Chapter 7

Electrodynamics

- 7.1 Electromotive Force
- 7.2 Electromagnetic Induction
- 7.3 Maxwell's Equations

Electromotive Force

Ohm's Law

For most substances, the current density J is proportional to the force per unit charge f

$$\boldsymbol{J} = \sigma \boldsymbol{f}$$

The proportionality factor σ called the conductivity of the medium

$$\text{resistivity } \rho = \frac{1}{\sigma}$$

An electromagnetic force

$$\boldsymbol{J} = \sigma(\boldsymbol{E} + v \times \boldsymbol{B})$$

Ordinarily, the velocity of the charges is sufficiently small that the second term can be ignored:

Ohm's law $\quad \boldsymbol{J} = \sigma \boldsymbol{E}$

Material	Resistivity	Material	Resistivity
Conductors:		*Semiconductors:*	
Silver	1.59×10^{-8}	Sea water	0.2
Copper	1.68×10^{-8}	Germanium	0.46
Gold	2.21×10^{-8}	Diamond	2.7
Aluminum	2.65×10^{-8}	Silicon	2500
Iron	9.61×10^{-8}	*Insulators:*	
Mercury	9.61×10^{-7}	Water (pure)	8.3×10^{3}
Nichrome	1.08×10^{-6}	Glass	$10^{9} - 10^{14}$
Manganese	1.44×10^{-6}	Rubber	$10^{13} - 10^{15}$
Graphite	1.6×10^{-5}	Teflon	$10^{22} - 10^{24}$

TABLE 7.1 Resistivities, in ohm-meters (all values are for 1 atm, 20° C). *Data from Handbook of Chemistry and Physics*, 91st ed. (Boca Raton, Fla.: CRC Press, 2010) and other references.

Example 7.1

A cylindrical resistor of cross-sectional area A and length L is made from material with conductivity σ. (See Fig. 7.1; as indicated, the cross section need not be circular) If we stipulate that the potential is constant over each end, and the potential difference between the ends is V, what current flows?

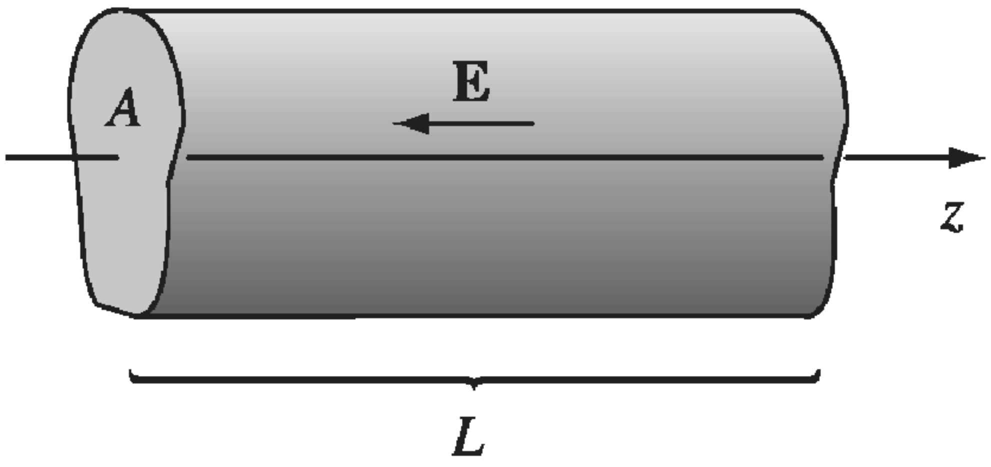


FIGURE 7.1

$$\boxed{I = JA = \sigma E A = \frac{\sigma A}{L} V}$$

Example 7.2

Two long coaxial metal cylinders (radius a and b) are separated by material of conductivity σ (Fig. 7 .2). If they are maintained at a potential difference V, what current flows from one to the other, in a length L?

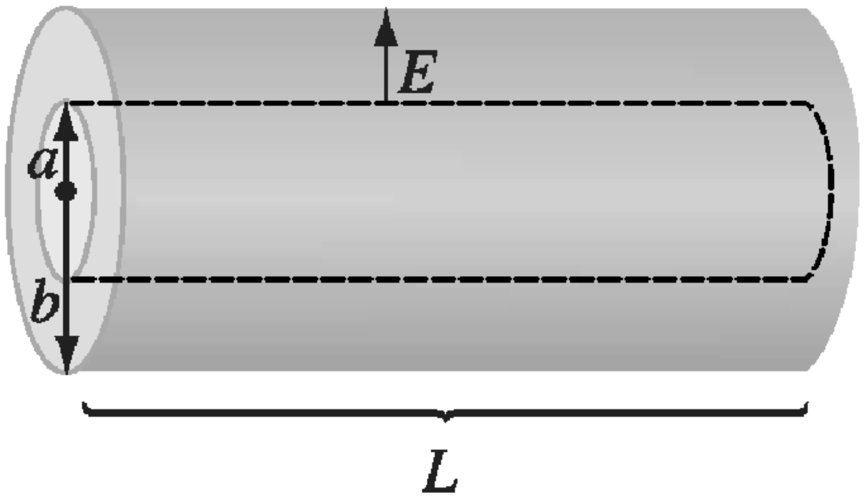


FIGURE 7.2

Solution

The field between the cylinders is

$$\boldsymbol{E} = \frac{\boldsymbol{\lambda}}{\boldsymbol{2\pi\epsilon_0 s}}\vec{\boldsymbol{s}}$$

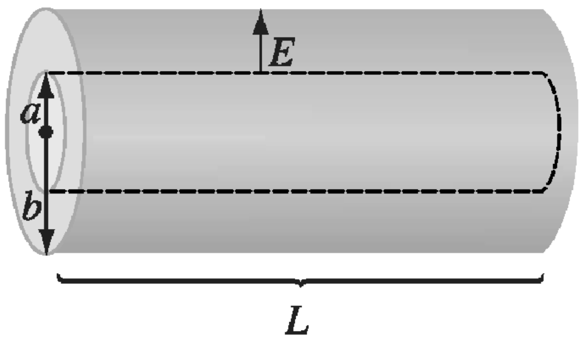


FIGURE 7.2

where λ is the charge per unit length on the inner cylinder, which leads to the potential difference. The current is therefore

$$I = \int \mathbf{J} \cdot d\mathbf{a} = \sigma \int \mathbf{E} \cdot d\mathbf{a} = \frac{\sigma}{\epsilon_0}\lambda L$$

(The integral is over any surface enclosing the inner cylinder.)
Meanwhile, the potential difference between the cylinders is

$$V = -\int_b^a \mathbf{E} \cdot d\mathbf{l} = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{b}{a}\right)$$

So

$$I = \frac{2\pi\sigma L}{\ln(b/a)} V$$

Conclusion

As these examples illustrate, the total current flowing from one electrode to the other is proportional to the potential difference between them:

$$\boldsymbol{V = IR}$$

This is a more familiar version of Ohm's law. *R* is called the resistance; it's a function of the geometry of the arrangement and the conductivity of the medium between the electrodes (In Example 1, $R = \frac{L}{\sigma A}$, in Example 2, $R = \ln(b/a)/_{2\pi\sigma L}$) Resistance is measured in **ohms** (Ω)

For steady currents and uniform conductivity

$$\nabla \cdot \mathbf{E} = \frac{1}{\sigma} \nabla \cdot \mathbf{J} = 0$$

therefore the charge density is zero; any unbalanced charge resides on the surface

Question: a given field $\boldsymbol{E}$ produces a force $q\boldsymbol{E}$ (on a charge q), and according to Newton's second law, the charge will accelerate. Ohm's law implies, on the contrary, that a constant field produces a constant current, which suggests a constant velocity.

Isn't that a contradiction to Newton's law?

In practice, the charges are already moving very fast because of their thermal energy. But the thermal velocities have random directions, and average to zero. The drift velocity we are concerned with is a tiny extra bit so the time between collisions is actually much shorter than we supposed; if we assume for the sake of argument that all charges travel the same distance λ between collisions, then

$$t = \frac{\lambda}{v_{\text{thermal}}}$$

Therefore $$v_{\text{ave}} = \frac{1}{2}at = \frac{a\lambda}{2v_{\text{thermal}}}$$

If there are n molecules per unit volume, and f free electrons per molecule, each with charge q and mass m, the current density is

$$\mathbf{J} = nfq\mathbf{v}_{\text{ave}} = \frac{nfq\lambda}{2v_{\text{thermal}}}\frac{\mathbf{F}}{m} = \left(\frac{nf\lambda q^2}{2mv_{\text{thermal}}}\right)\mathbf{E}$$

$$\mathbf{J} = nfq\mathbf{v}_{\text{ave}} = \frac{nfq\lambda}{2v_{\text{thermal}}}\frac{\mathbf{F}}{m} = \left(\frac{nf\lambda q^2}{2mv_{\text{thermal}}}\right)\mathbf{E}$$

It correctly predicts that conductivity is proportional to the density of the moving charges and (ordinarily) decreases with increasing temperature.

As a result of all the collisions, the work done by the electrical force is converted into heat in the resistor. Since the work done per unit charge is *V* and the charge flowing per unit time is *I*, the power delivered is

Joule heating law

$$P = VI = I^2R.$$

Note:
With I in amperes and R in ohms, P comes out in watts (Joules per second)

Electromotive Force

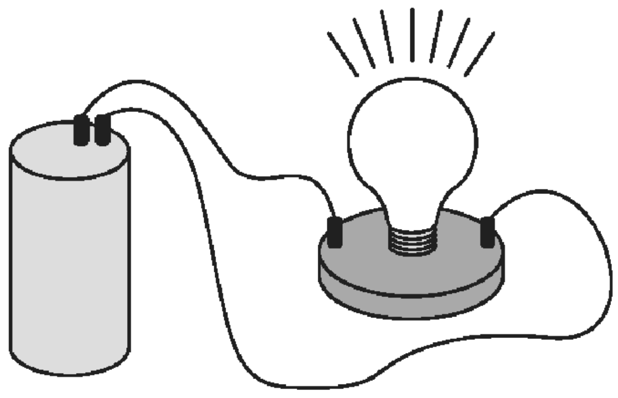


FIGURE 7.7

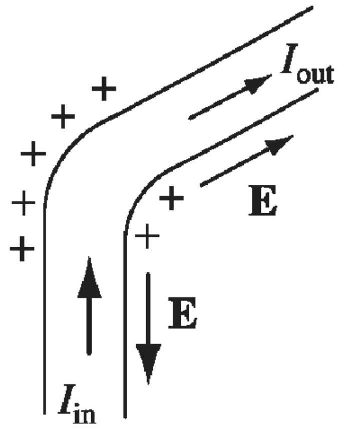


FIGURE 7.8

There are really two forces involved in driving current around a circuit: the source, f_s, which is ordinarily confined to one portion of the loop (a battery, say), and an electrostatic force, which serves to smooth out the flow and communicate the influence of the source to distant parts of the circuit

$$f = f_s + E$$

The physical agency responsible for f_s can be many different things: in a battery it's a chemical force; in a piezoelectric crystal mechanical pressure is converted into an electrical impulse; in a thermocouple it's a temperature gradient that does the job; in a photoelectric cell it's light. Whatever the *mechanism,* its net effect is determined by the line integral off around the circuit:

$$\mathcal{E} \equiv \oint \mathbf{f} \cdot d\mathbf{l} = \oint \mathbf{f}_s \cdot d\mathbf{l}$$

(Because $\oint \boldsymbol{E} \cdot dl = 0$ for electrostatic fields, it doesn't matter whether you use $\boldsymbol{f}$ or f_s) ε is called the *electromotive force*, or emf, of the circuit. It's a lousy term, since this is not a force at all —it is the integral of a force per unit charge.

Within an ideal source of emf (a resistanceless battery, for instance), the net force on the charges is zero. So $\boldsymbol{E} = -\boldsymbol{f}_{\boldsymbol{s}}$.

The potential difference between the terminals (a and b) in the source is therefore:

$$V = -\int_a^b \mathbf{E} \cdot d\mathbf{l} = \int_a^b \mathbf{f}_s \cdot d\mathbf{l} = \oint \mathbf{f}_s \cdot d\mathbf{l} = \mathcal{E}$$

The function of a battery, then, is to establish and maintain a voltage difference equal to the electromotive force. The resulting electrostatic field drives current around the rest of the circuit. Because it's the line integral of $\boldsymbol{f}_{s}$, $\boldsymbol{\varepsilon}$ can be interpreted as the work done per unit charge by the source.

Motional emf

Generator:
Generators exploit motional emfs, which arise when you move a wire through a magnetic field.

$$\mathcal{E} = \oint \mathbf{f}_{\text{mag}} \cdot d\mathbf{l} = vBh$$

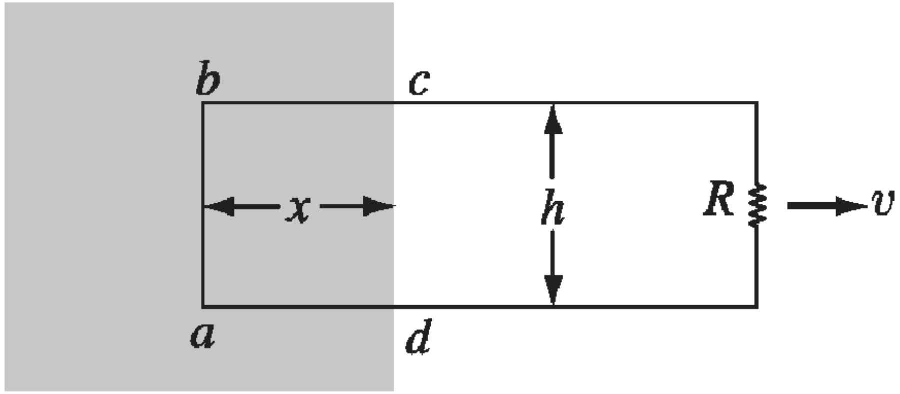


FIGURE 7.10

Note:
The integral you perform to calculate ε is carried out at one instant of time.

Question: In particular, although the magnetic force is responsible for establishing the emf, it is not doing any work - magnetic forces never do work. Who, then, is supplying the energy that heats the resistor?

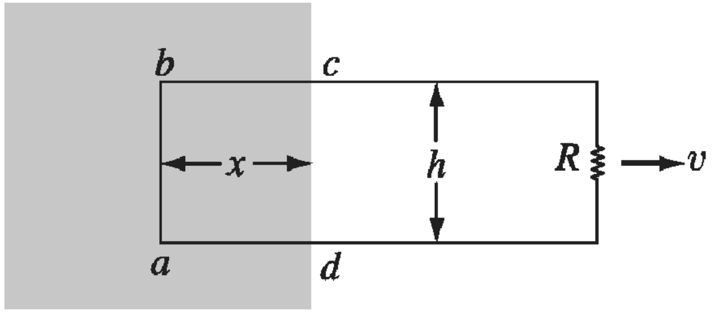


FIGURE 7.10

Answer: The person who's pulling on the loop. With the current flowing, the free charges in segment *ab* have a vertical velocity (call it u) in addition to the horizontal velocity v they inherit from the motion of the loop. Accordingly, the magnetic force has a component quB to the left. To counteract this, the person pulling on the wire must exert a force per unit charge

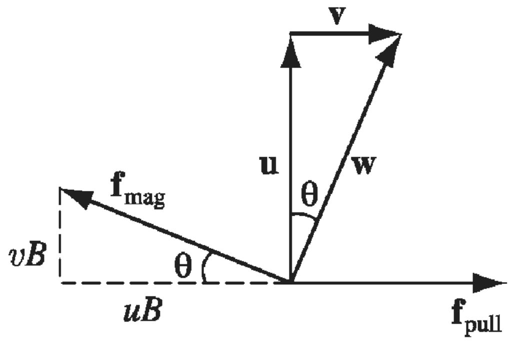
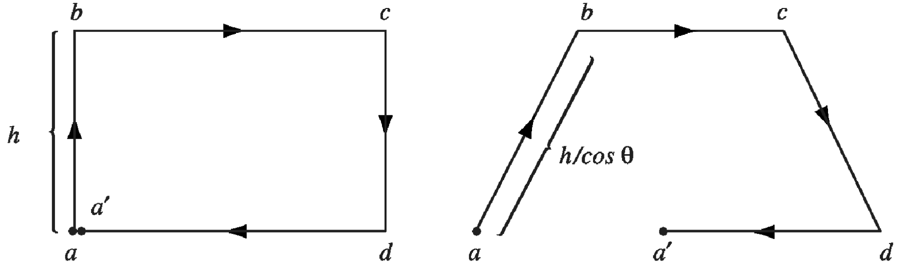


FIGURE 7.11

$$f_{pull} = uB$$

The work done per unit charge is therefore

$$\int \mathbf{f}_{\text{pull}} \cdot d\mathbf{l} = (uB)\left(\frac{h}{\cos\theta}\right)\sin\theta = vBh = \mathcal{E}$$



(a) Integration path for computing $\mathcal{E}$ (follow the wire at one instant of time).

(b) Integration path for calculating work done (follow the charge around the loop).

FIGURE 7.12

As it turns out, the *work done per unit charge is exactly equal to the emf,*

There is a particularly nice way of expressing the emf generated in a moving loop. Let Φ be the flux of ***B*** through the loop:

$$\Phi \equiv \int \mathbf{B} \cdot d\mathbf{a}.$$

For the rectangular loop

$$\Phi = Bhx$$

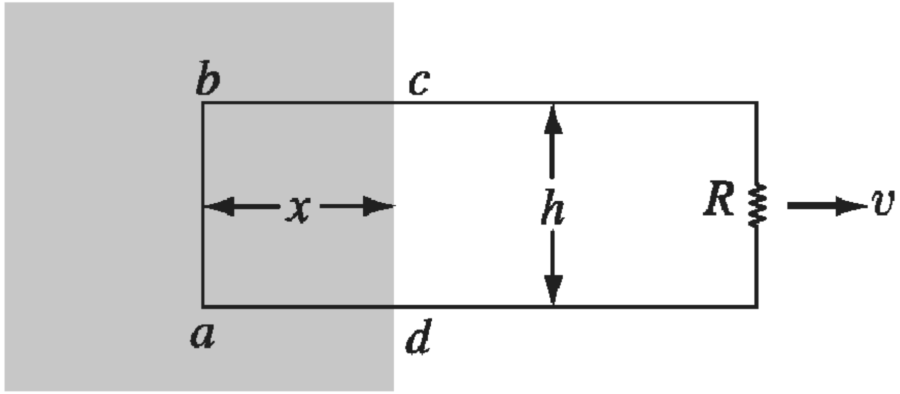


FIGURE 7.10

As the loop moves, the flux decreases

$$\frac{d\Phi}{dt} = Bh\frac{dx}{dt} = -Bhv$$

As $\quad \mathcal{E} = \oint \mathbf{f}_{\text{mag}} \cdot d\mathbf{l} = vBh.$

We have $\quad \boxed{\mathcal{E} = -\frac{d\Phi}{dt}}$

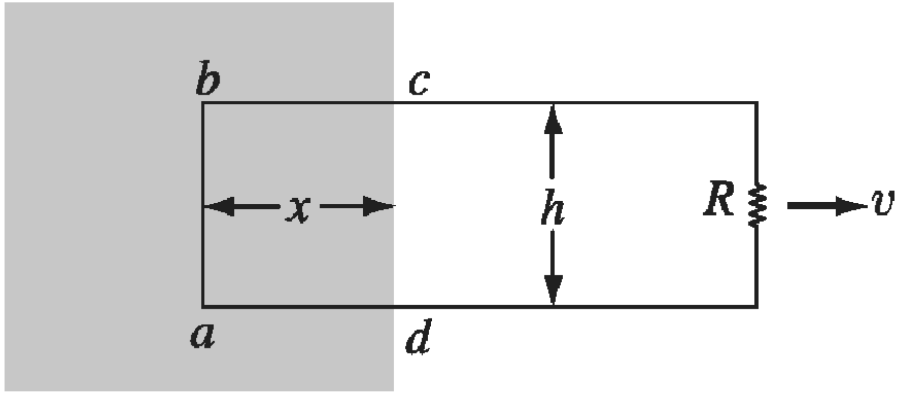


FIGURE 7.10

This is the flux rule for motional emf.

Apart from its delightful simplicity, the flux rule has the virtue of applying to nonrectangular loops moving in *arbitrary* directions through nonuniform magnetic fields; in fact, the loop need not even maintain a fixed shape.

Electromagnetic Induction

Faraday's Law

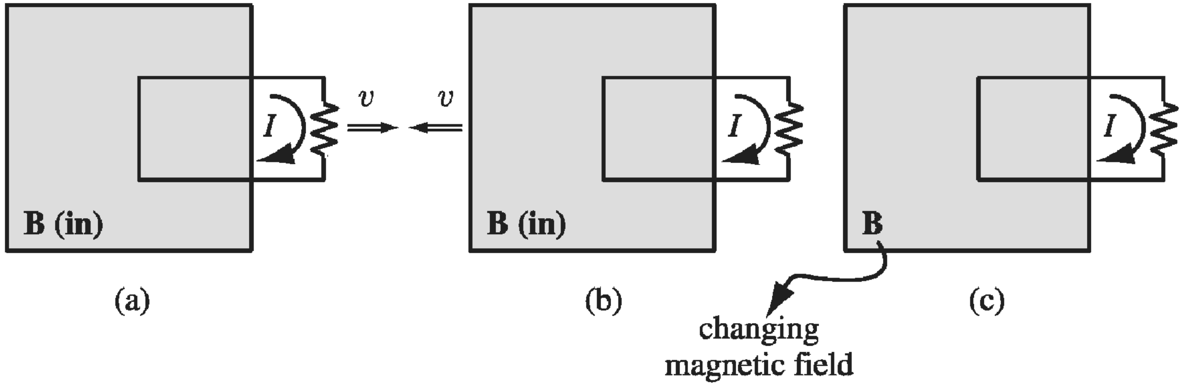


FIGURE 7.21

A changing magnetic field induces an electric field.

The emf is again equal to the rate of change of the flux

$$\mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt}$$

$$\mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt} \qquad \Phi \equiv \int \mathbf{B} \cdot d\mathbf{a}$$

then $\boldsymbol{E}$ is related to the change in $\boldsymbol{B}$ by the equation

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a}$$

This is Faraday's law, in integral form.
We can convert it to differential form
by applying Stokes' theorem:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Note :
Faraday's law reduces to the old rule $\oint \boldsymbol{E} \cdot dl = 0$ (or in differential form $\nabla \times \boldsymbol{E} = 0$) in the case of constant **B**

Example 7.5

A long cylindrical magnet of length L and radius a carries a uniform magnetization $\boldsymbol{M}$ parallel to its axis. It passes at constant velocity $\boldsymbol{v}$ through a circular wire ring of slightly larger diameter (Fig. 7.22). Graph the emf induced in the ring, as a function of time.

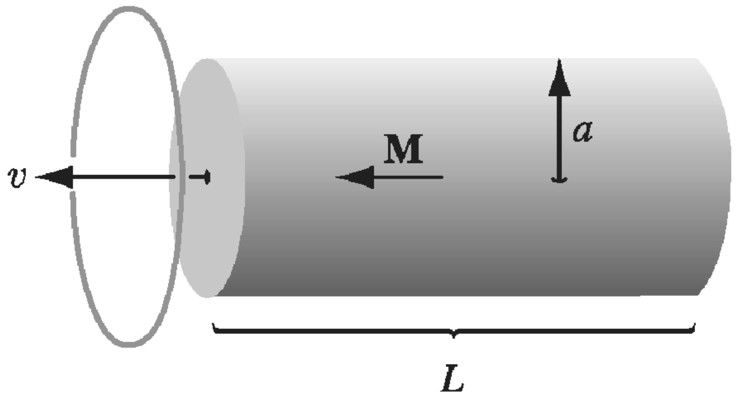


FIGURE 7.22

Solution

The magnetic field is the same as that of a long solenoid with surface current $K_b = M\vec{\Phi}$. So the field inside is $\boldsymbol{B} = \mu_0 \boldsymbol{M}$ except near the ends, where it starts to spread out. The flux through the ring is zero when the magnet is far away; it builds up to a maximum of $\mu_0 M\pi a^2$ as the leading end passes through; and it drops back to zero as the trailing end emerges (Fig. 7.23a). The emf is (minus) the derivative of Φ with respect to time, so it consists of two spikes, as shown in Fig. 7.23b.

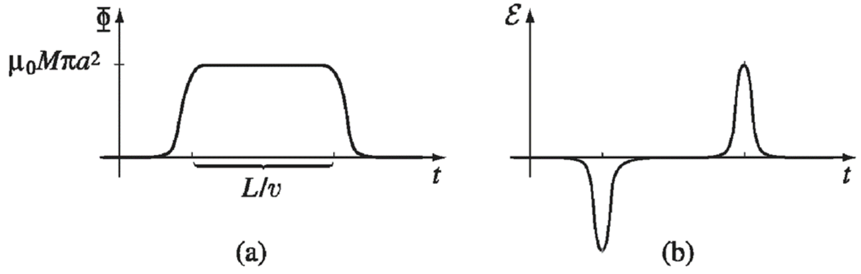


FIGURE 7.23

Lenz's law

Nature abhors a change in flux

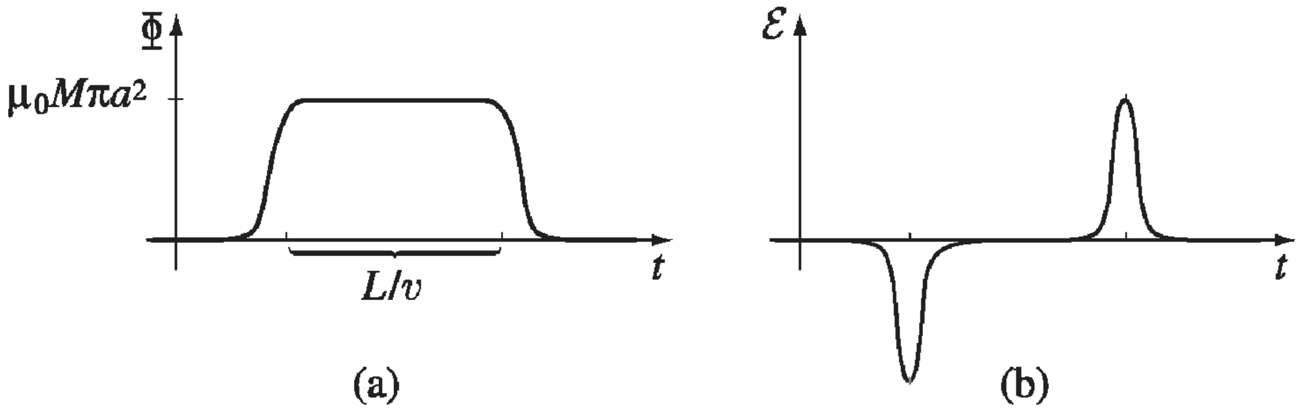


FIGURE 7.23

The induced current will flow in such a direction that the flux it produces tends to cancel the change. (As the front end of the magnet in Ex. 7.5 enters the ring, the flux increases, so the current in the ring must generate a field to the right. Notice that it is the *change* in flux, not the flux itself, that nature abhors (when the tail end of the magnet exits the ring, the flux *drops,* so the induced current flows *counterclockwise*

The Induced Electric Field

Example 7.7

A uniform magnetic field $\boldsymbol{B}(t)$, pointing straight up, fills the shaded circular region of Fig. 7.25. If $\boldsymbol{B}$ is changing with time, what is the induced electric field?

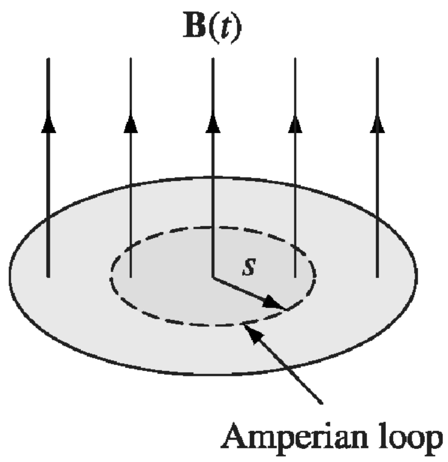


FIGURE 7.25

Solution

E points in the circumferential direction, just like the magnetic field inside a long straight wire carrying a uniform current density. Draw an Amperian loop of radius s, and apply Faraday's law:

$$\oint \mathbf{E} \cdot d\mathbf{l} = E(2\pi s) = -\frac{d\Phi}{dt} = -\frac{d}{dt}\left(\pi s^2 B(t)\right) = -\pi s^2 \frac{dB}{dt}.$$

$$\mathbf{E} = -\frac{s}{2}\frac{dB}{dt}\hat{\boldsymbol{\phi}}$$

If $\boldsymbol{B}$ is *increasing*, $\boldsymbol{E}$ runs *clockwise*, as viewed from above

Inductance

Suppose you have two loops of wire, at rest If you run a steady current I_1 around loop 1, it produces a magnetic field. Some of the field lines pass through loop 2; let Φ_2 be the flux of B_1 through 2

Biot-Savart law $$\mathbf{B}_1 = \frac{\mu_0}{4\pi} I_1 \oint \frac{d\mathbf{l}_1 \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

$$\Phi_2 = \int \mathbf{B}_1 \cdot d\mathbf{a}_2 \quad \Rightarrow \quad \Phi_2 = M_{21} I_1$$

where M_{21} is the constant of proportionality; it is known as the **mutual inductance** of the two loops

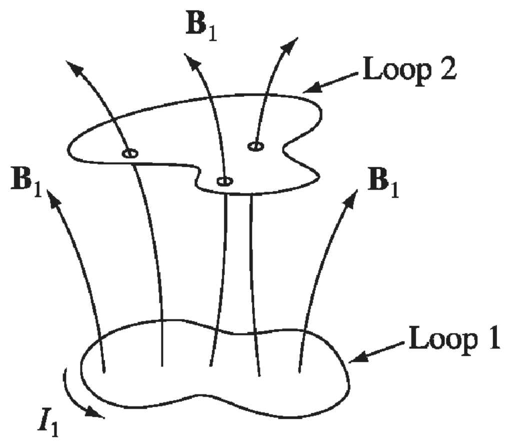


FIGURE 7.30

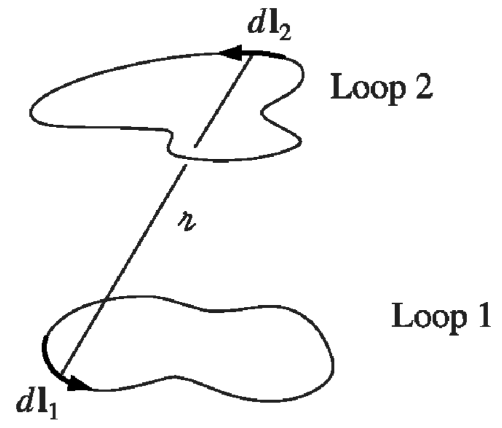


FIGURE 7.31

$$\Phi_2 = \int \mathbf{B}_1 \cdot d\mathbf{a}_2 = \int (\nabla \times \mathbf{A}_1) \cdot d\mathbf{a}_2 = \oint \mathbf{A}_1 \cdot d\mathbf{l}_2$$

$$\mathbf{A}_1 = \frac{\mu_0 I_1}{4\pi} \oint \frac{d\mathbf{l}_1}{\mathscr{r}}$$

$$\Phi_2 = \frac{\mu_0 I_1}{4\pi} \oint \left(\oint \frac{d\mathbf{l}_1}{\mathscr{r}} \right) \cdot d\mathbf{l}_2$$

Neumann formula $$M_{21} = \frac{\mu_0}{4\pi} \oint \oint \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{\mathscr{r}}$$

it involves a double line integral-one integration around loop 1, the other around loop 2 (Fig. 7.31). It's not very useful for practical calculations, but it does reveal two important things about mutual inductance:

- M_{21} is a purely geometrical quantity, having to do with the sizes, shapes, and relative positions of the two loops
- The integral in M_{21} is unchanged if we switch the roles of loops 1 and 2; it follows that $M_{21} = M_{12}$

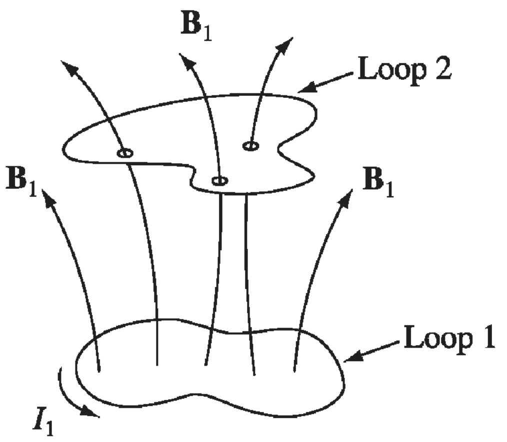


FIGURE 7.30

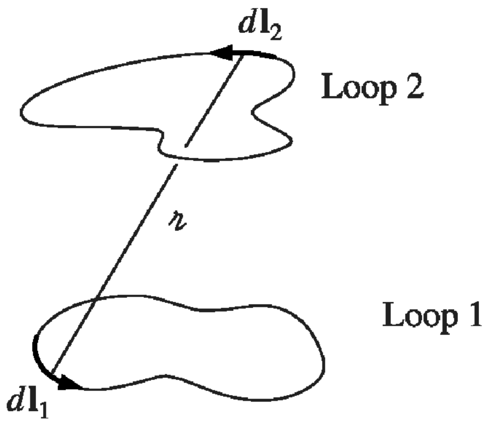


FIGURE 7.31

Conclusion

Whatever the shapes and positions of the loops, the flux through 2 when we run a current I around 1 is identical to the flux through 1 when we send the same current I around 2. We may as well drop the subscripts and call them both **M**.

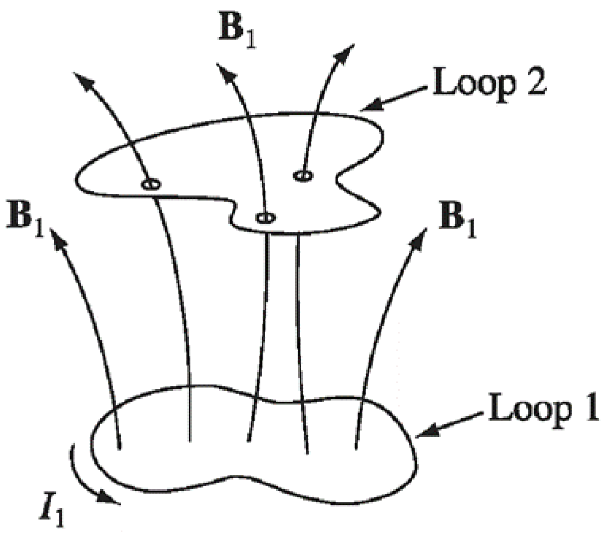


FIGURE 7.30

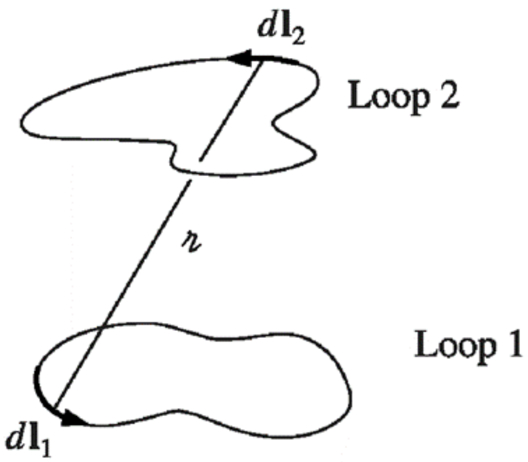


FIGURE 7.31

Example 7.10

A short solenoid (length l and radius a, with n_1 turns per unit length) lies on the axis of a very long solenoid (radius b, n_2 turns per unit length) as shown in Fig 7.32. Current I flows in the short solenoid. What is the flux through the long solenoid?

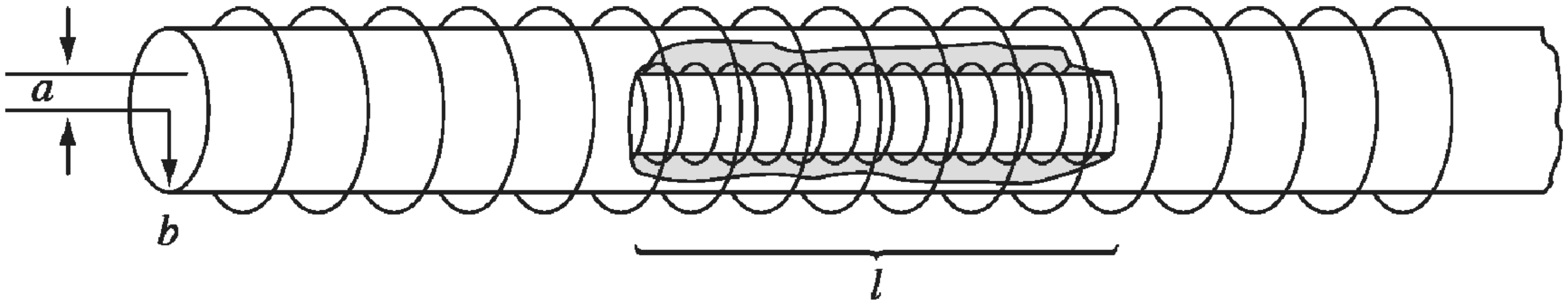


FIGURE 7.32

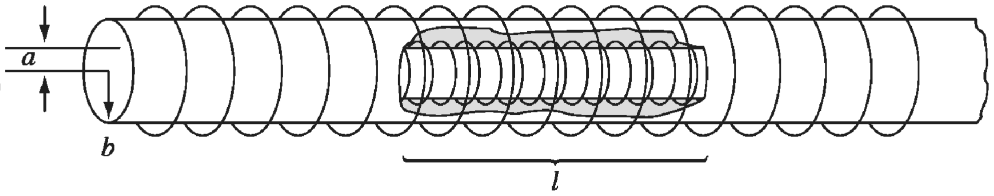


FIGURE 7.32

Solution

Since the inner solenoid is short, it has a very complicated field; moreover, it puts a different flux through each turn of the outer solenoid. It would be a *miserable* task to compute the total flux this way. However, if we exploit the equality of the mutual inductances, the problem becomes very easy. Just look at the reverse situation: run the current I through the *outer* solenoid and calculate the flux through the *inner* one. The field inside the long solenoid is constant:

$$B = \mu_0 n_2 I$$

so the flux through a single loop of the short solenoid is

$$B\pi a^2 = \mu_0 n_2 I \pi a^2$$

There are $n_1 l$ turns in all, so the total flux through the inner solenoid is

$$\Phi = \mu_0 \pi a^2 n_1 n_2 l I \qquad M = \mu_0 \pi a^2 n_1 n_2 l$$

Suppose, now, that you *vary* the current in loop 1, his changing flux will induce an emf in loop 2:

$$\mathcal{E}_2 = -\frac{d\Phi_2}{dt} = -M\frac{dI_1}{dt}$$

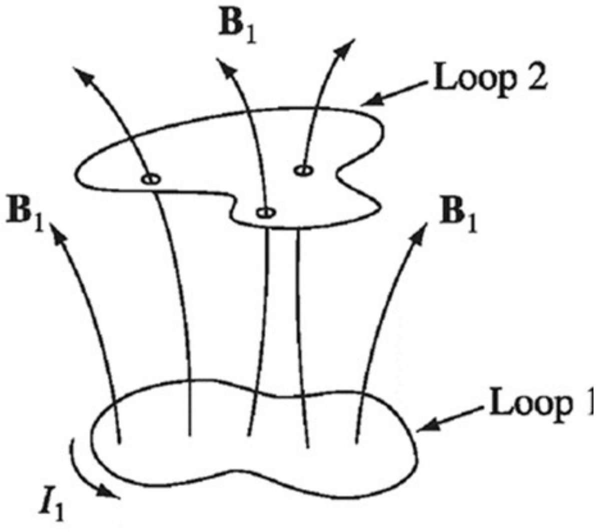


FIGURE 7.30

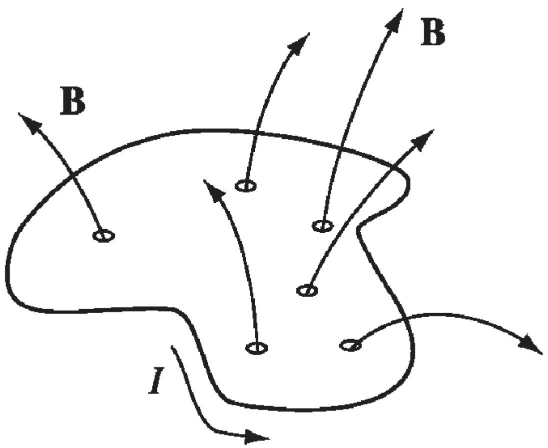


FIGURE 7.33

- Every time you change the current in loop 1, an induced current flows in loop 2---even though there are no wires connecting them! (Assuming that the currents change slowly *quasistatic*)
- It also induces an emf in the source loop *itself*. The constant of proportionality L is called the **self inductance** (or simply the **inductance)** of the loop. As with *M,* it depends on the geometry (size and shape) of the loop

$$\Phi = LI \qquad \mathcal{E} = -L\frac{dI}{dt}$$

henries (H); a henry is a volt-second per ampere

Example 7.11

Find the self-inductance of a toroidal coil with rectangular cross section (inner radius *a,* outer radius *b,* height *h*), that carries a total of *N* turns .

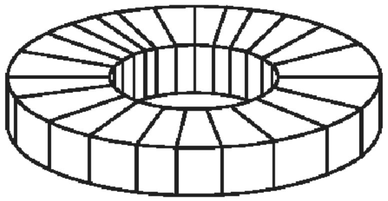


FIGURE 5.38

Solution

The magnetic field inside the toroid is (see Ex. 5.10 for details)

$$B = \frac{\mu_0 N I}{2\pi s}.$$

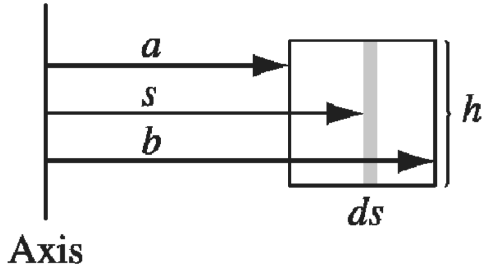


FIGURE 7.34

The flux in a single turn:

$$\int \mathbf{B} \cdot d\mathbf{a} = \frac{\mu_0 N I}{2\pi} h \int_a^b \frac{1}{s} \, ds = \frac{\mu_0 N I h}{2\pi} \ln\left(\frac{b}{a}\right)$$

The *total* flux is *N* times this, so the self-inductance

$$L = \frac{\mu_0 N^2 h}{2\pi} \ln\left(\frac{b}{a}\right)$$

Inductance (like capacitance) is an intrinsically *positive* quantity. that the emf is in such a direction as to *oppose* any *change in current.* it is called a back emf.

Example 7.12

- Suppose a current *I* is flowing around a loop, when someone suddenly cuts the wire. The current drops "instantaneously" to zero. This generates a whopping back emf, for although *I* may be small, dI/dt is enormous.

- That's why you sometimes draw a spark when you unplug an iron or toaster - electromagnetic induction is desperately trying to keep the current going, even if it has to jump the gap in the circuit.

- Nothing so dramatic occurs when you plug *in* a toaster or iron. In this case induction opposes the sudden *increase* in current, prescribing instead a smooth and continuous buildup.

- Suppose, for instance, that a battery (which supplies a constant emf ε_0) is connected to a circuit of resistance R and inductance L (Fig. 7 .35). What current flows?

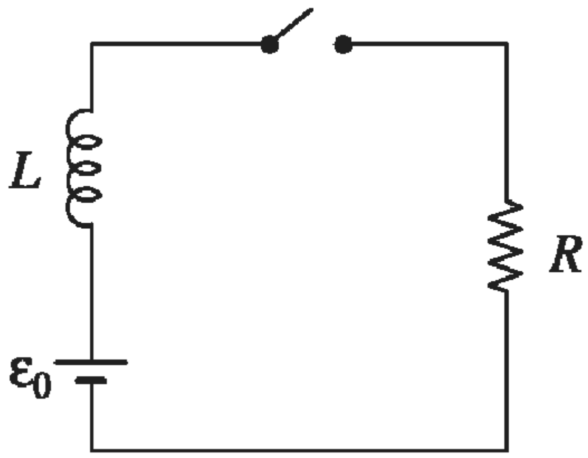


FIGURE 7.35

Solution

The total emf in this circuit is $\mathcal{E}_0$ from the battery plus $-L(^{dI}/_{dt})$ from the inductance. Ohm's law, then, says

$$\mathcal{E}_0 - L\frac{dI}{dt} = IR$$

The general solution

$$I(t) = \frac{\mathcal{E}_0}{R} + ke^{-(R/L)t}$$

where k is a constant to be determined by the initial conditions. In particular, if you close the switch at time $t = 0$, so $I(0) = 0$, then $k = -\varepsilon_0/R$, therefore

$$I(t) = \frac{\mathcal{E}_0}{R}\left[1 - e^{-(R/L)t}\right]$$

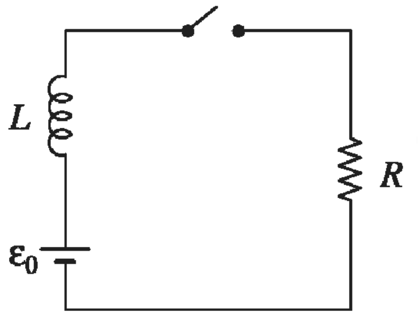


FIGURE 7.35

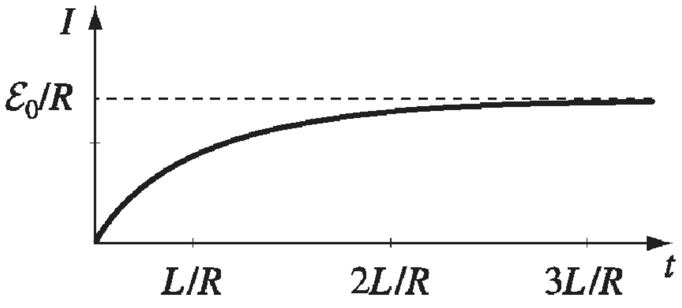


FIGURE 7.36

Note:
Had there been no inductance in the circuit, the current would have jumped immediately to $\frac{\varepsilon_0}{R}$. In practice, *every* circuit has *some* self-inductance, and the current approaches $\frac{\varepsilon_0}{R}$ asymptotically. The quantity $\tau \equiv {}^{L}/_{R}$ is the time constant.

Energy in Magnetic Fields

- It takes a certain amount of energy to start a current flowing in a circuit. Not the energy converted into heat in the resistors.
- We consider the work you must to do against the back emf to get the current going.
- This is a fixed amount, and it is recoverable: you get it back when the current is turned off.
- It can be regarded as energy stored in the magnetic field.

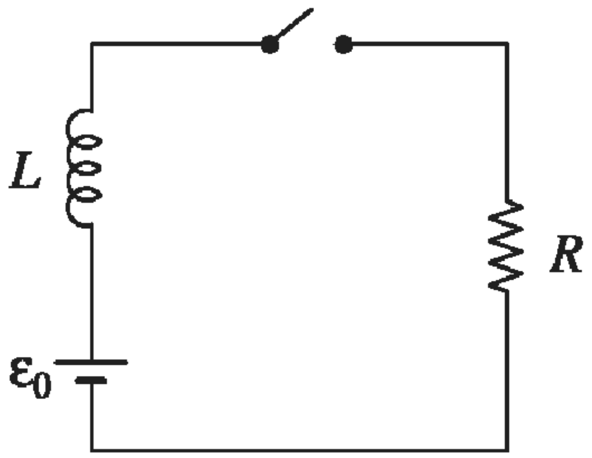


FIGURE 7.35

Energy in Magnetic Fields

The work done on a unit charge, against the back emf, in one trip around the circuit is - $\mathcal{E}_0$ (the minus sign records the fact that this is the work done by you **against** the emf, not the work done by the emf). The total work done per unit time is:

$$\frac{dW}{dt} = -\mathcal{E}I = LI\frac{dI}{dt} \quad \longleftarrow \quad \mathcal{E} = -L\frac{dI}{dt}$$

If we start with zero current and build it up to a final value I

$$W = \frac{1}{2}LI^2$$

There is a nicer way to write W, which has the advantage that it is readily generalized to surface and volume currents:

$$\Phi = LI \quad \Longleftrightarrow \quad \Phi = \int \mathbf{B} \cdot d\mathbf{a} = \int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$$

$$LI = \oint \mathbf{A} \cdot d\mathbf{l}$$

$$W = \frac{1}{2} L I^2 \quad \Longrightarrow \quad W = \frac{1}{2} I \oint \mathbf{A} \cdot d\mathbf{l} = \frac{1}{2} \oint (\mathbf{A} \cdot \mathbf{I})\, dl$$

In this form, the generalization to volume currents is obvious

$$W = \frac{1}{2} \int_{\mathcal{V}} (\mathbf{A} \cdot \mathbf{J})\, d\tau$$

We can do even better:

Ampere's law $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$ $\Longrightarrow$ $W = \dfrac{1}{2\mu_0} \displaystyle\int \mathbf{A} \cdot (\nabla \times \mathbf{B})\, d\tau$

product rule 6 states that $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

$$\mathbf{A} \cdot (\nabla \times \mathbf{B}) = \mathbf{B} \cdot \mathbf{B} - \nabla \cdot (\mathbf{A} \times \mathbf{B})$$

$$W = \frac{1}{2\mu_0}\left[\int B^2\, d\tau - \int \nabla \cdot (\mathbf{A} \times \mathbf{B})\, d\tau\right]$$

$$= \frac{1}{2\mu_0}\left[\int_{\mathcal{V}} B^2\, d\tau - \oint_{\mathcal{S}} (\mathbf{A} \times \mathbf{B}) \cdot d\mathbf{a}\right],$$

where S is the surface bounding the volume ν.

as the surface gets farther from the current, both A and B decrease.
if we agree to integrate over *all* space, then the surface integral goes to zero

$$W = \frac{1}{2\mu_0}\int_{\text{all space}} B^2\, d\tau$$

The energy is stored in the *current distribution,* in the amount $\frac{1}{2}(\mathrm{A} \cdot J)$ or $\frac{B^2}{2\mu_0}$ per unit volume

$$W_{\text{elec}} = \frac{1}{2}\int (V\rho)\, d\tau = \frac{\epsilon_0}{2}\int E^2\, d\tau,$$

$$W_{\text{mag}} = \frac{1}{2}\int (\mathbf{A}\cdot\mathbf{J})\, d\tau = \frac{1}{2\mu_0}\int B^2\, d\tau.$$

Maxwell's Equations

Electrodynamics Before Maxwell

$$
\begin{aligned}
&\text{(i)} && \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho && \text{(Gauss's law),} \\
&\text{(ii)} && \nabla \cdot \mathbf{B} = 0 && \text{(no name),} \\
&\text{(iii)} && \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} && \text{(Faraday's law),} \\
&\text{(iv)} && \nabla \times \mathbf{B} = \mu_0 \mathbf{J} && \text{(Ampère's law).}
\end{aligned}
$$

These equations represent the state of electromagnetic theory in the mid-nineteenth century, when Maxwell began his work.

There is a violation of the rule that divergence of curl is always zero

From (iii)

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{E}) = \boldsymbol{\nabla} \cdot \left(-\frac{\partial \mathbf{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\boldsymbol{\nabla} \cdot \mathbf{B})$$

From (iv)

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{B}) = \mu_0(\boldsymbol{\nabla} \cdot \mathbf{J})$$

The left side of second equation must be zero, but the right side, in general, is not zero

How Maxwell Fixed Ampere's Law

Applying the continuity Eq. (5.29) and Gauss's law, the offending term can be rewritten

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}(\epsilon_0 \nabla \cdot \mathbf{E}) = -\nabla \cdot \left(\epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right)$$

If we were to combine $\epsilon_0(\partial E/\partial t)$ with J, in Ampere's law, it would be just right to kill off the extra divergence:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Such a modification changes nothing, as far as magnetostatics is concerned: when E is constant, we still have $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$. In fact, Maxwell's term is hard to detect in ordinary electromagnetic experiments, where it must compete for attention with J - that's why Faraday and the others never discovered it in the laboratory.
However, it plays a crucial role in the propagation of electromagnetic waves, as we'll see in Chapter 9.

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

A changing electric field induces a magnetic field.

Maxwell called his extra term the displacement current:

$$\mathbf{J}_d \equiv \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Maxwell's Equations

(i) $\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho$ (Gauss's law),

(ii) $\nabla \cdot \mathbf{B} = 0$ (no name),

(iii) $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ (Faraday's law),

(iv) $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$ (Ampère's law with Maxwell's correction).

Together with the force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

They summarize the entire theoretical content of classical electrodynamics.
Even the continuity equation,

$$\boldsymbol{\nabla} \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$$

which is the mathematical expression of conservation of charge, can be derived from Maxwell's equations by applying the divergence to number(iv).

Maxwell's equations in the traditional way, which emphasizes that they specify the divergence and curl of ***E*** and ***B***.
It is logically preferable to write

$$\left.\begin{array}{llll} \text{(i)} & \nabla\cdot\mathbf{E}=\dfrac{1}{\epsilon_0}\rho, & \text{(iii)} & \nabla\times\mathbf{E}+\dfrac{\partial\mathbf{B}}{\partial t}=\mathbf{0}, \\[2ex] \text{(ii)} & \nabla\cdot\mathbf{B}=0, & \text{(iv)} & \nabla\times\mathbf{B}-\mu_0\epsilon_0\dfrac{\partial\mathbf{E}}{\partial t}=\mu_0\mathbf{J}, \end{array}\right\}$$

with the fields (***E*** and ***B***) on the left and the sources (ρ and ***J***) on the right. This notation emphasizes that all electromagnetic fields are ultimately attributable to charges and currents. Maxwell's equations tell you how charges produce fields; reciprocally, the force law tells you how fields affect charges.

Magnetic Charge

If we had

$$\text{(i)}\ \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho_e, \qquad \text{(iii)}\ \nabla \times \mathbf{E} = -\mu_0 \mathbf{J}_m - \frac{\partial \mathbf{B}}{\partial t},$$

$$\text{(ii)}\ \nabla \cdot \mathbf{B} = \mu_0 \rho_m, \qquad \text{(iv)}\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J}_e + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}.$$

Then ρ_m would represent the density of magnetic "charge," and ρ_e the density of electric charge; J_m would be the current of magnetic charge, and J_e the current of electric charge. Both charges would be conserved:

$$\nabla \cdot \mathbf{J}_m = -\frac{\partial \rho_m}{\partial t}, \quad \text{and} \quad \nabla \cdot \mathbf{J}_e = -\frac{\partial \rho_e}{\partial t}$$

The former follows by application of the divergence to (iii), the latter by taking the divergence of (iv).

Magnetic Charge

If we had

$$\left.\begin{aligned} &\text{(i)}\ \nabla\cdot\mathbf{E}=\frac{1}{\epsilon_0}\rho_e, &&\text{(iii)}\ \nabla\times\mathbf{E}=-\mu_0\mathbf{J}_m-\frac{\partial\mathbf{B}}{\partial t},\\ &\text{(ii)}\ \nabla\cdot\mathbf{B}=\mu_0\rho_m, &&\text{(iv)}\ \nabla\times\mathbf{B}=\mu_0\mathbf{J}_e+\mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}. \end{aligned}\right\}$$

Then ρ_m would represent the density of magnetic "charge," and ρ_e the density of electric charge; J_m would be the current of magnetic charge, and J_e the current of electric charge. Both charges would be conserved:

In a sense, Maxwell's equations *beg* for magnetic charge to exist-it would fit in so nicely. And yet, in spite of a diligent search, no one has ever found any. As far as we know, ρ_m is zero everywhere, and so is $\boldsymbol{J}_m$; $\boldsymbol{B}$ is *not* on equal footing with $\boldsymbol{E}$: there exist stationary sources for $\boldsymbol{E}$ (electric charges) but none for $\boldsymbol{B}$. (This is reflected in the fact that magnetic multi pole expansions have no monopole term, and magnetic dipoles consist of current loops, not separated north and south "poles.")

Maxwell's Equations in Matter

We have already learned, from the static case, that an electric polarization **P** produces a bound charge density

$$\rho_b = -\nabla \cdot \mathbf{P}$$

Likewise, a magnetic polarization (or "magnetization") **M** results in a bound current

$$\mathbf{J}_b = \nabla \times \mathbf{M}$$

There's just one new feature to consider in the nonstatic case: any change in the electric polarization involves a flow of (bound) charge (call it $\boldsymbol{J}_p$) included in the total current.

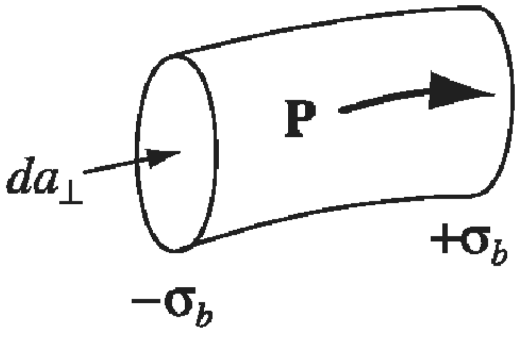


FIGURE 7.47

Suppose we examine a tiny chunk of polarized material. The polarization introduces a surface charge density $\boldsymbol{\sigma}_b = \boldsymbol{P}$ one end and -$\boldsymbol{\sigma}_b$ at the other.

If ***P*** increases a bit, there will be a net current

$$dI = \frac{\partial \sigma_b}{\partial t} da_\perp = \frac{\partial P}{\partial t} da_\perp . \quad \Longrightarrow \quad \mathbf{J}_p = \frac{\partial \mathbf{P}}{\partial t} \qquad \text{(Eq. 7.49)}$$

This polarization current J_p is the result of the linear motion of charge when the electric polarization changes.

Eq. 7.49 is consistent with the continuity equation

$$\nabla \cdot \mathbf{J}_p = \nabla \cdot \frac{\partial \mathbf{P}}{\partial t} = \frac{\partial}{\partial t}(\nabla \cdot \mathbf{P}) = -\frac{\partial \rho_b}{\partial t}$$

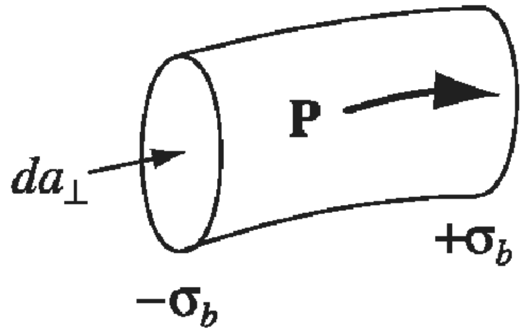


FIGURE 7.47

The total charge density can be separated into two parts

$$\rho = \rho_f + \rho_b = \rho_f - \nabla \cdot \mathbf{P}$$

Gauss's law becomes $$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}(\rho_f - \nabla \cdot \mathbf{P})$$

or $$\nabla \cdot \mathbf{D} = \rho_f$$

where, as in the static case $$\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P}$$

The current density: $$\mathbf{J} = \mathbf{J}_f + \mathbf{J}_b + \mathbf{J}_p = \mathbf{J}_f + \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t}$$

Ampere's law (with Maxwell's term) becomes

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J}_f + \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t} \right) + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

or $$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$$

$$\mathbf{H} \equiv \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}.$$

Faraday's law and $\nabla \cdot \boldsymbol{B} = 0$ are not affected by our separation of charge and current into free and bound parts, since they do not involve ρ or $\boldsymbol{J}$

$$\text{(i)} \quad \nabla \cdot \mathbf{D} = \rho_f, \qquad \text{(iii)} \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\text{(ii)} \quad \nabla \cdot \mathbf{B} = 0, \qquad \text{(iv)} \quad \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}.$$

$$\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P}$$

$$\mathbf{H} \equiv \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}.$$

A convenient division of charge and current into free and non free parts they have the disadvantage of hybrid notation, since they contain both $\boldsymbol{E}$ and $\boldsymbol{D}$, both $\boldsymbol{B}$ and $\boldsymbol{H}$.

For linear media $\quad \mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \quad \text{and} \quad \mathbf{M} = \chi_m \mathbf{H},$

$\epsilon \equiv \epsilon_0\,(1+\chi_e)$
$\mu \equiv \mu_0(1 + \chi_m)$

$$\mathbf{D} = \epsilon \mathbf{E}, \quad \text{and} \quad \mathbf{H} = \frac{1}{\mu}\mathbf{B},$$

Displacement current.

$$\mathbf{J}_d \equiv \frac{\partial \mathbf{D}}{\partial t}$$

$$\text{(i)}\ \nabla \cdot \mathbf{D} = \rho_f, \qquad \text{(iii)}\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\text{(ii)}\ \nabla \cdot \mathbf{B} = 0, \qquad \text{(iv)}\ \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}.$$

Boundary Conditions

In general, the fields ***E*, *B***, ***D***, and ***H*** will be discontinuous at a boundary between two different media, or at a surface that carries a charge density or a current density. The explicit form of these discontinuities can be deduced from Maxwell's equations in their integral form

$$\left.\begin{array}{ll} \text{(i)} & \oint_{\mathcal{S}} \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}} \\ \text{(ii)} & \oint_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{a} = 0 \end{array}\right\} \text{ over any closed surface } \mathcal{S}.$$

$$\left.\begin{array}{ll} \text{(iii)} & \oint_{\mathcal{P}} \mathbf{E} \cdot d\mathbf{l} = -\dfrac{d}{dt}\displaystyle\int_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{a} \\ \text{(iv)} & \oint_{\mathcal{P}} \mathbf{H} \cdot d\mathbf{l} = I_{f_{\text{enc}}} + \dfrac{d}{dt}\displaystyle\int_{\mathcal{S}} \mathbf{D} \cdot d\mathbf{a} \end{array}\right\} \begin{array}{l} \text{for any surface } \mathcal{S} \\ \text{bounded by the} \\ \text{closed loop } \mathcal{P}. \end{array}$$

Applying (i) to a tiny, wafer-thin Gaussian pillbox extending just slightly into the material on either side of the boundary (Fig. 7.48), we obtain:

$$\mathbf{D}_1 \cdot \mathbf{a} - \mathbf{D}_2 \cdot \mathbf{a} = \sigma_f \, a$$

The positive direction for ***a*** is from 2 toward l. The edge of the wafer contributes nothing in the limit as the thickness goes to zero; nor does any volume charge density.) Thus, the component of D that is perpendicular to the interface is discontinuous in the amount

$$D_1^{\perp} - D_2^{\perp} = \sigma_f$$

Identical reasoning, applied to equation (ii), yields

$$B_1^{\perp} - B_2^{\perp} = 0$$

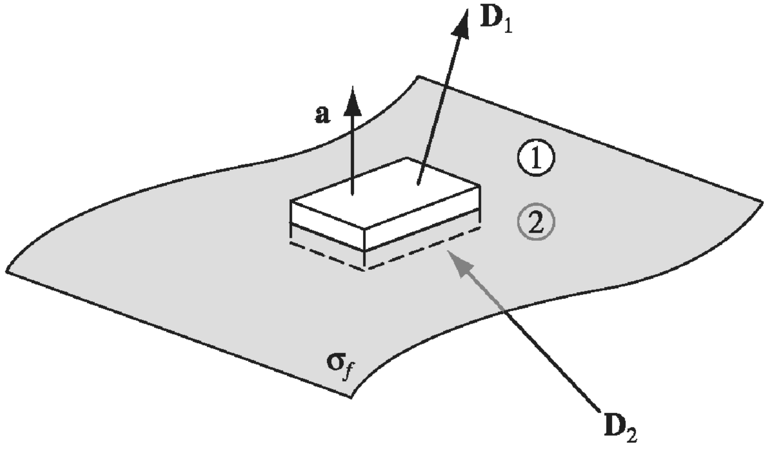


FIGURE 7.48

Turning to (iii), a very thin Amperian loop straddling the surface gives

$$\mathbf{E}_1 \cdot \mathbf{l} - \mathbf{E}_2 \cdot \mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{a}$$

But in the limit as the width of the loop goes to zero, the flux vanishes

$$\mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = \mathbf{0}$$

By the same token, (iv) implies

$$\mathbf{H}_1 \cdot \mathbf{l} - \mathbf{H}_2 \cdot \mathbf{l} = I_{f_{\mathrm{enc}}}$$

where I_{fenc} is the free current passing through the Amperian loop. No volume current density will contribute

$$I_{f_{\mathrm{enc}}} = \mathbf{K}_f \cdot (\hat{\mathbf{n}} \times \mathbf{l}) = (\mathbf{K}_f \times \hat{\mathbf{n}}) \cdot \mathbf{l}$$

$$\mathbf{H}_1^{\parallel} - \mathbf{H}_2^{\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}}$$

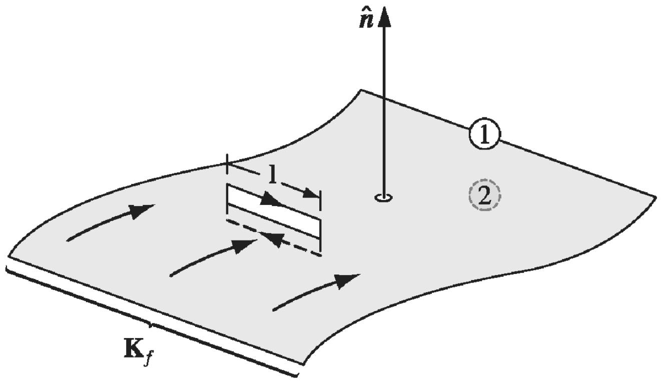


FIGURE 7.49

The general boundary conditions for electrodynamics in the case of linear media

$$\left.\begin{array}{ll}\text{(i)}\ \ \epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f, & \text{(iii)}\ \ \mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = \mathbf{0}, \\ \text{(ii)}\ \ B_1^{\perp} - B_2^{\perp} = 0, & \text{(iv)}\ \ \dfrac{1}{\mu_1}\mathbf{B}_1^{\parallel} - \dfrac{1}{\mu_2}\mathbf{B}_2^{\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}}.\end{array}\right\}$$

In particular, if there is no free charge or free current at the interface, then

$$\begin{array}{ll}\text{(i)}\ \ \epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = 0, & \text{(iii)}\ \ \mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = \mathbf{0}, \\ \text{(ii)}\ \ B_1^{\perp} - B_2^{\perp} = 0, & \text{(iv)}\ \ \dfrac{1}{\mu_1}\mathbf{B}_1^{\parallel} - \dfrac{1}{\mu_2}\mathbf{B}_2^{\parallel} = \mathbf{0}.\end{array}$$