第二章 分析力学(II)

(Analytical Mechanics)

§4. 运动积分的拉格朗日判据(Constants of the Motion in the Lagranglan Formulation)

一. 循环坐标

广义动量：$p_\alpha = \dfrac{\partial L}{\partial \dot{q}_\alpha} = \dfrac{\partial T}{\partial \dot{q}_\alpha} = p_\alpha(q.\dot{q}.t)$

循环坐标：如 $\dfrac{\partial L}{\partial q_\alpha} = 0$ $\Longrightarrow$ 则称q_α为循环坐标

$$\because \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = 0$$

$p_\alpha = p_{\alpha 0}$(conservation)

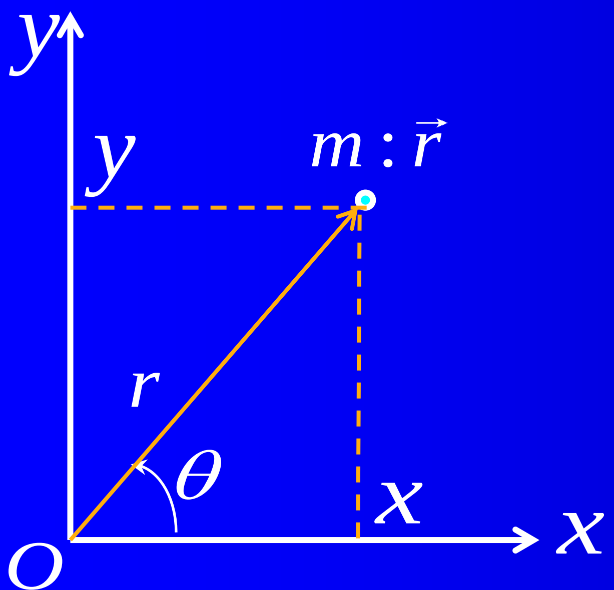
若q_α为线量 $\Rightarrow$ 动量守恒

若q_α为角量 $\Rightarrow$ 角动量守恒

Attention:

- 循环坐标是否出现及出现的多少是判断广义坐标是否合适的标志;
- 循环坐标是否出现与广义坐标选取有关;

例：有心力作用下质点运动。



$$L=\frac{1}{2}m(\dot{x}^2+\dot{y}^2)+\frac{mk^2}{\sqrt{x^2+y^2}}$$

$$\frac{\partial L}{\partial t}=0, \therefore E_0=const.$$

$$L=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)+\frac{mk^2}{r}$$

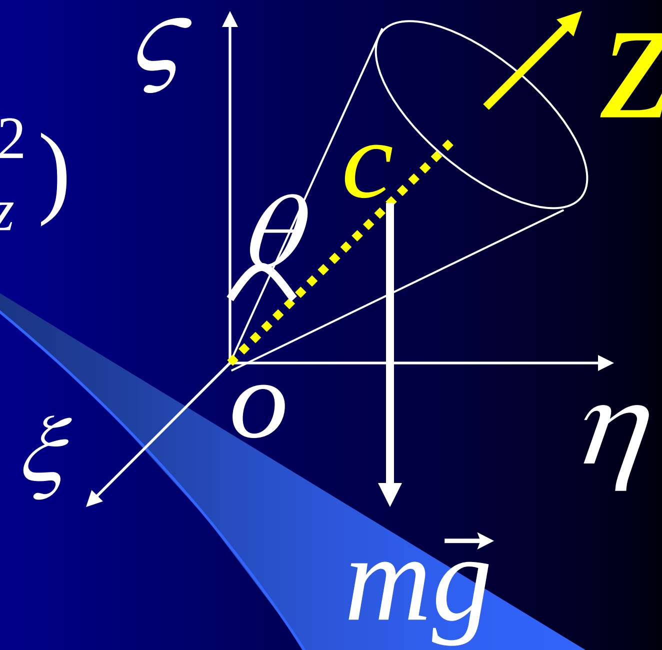
$$1. \frac{\partial L}{\partial t}=0, \therefore E_0=const.$$

$$2. \frac{\partial L}{\partial \theta}=0, \therefore p_\theta=\frac{\partial L}{\partial \dot{\theta}}=mr^2\dot{\theta}=const.$$

例： 使用拉氏方程求拉格朗日泊松刚体三个运动积分。

解：

$$T=\frac{1}{2}(I_x\omega_x^2+I_y\omega_y^2+I_z\omega_z^2)$$



$$\begin{cases}\omega_x=\dot{\varphi}\sin\theta\sin\psi+\dot{\theta}\cos\psi\\ \omega_y=\dot{\varphi}\sin\theta\cos\psi-\dot{\theta}\sin\psi\\ \omega_z=\dot{\varphi}\cos\theta+\dot{\psi}\end{cases}$$

$$T=\frac{1}{2}I_x(\dot{\theta}^2+\dot{\varphi}^2\sin^2\theta)+\frac{1}{2}I_z(\dot{\varphi}\cos\theta+\dot{\psi})^2$$

$$V=mgl\cos\theta$$

$$L=\frac{1}{2}I_x(\dot{\theta}^2+\dot{\varphi}^2\sin^2\theta)+\frac{1}{2}I_z(\dot{\varphi}\cos\theta+\dot{\psi})^2-mgl\cos\theta$$

$$L=\frac{1}{2}I_x(\dot\theta^2+\dot\varphi^2\sin^2\theta)+\frac{1}{2}I_z(\dot\varphi\cos\theta+\dot\psi)^2-mgl\cos\theta$$

$$\because \frac{\partial L}{\partial \varphi}=0$$

$$\therefore\ p_\varphi=\frac{\partial L}{\partial\dot\varphi}=I_x\dot\varphi\sin^2\theta+I_z(\dot\varphi\cos\theta+\dot\psi)\cos\theta=const$$

$$\because \frac{\partial L}{\partial \psi}=0$$

$$\therefore \frac{\partial L}{\partial\dot\psi}=I_z(\dot\varphi\cos\theta+\dot\psi)=const$$

$\because \frac{\partial L}{\partial t}=0$,系统保守稳定

$$\therefore H=T+V$$

$$H=\frac{1}{2}I_x(\dot\theta^2+\dot\varphi^2\sin^2\theta)+\frac{1}{2}I_z(\dot\varphi\cos\theta+\dot\psi)^2+mgl\cos\theta$$

二. 动能T表达式

数学补充：欧拉齐次函数定理

定义：如果f(x_1 x_2 x_3…… x_N)是x_1 x_2 x_3…… x_N 的n次齐次函数, 即对任意t,有：

$$f(tx_1\ tx_2\ tx_3.....tx_N) = t^n f(x_1\ x_2\ x_3.....x_N)$$

则称 $f(x_1\ x_2\ x_3.....x_N)$ ⇒ n次齐次函数。

Example:

$$f(x, y) = x^2 + xy$$

$$f(3x,3y) = (3x)^2 + (3x)(3y)$$

$$= 3^2(x^2 + xy)$$

二次齐次函数

欧拉齐次函数定理:

$$\sum_{i=1}^{N}\frac{\partial f}{\partial x_i}x_i = nf\left(x_1\ \ x_2\ \ x_3\ldots\ldots x_N\right)$$

证明

$$f\left(tx_1\ \ tx_2\ \ tx_3\ldots\ldots tx_N\right) = t^n f\left(x_1\ \ x_2\ \ x_3\ldots\ldots x_N\right)$$

两边对t求导

$$\sum_{i=1}^{N}\left\{\frac{\partial f\left(tx_1\ \ tx_2\ \ tx_3\ldots\ldots tx_N\right)}{\partial\left(tx_N\right)}\frac{\partial\left(tx_N\right)}{\partial t}\right\}$$

$$= nt^{n-1}f\left(x_1\ \ x_2\ \ x_3\ldots\ldots x_N\right)$$

$$\sum_{i=1}^{N}\left\{\frac{\partial f(tx_i)}{\partial(tx_i)}\frac{\partial(tx_i)}{\partial t}\right\}=nt^{n-1}f(x_1\ x_2\ x_3\ldots\ldots x_N)$$

令 $x_i^{'}=tx_i$

$$\sum_{i=1}^{N}\left\{\frac{\partial f(x_i^{'})}{\partial x_i^{'}}x_i\right\}=nt^{n-1}f(x_1\ x_2\ x_3\ldots\ldots x_N)$$

令 $t=1$

$$\sum_{i=1}^{N}\left\{\frac{\partial f(x_i)}{\partial x_i}x_i\right\}=nf(x_1\ x_2\ x_3\ldots\ldots x_N)$$

Example: $$f(x,y)=x^2+xy$$

$$f(2x,2y)=(2x)^2+(2x)(2y)$$

$$=2^2(x^2+xy)$$

二次齐次函数

欧拉齐次函数定理:

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf(x,y)$$

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=x(2x+y)+yx=2(x^2+xy)$$

动能表达式　系统有N个质点，自由度为S

$$T=\frac{1}{2}m_i\dot{\vec{r}}_i^{\,2}\qquad(i=1.2\ldots N)$$

$$\vec{r}_i=\vec{r}_i(q,t)\Rightarrow\dot{\vec{r}}=\frac{\partial\vec{r}_i}{\partial q_\alpha}\dot{q}_\alpha+\frac{\partial\vec{r}_i}{\partial t}$$

$$T=\frac{1}{2}m_i\frac{\partial\vec{r}_i}{\partial q_\alpha}\frac{\partial\vec{r}_i}{\partial q_\beta}\dot{q}_\alpha\dot{q}_\beta+m_i\frac{\partial\vec{r}_i}{\partial q_\alpha}\frac{\partial\vec{r}_i}{\partial t}\dot{q}_\alpha+\frac{1}{2}m_i\left(\frac{\partial\vec{r}_i}{\partial t}\right)^2$$

$$=\frac{1}{2}a_{\alpha\beta}\dot{q}_\alpha\dot{q}_\beta+b_\alpha\dot{q}_\alpha+c\qquad(\alpha,\beta=1.2\ldots S)$$

如何理解$\frac{\partial\vec{r}_i}{\partial t}=0$，即约束稳定！$\frac{\partial\vec{r}_i}{\partial t}=0$时，速度$\dot{\vec{r}}$仅来自$\dot{q}_\alpha$贡献！

$$a_{\alpha\beta} = m_i \frac{\partial \vec{r}_i}{\partial q_\alpha} \frac{\partial \vec{r}_i}{\partial q_\beta} = a_{\alpha\beta}(q, t)$$

$$b_\alpha = m_i \frac{\partial \vec{r}_i}{\partial q_\alpha} \frac{\partial \vec{r}_i}{\partial t} = b_\alpha(q, t)$$

$$C = \frac{1}{2} m_i \left(\frac{\partial \vec{r}_i}{\partial t}\right)^2 = C(q, t) \qquad (i = 1, 2, 3 \ldots N)$$

$$\boxed{T = T_2 + T_1 + T_0}$$

$$T_2 = \tfrac{1}{2} a_{\alpha\beta} \dot{q}_\alpha \dot{q}_\beta \qquad T_1 = b_\alpha \dot{q}_\alpha \qquad T_0 = C$$

$$(\alpha \ \beta = 1, 2, 3 \ldots s)$$

三.广义能量积分

$$\because L = L(q_\alpha, \dot{q}_\alpha, t)$$

$$\therefore \frac{dL}{dt} = \frac{\partial L}{\partial q_\alpha}\dot{q}_\alpha + \frac{\partial L}{\partial \dot{q}_\alpha}\ddot{q}_\alpha + \frac{\partial L}{\partial t} \qquad (\alpha = 1.2\ldots S)$$

完整保守系

$$\because \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = 0$$

$$= \underbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right)\dot{q}_\alpha + \frac{\partial L}{\partial \dot{q}_\alpha}\ddot{q}_\alpha}_{} + \frac{\partial L}{\partial t}$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\dot{q}_\alpha\right) + \frac{\partial L}{\partial t}$$

$$\frac{d}{dt}\left(-L + \frac{\partial L}{\partial \dot{q}_\alpha}\dot{q}_\alpha\right) = -\frac{\partial L}{\partial t}$$

定义广义能量

$$H = -L + \dot{q}_\alpha \frac{\partial L}{\partial \dot{q}_\alpha}$$

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

如果 $\frac{\partial L}{\partial t}=0 \Rightarrow \frac{dH}{dt}=0$

$$H=-L+\dot{q}_\alpha\frac{\partial L}{\partial \dot{q}_\alpha}=H_0(conservation)$$

一般 $V(q), \frac{\partial L}{\partial \dot{q}_\alpha}=\frac{\partial T}{\partial \dot{q}_\alpha},$

由欧拉齐次函数定理：$\dot{q}_\alpha\frac{\partial T}{\partial \dot{q}_\alpha}=2T_2+T_1$

$$H=-L+\dot{q}_\alpha\frac{\partial L}{\partial \dot{q}_\alpha}=-(T_2+T_1+T_0-V)+2T_2+T_1$$
$$=T_2-T_0+V$$

对完整，保守，稳定 $\Rightarrow \frac{\partial \vec{r}_i}{\partial t}=0$

$$H=T_2+V=E_0(\text{机械能守守恒})$$

非稳定保守系统，E不守恒??

$$m_i\ddot{\vec{r}}_i = \vec{F}_i + \vec{N}_i$$

$$-\frac{dV}{d\vec{r}_i} \qquad \vec{N} = \lambda\nabla_i f$$

$$\ddot{\vec{r}}_i = \frac{d\dot{\vec{r}}_i}{d\vec{r}_i}\frac{d\vec{r}_i}{dt} = \dot{\vec{r}}_i\frac{d\dot{\vec{r}}_i}{d\vec{r}_i}$$

$$m_i\dot{\vec{r}}_i\frac{d\dot{\vec{r}}_i}{d\vec{r}_i} = -\frac{dV}{d\vec{r}_i} + \lambda\nabla_i f$$

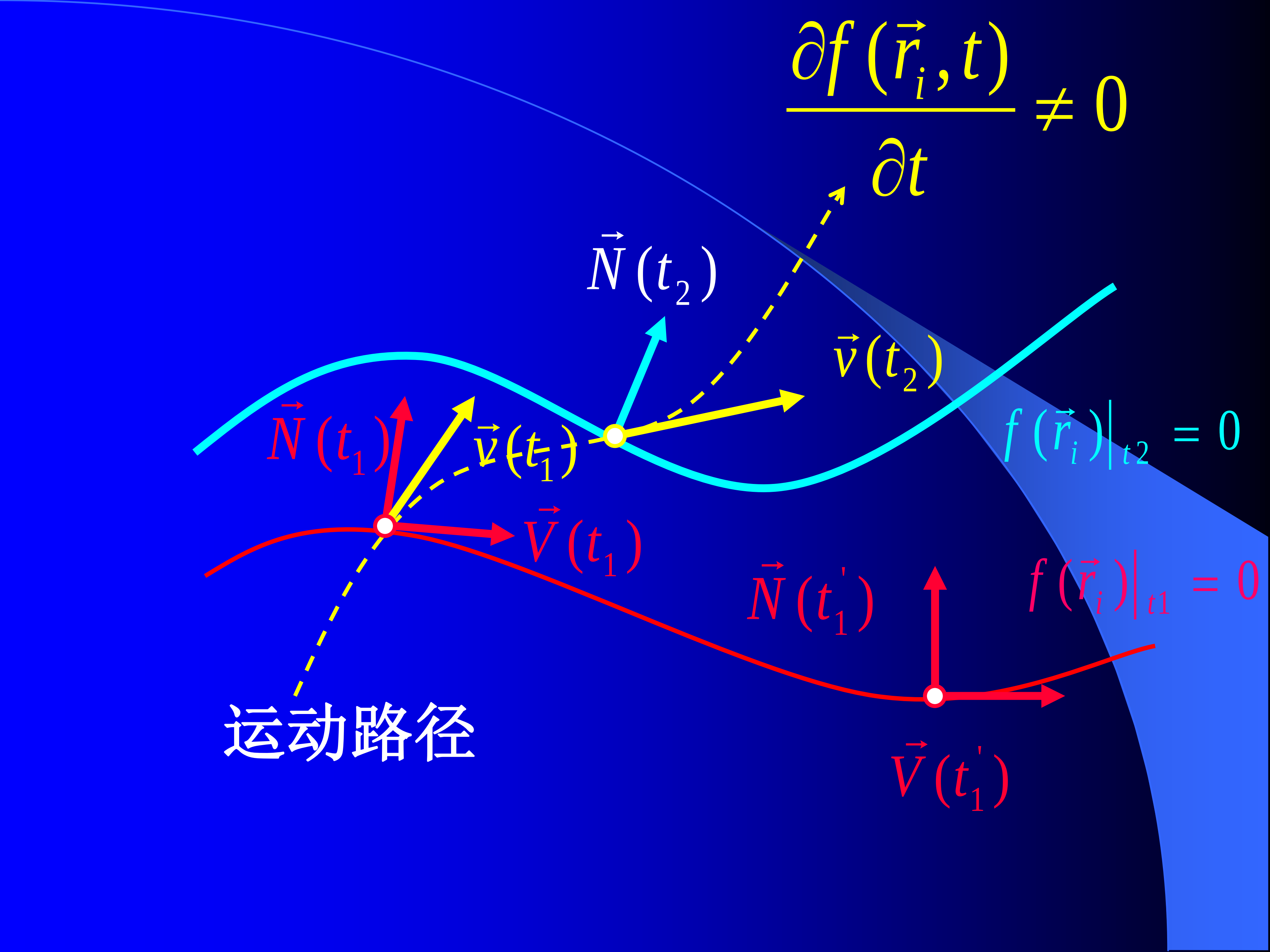
$$d\left(\tfrac{1}{2}m_i\dot{\vec{r}}_i^{\,2}\right) = -dV + \lambda\nabla_i f \bullet d\vec{r}_i$$

$$\frac{d(T+V)}{dt} = \lambda \nabla_i f \frac{d\vec{r}_i}{dt}$$

$$\because \quad f(\vec{r}_i, t) = 0$$

$$\therefore \nabla_i f d\vec{r}_i + \frac{\partial f}{\partial t} dt = 0$$

$$\frac{dE}{dt} = -\lambda \frac{\partial f}{\partial t} \neq 0$$



$\frac{\partial f(\vec{r}_i, t)}{\partial t} \neq 0$
$\vec{N}(t_2)$
$\vec{v}(t_2)$
$\vec{N}(t_1)$
$\vec{v}(t_1)$
$f(\vec{r}_i)|_{t2} = 0$
$\vec{V}(t_1)$
$\vec{N}(t_1^{'})$
$f(\vec{r}_i)|_{t1} = 0$
运动路径
$\vec{V}(t_1^{'})$

Summary:

- 循环坐标是否出现与广义坐标选取有关;
- 不要将 $\frac{\partial L}{\partial t}=0$ 与 $\frac{\partial \vec{r}}{\partial t}=0$ 混淆;

  $\frac{\partial \vec{r}}{\partial t}=0$, 有 $\frac{\partial L}{\partial t}=0$; 反之不真;
- 约束是否稳定与参照系有关;仅对完整、保守、稳定系,广义能量守恒即机械能守恒。
- 相对运动时,广义能量是否代表体系机械能与参照系有关;约束非稳定,**H**守恒**E**不一定守恒
- 一般情况下

$$H \neq T_2+V \quad H=-L+\dot{q}_\alpha p_\alpha(\alpha=1,2,3 \ldots S)$$

1. 完整；2. 保守系

$\frac{dL}{dt}$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right)-\frac{\partial L}{\partial q_\alpha}=0$$

$$\frac{dH}{dt}=-\frac{\partial L}{\partial t}$$

$$H=-L+\dot{q}_\alpha\frac{\partial L}{\partial \dot{q}_\alpha}=-L+\dot{q}_\alpha p_\alpha$$

欧拉齐次函数定理： $\dot{q}_\alpha\frac{\partial T}{\partial \dot{q}_\alpha}=2T_2+T_1$

定义广义能量： $H=T_2-T_0+V$

$\frac{\partial L}{\partial t}=0$：

3. L不显含 t

3. 稳定约束：

$\frac{\partial \vec{r}_i}{\partial t}=0$

$\frac{\partial f}{\partial t}=0$，

约束力不做功：

$\frac{dE}{dt}=-\lambda\frac{\partial f}{\partial t}=0$。

$\dot{H}=0$，广义能量守恒：

$H=T_2-T_0+V=E_0$

广义能量即为机械能

$H=T_2+V$

系统机械能 $H=T_2+V$ 守恒。

例题 一质量为m的小环套在一光滑抛物线金属丝$x^2=4ay$上滑动，金属丝以匀角速ω绕y轴转动，试写出L, H, E.

解: 在转动坐标系中

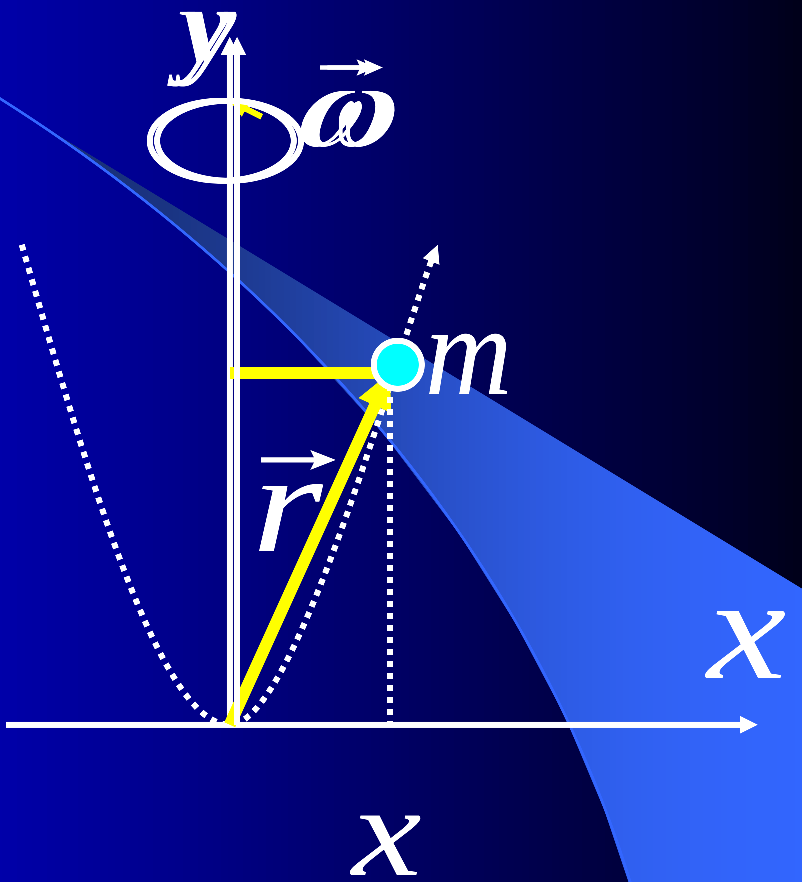
坐标数：2

约束：$x^2 = 4ay \quad (1)$

自由度：1

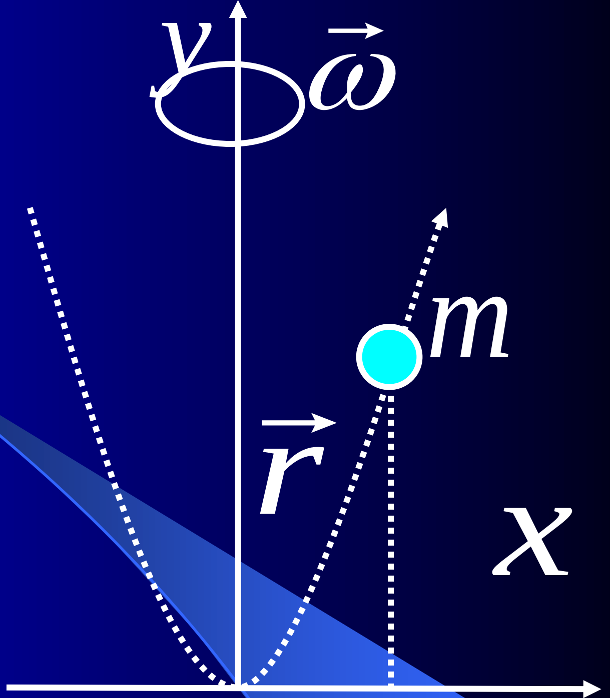
取如图所示 x 为广义坐标

$$\vec{r} = x\vec{i} + y\vec{j} \quad (2)$$



$$\vec{r} = x\vec{i} + y\vec{j} \quad (2)$$

$$\begin{cases} \vec{V} = \dfrac{d\vec{r}}{dt} = \dot{x}\vec{i} + \dot{y}\vec{j} \\ \because x^2 = 4ay \qquad \therefore \dot{y} = \dfrac{x\dot{x}}{2a} \end{cases}$$



$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m(1 + \frac{x^2}{4a^2})\dot{x}^2$$

$$V_g = mgy = mg\frac{x^2}{4a}$$

$\because$受惯性离心力$m\omega^2 x = -\dfrac{dV_e}{dx}$ $\qquad \therefore V_e = -\dfrac{1}{2}m\omega^2 x^2$

$$L = T - V = \frac{1}{2}m(1 + \frac{x^2}{4a^2})\dot{x}^2 + \frac{1}{2}m\omega^2 x^2 - mg\frac{x^2}{4a}$$

$$\vec{r} = x\vec{i} + y\vec{j} \quad (2)$$

$\frac{\partial \vec{r}}{\partial t} = 0$,稳定约束。

$$L = T - V$$

$$= \frac{1}{2}m(1+\frac{x^2}{4a^2})\dot{x}^2 - (-\frac{1}{2}m\omega^2 x^2 + mg\frac{x^2}{4a}) \quad (3)$$

$$T = T_2 = \underbrace{\frac{1}{2}m(1+\frac{x^2}{4a^2})\dot{x}^2}_{T_2} + \underbrace{0}_{T_1} + \underbrace{0}_{T_0}$$

$$H = T + V$$

$$= \frac{1}{2}m(1+\frac{x^2}{4a^2})\dot{x}^2 + (-\frac{1}{2}m\omega^2 x^2 + mg\frac{x^2}{4a}) \quad (4)$$

$\because \frac{\partial L}{\partial t} = 0 \quad \frac{\partial \vec{r}}{\partial t} = 0$

$$H = T_2 + V = E$$

Conclusion: $\because \dfrac{\partial L}{\partial t} = 0 \quad \dfrac{\partial \vec{r}}{\partial t} = 0$

在非惯性参照系中: 满足完整,保守,稳定

惯性离心力: $\because m\omega^2 x = -\dfrac{dV_e}{dx} \qquad \therefore V_e = -\dfrac{1}{2}m\omega^2 x^2$

动能: $T_r = \dfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \dfrac{1}{2}m(1+\dfrac{x^2}{4a^2})\dot{x}^2$

$$E_r = T_r + V_e + V = \frac{1}{2}m(1+\frac{x^2}{4a^2})\dot{x}^2 - \frac{1}{2}m\omega^2 x^2 + mg\frac{x^2}{4a}$$

$$= T_2 + V = H$$

$$E_r = \frac{1}{2}m\left(1+\frac{x^2}{4a^2}\right)\dot{x}^2 - \frac{1}{2}m\omega^2 x + mg\frac{x^2}{4a}$$

$$= E_0^r\,(Constant)$$

在静系中:

坐标数: 3

约束: $x^2 = 4ay \begin{cases} x^2 + z^2 = 4ay \quad (4) \\ \dfrac{x}{z} = tg\omega t \quad (5) \end{cases}$

自由度:1

取如图所示 $oA = R$ 为广义坐标

$$\vec{r} = R\sin\omega t\vec{i} + R\cos\omega t\vec{k} + \frac{R^2}{4a}\vec{j} \quad (6)$$

$$\vec{r} = R\sin\omega t\vec{i} + R\cos\omega t\vec{k} + \frac{R^2}{4a}\vec{j} \quad (6)$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \omega R\cos\omega t\vec{i} + \dot{R}\sin\omega t\vec{i}$$

$$-\omega R\sin\omega t\vec{k} + \dot{R}\cos\omega t\vec{k} + \frac{R\dot{R}}{2a}\vec{j}$$

$$T = \frac{1}{2}m[\omega^2 R^2 + (1+\frac{R^2}{4a^2})\dot{R}^2]$$

$$V = mgy = \frac{R^2}{4a}mg$$

$$L = \frac{1}{2}m[\omega^2 R^2 + (1+\frac{R^2}{4a^2})\dot{R}^2] - \frac{R^2}{4a}mg \quad (7)$$

上式与稳定约束下L一致

$$L = \frac{1}{2}m(1+\frac{x^2}{4a^2})\dot{x}^2 - (-\frac{1}{2}m\omega^2 x^2 + mg\frac{x^2}{4a}) \quad (3)$$

$$\vec{r} = R\sin\omega t\vec{i} + R\cos\omega t\vec{k} + \frac{R^2}{4a}\vec{j} \quad (6)$$

$$\because \frac{\partial \vec{r}}{\partial t} \neq 0 \qquad \therefore H = T_2 - T_0 + V \neq E$$

$$\begin{cases} T_2 = \frac{1}{2}m\left(\frac{\partial \vec{r}_i}{\partial q_\alpha}\right)^2 \dot{q}_\alpha^2 = \frac{1}{2}m\left(1+\frac{R^2}{4a}\right)\dot{R}^2 \\ T_0 = \frac{1}{2}m\left(\frac{\partial \vec{r}}{\partial t}\right)^2 = \frac{1}{2}m\omega^2 R^2 \neq 0 \end{cases}$$

$$H = \frac{1}{2}m\left(1+\frac{R^2}{4a^2}\right)\dot{R}^2 - \frac{1}{2}m\omega^2 R^2 + \frac{R^2}{4a}mg \quad (8)$$

上式与稳定约束下H形式一致:

$$H = T + V = \frac{1}{2}m\left(1+\frac{x^2}{4a^2}\right)\dot{x}^2 + \left(-\frac{1}{2}m\omega^2 x^2 + mg\frac{x^2}{4a}\right) \quad (4)$$

静系中的机械能E ≠H

$$E = T + V = \frac{1}{2} m[\omega^2 R^2 + (1+\frac{R^2}{4a^2})\dot{R}^2] + \frac{R^2}{4a} mg$$

$$H = \frac{1}{2} m(1+\frac{R^2}{4a^2})\dot{R}^2 - \frac{1}{2} m\omega^2 R^2 + mg\frac{R^2}{4a}$$

$$\because \frac{\partial \vec{r}}{\partial t} \neq 0, \quad H = T_2 - T_0 + V \neq E$$

Summary:

- L, H与参照系有关？ 否！
- 约束是否稳定与参照系有关
- 广义能量是否代表机械能亦与参照系有关
- 弄清 $\frac{\partial \vec{r}}{\partial t} = 0$ 与 $\frac{\partial L}{\partial t} = 0$ 间的关系

§ 5. 哈密顿正则方程 (Hamilton's Equation)

一. 勒让特变换

新变量

f= f (x, y)

$$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy=u(x,y)dx+v(x,y)dy$$

$$\begin{cases} u=\dfrac{\partial f}{\partial x}=u(x,y) \\ v=\dfrac{\partial f}{\partial y}=v(x,y) \end{cases}$$

旧方程

旧函数

旧变量

$$x=\frac{\partial(?)}{\partial u} \qquad y=\frac{\partial(?)}{\partial v}$$

$$
\begin{tikzcd}[row sep=small]
 & & u \\
 & x \arrow[ur, dash] \arrow[dr, dash] & \\
 & & v \\
f \arrow[uur, dash] \arrow[ddr, dash] & & \\
 & & u \\
 & y \arrow[ur, dash] \arrow[dr, dash] & \\
 & & v
\end{tikzcd}
$$

$$df = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u}du + \frac{\partial f}{\partial x}\frac{\partial x}{\partial v}dv + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}du + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}dv$$

$$= \left(u\frac{\partial x}{\partial u} + v\frac{\partial y}{\partial u}\right)du + \left(u\frac{\partial x}{\partial v} + v\frac{\partial y}{\partial v}\right)dv$$

$$\therefore \frac{\partial f}{\partial u} = u\frac{\partial x}{\partial u} + v\frac{\partial y}{\partial u} = \frac{\partial (xu)}{\partial u} - x + \frac{\partial (yv)}{\partial u}$$

$$\therefore \frac{\partial f}{\partial v} = u\frac{\partial x}{\partial v} + v\frac{\partial y}{\partial v} = \frac{\partial (xu)}{\partial v} + \frac{\partial (yv)}{\partial v} - y$$

$$\begin{cases} x = \dfrac{\partial}{\partial u}(-f + xu + yv) \\ y = \dfrac{\partial}{\partial v}(-f + xu + yv) \end{cases}$$

令 $F \equiv -f + xu + yv$ ⇒ 新函数

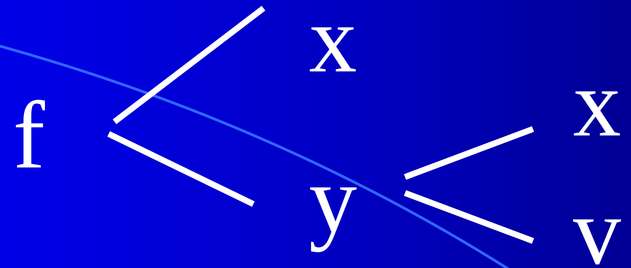
$$\begin{cases} x = \dfrac{\partial F}{\partial u} = x(u, v) \\ y = \dfrac{\partial F}{\partial v} = y(u, v) \end{cases}$$ ⇒ 新方程

$$\begin{cases} u = \dfrac{\partial f}{\partial x} = u(x, y) \\ v = \dfrac{\partial f}{\partial y} = v(x, y) \end{cases}$$ ⇒ 旧方程

保留变量
- 旧变量x
- 新变量v

去掉变量
- 旧变量y
- 新变量u

f
- x
- y
  - x
  - v



$$df = \underline{\left[\left(\frac{\partial f}{\partial x}\right)_y + \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial x}\right)_v\right]} dx + \left[\underline{\left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_x}\right] dv$$

$$\frac{\partial f}{\partial x} = u + v\left(\frac{\partial y}{\partial x}\right)_v$$

$$= u + \left(\frac{\partial yv}{\partial x}\right)_v \quad \Longrightarrow \quad u = -\frac{\partial}{\partial x}(-f + yv)$$

$$\frac{\partial f}{\partial v} = v\left(\frac{\partial y}{\partial v}\right)_x$$

$$= \left(\frac{\partial yv}{\partial v}\right)_x - y \quad \Longrightarrow \quad y = \frac{\partial}{\partial v}(-f + yv)$$

$$u = -\frac{\partial}{\partial x}(-f + yv)$$

$$y = \frac{\partial}{\partial v}(-f + yv)$$

令 F= (-f + y v)

新函数

$$u = -\frac{\partial F}{\partial x}$$

$$y = \frac{\partial F}{\partial v}$$

Summary:

$$\begin{cases} \text{新函数F} = -\text{旧函数f} + \text{去掉旧变量y}\dfrac{\partial(\text{旧函数f})}{\partial(\text{去掉旧变量y})} \\ \text{去掉旧变量y} = \dfrac{\partial(\text{新函数F})}{\partial(\text{保留新变量v})} \\ \text{去掉新变量u} = -\dfrac{\partial(\text{新函数F})}{\partial(\text{保留旧变量x})} \end{cases}$$

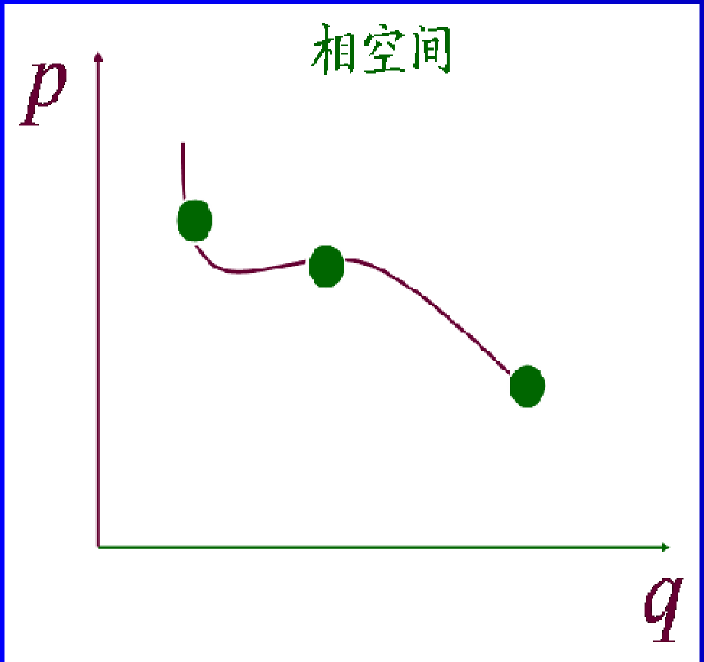
二. 相空间和正则方程

(q, p) ⟷ 一对正则变量

s个广义坐标$(q_1, q_2, q_3 \ldots q_s)$

s个广义动量$(p_1, p_2, p_3 \ldots p_s)$

⇒ 相空间



q, p为独立坐标:

H力学更完备！

旧函数 $L(\dot{q},q,t)$

旧变量：$(q\ \dot{q})$

$$p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha}$$

$$\dot{p}_\alpha = \frac{\partial L}{\partial q_\alpha}$$

新变量

新函数 $H(p,q,t)$

$$H = -L + \dot{q}_\alpha \frac{\partial L}{\partial \dot{q}_\alpha} = H(q.p.t)$$

去掉旧变量 $\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha}$

去掉新变量 $\dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}$

$$新函数F = -旧函数f + 去掉旧变量y\ \frac{\partial(旧函数f)}{\partial(去掉旧变量y)}$$

$$去掉旧变量y = \frac{\partial(新函数F)}{\partial(保留新变量v)}$$

$$去掉新变量u = -\frac{\partial(新函数F)}{\partial(保留旧变量x)}$$

$$H = -L + \dot{q}_\alpha \frac{\partial L}{\partial \dot{q}_\alpha} = -L + \dot{q}_\alpha p_\alpha = H(q.p.t)$$

保守系

正则方程

$$\begin{cases} \dot{q}_\alpha = \dfrac{\partial H}{\partial p_\alpha} \\ \dot{p}_\alpha = -\dfrac{\partial H}{\partial q_\alpha} \end{cases} \quad (\alpha = 1,2,3\ldots.S)$$

$$H = -T + \dot{q}_\alpha \frac{\partial T}{\partial \dot{q}_\alpha} = -T + \dot{q}_\alpha p_\alpha = H(q.p.t)$$

非保守系

正则方程

$$\begin{cases} \dot{q}_\alpha = \dfrac{\partial H}{\partial p_\alpha} \\ \dot{p}_\alpha = -\dfrac{\partial H}{\partial q_\alpha} + Q_\alpha \end{cases} \quad (\alpha = 1,2,3\ldots.S)$$

对保守系 $H=-L+\dot{q}_\alpha\frac{\partial L}{\partial \dot{q}_\alpha}=H(q.p.t)$

$$dH=-dL+\dot{q}\,dp_\alpha+p_\alpha d\dot{q}_\alpha$$

$$\because\ L=L(q,\dot{q},t)$$

$$\therefore dL=\frac{\partial L}{\partial q_\alpha}dq_\alpha+\frac{\partial L}{\partial \dot{q}_\alpha}d\dot{q}_\alpha+\frac{\partial L}{\partial t}dt$$

$$dH=-\frac{\partial L}{\partial q_\alpha}dq_\alpha-\frac{\partial L}{\partial \dot{q}_\alpha}d\dot{q}_\alpha-\frac{\partial L}{\partial t}dt+\dot{q}_\alpha dp_\alpha+p_\alpha d\dot{q}_\alpha$$

$$=-\dot{p}_\alpha dq_\alpha-p_\alpha d\dot{q}_\alpha-\frac{\partial L}{\partial t}dt+\dot{q}_\alpha dp_\alpha+p_\alpha d\dot{q}_\alpha$$

$$dH = -\dot{p}_\alpha dq_\alpha + \dot{q}_\alpha dp_\alpha - \frac{\partial L}{\partial t}dt$$

$$\because H = H(q, p, t)$$

$$\therefore dH = \frac{\partial H}{\partial q_\alpha}dq_\alpha + \frac{\partial H}{\partial p_\alpha}dp_\alpha + \frac{\partial H}{\partial t}dt$$

$$\dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha} \qquad \dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha} \qquad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

$$(\alpha = (1, 2, 3, \ldots\ldots S))$$

§6. 运动积分的哈密顿判据(H)(Constants of the motion in Hamiltonial formation)

一. 循环坐标

H=H(q , p ,t)中不显含的坐标 ⟹ 循环坐标

$$\frac{\partial H}{\partial q_\alpha}=0 \Rightarrow q_\alpha\text{循环坐标}$$

$$\frac{\partial H}{\partial p_\alpha}=0 \Rightarrow p_\alpha\text{循环坐标}$$

$$\dot{p}_\alpha=-\frac{\partial H}{\partial q_\alpha}=0 \quad p_\alpha=p_{\alpha 0}(conservation)$$

$$\dot{q}_\alpha=\frac{\partial H}{\partial p_\alpha}=0 \quad q_\alpha=q_{\alpha 0}(conservation)$$

二. 广义能量积分

$= ???$ $\quad H = H(q, p, t)$

$$\frac{dH}{dt} = \frac{\partial H}{\partial q_\alpha}\dot{q}_\alpha + \frac{\partial H}{\partial p_\alpha}\dot{p}_\alpha + \frac{\partial H}{\partial t} \qquad (\alpha = 1.2\ldots.S)$$

$$\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha} \qquad \dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

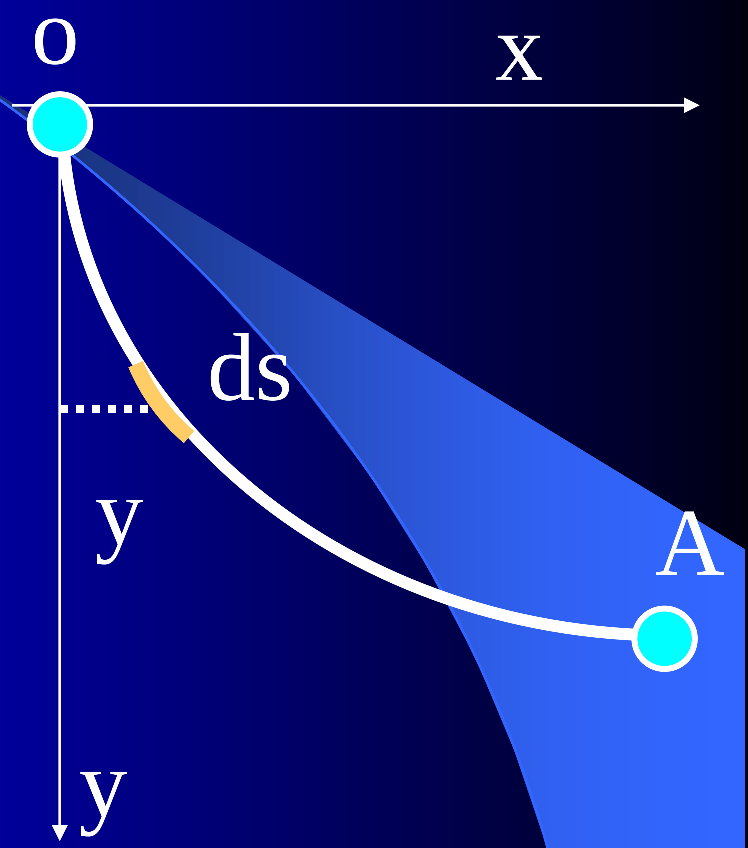
$$\frac{dH}{dt} = \frac{\partial H}{\partial t} \qquad \because \frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

$$\frac{\partial H}{dt} = -\frac{\partial L}{\partial t} = 0 \Rightarrow \text{广义能量守恒}$$

对完整 保守 稳定系统 $\Rightarrow$ $H=T+V=E_0$

§7.哈密顿原理(Hamiltional Principal)

一.变分法简介



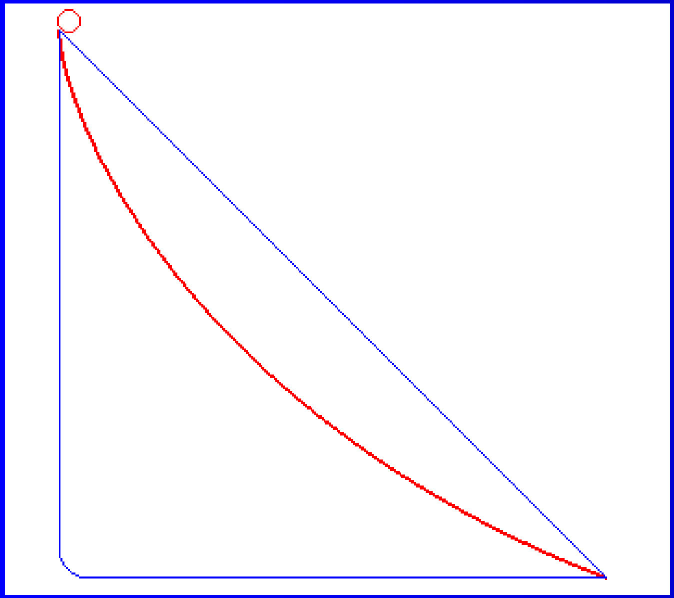
$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + y'^2}\, dx$$

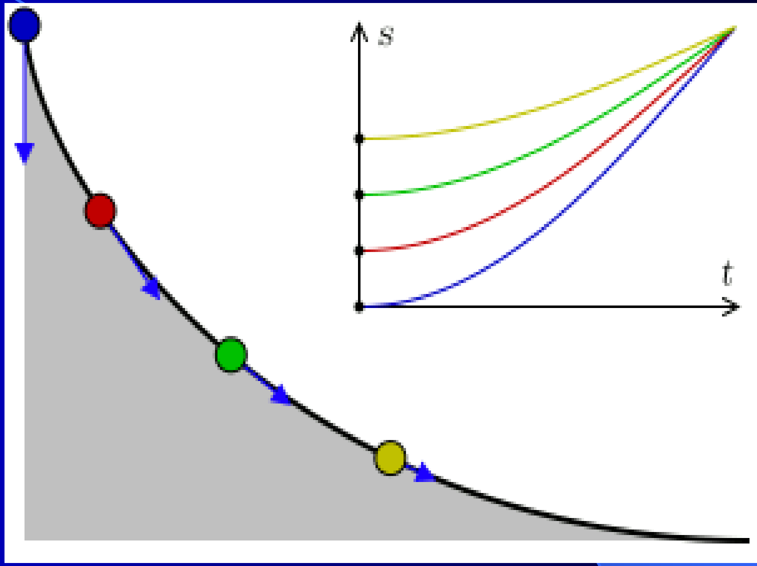
$$v = \frac{ds}{dt} = \frac{\sqrt{1 + y'^2}\, dx}{dt}$$

$$\frac{1}{2} m v^2 - m g y = 0$$

$$t = \int_0^t dt = \int_{x_o}^{x_A} \frac{\sqrt{1 + y'^2}\, dx}{v} = \int_{x_o}^{x_A} \frac{\sqrt{1 + y'^2}\, dx}{\sqrt{2gy}}$$

最速下降曲线





如何使下滑时间最短？（应用Skiing）

$$t=\int_{x_0}^{x_A}\frac{\sqrt{1+y'^2}\,dx}{\sqrt{2gy}}=T(y,y',x)\text{取极值！}$$

$T(y,y',x)$为$y(x)$的泛函，即泛函求极值问题。

泛函定义:

$$J[y(x)] = \int_{x_B}^{x_A} F(y', y, x)dx$$

Attention:

J与y(x)的函数形式有关

泛函的极值是变分法的核心

二. 变分法计算法则

变分法 $\begin{cases} \text{全变分}\Delta \\ \text{等自变量变分}\delta x=0 \end{cases}$

- 函数和泛函的变分

原函数: y(x)

变更后: $\begin{cases} \tilde{y}(x) = y(x) + \varepsilon\eta(x) \\ \varepsilon\text{与}x\text{无关小参数} \\ \eta(x)\big|_{x_A} = \eta(x)\big|_{x_B} = 0 \end{cases}$

定义函数变分:

$$\delta y = \tilde{y}(x) - y(x) = \varepsilon\eta(x)$$

$$\delta y\big|_{x_A} = \delta y\big|_{x_B} = 0$$

泛函变分:

$$\delta J[y(x)] = J[\tilde{y}(x)] - J[y(x)]$$

$$= \int_{x_A}^{x_B} F(\tilde{y}', \tilde{y}, x)dx - \int_{x_A}^{x_B} F(y', y, x)dx$$

- δ变分法基本运算法则:

$$\begin{cases} \delta(\varphi_1 + \varphi_2) = \delta\varphi_1 + \delta\varphi_2 \\ \delta(\varphi_1\varphi_2) = \varphi_1\delta\varphi_2 + \varphi_2\delta\varphi_1 \\ \delta\left(\dfrac{\varphi_1}{\varphi_2}\right) = \dfrac{\varphi_2\delta\varphi_1 - \varphi_1\delta\varphi_2}{\varphi_2^2} \end{cases}$$

• δ变分法基本对易关系

$$\begin{cases} \text{“}\delta\text{”与“}d\text{”对易} \\ \text{"}\delta\text{"与"}\dfrac{d}{dx}\text{"对易} \\ \text{"}\delta\text{"与"}\int(\ )dx\text{"对易} \end{cases}$$

证明:

$$\because \quad \delta y = \tilde{y}(x) - y(x) = \varepsilon\eta(x)$$

$$\begin{cases} \delta\left(\dfrac{dy}{dx}\right) = \tilde{y}'(x) - y'(x) = \varepsilon\eta'(x) \\ \dfrac{d}{dx}(\delta y) = \dfrac{d}{dx}[\tilde{y}(x) - y(x)] = \varepsilon\eta'(x) \end{cases}$$

$$\boxed{\delta\frac{d}{dx} = \frac{d}{dx}\delta}$$

由图可知:

$$y_2 = y_1 + dy_1$$

$$\tilde{y}_1 = y_1 + \delta y_1$$

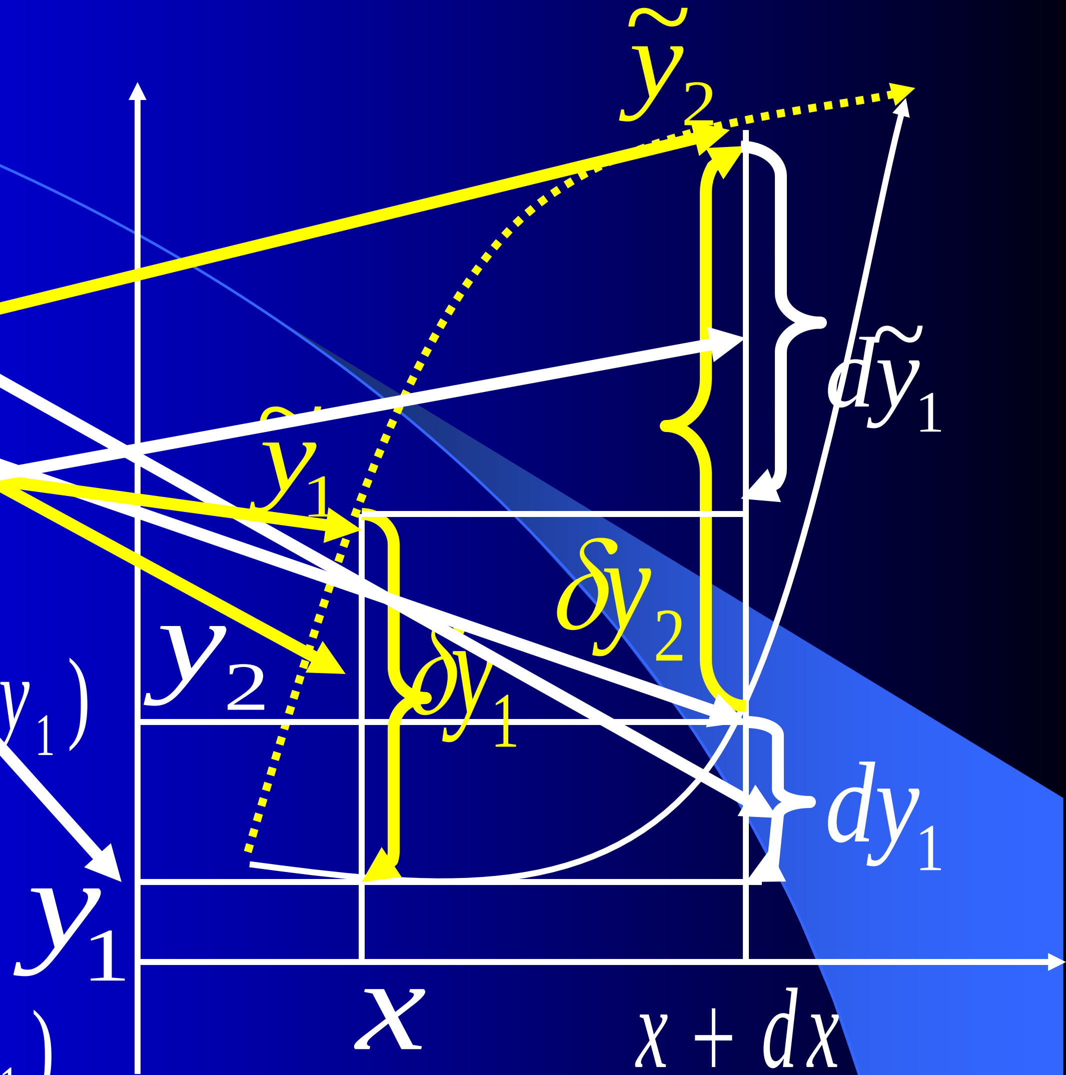
$$\tilde{y}_2 = \tilde{y}_1 + d\tilde{y}_1$$

$$= (y_1 + \delta y_1) + d(y_1 + \delta y_1)$$

$$= y_2 + \delta y_2$$

$$= (y_1 + dy_1) + \delta(y_1 + dy_1)$$

$$d\delta y_1 = \delta d y_1$$



$$\boxed{d\delta = \delta d, \quad [d \quad \delta] = 0}$$

$$\delta\int_{x_1}^{x_2}\varphi(x)dx=\int_{x_1}^{x_2}\tilde{\varphi}(x)dx-\int_{x_1}^{x_2}\varphi(x)dx$$

$$=\int_{x_1}^{x_2}[\tilde{\varphi}(x)-\varphi(x)]dx$$

$$=\int_{x_1}^{x_2}[\delta(\varphi(x))]dx$$

$$\boxed{\delta\int(\quad)dx=\int\delta(\quad)dx}$$

三.泛函极值条件

$$\because \quad \tilde{y}(x) = y(x) + \varepsilon\eta(x)$$

$$J[\tilde{y}(x)] = \int_{x_1}^{x_2} F(\tilde{y}', \tilde{y}, x)dx$$

是ε的函数

将被积函数在y(x)即ε=0处 附近展开

$$J[\varepsilon] = \int_{x_1}^{x_2} [F(y', y, x)dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial \tilde{y}'}(\tilde{y}' - y')dx + \int_{x_1}^{x_2} \frac{\partial F}{\partial \tilde{y}}(\tilde{y} - y) + ..]dx$$

与ε无关

$$\varepsilon\eta'$$

$$\varepsilon\eta$$

$$\left.\frac{dJ}{d\varepsilon}\right|_{\varepsilon=0} = \int_{x_1}^{x_2} (\eta' \frac{\partial F}{\partial y'} + \eta \frac{\partial F}{\partial y})dx = 0$$

$$\frac{dJ}{d\varepsilon}\Big|_{\varepsilon=0} = \int_{x_1}^{x_2} \left(\eta' \frac{\partial F}{\partial y'} + \eta \frac{\partial F}{\partial y}\right) dx = 0$$

$$\because \int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\eta' \, dx = \int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\frac{d\eta}{dx} \, dx$$

$$\because \int_{x_1}^{x_2}\left[\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\eta\right)\right]dx = \frac{\partial F}{\partial y'}\eta\Big|_{x_1}^{x_2} = 0$$

$$= \int_{x_1}^{x_2} -\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\eta \, dx$$

$$\therefore \frac{dJ}{d\varepsilon}\Big|_{\varepsilon=0} = \int_{x_1}^{x_2} \underbrace{[\frac{\partial F}{\partial y} - \frac{d}{dx}(\frac{\partial F}{\partial y'})]\eta(x)}\, dx = 0$$

$$[\frac{d}{dx}(\frac{\partial F}{\partial y'}) - \frac{\partial F}{\partial y}]\eta(x) = 0$$

$\eta(x)$在$(x_1\ x_2)$取值任意

$$\frac{d}{dx}(\frac{\partial F}{\partial y'}) - \frac{\partial F}{\partial y} = 0$$

欧拉方程

利用变分法运算法则求极值

$$\delta J = \int_{x_1}^{x_2} \delta F(y', y, x)dx = \int_{x_1}^{x_2} [\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y']dx = 0$$

$$\delta J = \int_{x_1}^{x_2} \{\frac{\partial F}{\partial y}\delta y$$

$$\delta y' = \delta(\frac{dy}{dx}) = \frac{d(\delta y)}{dx}$$

$$\delta J = \int_{x_1}^{x_2} \{\frac{\partial F}{\partial y}\delta y - \frac{d}{dx}(\frac{\partial F}{\partial y'})\delta y\}dx = 0$$

$$(\frac{d}{dx}(\frac{\partial F}{\partial y'}) - \frac{\partial F}{\partial y})\delta y = 0$$

$\because \delta y$取值任意

$$\frac{d}{dx}(\frac{\partial F}{\partial y'}) - \frac{\partial F}{\partial y} = 0$$

四. 位形空间中的哈密顿原理

$x \Rightarrow$ t

$\begin{cases} y \Rightarrow q(t) \\ y' \Rightarrow \dot{q} \end{cases}$

$F(y, y', x) \Rightarrow L(q, \dot{q}, t)$

定义哈密顿主函数

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t)dt$$

位形空间中的哈密顿原理(有势系统)

$$\delta S = \delta \int_{t_1}^{t_2} L(q, \dot{q}, t)dt = 0$$

位形空间中的哈密顿原理(非有势系统)

$$\therefore \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)-\frac{\partial T}{\partial q_\alpha}=Q_\alpha \qquad (\alpha=1.2\ldots S)$$

$$\int_{t_1}^{t_2}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)-\frac{\partial T}{\partial q_\alpha}-Q_\alpha\right]\delta q_\alpha dt=0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\delta q_\alpha\right)-\frac{\partial T}{\partial \dot{q}_\alpha}\delta \dot{q}_\alpha$$

$$\int_{t_1}^{t_2}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\delta q_\alpha\right)-\frac{\partial T}{\partial \dot{q}_\alpha}\delta \dot{q}_\alpha-\left(\frac{\partial T}{\partial q_\alpha}+Q_\alpha\right)\delta q_\alpha\right]dt=0$$

$$\int_{t_1}^{t_2}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\delta q_\alpha\right)-\frac{\partial T}{\partial \dot{q}_\alpha}\delta \dot{q}_\alpha-\left(\frac{\partial T}{\partial q_\alpha}+Q_\alpha\right)\delta q_\alpha\right]dt=0$$

$$\because \int_{t_1}^{t_2}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\delta q_\alpha\right)dt=\frac{\partial T}{\partial \dot{q}_\alpha}\delta q_\alpha\Big|_{t_1}^{t_2}=0$$

$$\delta q_\alpha\big|_{t_1}=\delta q_\alpha\big|_{t_2}=0$$

$$\int_{t_1}^{t_2}(\delta T+\delta W)dt=0$$

$$\delta w=Q_\alpha\delta q_\alpha \qquad (\alpha=1.2\ldots.S)$$

五. 相空间中的哈密顿原理

$$x \Rightarrow t \qquad y \Rightarrow \begin{cases} q(t) \\ P(t) \end{cases}$$

$$\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha} \qquad \dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}$$

$$\int_{t_1}^{t_2} [(\dot{q}_\alpha - \frac{\partial H}{\partial p_\alpha})\delta p_\alpha - (\dot{p}_\alpha + \frac{\partial H}{\partial q_\alpha})\delta q_\alpha]dt = 0$$

$$(\alpha = 1.2.3\ldots\ldots S)$$

$$\int_{t_1}^{t_2} [(\dot{q}_\alpha \delta p_\alpha - \dot{p}_\alpha \delta q_\alpha) - (\frac{\partial H}{\partial p_\alpha}\delta p_\alpha + \frac{\partial H}{\partial q_\alpha}\delta q_\alpha)]dt = 0$$

$$(\alpha = 1.2.3\ldots\ldots S)$$

$$\int_{t_1}^{t_2}[-\delta H+\delta(\dot{q}_\alpha p_\alpha)]dt=0$$

定义相空间的哈密顿主函数

$$\boxed{S=\int_{t_1}^{t_2}(-H+\dot{q}_\alpha p_\alpha)dt}$$

$$\boxed{\delta S=\delta\int_{t_1}^{t_2}(-H+\dot{q}_\alpha p_\alpha)dt=0}$$

利用位形空间中的哈密顿原理导出拉格朗日方程

$$\delta S=\delta\int_{t_1}^{t_2}L(q,\dot{q},t)dt=0$$

$$\delta S=\int_{t_1}^{t_2}\delta L(q,\dot{q},t)dt$$

$$=?0$$

$$=\int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q_\alpha}\delta q_\alpha+\frac{\partial L}{\partial \dot{q}_\alpha}\delta \dot{q}_\alpha+\frac{\partial L}{\partial t}\delta t\right)dt$$

$$(\alpha=1\ 2\ 3\ldots s)$$

$$\frac{d}{dt}(\delta q)$$

$$\frac{\partial L}{\partial \dot{q}_\alpha}\frac{d}{dt}(\delta q_\alpha)=\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\delta q_\alpha\right)-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right)\delta q_\alpha$$

$$\delta S = \int_{t_1}^{t_2} [\frac{\partial L}{\partial q_\alpha} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_\alpha})]\delta q_\alpha dt = 0$$

$$(\alpha = 1\ 2\ 3 \ldots s)$$

$$[\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_\alpha}) - \frac{\partial L}{\partial q_\alpha}]\delta q_\alpha = 0$$

$\because \delta q_\alpha$在（t_1, t_2）取值任意

$$\boxed{\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_\alpha}) - \frac{\partial L}{\partial q_\alpha} = 0}$$

利用相空间中的哈密顿原理导出哈密顿正则方程

$$\int_{t_1}^{t_2}[-\delta H+\delta(\dot{q}_\alpha p_\alpha)]dt=0$$

$\because H=H(q,p,t)$

$$\int_{t_1}^{t_2}\left[-\frac{\partial H}{\partial q_\alpha}\delta q_\alpha-\frac{\partial H}{\partial p_\alpha}\delta p_\alpha+\dot{q}_\alpha\delta p_\alpha+p_\alpha\delta\dot{q}_\alpha\right]dt=0$$

$$(\alpha=1\ 2\ 3\ldots s)$$

$$p_\alpha\frac{d}{dt}(\delta q_\alpha)=\frac{d}{dt}(p_\alpha\delta q_\alpha)-\dot{p}_\alpha\delta q_\alpha$$

$$\int_{t_1}^{t_2}\left[\left(-\frac{\partial H}{\partial q_\alpha}-\dot{p}_\alpha\right)\delta q_\alpha+\left(-\frac{\partial H}{\partial p_\alpha}+\dot{q}_\alpha\right)\delta p_\alpha\right]dt=0$$

$$\left(-\frac{\partial H}{\partial q_\alpha}-\dot{p}_\alpha\right)\delta q_\alpha+\left(-\frac{\partial H}{\partial p_\alpha}+\dot{q}_\alpha\right)\delta p_\alpha=0$$

δq_α和δp_α在（t_1　t_2）取值任意

$$\dot{p}_\alpha=-\frac{\partial H}{\partial q_\alpha}\qquad\qquad \dot{q}_\alpha=\frac{\partial H}{\partial p_\alpha}$$

Maupertuis –Lagrange Principle of Least Action

$$\Delta \mathrm{M} = \Delta \int_l p_\alpha \dot{q}_\alpha dt = 0,$$

$$\Delta q\big|_{t1} = \Delta q\big|_{t2} = 0, \quad \Delta \mathrm{H} = 0$$

$$p_\alpha \dot{q}_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha}\dot{q}_\alpha = \frac{\partial T}{\partial \dot{q}_\alpha}\dot{q}_\alpha = 2T$$

$$\Delta \int_l m v^2 dt = 0$$

$$\Delta \int_l m v ds = 0 \Rightarrow \Delta \int_l v ds = 0 \Rightarrow \Delta\left(t_2 - t_1\right) = 0$$

哈密顿正则方程的应用

正则方程解题步骤

- 分析约束, 确定自由度
- 选好广义坐标
- 写出系统的T,V,L
- 写出 $H = -L + \dot{q}_\alpha \dfrac{\partial L}{\partial \dot{q}_\alpha} \quad (\alpha = 1,2,3.\ldots\ldots)$
- 代入正则方程求解

H=H(q,p,t)

Attention:

广义动量 $p_\alpha = \dfrac{\partial L}{\partial \dot{q}_\alpha} = p_\alpha(q,\dot{q},t)$

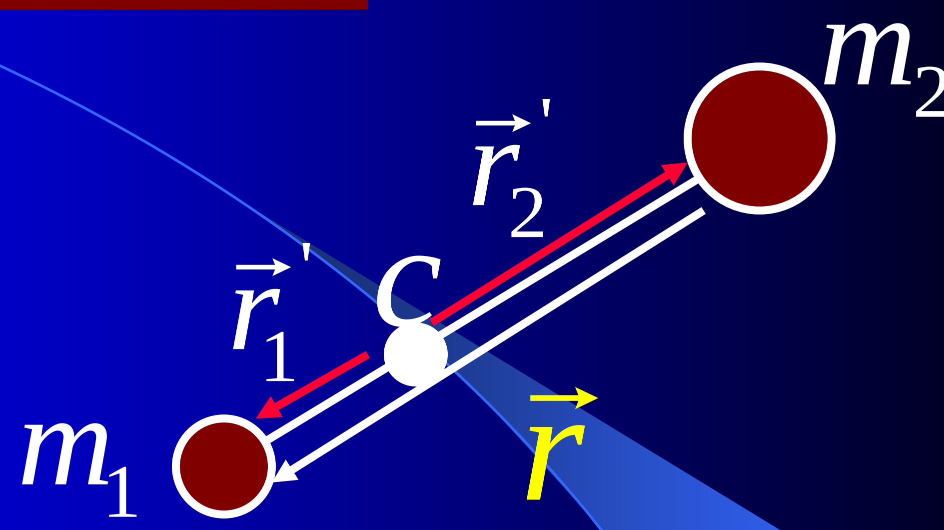
$\dot{q}_\alpha = \dot{q}_\alpha(q,p,t)$

例1：双原子分子哈密顿量

相对质心位矢

$$\vec{r}_1{}' \qquad \vec{r}_2{}'$$

相位矢 $\longrightarrow$ $\vec{r}$



$$\because m_1\vec{r}_1{}' + m_2\vec{r}_2{}' = 0 \qquad \vec{r} = \vec{r}_1{}' - \vec{r}_2{}'$$

$$\vec{r}_1{}' = \frac{m_2}{m_1+m_2}\vec{r} \qquad \vec{r}_2{}' = -\frac{m_1}{m_1+m_2}\vec{r}$$

分子运动

双原子

质心运动 $\Rightarrow$ (x, y, z)

相对质心运动 $\Rightarrow$ (r, θ, φ)

相对质心运动

$$m_1\ddot{\vec{r}}_1' = \vec{f}_{12} \quad (1) \qquad m_2\ddot{\vec{r}}_2' = \vec{f}_{21} \quad (2)$$

$$(1)\times m_2 - (2)\times m_1 \qquad \vec{f}_{12} = -\vec{f}_{21} = \vec{f}$$

$$\frac{m_1 m_2}{m_1 + m_2}\ddot{\vec{r}} = \vec{f} \qquad \mu = \frac{m_1 m_2}{m_1 + m_2} \Rightarrow \text{折合质量}$$

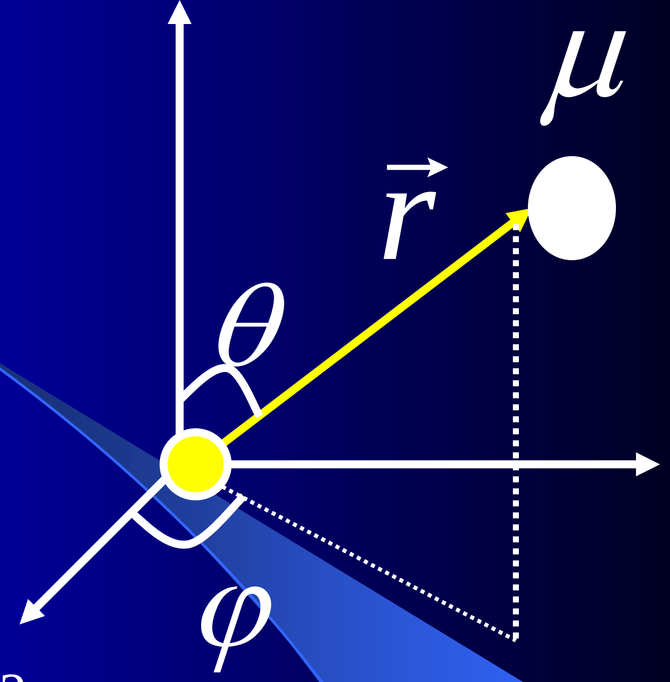
相对质心运动

$$\mu \ddot{\vec{r}} = \vec{f}$$

$$T' = \frac{1}{2}\mu \dot{\vec{r}}^{2}$$

相对质心

运动动能

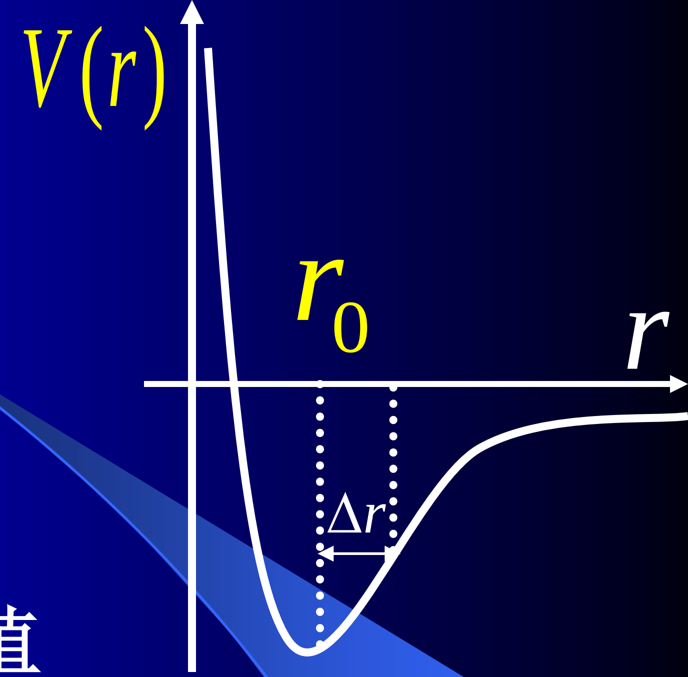


$$T' = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\varphi}^2 \sin^2\theta)$$

$$T_c = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \qquad m = m_1 + m_2$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\varphi}^2 \sin^2\theta)$$

简谐近似，V(r)在平衡位置r_0展开



$$V(r)=V(r_0)+\frac{\partial V}{\partial r}\Big|_{r_0}(r-r_0)$$

$$+\frac{1}{2}\frac{\partial^2 V}{\partial r^2}\Big|_{r_0}(r-r_0)^2+\cdots\cdots$$

取$V(r_0)=0$，在平衡位置势能取极小值

$$有\ \frac{\partial V}{\partial r}\Big|_{r_0}=0,\quad \therefore V(r)=\frac{1}{2}k\Delta r^2 \quad \Delta r=r-r_0$$

$$\boxed{L=T-V=\frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)-\frac{1}{2}k\Delta r^2}$$

$$\boxed{+\frac{1}{2}\mu(\dot{r}^2+r^2\dot{\theta}^2+r^2\dot{\varphi}^2\sin^2\theta)}$$

$$L = T - V = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{1}{2} k \Delta r^2$$

$$+ \frac{1}{2} \mu (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \dot{\varphi}^2 \sin^2 \theta)$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \Rightarrow \dot{x} = \frac{p_x}{m}$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} \Rightarrow \dot{y} = \frac{p_y}{m}$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \Rightarrow \dot{z} = \frac{p_z}{m}$$

$$L = T - V = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{1}{2}k\Delta r^2$$

$$+\frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\varphi}^2\sin^2\theta)$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = \mu\dot{r} \quad \Rightarrow \quad \dot{r} = \frac{p_r}{\mu}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2\dot{\theta} \quad \Rightarrow \quad \dot{\theta} = \frac{p_\theta}{\mu r^2}$$

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = \mu r^2\dot{\varphi}\sin^2\theta \quad \Rightarrow \quad \dot{\varphi} = \frac{p_\varphi}{\mu r^2\sin^2\theta}$$

$$H = -L + \dot{x}p_x + \dot{y}p_y + \dot{z}p_z + \dot{r}p_r + \dot{\theta}p_\theta + \dot{\varphi}p_\varphi$$

$$= \frac{1}{2m}(\dot{p}_x^2 + \dot{p}_y^2 + \dot{p}_z^2) + \frac{1}{2I}\left(p_\theta^2 + \frac{p_\varphi^2}{\sin^2\theta}\right) + \left[\frac{p_r^2}{2\mu} + V(r)\right]$$

$$I = \mu r^2$$

转动惯量

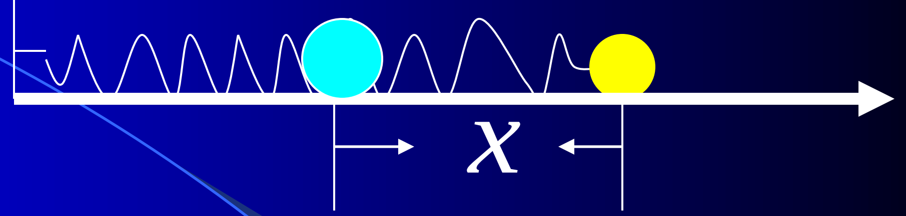
平动

转动

振动

例2　一维谐振子

demonstration



解: $v = \dfrac{dx}{dt} = \dot{x}$

动能　$T = \frac{1}{2} m v^2 = \frac{1}{2} m \dot{x}^2$

势能　$V = \dfrac{1}{2} k x^2$

$$L = T - V = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2$$

$$\frac{\partial L}{\partial x} = -kx \qquad \frac{\partial L}{\partial \dot{x}} = m\dot{x} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = m\ddot{x} + kx = 0$$

$$\boxed{\ddot{x} = -\frac{k}{m} x}$$

$$L = T - V \quad = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad \Rightarrow \quad \dot{x} = \frac{p_x}{m}$$

$$H = -L + \sum_{\alpha} \dot{q}_\alpha \frac{\partial L}{\partial \dot{q}_\alpha} = -L + \dot{x}\frac{\partial L}{\partial \dot{x}}$$

$$= -\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 + \dot{x}p_x$$

$$= -\frac{1}{2}m\left(\frac{p_x}{m}\right)^2 + \frac{1}{2}kx^2 + \frac{p_x}{m}p_x \quad = \frac{1}{2}kx^2 + \frac{p_x^2}{2m}$$

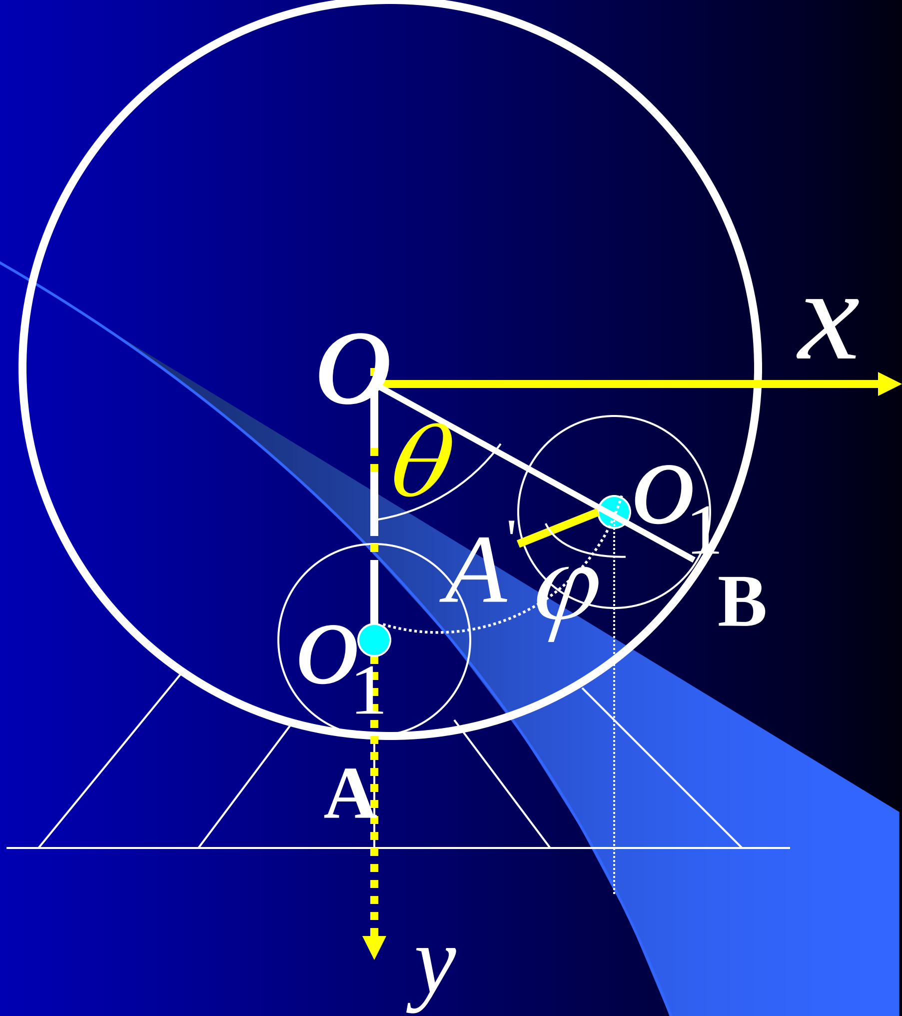
$$H = \frac{1}{2}kx^2 + \frac{p_x^2}{2m}$$

$$\left.\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p_x} = \frac{p_x}{m} \\ \dot{p}_x &= -\frac{\partial H}{\partial x} = -kx \end{aligned}\right\}$$ 根据正则方程

$$\ddot{x} = \frac{\dot{p}_x}{m} \Rightarrow \boxed{\ddot{x} = -\frac{k}{m}x}$$

例3

一半径为r，质量为m的实心圆柱体在一半径为R的大圆柱体内表面作纯滚动，试用哈密顿正则方程求其在平衡位置附近作微振动的周期。



demonstration

分析

$$\overset{\frown}{AB} = \overset{\frown}{A'B}$$

$$R\theta = r(\theta + \varphi)$$

$$\varphi = \frac{R-r}{r}\theta$$

坐标数 → 3

约束数 → 2

- $\varphi = \dfrac{R-r}{r}\theta$
- $oo_1 = R - r$

自由度 → 1

取θ为广义坐标

$$L = T - V = \frac{3}{4}m(R-r)^2\dot{\theta}^2 + mg(R-r)\cos\theta$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \frac{3}{2}m(R-r)\dot{\theta} \Rightarrow \dot{\theta} = \frac{2p_\theta}{3m(R-r)}$$

$$H = -L + \dot{\theta}p_\theta$$

$$= -\frac{3}{4}m(R-r)^2\dot{\theta}^2 - mg(R-r)\cos\theta + \dot{\theta}p_\theta$$

$$\boxed{H = \frac{p_\theta^2}{3m(R-r)^2} - mg(R-r)\cos\theta}$$

$$H=\frac{p_\theta^2}{3m(R-r)^2}-mg(R-r)\cos\theta$$

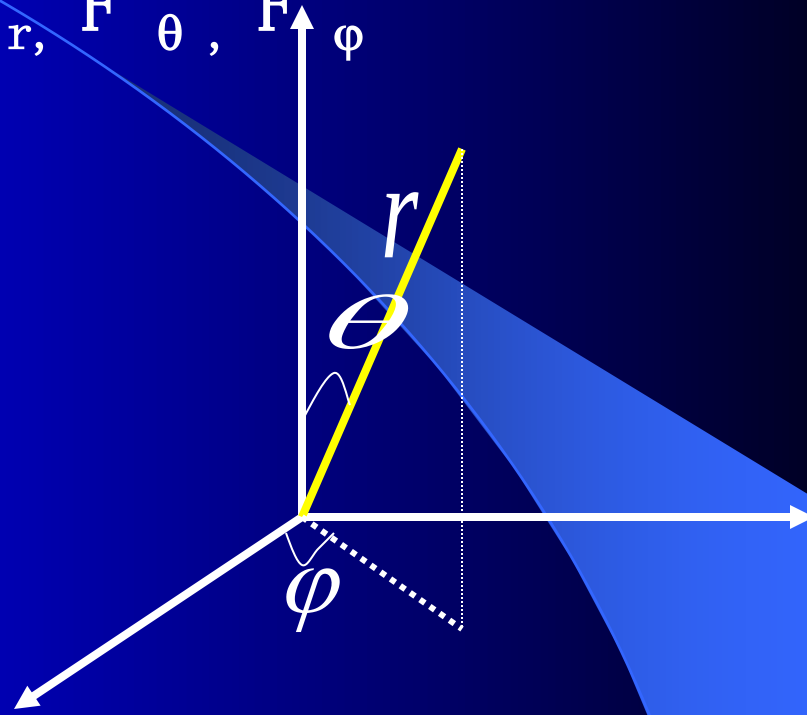
根据正则方程

$$\begin{cases}\dot{\theta}=\dfrac{\partial H}{\partial p_\theta}=\dfrac{2p_\theta}{3m(R-r)^2}\\ \dot{p}_\theta=-\dfrac{\partial H}{\partial \theta}=-mg(R-r)\sin\theta\end{cases}$$

$$\ddot{\theta}=\frac{2\dot{p}_\theta}{3m(R-r)^2}=-\frac{2g\sin\theta}{3(R-r)}\approx-\frac{2g}{3(R-r)}\theta$$

$$\omega=\sqrt{\frac{2g}{3(R-r)}}$$

例4 用哈密顿正则方程求自由质点在球坐标下加速度的表达式. 设其受力在r, θ, φ三个方向的分量分别为F_r, F_θ, F_φ



解: 广义力

非保守系拉氏方程

必先求动能

$$T=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2+r^2\sin^2\theta\dot{\varphi}^2)$$

$$p_r=\frac{\partial T}{\partial \dot{r}}=m\dot{r} \Rightarrow \dot{r}=\frac{p_r}{m}$$

$$p_\theta=\frac{\partial T}{\partial \dot{\theta}}=mr^2\dot{\theta} \Rightarrow \dot{\theta}=\frac{p_\theta}{mr^2}$$

$$p_\varphi=\frac{\partial T}{\partial \dot{\varphi}}=mr^2\dot{\varphi}\sin\theta \Rightarrow \dot{\varphi}=\frac{p_\varphi}{mr^2\sin^2\theta}$$

$$H=-T+\dot{r}p_r+\dot{\theta}p_\theta+\dot{\varphi}p_\varphi=\frac{1}{2m}(p_r^2+\frac{p_\theta^2}{r^2}+\frac{p_\varphi^2}{r^2\sin^2\theta})$$

根据正则方程

$$H=\frac{1}{2m}\left(p_r^2+\frac{p_\theta^2}{r^2}+\frac{p_\varphi^2}{r^2\sin^2\theta}\right)$$

$$\begin{cases}\dot{p}_r=-\dfrac{\partial H}{\partial r}+Q_r=\dfrac{p_\theta^2}{mr^3}+\dfrac{p_\varphi^2}{mr^3\sin^2\theta}+Q_r\\[2ex]\dot{p}_\theta=-\dfrac{\partial H}{\partial \theta}+Q_\theta=\dfrac{p_\varphi^2\cos\theta}{mr^2\sin^3\theta}+Q_\theta\\[2ex]\dot{p}_\varphi=-\dfrac{\partial H}{\partial \varphi}+Q_\varphi=Q_\varphi\end{cases}$$

$$\because Q_r=F_r\qquad Q_\theta=rF_\theta\qquad Q_\varphi=r\sin\theta F_\varphi$$

$$p_r = m\dot{r} \qquad \dot{p}_r = m\ddot{r}$$

$$\because \dot{p}_r = \frac{p_\theta^2}{mr^3} + \frac{\qquad}{m \cdots {}^2\theta} + Q_r$$

$$m\ddot{r} = \frac{(mr^2\dot{\theta})^2}{mr^3} + \frac{(mr^2\dot{\varphi}\sin^2\theta)^2}{mr^3\sin^2\theta} + F_r$$

$$a_r = \frac{F_r}{m} = \ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2\sin^2\theta$$

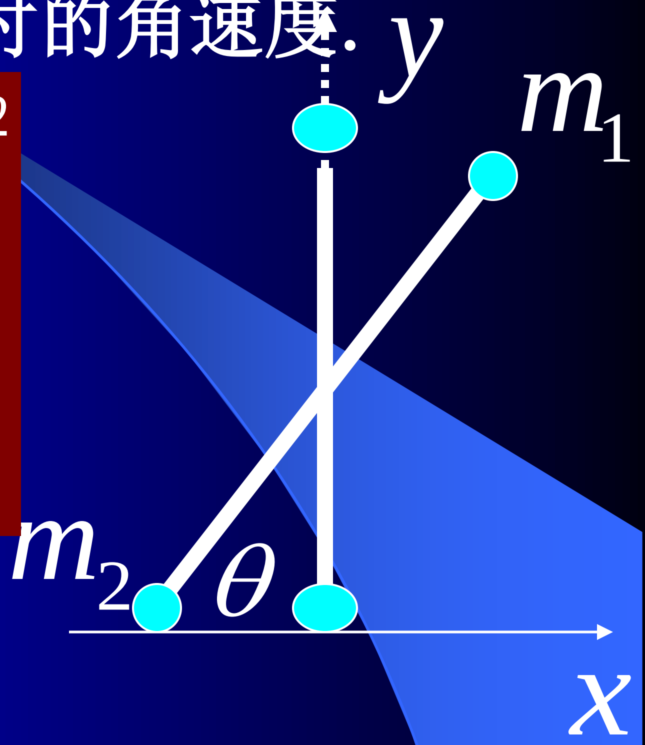
同理可得

$$a_{\varphi}=\frac{F_{\varphi}}{m}=r\ddot{\varphi}\sin\theta+2\dot{r}\dot{\varphi}\sin\theta+2r\dot{\theta}\dot{\varphi}\cos\theta$$

$$a_{\theta}=\frac{F_{\theta}}{m}=r\ddot{\theta}+2\dot{r}\dot{\theta}-r\dot{\varphi}^{2}\sin\theta\cos\theta$$

$$a_{r}=\frac{F_{r}}{m}=\ddot{r}-r\dot{\theta}^{2}-r\dot{\varphi}^{2}\sin^{2}\theta$$

例5 质量为m的相同二质点用一长为l的轻杆连接初始时直立静止在光滑水平面上, 以后任其倒下, 试用正则方程求杆落地时的角速度.



分析

$$(x_2 - x_1)^2 + y_1^2 = l^2$$

$$m_1 x_1 + m_2 x_2 = 0$$

$$x_1 = -x_2$$

坐标数 → 3

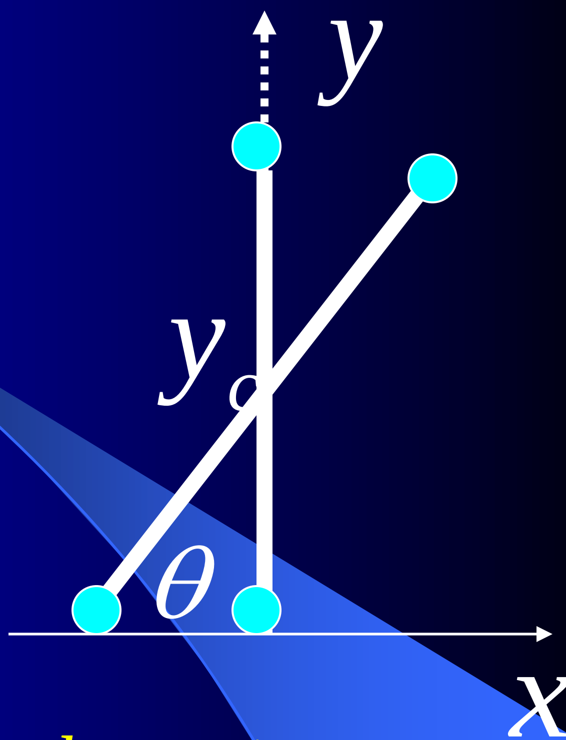
约束数 → 2

自由度数 → 1

取如图所示 θ 为广义坐标

$$y_c = \frac{l}{2}\sin\theta$$

$$\dot{y}_c = \frac{l}{2}\dot{\theta}\cos\theta$$



根据柯尼西定理

$$T = \frac{1}{2}2m\dot{y}_c^2 + \frac{1}{2}I_c\dot{\theta}^2$$

$$I_c = 2m\left(\frac{l}{2}\right)^2 = \frac{1}{2}ml^2$$

$$T = \frac{1}{4}ml^2\dot{\theta}^2(1+\cos^2\theta)$$

$$V = mgl\sin\theta$$

$$L = T - V = \frac{1}{4}ml^2\dot{\theta}^2(1+\cos^2\theta) - mgl\sin\theta$$

$$L = T - V = \frac{1}{4} m l^2 \dot{\theta}^2 (1 + \cos^2\theta) - mgl \sin\theta$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2} m l^2 \dot{\theta} (1 + \cos^2\theta)$$

$$\dot{\theta} = \frac{2 p_\theta}{m l^2 (1 + \cos^2\theta)}$$

$$H = -L + \dot{\theta} p_\theta$$

$$= -\frac{1}{4} m l^2 \dot{\theta}^2 (1 + \cos^2\theta) + mgl \sin\theta + \dot{\theta} p_\theta$$

$$\boxed{H = \frac{p_\theta^2}{m l^2 (1 + \cos^2\theta)} + mgl \sin\theta}$$

$$H=\frac{p_\theta^2}{ml^2(1+\cos^2\theta)}+mgl\sin\theta$$

$$\dot p_\theta=-\frac{\partial H}{\partial\theta}=\frac{2\sin\theta\cos\theta p_\theta^2}{ml^2(1+\cos^2\theta)^2}-mgl\cos\theta$$

$$p_\theta=\frac{1}{2}ml^2\dot\theta(1+\cos^2\theta)$$

$$\ddot\theta=\frac{2\dot p_\theta}{ml^2(1+\cos^2\theta)}+\frac{4p_\theta\dot\theta\cos\theta\sin\theta}{ml^2(1+\cos^2\theta)^2}$$

$$l\ \ddot\theta(1+\cos^2\theta)-l\ \dot\theta^2\sin\theta\cos\theta+2g\cos\theta=0$$

$$H=\frac{p_\theta^2}{ml^2(1+\cos^2\theta)}+mgl\sin\theta$$

$$\because \frac{\partial H}{\partial t}=0$$

$$(x_2-x_1)^2+y_1^2=l^2$$

$$m_1x_1+m_2x_2=0$$

$$x_1=-x_2$$

稳定约束

$$H=T+V$$

$$=\frac{1}{4}ml^2\dot{\theta}^2(1+\cos^2\theta)+mgl\sin\theta=E_0=mgl$$

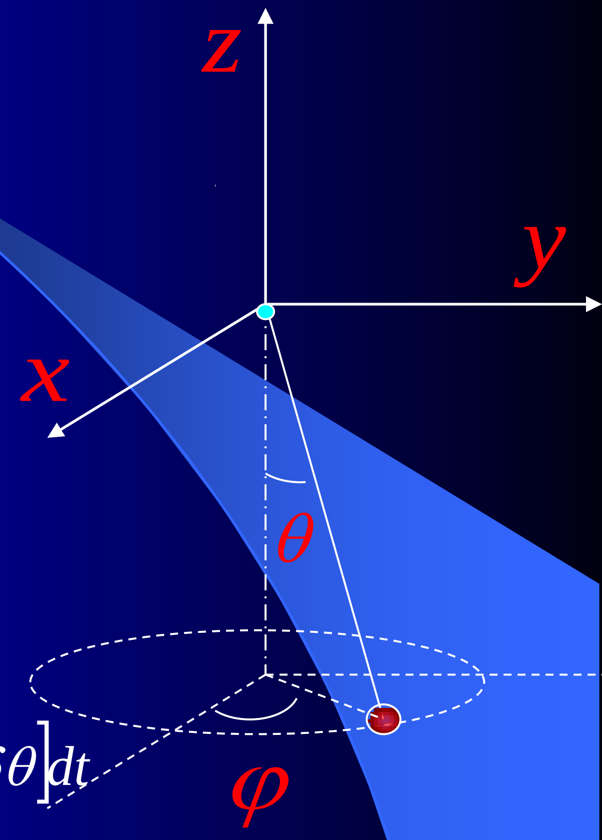
$$\dot{\theta}\Big|_{\theta=0}=\sqrt{\frac{2g}{l}}$$

Ex：质量为m,长为L的球面摆，经角为φ，摆线与竖直线夹角为θ,利用Hamilton原理求系统运动微分方程。

Solution:

The Lagrangian for the system is,

$$L=\frac{1}{2}ml^2(\dot\theta^2+\dot\varphi^2\sin^2\theta)+mgl\cos\theta,$$



The variation of the action is,

$$\delta S=\delta\int_0^t L dt=\int_0^t \delta\left[\frac{1}{2}ml^2(\dot\theta^2+\dot\varphi^2\sin^2\theta)+mgl\cos\theta\right]dt$$

$$=\int_0^t\left[ml^2(\dot\theta\delta\dot\theta+\dot\varphi\delta\dot\varphi\sin^2\theta+\dot\varphi^2\sin\theta\cos\theta\delta\theta)-mgl\sin\theta\delta\theta\right]dt$$

$$=\int_0^t\left\{ml^2\left[\frac{\mathrm{d}}{\mathrm{dt}}(\dot\theta\delta\theta)-\ddot\theta\delta\theta+\dot\varphi\delta\dot\varphi\sin^2\theta+\dot\varphi^2\sin\theta\cos\theta\delta\theta\right]-mgl\sin\theta\delta\theta\right\}dt$$

$$=\int_0^t\left\{ml^2\left[-\ddot\theta+\dot\varphi^2\sin\theta\cos\theta-\frac{g}{l}\sin\theta\right]\delta\theta-\frac{d}{dt}\left(ml^2\dot\varphi\sin^2\theta\right)\delta\varphi\right\}dt$$

$$\delta\theta(0,t) \neq 0, \delta\varphi(0,t) \neq 0$$

From the Hamilton principle, then we have the following equations for the dynamical system:

$$\begin{cases} ml^2\dot{\varphi}\sin^2\theta = const. \\ l\ddot{\theta} - l\dot{\varphi}^2\sin\theta\cos\theta + g\sin\theta = 0 \end{cases}$$

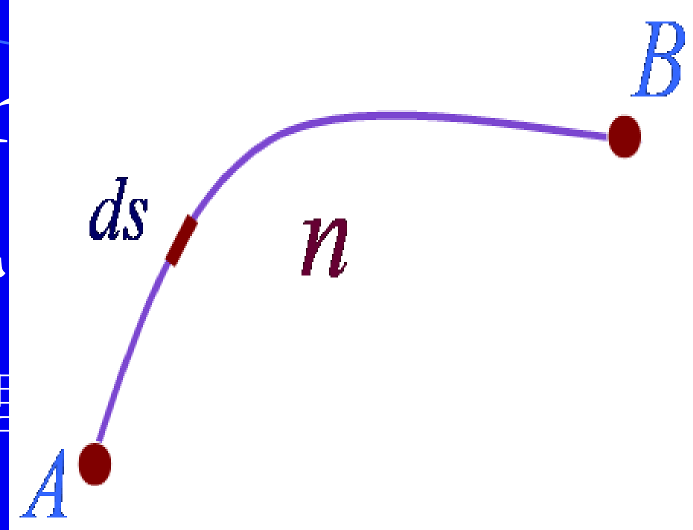
P.S., because $\dfrac{\partial L}{\partial t} = 0,$ **we have the conserved energy,**

$$H = \frac{1}{2}ml^2(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta) - mgl\cos\theta = E_0$$

Fermat Pr

光传播的几 之

故费马原理 的

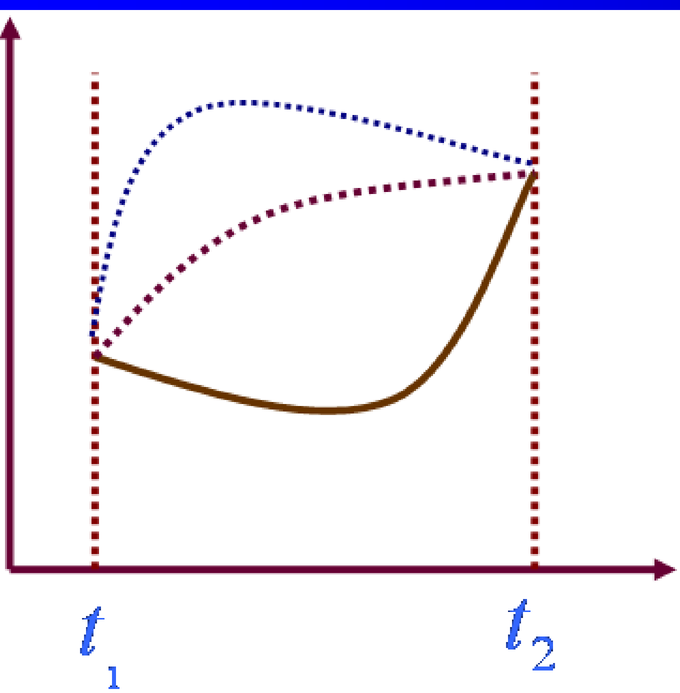


$$\frac{x}{\sqrt{x^2+a^2}} = \sin\theta_1$$

$$\frac{d-x}{\sqrt{(d-x)^2+b^2}} = \sin\theta_2$$

取极值

$$\delta \int_l n ds = 0$$



$$t(x) = \frac{\sqrt{x^2+a^2}}{v_1} + \frac{\sqrt{(d-x)^2+b^2}}{v_2}$$

$$(x) = \frac{x}{\sqrt{x^2+a^2}}\frac{1}{v_1} - \frac{d-x}{\sqrt{(d-x)^2+b^2}}\frac{1}{v_2} = 0$$

$$\delta S = \delta \int_{t_1}^{t_2} L dt = 0$$

为什么Broglie会萌发物质波粒二象性概念?

$$\delta\int_l mv\,ds = 0$$

$$\delta\int_l n\,ds = 0$$

$$E = h\nu$$

粒子性

Broglie关系

波动性

$$P = \frac{h}{\lambda}$$

$$S = \int_l p\,ds = \int_l \frac{h}{\lambda}ds = \frac{h}{\lambda_0}\int_l \frac{\lambda_0}{\lambda}ds = \frac{h}{\lambda_0}\int_l \frac{c}{u}ds = \frac{h}{\lambda_0}\int_l n\,ds,$$

λ_0: 真空中光的波长。

如速度v=5.0×10^{2}m/s飞行的子弹，质量为m=10^{-2}Kg，对应的德布罗意波长为:

$$\lambda = \frac{h}{mv} = 1.3 \times 10^{-25} nm$$

太小测不到！

如电子m=9.1×10^{-31}Kg，速度v=5.0×10^{7}m/s，对应的德布罗意波长为:

$$\lambda = \frac{h}{mv} = 1.4 \times 10^{-2} nm$$

X射线波段

Summary:

- 泛函概念
- 变分法概念、计算法则、对易关系
- 泛函的极值
- 位形空间中的哈密顿原理

$$\delta s = \delta \int_{t_1}^{t_2} L dt = 0 \qquad \int_{t_1}^{t_2} (\delta T + \delta W) dt = 0$$

- 相空间中的哈密顿原理 $\boxed{\because\ H = H(q, p, t)}$

$$\delta s = \delta \int_{t_1}^{t_2} (-H + \dot{q}_\alpha p_\alpha) dt = 0$$

§ 8.泊松括号及泊松定理

(Poisson Bracket and Poisson's Theorem)

一.泊松括号的定义

若f = f (q.P.t), φ= φ (q.P.t)

则定义 :$[f\ \varphi]=\dfrac{\partial f}{\partial q_\alpha}\dfrac{\partial \varphi}{\partial p_\alpha}-\dfrac{\partial f}{\partial p_\alpha}\dfrac{\partial \varphi}{\partial q_\alpha}$ ⇒ 泊松括号

$(\alpha = 1, 2, 3 \ldots\ldots s)$

二. 泊松括号的性质

- $[q_\alpha\ P_\beta]=\delta_{\alpha\beta}=\begin{cases}1, & \alpha=\beta\\ 0, & \alpha\neq\beta\end{cases}$

  $[q_\alpha\ q_\beta]=[P_\alpha\ P_\beta]=0$

- 若c为常数, 则$[c\ \ f\,]=0$
- 反对易性: $[f\ \ \varphi]=-[\varphi\ f\,]$
- 分配律: $[\Sigma\varphi_i\ \ \psi]=\Sigma[\varphi_i\ \ \psi]$
- 结合律: $[\varphi_1\varphi_2\ \ \ \varphi_3\,]=\varphi_1[\,\varphi_2\ \ \varphi_3\,]+\varphi_2[\,\varphi_1\ \ \varphi_3\,]$
- 求导运算

$$\frac{\partial}{\partial x}[\varphi_1\ \ \varphi_2]=\left[\frac{\partial\varphi_1}{\partial x}\ \ \varphi_2\right]+\left[\varphi_1\ \ \frac{\partial\varphi_2}{\partial x}\right]$$

- 线性: $[a\varphi_1+b\,\varphi_2\ \ \psi]= a[\varphi_1\ \ \psi]+b\,[\varphi_2\ \ \psi]$
- 雅可比关系:

$$[\varphi_1\,[\varphi_2\,\varphi_3\,]]+[\varphi_2\,[\varphi_3\,\varphi_1\,]]+[\varphi_3\,[\varphi_1\,\varphi_2\,]]=0$$

三. 用泊松括号表示的运动方程

如果 $\varphi=\varphi(q\ \ p\ \ t)$ t为参量则

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t}=\frac{\partial\varphi}{\partial t}+\frac{\partial\varphi}{\partial q_\alpha}\dot{q}_\alpha+\frac{\partial\varphi}{\partial p_\alpha}\dot{p}_\alpha\qquad(\alpha=1.2\ldots.s)$$

$$=\frac{\partial\varphi}{\partial t}+[\varphi\ \ H]$$

若 $\varphi=q_\alpha$ 则 $\dot{q}_\alpha=[q_\alpha\ \ H]$

$\varphi=p_\alpha$ 则 $\dot{p}_\alpha=[p_\alpha\ \ H]$

正则方程

四.

$$\frac{d\varphi}{dt}=\frac{\partial\varphi}{\partial t}+[\varphi\ H]=\frac{\partial\varphi}{\partial t}+\frac{\partial\varphi}{\partial q_\alpha}\frac{\partial H}{\partial p_\alpha}-\frac{\partial\varphi}{\partial p_\alpha}\frac{\partial H}{\partial q_\alpha}=0$$

一阶齐次线性偏微分方程　对应特征方程

$$\frac{dt}{1}=\frac{dq_1}{\frac{\partial H}{\partial p_1}}=\frac{dq_2}{\frac{\partial H}{\partial p_2}}\cdots\cdots=\frac{dq_s}{\frac{\partial H}{\partial p_s}}=\frac{dp_1}{\frac{\partial H}{\partial q_1}}=\frac{dp_2}{\frac{\partial H}{\partial q_2}}\cdots\cdots=\frac{dp_s}{\frac{\partial H}{\partial q_w}}$$

五. 泊

正则方程

也是一

$$P\frac{\partial z}{\partial x}+Q\frac{\partial z}{\partial y}=R(x,y,z)$$

特征方程

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

$$\frac{d}{dt}[\varphi_1\ \varphi \quad \varphi_2]=0$$

证明：根据雅可比关系有

$\because \varphi_2$是运动积分

$\because \dfrac{\partial \varphi_2}{\partial t} + [\varphi_2\ H] = 0$

$\because \varphi_1$是运动积分

$\because \dfrac{\partial \varphi_1}{\partial t} + [\varphi_1\ H] = 0$

$\varphi_1\ \varphi_2\ ]\ ]=0$

$$[\varphi_1\ [\varphi_2\ H\ ]\ ] + [\varphi_2\ [H\ \varphi_1\ ]\ ] = [\ [\varphi_1\ \varphi_2\ ]\ H\ ]$$

$$\left[\varphi_1\ -\frac{\partial \varphi_2}{\partial t}\right] + \left[\varphi_2\ \frac{\partial \varphi_1}{\partial t}\right]$$

$$-\frac{\partial}{\partial t}[\varphi_1\quad \varphi_2]$$

§9. 时空对称性和守恒定律(Symmetry and Conservation law)

一. 时空对称性,不可观测量和守恒定律互为因果关系

不可观测量	时空对称性	守恒定律
绝对位置or绝对坐标原点	空间平移不变性	动量守恒
绝对时间or绝对时间原点	时间平移	能量守恒
空间绝对方位	空间旋转不变性	角动量守恒
空间左右	空间反演不变性	宇称守恒
正反粒子不可区分	电荷共轭变换	C宇称守恒

二.经典力学中的对称性和守恒定律

时间平移与机械能守恒

Time Translation and

Energy Conservation

时间平移与机械能守恒

若拉氏函数具有时间平移不变性

t ⇒ t + δt ⇒ L ⇒ L + δL

$$\delta L = \frac{\partial L}{\partial t}\delta t = 0$$

$$\because \delta t \neq 0 \quad \Rightarrow \quad \frac{\partial L}{\partial t} = 0$$

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} = 0$$ ⇒ **能量守恒**

空间平移与动量守恒

(Spatial Translation and Momentum Conservation)

空间平移与动量守恒

将空间整体平移 $\delta\vec{r}$

$$\vec{r}_i \Longrightarrow \vec{r}_i+\delta\vec{r}$$

$$\delta L=\sum_{i=1}^{N}\frac{\partial L}{\partial\vec{r}_i}\delta\vec{r}=0$$

$$\because\ \delta\vec{r}\neq 0 \Longrightarrow \sum\frac{\partial L}{\partial\vec{r}_i}=0$$

$$\Downarrow$$

$$\sum\left[\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\vec{r}}_i}\right)-\frac{\partial L}{\partial\vec{r}_i}\right]=0 \Longrightarrow \sum\vec{P}_i=\vec{P}_0\text{(守恒量)}$$

空间旋转不变性与角动量守恒

(Space rotation Invariance and Conservation of Angular momentum)

$$\delta \vec{r}_i = \delta\vec{\varphi} \times \vec{r}_i$$

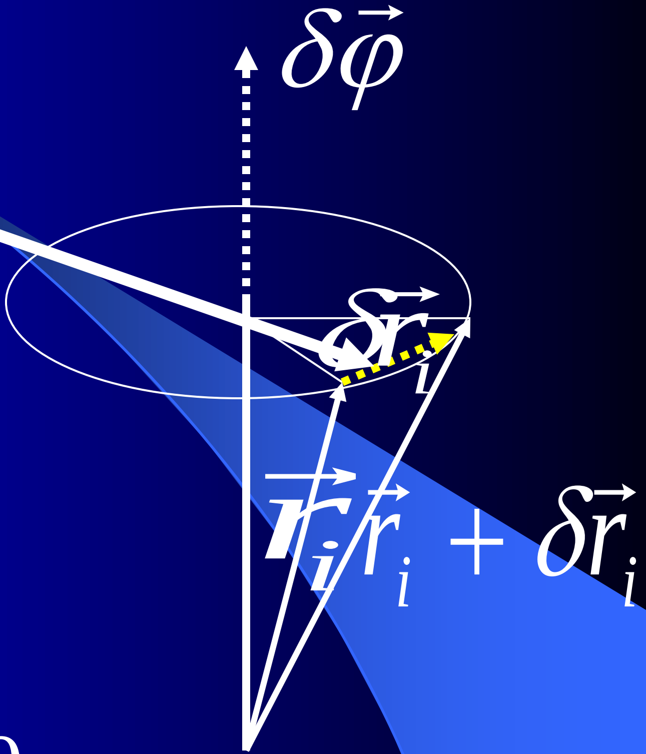
$$\delta \dot{\vec{r}}_i = \delta\vec{\varphi} \times \dot{\vec{r}}_i$$

$$\delta L = \frac{\partial L}{\partial \vec{r}_i}\delta\vec{r}_i + \frac{\partial L}{\partial \dot{\vec{r}}_i}\delta\dot{\vec{r}}_i = 0$$

$$\vec{a} \bullet (\vec{b} \times \vec{c}) = \vec{b} \bullet (\vec{c} \times \vec{a})$$

$$\delta L = \frac{\partial L}{\partial \vec{r}_i}\delta\vec{\varphi} \times \vec{r}_i + \frac{\partial L}{\partial \dot{\vec{r}}_i}\delta\vec{\varphi} \times \dot{\vec{r}}$$

$$= \delta\vec{\varphi}\left[\left(\vec{r}_i \times \frac{\partial L}{\partial \vec{r}_i}\right) + \left(\dot{\vec{r}} \times \frac{\partial L}{\partial \dot{\vec{r}}_i}\right)\right] = 0$$



$\because \delta\vec{\varphi} \neq 0$

$$\delta L = \delta\vec{\varphi} \bullet (\vec{r}_i \times \dot{\vec{P}}_i + \dot{\vec{r}}_i \times \vec{P}_i)$$

$$= \delta\vec{\varphi}\sum\frac{d}{dt}(\vec{r}_i \times \vec{P}_i) = \delta\vec{\varphi}\frac{d}{dt}\sum\vec{J}_i = 0$$

$\therefore \vec{J} = \sum_i \vec{J}_i$ 即为守恒量