Chapter 9

Electromagnetic Waves

Waves in One Dimension

The Wave Equation

What is a "wave"?

A wave is a disturbance of a continuous medium that propagates with a fixed shape at constant velocity. in the presence of absorption, the wave will diminish in size as it moves.

Given the initial shape of the string, $g(z) = f(z, 0)$

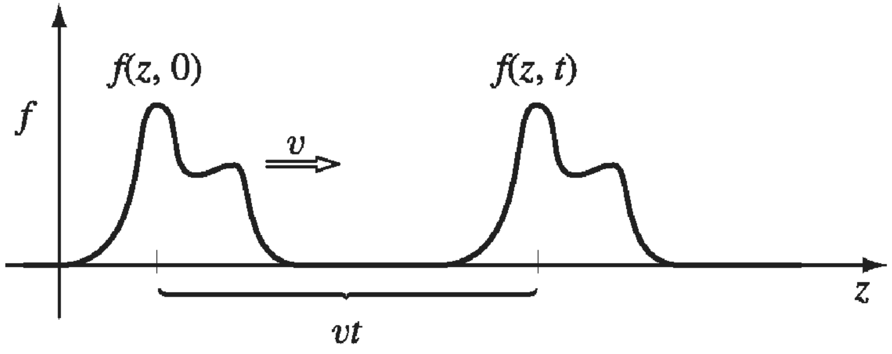

FIGURE 9.1

$$f(z,t) = f(z - vt, 0) = g(z - vt)$$

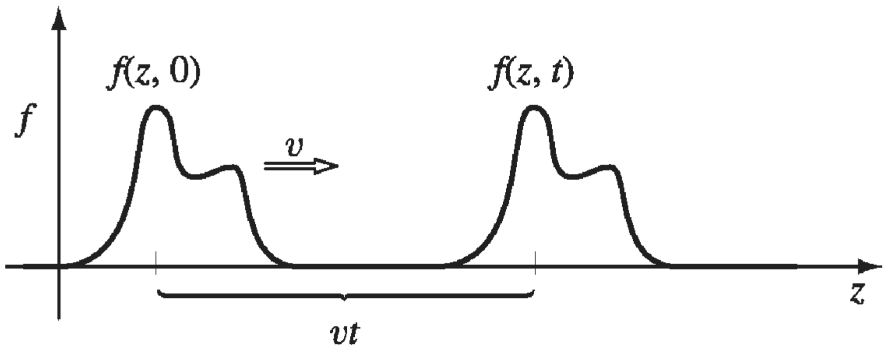

FIGURE 9.1

It tells us that the function $f(z,t)$, which might have depended on z and time, in fact depends on them only in the very special combination $z - vt$; when that is true, the function $f(z,t)$ represents a wave of fixed shape traveling in the z direction at speed v

$$f_1(z,t) = Ae^{-b(z-vt)^2}$$

$$f_2(z,t) = A\sin[b(z-vt)]$$

$$f_3(z,t) = \frac{A}{b(z-vt)^2+1}$$

all represent waves (with different shapes, of course), but

$$f_4(z,t) = Ae^{-b(bz^2+vt)} \qquad \text{and} \qquad f_5(z,t) = A\sin(bz)\cos(bvt)^3$$

do not.

Imagine a very long string under tension T. If it is displaced from equilibrium, the net transverse force on the segment between z and $z + \Delta z$ is

$$\Delta F = T\sin\theta' - T\sin\theta.$$

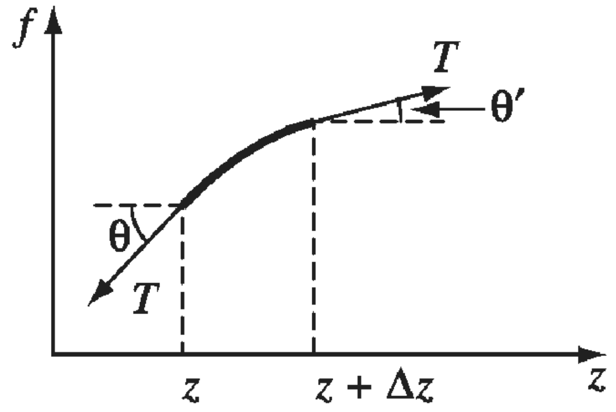

FIGURE 9.2

Provided that the distortion of the string is not too great, these angles are small and we can replace the sine by the tangent:

$$\Delta F \cong T(\tan\theta' - \tan\theta) = T\left(\left.\frac{\partial f}{\partial z}\right|_{z+\Delta z} - \left.\frac{\partial f}{\partial z}\right|_{z}\right) \cong T\frac{\partial^2 f}{\partial z^2}\Delta z.$$

If the mass per unit length is μ, Newton's second law says

$$\Delta F = \mu(\Delta z)\frac{\partial^2 f}{\partial t^2}$$

therefore

$$\frac{\partial^2 f}{\partial z^2} = \frac{\mu}{T}\frac{\partial^2 f}{\partial t^2}$$

Evidently, small disturbances on the string satisfy

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \qquad (9.2)$$

where v represents the speed of propagation is

$$\boldsymbol{v} = \sqrt{\frac{\boldsymbol{T}}{\boldsymbol{\mu}}}$$

Equation 9.2 is known as the (classical) **wave equation**, because it admits as solutions all functions of the form

$$f(z,t) = g(z - vt)$$

such functions represent waves propagating in the z direction with speed v

$$\boldsymbol{u \equiv z - vt} \qquad \frac{\partial f}{\partial z} = \frac{dg}{du}\frac{\partial u}{\partial z} = \frac{dg}{du} \qquad \frac{\partial f}{\partial t} = \frac{dg}{du}\frac{\partial u}{\partial t} = -v\frac{dg}{du}$$

and

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{dg}{du}\right) = \frac{d^2 g}{du^2}\frac{\partial u}{\partial z} = \frac{d^2 g}{du^2}$$

$$\frac{\partial^2 f}{\partial t^2} = -v\frac{\partial}{\partial t}\left(\frac{dg}{du}\right) = -v\frac{d^2 g}{du^2}\frac{\partial u}{\partial t} = v^2\frac{d^2 g}{du^2}$$

$$\frac{d^2 g}{du^2} = \frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

Note:
$g(u)$ can be any differentiable function.

$g(z - vt)$ are not the only solutions. The wave equation involves the square of v, so we can generate another class of solutions by simply changing the sign of the velocity:

$$f(z,t) = h(z + vt)$$

This represents a wave propagating in the *negative z* direction.

The most general solution to the wave equation is the sum of a wave to the right and a wave to the left:

$$f(z,t) = g(z - vt) + h(z + vt)$$

Note:
The wave equation is **linear**: The sum of any two solutions is itself a solution. Every solution to the wave equation can be expressed in this form.

Sinusoidal Waves

$(\boldsymbol{i})$ **Terminology**. Of all possible wave forms, the sinusoidal one

$$f(z,t) = A\cos[k(z - vt) + \delta]$$

$$A \rightarrow amplitude$$

$$\delta \rightarrow phase\ constant \ \ (0 \leq \delta \leq 2\pi)$$

Note:

For $z = vt - \delta/k$, the phase is zero, and this is called the "central maximum". ${}^{\delta}/_{k}$ is the distance by which the central maximum (and therefore the entire wave) is "delayed". k is the **wave number.**

It is related to the wavelength λ by the equation:

$$\lambda = \frac{2\pi}{k}$$

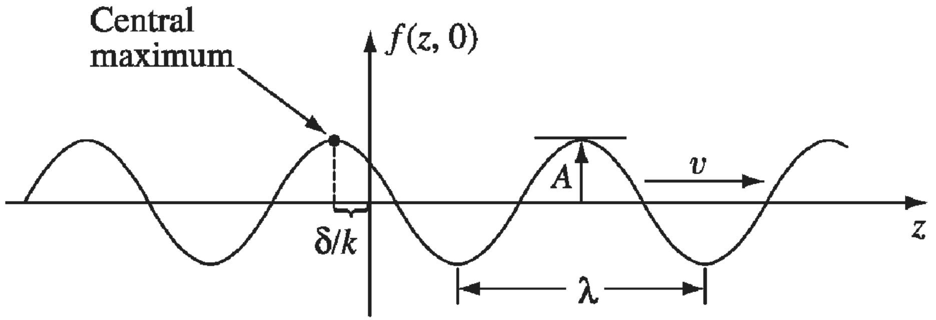

FIGURE 9.3

The period for speed v is

$$T = \frac{2\pi}{kv}$$

The frequency ν (number of oscillations per unit time) is

$$\upsilon = \frac{1}{T} = \frac{kv}{2\pi} = \frac{v}{\lambda}$$

a more convenient unit is the **angular frequency** ω

$$\omega = 2\pi\upsilon = kv$$

Ordinarily, sinusoidal waves in terms of ω

$$f(z, t) = A\cos(kz - \omega t + \delta)$$

Traveling to the left would be written

$$f(z,t) = A\cos(kz + \omega t - \delta)$$

The sign of the phase constant is chosen for consistency with our previous convention that $\frac{\delta}{k}$ shall represent the distance which the wave is "delayed“.

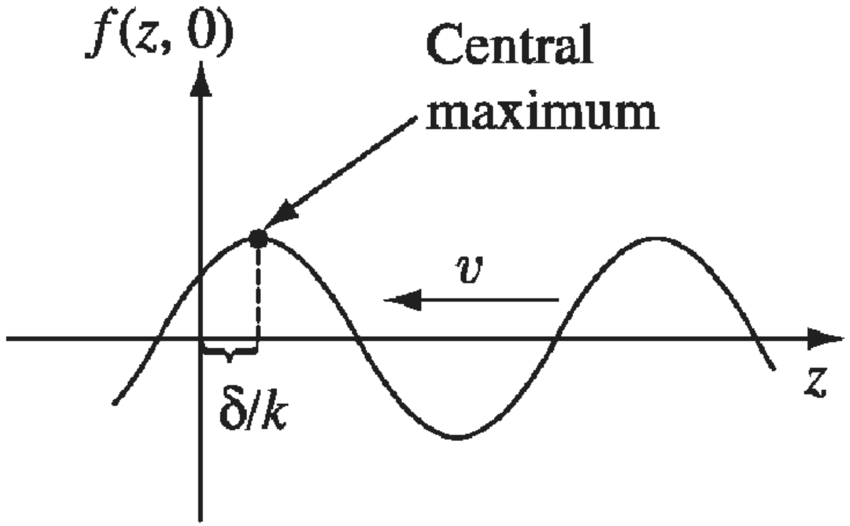

FIGURE 9.4

(ii) **Complex notation.** In view of **Euler's formula,**

$$e^{i\theta} = \cos\theta + i\sin\theta$$

The sinusoidal wave can be written

$$f(z,t) = \mathrm{Re}\left[Ae^{i(kz-\omega t+\delta)}\right]$$

introduce the complex wave function

$$\tilde{f}(z,t) \equiv \tilde{A}e^{i(kz-\omega t)}$$

with the complex amplitude $\tilde{A} \equiv Ae^{i\delta}$ absorbing the phase constant.

The actual wave function is the real part of $\tilde{f}$

$$f(z,t) = \mathrm{Re}[\tilde{f}(z,t)]$$

Note:
The advantage of the complex notation is that exponentials are much easier to manipulate than sines and cosines

(iii) Linear combinations of sinusoidal waves

Any wave can be expressed as a linear combination of sinusoidal ones

$$\tilde{f}(z,t) = \int_{-\infty}^{\infty} \tilde{A}(k) e^{i(kz-\omega t)}\, dk$$

Here ω is a function of k , k run through negative values in order to include waves going both directions

Note:

The formula for $\tilde{A}(k)$, in terms of the initial conditions $f(z,0)$ and $\dot{f}(z,0)$, can be obtained from the theory of ***Fourier transforms***

Electromagnetic Waves in Vacuum

The Wave Equation for *E* and *B*

In regions of space where there is no charge or current, Maxwell's equations read

$$\left.\begin{array}{llll} \text{(i)} & \nabla \cdot \mathbf{E} = 0, & \text{(iii)} & \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}, \\[2ex] \text{(ii)} & \nabla \cdot \mathbf{B} = 0, & \text{(iv)} & \nabla \times \mathbf{B} = \mu_0 \epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t}. \end{array}\right\}$$

They constitute a set of coupled, first-order, partial differential equations for E and B.
They can be decoupled by applying the curl to (iii) and (iv):

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = \nabla \times \left(-\frac{\partial \mathbf{B}}{\partial t}\right)$$

$$= -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2},$$

$$\nabla \times (\nabla \times \mathbf{B}) = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = \nabla \times \left(\mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right)$$

$$= \mu_0 \epsilon_0 \frac{\partial}{\partial t}(\nabla \times \mathbf{E}) = -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}.$$

Since $\nabla \cdot \boldsymbol{E} = 0$ and $\nabla \cdot \boldsymbol{B} = 0$,

$$\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}, \quad \nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

We now have separated equations for E and B, but they are of second order; that's the price you pay for decoupling them

Each Cartesian component of E and B satisfies the three dimensional wave equation

$$\nabla^2 f = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}$$

Maxwell's equations imply that empty space supports the propagation of electromagnetic waves, traveling at a speed

$$v = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = 3.00 \times 10^8 \text{ m/s}$$

Plane waves:

Different frequencies in the visible range correspond to different colors, such waves are called monochromatic. Moreover, the waves traveling in the z direction have no x or y dependence, because the fields are uniform over every plane perpendicular to the direction of propagation

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{i(kz-\omega t)}, \quad \tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 e^{i(kz-\omega t)}$$

where $\widetilde{\boldsymbol{E}_0}$ and $\widetilde{\boldsymbol{B}_0}$ are the (complex) amplitudes (the physical fields, of course, are the real parts of $\boldsymbol{E}$ and $\boldsymbol{B}$)

Every solution to Maxwell's equations (in empty space) must obey the wave equation, the converse is not true; Maxwell's equations impose extra constraints on $\widetilde{\boldsymbol{E}_0}$ and $\widetilde{\boldsymbol{B}_0}$.

Since $\nabla \cdot \boldsymbol{E} = 0$ and $\nabla \cdot \boldsymbol{B} = 0$ it follows that

$$(\tilde{E}_0)_z = (\tilde{B}_0)_z = 0$$

Note:

Because the real part of E differs from the imaginary part only in the replacement of sine by cosine, if the former obeys Maxwell's equations, so does the latter, and hence E as well.

That is, electromagnetic waves are transverse: the electric and magnetic fields are perpendicular to the direction of propagation. Moreover, Faraday's law, $\nabla \times E = -\frac{\partial \boldsymbol{B}}{\partial t}$, implies a relation between the electric and magnetic amplitudes, to wit:

$$-k(\tilde{E}_0)_y = \omega(\tilde{B}_0)_x, \quad k(\tilde{E}_0)_x = \omega(\tilde{B}_0)_y \qquad (9.45)$$

or, more compactly

$$\tilde{\mathbf{B}}_0 = \frac{k}{\omega}(\hat{\mathbf{z}} \times \tilde{\mathbf{E}}_0)$$

$\boldsymbol{E}$ and $\boldsymbol{B}$ are in phase and mutually perpendicular; their (real) amplitudes are related by

$$B_0 = \frac{k}{\omega}E_0 = \frac{1}{c}E_0$$

The fourth of Maxwell's equations, $\nabla \times \boldsymbol{B} = \mu_0\epsilon_0({}^{\partial \boldsymbol{E}}/{}_{\partial t})$, does not yield an independent condition; it simply reproduces $Eq. 9.45$

We can easily generalize to monochromatic plane waves traveling in an arbitrary direction with propagation (or wave) vector, **k**, pointing in the direction of propagation

$$\tilde{\mathbf{E}}(\mathbf{r}, t) = \tilde{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\,\hat{\mathbf{n}},$$

$$\tilde{\mathbf{B}}(\mathbf{r}, t) = \frac{1}{c}\tilde{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}(\hat{\mathbf{k}}\times\hat{\mathbf{n}}) = \frac{1}{c}\hat{\mathbf{k}}\times\tilde{\mathbf{E}},$$

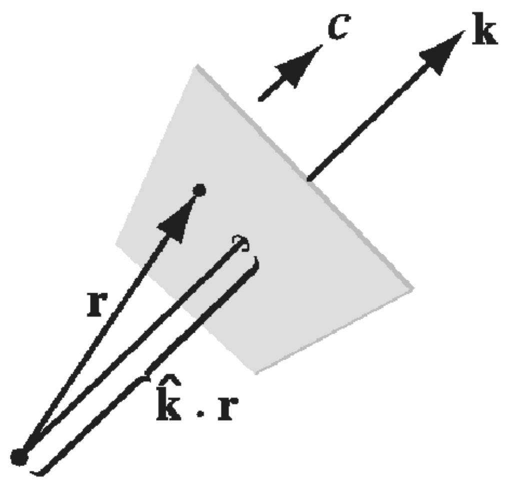

FIGURE 9.11

where $\hat{\boldsymbol{n}}$ is the polarization vector. Because **E** is transverse,

$$\hat{\mathbf{n}}\cdot\hat{\mathbf{k}} = 0$$

The actual (real) electric and magnetic fields in a monochromatic plane wave with propagation vector $\boldsymbol{k}$ and polarization $\hat{\boldsymbol{n}}$ are

$$\mathbf{E}(\mathbf{r}, t) = E_0\cos(\mathbf{k}\cdot\mathbf{r}-\omega t+\delta)\,\hat{\mathbf{n}},$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{1}{c}E_0\cos(\mathbf{k}\cdot\mathbf{r}-\omega t+\delta)(\hat{\mathbf{k}}\times\hat{\mathbf{n}}).$$

Energy and Momentum in Electromagnetic Waves

The energy per unit volume in electromagnetic fields is

$$u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$$

In the case of a monochromatic plane wave (Eq. 9.48)

$$B^2 = \frac{1}{c^2} E^2 = \mu_0 \epsilon_0 E^2$$

The electric and magnetic contributions are equal:

$$u = \epsilon_0 E^2 = \epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)$$

As the wave travels, it carries this energy along with it. The energy flux density (energy per unit area, per unit time) is given by the Poynting vector:

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$$

For monochromatic plane waves propagating in the *z* direction,

$$\mathbf{S} = c\epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)\,\hat{\mathbf{z}} = cu\,\hat{\mathbf{z}}$$

Electromagnetic fields not only carry energy, they also carry momentum, the momentum density stored in the fields is

$$\mathbf{g} = \frac{1}{c^2}\mathbf{S}$$

For monochromatic plane waves

$$\mathbf{g} = \frac{1}{c}\epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)\,\hat{\mathbf{z}} = \frac{1}{c}u\,\hat{\mathbf{z}}$$

Typically, all we want is the average value of cosine-squared over a complete cycle:

$$\langle u \rangle = \frac{1}{2}\epsilon_0 E_0^2 \qquad \langle \mathbf{S} \rangle = \frac{1}{2}c\epsilon_0 E_0^2\,\hat{\mathbf{z}} \qquad \langle \mathbf{g} \rangle = \frac{1}{2c}\epsilon_0 E_0^2\,\hat{\mathbf{z}}$$

Bracket $< >$ denotes the (time) average over a complete cycle (or many cycles)

Here is a cute trick that over a complete cycle the average of $sin^2\theta$ is equal to the average of $cos^2\theta$, so $\langle \sin^2 \rangle = \langle \cos^2 \rangle = 1/2$. More formally,

$$\frac{1}{T}\int_0^T \cos^2(kz - 2\pi t/T + \delta)\, dt = 1/2$$

The average power per unit area transported by an electromagnetic wave is called the intensity:

$$I \equiv \langle S \rangle = \frac{1}{2} c\epsilon_0 E_0^2$$

When light falls (at normal incidence) on a perfect absorber, it delivers its momentum to the surface. In a time Δt, the momentum transfer is $\Delta \mathbf{p} = \langle \mathbf{g} \rangle Ac\, \Delta t$,

so the radiation pressure (average force per unit area) is

$$P = \frac{1}{A}\frac{\Delta p}{\Delta t} = \frac{1}{2}\epsilon_0 E_0^2 = \frac{I}{c}$$

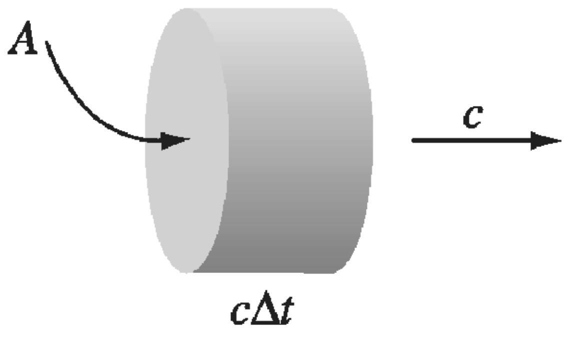

FIGURE 9.12

On a perfect reflector the pressure is twice as great, because the momentum switches direction, instead of simply being absorbed.

Electromagnetic Waves In Matter

Propagation in Linear Media

Inside matter, but in regions where there is no free charge or free current, Maxwell's equations become

$$\left.\begin{array}{llll} \text{(i)} & \nabla\cdot\mathbf{D}=0, & \text{(iii)} & \nabla\times\mathbf{E}=-\dfrac{\partial\mathbf{B}}{\partial t}, \\ \text{(ii)} & \nabla\cdot\mathbf{B}=0, & \text{(iv)} & \nabla\times\mathbf{H}=\dfrac{\partial\mathbf{D}}{\partial t}. \end{array}\right\}$$

If the medium is linear $\mathbf{D}=\epsilon\mathbf{E}, \quad \mathbf{H}=\dfrac{1}{\mu}\mathbf{B}$

and homogeneous, they reduce to

(Eq. 9.67)

$$\left.\begin{array}{llll} \text{(i)} & \nabla\cdot\mathbf{E}=0, & \text{(iii)} & \nabla\times\mathbf{E}=-\dfrac{\partial\mathbf{B}}{\partial t}, \\ \text{(ii)} & \nabla\cdot\mathbf{B}=0, & \text{(iv)} & \nabla\times\mathbf{B}=\mu\epsilon\dfrac{\partial\mathbf{E}}{\partial t}, \end{array}\right\}$$

Note:
These equations differ from the vacuum analogs (Eqs. 9.40) only in the replacement of $\mu_0\epsilon_0$ by $\mu\epsilon$

Evidently electromagnetic waves propagate through a linear homogeneous medium at a speed

$$v = \frac{1}{\sqrt{\epsilon\mu}} = \frac{c}{n}$$

$$n \equiv \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}}$$

n is the index of refraction of the substance

Note:
This observation is mathematically pretty trivial, As the wave passes through, the fields busily polarize and magnetize all the molecules, resulting their own electric and magnetic fields. These combine with the original fields in such but a different speed. This extraordinary conspiracy the phenomenon of transparency. It is a distinctly nontrivial consequence of linearity.

For most materials μ_0 is very close to μ

So $$n \cong \sqrt{\epsilon_r}$$

where ϵ_r is the dielectric constant. Since ϵ_r is almost always greater than 1, light travels more slowly through matter, a fact that is well known from optics.

In the following, as you will see that all of our previous results carry over, with the simple transcription:

$\epsilon_0 \rightarrow \epsilon \;\; \mu_0 \rightarrow \mu$ and hence $c \rightarrow v$

The energy density is

$$u = \frac{1}{2}\left(\epsilon E^2 + \frac{1}{\mu}B^2\right)$$

The Poynting vector is

$$\mathbf{S} = \frac{1}{\mu}(\mathbf{E} \times \mathbf{B})$$

The intensity is

$$I = \frac{1}{2}\epsilon v E_0^2$$

Note:
The interesting question is this: What happens when a wave passes from one transparent medium into another. The details depend on the exact nature of the electrodynamic boundary conditions, which we derived in Chapter 7:

$$\boxed{\left.\begin{array}{llll} \text{(i)} & \epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp}, & \text{(iii)} & \mathbf{E}_1^{\parallel} = \mathbf{E}_2^{\parallel}, \\ \text{(ii)} & B_1^{\perp} = B_2^{\perp}, & \text{(iv)} & \frac{1}{\mu_1}\mathbf{B}_1^{\parallel} = \frac{1}{\mu_2}\mathbf{B}_2^{\parallel}. \end{array}\right\}}$$

In the following sections, we use them to deduce the laws governing reflection and refraction of electromagnetic waves.

Reflection and Transmission at Normal Incidence

Suppose the xy plane forms the boundary between two linear media. A plane wave of frequency ω, traveling in the z direction and polarized in the x direction, approaches the interface from the left (Fig. 9.13):

$$\left.\begin{aligned}\tilde{\mathbf{E}}_I(z,t) &= \tilde{E}_{0_I} e^{i(k_1 z - \omega t)}\,\hat{\mathbf{x}},\\ \tilde{\mathbf{B}}_I(z,t) &= \frac{1}{v_1}\tilde{E}_{0_I} e^{i(k_1 z - \omega t)}\,\hat{\mathbf{y}}.\end{aligned}\right\}$$

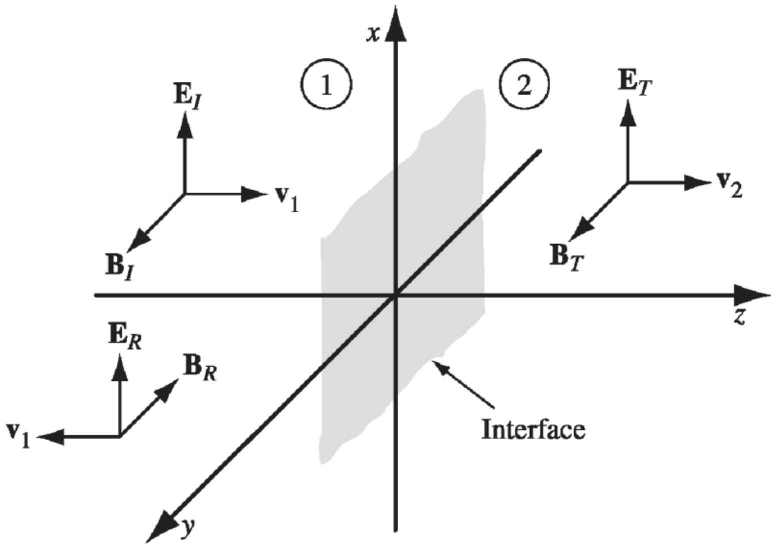

FIGURE 9.13

It gives rise to a reflected wave

$$\left.\begin{aligned}\tilde{\mathbf{E}}_R(z,t) &= \tilde{E}_{0_R} e^{i(-k_1 z - \omega t)}\,\hat{\mathbf{x}},\\ \tilde{\mathbf{B}}_R(z,t) &= -\frac{1}{v_1}\tilde{E}_{0_R} e^{i(-k_1 z - \omega t)}\,\hat{\mathbf{y}},\end{aligned}\right\}$$

which travels back to the left in medium (1), and a transmitted wave

$$\left.\begin{aligned}\tilde{\mathbf{E}}_T(z,t) &= \tilde{E}_{0_T} e^{i(k_2 z-\omega t)}\,\hat{\mathbf{x}},\\ \tilde{\mathbf{B}}_T(z,t) &= \frac{1}{v_2}\tilde{E}_{0_T} e^{i(k_2 z-\omega t)}\,\hat{\mathbf{y}},\end{aligned}\right\}$$

which continues on to the right in medium (2). Note the minus sign in $\widetilde{\boldsymbol{B}}_{\boldsymbol{R}}$. The Poynting vector aims in the direction of propagation.

In this case there are no components perpendicular to the surface, so (i) and (ii) are trivial. However, (iii) requires that

$$\tilde{E}_{0_I} + \tilde{E}_{0_R} = \tilde{E}_{0_T} \qquad (9.78)$$

$$\left.\begin{aligned}&\text{(i)}\ \ \epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp}, &&\text{(iii)}\ \ \mathbf{E}_1^{\parallel} = \mathbf{E}_2^{\parallel},\\ &\text{(ii)}\ \ B_1^{\perp} = B_2^{\perp}, &&\text{(iv)}\ \ \frac{1}{\mu_1}\mathbf{B}_1^{\parallel} = \frac{1}{\mu_2}\mathbf{B}_2^{\parallel}.\end{aligned}\right\}$$

while (iv) says
$$\frac{1}{\mu_1}\left(\frac{1}{v_1}\tilde{E}_{0_I} - \frac{1}{v_1}\tilde{E}_{0_R}\right) = \frac{1}{\mu_2}\left(\frac{1}{v_2}\tilde{E}_{0_T}\right)$$

or $$\tilde{E}_{0_I} - \tilde{E}_{0_R} = \beta \tilde{E}_{0_T} \qquad (9.80)$$

where $$\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1}$$

Therefore

$$\tilde{E}_{0_R} = \left(\frac{1-\beta}{1+\beta}\right)\tilde{E}_{0_I}, \quad \tilde{E}_{0_T} = \left(\frac{2}{1+\beta}\right)\tilde{E}_{0_I}$$

If the permeabilities μ are close to their values in vacuum , then $\beta = v_1/v_2$, and we have

$$\tilde{E}_{0_R} = \left(\frac{v_2 - v_1}{v_2 + v_1}\right)\tilde{E}_{0_I}, \quad \tilde{E}_{0_T} = \left(\frac{2v_2}{v_2 + v_1}\right)\tilde{E}_{0_I}$$

In terms of the indices of refraction

$$E_{0_R} = \left|\frac{n_1 - n_2}{n_1 + n_2}\right| E_{0_I}, \quad E_{0_T} = \left(\frac{2n_1}{n_1 + n_2}\right) E_{0_I}$$

What fraction of the incident energy is reflected, and what fraction is transmitted? According to Eq. 9.73, the intensity (average power per unit area) is:

$$I = \frac{1}{2}\epsilon v E_0^2.$$

If (again) $\mu_1 = \mu_2$, then the ratio of the reflected intensity to the incident intensity is

$$R \equiv \frac{I_R}{I_I} = \left(\frac{E_{0_R}}{E_{0_I}}\right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2.$$

whereas the ratio of the transmitted intensity to the incident intensity is

$$T \equiv \frac{I_T}{I_I} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\left(\frac{E_{0_T}}{E_{0_I}}\right)^2 = \frac{4n_1 n_2}{(n_1 + n_2)^2}$$

R is called the reflection coefficient and ***T*** the transmission coefficient

$$R + T = 1$$

For instance, when light passes from air (n_1 = 1) into glass (n_2 = 1.5), R = 0.04 and T = 0.96. No surprise: most of the light is transmitted.

Reflection and Transmission at Oblique Incidence

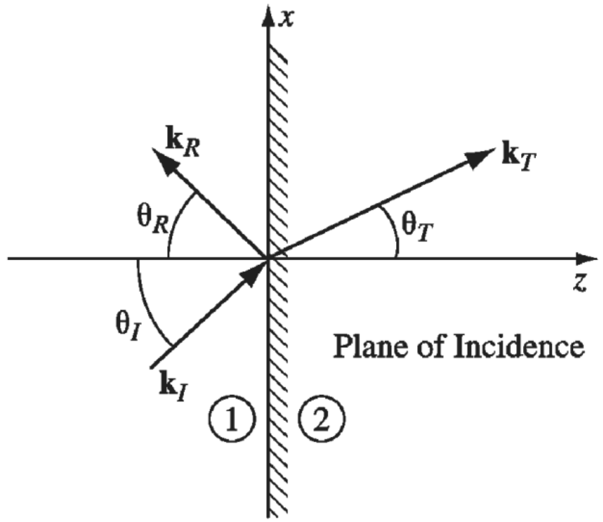

FIGURE 9.14

Suppose a monochromatic plane wave

$$\tilde{\mathbf{E}}_I(\mathbf{r}, t) = \tilde{\mathbf{E}}_{0_I} e^{i(\mathbf{k}_I \cdot \mathbf{r} - \omega t)}, \quad \tilde{\mathbf{B}}_I(\mathbf{r}, t) = \frac{1}{v_1}(\hat{\mathbf{k}}_I \times \tilde{\mathbf{E}}_I)$$

approaches from the left, giving rise to a reflected wave,

$$\tilde{\mathbf{E}}_R(\mathbf{r}, t) = \tilde{\mathbf{E}}_{0_R} e^{i(\mathbf{k}_R \cdot \mathbf{r} - \omega t)}, \quad \tilde{\mathbf{B}}_R(\mathbf{r}, t) = \frac{1}{v_1}\left(\hat{\mathbf{k}}_R \times \tilde{\mathbf{E}}_R\right)$$

and a transmitted wave

$$\tilde{\mathbf{E}}_T(\mathbf{r}, t) = \tilde{\mathbf{E}}_{0_T} e^{i(\mathbf{k}_T \cdot \mathbf{r} - \omega t)}, \quad \tilde{\mathbf{B}}_T(\mathbf{r}, t) = \frac{1}{v_2}\left(\hat{\mathbf{k}}_T \times \tilde{\mathbf{E}}_T\right)$$

All three waves have the same frequency ω that is determined once and for all at the source.
The three wave numbers are related:

$$k_I v_1 = k_R v_1 = k_T v_2 = \omega, \quad \text{or} \quad k_I = k_R = \frac{v_2}{v_1} k_T = \frac{n_1}{n_2} k_T$$

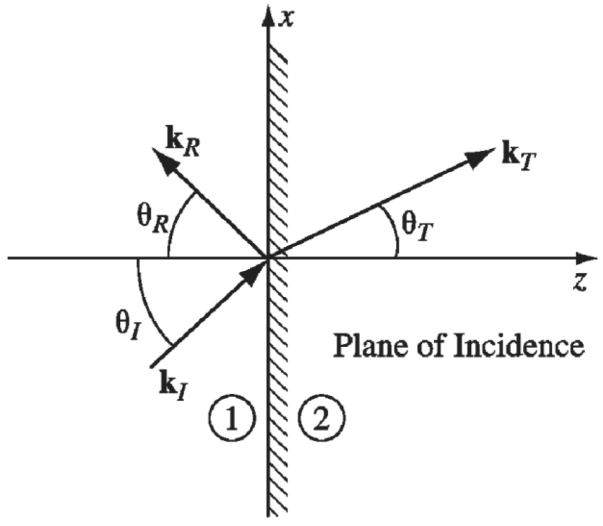

FIGURE 9.14

The boundary conditions ($Eq. 9.74$) all share the generic structure

$$(\ )e^{i(\mathbf{k}_I \cdot \mathbf{r} - \omega t)} + (\ )e^{i(\mathbf{k}_R \cdot \mathbf{r} - \omega t)} = (\ )e^{i(\mathbf{k}_T \cdot \mathbf{r} - \omega t)}, \quad \text{at } z = 0$$

Because the boundary conditions must hold at all points on the plane, and for all times, these exponential factors must be equal (when $z = 0$). Otherwise, a slight change in x, say, would destroy the equality. the time factors are already equal.

As for the spatial terms, evidently

$$\mathbf{k}_I \cdot \mathbf{r} = \mathbf{k}_R \cdot \mathbf{r} = \mathbf{k}_T \cdot \mathbf{r}, \quad \text{when } z = 0$$

or, more explicitly

$$x(k_I)_x + y(k_I)_y = x(k_R)_x + y(k_R)_y = x(k_T)_x + y(k_T)_y$$

for all x and all y.

for if $x = 0$, we get $\quad (k_I)_y = (k_R)_y = (k_T)_y \qquad$ (Eq. 9.96)

while $y = 0$ gives $\quad (k_I)_x = (k_R)_x = (k_T)_x \qquad$ (Eq. 9.97)

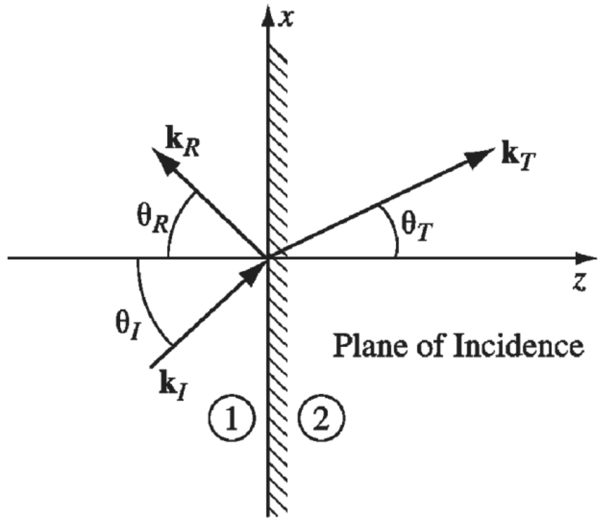

FIGURE 9.14

We may as well orient our axes so that $\boldsymbol{k_I}$ lies in the xz plane. According to Eq. 9.96, so too will $\boldsymbol{k_R}$ and $\boldsymbol{k_T}$

Conclusion:

First Law: The incident, reflected, and transmitted wave vectors form a plane (called the **plane of incidence),** which also includes the normal to the surface (here, the z axis).

Meanwhile, Eq. 9.97 implies that

$$k_I \sin\theta_I = k_R \sin\theta_R = k_T \sin\theta_T$$

where θ_R is the angle of incidence, θ_R is the angle of reflection, and θ_T is the angle of transmission (more commonly known as the angle of refraction), all of them measured with respect to the normal.

Second Law:
The angle of incidence is equal to the angle of reflection,

$$\theta_I = \theta_R.$$

This is the law of reflection.

Third Law:

$$\frac{\sin\theta_T}{\sin\theta_I} = \frac{n_1}{n_2}$$

This is the **law of refraction-Snell's law**

These are the three fundamental laws of geometrical optics. It is remarkable how little actual electrodynamics went into them: all we used was their generic form. Therefore, any *other* waves (water waves, for instance, or sound waves) can be expected to obey the same "optical" laws when they pass from one medium into another.

Now, the exponential factors cancel, the boundary conditions become

$$\left.\begin{aligned}
&\text{(i)}\quad \epsilon_1\left(\tilde{\mathbf{E}}_{0_I}+\tilde{\mathbf{E}}_{0_R}\right)_z=\epsilon_2\left(\tilde{\mathbf{E}}_{0_T}\right)_z\\
&\text{(ii)}\quad \left(\tilde{\mathbf{B}}_{0_I}+\tilde{\mathbf{B}}_{0_R}\right)_z=\left(\tilde{\mathbf{B}}_{0_T}\right)_z\\
&\text{(iii)}\quad \left(\tilde{\mathbf{E}}_{0_I}+\tilde{\mathbf{E}}_{0_R}\right)_{x,y}=\left(\tilde{\mathbf{E}}_{0_T}\right)_{x,y}\\
&\text{(iv)}\quad \frac{1}{\mu_1}\left(\tilde{\mathbf{B}}_{0_I}+\tilde{\mathbf{B}}_{0_R}\right)_{x,y}=\frac{1}{\mu_2}\left(\tilde{\mathbf{B}}_{0_T}\right)_{x,y}
\end{aligned}\right\}$$

where $\tilde{\mathbf{B}}_0=(1/v)\hat{\mathbf{k}}\times\tilde{\mathbf{E}}_0$ in each case

$$\left.\begin{array}{ll}\text{(i)} & \epsilon_1\left(\tilde{\mathbf{E}}_{0_I}+\tilde{\mathbf{E}}_{0_R}\right)_z=\epsilon_2\left(\tilde{\mathbf{E}}_{0_T}\right)_z \\ \text{(ii)} & \left(\tilde{\mathbf{B}}_{0_I}+\tilde{\mathbf{B}}_{0_R}\right)_z=\left(\tilde{\mathbf{B}}_{0_T}\right)_z \\ \text{(iii)} & \left(\tilde{\mathbf{E}}_{0_I}+\tilde{\mathbf{E}}_{0_R}\right)_{x,y}=\left(\tilde{\mathbf{E}}_{0_T}\right)_{x,y} \\ \text{(iv)} & \frac{1}{\mu_1}\left(\tilde{\mathbf{B}}_{0_I}+\tilde{\mathbf{B}}_{0_R}\right)_{x,y}=\frac{1}{\mu_2}\left(\tilde{\mathbf{B}}_{0_T}\right)_{x,y}\end{array}\right\}$$

Suppose the polarization of the incident wave is parallel to the plane of incidence (the xz plane); the reflected and transmitted waves are also polarized in this plane

Then (i) reads

$$\epsilon_1\left(-\tilde{E}_{0_I}\sin\theta_I+\tilde{E}_{0_R}\sin\theta_R\right)=\epsilon_2\left(-\tilde{E}_{0_T}\sin\theta_T\right)$$

(ii) adds nothing (0 = 0), since the magnetic fields have no z components; (iii) becomes

$$\tilde{E}_{0_I}\cos\theta_I+\tilde{E}_{0_R}\cos\theta_R=\tilde{E}_{0_T}\cos\theta_T$$

and (iv) says

$$\frac{1}{\mu_1 v_1}\left(\tilde{E}_{0_I}-\tilde{E}_{0_R}\right)=\frac{1}{\mu_2 v_2}\tilde{E}_{0_T}$$

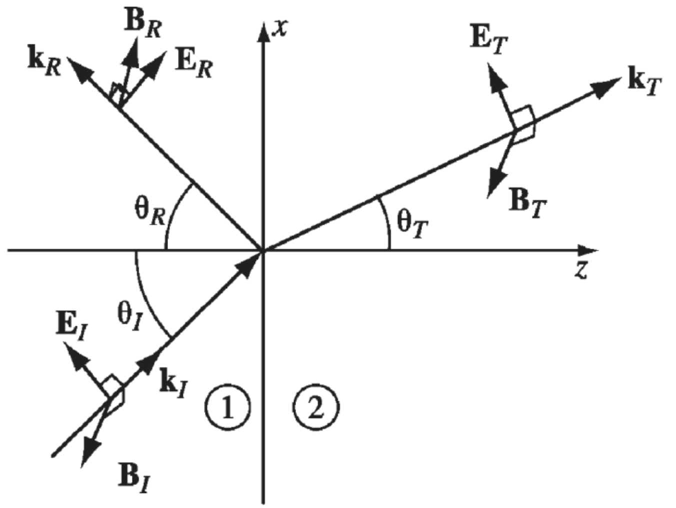

FIGURE 9.15

Given the laws of reflection and refraction, the first two equations both reduce to

$$\tilde{E}_{0_I} - \tilde{E}_{0_R} = \beta \tilde{E}_{0_T}$$

where (as before)

$$\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1}$$

and the third says

$$\tilde{E}_{0_I} + \tilde{E}_{0_R} = \alpha \tilde{E}_{0_T}$$

where

$$\alpha \equiv \frac{\cos\theta_T}{\cos\theta_I}$$

We obtain

$$\tilde{E}_{0_R} = \left(\frac{\alpha - \beta}{\alpha + \beta}\right)\tilde{E}_{0_I}, \quad \tilde{E}_{0_T} = \left(\frac{2}{\alpha + \beta}\right)\tilde{E}_{0_I}$$

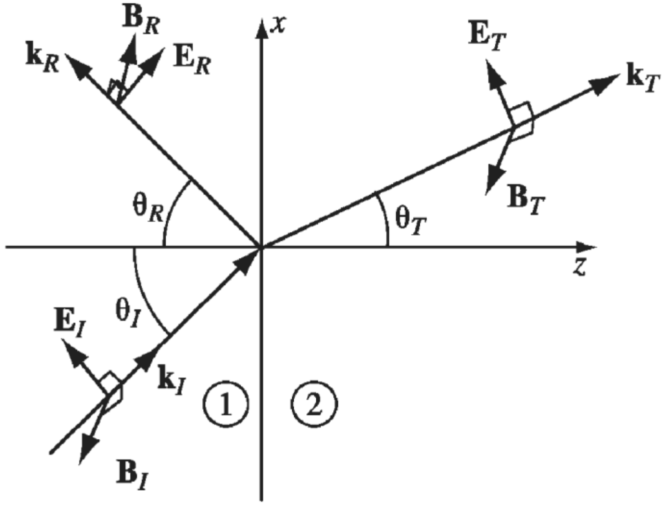

FIGURE 9.15

These are known as Fresnel's equations, and there are two other Fresnel equations, giving the reflected and transmitted amplitudes when the polarization is *perpendicular* to the plane of incidence.
See Prob. 9.16.

The amplitudes of the transmitted and reflected waves depend on the angle of incidence

$$\alpha = \frac{\sqrt{1-\sin^2\theta_T}}{\cos\theta_I} = \frac{\sqrt{1-[(n_1/n_2)\sin\theta_I]^2}}{\cos\theta_I}$$

In the case of normal incidence ($\theta_I = 0$), $\alpha = 1$, and we recover $Eq. 9.82$

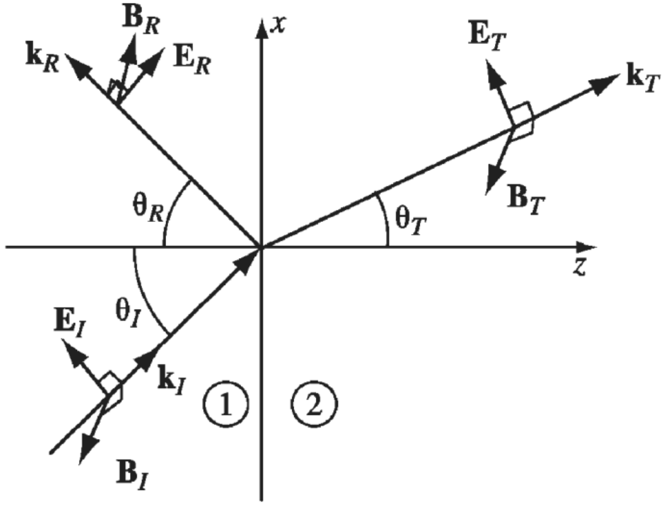

FIGURE 9.15

$$\tilde{E}_{0_R} = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)\tilde{E}_{0_I}, \quad \tilde{E}_{0_T} = \left(\frac{2}{\alpha+\beta}\right)\tilde{E}_{0_I}$$

Interestingly, there is an intermediate angle, θ_B (called Brewster's angle), at which the reflected wave is completely extinguished. According to $Eq.$ 9.109, this occurs when $\alpha = \beta$, or

$$\sin^2\theta_B = \frac{1-\beta^2}{(n_1/n_2)^2 - \beta^2}$$

For the typical case $\mu_1 \cong \mu_s$, so $\beta \cong {}^{n_2}/_{n_1}$, $sin^2\theta_B \cong \beta^2/(1+\beta^2)$ and hence

$$\tan\theta_B \cong \frac{n_2}{n_1}$$

$$\tilde{E}_{0_R} = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)\tilde{E}_{0_I}, \quad \tilde{E}_{0_T} = \left(\frac{2}{\alpha+\beta}\right)\tilde{E}_{0_I}$$

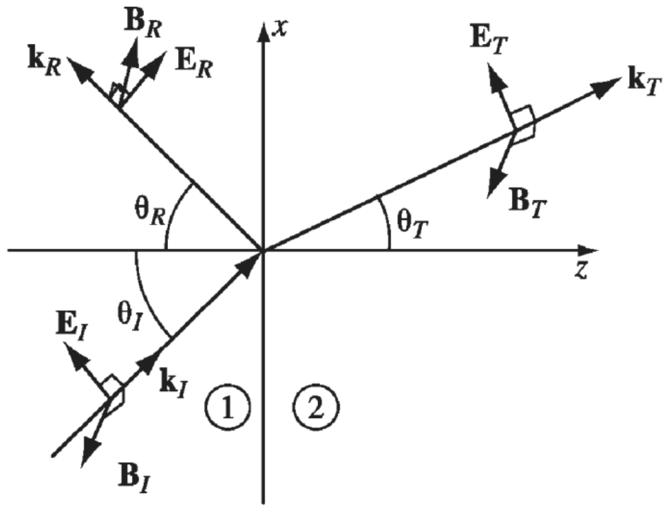

FIGURE 9.15

Figure 9.16 shows a plot of the transmitted and reflected amplitudes as functions of θ_I, for light incident on glass ($n_2 = 1.5$) from air ($n_1 = 1$). (On the graph, a negative number indicates that the wave is 180° out of phase with the incident beam-the amplitude itself is the absolute value.)

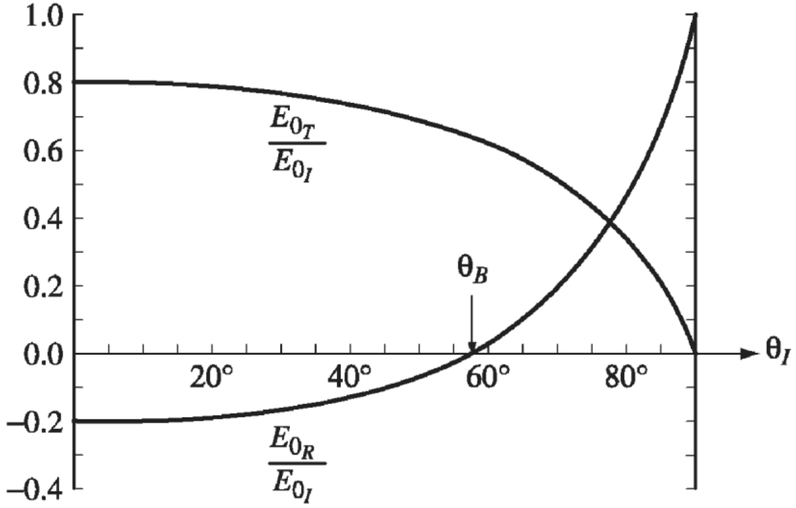

FIGURE 9.16

Thus the incident intensity is

$$I_I = \frac{1}{2}\epsilon_1 v_1 E_{0_I}^2 \cos\theta_I$$

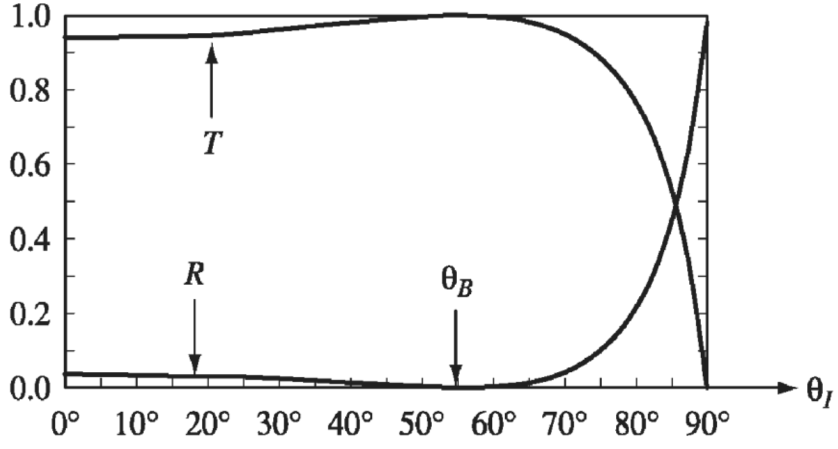

FIGURE 9.17

while the reflected and transmitted intensities are

$$I_R = \frac{1}{2}\epsilon_1 v_1 E_{0_R}^2 \cos\theta_R, \quad \text{and} \quad I_T = \frac{1}{2}\epsilon_2 v_2 E_{0_T}^2 \cos\theta_T$$

$$R \equiv \frac{I_R}{I_I} = \left(\frac{E_{0_R}}{E_{0_I}}\right)^2 = \left(\frac{\alpha - \beta}{\alpha + \beta}\right)^2 \qquad T \equiv \frac{I_T}{I_I} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\left(\frac{E_{0_T}}{E_{0_I}}\right)^2 \frac{\cos\theta_T}{\cos\theta_I} = \alpha\beta\left(\frac{2}{\alpha + \beta}\right)^2$$

Electromagnetic Waves in Conductors

$$\mathbf{J}_f = \sigma \boldsymbol{E}$$

Maxwell's equations for linear media assume the form

$$\left.\begin{array}{ll} \text{(i)}\ \nabla \cdot \mathbf{E} = \frac{1}{\epsilon}\rho_f, & \text{(iii)}\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \\ \text{(ii)}\ \nabla \cdot \mathbf{B} = 0, & \text{(iv)}\ \nabla \times \mathbf{B} = \mu\sigma\mathbf{E} + \mu\epsilon\frac{\partial \mathbf{E}}{\partial t}. \end{array}\right\}$$

Now, the continuity equation for free charge,

$$\nabla \cdot \mathbf{J}_f = -\frac{\partial \rho_f}{\partial t}$$

together with Ohm's law and Gauss's law (i), gives

$$\frac{\partial \rho_f}{\partial t} = -\sigma(\nabla \cdot \mathbf{E}) = -\frac{\sigma}{\epsilon}\rho_f$$

for a homogeneous linear medium, from which it follows that

$$\rho_f(t) = e^{-(\sigma/\epsilon)t}\rho_f(0)$$

Thus any initial free charge $\rho_f(0)$ dissipates in a characteristic time $\tau \equiv {}^{\epsilon}/_{\sigma}$. This reflects the familiar fact that if you put some free charge on a conductor, it will flow out to the edges.

From then on $\rho_f = 0$, and we have

$$\left.\begin{array}{ll} \text{(i)} \quad \nabla \cdot \mathbf{E} = 0, & \text{(iii)} \quad \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}, \\ \text{(ii)} \quad \nabla \cdot \mathbf{B} = 0, & \text{(iv)} \quad \nabla \times \mathbf{B} = \mu\epsilon\dfrac{\partial \mathbf{E}}{\partial t} + \mu\sigma\mathbf{E}. \end{array}\right\} \qquad \text{(Eq. 9.121)}$$

These differ from the corresponding equations for nonconducting media $(Eq. 9.67)$ only in the last term in (iv).

Applying the curl to (iii) and (iv), as before, we obtain modified wave equations for E and B:

$$\nabla^2\mathbf{E} = \mu\epsilon\frac{\partial^2\mathbf{E}}{\partial t^2} + \mu\sigma\frac{\partial\mathbf{E}}{\partial t}, \quad \nabla^2\mathbf{B} = \mu\epsilon\frac{\partial^2\mathbf{B}}{\partial t^2} + \mu\sigma\frac{\partial\mathbf{B}}{\partial t}. \qquad (\text{Eq. } 9.122)$$

These equations still admit plane-wave solutions,

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{i(\tilde{k}z-\omega t)}, \quad \tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 e^{i(\tilde{k}z-\omega t)} \qquad (\text{Eq. } 9.123)$$

but this time the "wave number" $\tilde{k}$ is complex:

$$\tilde{k}^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega$$

as you can easily check by plugging $Eq. 9.123$ into $Eq. 9.122$, which gives

$$\tilde{k} = k + i\kappa$$

where $$k \equiv \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}+1\right]^{1/2}, \quad \kappa \equiv \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}-1\right]^{1/2}$$

The imaginary part of $\tilde{k}$ results in an attenuation of the wave (decreasing amplitude with increasing z):

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{-\kappa z} e^{i(kz-\omega t)}, \quad \tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 e^{-\kappa z} e^{i(kz-\omega t)}$$

The distance it takes to reduce the amplitude by a factor of 1/e (about a third) is called the skin depth :

$$d \equiv \frac{1}{\kappa}$$

it is a measure of how far the wave penetrates into the conductor. the real part of $\tilde{k}$ in the usual way:

$$\lambda = \frac{2\pi}{k}, \quad v = \frac{\omega}{k}, \quad n = \frac{ck}{\omega}$$

Maxwell's equations (9.121) impose further constraints, (i) and (ii) rule out any z components: the fields are *transverse*. We may as well orient our axes so that E is polarized along the x direction:

$$\tilde{\mathbf{E}}(z, t) = \tilde{E}_0 e^{-\kappa z} e^{i(kz-\omega t)}\,\hat{\mathbf{x}}$$

Then (iii) gives

$$\tilde{\mathbf{B}}(z, t) = \frac{\tilde{k}}{\omega} \tilde{E}_0 e^{-\kappa z} e^{i(kz-\omega t)}\,\hat{\mathbf{y}}$$

Reflection at a Conducting Surface

The boundary conditions we used to analyze reflection and refraction at an interface between two dielectrics do not hold in the presence of free charges and currents. Instead, we have the more general relations:

$$\left.\begin{array}{llll} \text{(i)} & \epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f, & \text{(iii)} & \mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = \mathbf{0}, \\ \text{(ii)} & B_1^{\perp} - B_2^{\perp} = 0, & \text{(iv)} & \dfrac{1}{\mu_1}\mathbf{B}_1^{\parallel} - \dfrac{1}{\mu_2}\mathbf{B}_2^{\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}}, \end{array}\right\}$$

Where σ_f is the free surface charge, K_f is the free surface current, and $\hat{n}$ is a unit vector perpendicular to the surface, pointing from medium (2) into medium (1).

Suppose now that the xy plane forms the boundary between a nonconducting linear medium (1) and a conductor (2). A monochromatic plane wave, traveling in the z direction and polarized in the x direction, approaches from the left, as in $Fig. 9.13$:

$$\tilde{\beta} \equiv \frac{\mu_1 v_1}{\mu_2 \omega} \tilde{k}_2$$

It follows that $$\tilde{E}_{0_R} = \left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right) \tilde{E}_{0_I}, \quad \tilde{E}_{0_T} = \left(\frac{2}{1+\tilde{\beta}}\right) \tilde{E}_{0_I}$$

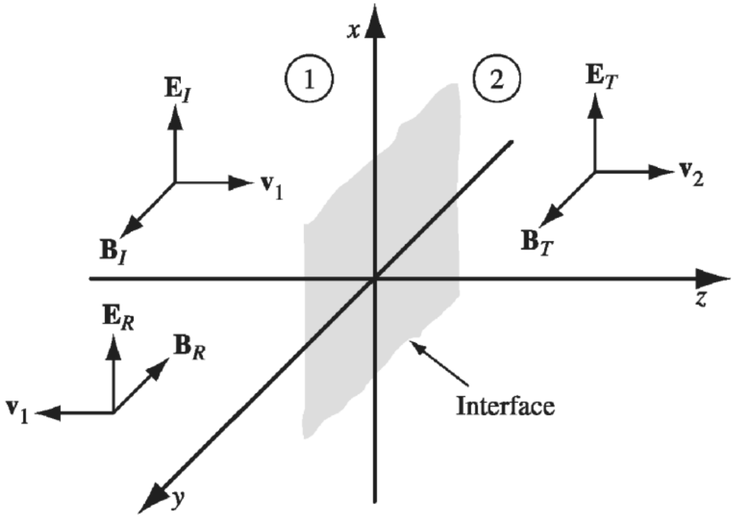

FIGURE 9.13

These results are formally identical to the ones that apply at the boundary between nonconductors $(Eq. 9.82)$, but the resemblance is deceptive since $\tilde{\beta}$ is now a complex number.

For a perfect conductor $(\sigma = \infty)$, $k_2 = \infty$ so $\tilde{\beta}$ is infinite, and

$$\tilde{E}_{0_R} = -\tilde{E}_{0_I}, \quad \tilde{E}_{0_T} = 0$$

Note:
In this case the wave is totally reflected, with a 180° phase shift. (That's why excellent conductors make good mirrors)

The Frequency Dependence of Permittivity

In the preceding sections, we have seen that the propagation of electromagnetic waves through matter is governed by three properties of the material: the permittivity ϵ, the permeability μ, and the conductivity σ. Actually, each of these parameters depends to some extent on the frequency of the waves you are considering.

Indeed, it is well known from optics that $n \cong \sqrt{\epsilon_r}$ is a function of wavelength ($Fig.$ 9.19 shows the graph for a typical glass). A prism or a raindrop bends blue light more sharply than red, and spreads white light out into a rainbow of colors. This phenomenon is called dispersion. By extension, whenever the speed of a wave depends on its frequency, the supporting medium is called dispersive.

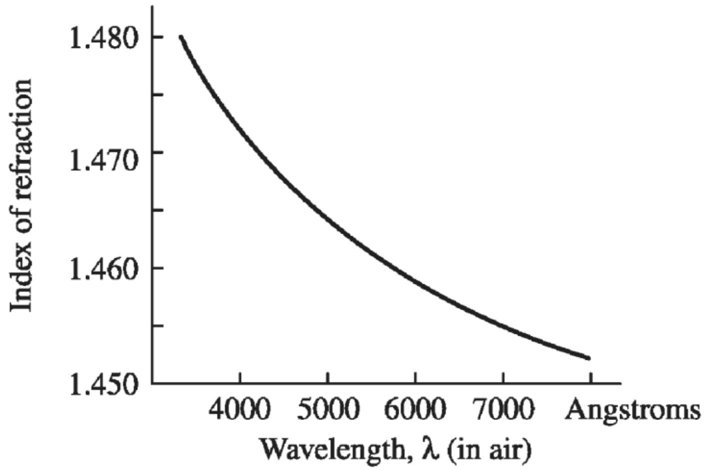

FIGURE 9.19

Purpose in this section is to account for the frequency dependence of ϵ in dielectrics, using a simplified model for the behavior of the electrons

Because waves of different frequency travel at different speeds in a dispersive medium, a wave form that incorporates a range of frequencies will change shape as it propagates. A sharply peaked wave typically flattens out, and whereas each sinusoidal component travels at the ordinary **wave** (or **phase**) **velocity** ,

$$v = \frac{\omega}{k}$$

the packet as a whole (the "envelope") moves at the so-called **group velocity**

$$v_g = \frac{d\omega}{dk}$$

the energy carried by a wave packet in a dispersive medium does not travel at the phase velocity

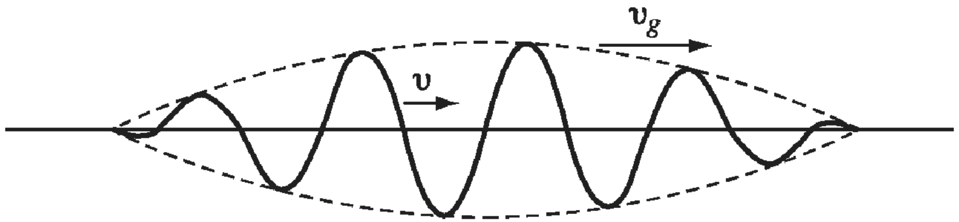

FIGURE 9.20

The electrons in a nonconductor are bound to specific molecules. We picture each electron as attached to the end of a spring, with force constant k_{sping}

$$F_{\text{binding}} = -k_{\text{spring}}x = -m\omega_0^2 x$$

where x is displacement from equilibrium, m is the electron's mass, and

ω_0 is the natural oscillation frequency $\sqrt{k_{spring}/m}$

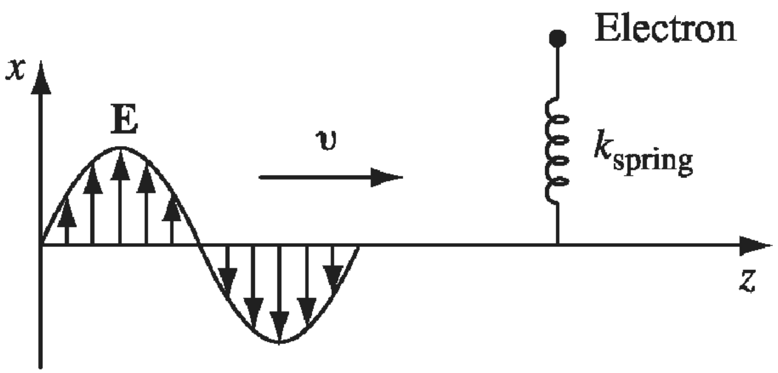

FIGURE 9.21

Meanwhile, there will presumably be some damping force on the electron, the simplest possible form

$$F_{\text{damping}} = -m\gamma \frac{dx}{dt}$$

The damping must be opposite in direction to the velocity, and making it proportional to the velocity is the easiest way to accomplish this. The cause of the damping does not concern us here among other things, an oscillating charge radiates, and the radiation siphons off Energy.

In the presence of an electromagnetic wave of frequency w, polarized in the x direction (Fig. 9.21), the electron is subject to a driving force

$$F_{\text{driving}} = qE = qE_0 \cos(\omega t)$$

Newton's second law gives

$$m\frac{d^2x}{dt^2} = F_{\text{tot}} = F_{\text{binding}} + F_{\text{damping}} + F_{\text{driving}}$$

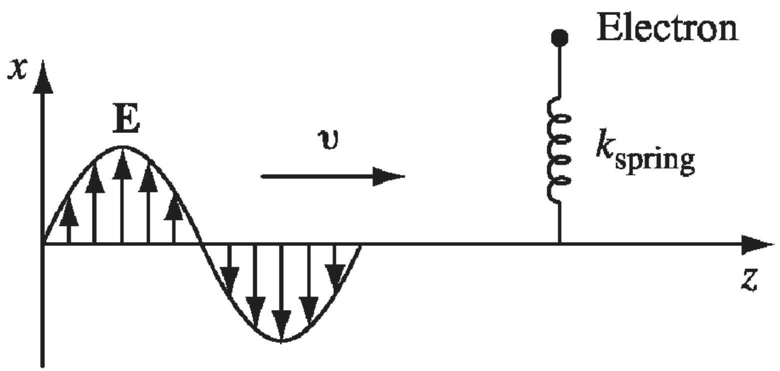

FIGURE 9.21

or

$$m\frac{d^2x}{dt^2} + m\gamma\frac{dx}{dt} + m\omega_0^2 x = qE_0 \cos(\omega t)$$

Equation 9.154 is easier to handle if we regard it as the real part of a complex equation:

$$\frac{d^2\tilde{x}}{dt^2} + \gamma\frac{d\tilde{x}}{dt} + \omega_0^2\tilde{x} = \frac{q}{m}E_0 e^{-i\omega t}$$

In the steady state, the system oscillates at the driving frequency

$$\tilde{x}(t) = \tilde{x}_0 e^{-i\omega t}$$

Inserting this into Eq. 9.155, we obtain

$$\tilde{x}_0 = \frac{q/m}{\omega_0^2 - \omega^2 - i\gamma\omega} E_0$$

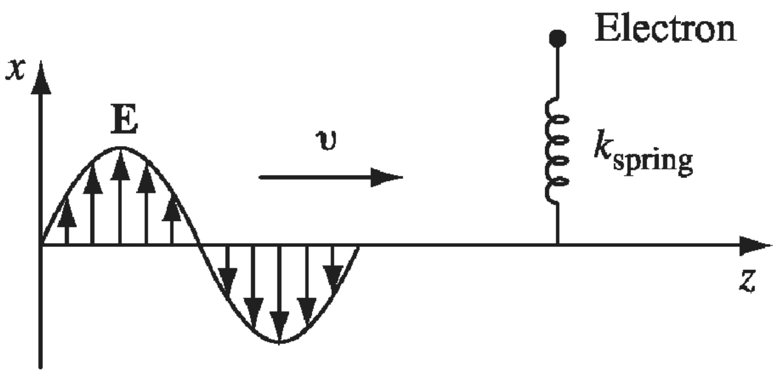

FIGURE 9.21

The resulting dipole moment is the real part of

$$\tilde{p}(t) = q\tilde{x}(t) = \frac{q^2/m}{\omega_0^2 - \omega^2 - i\gamma\omega} E_0 e^{-i\omega t}$$

The imaginary term in the denominator means that p is *out of phase* with E—lagging behind by an angle $\tan^{-1}[\gamma\omega/(\omega_0^2 - \omega^2)]$ that is very small when $\omega \ll \omega_0$ and rises to π when $\omega \gg \omega_0$.

Let's say there are f_i electrons with frequency ω_j and damping γ_j in each molecule. If there are N molecules per unit volume, the polarization **P** is given by the real part of

$$\tilde{\mathbf{P}} = \frac{Nq^2}{m}\left(\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right)\tilde{\mathbf{E}}$$

In the present case, **P** is not proportional to E (this is not, strictly speaking, a linear medium) because of the difference in phase. However, the complex polarization $\tilde{\boldsymbol{P}}$ is proportional to the complex field E, and this suggests that we introduce a complex susceptibility $\widetilde{\chi_e}$:

$$\tilde{\mathbf{P}} = \epsilon_0 \tilde{\chi}_e \tilde{\mathbf{E}}$$

In particular, the proportionality between $\tilde{\boldsymbol{D}}$ and $\tilde{\boldsymbol{E}}$ is the complex permittivity

$$\tilde{\epsilon} = \epsilon_0(1 + \tilde{\chi}_e)$$

and the complex dielectric constant (in this model) is

$$\tilde{\epsilon}_r = \frac{\tilde{\epsilon}}{\epsilon_0} = 1 + \frac{Nq^2}{m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}$$

Guided Waves

Wave Guides

So far, we have dealt with plane waves of infinite extent; now we consider electromagnetic waves confined to the interior of a hollow pipe, or wave guide(Fig. 9.23). We'll assume the wave guide is a perfect conductor, so that $\boldsymbol{E} = 0$ and $\boldsymbol{B} = 0$ inside the material itself, and hence the boundary conditions at the inner wall are :

(Eq. 9.175)

$$\left.\begin{array}{ll} \text{(i)} & \mathbf{E}^{\parallel} = 0, \\ \text{(ii)} & B^{\perp} = 0. \end{array}\right\}$$

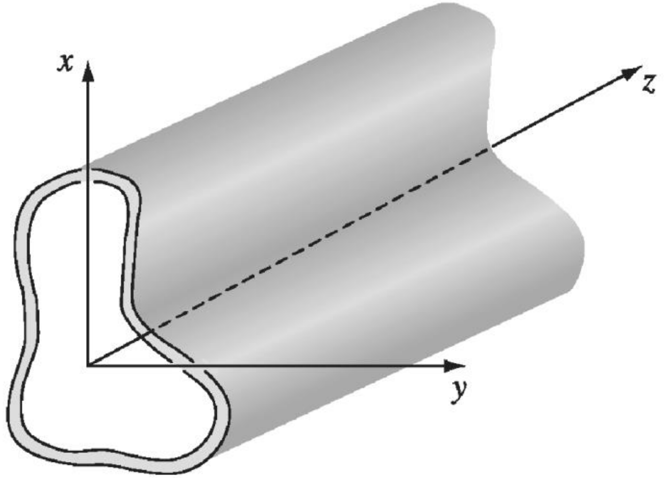

FIGURE 9.23

Free charges and currents will be induced on the surface.
We are interested in monochromatic waves that propagate down the tube, so $\boldsymbol{E}$ and $\boldsymbol{B}$ have the generic form

$$\left.\begin{array}{ll} \text{(i)} & \tilde{\mathbf{E}}(x, y, z, t) = \tilde{\mathbf{E}}_0(x, y)e^{i(kz-\omega t)}, \\ \text{(ii)} & \tilde{\mathbf{B}}(x, y, z, t) = \tilde{\mathbf{B}}_0(x, y)e^{i(kz-\omega t)}. \end{array}\right\}$$

The electric and magnetic fields must, of course, satisfy Maxwell's equations, in the interior of the wave guide:

$$\left.\begin{aligned} &\text{(i)}\quad \nabla \cdot \mathbf{E} = 0, &\qquad &\text{(iii)}\quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \\ &\text{(ii)}\quad \nabla \cdot \mathbf{B} = 0, &\qquad &\text{(iv)}\quad \nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}. \end{aligned}\right\} \qquad \text{(Eq. 9.177)}$$

in order to fit the boundary conditions we shall have to include longitudinal components (E_z *and* B_z):

$$\tilde{\mathbf{E}}_0 = E_x\,\hat{\mathbf{x}} + E_y\,\hat{\mathbf{y}} + E_z\,\hat{\mathbf{z}}, \qquad \tilde{\mathbf{B}}_0 = B_x\,\hat{\mathbf{x}} + B_y\,\hat{\mathbf{y}} + B_z\,\hat{\mathbf{z}}, \qquad \text{(Eq. 9.178)}$$

where each of the components is a function of x and y. Putting this into Maxwell's equations (iii) and (iv), we obtain

$$\left.\begin{array}{llll}
\text{(i)} & \dfrac{\partial E_y}{\partial x}-\dfrac{\partial E_x}{\partial y}=i\omega B_z, & \text{(iv)} & \dfrac{\partial B_y}{\partial x}-\dfrac{\partial B_x}{\partial y}=-\dfrac{i\omega}{c^2}E_z, \\
\text{(ii)} & \dfrac{\partial E_z}{\partial y}-ikE_y=i\omega B_x, & \text{(v)} & \dfrac{\partial B_z}{\partial y}-ikB_y=-\dfrac{i\omega}{c^2}E_x, \\
\text{(iii)} & ikE_x-\dfrac{\partial E_z}{\partial x}=i\omega B_y, & \text{(vi)} & ikB_x-\dfrac{\partial B_z}{\partial x}=-\dfrac{i\omega}{c^2}E_y.
\end{array}\right\} \qquad \text{(Eq. 9.179)}$$

Equations $(ii), (iii), (v)$, and (vi) can be solved for E_x , E_y , B_x and B_y:

$$\left.\begin{aligned}
&\text{(i)} \quad E_x = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial E_z}{\partial x} + \omega\frac{\partial B_z}{\partial y}\right), \\
&\text{(ii)} \quad E_y = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial E_z}{\partial y} - \omega\frac{\partial B_z}{\partial x}\right), \\
&\text{(iii)} \quad B_x = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial B_z}{\partial x} - \frac{\omega}{c^2}\frac{\partial E_z}{\partial y}\right), \\
&\text{(iv)} \quad B_y = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial B_z}{\partial y} + \frac{\omega}{c^2}\frac{\partial E_z}{\partial x}\right).
\end{aligned}\right\} \qquad \text{(Eq. 9.180)}$$

Inserting $Eq. 9.180$ into the remaining Maxwell equations (Prob. 9.27b) yields uncoupled equations for E_z and B_z:

$$\left.\begin{aligned} &\text{(i)} \quad \left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+(\omega/c)^2-k^2\right]E_z=0, \\ &\text{(ii)} \quad \left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+(\omega/c)^2-k^2\right]B_z=0. \end{aligned}\right\}$$

If $Ez = 0$, we call these **TE** ("transverse electric") **waves**; if $Bz = 0$, they are called TM ("transverse magnetic") waves; if both $Ez = 0$ and $Bz = 0$, we call them TEM waves. It turns out that TEM waves cannot occur in a hollow wave guide.

Proof If $Ez = 0$, Gauss's law ($Eq. 9.177i$) says

$$\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}=0$$

and if $Bz = 0$, Faraday's law ($Eq. 9.177iii$) says

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0$$

Indeed, the vector $\widetilde{E_0}$ in $Eq. 9.178$ has zero divergence and zero curl. It can therefore be written as the gradient of a scalar potential that satisfies Laplace's equation. But the boundary condition on $\boldsymbol{E}$ ($Eq. 9.175$) requires that the surface be an equipotential, and since Laplace's equation admits no local maxima or minima ($Sect. 3.1.4$), this means that the potential is constant throughout, and hence the electric field is zero - no wave at all.

$$\left.\begin{array}{l}\text{(i)}\;E_x=\frac{i}{(\omega/c)^2-k^2}\left(k\frac{\partial E_z}{\partial x}+\omega\frac{\partial B_z}{\partial y}\right),\\ \text{(ii)}\;E_y=\frac{i}{(\omega/c)^2-k^2}\left(k\frac{\partial E_z}{\partial y}-\omega\frac{\partial B_z}{\partial x}\right),\\ \text{(iii)}\;B_x=\frac{i}{(\omega/c)^2-k^2}\left(k\frac{\partial B_z}{\partial x}-\frac{\omega}{c^2}\frac{\partial E_z}{\partial y}\right),\\ \text{(iv)}\;B_y=\frac{i}{(\omega/c)^2-k^2}\left(k\frac{\partial B_z}{\partial y}+\frac{\omega}{c^2}\frac{\partial E_z}{\partial x}\right).\end{array}\right\}$$

$$\left.\begin{array}{ll}\text{(i)} & \mathbf{E}^{\parallel}=0,\\ \text{(ii)} & B^{\perp}=0.\end{array}\right\}$$

TE Waves in a Rectangular Wave Guide

Suppose we have a wave guide of rectangular shape ($Fig.$ 9.24), with height a and width b, and we are interested in the propagation of TE waves. The problem is to solve $Eq.$ 9.181ii, subject to the boundary condition 9.175ii. We'll do it by separation of variables. Let

$$B_z(x,y)=X(x)Y(y)$$

$$Y\frac{d^2X}{dx^2}+X\frac{d^2Y}{dy^2}+\left[(\omega/c)^2-k^2\right]XY=0$$

Divide by XY,

$$\text{(i)}\;\frac{1}{X}\frac{d^2X}{dx^2}=-k_x^2,\quad\text{(ii)}\;\frac{1}{Y}\frac{d^2Y}{dy^2}=-k_y^2$$

$$-k_x^2-k_y^2+(\omega/c)^2-k^2=0$$

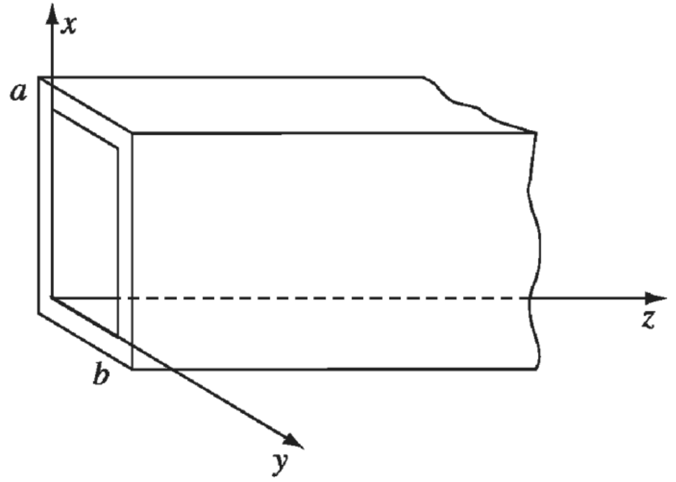

FIGURE 9.24

The general solution to $Eq. 9.182i$ is

$$X(x) = A\sin(k_x x) + B\cos(k_x x)$$

the boundary conditions require that Bx, and hence also $(Eq. 9.180iii)$ ${}^{dX}/_{dx}$, vanishes at $x = 0$ and $x = a$. So $A = 0$, and

$$k_x = m\pi/a, \quad (m = 0, 1, 2, \ldots) \qquad \text{(Eq. 9.184)}$$

The same goes for Y, with

$$k_y = n\pi/b, \quad (n = 0, 1, 2, \ldots) \qquad \text{(Eq. 9.185)}$$

and we conclude that

$$B_z = B_0 \cos(m\pi x/a)\cos(n\pi y/b)$$

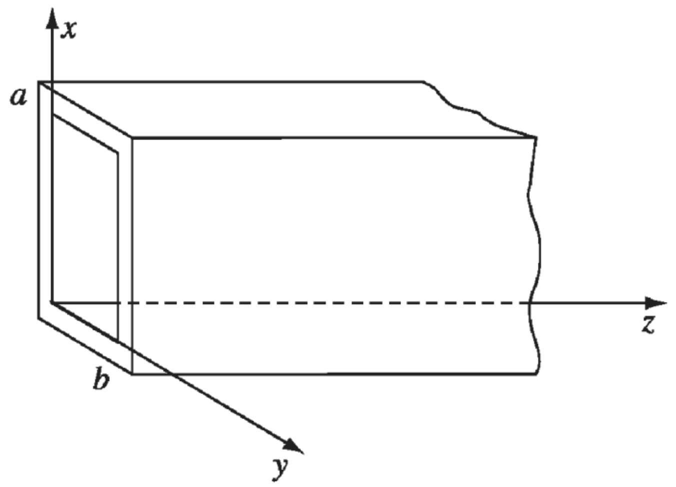

FIGURE 9.24

This solution is called the ***TEmn*** mode.

The wave number (k) is obtained by putting $Eqs.$ 9.184 and 9.185 into $Eq.$ 9.183

$$k = \sqrt{(\omega/c)^2 - \pi^2[(m/a)^2 + (n/b)^2]}$$

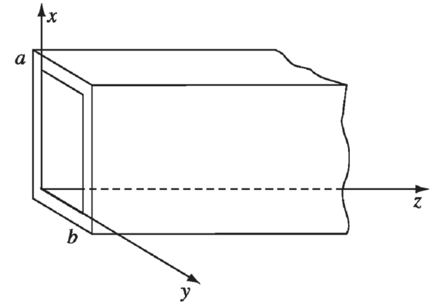

FIGURE 9.24

If $$\omega < c\pi\sqrt{(m/a)^2 + (n/b)^2} \equiv \omega_{mn}$$

the wave number is imaginary, and instead of a traveling wave we have exponentially attenuated fields. For this reason, ω_{mn} is called the **cutoff** frequency for the mode in question. The lowest cutoff frequency for a given wave guide occurs for the mode $\boldsymbol{TE_{10}}$:

$$\omega_{10} = c\pi/a$$

Frequencies less than this will not propagate at all

The wave number can be written more simply in terms of the cutoff frequency:

$$k = \frac{1}{c}\sqrt{\omega^2 - \omega_{mn}^2}$$

The wave velocity is

$$v = \frac{\omega}{k} = \frac{c}{\sqrt{1 - (\omega_{mn}/\omega)^2}}$$

which is greater than c. However , the energy carried by the wave travels at the group velocity

which is greater than c. However , the energy carried by the wave travels at the group velocity

$$v_g = \frac{1}{dk/d\omega} = c\sqrt{1 - (\omega_{mn}/\omega)^2} < c$$

To visualize the propagation, consider an ordinary *plane* wave, traveling at an angle θ to the z axis, and reflecting perfectly off each conducting surface. The propagation vector for the "original" plane wave is therefore

$$\mathbf{k}' = \frac{\pi m}{a}\,\hat{\mathbf{x}} + \frac{\pi n}{b}\,\hat{\mathbf{y}} + k\,\hat{\mathbf{z}}$$

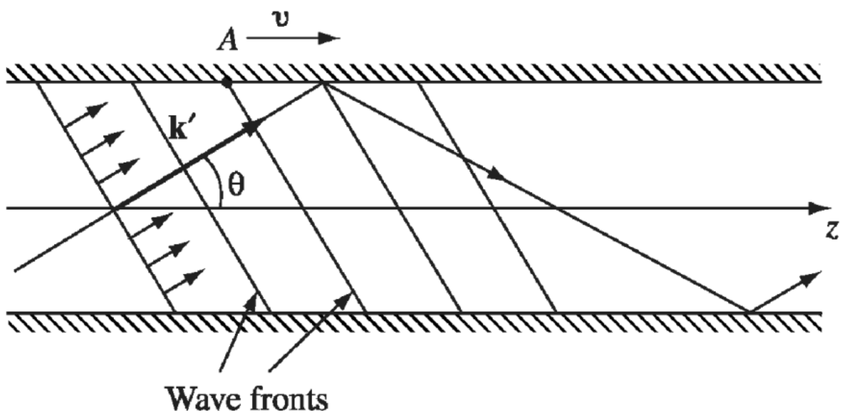

FIGURE 9.25

and the frequency is

$$\omega = c|\mathbf{k}'| = c\sqrt{k^2 + \pi^2[(m/a)^2 + (n/b)^2]} = \sqrt{(ck)^2 + (\omega_{mn})^2}$$

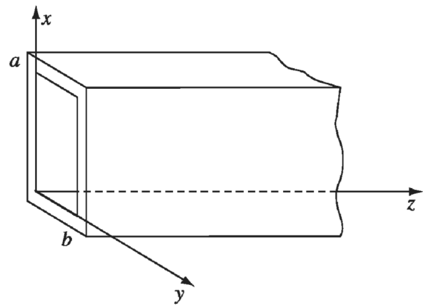

FIGURE 9.24

Only certain angles will lead to one of the allowed standing wave patterns

$$\cos\theta = \frac{k}{|\mathbf{k}'|} = \sqrt{1 - (\omega_{mn}/\omega)^2}$$

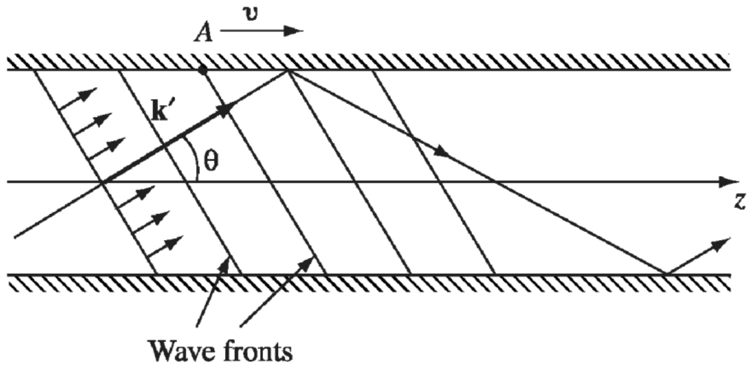

FIGURE 9.25

its net velocity down the wave guide is

$$v_g = c\cos\theta = c\sqrt{1 - (\omega_{mn}/\omega)^2}.$$

The wave velocity, on the other hand, is the speed of the wave fronts (A, say, in $Fig. 9.25$) down the pipe

$$v = \frac{c}{\cos\theta} = \frac{c}{\sqrt{1 - (\omega_{mn}/\omega)^2}}$$

The Coaxial Transmission Line

A hollow wave guide cannot support TEM waves. But a coaxial transmission line, consisting of a long straight wire of radius a, surrounded by a cylindrical conducting sheath of radius b (Fig. 9.26), does admit modes with $Ez = 0$ and $Bz = 0$. In this case Maxwell's equations (Eq. 9.179) yield

$$k = \omega/c$$

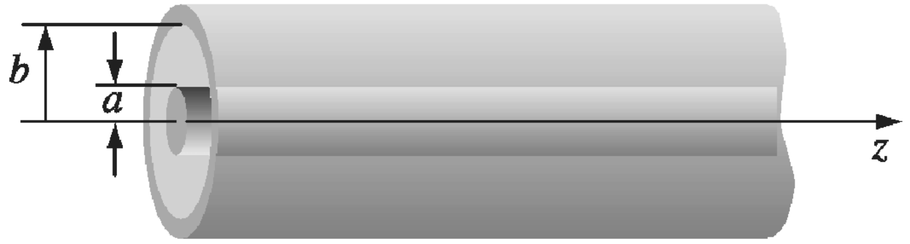

FIGURE 9.26

so the waves travel at speed c, and are nondispersive

$$cB_y = E_x \quad \text{and} \quad cB_x = -E_y$$

(so E and B are mutually perpendicular), and (together with $\nabla \cdot E = 0, \nabla \cdot B = 0$)

$$\left.\begin{array}{ll} \dfrac{\partial E_x}{\partial x} + \dfrac{\partial E_y}{\partial y} = 0, & \dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y} = 0, \\ \dfrac{\partial B_x}{\partial x} + \dfrac{\partial B_y}{\partial y} = 0, & \dfrac{\partial B_y}{\partial x} - \dfrac{\partial B_x}{\partial y} = 0. \end{array}\right\}$$

These are precisely the equations of electrostatics and magnetostatics, for empty space, in two dimensions; the solution with cylindrical symmetry can be borrowed directly from the case of an infinite line charge and an infinite straight current, respectively:

$$\mathbf{E}_0(s,\phi) = \frac{A}{s}\,\hat{\mathbf{s}}, \qquad \mathbf{B}_0(s,\phi) = \frac{A}{cs}\,\hat{\boldsymbol{\phi}}$$

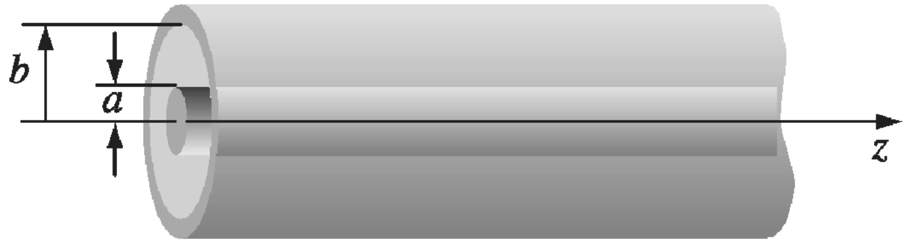

FIGURE 9.26

$$\left.\begin{aligned} \mathbf{E}(s,\phi,z,t) &= \frac{A\cos(kz-\omega t)}{s}\,\hat{\mathbf{s}}, \\ \mathbf{B}(s,\phi,z,t) &= \frac{A\cos(kz-\omega t)}{cs}\,\hat{\boldsymbol{\phi}}. \end{aligned}\right\}$$