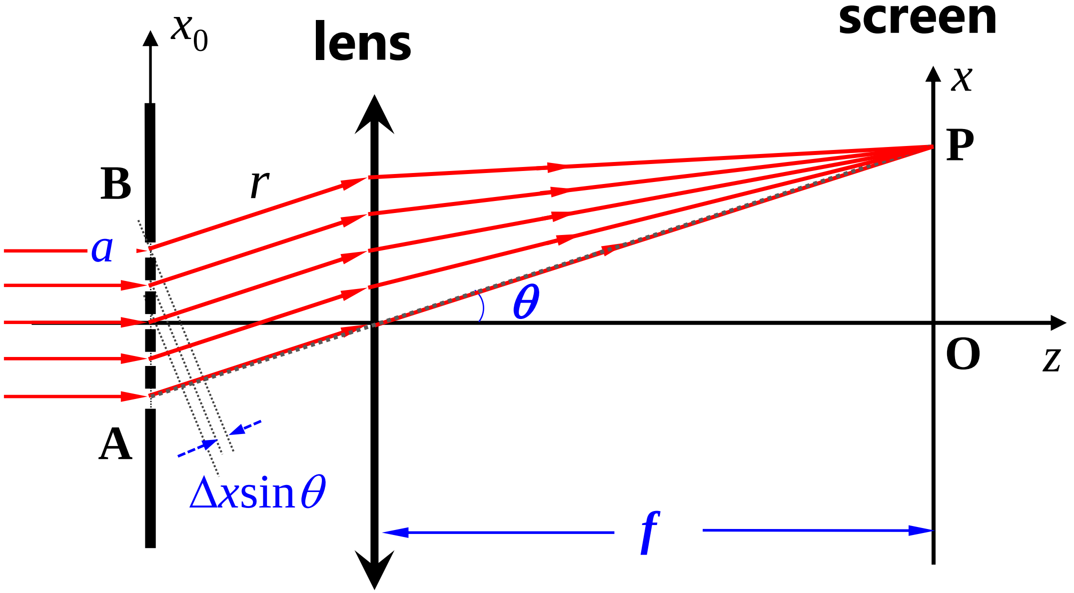
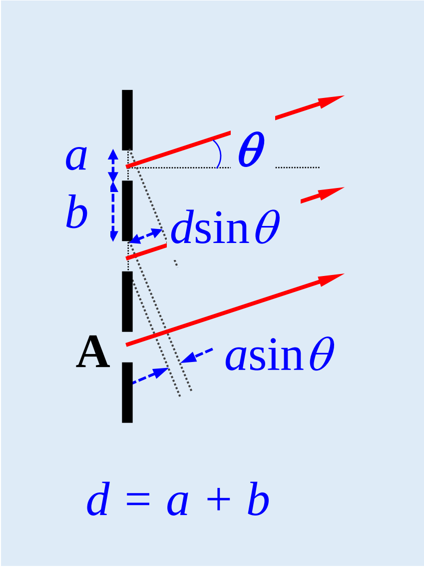
§ 7.7 Diffraction by many slits

- Assume that there is N slits. The width of the transparent part is a, the opaque part b.
- **Grating constant**: period of the grating: $d = a + b$





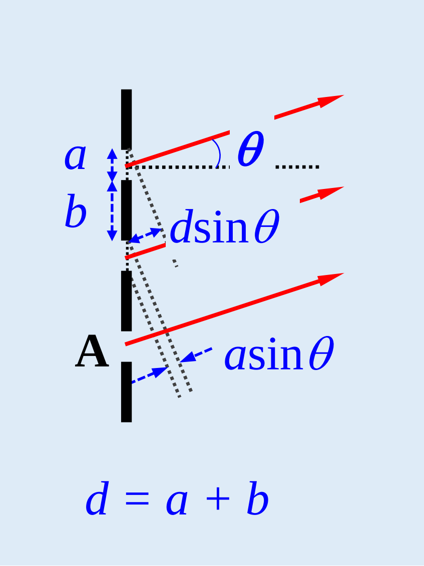
- The diffracted light of each slit is regarded as a beam of light. Then we have multi-beam interference.

Intensity distribution

① Write the complex amplitude $\tilde{E}_m(\theta)$ produced by each slit at point P_θ;

② Find the phase relationship of $\tilde{E}_m(\theta)$ by each slit;

③ Superimpose all $\tilde{E}_m(\theta)$.



① The difference of OPL and phase at point P_θ between adjacent two slits centers

$$\Delta = d\sin\theta, \qquad \delta = k\Delta = (2\pi/\lambda)d\sin\theta$$

② Complex amplitude of vibration generated by each slit at P_θ

$$\tilde{E}_1(\theta) = \tilde{A}_0\frac{\sin\alpha}{\alpha} \quad \tilde{E}_2(\theta) = \tilde{A}_0\frac{\sin\alpha}{\alpha}e^{i\delta} \quad \cdots \quad \tilde{E}_N(\theta) = \tilde{A}_0\frac{\sin\alpha}{\alpha}e^{i(N-1)\delta}$$

Intensity distribution

③ Superimpose all $\tilde{E}_m(\theta)$.

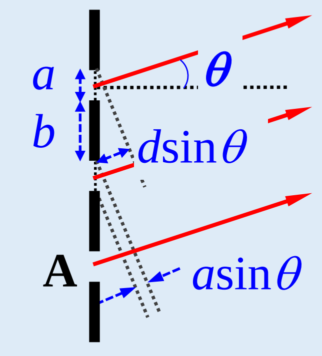
$$\tilde{E}(\theta)=\sum_{m=1}^{N}\tilde{E}_m(\theta)$$

$$=\tilde{A}_0\frac{\sin\alpha}{\alpha}\left[1+e^{i\delta}+\cdots+e^{i(N-1)\delta}\right]$$

$$=\tilde{A}_0\frac{\sin\alpha}{\alpha}\frac{1-e^{iN\delta}}{1-e^{i\delta}}$$

$$=\tilde{A}_0\frac{\sin\alpha}{\alpha}\frac{\left(e^{-iN\delta/2}-e^{iN\delta/2}\right)e^{iN\delta/2}}{\left(e^{-i\delta/2}-e^{i\delta/2}\right)e^{i\delta/2}}$$

$$=\tilde{A}_0\frac{\sin\alpha}{\alpha}\frac{\sin(N\delta/2)}{\sin(\delta/2)}e^{i(N-1)\delta/2}$$



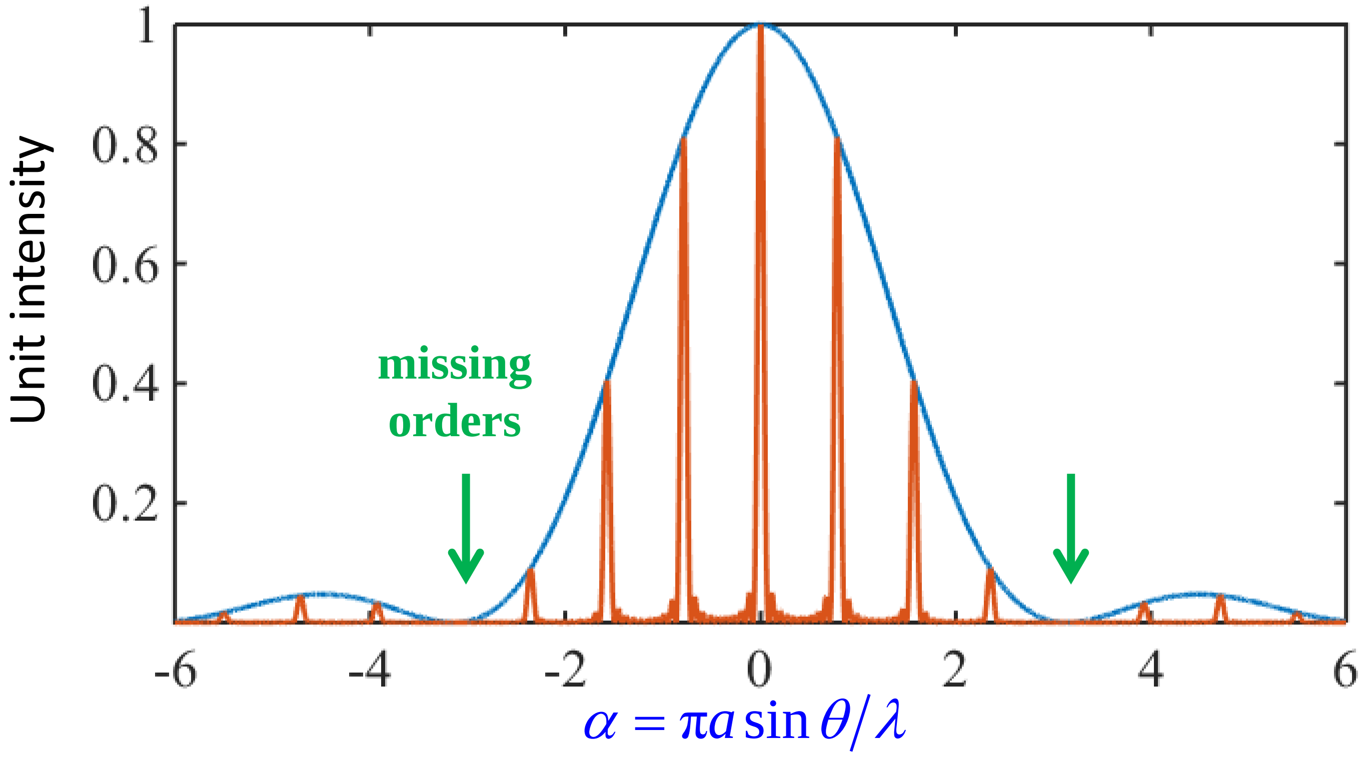
$$\delta=\frac{2\pi}{\lambda}d\sin\theta$$

$$\boxed{\begin{aligned}I(\theta)&=\tilde{E}(\theta)\cdot\tilde{E}^*(\theta)\\&=\left|\tilde{A}_0\right|^2\left(\frac{\sin\alpha}{\alpha}\right)^2\left[\frac{\sin(N\delta/2)}{\sin(\delta/2)}\right]^2\end{aligned}}$$

Intensity distribution

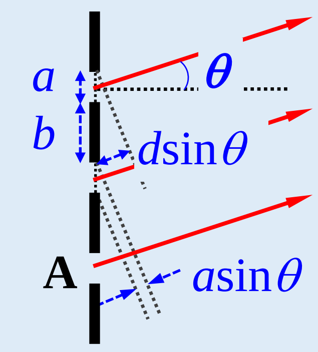
$$I(\theta) = \left|\tilde{A}_0\right|^2 \underbrace{\left(\frac{\sin\alpha}{\alpha}\right)^2}_{\text{slit diffraction}} \underbrace{\left[\frac{\sin(N\beta)}{\sin\beta}\right]^2}_{\text{slits interference}} \qquad \beta \equiv \frac{\delta}{2}$$

(Equal amplitude interference)



Intensity distribution

$$I(\theta)=\left|\tilde{A}_0\right|^2\left(\frac{\sin\alpha}{\alpha}\right)^2\left[\frac{\sin(N\beta)}{\sin\beta}\right]^2$$



$$\delta=\frac{2\pi}{\lambda}d\sin\theta$$

Modulation of one function to another.

Interference factor $\left(\dfrac{\sin N\beta}{\sin\beta}\right)^2$ $\quad \beta\equiv\dfrac{\delta}{2}=\dfrac{\pi}{\lambda}d\sin\theta$

① Interference maximal

When $\begin{cases}\beta=m\pi \text{ then } \boxed{m\lambda=d\sin\theta} \\ \lim\limits_{\beta\to m\pi}\dfrac{\sin N\beta}{\sin\beta}=\pm N \quad I=N^2I_0\end{cases}$ **Grating equation**

$$=\lim_{\beta\to m\pi}\frac{\sin N\beta-\sin Nm\pi}{\sin\beta-\sin m\pi}$$

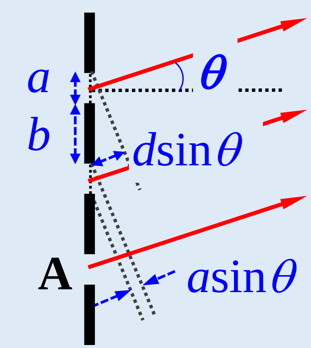
$$=\lim_{\beta\to m\pi}\frac{\cos Nm\pi\cdot(N\beta-Nm\pi)}{\cos m\pi\cdot(\beta-m\pi)}=\pm\frac{N(\beta-m\pi)}{(\beta-m\pi)}=\pm N$$

Intensity distribution

② Interference minimum

When $\begin{cases} \sin\beta \neq 0 \\ \sin N\beta = 0 \end{cases}$, $\left(\dfrac{\sin N\beta}{\sin\beta}\right)^2 = 0$

$$\beta = \left(m + \frac{m'}{N}\right)\pi \qquad m' = 1,\ 2,\ 3, \cdots N-1$$



$$\delta = \frac{2\pi}{\lambda} d \sin\theta$$

There is an interference minimum.

There are N-1 minimum values between the two main maxima, N-2 subsidiary maxima.

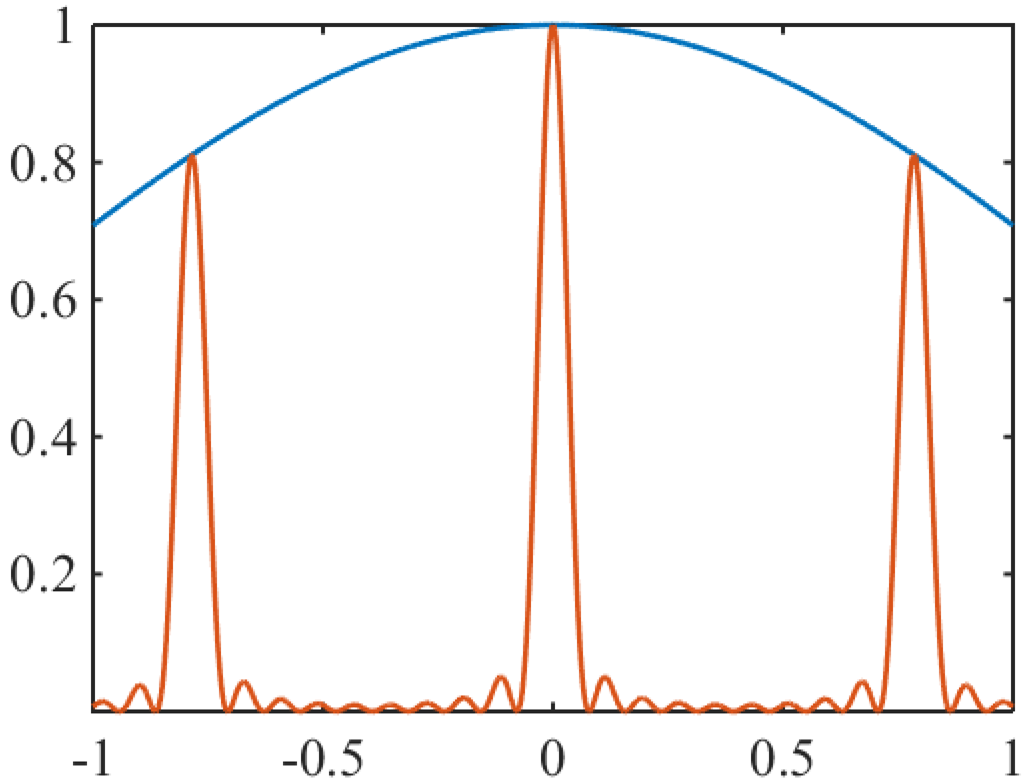
③ Subsidiary maximum

Approximately, when $\begin{cases} \sin\beta \neq 0 \\ \sin N\beta = 1 \end{cases}$, $\beta = \dfrac{\pi}{2N}(1+2m)$

$$m = 1,\ 2,\ 3, \cdots$$

Intensity distribution

$$I(\theta) = \left|\tilde{A}_0\right|^2 \left(\frac{\sin\alpha}{\alpha}\right)^2 \left[\frac{\sin(N\beta)}{\sin\beta}\right]^2 \qquad \beta \equiv \frac{\delta}{2}$$



$\alpha = \pi a \sin\theta / \lambda$

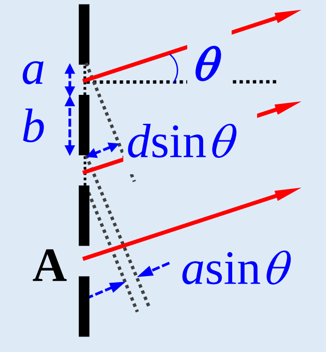
N = 10 (9 minima, 8 subsidiary maxima)

Intensity distribution

④ Maximal half-width

Half angular width: The angular distance between the main center and the **first** minimum.

Main max center $\beta = m\pi$ $\qquad \beta = \dfrac{\pi}{\lambda} d\sin\theta$



$$\delta = \frac{2\pi}{\lambda} d\sin\theta$$

First minimum $\beta = \left(m + \dfrac{m'}{N}\right)\pi \qquad m' = 1$

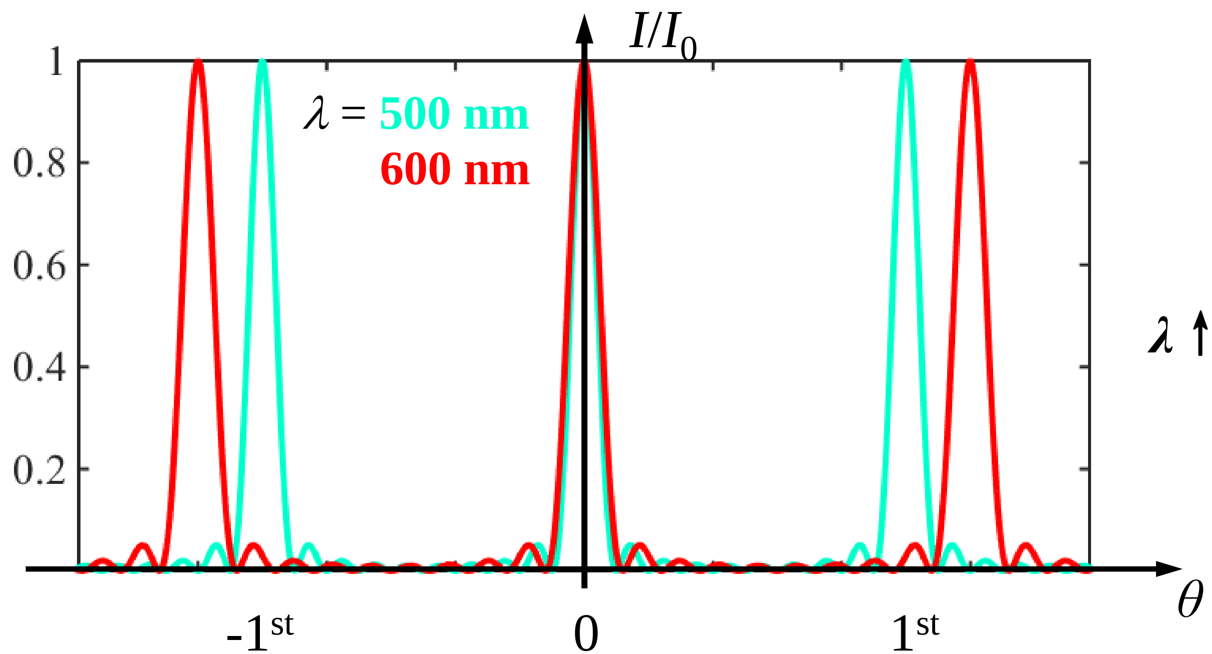
$$\Delta\beta = \frac{\pi}{N} = \frac{\pi}{\lambda} d\cos\theta \cdot \Delta\theta \quad \Rightarrow \quad \boxed{\Delta\theta = \frac{\lambda}{Nd\cos\theta}}$$

Nd is the grating width, When d is given, $N\uparrow$, $\Delta\theta\downarrow$, the fringes is more narrow.

Gratings

Grating equation $\boxed{m\lambda = d\sin\theta}$

How it works? For an incident white light, except the 0th order, the principal maximums for different color appear in different angles (directions).



$\lambda\uparrow \rightarrow \theta_m\uparrow$

Gratings

Multi-slit as transmission type grating.

The drawback: Large dispersion happen for high order diffraction ($m\uparrow$).

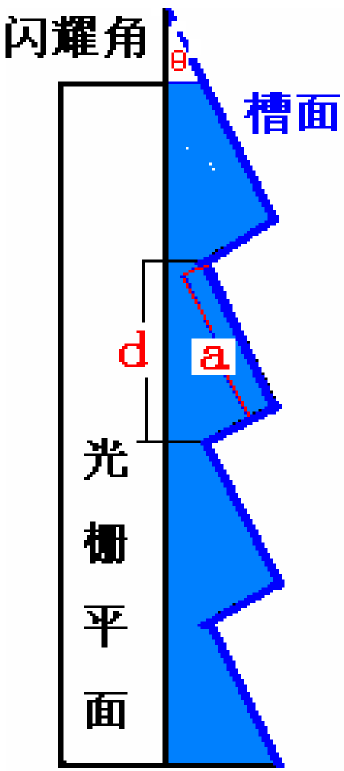
$$D_\theta = \frac{d\theta}{d\lambda} = \frac{m}{d\cos\theta}$$

There is a waste of measured signal!

Unpractical when the incident light is very weak.

We need an efficient grating to disperse the white light without significant loss!

Blazed gratings



Blazed grating: It has a periodic spatial structure that causes periodic changes in the **phase** of the reflected light wave.
(No modulation of the amplitude because the reflectivity is the same)

Grooved surface a: corresponds to a slit

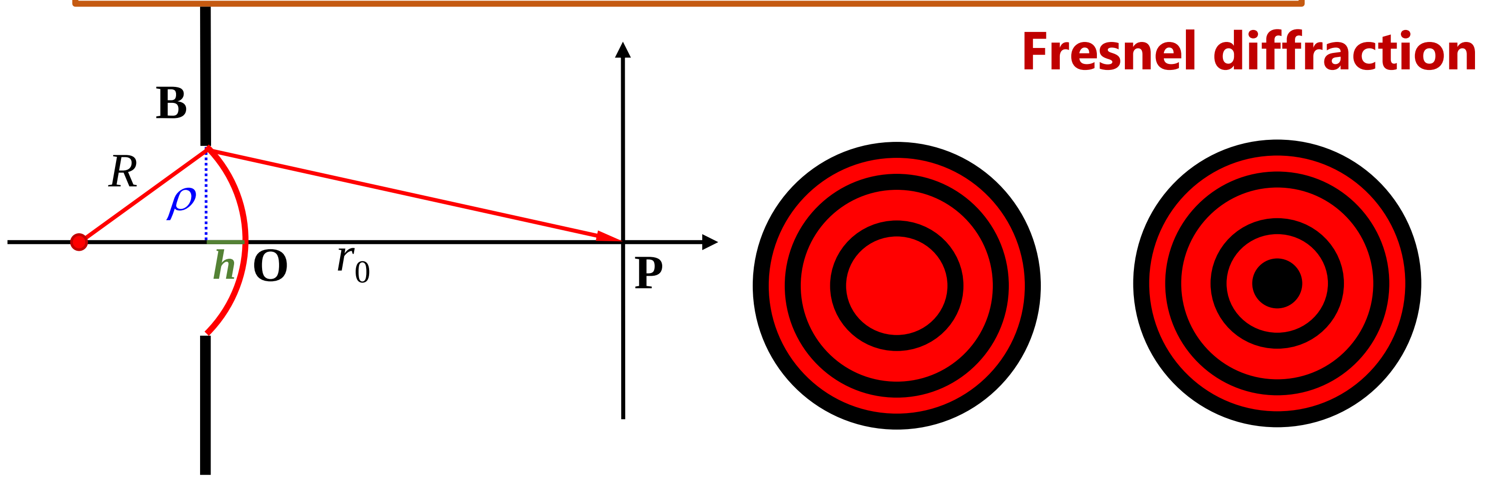
Grating constants d

The blazed grating is a **planar, reflective, phase-type** grating.

§ 7.8 Fresnel diffraction

- Fresnel diffraction: integral method, half-period zone.

$$\boxed{\tilde{E}(x,y)=\frac{e^{ikz}}{i\lambda z}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{E}(x_0,y_0)\,\mathrm{e}^{\frac{ik}{2z}\left[(x-x_0)^2+(y-y_0)^2\right]}\mathrm{d}x_0\mathrm{d}y_0}$$

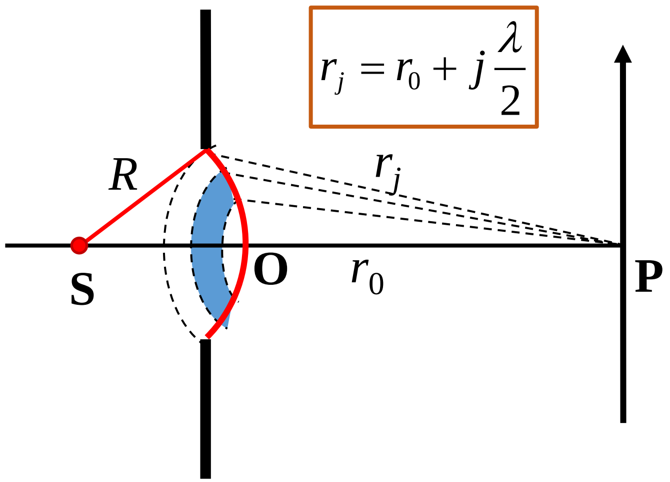


- R, r_0, ρ can influence the intensity at P.
  R influences ***h***, equal to changing the phase and amplitude at the screen.

Half-period zone

- Use point P as the center, $r_j = r_0 + j\,\lambda/2$ as radius, draw spheres. The spherical surfaces divide the wavefront into many rings $(j = 1, 2, 3 \ldots n)$

- The averaged distance from adjacent rings to point P differs by half a wavelength, the phase difference is π. so called **Fresnel (half-period) zone**.



Two adjacent Fresnel zones have opposite phases. The combined amplitude of point P is the sum of the amplitude of all Fresnel zones at point P.

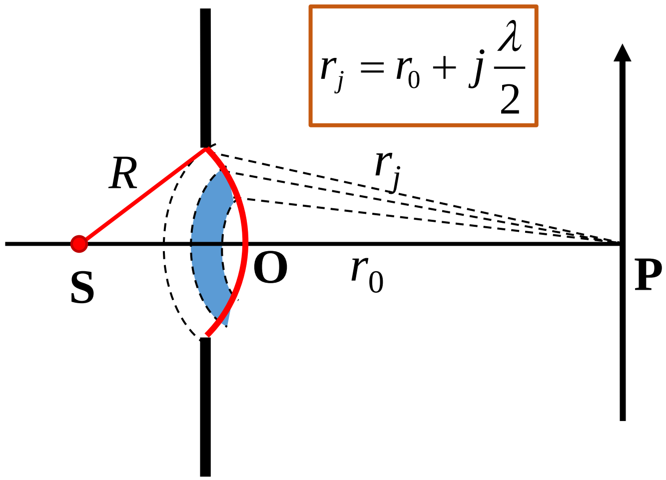
Half-period zone

- The amplitude of each half-wave zone at point P is $a_1, a_2 \ldots a_n$, so

$$A_n = a_1 - a_2 + a_3 - \cdots + (-1)^{n-1} a_n$$

- According to the Huygens-Fresnel principle,

$$a_j \propto (1+\cos\theta_j)\frac{\Delta\sigma_j}{r_j}$$



Inclination factor

$$F(\theta) = \frac{\cos\theta_0 + \cos\theta}{2}$$

When the source S is at the axis, $\theta_0 = 0$.

$\Delta\sigma_j$: The area of the *j*th half wave zone.

r_j: Average distance from the *j*th half wave to the P point.

θ_j: The average angle between the surface normal of the *j*th zone to the line connecting the point P and the *j*th zone.

Half-period zone

Prove that $\Delta\sigma_j / r_j$ is a constants approximately,

$$\Delta\sigma_j = \frac{\pi R\lambda}{R+r_0} r_j$$

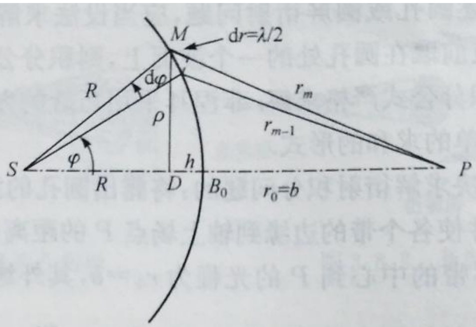


图 3.8.7 半波带面积的计算

对于图 3.8.7 中的△SMP，利用余弦公式，得到 $\cos\varphi=\frac{R^2+(R+r_0)^2-r_m^2}{2R(R+r_0)}$，两端求微分，则有

$$\sin\varphi \mathrm{d}\varphi=\frac{r_m}{R(R+r_0)}\mathrm{d}r_m \qquad (3.8.2)$$

将式(3.8.1)和式(3.8.2)结合，得到

$$\mathrm{d}S=2\pi R^2\frac{r_m}{R(R+r_0)}\mathrm{d}r_m$$

按照半波带的定义，当 $\mathrm{d}r_m=\lambda/2$ 时，$\mathrm{d}S=\Sigma_m$，Σ_m 为第 m 个半波带的面积. 于是上式变为 $\Sigma_m=\frac{\pi R r_m}{R+r_0}\lambda$，即

$$\frac{\Sigma_m}{r_m}=\frac{\pi R}{R+r_0}\lambda \qquad (3.8.3)$$

可见，对于各个不同的半波带，其面积与到场点的光程之比为常数.

3. 单个半波带在轴上场点的复振幅

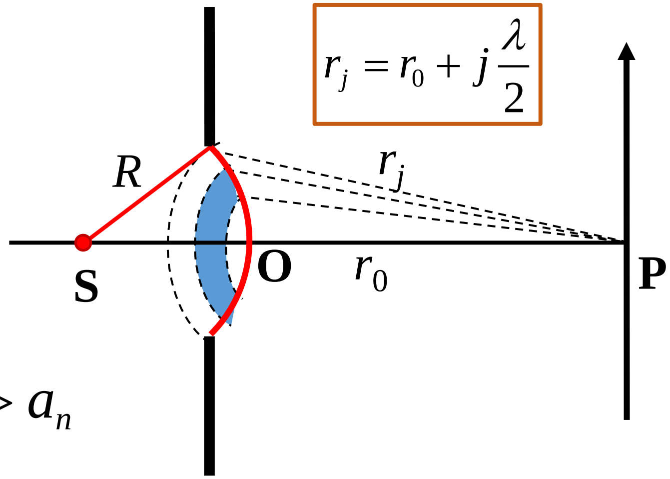
Equation 3.8.1

$$\mathrm{d}S = 2\pi R^2 \sin\varphi \mathrm{d}\varphi$$

Half-period zone

$$a_j \propto \left(1+\cos\bar{\theta}_j\right)\cancel{\frac{\Delta\sigma_j}{r_j}}$$

- It is shown that the amplitude of each zone at point P is only related to the angle.



- when $j\uparrow \rightarrow \theta_j \uparrow \rightarrow a_j \downarrow$: $a_1 > a_2 > a_3 > \cdots > a_n$

- Since λ is small, the change of θ for adjacent zones are small, and the difference between a_j and a_{j+1} is small.

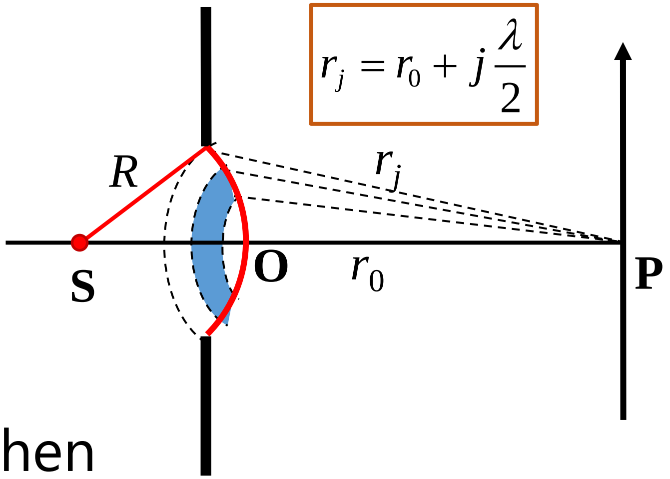
$$a_j \approx \frac{a_{j-1}+a_{j+1}}{2}$$

$$A_n = a_1 - a_2 + a_3 - \cdots + (-1)^{n-1} a_n$$

$$= \frac{a_1}{2} + \left(\frac{a_1}{2} - a_2 + \frac{a_3}{2}\right) + \left(\frac{a_3}{2} - a_4 + \frac{a_5}{2}\right) + \cdots = \begin{cases} \dfrac{a_1}{2} + \dfrac{a_n}{2} & n \text{ is odd} \\ \dfrac{a_1}{2} + \dfrac{a_{n-1}}{2} - a_n & n \text{ is even} \end{cases}$$

Half-period zone

$$A_n = \begin{cases} \dfrac{a_1}{2} + \dfrac{a_n}{2} & n \text{ is odd} \\ \dfrac{a_1}{2} - \dfrac{a_n}{2} & n \text{ is even} \end{cases}$$



- If n is a small integer, the difference between a_1 and a_n the is very small, then

$$A_n = \begin{cases} a_1 & n \text{ is odd, P is bright} \\ 0 & n \text{ is even, P is dark} \end{cases}$$

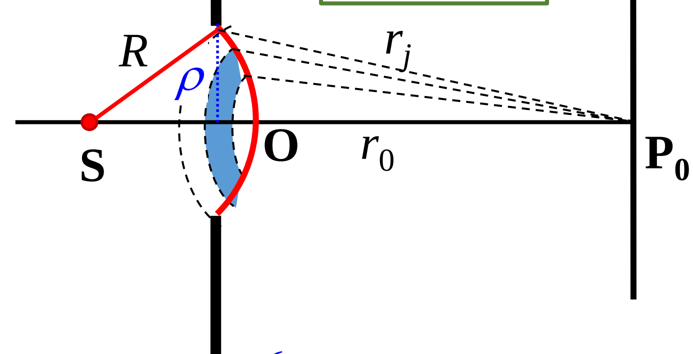
- (You&Yu's Optics, P128) The number of half-wave zones exposed at the circular holes

$$n = \frac{\rho^2}{\lambda}\left(\frac{1}{r_0} + \frac{1}{R}\right)$$

Half-period zone

$$n = \frac{\rho^2}{\lambda}\left(\frac{1}{r_0} + \frac{1}{R}\right)$$

$$\boxed{r_j = r_0 + j\frac{\lambda}{2}}$$



① R, ρ is fixed, change $r_0 \uparrow \Rightarrow n \downarrow$

Move the screen (P_0), the diffraction pattern changes **reversibly bright and dark**;

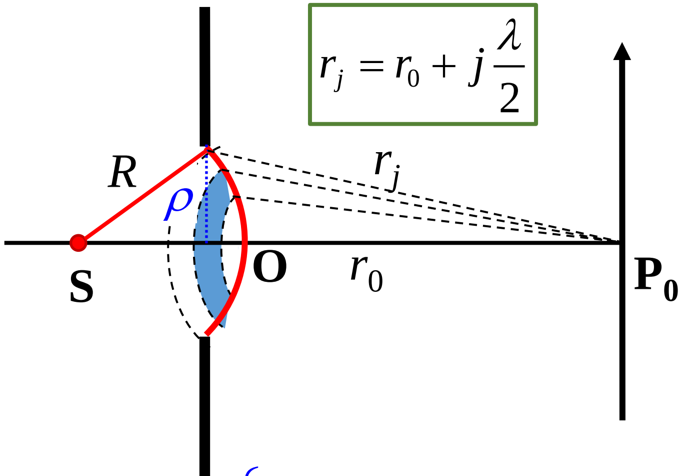
$$A_n = \begin{cases} \dfrac{a_1}{2} + \dfrac{a_n}{2} & n \text{ is odd} \\ \dfrac{a_1}{2} - \dfrac{a_n}{2} & n \text{ is even} \end{cases}$$

② When R, r_0 is given, change $\rho \uparrow \Rightarrow n \uparrow$

When the size of the diffraction hole changes, the parity of n changes, and the intensity of P_0 also changes.
When the circular hole ρ is small, then n is small, so the diffraction is more remarkable.

Half-period zone

$$n = \frac{\rho^2}{\lambda}\left(\frac{1}{r_0} + \frac{1}{R}\right)$$



③ when ρ is large enough and R、r_0 is finite, $n \to \infty$,

$$a_n \to 0 \quad A_{n\to\infty} = \frac{a_1}{2}$$

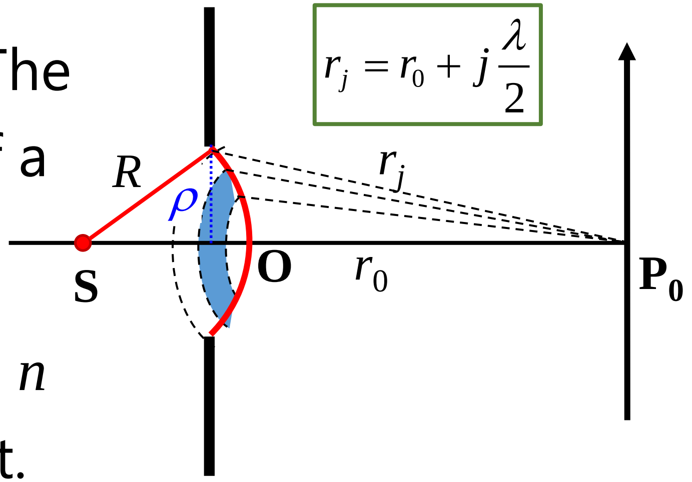
Regardless of n, if you change r_0 at this time, the P point is always a bright spot.

$$A_n = \begin{cases} \frac{a_1}{2} + \frac{a_n}{2} & n \text{ is odd} \\ \frac{a_1}{2} - \frac{a_n}{2} & n \text{ is even} \end{cases}$$

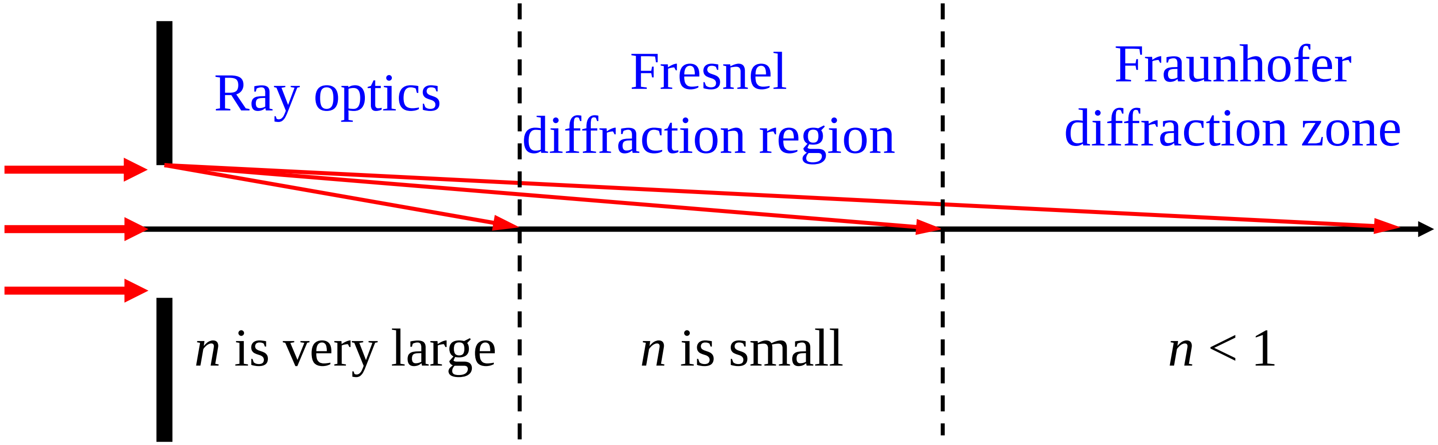
- When light travels freely (no obstacles), $n \to \infty$, point P is always bright. >>Geometric optics is the limit of wave optics when the number of half-period zone contribute to the observed point $n \to \infty$.

Half-period zone

④when R , r_0 is large enough, $n < 1$. The small hole only allows a small part of a half-period zone to transmit.



In this case, when increase R, r_0, then $n < 1$ still holds. Point P is always bright. >>Entering the Fraunhofer diffraction.



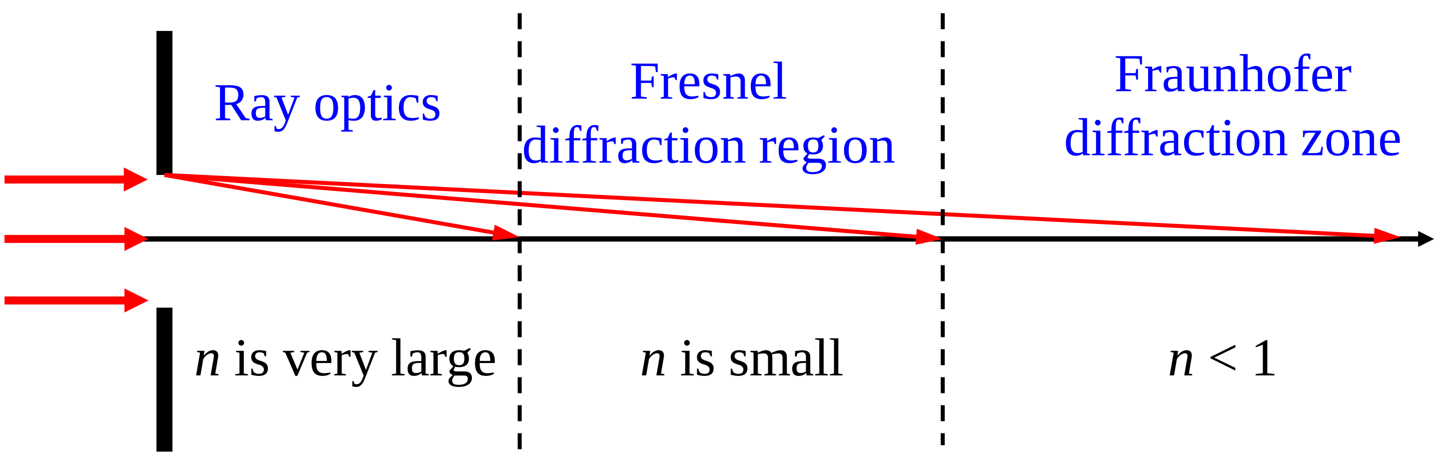
Half-period zone

If $R \to \infty$ (parallel light) , so

$$n = \frac{\rho^2}{\lambda}\left(\frac{1}{r_0} + \frac{1}{R}\right)$$

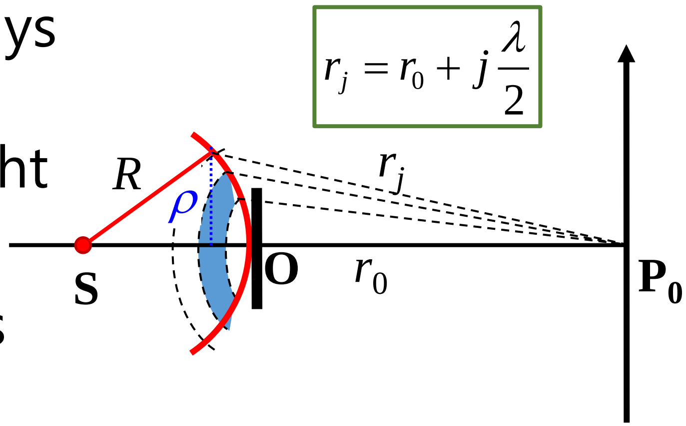
$n = \rho^2/\lambda r_0 << 1$， then $r_0 >> \rho^2/\lambda$

The conditions that should be satisfied for the Fraunhofer diffraction zone.



Circular screen

①The center of the shadow is always bright (**Poisson's spot**);
②There are very few circles of bright concentric rings around the spot.



- Supposed that a small disc blocks the first m half-period zones

$$A = \underline{a_1 - a_2 + \cdots + (-1)^{m-1} a_m} + \cdots + a_{n\to\infty}$$

$$= \begin{cases} -\dfrac{a_{m+1}}{2} + \left(-\dfrac{a_{m+1}}{2} + a_{m+2} - \dfrac{a_{m+3}}{2}\right) + \cdots + \dfrac{a_{m\to\infty}}{2} \\ \dfrac{a_{m+1}}{2} + \left(\dfrac{a_{m+1}}{2} - a_{m+2} + \dfrac{a_{m+3}}{2}\right) + \cdots - \dfrac{a_{m\to\infty}}{2} \end{cases} = \begin{cases} -\dfrac{a_{m+1}}{2} \\ \dfrac{a_{m+1}}{2} \end{cases}$$

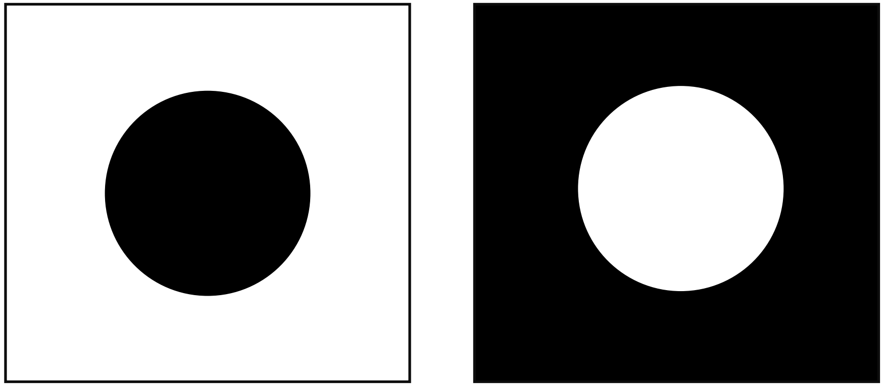
$$I = |A|^2 = a_{m+1}^2/4$$

No matter m is even or odd, the center point is always bright!

Babinet principle

Complementary screen

$$\tilde{t}_{\mathrm{a}} + \tilde{t}_{\mathrm{b}} = 1$$

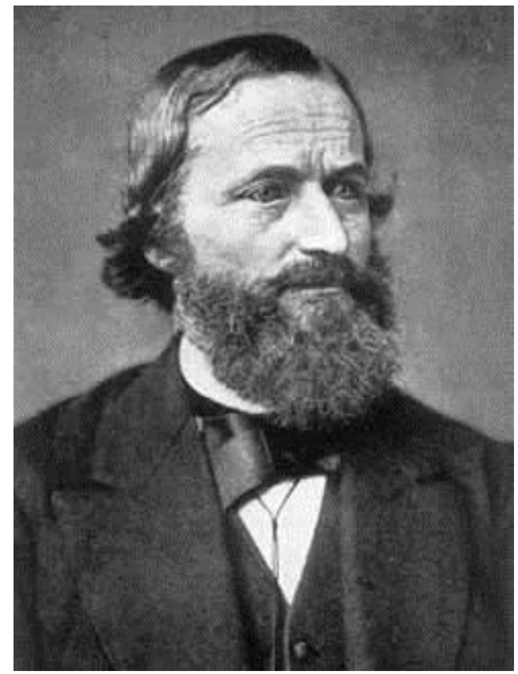


- **Babinet principle**: The sum of the complex amplitudes of the diffracted fields caused by the two complementary screens is equal to the complex amplitude of the lightwave in free space.
- The diffraction field of the complementary screen can be obtained from the diffraction field of a screen by the Babinet principle.
- **Application: The diffraction pattern of the same width filament can be known from the single slit Fraunhofer diffraction pattern. And get the size of the filament.**

Joseph von Fraunhofer

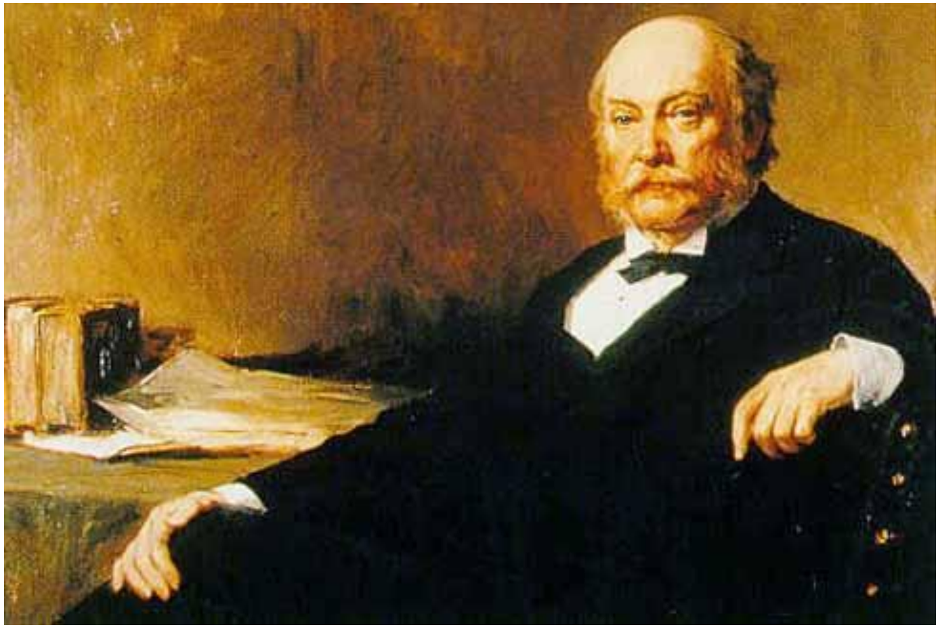
Born: 6 March 1787 in Straubing, Bavaria
Died: 7 June 1826 in Munich

Gustav Robert Kirchhoff



Born: 12 March 1824 in Königsberg, Prussia (now Kaliningrad, Russia)
Died: 17 Oct 1887 in Berlin, Germany

John William Strutt Lord Rayleigh



Born: 12 Nov 1842 in Langford Grove (near Maldon), Essex, England

Died: 30 June 1919 in Terling Place, Witham, Essex, England