"经典力学" 第一章 习题

1-1: 第一球上升高度：$h=\frac{V_0^2}{2g}$，$t=\frac{V_0}{g}$. 然后下降：

$S_1=\frac{1}{2}gt_1^2$ ，第二球上升：$S_2=V_0t_2-\frac{1}{2}gt_2^2$.

t_1和t_2的关系：$t_1=t_2-\frac{V_0}{g}+t_0$ （或 $t_2=t_1+\frac{V_0}{g}-t_0$）.

$h=S_1+S_2$，$S_2=h-S_1$

$$h=S_1+S_2=\frac{1}{2}gt_1^2+V_0t_2-\frac{1}{2}gt_2^2=\frac{1}{2}gt_1^2+V_0\left(t_1+\frac{V_0}{g}-t_0\right)-\frac{1}{2}g\left(t_1+\frac{V_0}{g}-t_0\right)^2=\frac{V_0^2}{2g}$$

由上式解出 $t_1=\frac{1}{2}t_0$.

∴ 相遇高度 $S_2=h-S_1=\frac{V_0^2}{2g}-\frac{1}{2}g\left(\frac{t_0}{2}\right)^2=\frac{V_0^2}{2g}-\frac{gt_0^2}{8}$.

1-2: (1). $\dot{x}=-6e^{-2t}$，$\dot{y}=12\cos 3t$. $\dot{z}=-15\sin 3t$

$\vec{V}=-6e^{-2t}\vec{i}+12\cos 3t\,\vec{j}-15\sin 3t\,\vec{k}$.

$\ddot{x}=12e^{-2t}$，$\ddot{y}=-36\sin 3t$. $\ddot{z}=-45\cos 3t$

$\vec{a}=12e^{-2t}\vec{i}-36\sin 3t\,\vec{j}-45\cos 3t\,\vec{k}$.

(2) $t=0$. $\vec{V}=-6\vec{i}+12\vec{j}$ $\vec{a}=12\vec{i}-45\vec{k}$.

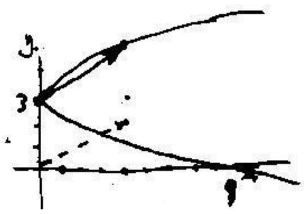
1-3: $\vec{V}=-a\omega\sin\omega t\,\vec{i}+a\omega\cos\omega t\,\vec{j}+b\vec{k}$

$\vec{a}=-a\omega^2\cos\omega t\,\vec{i}-a\omega^2\sin\omega t\,\vec{j}$ 作匀速上升的螺旋运动.

1-4: (1). 由题设可得质点运动的参数方程.

$x=4t^2$，$y=2t+3$. ，消除参数t，$t=\frac{y-3}{2}$

∴ $x=(y-3)^2$. 这是抛物线方程.



(2). $\Delta\vec{r}=\vec{r}(t=1)-\vec{r}(t=0)$

$=4\vec{i}+5\vec{j}-3\vec{j}=4\vec{i}+2\vec{j}$

(3). $\vec{V}=8t\vec{i}+2\vec{j}$.

$\vec{V}(1)=8\vec{i}+2\vec{j}$，$\vec{V}(0)=2\vec{j}$.

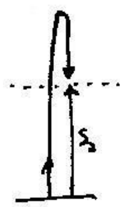
(4). $\vec{a}=\dot{\vec{V}}=8\vec{i}$ $\vec{a}(1)=\vec{a}(0)=8\vec{i}$.

1-1: 方法二: 设在第二物体抛出后 t_2 秒相遇. 第一物体上升后下落，如图：

$V_0(t_2+t_0)-\frac{1}{2}g(t_2+t_0)^2=V_0t_2-\frac{1}{2}gt_2^2$

解得 $t_2=\frac{V_0}{g}-\frac{t_0}{2}$

$S_2=V_0\left(\frac{V_0}{g}-\frac{t_0}{2}\right)-\frac{g}{2}\left(\frac{V_0}{g}-\frac{t_0}{2}\right)^2=\frac{V_0^2}{2g}-\frac{gt_0^2}{8}$.



1-5. 证明：因作直线运动，下面的常用积分公式.

(一). $a=\frac{dv}{dt}=-f(t)$ $\quad dv=-f(t)dt$. $\quad v(t)-v(0)=-\int_0^t f(t')dt'$ (1)

按题意 $t=T$ 时 $v(T)=0$. $\quad \therefore v(0)=\int_0^T f(t')dt'$ (2)

任一时刻 t 有 $v(t)=$ (2) 代入 (1) $=v(t)=\int_0^T f(t')dt'-\int_0^t f(t')dt'$

$ds=v(t)dt$, $\quad S=\int_0^S ds=\int_0^T v(t)dt$

$$S=\int_0^T\Big[\underbrace{\int_0^T f(t')dt'}_{\text{常数}}-\int_0^t f(t')dt'\Big]dt$$

$$=\Big[t\int_0^T f(t')dt'\Big]_0^T-\int_0^T\Big[\int_0^t f(t')dt'\Big]dt \quad \text{分部积分}$$

$$=T\int_0^T f(t')dt'-\Big[t\int_0^t f(t')dt'\Big]_0^T+\int_0^T t\, d\Big[\int_0^t f(t')dt'\Big]$$

$=$ 应用积分求导公式 $\frac{d}{dt}\int_a^{b(t)} f(x)dx=f[b(t)]\frac{db(t)}{dt}$

$$=\int_0^T t\, f(t)\frac{dt}{dt}dt=\int_0^T t f(t)dt.$$

(二). 方法二:

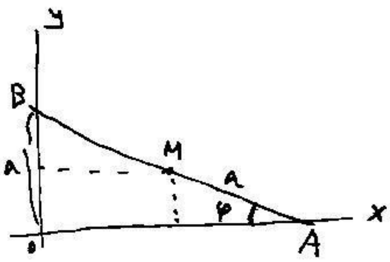
$dv=-f(t)dt$. 两边乘 $-t$ 再从0到T积分:

$\int_0^T -t\,dv=\int_0^T t f(t)dt$. $\quad$ 而

$$-\int_0^T t\,dv=-tv\Big|_0^T+\int_0^T v\,dt \overset{t=T,\ v=0}{=\!=} \int_0^T v\,dt=S.$$

$\therefore S=\int_0^T t f(t)dt$.

1-6. (一). 轨迹方程: M点坐标 x_M. y_M.



$$x_M=x_A-a\cos\varphi=a\,\mathrm{ctg}\,\varphi-a\cos\varphi$$

$$y_M=a\sin\varphi.$$

消去参数 φ:

$$x_M=a\cos\varphi\left(\frac{1}{\sin\varphi}-1\right)=a\sqrt{1-\sin^2\varphi}\left(\frac{1}{\sin\varphi}-1\right)$$

将 $\sin\varphi=\frac{y_M}{a}$ 代入上式

$$x_M=a\sqrt{1-y_M^2/a^2}\left(\frac{a}{y_M}-1\right)$$

$$=\sqrt{a^2-y_M^2}\left(\frac{a}{y_M}-1\right).$$

(二).
$$\begin{cases}\dot{x}_M = \dot{x}_A + a\sin\varphi\dot{\varphi} & (1)\\ \dot{y}_M = a\cos\varphi\dot{\varphi} & (2)\end{cases}$$

利用 $\dot{x}_A = V_0$, $x_A = a\,ctg\varphi = a\frac{\cos\varphi}{\sin\varphi}$

$V_0 = a\frac{-(\sin^2\varphi+\cos^2\varphi)}{\sin^2\varphi}\dot{\varphi} = -\frac{a}{\sin^2\varphi}\dot{\varphi}$. $\dot{\varphi} = -\frac{V_0}{a}\sin^2\varphi$ 代入①.②式

$\dot{x}_M = V_0(1-\sin^3\varphi)$, $\dot{y}_M = -V_0(\cos\varphi-\cos^3\varphi) = -V_0\cos\varphi\sin^2\varphi$

$V^2 = \dot{x}_M^2 + \dot{y}_M^2 = V_0^2(1-2\sin^3\varphi+\sin^4\varphi)$.

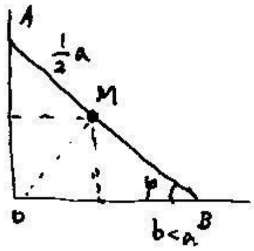
(三). $\ddot{x}_M = -V_0 3\sin^2\varphi\cos\varphi\dot{\varphi} = \frac{3V_0^2}{a}\sin^4\varphi\cos\varphi$

$\ddot{y}_M = -V_0(-\sin\varphi\dot{\varphi} + 3\cos^2\varphi\sin\varphi\dot{\varphi}) = \frac{V_0^2}{a}\sin^3\varphi(3\cos^2\varphi-1)$.

$a_M^2 = \ddot{x}_M^2 + \ddot{y}_M^2 = -6+6\sin^2\varphi+9\sin^6\varphi-9\sin^8\varphi$.

为什么不能用 $a=\frac{dv}{dt}$ 求出?

1-7:(一) 由几何分析可知:$OM=\frac{1}{2}a$.



$x_M = \frac{a}{2}\cos\varphi$. $y_M = \frac{a}{2}\sin\varphi$

$\therefore\ x_M^2 + y_M^2 = \frac{a^2}{4}$

(二). $\dot{x}_M = -\frac{a}{2}\sin\varphi\dot{\varphi}$ — $\dot{\varphi} = -\frac{V_0}{a\sin\varphi}$

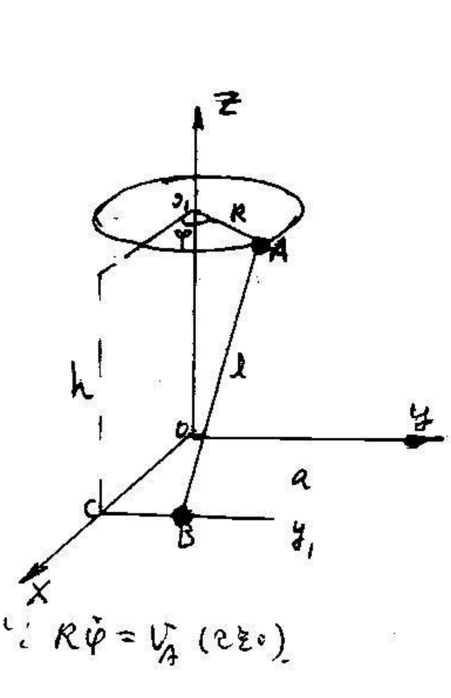
$\dot{y}_M = \frac{a}{2}\cos\varphi\dot{\varphi}$

$\therefore\ \dot{x}_M = \frac{V_0}{2}$, $\dot{y}_M = -\frac{V_0}{2}ctg\varphi$, $ctg\varphi = \frac{b}{\sqrt{a^2-b^2}}$

$V_M^2 = \dot{x}_M^2 + \dot{y}_M^2 = \frac{V_0^2}{4} + \frac{V_0^2}{4}\frac{b^2}{a^2-b^2} = \frac{V_0^2}{4}(1+\frac{b^2}{a^2-b^2})$

$V_M = \frac{V_0}{2}\frac{a}{\sqrt{a^2-b^2}}$.

1-8. 该题关键在于A.B两点在运动中始终两者之间距离保持l不变。引入直角坐标系:xyz.
A.B两点的坐标(或位矢).



$x_A = R\cos\varphi$, $y_A = R\sin\varphi$. $z_A = h$.

$x_B = a$, $y_B = ?$ $z_B = 0$.

约束方程:$(a-R\cos\varphi)^2 + (y_B - R\sin\varphi)^2 + h^2 = l^2$.

$\therefore\ y_B = \sqrt{l^2-h^2-(a-R\cos\varphi)^2} + R\sin\varphi$

$V_B = \dot{y}_B = \frac{1}{2}\frac{-2(a-R\cos\varphi)^2\dot{\varphi}R\sin\varphi}{\sqrt{l^2-h^2-(a-R\cos\varphi)^2}} + R\dot{\varphi}\cos\varphi$

$= V_A\left[\cos\varphi - \frac{(a-R\cos\varphi)\sin\varphi}{\sqrt{l^2-h^2-(a-R\cos\varphi)^2}}\right]$

$\because R\dot{\varphi} = V_A$ (已知)

1-9 (1). $\omega_0 = 2\times3.14\times60$/秒, $\beta = -5$/秒^2

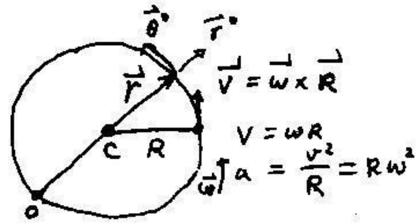
$\therefore t = \frac{\omega_0}{\beta} = \frac{2\times3.14\times60}{5}$秒 $= 75.4$秒.

(2). $S = V_0 t - \frac{1}{2}a t^2 = \omega_0 r t - \frac{1}{2}\beta r t^2$, $r = 1$米.

$= 60\times6.28\times75.4 - \frac{1}{2}\times5\times75.4^2$

$= 1.42\times10^4$米.

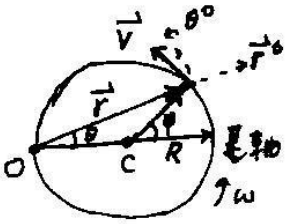
1-10. 方法一：$\vec{V}$沿切线方向，$\vec{a}$指向圆心



如图：$\vec{V} = R\omega\cos\theta\,\vec{\theta}^0 - R\omega\sin\theta\,\vec{r}^0$

$= -R\omega\sin\frac{\omega t}{2}\,\vec{r}^0 + R\omega\cos\frac{\omega t}{2}\,\vec{\theta}^0$

$\vec{a} = -R\omega^2\cos\frac{\omega t}{2}\,\vec{r}^0 - R\omega^2\sin\frac{\omega t}{2}\,\vec{\theta}^0$



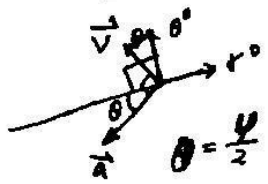
方法二：写出新坐标系中的位矢表达式：

$\vec{r} = 2R\cos\theta\,\vec{r}^0 = 2R\cos\frac{\omega t}{2}\,\vec{r}^0$

$\therefore \vec{V} = \frac{d\vec{r}}{dt} = -R\omega\sin\frac{\omega t}{2}\vec{r}^0 + 2R\cos\frac{\omega t}{2}\,\frac{d\vec{r}^0}{dt}$

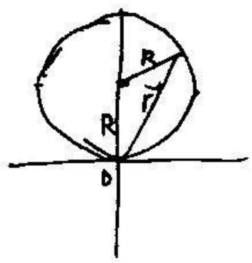
$= -R\omega\sin\frac{\omega t}{2}\vec{r}^0 + 2R\cos\frac{\omega t}{2}\,\frac{d\theta}{dt}\,\vec{\theta}^0$

$= -R\omega\sin\frac{\omega t}{2}\vec{r}^0 + R\omega\cos\frac{\omega t}{2}\,\vec{\theta}^0$



$\vec{a} = \frac{d\vec{V}}{dt} = -R\omega^2\left(\cos\frac{\omega t}{2}\,\vec{r}^0 + \sin\frac{\omega t}{2}\,\vec{\theta}^0\right)$.

1-11. 与上题情况一样



$\vec{V} = -R\omega\sin\frac{\omega t}{2}\,\vec{r}^0 + R\omega\cos\frac{\omega t}{2}\,\vec{\theta}^0$

$= -R\omega\left(\sin\omega' t\,\vec{r}^0 - \cos\omega'\,\theta^0\right)$

$\omega' = \frac{\omega}{2}$.

1-12. 在极坐标系中 $\vec{a} = (\ddot{\rho} - \rho\dot{\varphi}^2)\vec{\rho}^0 + (2\dot{\varphi}\dot{\rho} + \rho\ddot{\varphi})\vec{\varphi}^0 = (\ddot{\rho} - \rho\dot{\varphi}^2)\vec{\rho}^0 + \frac{1}{\rho}\frac{d}{dt}(\rho^2\dot{\varphi})\vec{\varphi}^0$

$\therefore a_\varphi = \frac{1}{\rho}\frac{d}{dt}(\rho^2\dot{\varphi}) = \frac{1}{\rho}\frac{d}{dt}c = 0$. ∴质点加速度只有径向分量.

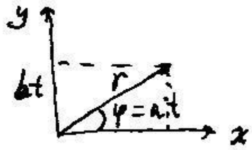
$\rho = \frac{p}{1+e\cos\varphi}$ $\quad \dot{\rho} = \frac{+pe\sin\varphi\,\dot{\varphi}}{(1+e\cos\varphi)^2} = \frac{+\rho^2 e\sin\varphi\,\dot{\varphi}}{p} = +\frac{ce\sin\varphi}{p}$.

$\ddot{\rho} = +\frac{ce\cos\varphi\,\dot{\varphi}}{p} = +\frac{c^2 e\cos\varphi}{p\rho^2}$

$a = \ddot{\rho} - \rho\dot{\varphi}^2 = \frac{c^2 e\cos\varphi}{p\rho^2} - \rho\left(\frac{c}{\rho^2}\right)^2 = \frac{-c^2(p - e\rho\cos\varphi)}{p\rho^3} = \frac{-c^2}{p\rho^3}\,\frac{p}{1+e\cos\varphi}$

$= -\frac{c^2}{p}\frac{1}{\rho^2}$. ∴ a的大小与 ρ^{-2} 成比例，方向指向极点.

1-13. 已知 $y=bt$. $\varphi=at$.



1. 直角坐标系:

$x = bt\,\mathrm{ctg}\,at$, $\quad y = bt$.

$\therefore \dfrac{x}{y} = \mathrm{ctg}\,at \qquad t = \dfrac{y}{b} \qquad \therefore x = y\,\mathrm{ctg}\left(\dfrac{ay}{b}\right)$.

2. 极坐标系:

$r = \dfrac{bt}{\sin at}$. $\qquad \varphi = at$, $\quad t = \dfrac{\varphi}{a}$

$\therefore r = \dfrac{b\varphi}{a\sin\varphi}$.

1-14. (1), $\vec{r} = 3\cos 2t\,\vec{i} + 3\sin 2t\,\vec{j} + (8t-4)\vec{k}$.

$d\vec{r} = (-6\sin 2t\,\vec{i} + 6\cos 2t\,\vec{j} + 8\vec{k})\,dt$

$\therefore ds = \sqrt{(-6\sin 2t)^2 + (6\cos 2t)^2 + 64}\,dt = \sqrt{100}\,dt = 10\,dt$.

$d\vec{r} = ds\,\vec{\tau}$, $\qquad \vec{\tau} = \dfrac{d\vec{r}}{ds}$

$\vec{\tau} = \dfrac{d\vec{r}}{dt}\cdot\dfrac{dt}{ds} = (-6\sin 2t\,\vec{i} + 6\cos 2t\,\vec{j} + 8\vec{k})/10$

$= \dfrac{1}{5}(-3\sin 2t\,\vec{i} + 3\cos 2t\,\vec{j} + 4\vec{k})$.

(2), $\vec{V} = \dfrac{d\vec{r}}{dt} \boxed{= \dfrac{ds}{dt}\vec{\tau} = v\vec{\tau}}$

$= (-6\sin 2t\,\vec{i} + 6\cos 2t\,\vec{j} + 8\vec{k})$

$= \dfrac{10}{5}(-3\sin 2t\,\vec{i} + 3\cos 2t\,\vec{j} + 4\vec{k}) = \dfrac{ds}{dt}\cdot\vec{\tau} = v\vec{\tau}$.

1-15.

$$\vec{r} = a\cos\omega t\,\vec{i} + b\sin\omega t\,\vec{j}$$

$$\vec{V} = -a\omega\sin\omega t\,\vec{i} + b\omega\cos\omega t\,\vec{j}$$

$$V = \omega\sqrt{a^2\sin^2\omega t + b^2\cos^2\omega t}$$

$$\vec{a} = \frac{dv}{dt}\vec{\tau} + \frac{v^2}{\rho}\vec{n}$$

$$a_\tau = \frac{dv}{dt} = \omega^2(a^2-b^2)\sin\omega t\cos\omega t\Big/\sqrt{a^2\sin^2\omega t + b^2\cos^2\omega t}.$$

曲率公式 $K = |f'\varphi'' - \varphi' f''|\big/(\varphi'^2 + f'^2)^{3/2}$

这里 $f' = \dot{x} = -a\omega\sin\omega t$, $f'' = \ddot{x} = -a\omega^2\cos\omega t$

$\varphi' = \dot{y} = b\omega\cos\omega t$, $\varphi'' = \ddot{y} = -b\omega^2\sin\omega t$.

$$\therefore \frac{1}{\rho} = K = (ab\omega^3\sin^2\omega t + ab\omega^3\cos^2\omega t)\big/(a^2\omega^2\sin^2\omega t + b^2\omega^2\cos^2\omega t)^{3/2}$$

$$= ab\big/(a^2\sin^2\omega t + b^2\cos^2\omega t)^{3/2}$$

$$a_n = \frac{v^2}{\rho} = \frac{ab\omega^2(a^2\sin^2\omega t + b^2\cos^2\omega t)}{(a^2\sin^2\omega t + b^2\cos^2\omega t)^{3/2}}$$

$$= \frac{ab\omega^2}{(a^2\sin^2\omega t + b^2\cos^2\omega t)^{1/2}}.$$

$$a^2 = a_\tau^2 + a_n^2 = \frac{\omega^4(a^2-b^2)^2\sin^2\omega t\cos^2\omega t + a^2b^2\omega^4}{a^2\sin^2\omega t + b^2\cos^2\omega t}$$

1-16.

$\dot{x}^2 + \dot{y}^2 = v^2$ $\quad \dot{y} = c \quad \therefore \quad v^2 = \dot{x}^2 + c^2$.

$v\frac{dv}{dt} = \dot{x}\ddot{x} = a\dot{x}$ $\quad$ ($\because \ddot{y} = 0$, $\therefore \ddot{x} = a$.

$$\frac{dv}{dt} = \frac{a}{v}\sqrt{v^2 - c^2}$$

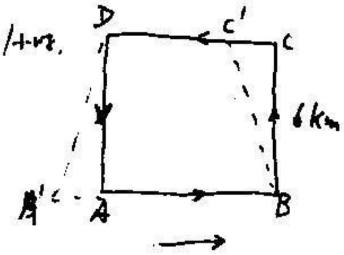
又: $\frac{dv}{dt} = \sqrt{a^2 - a_n^2} = \sqrt{a^2 - \left(\frac{v^2}{\rho}\right)^2}$

$$\therefore \quad \frac{a^2(v^2-c^2)}{v^2} = a^2 - \frac{v^4}{\rho^2}$$

$$a = \frac{v^3}{c\rho}$$

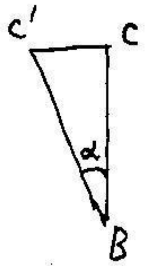
1-17.
$$\vec{V}\times\vec{a} = V\vec{\tau}\times(a_\tau\vec{\tau}+a_n\vec{n}) = Va_n(\vec{\tau}\times\vec{n})$$
$$= \frac{V^3}{\rho}(\vec{\tau}\times\vec{n}).$$
$$\therefore |\vec{V}\times\vec{a}| = \frac{V^3}{\rho}$$
$$\therefore \rho = \frac{V^3}{|\vec{V}\times\vec{a}|}.$$

1-18. 相对速度 $V_{相}$ 是相对静止空气 $V_{相}=100$ km/小时.
风速为牵连速度 $V_{牵}$ 28 km/小时.

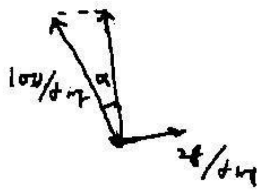


$$\therefore AB:\ \frac{6km}{(100+28)\,km/小时}$$
$$CD:\ \frac{6km}{(100-28)\,km/小时}.$$

若由B到C, 则飞机应朝BC'飞行才能到达C点.



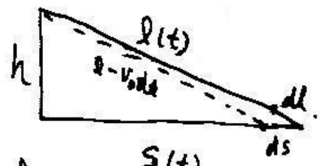
$$\sin\alpha = \frac{28}{100},\quad BC = BC'\cos\alpha$$
$$\therefore BC' = \frac{BC}{\cos\alpha} = \frac{6}{\sqrt{1-\left(\frac{28}{100}\right)^2}} = \frac{25}{4}.$$
$$\therefore B\to C:\ \frac{25/4\ km}{100\ km/小时}$$



同理可得 $D\to A:\ \frac{25}{4\times100}$ 小时.

∴ 总飞行时间
$$t = \frac{6}{128}+\frac{6}{72}+2\times\frac{25}{4\times100} = \frac{3}{64}+\frac{1}{12}+\frac{1}{8} \approx 0.255\ 小时.$$

1-19. 绳长 l 是时间的函数: $V_0 = \frac{dl}{dt}$.



$$V = \frac{ds}{dt}\quad \therefore \frac{V}{V_0} = \frac{ds}{dl}.$$
$$s^2+h^2 = l^2\quad \therefore 2s\,ds = 2l\,dl,\ \frac{ds}{dl} = \frac{l}{s}$$
$$\therefore V = V_0\frac{ds}{dl} = V_0\frac{l}{s} = \frac{V_0\sqrt{h^2+s^2}}{s}.$$

船的速率与船离岸的距离有关.

1-20. $\vec{r}_1 = 2\vec{i} - 2t\vec{j} + (6t-4)\vec{k}$

$\vec{r}_2 = (10t-12)\vec{i} + 3t^2\vec{j} - 3\vec{k}$.

相对速度 $\vec{V}_{21}\Big|_{t=2} = (10t-14)\vec{i} + (2t+3t^2)\vec{j} + (-3-6t+4)\vec{k}\Big|_{t=2}$

$= 6\vec{i} + 16\vec{j} - 11\vec{k}$.

$\ddot{\vec{r}}_1 = -2\vec{j} + 6\vec{k}$ $\qquad$ $\ddot{\vec{r}}_2 = 10\vec{i} + 6t\vec{j}$.

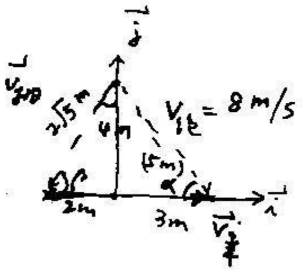
相对加速度: $\vec{a}_{21}\Big|_{t=2} = 10\vec{i} + 14\vec{j} - 6\vec{k}$.

1-21. $\vec{V}_{A0} = V_{A0}\cos\alpha\,\vec{i} + (V_{A0}\sin\alpha - \frac{1}{2}gt^2)\vec{j}$

$\vec{V}_{B0} = V_{B0}\cos\beta\,\vec{i} + (V_{B0}\sin\beta - \frac{1}{2}gt^2)\vec{j}$

$\vec{V}_{B0} - \vec{V}_{A0} = (V_{B0}\cos\beta - V_{A0}\cos\alpha)\vec{i} + (V_{B0}\sin\beta - V_{A0}\sin\alpha)\vec{j} =$ 常矢量.

1-22. (1). 由速度矢量三角形可知:



$\vec{V}_{绝} = \vec{V}_{牵} + \vec{V}_{相}$

由图可知: $|\vec{V}_{绝}| = |\vec{V}_{牵}|$. $(5m = 2m+3m)$

$\therefore V_{牵} = 8$ m/秒.

(2). 取静止的陆地为静系. $(\vec{i}, \vec{j})$.

雨相对静系速度 $\vec{V}_{绝} = 8\cos\alpha\,\vec{i} - 8\sin\alpha\,\vec{j} = 8(\frac{3}{5}\vec{i} - \frac{4}{5}\vec{j})$

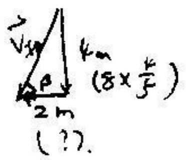
船相对静系速度 $\vec{V}_{牵} = ?$

雨相对运动的船的速度: $\cancel{\vec{V}_{相} = -8\cos\beta\,\vec{i} - 8\sin\beta\,\vec{j} = 8(\frac{2}{2\sqrt{5}} - \frac{4}{2\sqrt{5}})}$

由于船只有 $\vec{i}$ 方向速度, 所以雨在 $\vec{j}$ 方向的相对速度与绝对速度一样

于在 $\vec{j}$ 方向: $V_{相}\vec{j} = -8\times\frac{4}{5}\vec{j}$.

$\vec{i}$ 方向: $V_{相}\vec{i} = -8\times\frac{4}{5}\operatorname{ctg}\beta = -8\times\frac{4}{5}\times\frac{2}{4}$



$\therefore \vec{V}_{相} = -\frac{16}{5}\vec{i} - 8\times\frac{4}{5}\vec{j}$

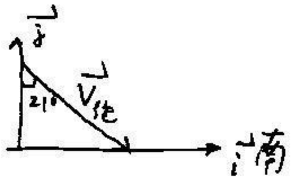
$\vec{V}_{牵} = \vec{V}_{绝} - \vec{V}_{相} = (8\times\frac{3}{5} + \frac{16}{5})\vec{i} + (-\frac{32}{5} + \frac{32}{5})\vec{j}$

$= 8\,(m/秒)\,\vec{i}$.

1-23. $\vec V_{绝} = v(\sin 21°\,\vec i - \cos 21°\,\vec j)$.

$\vec V_{牵} = 30\,\vec i$

$\vec V_{相} = -V_{相}\,\vec j$.



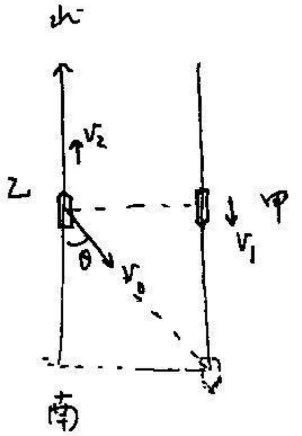
在 $\vec i$ 方向有：$v\sin 21° - 30 = 0$.

$\therefore v = \frac{30}{\sin 21°} \doteq 83.7$ m/秒.

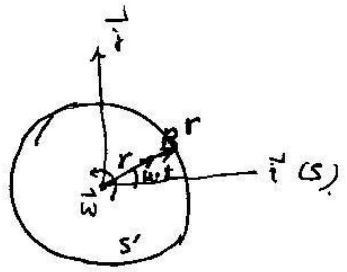
1-24. 以乙船为参照物，则甲船相对乙船的速度为 $V_1 + V_2$.

鱼雷相对乙船的速度为 V_0。可见，要击中甲船，应有 $V_0\cos\theta = V_1 + V_2$.

$\therefore \theta = \cos^{-1}\frac{V_1+V_2}{V_0}$.



1-25. 选圆盘为动系 S′



方法一：选静系 $\vec i\,\vec j$ 如图 S.（用静系计算）

在静系中质点的位矢：

$\vec r = r(\cos\omega t\,\vec i + \sin\omega t\,\vec j)$

$= R(\sin\omega t\cos\omega t\,\vec i + \sin^2\omega t\,\vec j)$.

$= R(\frac{1}{2}\sin 2\omega t\,\vec i + \sin^2\omega t\,\vec j)$.

$\vec V = \dot{\vec r} = R(\frac{1}{2}\cos 2\omega t\cdot 2\omega\,\vec i + 2\sin\omega t\cos\omega t\cdot\omega\,\vec j)$

$= R\omega(\cos 2\omega t\,\vec i + \sin 2\omega t\,\vec j)$.

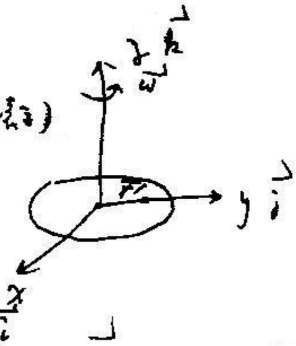
$v = |\vec V| = R\omega$.

$\vec a = \dot{\vec V} = R\omega(-2\omega\sin 2\omega t\,\vec i + 2\omega\cos 2\omega t\,\vec j)$

$= 2R\omega^2(-\sin 2\omega t\,\vec i + \cos 2\omega t\,\vec j)$

$a = |\vec a| = 2R\omega^2$.

（用动系计算）



方法二：选动系 $\vec i\,\vec j\,\vec k$ 与圆盘一起转动，且质点沿 $\vec j$ 方向运动

在这个转动系中 $\vec\omega = \omega\vec k$, $\vec r' = R\sin\omega t\,\vec j$.

相对速度 $\vec V' = \frac{d\vec r'}{dt} = R\omega\cos\omega t\,\vec j$

牵连速度 $\vec V_{牵} = \vec\omega\times\vec r' = \omega\vec k\times R\sin\omega t\,\vec j = -R\omega\sin\omega t\,\vec i$

绝对速度 $\vec{V}=\vec{V}'+\vec{\omega}\times\vec{r}' = +R\omega(-\sin\omega t\,\vec{i}+\cos\omega t\,\vec{j})$

$V=R\omega$.

$$\vec{a}=\frac{d\vec{V}}{dt}=R\omega\frac{d}{dt}(-\sin\omega t\,\vec{i}+\cos\omega t\,\vec{j})$$

$$=R\omega\left[-\omega\cos\omega t\,\vec{i}-\sin\omega t\,\dot{\vec{i}}-\omega\sin\omega t\,\vec{j}+\cos\omega t\,\dot{\vec{j}}\right]$$

$$=R\omega\left[-\omega\cos\omega t\,\vec{i}-\omega\sin\omega t\,\vec{j}-\omega\sin\omega t\,\vec{j}-\omega\cos\omega t\,\vec{i}\right]$$

$$=R\omega^2\left[-2\cos\omega t\,\vec{i}-2\sin\omega t\,\vec{j}\right]$$

$$=2R\omega^2\left[-\cos\omega t\,\vec{i}-\sin\omega t\,\vec{j}\right]$$

$a=|\vec{a}|=2R\omega^2$.

方法三：直接套公式

$$\vec{V}=\frac{d\vec{r}'}{dt}+\vec{\omega}\times\vec{r}'$$

$$\vec{a}=\frac{d\vec{V}'}{dt}+\vec{\omega}\times\vec{V}'+\dot{\vec{\omega}}\times\vec{r}'+\vec{\omega}\times\vec{V}'+\omega\times(\vec{\omega}\times\vec{r}')$$

$$\vec{V}=R\omega\cos\omega t\,\vec{j}+\omega\vec{k}\times R\sin\omega t\,\vec{j}=R\omega(-\sin\omega t\,\vec{i}+\cos\omega t\,\vec{j})$$

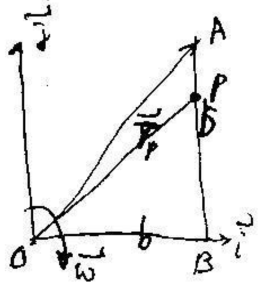
$$\vec{a}=R\omega^2(\quad-\sin\omega t\,\vec{j})+2(\omega\vec{k}\times R\omega\cos\omega t\,\vec{j})$$

$$+\omega\vec{k}\times(\omega\vec{k}\times R\sin\omega t\,\vec{j})$$

$$=R\omega^2(\quad-\sin\omega t\,\vec{j})-2R\omega^2\cos\omega t\,\vec{i}-R\omega^2\sin\omega t\,\vec{j}$$

$$=-2R\omega^2(\cos\omega t\,\vec{i}+\sin\omega t\,\vec{j})$$

1-26 $\vec{\omega}=-\omega\vec{k}$



P点的相对速度 $\vec{V}_p'=-V_0\vec{j}$

由题意知 $\omega t=2\pi$ 时 $V_0 t=b$ $\therefore V_0=\frac{\omega b}{2\pi}$.

$$\vec{r}_p=b\vec{i}+y\vec{j}$$

$$\vec{V}_p=\dot{\vec{r}}_p=b(-\omega\vec{k})\times\vec{i}+\dot{y}\vec{j}+y(-\omega\vec{k})\times\vec{j}$$

$$=-\omega b\vec{j}-\frac{\omega b}{2\pi}\vec{j}+\omega y\vec{i}=\left[\omega b\left(\vec{i}-\frac{2\pi+1}{2\pi}\vec{j}\right)\right]_{y=b}$$

$$V_p\Big|_{y=b}=|\vec{V}_p|_{y=b}=\sqrt{\left(\omega b+\frac{\omega b}{2\pi}\right)^2+\omega^2b^2}=\frac{\omega b}{2\pi}\sqrt{1+4\pi+8\pi^2}$$

$$\vec a_p=\dot{\vec V}_p=\omega\dot y\,\vec i+\omega y\,\dot{\vec i}-\left(\omega b+\frac{\omega b}{2\pi}\right)\dot{\vec j}$$

$$=-\frac{b\omega^2}{2\pi}\vec i+\omega y(-\omega\vec k)\times\vec i-\left(\omega b+\frac{\omega b}{2\pi}\right)(-\omega\vec k)\times\vec j$$

$$=-\frac{b\omega^2}{2\pi}\vec i-\omega^2 y\,\vec j-\omega^2 b\left(1+\frac{1}{2\pi}\right)\vec i$$

$$=-b\omega^2\left(1+\frac{1}{\pi}\right)\vec i-\omega^2 y\,\vec j$$

$$\vec a_p\Big|_{y=b}=-b\omega^2\left[\frac{1+\pi}{\pi}\vec i+\vec j\right]$$

$$a_p\Big|_{y=b}=b\omega^2\left(\frac{1+2\pi+\pi^2}{\pi^2}+1\right)^{\frac12}=\frac{b\omega^2}{\pi}\sqrt{1+2\pi+2\pi^2}$$

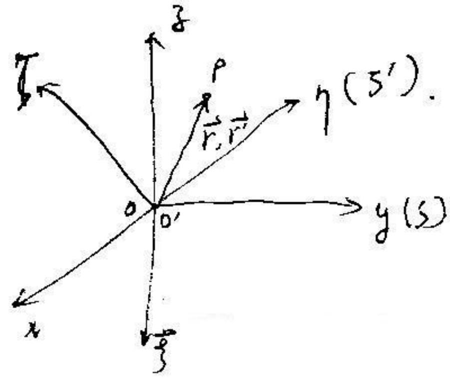
1-27. 为简单而又不失一般性，

设 S'系与 S系的原点重合

P点在 S'系中的位矢为 $\vec r'$，

在 S系中的位矢为 $\vec r$.

因此二者重合. $\vec r=\vec r'$



则 $\vec r=x\vec i+y\vec j+z\vec k$

$\vec r'=\xi\vec i'+\eta\vec j'+\zeta\vec k'$

设静系为 $S(\vec r)$. 动系为 S'. S'相对于 S 系转动 $\vec\omega'$

则 $$\frac{d\vec r'}{dt}=\frac{\tilde d\vec r'}{dt}+\vec\omega'\times\vec r'=\underline{(\dot\xi\vec i'+\dot\eta\vec j'+\dot\zeta\vec k')}+\vec\omega'\times\vec r' \quad ①$$

而 $$\frac{d\vec r}{dt}=(\dot x\vec i+\dot y\vec j+\dot z\vec k) \quad ②$$ $(\vec i\,\vec j\,\vec k)$为静系的坐标系，与时间无关.

换系为 $S'(\vec r')$ 动系为S静. S相对于 S'系转动 $\vec\omega$.

$$\frac{d\vec r}{dt}=\frac{\tilde d\vec r}{dt}+\vec\omega\times\vec r=(\dot x\vec i+\dot y\vec j+\dot z\vec k)+\vec\omega\times\vec r \quad ③$$

$(\vec i\,\vec j\,\vec k)$为动系坐标与时间有关.

而 $$\frac{d\vec r'}{dt}=(\dot\xi\vec i'+\dot\eta\vec j'+\dot\zeta\vec k') \quad ④$$

在数值上相等，但意义不一样.

①+③ ∴ $$\underbrace{\frac{d\vec r'}{dt}}_{a}+\underbrace{\frac{d\vec r}{dt}}_{b}=\underbrace{(\dot\xi\vec i'+\dot\eta\vec j'+\dot\zeta\vec k')}_{a'}+\underbrace{(\dot x\vec i+\dot y\vec j+\dot z\vec k)}_{b'}+(\vec\omega+\vec\omega')\times\vec r$$

由②和④式可知，在数值上有如下关系：$a=a'$，$b=b'$

∴ $0=(\vec\omega+\vec\omega')\times\vec r$ $\qquad$ r为任意位矢. ∴ $\vec r\neq0$，$\vec r\nparallel(\vec\omega+\vec\omega')$

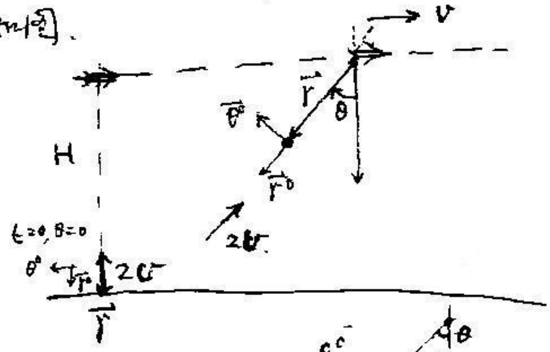
∴ 只有 $\vec\omega=-\vec\omega'$

1-28. 选飞机为平动参照系，讨论导弹相对飞机的运动。

选取与飞机一起作平动的极坐标系(r,θ)如图。

$t=0$时，$\theta=0$ 导弹发出，$r=H$。

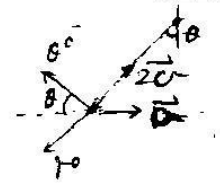
$t=T$时，$\theta=\frac{\pi}{2}$ 导弹击中，$r=0$。



飞机的速度是牵连速度，沿水平向右，

将其投影到极坐标系中：

$$\vec{V}_{牵} = -v\cos\theta\,\vec{\theta}^0 - v\sin\theta\,\vec{r}^0$$



导弹相对地面的速度是绝对速度，其方向始终指向飞机，

用极坐标表示为 $\vec{V}_{绝} = -2v\,\vec{r}^0$

导弹相对飞机的速度是相对速度，根据：$\vec{V}_{绝} = \vec{V}_{相} + \vec{V}_{牵}$

$$\therefore \vec{V}_{相} = \vec{V}_{绝} - \vec{V}_{牵} = [-2v-(-v\sin\theta)]\vec{r}^0 + v\cos\theta\,\vec{\theta}^0 = \dot{r}\,\vec{r}^0 + r\dot{\theta}\,\vec{\theta}^0$$

$$\therefore \frac{dr}{dt} = v(\sin\theta - 2) \quad ①$$

$$r\frac{d\theta}{dt} = v\cos\theta \quad ②$$

由$\frac{①}{②}$得 $\frac{dr}{r\,d\theta} = tg\theta - \frac{2}{\cos\theta}$

$$\frac{dr}{r} = tg\theta\,d\theta - \frac{2d\theta}{\cos\theta}$$

不定积分，查表得：

$$\ln r = -\ln\cos\theta - \ln(\sec\theta + tg\theta)^2 + C$$

$\because \theta=0$, $r=H$. $\therefore C = \ln H$.

$$r = \frac{H\cos\theta}{\cos^2\theta(\sec\theta + tg\theta)^2} = \frac{H\cos\theta}{(1+\sin\theta)^2}$$

代入②则

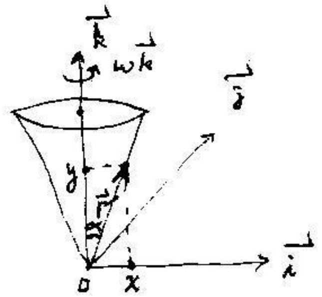
$$\frac{d\theta}{(1+\sin\theta)^2} = \frac{v}{H}dt.$$

$t=0$, $\theta=0$; $t=T$, $\theta=\frac{\pi}{2}$ 积分

$$\frac{vT}{H} = \int_0^{\frac{\pi}{2}} \frac{d\theta}{(1+\sin\theta)^2} = \frac{2}{3} \qquad \therefore T = \frac{2H}{3v}$$

1-29. 选动系 oxyz 一起随圆锥旋转.

z轴为圆锥轴, 质点运动的母线始终在 xoz 平面内. 所以质点在动系中的位矢:



$$\vec{r}' = ut\sin\alpha\,\vec{i} + ut\cos\alpha\,\vec{k}\,; \quad \vec{\omega} = \omega\vec{k}$$

所以 $\vec{V} = \dfrac{d\vec{r}'}{dt} = \dfrac{\tilde{d}\vec{r}'}{dt} + \vec{\omega}\times\vec{r}' \Rightarrow$ 相对+牵连.

$$= u(\sin\alpha\,\vec{i} + \cos\alpha\,\vec{k}) + \omega\vec{k}\times(ut\sin\alpha\,\vec{i} + ut\cos\alpha\,\vec{k})$$

$$= u(\sin\alpha\,\vec{i} + \cos\alpha\,\vec{k}) + u\omega t\sin\alpha\,\vec{j}$$

$$= u\sin\alpha\,\vec{i} + u\omega t\sin\alpha\,\vec{j} + u\cos\alpha\,\vec{k}$$

$$\therefore V = \sqrt{u^2\sin^2\alpha + u^2\cos^2\alpha + u^2\omega^2t^2\sin^2\alpha}$$

$$= u\sqrt{1+\omega^2t^2\sin^2\alpha}\,.$$

$$\vec{a} = \frac{d\vec{V}}{dt} = u\sin\alpha\,\omega\vec{j} + u\omega\sin\alpha\,\vec{j} - u\omega^2t\sin\alpha\,\vec{i}$$

$$= 2u\omega\sin\alpha\,\vec{j} - u\omega^2t\sin\alpha\,\vec{i}$$

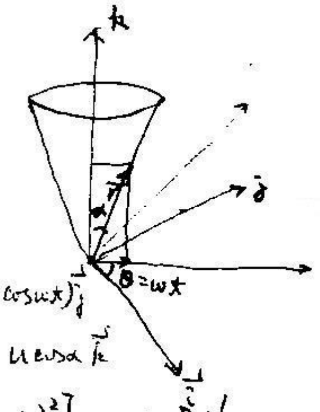
$$a = \sqrt{4u^2\omega^2\sin^2\alpha + u^2\omega^4t^2\sin^2\alpha} = u\omega\sin\alpha\sqrt{4+\omega^2t^2}\,.$$

若用静系表达:

$$\vec{r} = r\sin\alpha\cos\omega t\,\vec{i} + r\sin\alpha\sin\omega t\,\vec{j} + r\cos\alpha\,\vec{k}$$

$$= ut(\sin\alpha\cos\omega t\,\vec{i} + \sin\alpha\sin\omega t\,\vec{j} + \cos\alpha\,\vec{k})$$

$$\vec{V} = \frac{d\vec{r}}{dt} = u(\sin\alpha\cos\omega t\,\vec{i} + \sin\alpha\sin\omega t\,\vec{j} + \cos\alpha\,\vec{k})$$
$$+ ut(-\omega\sin\alpha\sin\omega t\,\vec{i} + \omega\sin\alpha\cos\omega t\,\vec{j})$$



$$= u\sin\alpha(\cos\omega t - \omega t\sin\omega t)\vec{i} + u\sin\alpha(\sin\omega t + \omega t\cos\omega t)\vec{j}$$
$$+ u\cos\alpha\,\vec{k}$$

$$U = u\left\{\sin^2\alpha\left[(\cos\omega t - \omega t\sin\omega t)^2 + (\sin\omega t + \omega t\cos\omega t)^2\right] + \cos^2\alpha\right\}^{\frac{1}{2}}$$

$$= u\left\{\sin^2\alpha\left[\cos^2\omega t + \omega^2t^2\sin^2\omega t \cancel{- 2\omega t\sin\omega t\cos\omega t} + \sin^2\omega t + \omega^2t^2\cos^2\omega t \cancel{+ 2\omega t\sin\omega t\cos\omega t}\right] + \cos^2\alpha\right\}^{\frac{1}{2}}$$

$$= u\left\{\sin^2\alpha\left[1+\omega^2t^2\right] + \cos^2\alpha\right\}^{\frac{1}{2}} = u\left\{1+\sin^2\alpha\cdot\omega^2t^2\right\}$$

$$= u\sqrt{1+\omega^2t^2\sin^2\alpha}\,.$$