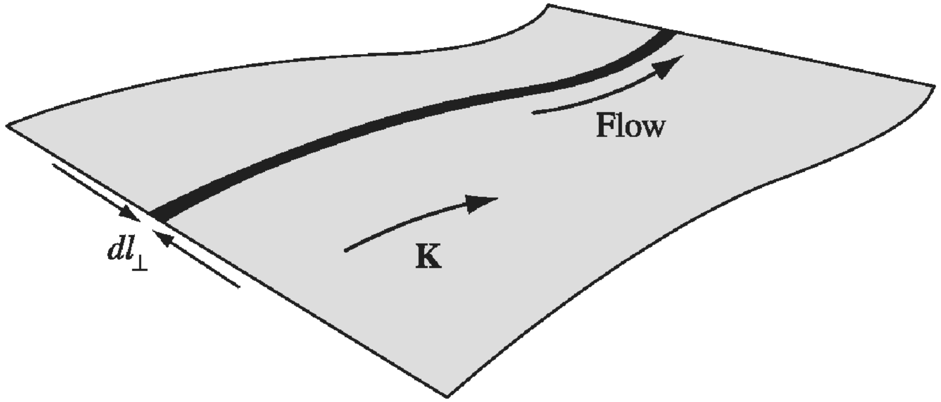


FIGURE 5.13

K is the *current per unit width*

$$\mathbf{K} \equiv \frac{d\mathbf{I}}{dl_\perp}$$

The magnetic force on the surface current is

If the (mobile) surface charge density is a and its velocity is v

$$\mathbf{K} = \sigma \mathbf{v}$$

$$\mathbf{F}_{mag} = \int (\mathbf{v} \times \mathbf{B})\sigma \, da = \int (\mathbf{K} \times \mathbf{B}) \, da$$

When the flow of charge is distributed throughout a three-dimensional region, we describe it by the volume current density, **J**

$$\mathbf{J} \equiv \frac{d\mathbf{I}}{da_{\perp}}$$

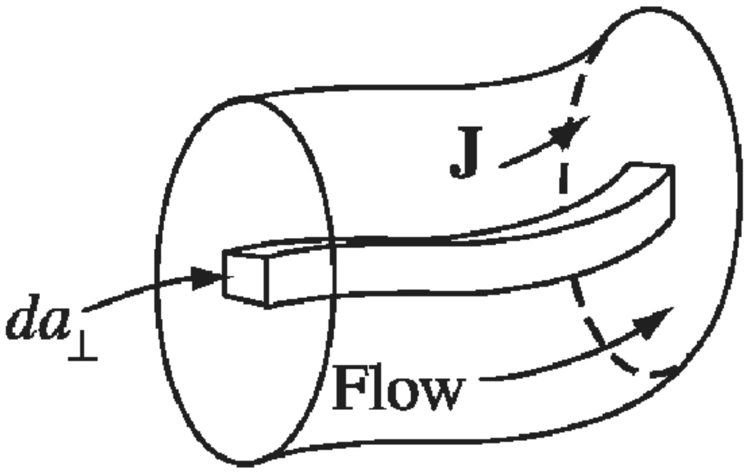


FIGURE 5.14

J is the *current per unit area-perpendicular-to-flow*

$$\mathbf{J} = \rho \mathbf{v}$$

The magnetic force on a volume current is therefore:

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B}) \rho \, d\tau = \int (\mathbf{J} \times \mathbf{B}) \, d\tau$$

The total current crossing a surface S can be written as

$$I = \int_S J\, da_\perp = \int_S \mathbf{J} \cdot d\mathbf{a}$$

In particular, the charge per unit time leaving a volume $\mathcal{V}$ is

$$\oint_S \mathbf{J} \cdot d\mathbf{a} = \int_{\mathcal{V}} (\nabla \cdot \mathbf{J})\, d\tau$$

Because charge is conserved:

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{J})\, d\tau = -\frac{d}{dt} \int_{\mathcal{V}} \rho\, d\tau = -\int_{\mathcal{V}} \left(\frac{\partial \rho}{\partial t}\right) d\tau$$

Continuity equation: $\nabla \cdot \mathbf{J} = -\dfrac{\partial \rho}{\partial t}$

Dictionary

Translating equations into the forms appropriate to point, line, surface, and volume currents:

$$\sum_{i=1}^{n} (\quad) q_i \mathbf{v}_i \sim \int_{\text{line}} (\quad) \mathbf{I}\, dl \sim \int_{\text{surface}} (\quad) \mathbf{K}\, da \sim \int_{\text{volume}} (\quad) \mathbf{J}\, d\tau$$

$q \sim \lambda dl \sim \sigma da \sim \rho d\tau$ for the various charge distributions

The Biot-Savart Law

Steady Currents

Stationary charges	$\Rightarrow$	**constant electric fields: electrostatics.**
Steady currents	$\Rightarrow$	**constant magnetic fields: magnetostatics.**

By steady current I mean a continuous flow that has been going on forever, without change and without charge piling up anywhere

Note: a moving point charge cannot possibly constitute a steady current

When a steady current flows in a wire, its magnitude I must be the same all along the line:

$$\nabla \cdot \mathbf{J} = 0$$

The Magnetic Field of a Steady Current

The magnetic field of a steady line current is given by the **Biot-Savart law**

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \, dl' = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}' \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

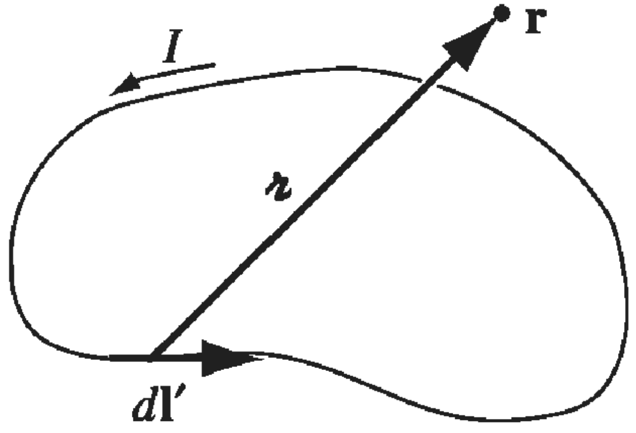


FIGURE 5.17

The constant μ_0 is called the permeability of free space:

$$\mu_0 = 4\pi \times 10^{-7} \, {}^{N}/_{A^2}$$

B itself comes out in newtons per ampere-meter or **teslas**(T)

$$1T = 1 \, {}^{N}/_{(A \cdot m)}$$

Example

Find the magnetic field a distance s from a long straight wire carrying a steady current ***I*** (Fig. 5.18).

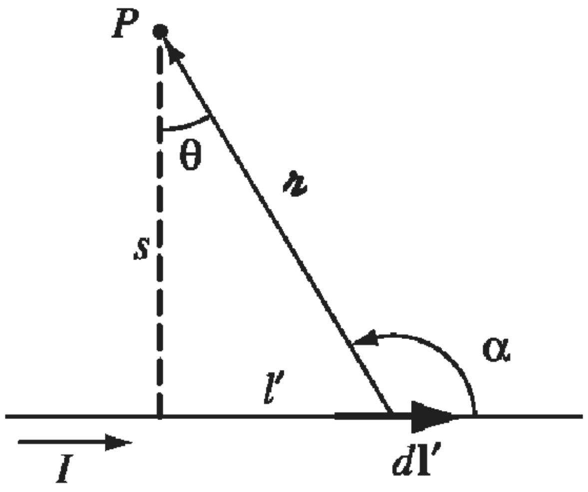


FIGURE 5.18

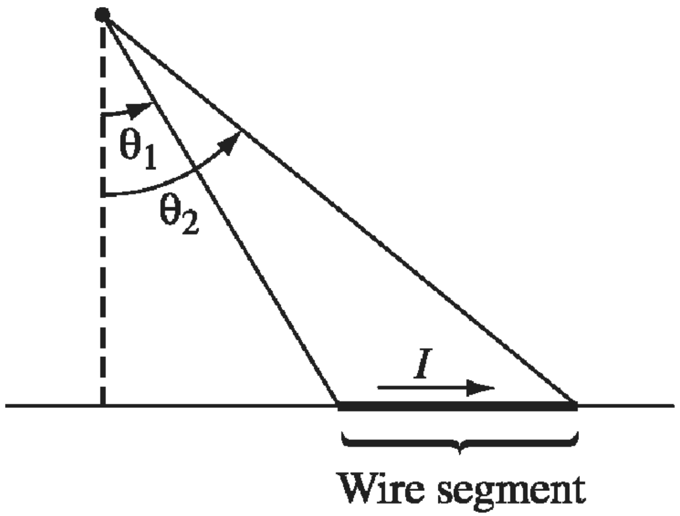


FIGURE 5.19

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I}\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,dl' = \frac{\mu_0}{4\pi}I\int \frac{d\mathbf{l}'\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

Solution

In the diagram, $(d\boldsymbol{l}' \times \hat{\boldsymbol{\mathscr{r}}})$points out of the page, and has the magnitude

$$dl'\sin\alpha = dl'\cos\theta$$

Also, $l' = s\tan\theta$

$$dl' = \frac{s}{\cos^2\theta}\,d\theta$$

And, $s = \mathscr{r}\cos\theta$

$$\frac{1}{\mathscr{r}^2} = \frac{\cos^2\theta}{s^2}$$

Thus:

$$B = \frac{\mu_0 I}{4\pi}\int_{\theta_1}^{\theta_2}\left(\frac{\cos^2\theta}{s^2}\right)\left(\frac{s}{\cos^2\theta}\right)\cos\theta\,d\theta$$

$$= \frac{\mu_0 I}{4\pi s}\int_{\theta_1}^{\theta_2}\cos\theta\,d\theta = \frac{\mu_0 I}{4\pi s}(\sin\theta_2 - \sin\theta_1)$$

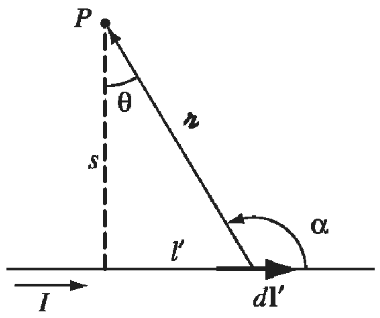


FIGURE 5.18

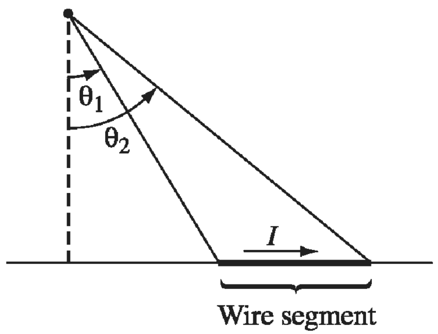


FIGURE 5.19

In the case of an infinite wire,

θ_1=-$\frac{\pi}{2}$ $and\ \theta_2$=$\frac{\pi}{2}$

We obtain:

$$B = \frac{\mu_0 I}{2\pi s}$$

For surface and volume currents, the ***Biot-Savart law*** becomes

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}(\mathbf{r}') \times \hat{\boldsymbol{\imath}}}{\imath^2} \, da'$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\imath}}}{\imath^2} \, d\tau',$$

For a moving point charge

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{q\mathbf{v} \times \hat{\boldsymbol{\imath}}}{\imath^2}.$$

The Divergence and Curl of B

Straight-Line Currents

The magnetic field of an infinite straight wire is shown in Fig. 5.27 has a nonezero curl

$$\oint \mathbf{B} \cdot d\mathbf{l} = \oint \frac{\mu_0 I}{2\pi s} dl = \frac{\mu_0 I}{2\pi s} \oint dl = \mu_0 I.$$

For a bundle of straight wires

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$$

where *Ienc* stands for the total current enclosed by the integration path. If the flow of charge is represented by a volume current density ***J***, the enclosed current is

$$I_{\text{enc}} = \int \mathbf{J} \cdot d\mathbf{a}$$

Applying Stoke's theorem

$$\int (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a} \longrightarrow \nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

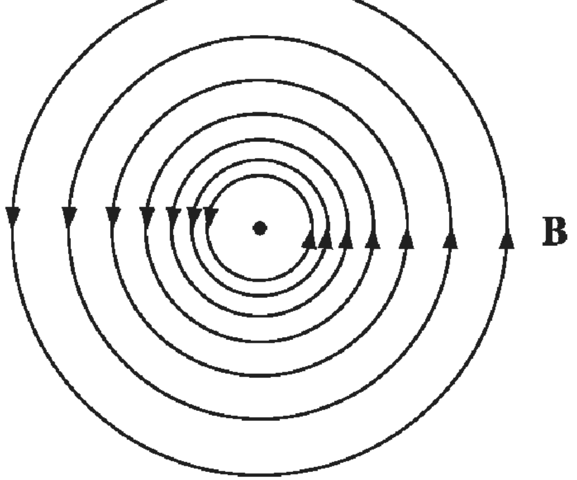


FIGURE 5.27

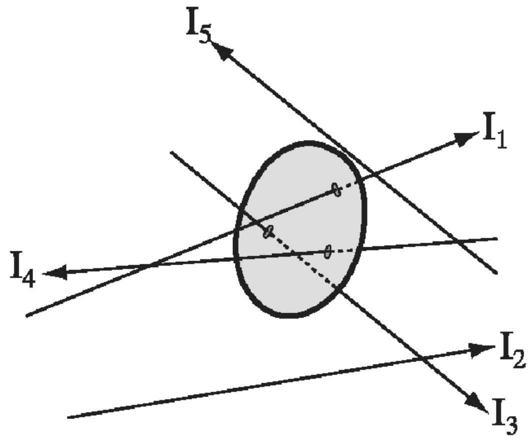


FIGURE 5.29

The Divergence and Curl of B

The formal derivation of the divergence and curl of ***B*** from the Biot-Savart law itself

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \, d\tau'$$

$\mathbf{B}$ is a function of (x, y, z),

$\mathbf{J}$ is a function of (x', y', z'),

$$\boldsymbol{\mathscr{r}} = (x - x')\,\hat{\mathbf{x}} + (y - y')\,\hat{\mathbf{y}} + (z - z')\,\hat{\mathbf{z}}$$

$$d\tau' = dx'\,dy'\,dz'.$$

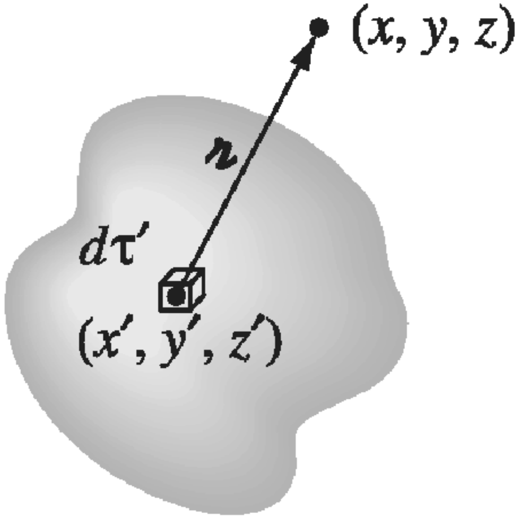


FIGURE 5.30

(1) Applying the divergence

$$\nabla \cdot \mathbf{B} = \frac{\mu_0}{4\pi} \int \nabla \cdot \left(\mathbf{J} \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) d\tau'$$

Invoking product rule number 6

$$\nabla \cdot \left(\mathbf{J} \times \frac{\hat{\boldsymbol{\imath}}}{\imath^2}\right) = \frac{\hat{\boldsymbol{\imath}}}{\imath^2} \cdot (\nabla \times \mathbf{J}) - \mathbf{J} \cdot \left(\nabla \times \frac{\hat{\boldsymbol{\imath}}}{\imath^2}\right)$$

But $\nabla \times \boldsymbol{J} = \mathbf{0}$, because J doesn't depend on the unprimed variables (x, y, z), whereas

$$\nabla \times (\hat{\boldsymbol{\imath}}/\imath^2) = \mathbf{0}$$

$$\nabla \cdot \mathbf{B} = 0.$$

Note: divergence of the magnetic field is zero

(2) Applying the curl

$$\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi} \int \nabla \times \left(\mathbf{J} \times \frac{\hat{\boldsymbol{\imath}}}{\imath^2} \right) d\tau'$$

using the appropriate product rule-in this case number 8

$$\nabla \times \left(\mathbf{J} \times \frac{\hat{\boldsymbol{\imath}}}{\imath^2} \right) = \mathbf{J} \left(\nabla \cdot \frac{\hat{\boldsymbol{\imath}}}{\imath^2} \right) - (\mathbf{J} \cdot \nabla) \frac{\hat{\boldsymbol{\imath}}}{\imath^2}$$

The first term:

$$\nabla \cdot \left(\frac{\hat{\boldsymbol{\imath}}}{\imath^2} \right) = 4\pi \delta^3(\boldsymbol{\imath})$$

The second term:

Because the derivative acts only on $\hat{\boldsymbol{\imath}}/\imath^2$ we can switch from ∇ *to* ∇'

$$-(\mathbf{J} \cdot \nabla) \frac{\hat{\boldsymbol{\imath}}}{\imath^2} = (\mathbf{J} \cdot \nabla') \frac{\hat{\boldsymbol{\imath}}}{\imath^2}$$

The x component, in particular, is

$$(\mathbf{J}\cdot\nabla')\left(\frac{x-x'}{\mathscr{r}^3}\right) = \nabla'\cdot\left[\frac{(x-x')}{\mathscr{r}^3}\mathbf{J}\right] - \left(\frac{x-x'}{\mathscr{r}^3}\right)(\nabla'\cdot\mathbf{J})$$

For steady currents the divergence of **J** is zero:

$$\left[-(\mathbf{J}\cdot\nabla)\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right]_x = \nabla'\cdot\left[\frac{(x-x')}{\mathscr{r}^3}\mathbf{J}\right]$$

$$\int_{\mathcal{V}}\nabla'\cdot\left[\frac{(x-x')}{\mathscr{r}^3}\mathbf{J}\right]d\tau' = \oint_{\mathcal{S}}\frac{(x-x')}{\mathscr{r}^3}\mathbf{J}\cdot d\mathbf{a}'$$

You can make the integration over a surface that J=0

Finally:

$$\nabla\times\mathbf{B} = \frac{\mu_0}{4\pi}\int\mathbf{J}(\mathbf{r}')4\pi\delta^3(\mathbf{r}-\mathbf{r}')\,d\tau' = \mu_0\mathbf{J}(\mathbf{r})$$ Ampere's law

Application of Ampere's Law

The equation for the curl of ***B***

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$ Ampere's law

Stokes' theorem

$$\int (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a}$$

Note : $\int \boldsymbol{J} \cdot d\boldsymbol{a}$ is the total current passing through the surface

$Ienc$ (the current enclosed by the Amperian loop)

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$$

The direction through the surface corresponds to a "positive" current:

right-hand rule: If the fingers of your right hand indicate the direction of integration around the boundary, then your thumb defines the direction of a positive current.

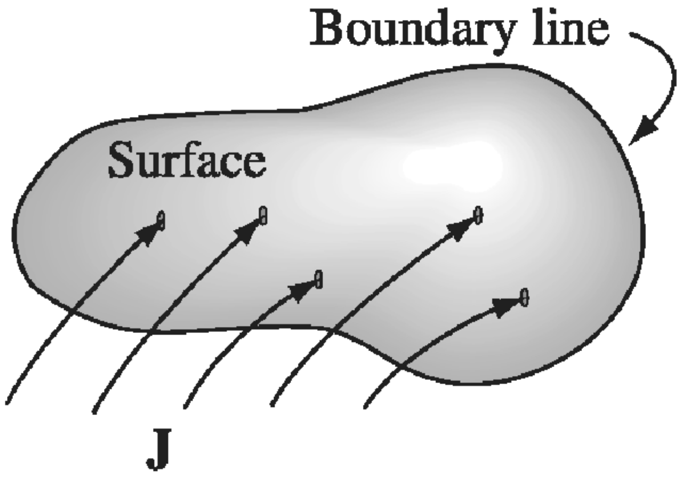


FIGURE 5.31

Just as the Biot-Savart law plays a role in magnetostatics that Coulomb's law assumed in electrostatics, Ampere’s law plays the part of Gauss’s law:

$$\begin{cases} \text{Electrostatics :} & \text{Coulomb} & \rightarrow & \text{Gauss,} \\ \text{Magnetostatics :} & \text{Biot}-\text{Savart} & \rightarrow & \text{Ampère.} \end{cases}$$

Example

Find the magnetic field a distance s from a long straight wire (Fig. 5.32), carrying a steady current ***I*** (the same problem we solved in Ex. 5.5,using the Biot-Savart law)

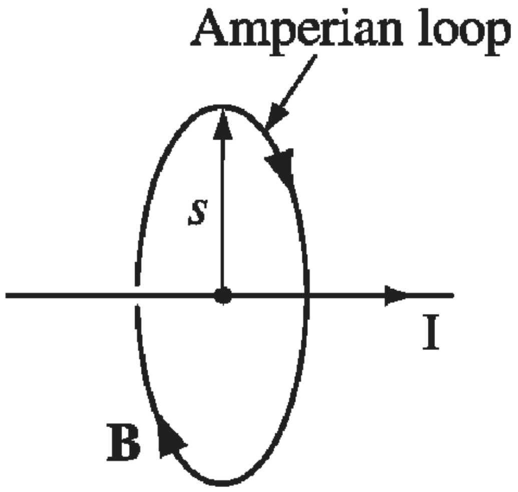


FIGURE 5.32

Solution

We know the direction of ***B*** is "circumferential," circling around the wire as indicated by the right-hand rule. By symmetry, the magnitude of B is constant around an Amperian loop of radius ***s***, centered on the wire. So Ampere's law gives:

$$\oint \mathbf{B} \cdot d\mathbf{l} = B \oint dl = B 2\pi s = \mu_0 I_{\text{enc}} = \mu_0 I$$

or $$B = \frac{\mu_0 I}{2\pi s}$$

Like Gauss's law, Ampere's law is always true (for steady currents), but it is not always useful. Only when the symmetry of the problem enables you to pull $\boldsymbol{B}$ outside the integral $\oint \boldsymbol{B} \cdot d\boldsymbol{l}$ can you calculate the magnetic field from Ampere's law. When it does work, it's by far the fastest method; when it doesn't, you have to fall back on the *Biot-Savart law*. The current configurations that can be handled by Ampere's law in Griffiths' book are:

1. Infinite straight lines (prototype: Ex. 5.7).
2. Infinite planes (prototype: Ex. 5.8).
3. Infinite solenoids (prototype: Ex. 5.9).
4. Toroids (prototype: Ex. 5.10).

Comparison of Magnetostatics and Electrostatics

The divergence and curl of the electrostatic field are

$$\begin{cases} \nabla \cdot \mathbf{E} = \dfrac{1}{\epsilon_0}\rho, & \text{(Gauss's law);} \\ \nabla \times \mathbf{E} = \mathbf{0}, & \text{(no name).} \end{cases}$$

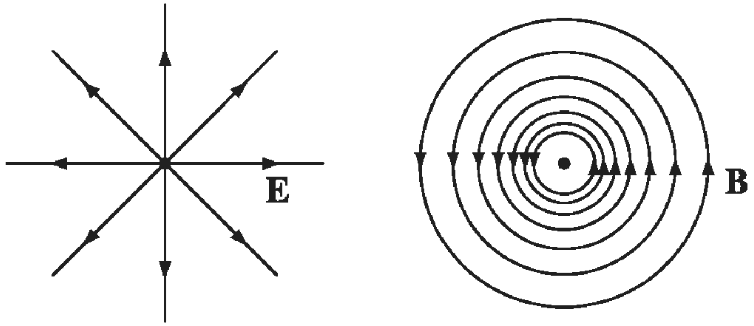
These are Maxwell's equations for electrostatics. Together with the boundary condition $\boldsymbol{E} \to 0$ far from all charges, Maxwell's equations determine the field, if the source charge density p is given; they contain essentially the same information as Coulomb's law plus the principle of superposition. The divergence and curl of the magnetostatic field are:

$$\begin{cases} \nabla \cdot \mathbf{B} = 0, & \text{(no name);} \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J}, & \text{(Ampère's law).} \end{cases}$$

These are Maxwell's equations for magnetostatics. Again, together with the boundary condition $\mathbf{B} \to 0$ far from all currents, Maxwell's equations determine the magnetic field; they are equivalent to the Biot-Savart law (plus superposition).

Maxwell's equations and the force law:

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$



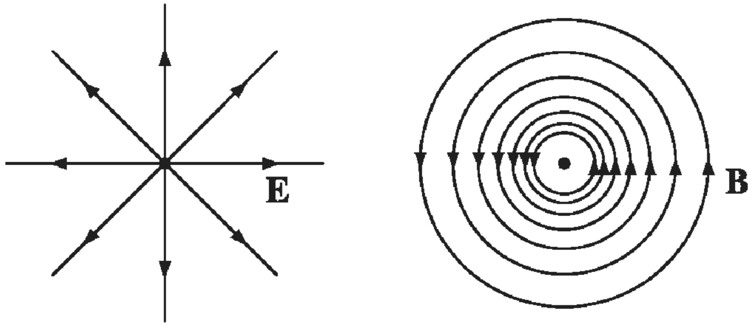
(a) Electrostatic field of a point charge (b) Magnetostatic field of a long wire

FIGURE 5.44

- The electric field diverges away from a (positive) charge; the magnetic field line *curls around a current* (Fig. 5.44).
- Electric field lines originate on positive charges and terminate on negative ones; magnetic field lines do not begin or end anywhere-to do so would require a nonzero divergence. They typically form closed loops or extend out to infinity.
- To put it another way, *there are no point sources* for **B**, as there are for E; there exists no magnetic analog to electric charge. This is the physical content of the statement $\nabla \cdot \boldsymbol{B} = 0$. Coulomb and others believed that magnetism was produced by **magnetic charges** (**magnetic monopoles**, as we would now call them), and in some older books you will still find references to a magnetic version of Coulomb's law, giving the force of attraction or repulsion between them. It was Ampere who first speculated that all magnetic effects are attributable to electric charges in motion (currents).

- As far as we know, Ampere was right; nevertheless, it remains an open experimental question whether magnetic monopoles exist in nature (they are obviously pretty rare, or somebody would have found one.

- For our purposes, though, **B** is divergence less, and there are no magnetic monopoles. It takes a moving electric charge to produce a magnetic field, and it takes another moving electric charge to "feel" a magnetic field.



(a) Electrostatic field of a point charge

(b) Magnetostatic field of a long wire

FIGURE 5.44

Magnetic Vector Potential

The Vector Potential

Just as $\nabla \times \boldsymbol{E} = 0$ permitted us to introduce a scalar potential (V) in electrostatics

$$\mathbf{E} = -\nabla V$$

So $\nabla \cdot \boldsymbol{B} = 0$ invites the introduction of a *vector* potential **A** in magnetostatics

$$\mathbf{B} = \nabla \times \mathbf{A}.$$

The potential formulation automatically takes care of $\nabla \cdot \boldsymbol{B} = 0$ (since the divergence of a curl is always zero); there remains Ampere's law:

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}$$

- Add to **A** any function whose curl vanishes with no effect on **B**.
- We can exploit this freedom to eliminate the divergence of **A** :

$$\nabla \cdot \mathbf{A} = 0. \qquad \nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}.$$

Poisson's equation

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$

$$V = \frac{1}{4\pi\epsilon_0} \int \frac{\rho}{r} \, d\tau',$$

$$\nabla^2 \varphi = f.$$

In three-dimensional Cartesian coordinates, it takes the form

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \varphi(x, y, z) = f(x, y, z).$$

When $f = 0$ identically we obtain Laplace's equation.

Poisson's equation may be solved using a Green's function:

$$\varphi(\mathbf{r}) = -\iiint \frac{f(\mathbf{r}')}{4\pi|\mathbf{r} - \mathbf{r}'|} \, \mathrm{d}^3 r',$$

- Add to **A** any function whose curl vanishes with no effect on **B**.
- We can exploit this freedom to eliminate the divergence of **A** :

$$\nabla \cdot \mathbf{A} = 0 \qquad \nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$$

Poisson's equation

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$

Assuming **J** goes to zero at infinity, we can read off the solution:

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I}}{\imath} dl' = \frac{\mu_0 I}{4\pi} \int \frac{1}{\imath} d\mathbf{l}' \qquad \mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}}{\imath} da' \qquad V = \frac{1}{4\pi\epsilon_0} \int \frac{\rho}{\imath} d\tau'$$

If the current does not go to zero at infinity, we have to find other ways to get **A**; some of these are explored in Ex. 5.12 and in the problems at the end of the section.

It must be said that **A** is not as useful as **V** For one thing, it's still a *vector.* Nevertheless, the vector potential has substantial theoretical importance, as we shall see in Chapter 10.

Example

A spherical shell of radius R, carrying a uniform surface charge a, is set spinning at angular velocity (J). Find the vector potential it produces at point **r** (Fig. 5.45)

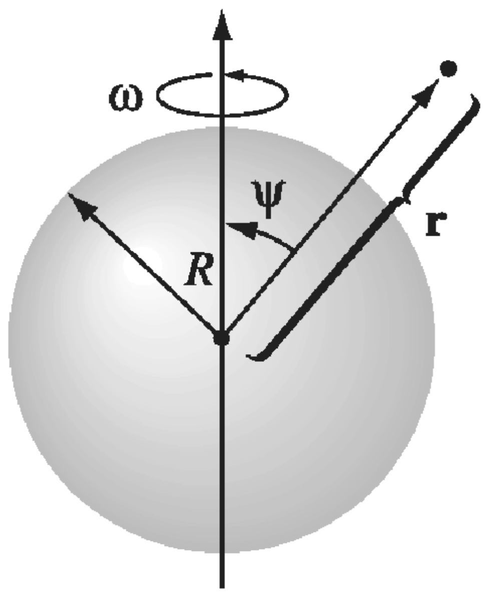


FIGURE 5.45

Solution

It might seem natural to set the polar axis along ω, but in fact the integration is easier if we let **r** lie on the z axis, so that ω is tilted at an angle ψ. We may as well orient the x axis so that ω lies in the xz plane, as shown in Fig. 5.46.

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{K}(\mathbf{r}')}{\mathscr{r}}\,da'$$

where $\mathbf{K} = \sigma\mathbf{v}$, $\mathscr{r} = \sqrt{R^2 + r^2 - 2Rr\cos\theta'}$, and $da' = R^2\sin\theta'\,d\theta'\,d\phi'$. Now the velocity of a point $\mathbf{r}'$ in a rotating rigid body is given by $\boldsymbol{\omega}\times\mathbf{r}'$; in this case,

$$\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}' = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \omega\sin\psi & 0 & \omega\cos\psi \\ R\sin\theta'\cos\phi' & R\sin\theta'\sin\phi' & R\cos\theta' \end{vmatrix}$$

$$= R\omega\left[-(\cos\psi\sin\theta'\sin\phi')\,\hat{\mathbf{x}} + (\cos\psi\sin\theta'\cos\phi' - \sin\psi\cos\theta')\,\hat{\mathbf{y}}\right.$$

$$\left. + (\sin\psi\sin\theta'\sin\phi')\,\hat{\mathbf{z}}\right].$$

Notice that each of these terms, save one, involves either $\sin\phi'$ or $\cos\phi'$. Since

$$\int_0^{2\pi}\sin\phi'\,d\phi' = \int_0^{2\pi}\cos\phi'\,d\phi' = 0,$$

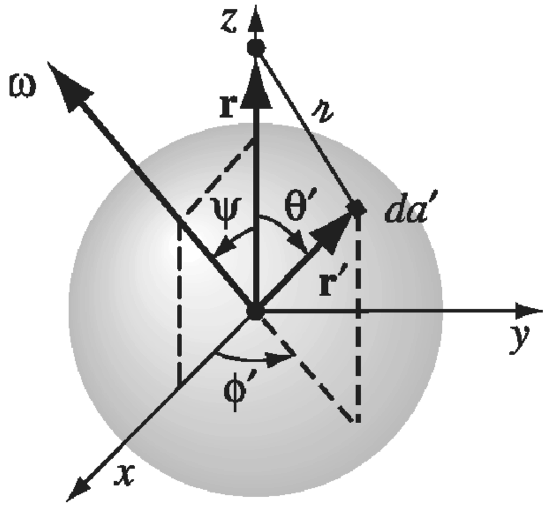


FIGURE 5.46

such terms contribute nothing. There remains

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0 R^3 \sigma \omega \sin\psi}{2} \left(\int_0^{\pi} \frac{\cos\theta' \sin\theta'}{\sqrt{R^2 + r^2 - 2Rr\cos\theta'}} \, d\theta' \right) \hat{\mathbf{y}}.$$

Letting $u \equiv \cos\theta'$, the integral becomes

$$\int_{-1}^{+1} \frac{u}{\sqrt{R^2 + r^2 - 2Rru}} \, du = -\frac{(R^2 + r^2 + Rru)}{3R^2r^2} \sqrt{R^2 + r^2 - 2Rru} \, \Bigg|_{-1}^{+1}$$

$$= -\frac{1}{3R^2r^2} \big[(R^2 + r^2 + Rr)|R - r| - (R^2 + r^2 - Rr)(R + r) \big].$$

If the point $\mathbf{r}$ lies *inside* the sphere, then $R > r$, and this expression reduces to $(2r/3R^2)$; if $\mathbf{r}$ lies *outside* the sphere, so that $R < r$, it reduces to $(2R/3r^2)$. Noting that $(\boldsymbol{\omega} \times \mathbf{r}) = -\omega r \sin\psi\, \hat{\mathbf{y}}$, we have, finally,

$$\mathbf{A}(\mathbf{r}) = \begin{cases} \dfrac{\mu_0 R \sigma}{3} (\boldsymbol{\omega} \times \mathbf{r}), & \text{for points } \textit{inside} \text{ the sphere,} \\[2ex] \dfrac{\mu_0 R^4 \sigma}{3r^3} (\boldsymbol{\omega} \times \mathbf{r}), & \text{for points } \textit{outside} \text{ the sphere.} \end{cases} \tag{5.68}$$

Boundary Conditions

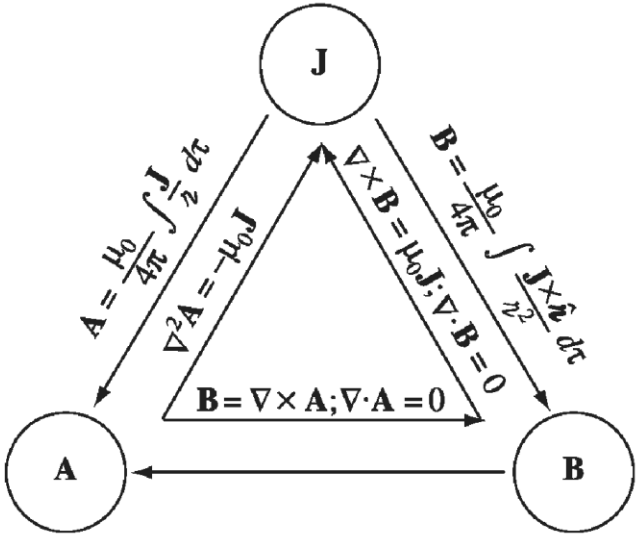


FIGURE 5.48

Just as the electric field suffers a discontinuity at a surface charge, so the magnetic field is discontinuous at a *surface current.*

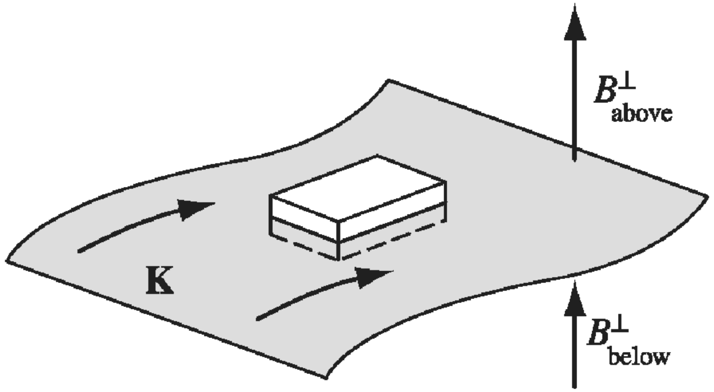


FIGURE 5.49

$$\oint \mathbf{B}\cdot d\mathbf{a} = 0.$$

To a wafer-thin pillbox straddling the surface

$$B^{\perp}_{\text{above}} = B^{\perp}_{\text{below}}$$

As for the tangential components, an Amperian loop running perpendicular to the current (Fig. 5.50) yields

$$\oint \mathbf{B} \cdot d\mathbf{l} = \left(B^{\parallel}_{\text{above}} - B^{\parallel}_{\text{below}}\right) l = \mu_0 I_{\text{enc}} = \mu_0 K l$$

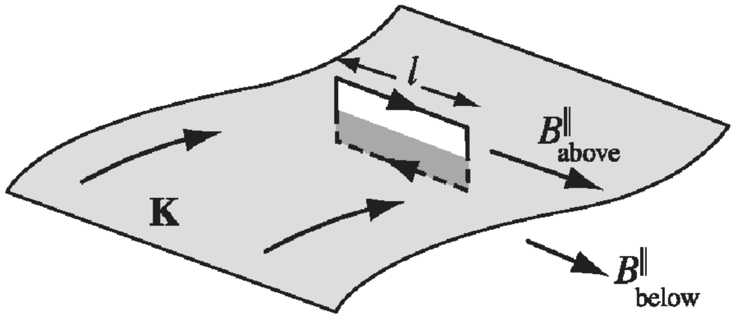


FIGURE 5.50

or $$B^{\parallel}_{\text{above}} - B^{\parallel}_{\text{below}} = \mu_0 K$$

$$\mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \mu_0 (\mathbf{K} \times \hat{\mathbf{n}})$$

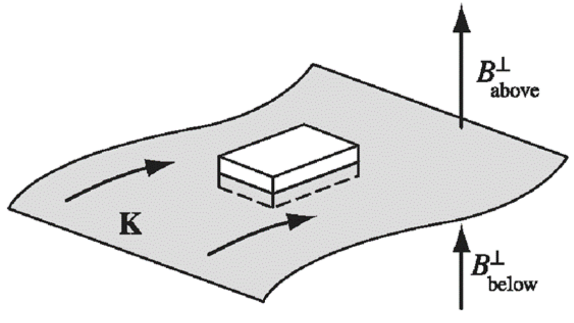


FIGURE 5.49

where $\boldsymbol{n}$ is a unit vector perpendicular to the surface, pointing "upward."

Like the scalar potential in electrostatics, the vector potential is continuous across any boundary:

$$\mathbf{A}_{\text{above}} = \mathbf{A}_{\text{below}}$$

for $\nabla \cdot \boldsymbol{A} = 0$ guarantees that the *normal* component is continuous; and $\nabla \times \boldsymbol{A} = \boldsymbol{B}$, in the form:

$$\oint \mathbf{A} \cdot d\mathbf{l} = \int \mathbf{B} \cdot d\mathbf{a} = \Phi$$

means that the tangential components are continuous (the flux through an Amperian loop of vanishing thickness is zero). But the derivative of **A** inherits the discontinuity of **B:**

$$\frac{\partial \mathbf{A}_{\text{above}}}{\partial n} - \frac{\partial \mathbf{A}_{\text{below}}}{\partial n} = -\mu_0 \mathbf{K}$$

Multipole Expansion of the Vector Potential

An approximate formula for the vector potential of a localized current distribution, valid at distant points, a multipole expansion is in order. Remember: the idea of a multipole expansion is to write the potential in the form of a power series in 1/r, where r is the distance to the point in question; if **r** is sufficiently large, the series will be dominated by the lowest nonvanishing contribution, and the higher terms can be ignored.

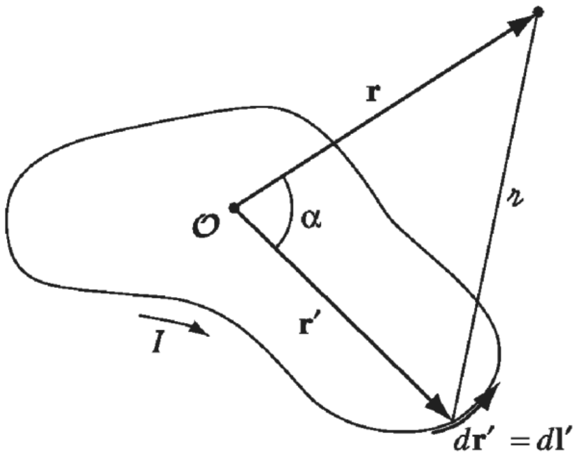


FIGURE 5.51

$$\frac{1}{\mathscr{r}} = \frac{1}{\sqrt{r^2 + (r')^2 - 2rr'\cos\alpha}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha)$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\oint \frac{1}{\mathscr{r}}\, d\mathbf{l}' = \frac{\mu_0 I}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\oint (r')^n P_n(\cos\alpha)\, d\mathbf{l}'$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\left[\frac{1}{r}\oint d\mathbf{l}' + \frac{1}{r^2}\oint r'\cos\alpha\, d\mathbf{l}' + \frac{1}{r^3}\oint (r')^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right) d\mathbf{l}' + \cdots\right]$$

The magnetic monopole term is always zero, for the integral is just the total vector displacement around a closed loop

$$\oint d\mathbf{l}' = \mathbf{0}.$$

In the absence of any monopole contribution, the dominant term is the dipole (except in the rare case where it, too, vanishes):

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0 I}{4\pi r^2}\oint r' \cos\alpha \, d\mathbf{l}' = \frac{\mu_0 I}{4\pi r^2}\oint (\hat{\mathbf{r}} \cdot \mathbf{r}')\, d\mathbf{l}'$$

$$\oint (\hat{\mathbf{r}} \cdot \mathbf{r}')\, d\mathbf{l}' = -\hat{\mathbf{r}} \times \int d\mathbf{a}'$$

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2}$$ **m** is the ***magnetic dipole moment*** $$\mathbf{m} \equiv I \int d\mathbf{a} = I\mathbf{a}$$

Here *a* is the "vector area" of the loop (Pro b. 1.62); if the loop is flat, a is the ordinary area enclosed, with the direction assigned by the usual right-hand rule(fingers in the direction of the current).