Chapter 3

Potential

- 3.1 Laplace's Equation
- 3.2 The Method of Images
- 3.3 Separation of Variables
- 3.4 Multipole Expansion

Laplace's Equation

The primary task of electrostatics is to find the electric field of a given stationary charge distribution.

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\rho(\mathbf{r}')\,d\tau'$$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}}\rho(\mathbf{r}')\,d\tau'$$

Easier

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho$$

Poisson's equation reduces to Laplace's equation:

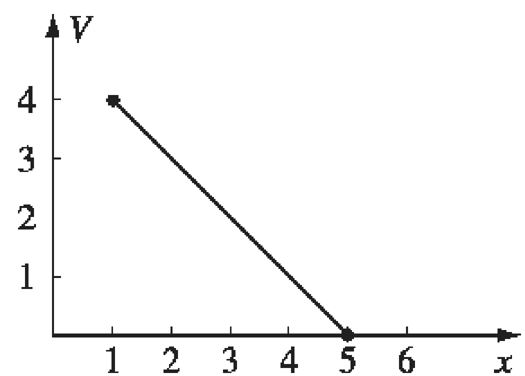
$$\nabla^2 V = 0$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Laplace's Equation in one Dimension

Use one dimension to show some basic properties:

$$\frac{d^2V}{dx^2}=0 \qquad V(x) = mx + b$$



1. V(x) is the average of V(x+a) and V(x-a)

2. Laplace's equation tolerates no local maxima or minima.

First uniqueness theorem :

The solution to Laplace's equation in some volume ν is uniquely determined if V is specified on the boundary surface S

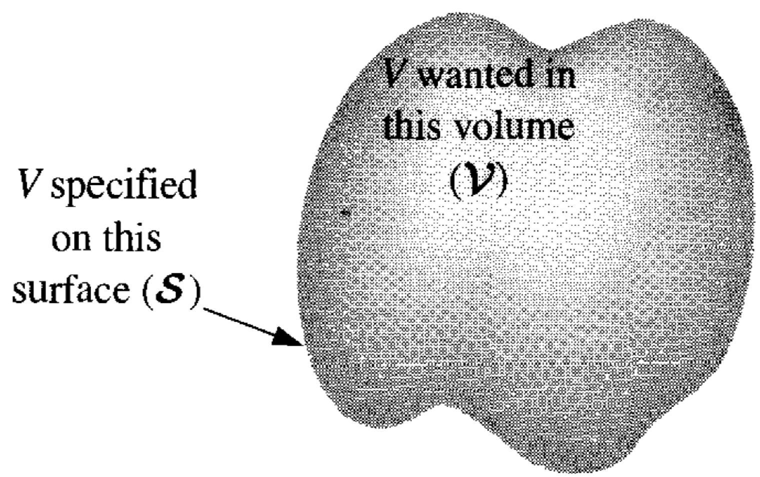


Figure 3.5

Second uniqueness theorem :

In a volume surrounded by conductors and containing a specified charge density, the electric field is uniquely determined if the total charge on each conductor is given .

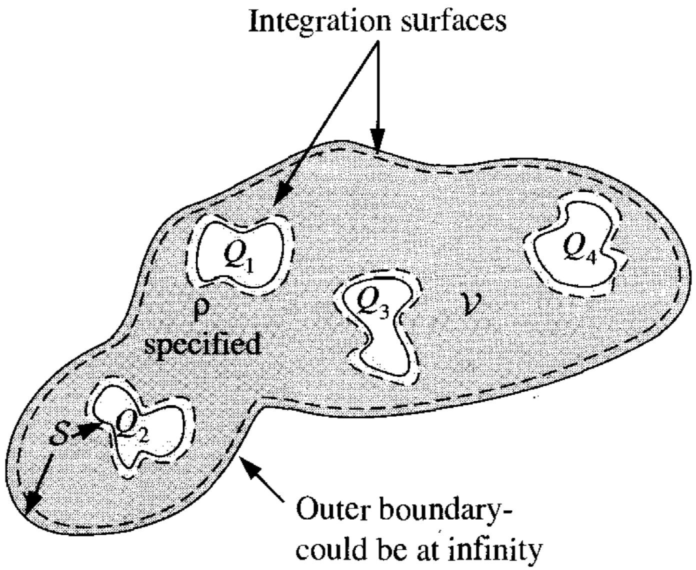
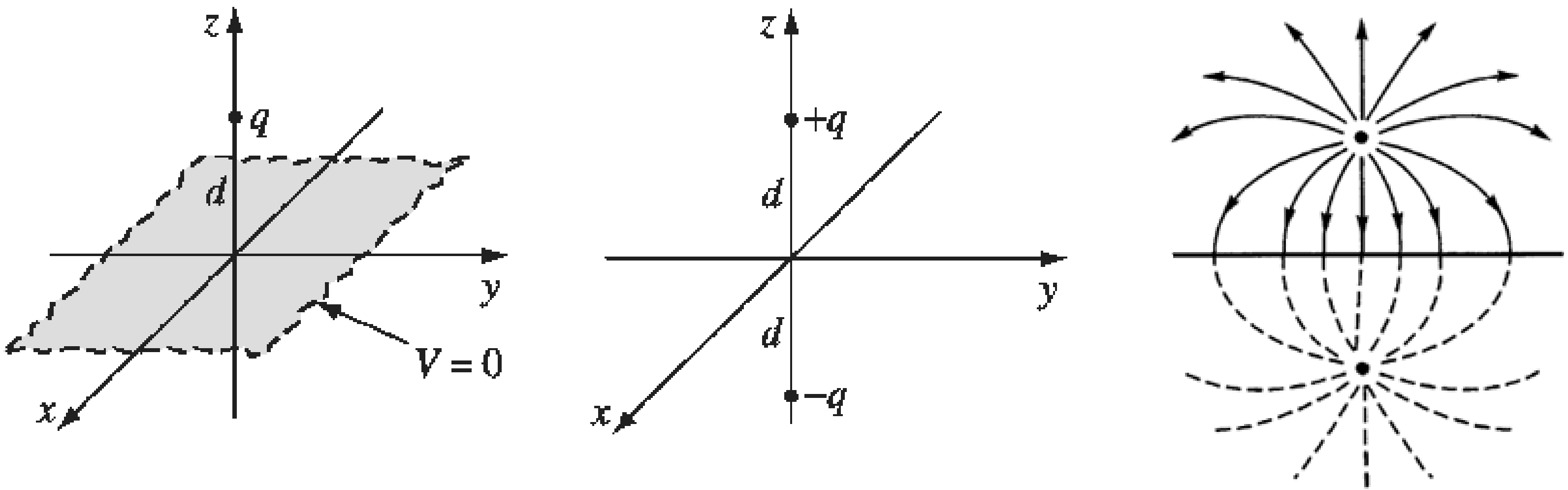


Figure 3.6

The Classic Image Problem

Suppose a point charge is held a distance d above an infinite grounded conducting plane, what the potential in the region above the plane?

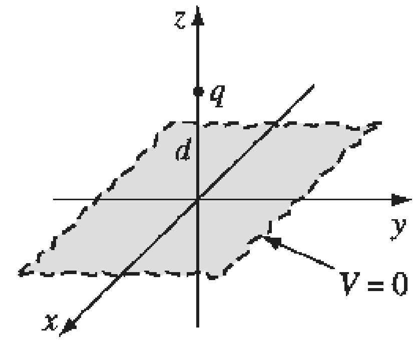


1. V=0 when z=0 (since the conducting plane is grounded)
2. $V \rightarrow 0$ far from the charge (that is for $x^2 + y^2 + z^2 \gg d^2$)

$$V(x, y, z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2 + y^2 + (z-d)^2}} - \frac{q}{\sqrt{x^2 + y^2 + (z+d)^2}}\right]. \qquad (3.9)$$

Induced Surface Charge

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial n}$$



$$\sigma = -\epsilon_0 \frac{\partial V}{\partial z}\bigg|_{z=0}$$

$$\frac{\partial V}{\partial z} = \frac{1}{4\pi\epsilon_0}\left\{\frac{-q(z-d)}{[x^2+y^2+(z-d)^2]^{3/2}} + \frac{q(z+d)}{[x^2+y^2+(z+d)^2]^{3/2}}\right\}$$

$$\sigma(r) = \frac{-qd}{2\pi(r^2+d^2)^{3/2}}$$

$$Q = \int \sigma\, da$$

$$Q = \int_0^{2\pi}\int_0^{\infty} \frac{-qd}{2\pi(r^2+d^2)^{3/2}} r\, dr\, d\phi = \frac{qd}{\sqrt{r^2+d^2}}\bigg|_0^{\infty} = -q$$

Force and Energy

$$\mathbf{F} = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{(2d)^2}\hat{\mathbf{z}}$$

Same as the "image"

$$W = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{2d}$$

Not the same!

$$W = \frac{\epsilon_0}{2}\int E^2\,d\tau$$

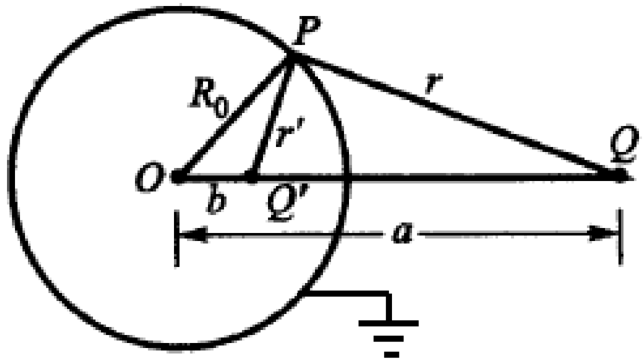
Note: only the upper region contains a nonzero field

one could also determine the energy by calculating the work required to bring q in from infinity

$$\begin{aligned} W &= \int_\infty^d \mathbf{F}\cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0}\int_\infty^d \frac{q^2}{4z^2}\,dz \\ &= \frac{1}{4\pi\epsilon_0}\left(-\frac{q^2}{4z}\right)\Bigg|_\infty^d = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{4d} \end{aligned}$$

Other Image Problem

Suppose a point charge Q is situated a distance a from the center of a grounded conducting sphere of radius R0. Find the potential outside the sphere.



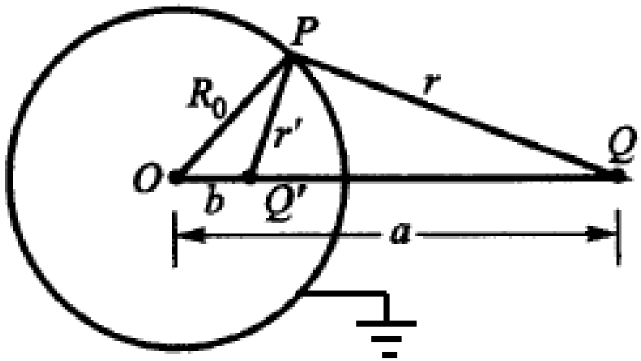
$$\varphi = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{r} + \frac{Q'}{r'}\right]$$

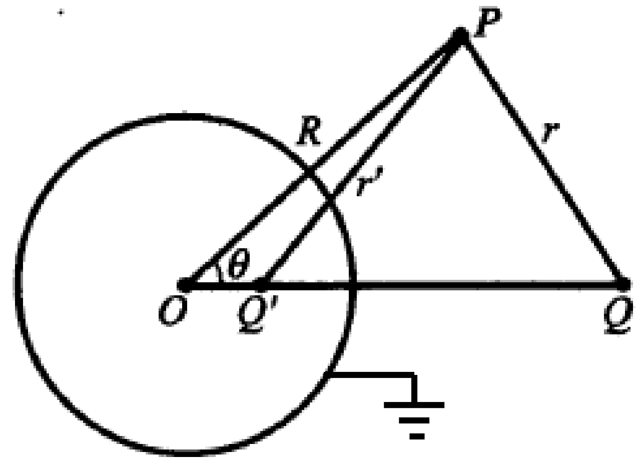
$$\varphi\Big|_{R_0} = 0 \rightarrow \left.\frac{Q}{r} + \frac{Q'}{r'}\right|_{R=R_0} = 0 \rightarrow \left.\frac{Q^2}{r^2}\right|_{R=R_0} = \left.\frac{Q'^2}{r'^2}\right|_{R=R_0}$$

$$r' = \sqrt{R^2 + b^2 - 2Rb\cos\theta}$$

$$r = \sqrt{R^2 + a^2 - 2Ra\cos\theta}$$

cosθ=1
cosθ=-1
cosθ=0





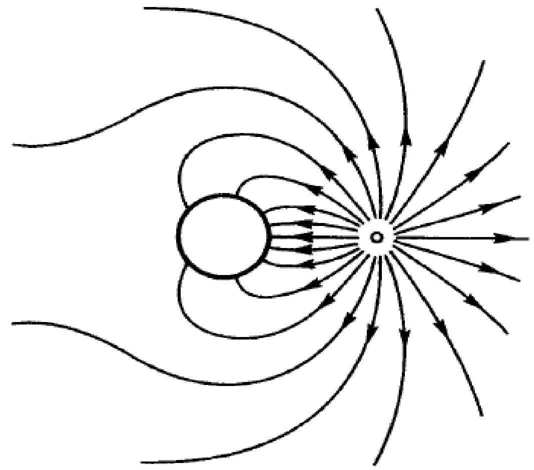
$$\varphi=\frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{r}+\frac{Q'}{r'}\right]\qquad Q'=-\frac{R_0Q}{a}\qquad b=\frac{R_0^2}{a}$$

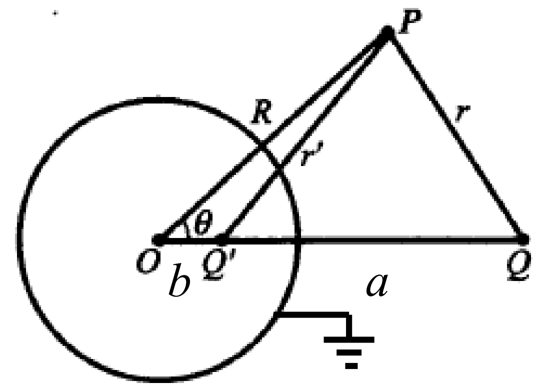
$$\begin{cases}\varphi=\dfrac{1}{4\pi\varepsilon_0}\left(\dfrac{1}{\sqrt{R^2+a^2-2Ra\cos\theta}}-\dfrac{\frac{R_0}{a}}{\sqrt{R^2+b^2-2Rb\cos\theta}}\right)(R>R_0)\\ \varphi=0\quad (R\le R_0)\end{cases}$$

Some discussions:

$$Q' = -\frac{R_0}{a} Q$$

$$|Q'| < |Q|$$





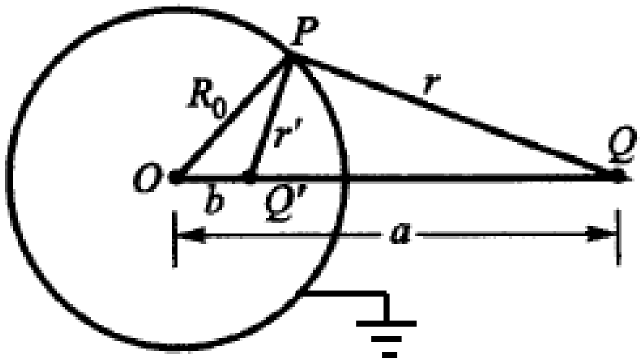
$$\sigma = -\varepsilon_0 \frac{\partial \phi}{\partial R}\bigg|_{R=R_0} = -\frac{Q}{4\pi} \frac{a^2 - R_0^2}{R_0 (a^2 - R_0^2 - 2R_0 a \cos\theta)^{3/2}}$$

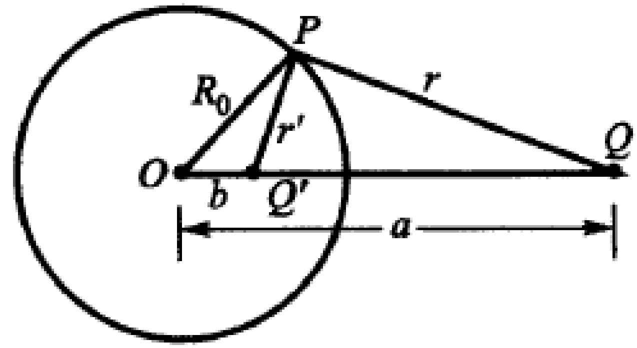
$$\oint_{R=R_0} \sigma \mathrm{d}S = \oint_{R=R_0} \sigma R_0^2 \sin\theta \, \mathrm{d}\theta \mathrm{d}\phi = -\frac{R_0 Q}{a}$$

Charge distributed only on the surface

Surface charge

What if the conducting sphere is not connected to the ground?





$$\phi|_{R=R_0} \text{ is a constant (unknown)}$$

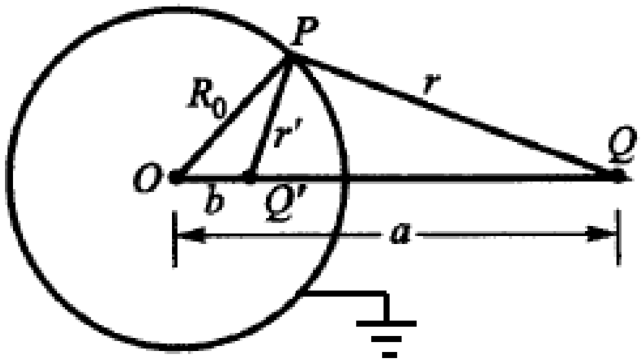
To keep the sphere neutral we need another charge inside the sphere:

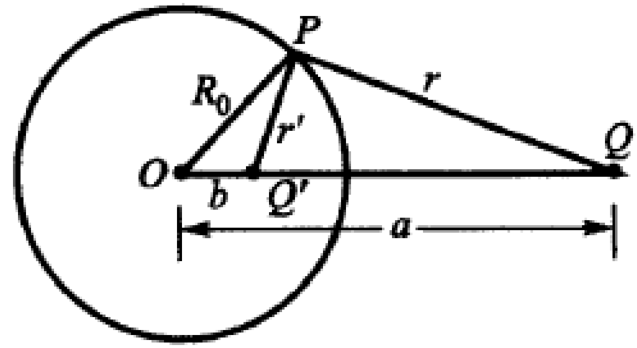
$$Q' = -\frac{R_0 Q}{a} \qquad\qquad Q'' = \frac{R_0}{a} Q$$

From the symmetry, the extra charge can be placed at the center of the sphere.

$$\phi_1 = \phi + \frac{Q''}{4\pi\varepsilon_0 R}$$

What if the conducting sphere contains uniformly distributed charge Q_0 ?





$$\phi_2 = \phi + \frac{Q''}{4\pi\varepsilon_0 R} + \frac{Q_0}{4\pi\varepsilon_0 R}$$

Separation of Variables

- Separation of variables is the physicist's favorite tool for solving partial differential equations.

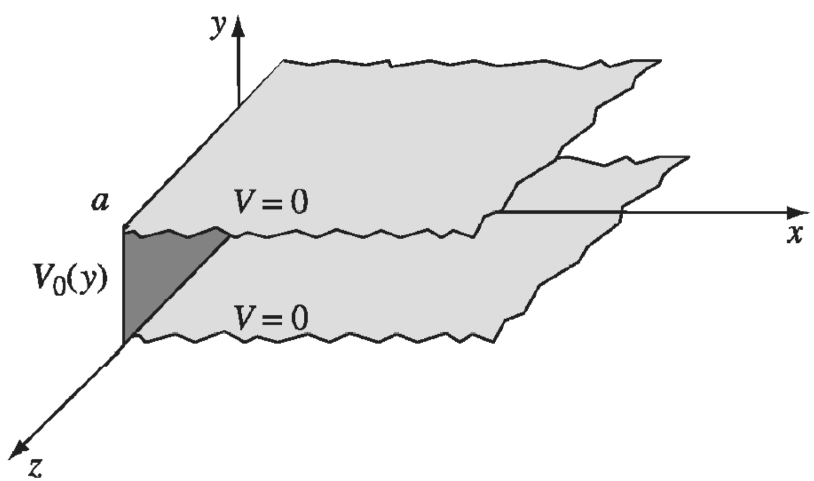
- The method is applicable in circumstances where the potential (V) or the charge density (σ) is specified on the boundaries of some region, and we are asked to find the potential in the interior.

- The basic strategy is very simple: We look for solutions that are products of functions, each of which depends on only one of the coordinates.

Cartesian Coordinates

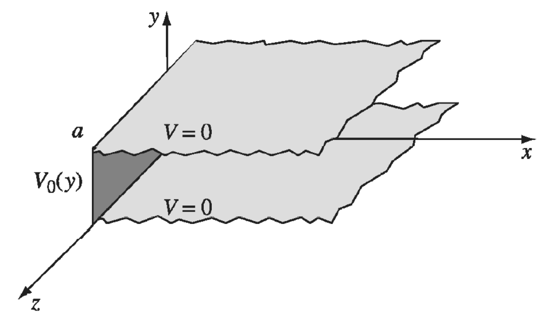
Example

Two infinite grounded metal plates lie parallel to the xz plane, one at y = 0, the other at y =a. The left end, at x = 0, is closed off with an infinite strip insulated from the two plates, and maintained at a specific potential $V_0(y)$. Find the potential inside this "slot."



The configuration is independent of z, so this is really a two-dimensional problem. In mathematical terms, we must solve Laplace's equation.

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0,$$



subject to the boundary conditions:

$$\left.\begin{array}{ll} \text{(i)} & V = 0 \text{ when } y = 0, \\ \text{(ii)} & V = 0 \text{ when } y = a, \\ \text{(iii)} & V = V_0(y) \text{ when } x = 0, \\ \text{(iv)} & V \to 0 \text{ as } x \to \infty. \end{array}\right\}$$

Try a solution like:

$$V(x, y) = X(x)Y(y) \qquad Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0$$

$$\frac{1}{X}\frac{d^2X}{dx^2} = C_1 \quad \text{and} \quad \frac{1}{Y}\frac{d^2Y}{dy^2} = C_2, \quad \text{with} \quad C_1 + C_2 = 0.$$

$$\frac{d^2X}{dx^2} = k^2X, \qquad \frac{d^2Y}{dy^2} = -k^2Y.$$

$$X(x) = Ae^{kx} + Be^{-kx}, \qquad Y(y) = C\sin ky + D\cos ky,$$

$$V(x, y) = (Ae^{kx} + Be^{-kx})(C\sin ky + D\cos ky)$$

$$V(x, y) = (Ae^{kx} + Be^{-kx})(C \sin ky + D \cos ky)$$

(iv) requires that A equal zero

$$V(x, y) = e^{-kx}(C \sin ky + D \cos ky)$$

Condition (i) now demands that D equal zero, so

$$V(x, y) = Ce^{-kx} \sin ky.$$

$$\left.\begin{array}{ll} \text{(i)} & V = 0 \text{ when } y = 0, \\ \text{(ii)} & V = 0 \text{ when } y = a, \\ \text{(iii)} & V = V_0(y) \text{ when } x = 0, \\ \text{(iv)} & V \to 0 \text{ as } x \to \infty. \end{array}\right\}$$

Meanwhile (ii) yields $\sin ka = 0$, from which it follows that

$$k = \frac{n\pi}{a}, \qquad (n = 1, 2, 3, \ldots).$$

Laplace Equation is linear:

$$V(x, y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a} \sin(n\pi y/a)$$

$$V(x, y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a} \sin(n\pi y/a)$$

$$\left.\begin{array}{ll} \text{(i)} & V = 0 \text{ when } y = 0, \\ \text{(ii)} & V = 0 \text{ when } y = a, \\ \text{(iii)} & V = V_0(y) \text{ when } x = 0, \\ \text{(iv)} & V \to 0 \text{ as } x \to \infty. \end{array}\right\}$$

$$V(0, y) = \sum_{n=1}^{\infty} C_n \sin(n\pi y/a) = V_0(y)$$

$$\sum_{n=1}^{\infty} C_n \int_0^a \sin(n\pi y/a)\ \sin(n'\pi y/a)\, dy = \int_0^a V_0(y) \sin(n'\pi y/a)\, dy$$

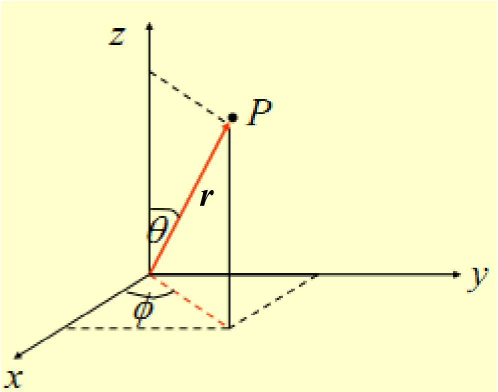
$$\int_0^a \sin(n\pi y/a)\ \sin(n'\pi y/a)\, dy = \begin{cases} 0, & \text{if } n' \neq n, \\ \dfrac{a}{2}, & \text{if } n' = n. \end{cases}$$

$$C_n = \frac{2}{a} \int_0^a V_0(y) \sin(n\pi y/a)\, dy$$

$$V(x, y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a} \sin(n\pi y/a)$$

Spherical Coordinates

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$



$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\varphi(\vec{x}) = 0$$

Introduce $\varphi(\vec{x}) = R(r)Y(\theta,\phi)$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = -\frac{1}{Y}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right] = l\left(l+1\right)$$

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}r}(r^2\dfrac{\mathrm{d}R}{\mathrm{d}r}) - l(l+1)R = 0 \\ \dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\theta}(\sin\theta\dfrac{\partial Y}{\partial\theta}) + \dfrac{1}{\sin^2\theta}\dfrac{\partial^2 Y}{\partial\phi^2} + l(l+1)Y = 0 \end{cases}$$

$$Y(\theta,\phi) = \Theta(\theta)\Phi(\phi)$$

$$\Big\downarrow \ \frac{\sin^2\theta}{\Theta\Phi}$$

$$\frac{\sin\theta}{\Theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) + l(l+1)\sin^2\theta = -\frac{1}{\Phi}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2} = m^2$$

$$\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) + \left(l(l+1)\sin^2\theta - m^2\right)\Theta = 0$$

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2} + m^2\Phi = 0$$

$$x=\cos\theta \qquad \theta=\arccos x,\ \ \Theta(\theta)=\Theta(x)$$

$$\frac{d}{d\theta}=\frac{dx}{d\theta}\frac{d}{dx}=-\sin\theta\frac{d}{dx}$$

$$\sin\theta\frac{d}{d\theta}(\sin\theta\frac{d}{d\theta})=\sin^2\theta\frac{d}{dx}(\sin^2\theta\frac{d}{dx})=(1-x^2)\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\right]$$

$$=(1-x^2)\left[(1-x^2)\frac{d^2}{dx^2}-2x\frac{d}{dx}\right]$$

Laplace Equation becomes:

$$\begin{cases}\frac{d}{dr}(r^2\frac{dR}{dr})-l(l+1)R=0 \Rightarrow \boxed{r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}-l(l+1)R=0} & \text{Euler Equation}\\ \boxed{(1-x^2)\frac{d^2\Theta}{dx^2}-2x\frac{d\Theta}{dx}+\left[l(l+1)-\frac{m^2}{1-x^2}\right]\Theta=0} & \text{Legendre Equation}\\ \frac{d^2\Phi}{d\phi^2}+m^2\Phi=0 \end{cases}$$

The solution of the Legendre Equations is：

$$\Theta(x) = P_l^m(x) = \left(1-x^2\right)^{\frac{m}{2}} \frac{\mathrm{d}^m P_l\left(x\right)}{\mathrm{d}x^m} = \left(1-x^2\right)^{\frac{m}{2}} P_l^{(m)}(x)$$

associated Legendre polynomial

ordinary Legendre polynomials

$$P_0\left(x\right) = 1$$

$$P_1\left(x\right) = x$$

$$\vdots$$

$$P_l\left(x\right) = \frac{1}{2^l l!} \frac{\mathrm{d}^l}{\mathrm{d}x^l}\left(x^2-1\right)^l, \quad l = 0,1,2,\cdots$$

$$\int_{-1}^{1} P_l^m(x) P_k^m(x) dx = \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!} \delta_{lk}$$

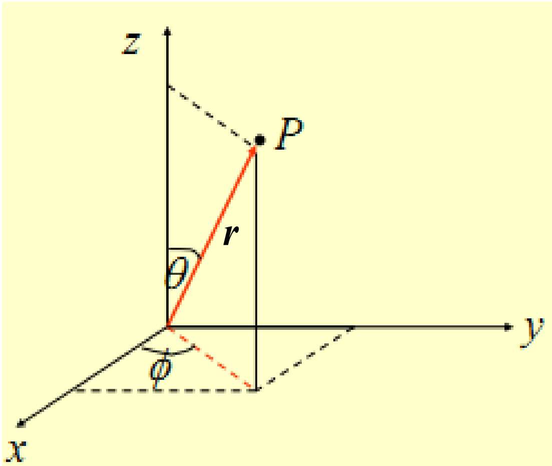
The solution of the Euler Equation is

$$R(r) = a_l r^l + \frac{b_l}{r^{l+1}}$$

$$\Phi(\phi) = Ae^{im\phi}, m = -l, \cdots, 0, \cdots, l$$

$$\boxed{\nabla^2\varphi = 0} \Leftarrow \boxed{\varphi(r,\theta,\phi) = \sum_{l,m}\left(a_{lm}r^l + \frac{b_{lm}}{r^{l+1}}\right)P_l^m(\cos\theta)e^{im\phi}}$$

$$\varphi(r,\theta,\phi)$$
$$= \sum_{l,m}\left(a_{lm}r^l + \frac{b_{lm}}{r^{l+1}}\right)P_l^m(\cos\theta)\cos(m\phi) + \sum_{l,m}\left(c_{lm}r^l + \frac{d_{lm}}{r^{l+1}}\right)P_l^m(\cos\theta)\sin(m\phi)$$



Example (3.9)

A specified charge density $\sigma_0(\theta)$ is glued over the surface of a spherical shell of radius R. Find the resulting potential inside and outside the sphere .

Solution

You could, of course, do this by direct integration:

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma_0}{\mathscr{r}}\, da,$$

but separation of variables is often easier. For the interior region, we have

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \qquad (r \le R) \tag{3.78}$$

In the exterior region we have

$$V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) \qquad (r \geq R) \tag{3.79}$$

These two functions must be joined together by the appropriate boundary conditions at the surface itself. First, the potential is continuous at $r = R$ (Eq. 2.34):

$$\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta). \tag{3.80}$$

It follows that the coefficients of like Legendre polynomials are equal:

$$B_l = A_l R^{2l+1}. \tag{3.81}$$

(To prove that formally, multiply both sides of Eq. 3.80 by $P_{l'}(\cos\theta)\sin\theta$ and integrate from 0 to π, using the orthogonality relation 3.68.) Second, the radial derivative of V suffers a discontinuity at the surface (Eq. 2.36):

$$\left(\frac{\partial V_{\text{out}}}{\partial r} - \frac{\partial V_{\text{in}}}{\partial r}\right)\bigg|_{r=R} = -\frac{1}{\epsilon_0}\sigma_0(\theta). \tag{3.82}$$

Thus $$-\sum_{l=0}^{\infty}(l+1)\frac{B_l}{R^{l+2}}P_l(\cos\theta) - \sum_{l=0}^{\infty} l A_l R^{l-1} P_l(\cos\theta) = -\frac{1}{\epsilon_0}\sigma_0(\theta)$$

or, using $Eq. 3.81$:

$$B_l = A_l R^{2l+1}. \qquad \sum_{l=0}^{\infty}(2l+1)A_l R^{l-1} P_l(\cos\theta) = \frac{1}{\epsilon_0}\sigma_0(\theta)$$

$$\int_{-1}^{1} P_l^m(x) P_k^m(x) dx = \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!} \delta_{lk}$$

From here, the coefficients can be determined using Fourier's trick

$$A_l = \frac{1}{2\epsilon_0 R^{l-1}} \int_0^{\pi} \sigma_0(\theta) P_l(\cos\theta) \sin\theta \, d\theta. \tag{3.84}$$

Equations 3.78 and 3.79 constitute the solution to our problem, with the coefficients given by $Eqs.$ 3.81 and 3.84.

For instance, if
$$\sigma_0(\theta) = k\cos\theta = k P_1(\cos\theta)$$

for some constant k, then all the A_l's are zero except for $l = 1$, and

$$A_1 = \frac{k}{2\epsilon_0} \int_0^{\pi} [P_1(\cos\theta)]^2 \sin\theta \, d\theta = \frac{k}{3\epsilon_0}$$

The potential inside the sphere is therefore

$$V(r,\theta) = \frac{k}{3\epsilon_0} r\cos\theta \qquad (r \leq R), \tag{3.86}$$

Whereas outside the sphere

$$V(r,\theta) = \frac{kR^3}{3\epsilon_0}\frac{1}{r^2}\cos\theta \qquad (r \geq R). \tag{3.87}$$

In particular, if $\sigma_0(\theta)$ is the induced charge on a metal sphere in an external field $E_0\hat{z}$, so that $k = 3\epsilon_0 E_0$ (Eq. 3.77), then the potential inside is $E_0 r\cos\theta = E_0 z$, and the field is $E_0\hat{z}$---exactly right to cancel off the external field, as of course it should be. Outside the sphere the potential due to this surface charge is

$$E_0 \frac{R^3}{r^2}\cos\theta$$

Chapter 3

Special Techniques

- 2.1 Laplace's Equation
- 2.2 The Method of Images
- 2.3 Separation of Variables
- 2.4 Multipole Expansion

Approximate Potential at Large Distances

If you are very far away from a localized charge distribution, it "looks" like a point charge, and the potential is—to good approximation—$\frac{Q}{4\pi\varepsilon_0 r}$, where Q is the total charge.

What if Q is *zero*?

Multipole Expansion

Example

A (physical) **electric dipole** consists of two equal and opposite charges ($\pm$q) separated by a distance d. Find the approximate potential at points far from the dipole

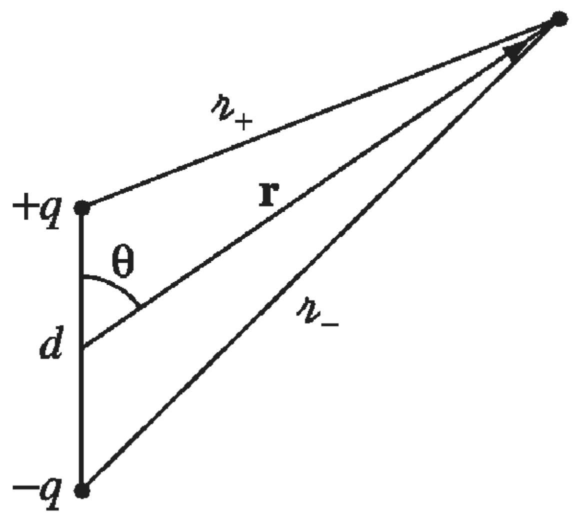


FIGURE 3.26

Solution:

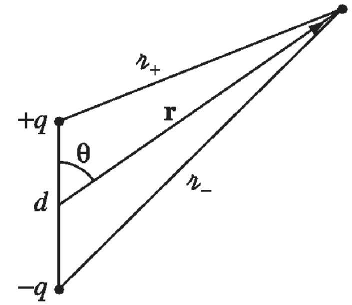


FIGURE 3.26

Let $\mathscr{r}_-$ be the distance from $-q$ and $\mathscr{r}_+$ the distance from $+q$ (Fig. 3.26). Then

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}_+} - \frac{q}{\mathscr{r}_-}\right),$$

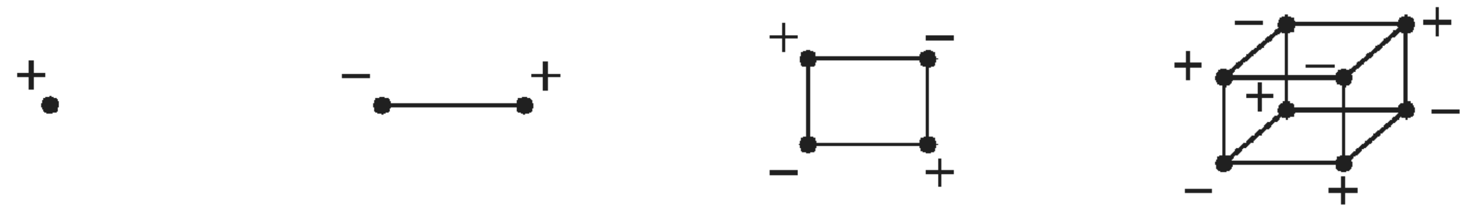
From the law of cosines :

$$\mathscr{r}_\pm^2 = r^2 + (d/2)^2 \mp rd\cos\theta = r^2\left(1 \mp \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right)$$

r >>d: $$\frac{1}{\mathscr{r}_\pm} \cong \frac{1}{r}\left(1 \mp \frac{d}{r}\cos\theta\right)^{-1/2} \cong \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right)$$

$$\frac{1}{\mathscr{r}_+} - \frac{1}{\mathscr{r}_-} \cong \frac{d}{r^2}\cos\theta$$

hence : $$V(\mathbf{r}) \cong \frac{1}{4\pi\epsilon_0}\frac{qd\cos\theta}{r^2}.$$



Monopole ($V \sim 1/r$) | Dipole ($V \sim 1/r^2$) | Quadrupole ($V \sim 1/r^3$) | Octopole ($V \sim 1/r^4$)

Example

For the potential of any localized charge distribution, in powers of $\frac{1}{r}$. Figure 3.28 defines the relevant variables; the potential at r is given by

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}}\rho(\mathbf{r}')\,d\tau'$$

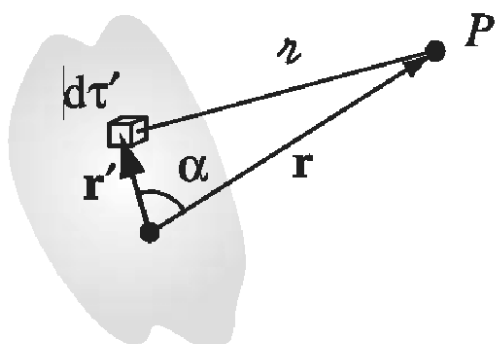


FIGURE 3.28

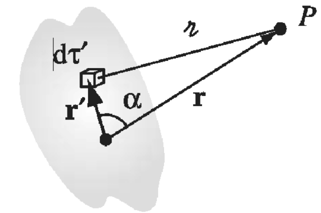


FIGURE 3.28

Using the law of cosines:

$$\mathscr{r}^2 = r^2 + (r')^2 - 2rr'\cos\alpha = r^2\left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\alpha\right]$$

$$\mathscr{r} = r\sqrt{1+\epsilon}$$

$$\epsilon \equiv \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right)$$

$$\frac{1}{\mathscr{r}} = \frac{1}{r}(1+\epsilon)^{-1/2} = \frac{1}{r}\left(1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \ldots\right)$$

$$\frac{1}{\mathscr{r}} = \frac{1}{r}(1+\epsilon)^{-1/2} = \frac{1}{r}\left(1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \dots\right)$$

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\left[1 - \frac{1}{2}\left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right) + \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r} - 2\cos\alpha\right)^2\right.$$

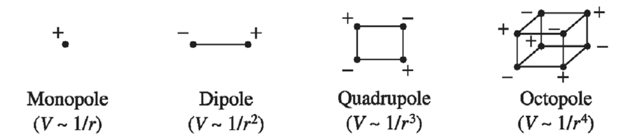
$$\left. - \frac{5}{16}\left(\frac{r'}{r}\right)^3\left(\frac{r'}{r} - 2\cos\alpha\right)^3 + \dots\right]$$

$$= \frac{1}{r}\left[1 + \left(\frac{r'}{r}\right)(\cos\alpha) + \left(\frac{r'}{r}\right)^2\left(\frac{3\cos^2\alpha - 1}{2}\right)\right.$$

$$\left. + \left(\frac{r'}{r}\right)^3\left(\frac{5\cos^3\alpha - 3\cos\alpha}{2}\right) + \dots\right].$$

$$\boxed{\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha)}$$

Legendre polynomials!



$$\frac{1}{\mathscr{r}} = \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r'}{r}\right)^n P_n(\cos\alpha)$$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{(n+1)}} \int (r')^n P_n(\cos\alpha)\rho(\mathbf{r}')\,d\tau'$$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\int \rho(\mathbf{r}')\,d\tau' + \frac{1}{r^2}\int r'\cos\alpha\,\rho(\mathbf{r}')\,d\tau' + \frac{1}{r^3}\int (r')^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)\rho(\mathbf{r}')\,d\tau' + \ldots\right]$$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}}\rho(\mathbf{r}')\,d\tau'$$

The Monopole and Dipole Terms

$$V_{\text{mon}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$$

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int r'\cos\alpha\,\rho(\mathbf{r}')\,d\tau'$$

$$r'\cos\alpha = \hat{\mathbf{r}}\cdot\mathbf{r}'$$

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\hat{\mathbf{r}}\cdot\int \mathbf{r}'\rho(\mathbf{r}')\,d\tau'$$

Dipole moment of the charge distribution

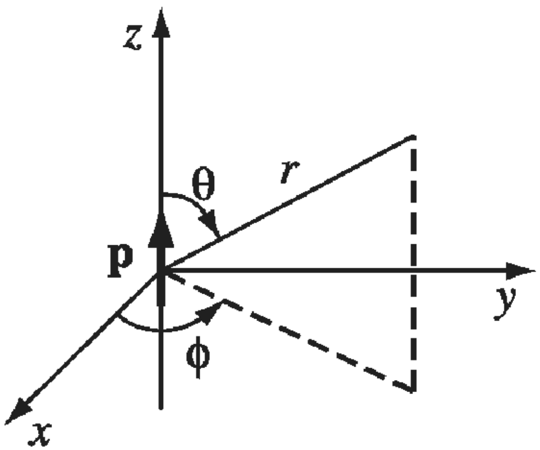
$$\mathbf{p} \equiv \int \mathbf{r}'\rho(\mathbf{r}')\,d\tau',$$

$$\mathbf{p} = \sum_{i=1}^{n} q_i \mathbf{r}'_i$$

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$$

$$\mathbf{p} = q\mathbf{r}'_+ - q\mathbf{r}'_- = q(\mathbf{r}'_+ - \mathbf{r}'_-) = q\mathbf{d}$$

The Electric Field of a pure Dipole



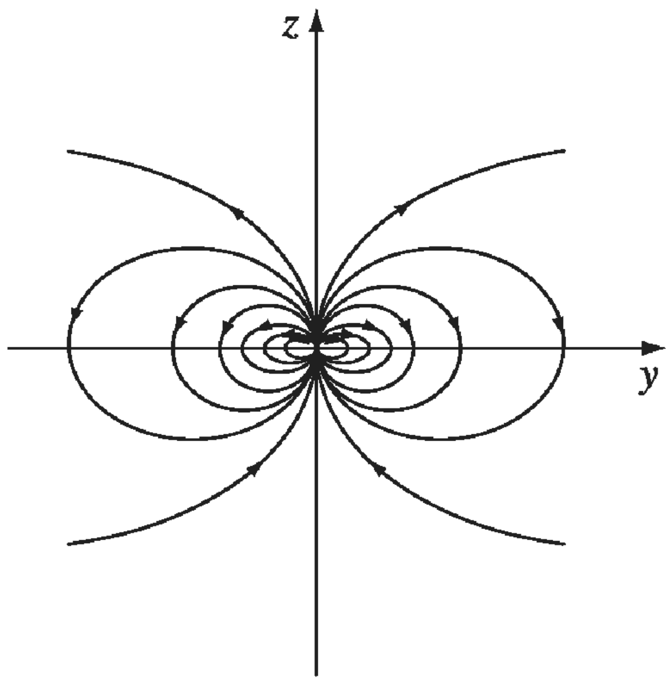
$$V_{\text{dip}}(r,\theta) = \frac{\hat{\mathbf{r}} \cdot \mathbf{p}}{4\pi\epsilon_0 r^2} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$$

$$E_r = -\frac{\partial V}{\partial r} = \frac{2p\cos\theta}{4\pi\epsilon_0 r^3},$$

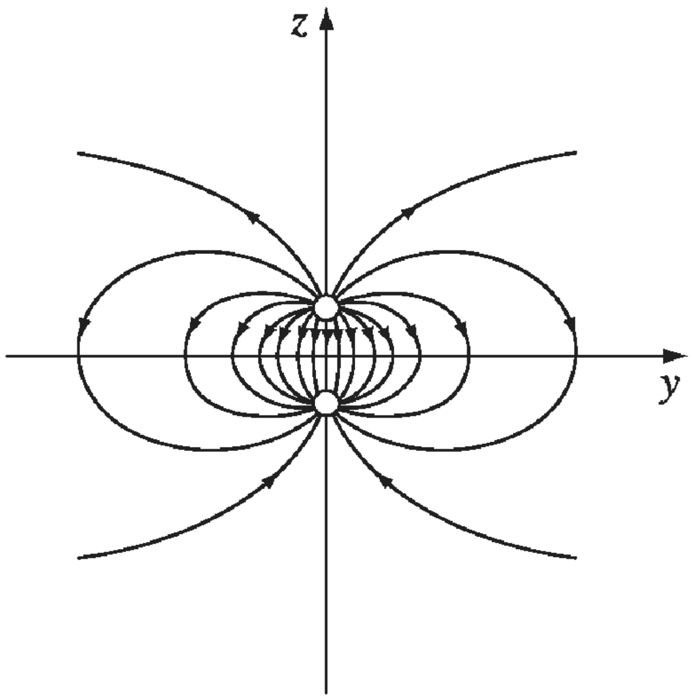
$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{p\sin\theta}{4\pi\epsilon_0 r^3},$$

$$E_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} = 0.$$

$$\mathbf{E}_{\text{dip}}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})$$



Field of a 'pure' dipole



Field of a 'physical' dipole