Chapter 8

Conservation Laws

- 8.1 Charge and energy
- 8.2 Momentum

Charge and Energy

The Continuity Equation

In this chapter we study conservation of energy, momentum, and angular momentum, in electrodynamics.
The charge in a volume $\mathcal{V}$ is

$$Q(t) = \int_{\mathcal{V}} \rho(\mathbf{r}, t)\, d\tau$$

The current flowing out through the boundary s is $\oint_s J \cdot da$ so local conservation of charge says

$$\frac{dQ}{dt} = -\oint_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{a}.$$

$$\int_{\mathcal{V}} \frac{\partial \rho}{\partial t}\, d\tau = -\int_{\mathcal{V}} \nabla \cdot \mathbf{J}\, d\tau$$

since this is true for any volume, it follows that

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}$$

Note:
It can be derived from Maxwell's equations. Conservation of charge is not an independent assumption; it is built into the laws

Poynting's Theorem

In Chapter 2, we found that the work necessary to assemble a static charge distribution (against the Coulomb repulsion of like charges) is

$$W_e = \frac{\epsilon_0}{2} \int E^2 \, d\tau. \qquad (8.6)$$

The work required to get currents going (against the back emf) is

$$W_m = \frac{1}{2\mu_0} \int B^2 \, d\tau.$$

This suggests that the total energy stored in electromagnetic fields, per unit volume, is

$$u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$$

Suppose we have some charge and current configuration which, at time t, produces fields **E** and **B**

According to the Lorentz force law, the work done on a charge q is

$$\mathbf{F} \cdot d\mathbf{l} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \mathbf{v}\, dt = q\mathbf{E} \cdot \mathbf{v}\, dt$$

$$q \rightarrow \rho\, d\tau \qquad \rho\mathbf{v} \rightarrow \mathbf{J}$$

so the rate at which work is done on all the charges in a volume $\mathcal{V}$ is

$$\frac{dW}{dt} = \int_{\mathcal{V}} (\mathbf{E} \cdot \mathbf{J})\, d\tau$$

$\boldsymbol{E} \cdot \boldsymbol{J}$ is the power delivered per unit volume

We can express this quantity in terms of the fields alone, using the Ampere-Maxwell law to eliminate J

$$\mathbf{E} \cdot \mathbf{J} = \frac{1}{\mu_0}\mathbf{E} \cdot (\nabla \times \mathbf{B}) - \epsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t}$$

From product rule 6, $\nabla \cdot (\mathbf{E} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{B})$

Invoking Faraday's law $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$

$$\mathbf{E} \cdot (\nabla \times \mathbf{B}) = -\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} - \nabla \cdot (\mathbf{E} \times \mathbf{B})$$

Meanwhile

$$\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(B^2) \qquad \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(E^2)$$

$$\mathbf{E} \cdot \mathbf{J} = -\frac{1}{2}\frac{\partial}{\partial t}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \frac{1}{\mu_0}\nabla \cdot (\mathbf{E} \times \mathbf{B})$$

Applying the divergence theorem to the second term

$$\frac{dW}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) d\tau - \frac{1}{\mu_0}\oint_{\mathcal{S}} (\mathbf{E} \times \mathbf{B}) \cdot d\mathbf{a}$$

This is Poynting's theorem; it is the "work energy theorem" of electrodynamics. The first integral on the right is the total energy stored in the fields $\boldsymbol{U_{em}}$. The second term evidently represents the rate at which energy is transported out of $\mathcal{V}$, across its boundary surface, by the electromagnetic fields. Poynting's theorem says the work done on the charges by the electromagnetic force is equal to the decrease in energy remaining in the fields, less the energy that flowed out through the surface.

The energy per unit time, per unit area, transported by the fields is called the **Poynting vector:**

$$\mathbf{S} \equiv \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$$

$\boldsymbol{S} \cdot d\boldsymbol{a}$ is the energy per unit time crossing the infinitesimal surface $d\boldsymbol{a}$ is the energy flux, $\boldsymbol{S}$ is the energy flux density.

Poynting's theorem more compactly

$$\frac{dW}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} u\, d\tau - \oint_{\mathcal{S}} \mathbf{S} \cdot d\mathbf{a}$$

In the case that no work is done on the charges in *V:*

$$dW/dt = 0$$

$$\int \frac{\partial u}{\partial t}\, d\tau = -\oint \mathbf{S} \cdot d\mathbf{a} = -\int (\nabla \cdot \mathbf{S})\, d\tau,$$

$$\frac{\partial u}{\partial t} = -\nabla \cdot \mathbf{S}$$

Note:
This is the "continuity equation" for energy: u (energy density) plays the role of ρ (charge density), and ***S*** takes the part of ***J*** (current density).

Example

When current flows down a wire, work is done, which shows up as Joule heating of the wire (Eq. 7.7) and can be calculated using the Poynting vector. Assuming it's uniform:

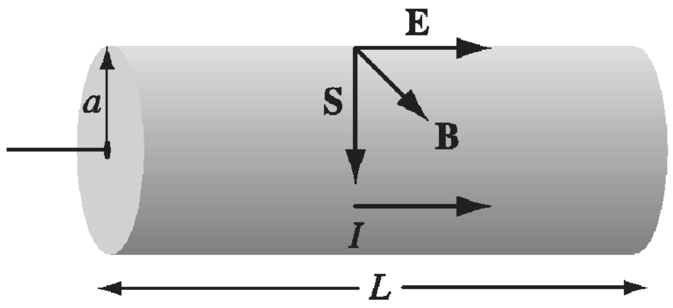


FIGURE 8.1

$$E = \frac{V}{L}$$

$$B = \frac{\mu_0 I}{2\pi a}$$

Accordingly, the magnitude of the Poynting vector is

$$S = \frac{1}{\mu_0}\frac{V}{L}\frac{\mu_0 I}{2\pi a} = \frac{VI}{2\pi a L}$$

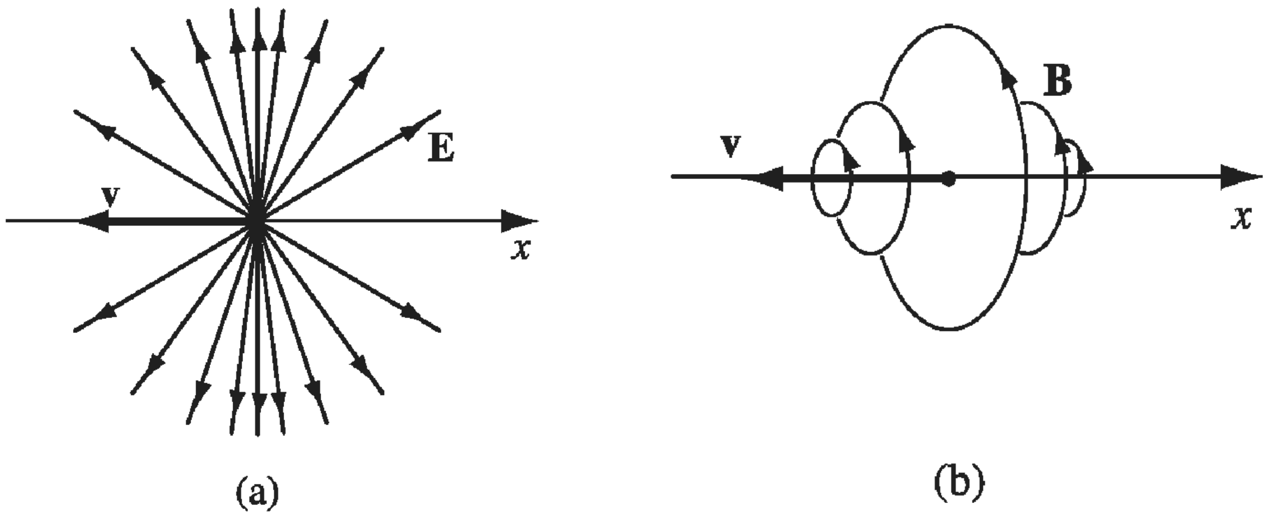
$$\mathbf{S} \equiv \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$$

The energy per unit time passing in through the surface of the wire is therefore

$$\int \mathbf{S} \cdot d\mathbf{a} = S(2\pi a L) = VI$$

Momentum

Newton's Third Law in Electrodynamics



Imagine a point charge q traveling in along the x axis at a constant speed $\boldsymbol{v}$. Because it is moving, its electric field is not given by Coulomb's law; nevertheless, $\boldsymbol{E}$ still points radially outward from the instantaneous position of the charge; a moving point charge does not constitute a steady current, its magnetic field is *not* given by the *Biot-Savart law*. Nevertheless, it's a fact that **B** still circles around the axis in a manner suggested by the right-hand rule.

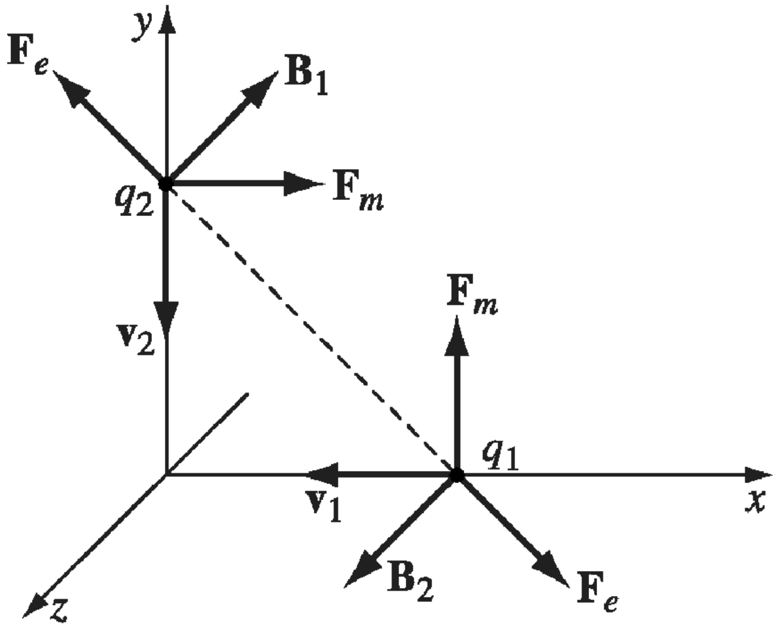


FIGURE 8.3

The net electromagnetic force of q_1 on q_2 is equal but not opposite to the force of q_2 on q_1. in violation of Newton's third law. In electrostatics and magnetostatics, the third law holds, but in electrodynamics it does not.

The proof of conservation of momentum rests on the cancellation of internal forces, which follows from the third law. Momentum conservation is rescued, in electrodynamics, by the realization that the fields themselves carry momentum. You’ll see how this works out quantitatively in the following sections.

Maxwell's Stress Tensor

Total electromagnetic force on the charges in volume $\boldsymbol{\nu}$

$$\mathbf{F} = \int_{\mathcal{V}} (\mathbf{E} + \mathbf{v} \times \mathbf{B})\rho \, d\tau = \int_{\mathcal{V}} (\rho\mathbf{E} + \mathbf{J} \times \mathbf{B}) \, d\tau$$

The force per unit volume is

$$\mathbf{f} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B}$$

eliminating ρ and $\boldsymbol{J}$ by using Maxwell's equations (i) and (iv):

$$\mathbf{f} = \epsilon_0(\nabla \cdot \mathbf{E})\mathbf{E} + \left(\frac{1}{\mu_0}\nabla \times \mathbf{B} - \epsilon_0\frac{\partial \mathbf{E}}{\partial t}\right) \times \mathbf{B}$$

$$\frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) = \left(\frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B}\right) + \left(\mathbf{E} \times \frac{\partial \mathbf{B}}{\partial t}\right)$$

and Faraday's law says $$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

so $$\frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B} = \frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) + \mathbf{E} \times (\nabla \times \mathbf{E})$$

Thus $$\mathbf{f} = \epsilon_0 [(\nabla \cdot \mathbf{E})\mathbf{E} - \mathbf{E} \times (\nabla \times \mathbf{E})] - \frac{1}{\mu_0}[\mathbf{B} \times (\nabla \times \mathbf{B})] - \epsilon_0 \frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B})$$

To make things look more symmetrical, throw in a "zero" term $(\nabla \cdot \mathbf{B})\mathbf{B}$ (since $\nabla \cdot \boldsymbol{B} = \mathbf{0}$)

Meanwhile, product rule 4 says

$$\nabla(E^2) = 2(\mathbf{E} \cdot \nabla)\mathbf{E} + 2\mathbf{E} \times (\nabla \times \mathbf{E})$$

$$\mathbf{E} \times (\nabla \times \mathbf{E}) = \frac{1}{2}\nabla(E^2) - (\mathbf{E} \cdot \nabla)\mathbf{E}$$

The same goes for **B**. Therefore

$$\mathbf{f} = \epsilon_0 [(\nabla \cdot \mathbf{E})\mathbf{E} + (\mathbf{E} \cdot \nabla)\mathbf{E}] + \frac{1}{\mu_0}[(\nabla \cdot \mathbf{B})\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{B}]$$

$$-\frac{1}{2}\nabla\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \epsilon_0 \frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}).$$

It can be simplified by introducing the Maxwell stress tensor,

$$T_{ij} \equiv \epsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\delta_{ij}B^2\right)$$

The indices i and j refer to the coordinates x, y, and z, for examples

$$T_{xx} = \frac{1}{2}\epsilon_0 \left(E_x^2 - E_y^2 - E_z^2\right) + \frac{1}{2\mu_0}\left(B_x^2 - B_y^2 - B_z^2\right)$$

$$T_{xy} = \epsilon_0(E_x E_y) + \frac{1}{\mu_0}(B_x B_y),$$

$$
T_{ij} = \begin{pmatrix}
\epsilon_0 (E_x^2 - E^2/2) + \frac{1}{\mu_0}(B_x^2 - B^2/2) & \epsilon_0 E_x E_y + \frac{1}{\mu_0}(B_x B_y) & \epsilon_0 E_x E_z + \frac{1}{\mu_0}(B_x B_z) \\
\epsilon_0 E_x E_y + \frac{1}{\mu_0}(B_x B_y) & \epsilon_0 (E_y^2 - E^2/2) + \frac{1}{\mu_0}(B_y^2 - B^2/2) & \epsilon_0 E_y E_z + \frac{1}{\mu_0}(B_y B_z) \\
\epsilon_0 E_x E_z + \frac{1}{\mu_0}(B_x B_z) & \epsilon_0 E_y E_z + \frac{1}{\mu_0}(B_y B_z) & \epsilon_0 (E_z^2 - E^2/2) + \frac{1}{\mu_0}(B_z^2 - B^2/2)
\end{pmatrix}
$$

- The element *ij* of the Maxwell stress tensor has units of momentum per unit of area per unit time and gives the flux of momentum parallel to the *i*-th axis crossing a surface normal to the *j*-th axis (in the negative direction) per unit of time.
- These units can also be seen as units of force per unit of area (negative pressure), and the *ij* element of the tensor can also be interpreted as the force parallel to the *i*-th axis suffered by a surface normal to the j-th axis per unit of area.
- The diagonal elements give the tension (pulling) acting on a differential area element normal to the corresponding axis.
- An area element in the electromagnetic field also feels a force in a direction that is not normal to the element. This shear is given by the off-diagonal elements of the stress tensor.

$\boldsymbol{T_{ij}}$ is sometimes written with a double arrow: $\overleftrightarrow{\boldsymbol{T}}$. One can form the dot product of $\overleftrightarrow{\boldsymbol{T}}$ with a vector $\boldsymbol{a}$

$$\left(\mathbf{a}\cdot\overleftrightarrow{\mathbf{T}}\right)_j = \sum_{i=x,y,z} a_i T_{ij}, \quad \left(\overleftrightarrow{\mathbf{T}}\cdot\mathbf{a}\right)_j = \sum_{i=x,y,z} T_{ji} a_i$$

The resulting object, which has one remaining index, is itself a vector.

The divergence of $\overleftrightarrow{\boldsymbol{T}}$ has as its jth component

$$\left(\nabla\cdot\overleftrightarrow{\mathbf{T}}\right)_j = \epsilon_0\left[(\nabla\cdot\mathbf{E})E_j + (\mathbf{E}\cdot\nabla)E_j - \frac{1}{2}\nabla_j E^2\right] + \frac{1}{\mu_0}\left[(\nabla\cdot\mathbf{B})B_j + (\mathbf{B}\cdot\nabla)B_j - \frac{1}{2}\nabla_j B^2\right]$$

Thus the force per unit volume can be written in the much tidier form

$$\mathbf{f} = \nabla\cdot\overleftrightarrow{\mathbf{T}} - \epsilon_0\mu_0\frac{\partial\mathbf{S}}{\partial t}$$ where $\boldsymbol{S}$ is the Poynting vector

The total electromagnetic force on the charges in $\mathcal{V}$

$$\mathbf{F} = \oint_{\mathcal{S}} \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a} - \epsilon_0 \mu_0 \frac{d}{dt} \int_{\mathcal{V}} \mathbf{S} \, d\tau$$

In the static case the second term drops out, and the electromagnetic force on the charge configuration can be expressed entirely in terms of the stress tensor at the boundary:

$$\mathbf{F} = \oint_{\mathcal{S}} \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a} \quad \text{(static)}$$

Physically, $\overleftrightarrow{\boldsymbol{T}}$ is the force per unit area (or **stress**) acting on the surface. More precisely, T_{ij} is the force (per unit area) in the ith direction acting on an element of surface oriented in the jth direction.

$$\mathbf{F} = \oint_S \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a} \quad \text{(static)}$$

Example

Determine the net force on the "northern" hemisphere of a uniformly charged solid sphere of radius $\boldsymbol{R}$ and charge $\boldsymbol{Q}$ (the same as Prob. 2.47, only this time we'll use the Maxwell stress tensor and Eq. 8.21).

Solution

The boundary surface consists of two parts - a hemispherical "bowl" at radius R, and a circular disk at $\theta = \frac{\pi}{2}$ (Fig. 8.4). For the bowl,

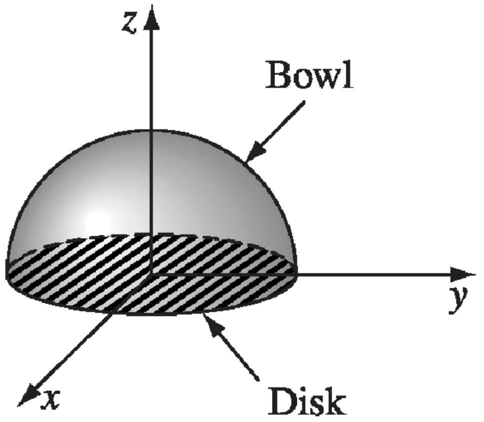


FIGURE 8.4

$$d\mathbf{a} = R^2 \sin\theta \, d\theta \, d\phi \, \hat{\mathbf{r}}$$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^2}\,\hat{\mathbf{r}}$$

$$\mathbf{F} = \oint_S \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a} \quad \text{(static)}$$

In Cartesian components,

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}$$

$$T_{zx} = \epsilon_0 E_z E_x = \epsilon_0\left(\frac{Q}{4\pi\epsilon_0 R^2}\right)^2 \sin\theta\cos\theta\cos\phi,$$

$$T_{zy} = \epsilon_0 E_z E_y = \epsilon_0\left(\frac{Q}{4\pi\epsilon_0 R^2}\right)^2 \sin\theta\cos\theta\sin\phi,$$

$$T_{zz} = \frac{\epsilon_0}{2}\left(E_z^2 - E_x^2 - E_y^2\right) = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R^2}\right)^2 (\cos^2\theta - \sin^2\theta)$$

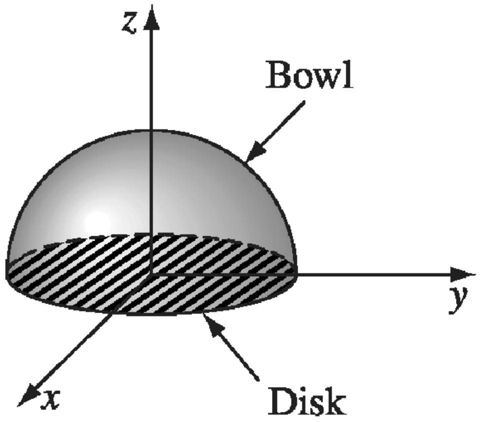


FIGURE 8.4

The net force is obviously in the z-direction, so it suffices to calculate

$$\left(\overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a}\right)_z = T_{zx}\,da_x + T_{zy}\,da_y + T_{zz}\,da_z = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R}\right)^2 \sin\theta\cos\theta\,d\theta\,d\phi$$

The force on the "bowl" is therefore

$$\mathbf{F} = \oint_S \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a} \quad \text{(static)}$$

$$F_{\text{bowl}} = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R}\right)^2 2\pi \int_0^{\pi/2} \sin\theta\cos\theta\, d\theta = \frac{1}{4\pi\epsilon_0}\frac{Q^2}{8R^2}$$

Meanwhile, for the equatorial disk

$$d\mathbf{a} = -r\, dr\, d\phi\, \hat{\mathbf{z}}$$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}\mathbf{r} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^3} r(\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}})$$

$$T_{zz} = \frac{\epsilon_0}{2}(E_z^2 - E_x^2 - E_y^2) = -\frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R^3}\right)^2 r^2$$

$$\left(\overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a}\right)_z = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R^3}\right)^2 r^3\, dr\, d\phi$$

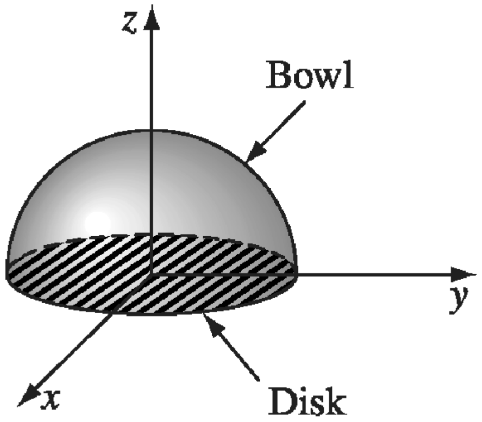


FIGURE 8.4

$$\mathbf{F} = \oint_{\mathcal{S}} \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a} \quad \text{(static)}$$

$$\left(\overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a}\right)_z = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R^3}\right)^2 r^3\, dr\, d\phi$$

$$F_{\text{disk}} = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R^3}\right)^2 2\pi \int_0^R r^3 dr = \frac{1}{4\pi\epsilon_0}\frac{Q^2}{16R^2}$$

$$F = \frac{1}{4\pi\epsilon_0}\frac{3Q^2}{16R^2}$$

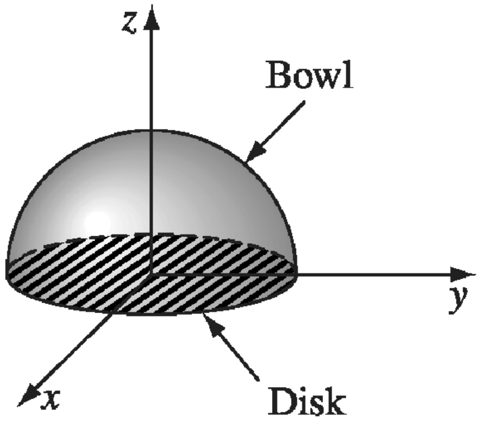


FIGURE 8.4

Conservation of Momentum

According to Newton's second law, the force on an object is equal to the rate of change of its momentum:

$$\mathbf{F} = \frac{d\mathbf{p}_{\text{mech}}}{dt}$$

Therefore

$$\frac{d\mathbf{p}_{\text{mech}}}{dt} = -\epsilon_0\mu_0\frac{d}{dt}\int_{\mathcal{V}} \mathbf{S}\,d\tau + \oint_{\mathcal{S}} \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a}, \qquad (8.28)$$

Where P_{mech} is the (mechanical) momentum of the particles in volume ν. The first integral represents *momentum stored in the fields:*

$$\mathbf{p} = \mu_0\epsilon_0 \int_{\mathcal{V}} \mathbf{S}\,d\tau$$

while the second integral is the momentum per unit time flowing in through the surface.

Equation 8.28 is the statement of conservation of momentum in electrodynamics: If the mechanical momentum increases, either the field momentum decreases, or else the fields are carrying momentum into the volume through the surface.

The momentum density in the fields is evidently

$$\mathbf{g} = \mu_0\epsilon_0\mathbf{S} = \epsilon_0(\mathbf{E} \times \mathbf{B})$$

If the mechanical momentum in $\boldsymbol{\mathcal{V}}$ is not changing:

$$\int \frac{\partial \mathbf{g}}{\partial t}\, d\tau = \oint \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a} = \int \nabla \cdot \overleftrightarrow{\mathbf{T}}\, d\tau.$$

$$\frac{\partial \mathbf{g}}{\partial t} = \nabla \cdot \overleftrightarrow{\mathbf{T}}$$

Note:

- This is the "continuity equation" for electromagnetic momentum, with $\boldsymbol{g}$ (momentum density) in the role of ρ (charge density) and $\overleftrightarrow{\boldsymbol{T}}$ playing the part of $\boldsymbol{J}$; it expresses the local conservation of field momentum. But in general (when there are charges around) the field momentum by itself, and the mechanical momentum by itself, are not conserved--charges and fields exchange momentum, and only the total is conserved.
- The Poynting vector has appeared in two quite different roles: $\boldsymbol{S}$ itself is the energy per unit area, per unit time, transported by the electromagnetic fields, while $\mu_0\epsilon_0\boldsymbol{S}$ is the momentum per unit volume stored in those fields.
- Similarly, $\overleftrightarrow{\boldsymbol{T}}$ plays a dual role : $\overleftrightarrow{\boldsymbol{T}}$ itself is the electromagnetic stress (force per unit area) acting on a surface, and $\overleftrightarrow{\boldsymbol{T}}$ describes the flow of momentum (it is the momentum current density) carried by the fields

$$\frac{\partial \mathbf{g}}{\partial t} = \nabla \cdot \overleftrightarrow{\mathbf{T}}$$

Example 8.3

A long coaxial cable, of length l, consists of an inner conductor (radius a) and an outer conductor (radius b). It is connected to a battery at one end and a resistor at the other (Fig. 8.5). The inner conductor carries a uniform charge per unit length λ, and a steady current I to the right; the outer conductor has the opposite charge and current. What is the electromagnetic momentum stored in the fields?

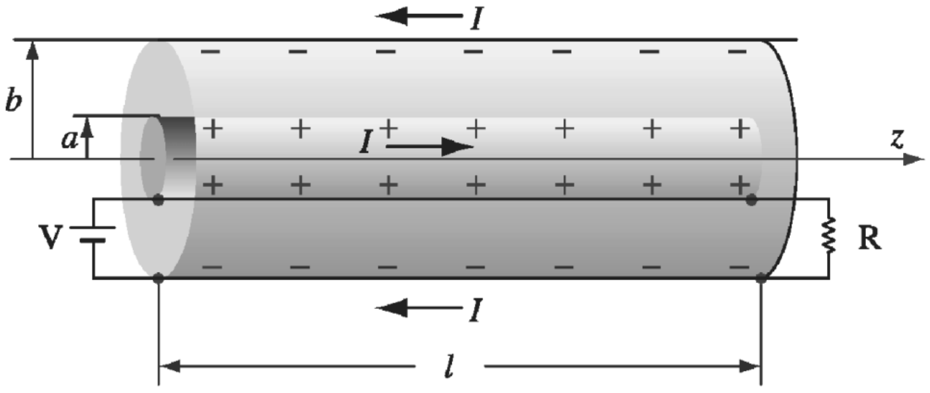


FIGURE 8.5

Solution

The fields are

$$\mathbf{E} = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{s}\,\hat{\mathbf{s}} \qquad \mathbf{B} = \frac{\mu_0}{2\pi}\frac{I}{s}\,\hat{\boldsymbol{\phi}}$$

The Poynting vector is therefore

$$\mathbf{S} = \frac{\lambda I}{4\pi^2\epsilon_0 s^2}\,\hat{\mathbf{z}}.$$

So energy is flowing down the line, from the battery to the resistor. In fact, the power transported is

$$P = \int \mathbf{S} \cdot d\mathbf{a} = \frac{\lambda I}{4\pi^2 \epsilon_0} \int_a^b \frac{1}{s^2} 2\pi s \, ds = \frac{\lambda I}{2\pi \epsilon_0} \ln(b/a) = IV$$

The momentum in the fields is

$$\mathbf{p} = \mu_0 \epsilon_0 \int \mathbf{S} \, d\tau = \frac{\mu_0 \lambda I}{4\pi^2} \hat{\mathbf{z}} \int_a^b \frac{1}{s^2} l 2\pi s \, ds = \frac{\mu_0 \lambda I l}{2\pi} \ln(b/a) \, \hat{\mathbf{z}} = \frac{IVl}{c^2} \hat{\mathbf{z}}$$

Angular Momentum

The electromagnetic fields have taken on a life of their own. They carry energy

$$u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$$

momentum

$$\mathbf{g} = \epsilon_0(\mathbf{E} \times \mathbf{B})$$

angular momentum

$$\boldsymbol{\ell} = \mathbf{r} \times \mathbf{g} = \epsilon_0\left[\mathbf{r} \times (\mathbf{E} \times \mathbf{B})\right]$$

Static fields can harbor momentum and angular momentum, as long as $\boldsymbol{E} \times \boldsymbol{B}$ is nonzero.

Example 8.4

Imagine a very long solenoid with radius R, n turns per unit length, and current I. Coaxial with the solenoid are two long cylindrical (nonconducting) shells of length l, one, inside the solenoid at radius a, carries a charge $+Q$, uniformly distributed over its surface; the other, outside the solenoid at radius b, carries charge $-Q$ (see Fig. 8.7; l is supposed to be much greater than b).

When the current in the solenoid is gradually reduced, the cylinders begin to rotate, as we found in Ex.7.8.

Question: Where does the angular momentum come from?

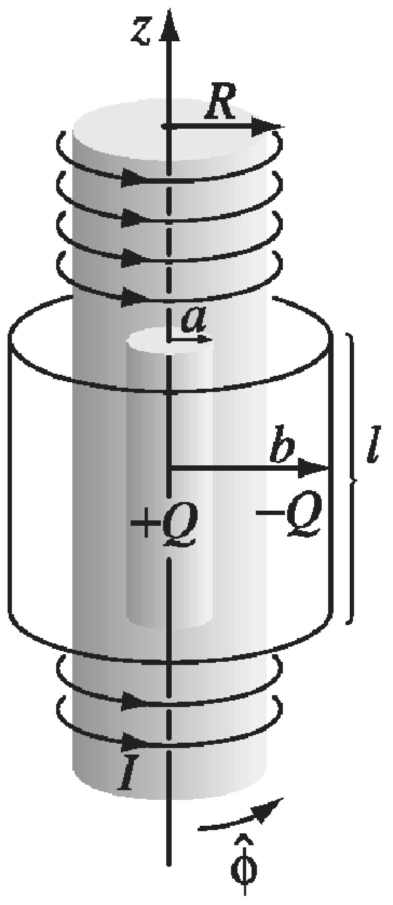


FIGURE 8.7

Solution

It was initially stored in the fields. Before the current was switched off, there was an electric field

$$\mathbf{E} = \frac{Q}{2\pi\epsilon_0 l}\frac{1}{s}\,\hat{\mathbf{s}} \quad (a < s < b)$$

in the region between the cylinders, and a magnetic field

$$\mathbf{B} = \mu_0 n I\,\hat{\mathbf{z}} \quad (s < R)$$

inside the solenoid. The momentum density (Eq. 8.29) was therefore (in the region $a < s < R$)

$$\mathbf{g} = \epsilon_0(\mathbf{E}\times\mathbf{B}) \qquad \mathbf{g} = -\frac{\mu_0 n I Q}{2\pi l s}\,\hat{\boldsymbol{\phi}}.$$

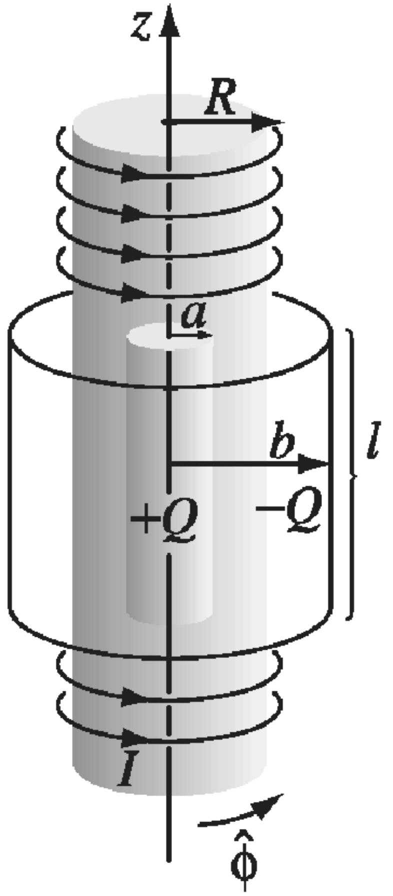


FIGURE 8.7

The z component of the *angular* momentum density was

$$\mathbf{g} = -\frac{\mu_0 n I Q}{2\pi l s}\hat{\boldsymbol{\phi}} \Rightarrow (\mathbf{r}\times\mathbf{g})_z = -\frac{\mu_0 n I Q}{2\pi l}$$

To get the total angular momentum in the fields, we simply multiply by the volume, $\pi(R^2 - a^2)l$:

$$\mathbf{L} = -\frac{1}{2}\mu_0 n I Q (R^2 - a^2)\,\hat{\mathbf{z}}$$

When the current is turned off, the changing magnetic field induces a circumferential electric field, given by Faraday's law

$$\oint \mathbf{E}\cdot d\mathbf{l} = E(2\pi s) = -\frac{d\Phi}{dt} = -\frac{d}{dt}\left(\pi s^2 B(t)\right)$$

$$\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$$

$$\mathbf{E} = \begin{cases} -\frac{1}{2}\mu_0 n \frac{dI}{dt}\frac{R^2}{s}\hat{\boldsymbol{\phi}}, & (s > R) \\ -\frac{1}{2}\mu_0 n \frac{dI}{dt} s\,\hat{\boldsymbol{\phi}}, & (s < R) \end{cases}$$

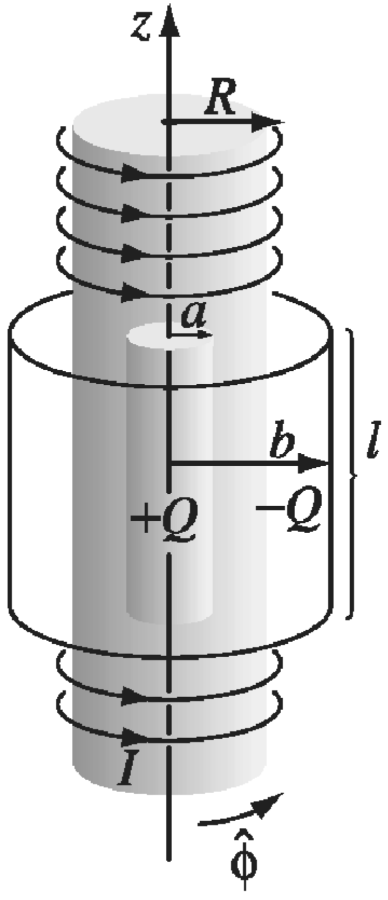


FIGURE 8.7

Thus the torque on the outer cylinder is

$$\mathbf{N}_b = \mathbf{r} \times (-Q\mathbf{E}) = \frac{1}{2}\mu_0 n Q R^2 \frac{dI}{dt}\hat{\mathbf{z}}.$$

and it picks up an angular momentum

$$\mathbf{L}_b = \frac{1}{2}\mu_0 n Q R^2\,\hat{\mathbf{z}} \int_I^0 \frac{dI}{dt}dt = -\frac{1}{2}\mu_0 n I Q R^2\,\hat{\mathbf{z}}.$$

Similarly, the torque on the inner cylinder is

$$\mathbf{N}_a = -\frac{1}{2}\mu_0 n Q a^2 \frac{dI}{dt}\,\hat{\mathbf{z}}$$

and its angular momentum increase is

$$\mathbf{L}_a = \frac{1}{2}\mu_0 n I Q a^2\,\hat{\mathbf{z}}$$

$$\mathbf{L} = -\frac{1}{2}\mu_0 n I Q (R^2 - a^2)\,\hat{\mathbf{z}}.$$

Note:
It all works out: $\boldsymbol{L} = \boldsymbol{L_a} + \boldsymbol{L_b}$.The angular momentum lost by the fields is precisely equal to the angular momentum gained by the cylinders, and the total angular momentum (fields plus matter) is conserved.