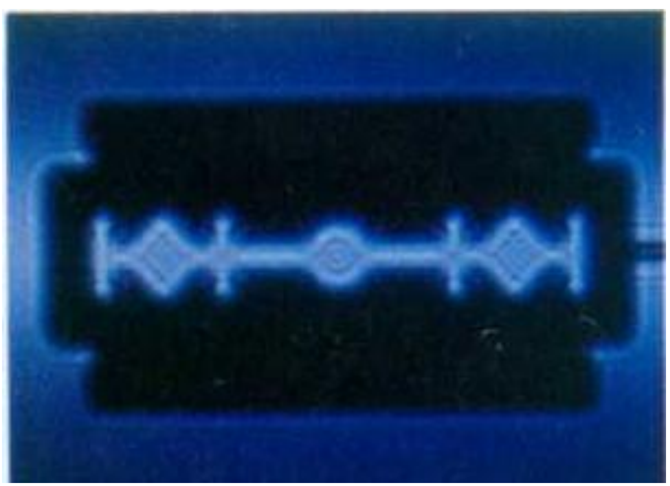
C7 Diffraction

- Diffraction: A phenomenon in which a wave deviates from rectilinear propagation when it meets an obstacle during propagation.
- Diffraction is another important feature of waves and is also the result of coherent superposition.
- Huygens-Fresnel principle;
- How to quantitatively analyze the diffraction of light;
- How the diffraction of light and the linear propagation of light are unified.

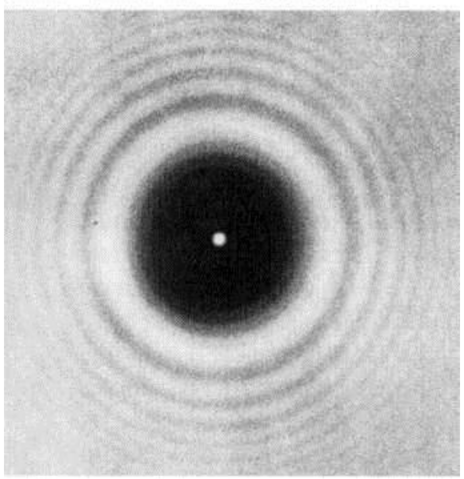
Diffraction is the commonality of waves. Waves with longer wavelengths are easier to observe their diffraction, such as water waves, sound waves, and radio waves.

The de Broglie's assumption of the matter wave in 1924 was confirmed by electron diffraction experiments.

Diffraction



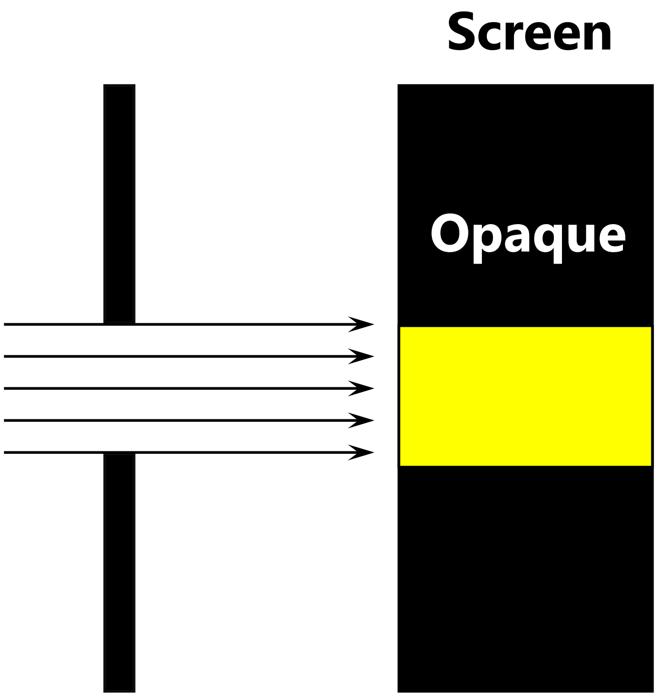
Edge of the blade

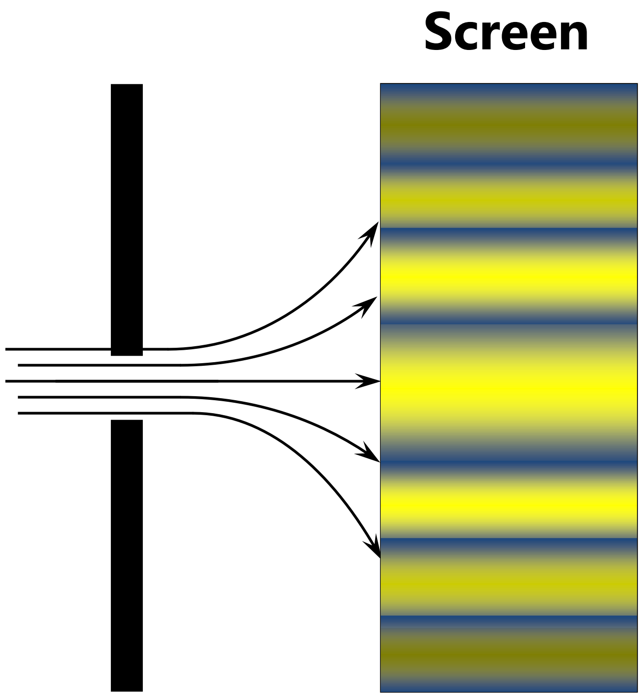


Circular obstacles

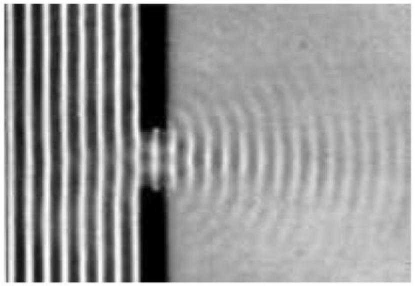
(Poisson's spot)

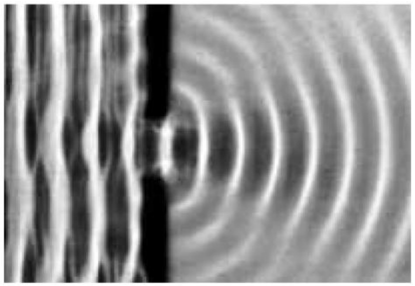
The smaller the slit is, the more obvious the diffraction is.





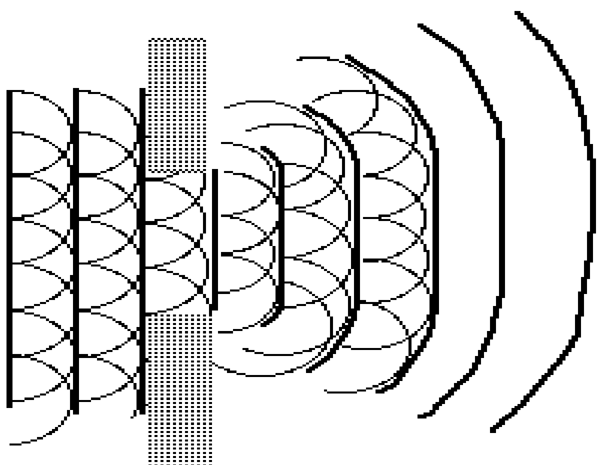
Ripple tank





§ 7.1 Huygens-Fresnel Principle

- **Huygens's principle**: Every point on a propagating wavefront serves as the source of spherical secondary wavelets, such that the wavefront at some later time is the envelope of these wavelets.

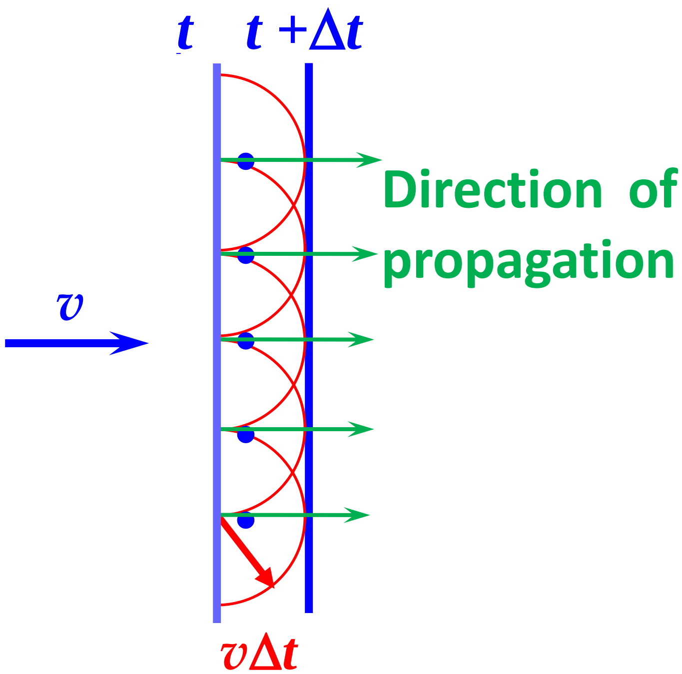


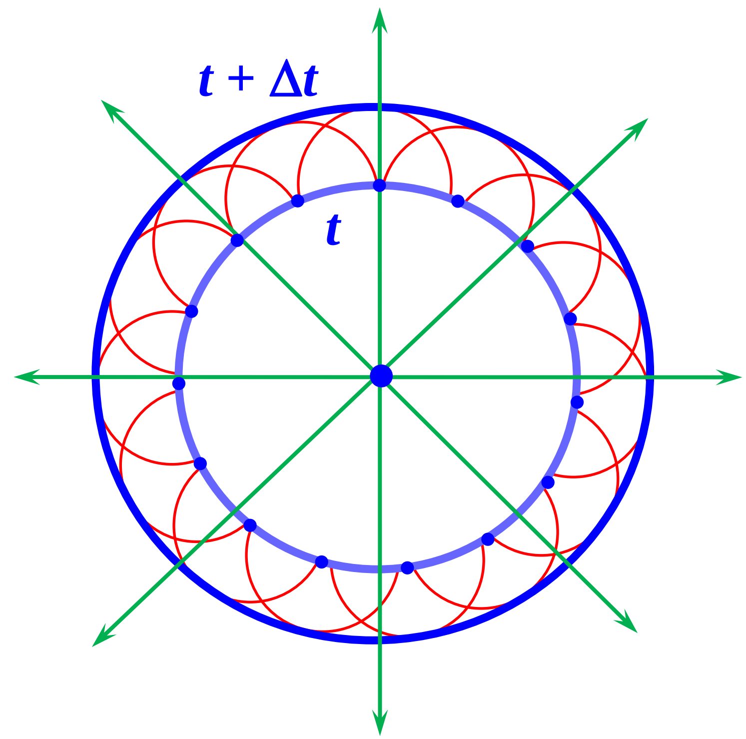
Christiaan Huygens
1629~1695,
Netherlands

- In principle, the position of the wavefront at the next moment can be mapped out by the graphing according to Huygens's principle.

The Huygens's Principle

① Explain the propagation of light in isotropic medium





The Huygens's Principle

② Laws of reflection and refraction

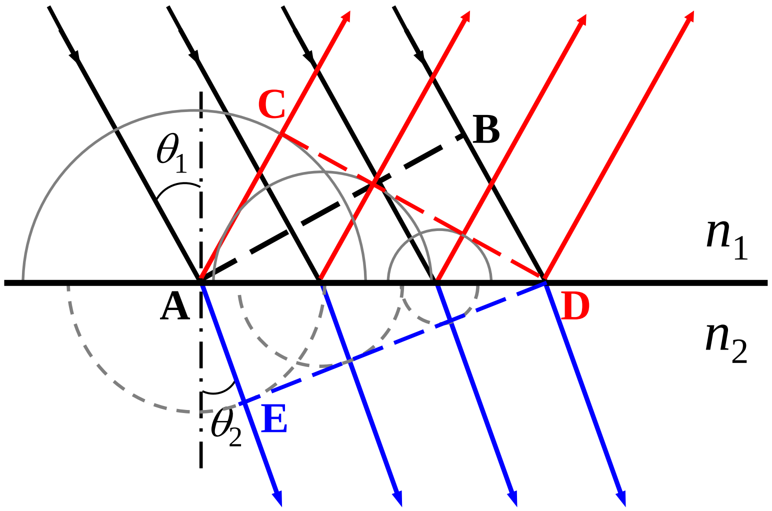
Light is incident from n_1 to n_2, and $n_1 < n_2$, incident angle is θ_1.

- For triangles ABD:

  $\sin\theta_1 = BD/AD = AC/AD$

- For triangles AED:

  $\sin\theta_2 = AE/AD$



$\therefore \quad \sin\theta_1 / \sin\theta_2 = AC/AE$

$= v_1\Delta t / v_2\Delta t = v_1 / v_2$

That is, the ratio of the sine of the incident angle to the sine of the angle of refraction is a constant.
Using the definition of refractive index, we have:

$$n_1\sin\theta_1 = n_2\sin\theta_2$$

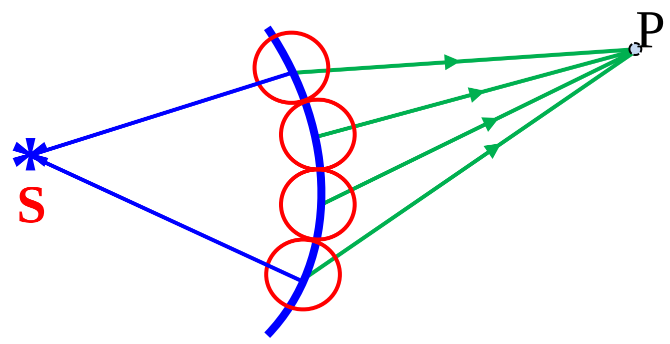
Huygens-Fresnel Principle

- **The limitations of the Huygens's principle:**

  ① Impossible to distinguish the conditions under which rectilinear propagation of light and the diffraction;

- ② Unable to determine the intensity distribution in the diffraction field;

  ③ A reverse wave was predicted.



Huygens-Fresnel Principle

Fresnel corrected the Huygens's principle in 1818 to:

1) Each point on the wavefront is a wave source of spherical waves;
2) Each wavelet is coherently superposition at every point in space.

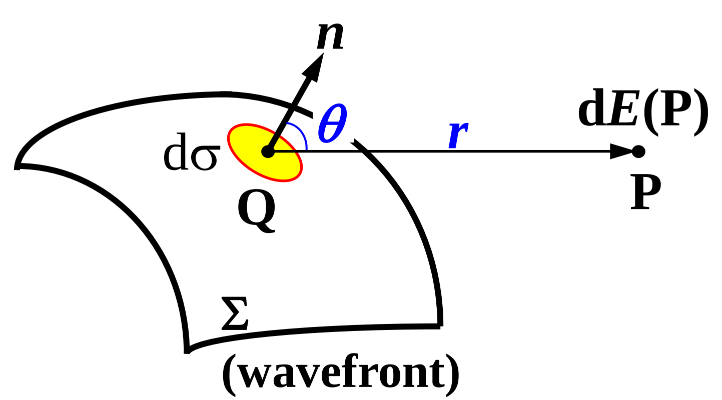
- **Huygens-Fresnel Principle**: Every unobstructed point of a wavefront, at a given instant, serves as a source of spherical secondary wavelets (with the same frequency as that of the primary wave). The amplitude of the optical field at any point beyond is the superposition of all these wavelets (considering their amplitudes and relative phases).

Huygens-Fresnel Principle

- **Diffraction becomes an interference of an infinite number of beams.**
- **The key**: calculate the difference of OPL between the source to each sub-unit and between each sub-unit to the field point.

Fresnel's hypothesis：

The contribution of the each sub-unit dσ on the wavefront to the P point



$d\tilde{E}(P) \propto d\sigma$ Area at Q

$\propto \tilde{E}(Q)$ Complex amplitude at Q

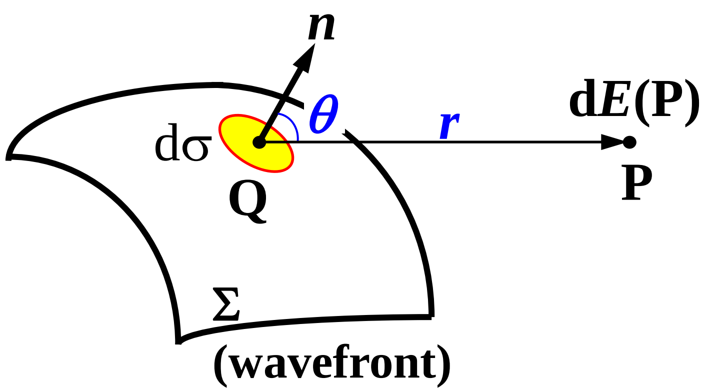
$\propto \frac{e^{ikr}}{r}$ Secondary wave is a spherical wave $e^{i\mathbf{k}\cdot\mathbf{r}} = e^{ikr}$

$\propto F(\theta)$ Tilt factor, a function related to θ

Huygens-Fresnel Principle

Tilt factor: **there is no secondary wave that is backward.**

$$F(\theta)=\begin{cases}\cos\theta & 0\le\theta\le\pi/2\\ 0 & \theta\ge\pi/2\end{cases}$$



$$d\tilde{E}(P)\propto\tilde{E}(Q)\frac{e^{ikr}}{r}F(\theta)\mathrm{d}\sigma$$

$$\tilde{E}(P)=\iint_{\Sigma}d\tilde{E}(P)$$

Infinite beam interference

$$=\iint_{\Sigma}C\cdot\tilde{E}(Q)\frac{e^{ikr}}{r}F(\theta)\mathrm{d}\sigma$$

where C is a coefficient independent of r, θ, and Q. The above formula, in principle, solves the problem of how to calculate the diffraction intensity, and can be extended to any wavefront Σ

Huygens-Fresnel Principle

The **limitations** of Huygens-Fresnel Principle

① The secondary wave is introduced in a hypothetical form.

② The introduction of the tilt factor is empirical and has no specific form. The use of the form

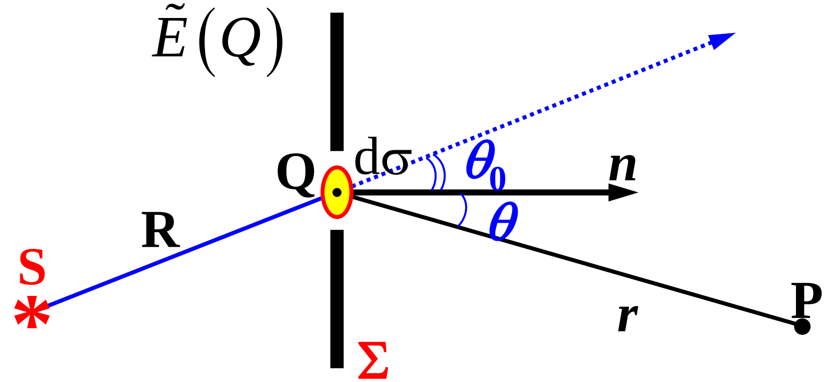
$$F(\theta)=\begin{cases}\cos\theta & 0\leq\theta\leq\pi/2\\ 0 & \theta\geq\pi/2\end{cases}$$

to eliminate the regressive wave, is not correct.

Kirchhoff integral theorem

- In 1882, G. Kirchhoff solved the wave equation and also obtained the expression of E(P), which provides the theoretical base for the Huygens-Fresnel Principle.

Σ is the wavefront;
R、$r >> \lambda$;



$$\tilde{E}(P) = \frac{1}{i\lambda} \iint_{\Sigma} \tilde{E}(Q) \frac{e^{ikr}}{r} \frac{\cos\theta_0 + \cos\theta}{2} \mathrm{d}\sigma$$

Fresnel-Kirchhoff diffraction formula (the complex amplitude of E-field at point P)

Kirchhoff integral theorem

Comparison:

$$\begin{cases} \tilde{E}(P) = \iint\limits_{\Sigma} C \cdot \tilde{E}(Q) \dfrac{e^{ikr}}{r} F(\theta) \mathrm{d}\sigma \\ \tilde{E}(P) = \dfrac{1}{i\lambda} \iint\limits_{\Sigma} \tilde{E}(Q) \dfrac{e^{ikr}}{r} \dfrac{\cos\theta_0 + \cos\theta}{2} \mathrm{d}\sigma \end{cases}$$

$$\Rightarrow \quad C = \frac{1}{i\lambda} \qquad F(\theta) = \frac{\cos\theta_0 + \cos\theta}{2}$$

So, $F(\theta)$ is related to θ_0, θ

When $\theta_0 = 0$ $\quad F(\theta) = \dfrac{1}{2}(1 + \cos\theta)$

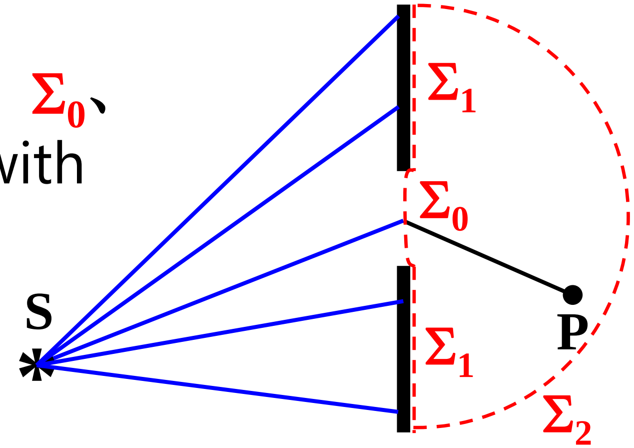
When $\theta = \pi$ $\quad F(\theta) = 0$ $\quad$ >> there is no backward wave.

Kirchhoff integral theorem

$$\boxed{\tilde{E}(P)=\frac{1}{i\lambda}\iint_{\Sigma}\tilde{E}(Q)\frac{e^{ikr}}{r}\frac{\cos\theta_0+\cos\theta}{2}\mathrm{d}\sigma}$$

Integral closed surface Σ consists of: Σ_0、Σ_1、Σ_2(taken on a spherical surface with infinitely large radius).

$$\Sigma=\Sigma_0+\Sigma_1+\Sigma_2$$



Boundary conditions:

$\tilde{E}(Q)$ On Σ_0, using the value as the incident (free propagation)

On Σ_1, $E=0$;

Kirchhoff proved that the integration over Σ_2 turned out to be 0, and it can be ignored.

Kirchoff's diffraction formula

$$\tilde{E}(P)=\frac{1}{i\lambda}\iint_{\Sigma}\tilde{E}(Q)\frac{e^{ikr}}{r}\frac{\cos\theta_0+\cos\theta}{2}\mathrm{d}\sigma$$

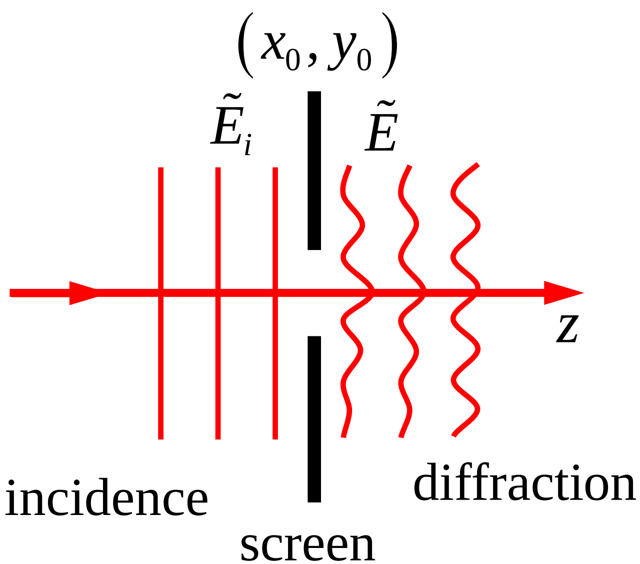
According to the formula:

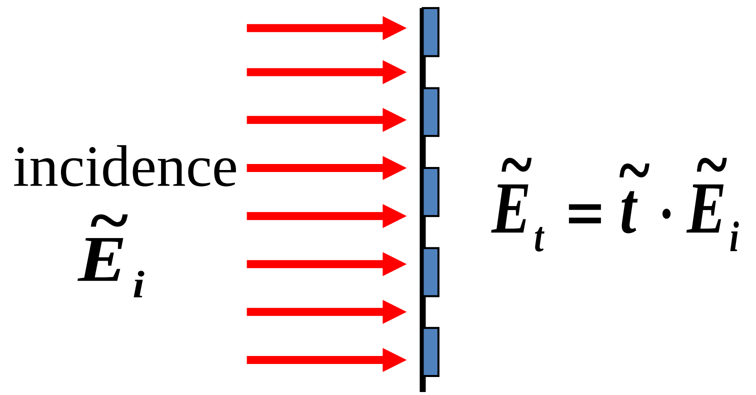
- The distribution of the light field on the Σ surface determines the spatial distribution of the light field after the screen.
- When $\tilde{E}(Q)$ (the distribution of the optical field on Σ) changes, the distribution of the light field after the screen changes.

The light field behaves as a whole. Different parts in the space are connected.

Screen function

What's the role of a screen?





The **amplitude** become position dependent.
>>Amplitude-type diffraction screen

The **wavefront** become position dependent.
>>Phase type diffraction screen

Screen function

Diffraction screen: an obstacle that can change the complex amplitude of the wavefront.

Screen function: The role of a diffraction screen can be represented by a screen function (also called a Aperture function).

$$\tilde{E}_t\left(x,y\right)=A'e^{i\delta'}=\tilde{t}\cdot\tilde{E}_i\left(x,y\right)$$
$$=\tilde{t}\left(x,y\right)A_0e^{i\delta_0}$$

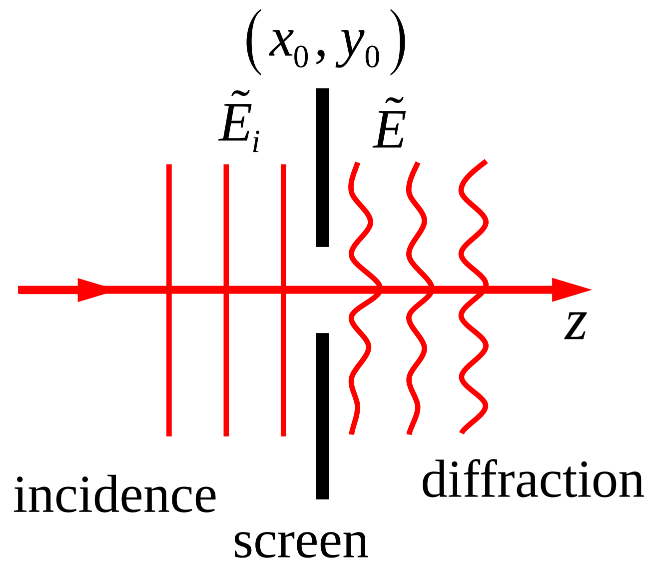
$\tilde{t}\left(x,y\right)$ is a complex function.

$|t|$ changes the amplitude of the field, angle φ changes the phase of the field.

Screen function

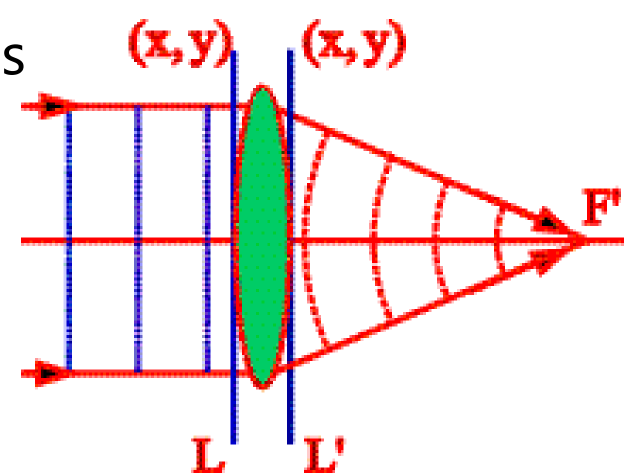
Amplitude-type

$$\tilde{t}(x,y)=\frac{\tilde{E}_t(x,y)}{\tilde{E}_i(x,y)}=\begin{cases}1\\0\end{cases}$$



Phase-type

Neglect absorption and phase shift upon reflection, screen function of a lens



$$\tilde{t}(x,y)=\exp\left(-\frac{ik\left(x^2+y^2\right)}{2f'}\right)$$

f' is the image focal length.

§ 7.2 Classification of Diffraction

- Kirchoff's diffraction formula:

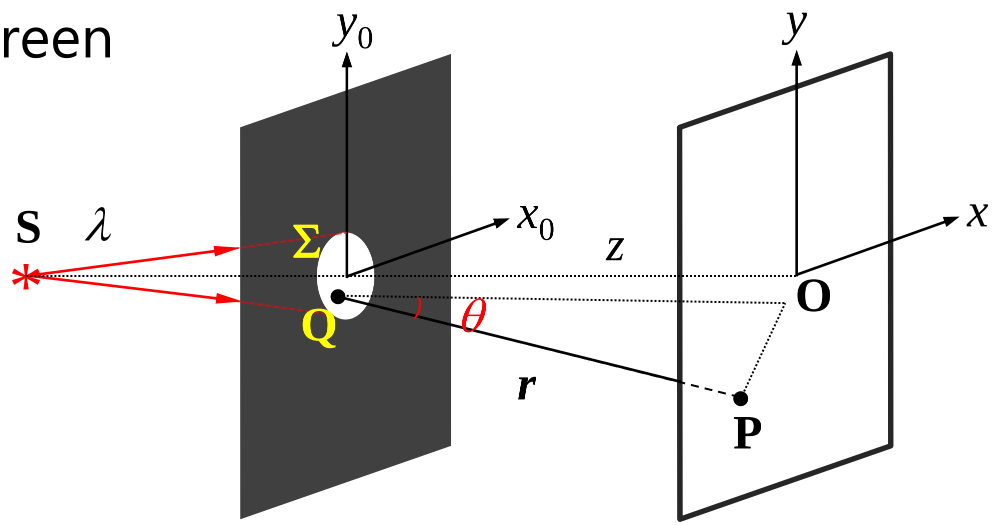
$$\boxed{\tilde{E}(P)=\frac{1}{i\lambda}\iint_{\Sigma}\tilde{E}(Q)\frac{e^{ikr}}{r}\frac{\cos\theta_0+\cos\theta}{2}\mathrm{d}\sigma}$$

- Applicable range (**scalar approximation**): The distance between the light source (the observation point) and the diffraction screen is long enough, the width of the diffraction aperture is larger than the wavelength. The offset of the light source or the observation point to the optical axis is not too large.

$$R、r >> \lambda; \quad D > \lambda;$$

- But, the Kirchhoff diffraction integral formula is too complicated. Can it be further simplified?

① If the source **S** is perpendicular to the center of the screen (hole). Then $\theta_0 = 0$.



$$F(\theta) = \frac{1}{2}(1+\cos\theta)$$

$$\Rightarrow \quad \tilde{E}(P) = \frac{1}{i\lambda}\iint_{\Sigma} \tilde{E}(Q)\frac{e^{ikr}}{r}\frac{1+\cos\theta}{2}\mathrm{d}\sigma$$

That is , $\tilde{E}(x,y) = \dfrac{1}{i\lambda}\displaystyle\iint_{\Sigma} \tilde{E}(x_0,y_0)\frac{e^{ikr}}{r}\frac{1+\cos\theta}{2}\mathrm{d}x_0\mathrm{d}y_0$

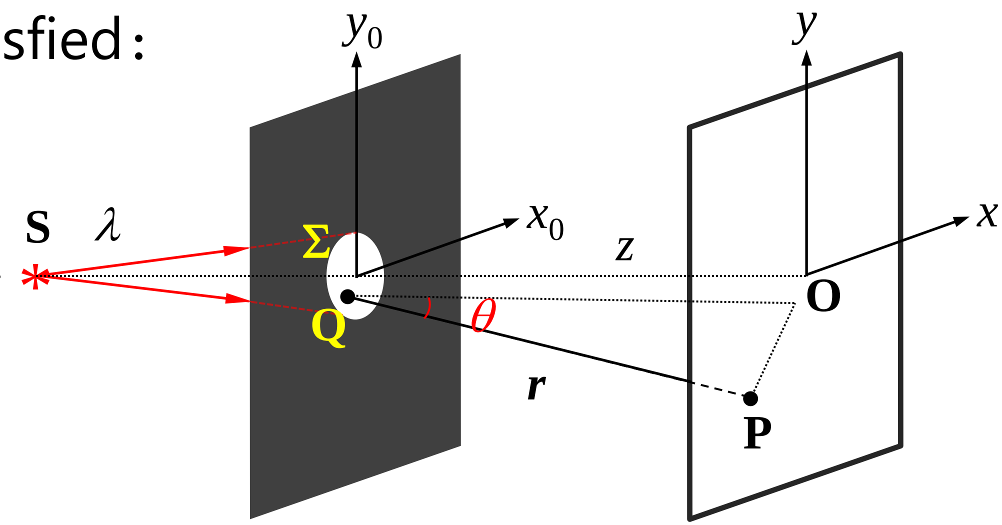
Paraxial approximation

② When the paraxial approximation is satisfied: $\theta \approx 0$

$$\overline{\mathrm{Q'P}} \equiv \rho$$

$$= \sqrt{(x-x_0)^2 + (y-y_0)^2}$$

$$r = \sqrt{z^2+\rho^2} = z\sqrt{1+\left(\frac{\rho}{z}\right)^2}$$



Since,

$$\theta \to 0 \qquad \frac{\rho}{z} \to 0$$

$$(1+x)^a = 1 + \frac{a}{1!}x + \frac{a(a-1)}{2!}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + \cdots$$

$$\Rightarrow \quad r = z\left[1 + \frac{1}{2}\left(\frac{\rho}{z}\right)^2 - \frac{1}{8}\left(\frac{\rho}{z}\right)^4 + \cdots\right]$$

Cylindrical coordinate

Paraxial approximation

$$r = z\left[1+\frac{1}{2}\left(\frac{\rho}{z}\right)^2-\frac{1}{8}\left(\frac{\rho}{z}\right)^4+\cdots\right]$$

$$\frac{e^{ikr}}{r}\begin{cases} r \approx z & \text{in denominator} \\ r \neq z & \text{in the phase term, since } k \text{ is not small.}\end{cases}$$

$\because \; r - z \approx \dfrac{\rho^2}{2z} \sim \lambda$ $\quad r - z$ can be larger or compared to λ.

$\therefore \; k(r-z)$ is not small, and phase change can be large

The amplitude is insensitive to changes in r but the phase is sensitive to changes in r.

— Chapter 6 Interference

Paraxial approximation

So, in paraxial approximation

$$\begin{cases} \dfrac{e^{ikr}}{r} & r \approx z \quad \text{in denominator} \\ \cos\theta \approx 1 \end{cases}$$

Kirchoff's diffraction formula:

$$\tilde{E}(x,y) = \frac{1}{i\lambda} \iint_{\Sigma} \tilde{E}(x_0, y_0) \frac{e^{ikr}}{r} \frac{1+\cos\theta}{2} \mathrm{d}x_0 \mathrm{d}y_0$$

can be simplified as:

$$\boxed{\tilde{E}(x,y) = \frac{1}{i\lambda z} \iint_{\Sigma} \tilde{E}(x_0, y_0) e^{ikr} \mathrm{d}x_0 \mathrm{d}y_0}$$

Fresnel approximation

③ Fresnel approximation

$$r = z\left[1+\frac{1}{2}\left(\frac{\rho}{z}\right)^2-\frac{1}{8}\left(\frac{\rho}{z}\right)^4+\cdots\right] \qquad \rho=\sqrt{\left(x-x_0\right)^2+\left(y-y_0\right)^2}$$

$$r \approx z\left[1+\frac{1}{2}\frac{\left(x-x_0\right)^2+\left(y-y_0\right)^2}{z^2}\right]$$

$$=\left(z+\frac{x^2+y^2}{2z}\right)+\left[\frac{x_0^2+y_0^2}{2z}-\frac{xx_0+yy_0}{z}\right]$$

Fresnel diffraction

$$\Rightarrow \boxed{\tilde{E}\left(x,y\right)=\frac{e^{ikz}}{i\lambda z}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{E}\left(x_0,y_0\right)\mathrm{e}^{\frac{ik}{2z}\left[\left(x-x_0\right)^2+\left(y-y_0\right)^2\right]}\mathrm{d}x_0\mathrm{d}y_0}$$

Fraunhofer approximation

④ Fraunhofer approximation

$$r = \left(z + \frac{x^2+y^2}{2z}\right) + \left[\cancel{\frac{x_0^2+y_0^2}{2z}} - \frac{xx_0+yy_0}{z}\right]$$

Fresnel approximation

If $$\frac{k\left(x_0^2+y_0^2\right)_{\max}}{2} << z$$

$$\Rightarrow \frac{k\left(x_0^2+y_0^2\right)_{\max}}{2z} \approx 0$$

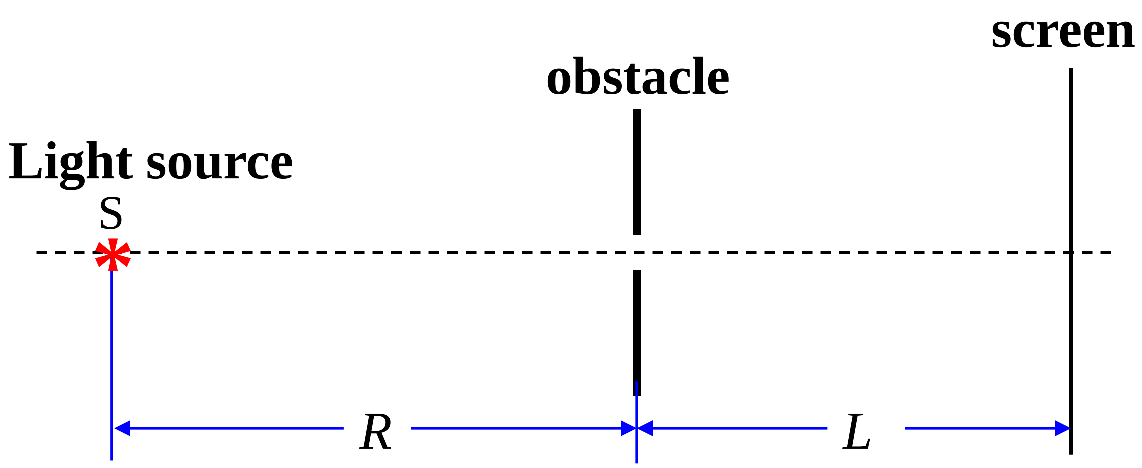
Fraunhofer approximation

Substituting diffraction formula **Fraunhofer diffraction**

$$\Rightarrow \boxed{\tilde{E}(x,y) = \frac{1}{i\lambda z} e^{ikz} \mathrm{e}^{\frac{ik}{2z}\left(x^2+y^2\right)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{E}(x_0,y_0)\,\mathrm{e}^{-\frac{ik}{z}\left(xx_0+yy_0\right)} \mathrm{d}x_0 \mathrm{d}y_0}$$

Diffraction classification

- **Classification:**



(1) Fresnel diffraction — near-field diffraction

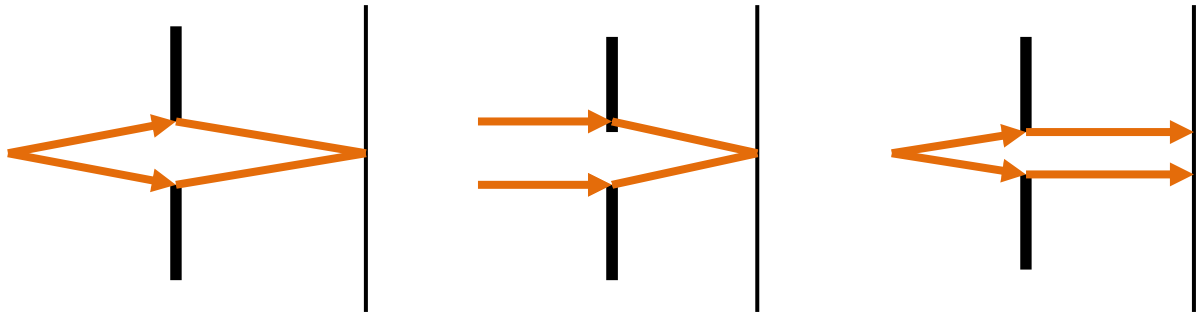
At least one of R and L is a finite value.

(2) Fraunhofer diffraction — far-field diffraction

Both R and L are infinite (also available with a lens).

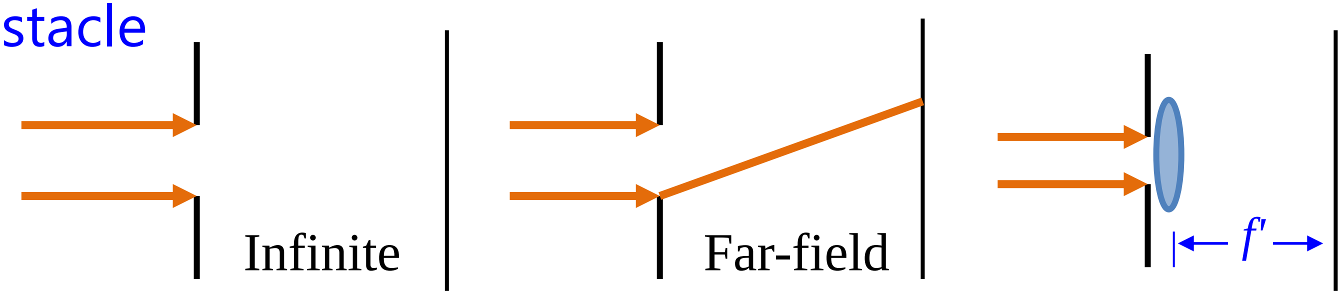
Diffraction classification

- Fresnel diffraction

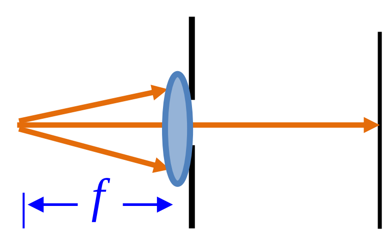


- Fraunhofer diffraction

① light source and the receiving screen are infinitely far from obstacle

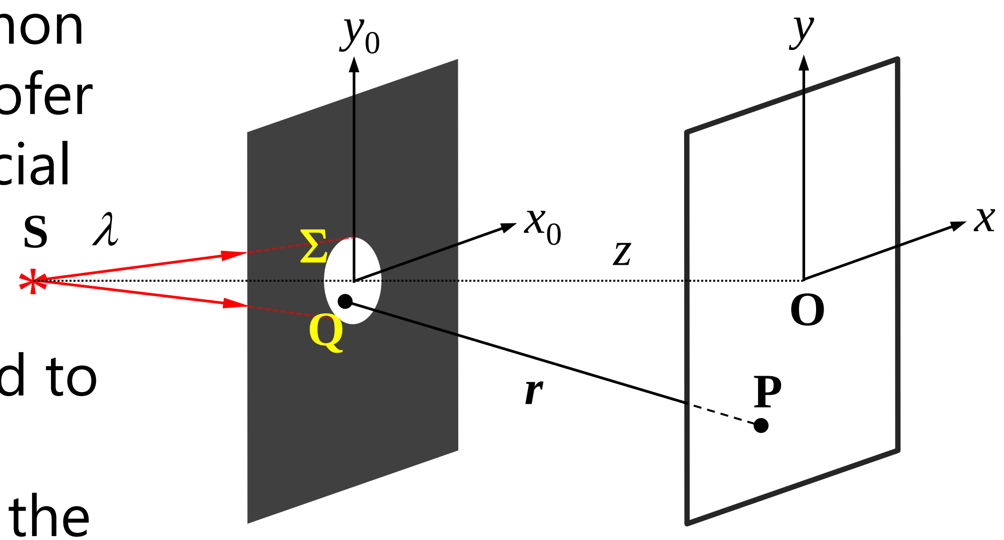


② on the conjugate image plane of the illumination source



Diffraction classification

- Fresnel diffraction is common phenomenon, and Fraunhofer diffraction is only one special case of it.



- "Far" and "near" are related to the relative sizes of the diffraction aperture D and the wavelength λ :
  - → For a given λ, If D is smaller, the diffraction phenomenon is more pronounced.
  - → When λ/D→0, Wave optics→geometric optics.

$$\frac{k\left(x_0^2 + y_0^2\right)_{\max}}{2} << z$$

$$D = \sqrt{\left(x_0^2 + y_0^2\right)_{\max}}$$

§ 7.3 Fourier transform

Chapter 2

- One-dimensional Fourier transform

$$g(t)=\int_{-\infty}^{\infty}G(\nu)e^{i2\pi\nu t}\mathrm{d}\nu$$

$$G(\nu)=\int_{-\infty}^{\infty}g(t)e^{-i2\pi\nu t}\mathrm{d}t$$

Time $\Leftrightarrow$ Frequency

Two-dimensional Fourier transform in optics

$$\begin{cases} g(x,y)=\iint G(f_x,f_y)e^{i2\pi(f_x x+f_y y)}\mathrm{d}f_x\mathrm{d}f_y \\ G(f_x,f_y)=\iint g(x,y)e^{-i2\pi(f_x x+f_y y)}\mathrm{d}x\mathrm{d}y \end{cases}$$

Space

$\Updownarrow$

Spatial frequency

For plane wave

$$f_x=\frac{k_x}{2\pi}\qquad f_y=\frac{k_y}{2\pi}\qquad f_z=\frac{k_z}{2\pi}\qquad \mathbf{f}=\frac{\mathbf{k}}{2\pi}$$

Compared with Fraunhofer diffraction:

$$\boxed{\tilde{E}(x,y)=\frac{1}{i\lambda z}e^{ikz}\mathrm{e}^{\frac{ik}{2z}\left(x^2+y^2\right)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{E}(x_0,y_0)\mathrm{e}^{-\frac{ik}{z}(xx_0+yy_0)}\mathrm{d}x_0\mathrm{d}y_0}$$

So,

$$G\left(f_x,f_y\right)=\iint g(x,y)e^{-i2\pi\left(f_x x+f_y y\right)}\mathrm{d}x\mathrm{d}y$$

The Fourier transform of the diffraction screen

can be written as

$$G\left(f_x,f_y\right)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g(x_0,y_0)\mathrm{e}^{-i2\pi\left(f_x x_0+f_y y_0\right)}\mathrm{d}x_0\mathrm{d}y_0$$

$$\boxed{\tilde{E}\left(\frac{x}{\lambda z},\frac{y}{\lambda z}\right)=\tilde{C}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{E}(x_0,y_0)\mathrm{e}^{-i2\pi\left(\frac{x}{\lambda z}x_0+\frac{y}{\lambda z}y_0\right)}\mathrm{d}x_0\mathrm{d}y_0}$$

Assume that: incidence is a plane wave with a unit amplitude.

$$\tilde{E}(x_0, y_0) = \tilde{E}_\mathrm{i}(x_0, y_0)\cdot\tilde{t}(x_0, y_0) = \tilde{t}(x_0, y_0)$$

$$\tilde{E}\left(\underbrace{\frac{x}{\lambda z}}_{f_x}, \underbrace{\frac{y}{\lambda z}}_{f_y}\right) = \tilde{C}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{E}(x_0, y_0)\,\mathrm{e}^{-i2\pi\left(\frac{x}{\lambda z}x_0+\frac{y}{\lambda z}y_0\right)}\mathrm{d}x_0\mathrm{d}y_0$$

$$\underbrace{\tilde{E}(f_x, f_y)}_{\text{Diffraction field}} = \tilde{C}\underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{t}(x_0, y_0)\,\mathrm{e}^{-i2\pi(f_x x_0+f_y y_0)}\mathrm{d}x_0\mathrm{d}y_0}_{\text{Fourier transform of the screen}}$$

$$\tilde{C} = \frac{1}{i\lambda z}e^{ikz}\mathrm{e}^{\frac{ik}{2z}\left(x^2+y^2\right)}$$

A complex factor that can be ignored when considering the distribution

Spatial frequency (f_x, f_y, f_z) correspond to a monochromatic plane wave propagating in a certain direction

$$f_x = \frac{k_x}{2\pi} \qquad f_y = \frac{k_y}{2\pi} \qquad f_z = \frac{k_z}{2\pi} \qquad \mathbf{f} = \frac{\mathbf{k}}{2\pi}$$

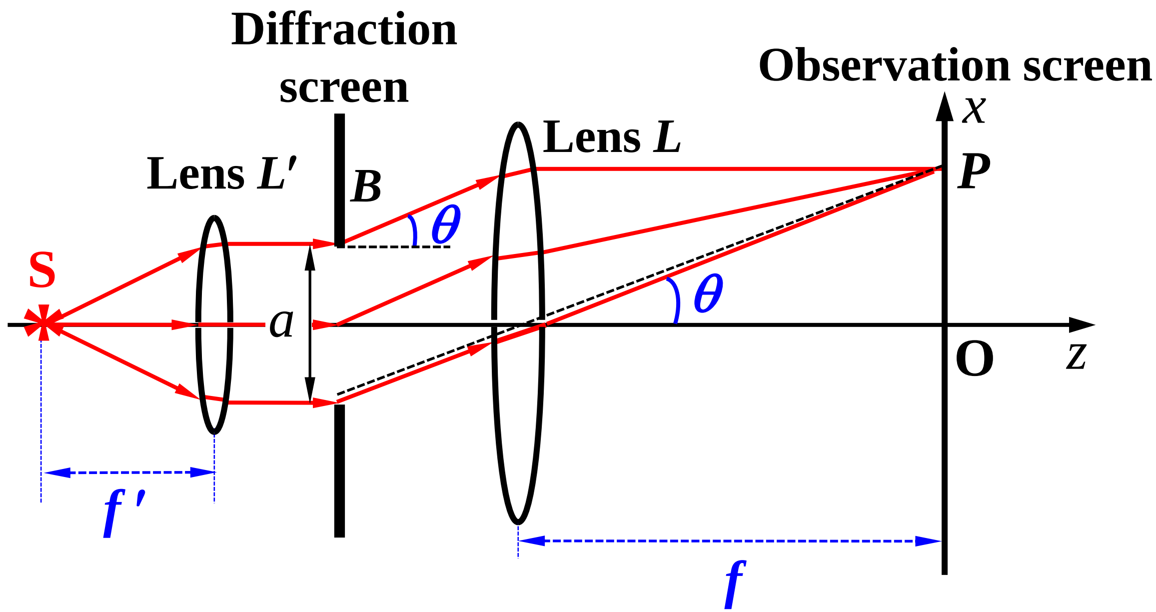
$$\tilde{E}\left(f_x, f_y\right) = \tilde{C}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{t}\left(x_0, y_0\right)e^{-i2\pi\left(f_x x_0 + f_y y_0\right)}dx_0 dy_0$$

It indicates that the entire diffraction field is the superposition of many plane wave of different amplitudes and different propagation directions. These diffracted waves are superimposed on each other in the **near field**, and they are separated from each other in the **far field**. >>spatial frequency division.

Fraunhofer diffraction is a spatial spectrum analyzer and is the physical implementation of the Fourier transform.

$(f_x, f_y, f_z) \Leftrightarrow$ direction of propagation $\xRightarrow{\text{lens}}$ a point on the screen

Chapter 4



So, we can get that (*xz* plane)

$$f_x = \frac{x}{z\lambda} = \frac{k_x}{2\pi} = \frac{k}{2\pi}\sin\theta \approx \frac{k}{2\pi}\tan\theta \qquad f_y = \frac{y}{z\lambda} = \frac{k_y}{2\pi} \qquad f_z = \frac{k_z}{2\pi}$$

The Fraunhofer diffraction pattern directly maps the spatial spectrum of the light field distribution at the diffraction screen.

$$\underline{\tilde{E}\left(f_x, f_y\right)} = \tilde{C}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\underline{\tilde{t}\left(x_0, y_0\right)}\mathrm{e}^{-i2\pi\left(f_x x_0 + f_y y_0\right)}\mathrm{d}x_0\mathrm{d}y_0$$

The distribution $\tilde{E}\left(f_x, f_y\right)$ of the light field on the back focal plane is a **spectrum** of the screen function $\tilde{t}\left(x_0, y_0\right)$, reflecting the structure of the diffraction screen.

Different screens (slits, circles, grating etc. in this chapter)>> different pattern!

X-Ray Diffraction (XRD): $\lambda = 0.001 \sim 10\ \mathrm{nm}$, used to analyze crystal structure (lattice constant, symmetry, etc.).

Homework

Prove the screen function of a convex lens.

Homework*

You are free.

Next week

Sinusoidal grating, single slit, circles, resolution limit etc.

Sections 10.2.1、10.2.5-10.2.6