1. A spaceship travels from planet A to planet B with a speed of $\frac{4}{5}c$, where c is the speed of light in the vacuum. The captain on the spaceship finds that it took 90 hours from planet A to planet B according to the watch on the spaceship. He sends immediately a message with light to planet A when he is passing planet B.
   (a) According to clocks on planet A, when was the message sent?
   (b) According to clocks on planet B, how long did the message arrive on planet A?
   (c) For the captain on the spaceship, how long did the message arrive on planet A after he passed planet A?

(20 points)

2. For an electromagnetic wave in the vacuum, the magnetic field is given by

$$\boldsymbol{B} = -\frac{2B_0\cos\theta}{\omega r^2}\left(\sin u + \frac{1}{kr}\cos u\right)\hat{\boldsymbol{r}} + \frac{B_0\sin\theta}{\omega r}\left(k\cos u - \frac{1}{kr^2}\cos u - \frac{1}{r}\sin u\right)\hat{\boldsymbol{\theta}}$$

where $u = kr - \omega t$ and $k = \omega/c$.
   (a) Find the associated electric field. (Hint: use one of Maxwell's equations)
   (b) Check that the associated electric field satisfies Gauss's law.
   (c) Calculate the Poynting vector.
   (d) Find the intensity of the electromagnetic wave. (Hint: average the Poynting vector over a complete cycle).
   (e) Find the total power radiated by the electromagnetic wave.

(40 points)

3. For a source of time-dependent current density

$$\boldsymbol{J}(\boldsymbol{r},t) = \boldsymbol{J}(\boldsymbol{r})e^{-i\omega t}, \qquad (1)$$

the vector potential in the far (radiation) zone ($kr \gg 1$ where $k = \omega/c$) can be simplified as

$$\boldsymbol{A}(\boldsymbol{r},t) = \boldsymbol{A}(\boldsymbol{r})e^{-i\omega t}, \qquad (2)$$

where

$$\boldsymbol{A}(\boldsymbol{r}) \approx \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \boldsymbol{J}(\boldsymbol{r}')e^{-ik\hat{\boldsymbol{r}}\cdot\boldsymbol{r}'}d^3r'\,. \qquad (3)$$

Now consider a system shown in the figure, a thin and linear antenna of length *2d* which is excited across a small gap at the midpoint. The current density is symmetric on the two arms of the antenna and vanishes at the two ends of the antenna, which can be expressed by Eq. (1) with

$$\boldsymbol{J}(\boldsymbol{r}) = I\sin(kd - k|z|)\,\delta(x)\delta(y)\hat{\boldsymbol{z}}, \qquad (4)$$

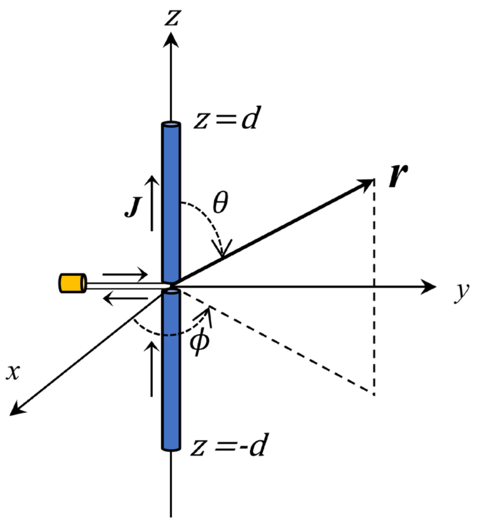
where $-d \le z \le d$. In the far zone, the vector potential generated by the current in two arms of the antenna can be calculated from Eq. (3) (neglect the connecting wires):

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\frac{2Ie^{ikr}}{kr}\left[\frac{\cos(kd\cos\theta) - \cos(kd)}{\sin^2\theta}\right]\hat{\boldsymbol{z}}. \qquad (5)$$

(a) Find the electric and magnetic fields far away from the antenna in the radiation zone.

(b) Show that the time-averaged power radiated per unit solid angle is

$$\frac{dP}{d\Omega} = \frac{\mu_0 cI^2}{8\pi^2}\frac{e^{ikr}}{kr}\left[\frac{\cos(kd\cos\theta) - \cos(kd)}{\sin\theta}\right]^2$$



(40 points)

4. **Extra Question**: Derive Eq. (5).
Note: You can receive maximum 5 bonus points until you get full (100) marks. First finish questions 1-3 and check your answers. If you have time left, try this one.