电动力学-第十一次作业

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Problem 11.12

Answer:

The acceleration of the particle is $a = g$ (downwards), so the power radiated is:

$$P = \frac{\mu_0 q^2 a^2}{6\pi c} = \frac{\mu_0 q^2 g^2}{6\pi c}$$

The total energy radiated in the time needed to traverse 1cm is:

$$U = \int_0^{t_f} P dt = P t_f$$

$$y = \frac{1}{2} g t_f^2 \Longrightarrow t_f = \sqrt{\frac{2y}{g}}$$

$$U = P\sqrt{\frac{2y}{g}} = \frac{\mu_0 q^2 g^2}{6\pi c}\sqrt{\frac{2y}{g}}$$

The ratio of this energy versus the potential energy lost by traversing that distance is:

$$\eta = \frac{U}{U_{pot}} = \frac{\frac{\mu_0 q^2 g^2}{6\pi c}\sqrt{\frac{2y}{g}}}{mgy} = \frac{\mu_0 q^2}{6\pi \mathrm{cm}}\sqrt{\frac{2g}{y}}$$

Numerically this ratio is equal to, with $m = 9.11 \cdot 10^{-31}\mathrm{kg}$ and $q = 1.609 \cdot 10^{-19}\mathrm{C}$:

$$\eta \approx 2.8 \cdot 10^{-22}$$

Problem 11.20

Answer:

(c)

We want to sum all of the interaction terms, and hopefully that will reproduce the 11.100). Now:

$$dF^{int} = \frac{\mu_0 \dot{a}}{12\pi c} dq_1 dq_2$$
$$dq_1 = 2\lambda dy_1 \quad dq_2 = 2\lambda dy_2$$
$$dF^{int} = \frac{\mu_0 \dot{a}\lambda^2}{3\pi c} dy_1 dy_2$$

Let $y_1 > y_2$ and let the bar run from $y = 0$ to $y = L$. First we integrate from $y_2 = 0$ to $y_2 = y_1$ to find the interaction of everything below y_1 with the dq_1 at y_1

$$dF^{int} = \frac{\mu_0 \dot{a}\lambda^2}{3\pi c} \left(\int_0^{y_1} dy_2 \right) dy_1 = \frac{\mu_0 \dot{a}\lambda^2}{3\pi c} y_1 dy_1$$

Then just integrate over all y_1 over the entire bar:

$$F^{int} = \frac{\mu_0 \dot{a}\lambda^2}{3\pi c} \int_0^L y_1 dy_1 = \frac{\mu_0 \dot{a}\lambda^2}{6\pi c} L^2 = \frac{\mu_0 \dot{a} q^2}{6\pi c}$$

Problem 11.24

Answer:

Refer to the Figure 11.19). The potentials of a dipole are:

$$V = -\frac{p_0\omega}{4\pi\epsilon_0 c} \frac{\cos\theta}{r} \sin[\omega(t - r/c)] \quad \vec{A} = -\frac{\mu_0 p_0 \omega}{4\pi r} \sin[\omega(t - r/c)]$$

Now, the dipoles are not at the origin, so:

$$l_\pm = \sqrt{r^2 + d^2/4 \mp rd\cos\theta} \approx r\left(1 \mp \frac{d}{2r}\cos\theta\right) \Rightarrow \frac{1}{l_\pm} \approx \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right)$$

The angles $\theta_\pm$ are different, as well:

$$r\cos\theta = \tfrac{d}{2} + l_+\cos\theta_+ \quad l_-\cos\theta_- = r\cos\theta + \tfrac{d}{2}$$
$$l_\pm \cos\theta_\pm = r\cos\theta \mp \tfrac{d}{2}$$

$$\Longrightarrow \cos\theta_\pm = \tfrac{1}{l_\pm}(r\cos\theta \mp d/2) \approx \cos\theta \pm \tfrac{d}{2r}\cos^2\theta \mp \tfrac{d}{2r}$$
$$= \cos\theta \mp \tfrac{d}{2r}\left(1 - \cos^2\theta\right) = \cos\theta \mp \tfrac{d}{2r}\sin^2\theta$$

With these results the potenials are (via a lenghty process) easy to get. Let us start with the vector potential:

$$\vec{A} = -\tfrac{\mu_0 p_0 \omega}{4\pi}\left[\tfrac{\sin[\omega(t - l_+/c)]}{l_+} - \tfrac{\sin[\omega(t - l_-/c)]}{l_-}\right]\hat{z}$$
$$\tfrac{\sin[\omega(t - l_\pm/c)]}{l_\pm} \approx \tfrac{1}{r}\left(1 \pm \tfrac{d}{2r}\cos\theta\right)\sin\left[t_0 \pm \tfrac{d}{2c}\cos\theta\right]$$

Taylor-expand the sine term:

$$\sin\left[\omega t_0 \pm \tfrac{\omega d}{2c}\cos\theta\right] \approx \sin(\omega t_0) \pm \cos(\omega t_0)\tfrac{\omega d}{2c}\cos\theta$$
$$\vec{A} = -\tfrac{\mu_0 p_0 \omega}{4\pi}\left\{\tfrac{1}{r}\left(1+\tfrac{d}{2r}\cos\theta\right)\left[\sin(\omega t_0)+\cos(\omega t_0)\tfrac{\omega d}{2c}\cos\theta\right]\right]$$
$$-\tfrac{1}{r}\left(1-\tfrac{d}{2r}\cos\theta\right)\left[\sin(\omega t_0)-\cos(\omega t_0)\tfrac{\omega d}{2c}\cos\theta\right]\Big\}\hat{z}$$

Add up all the terms and neglect all $\mathcal{O}(d^2)$ terms:

$$\begin{aligned}\vec{A} &= -\frac{\mu_0 p_0 \omega^2 d}{4\pi r c}\left[\cos\theta\cos(\omega t_0)+\frac{\omega}{rc}\cos\theta\sin(\omega t_0)\right]\hat{z}\\ &\approx -\frac{\mu_0 p_0 \omega^2 d}{4\pi r c}\cos\theta\cos(\omega t_0)\hat{z}\\ &= -\frac{\mu_0 p_0 \omega^2 d}{4\pi r c}\cos\theta\cos(\omega t_0)(\cos\theta\hat{r}-\sin\theta\hat{\theta})\end{aligned}$$

Now, the scalar potential:

$$\begin{aligned}V =& -\frac{p_0\omega}{4\pi\epsilon_0 c}\left\{\frac{\cos_+}{l_+}\sin\left[\omega\left(t-l_+/c\right)\right]-\frac{\cos_-}{l_-}\sin\left[\omega\left(t-l_-/c\right)\right]\right\}\\ =& -\frac{p_0\omega}{4\pi\epsilon_0 c}\left\{\frac{1}{r}\left(1+\frac{d}{2r}\cos\theta\right)\left[\sin(\omega t_0)+\cos(\omega t_0)\cos\theta\frac{\omega d}{2c}\right]\left(\cos-\frac{d}{2r}\sin^2\theta\right)\right.\\ &\left.-\frac{1}{r}\left(1-\frac{d}{2r}\cos\theta\right)\left[\sin(\omega t_0)-\cos(\omega t_0)\cos\theta\frac{\omega d}{2c}\right]\left(\cos+\frac{d}{2r}\sin^2\theta\right)\right\}\end{aligned}$$

Again, by collecting all the terms and neglecting $\mathcal{O}(d^2)$ terms:

$$\begin{aligned}V &= -\frac{\mu_0 p_0 \omega^2 d}{4\pi r}\left[\cos^2\theta\cos(\omega t_0)+\frac{c}{r\omega}\left(\cos^2\theta-\sin^2\theta\right)\sin(\omega t_0)\right]\\ &\approx -\frac{\mu_0 p_0 \omega^2 d}{4\pi r}\cos^2\theta\cos(\omega t_0)\end{aligned}$$

(b)

We have the potentials. The fields are then:

$$\begin{aligned}\vec{B} &= \nabla\times\vec{A} = \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta)-\frac{\partial A_r}{\partial\theta}\right)\hat{\phi}\\ &= \frac{\mu_0 p_0 \omega^2 d}{4\pi c r}\cos\theta\sin\theta\sin(\omega t_0)\frac{\omega}{c}\hat{\phi}-\frac{\mu_0 p_0 \omega^2 d}{2\pi c r^2}\sin\theta\cos\theta\cos(\omega t_0)\hat{\phi}\\ &\approx \frac{\mu_0 p_0 \omega^3 d}{4\pi c^2 r}\cos\theta\sin\theta\sin(\omega t_0)\hat{\phi}\end{aligned}$$

$$\vec{E} = -\nabla V - \frac{\partial\vec{A}}{\partial t}$$
$$\begin{aligned}\nabla V =& -\frac{\mu_0 p_0 \omega^2 d}{4\pi}\cos^2\theta\left(-\frac{1}{r^2}\cos(\omega t_0)+\frac{\omega}{rc}\sin(\omega t_0)\right)\hat{r}\\ &+\frac{\mu_0 p_0 \omega^2 d}{4\pi r^2}2\cos\theta\sin\theta\cos(\omega t_0)\hat{\theta}\end{aligned}$$

$$\approx -\frac{\mu_0 p_0 \omega^3 d}{4\pi r c}\cos^2\theta \sin(\omega t_0)\,\hat{r}$$

$$\frac{\partial \vec{A}}{\partial t} = \frac{\mu_0 p_0 \omega^3 d}{4\pi r c}\cos\theta \sin(\omega t_0)\left(\cos\theta \hat{r} - \sin\theta \hat{\theta}\right)$$

$$\vec{E} = \frac{\mu_0 p_0 \omega^3 d}{4\pi r c}\cos\theta \sin\theta \sin(\omega t_0)\,\hat{\theta}$$

(c)

The Poynting vector, the intensity and the total power radiated are:

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$$

$$= \frac{\mu_0}{c}\left(\frac{p_0\omega^3 d}{4\pi c r}\right)^2 \sin^2\theta \cos^2\theta \sin^2(\omega t_0)\,\hat{r}$$

$$\vec{I} = \langle\vec{S}\rangle$$

$$= \frac{\mu_0}{2c}\left(\frac{p_0\omega^3 d}{4\pi c r}\right)^2 \sin^2\theta\cos^2\theta\hat{r} = \sqrt{\frac{\mu_0}{8c}\left(\frac{p_0\omega^3 d}{4\pi c r}\right)^2 \sin^2(2\theta)\hat{r}}$$

$$P = \tfrac{\mu_0}{2c}\left(\tfrac{p_0\omega^3 d}{4\pi c}\right)^2 \int_0^{2\pi}\int_0^{\pi} \sin^2\theta\cos^2\theta\sin\theta d\theta d\phi$$

$$\cos\theta = x \quad dx = -\sin\theta d\theta$$

$$P = \frac{\mu_0}{2c}\left(\frac{p_0\omega^3 d}{4\pi c}\right)^2 \int_0^{2\pi}\int_0^{\pi} \sin^2\theta\cos^2\theta\sin\theta d\theta d\phi$$

$$\cos\theta = x \quad dx = -\sin\theta d\theta$$

$$= \frac{\mu_0\pi}{c}\left(\frac{p_0\omega^3 d}{4\pi c}\right)^2 \int_{-1}^{1}\left(1-x^2\right)x^2 dx = \frac{\mu_0\pi}{c}\left(\frac{p_0\omega^3 d}{4\pi c}\right)^2 \frac{4}{15}$$

$$= \frac{\mu_0}{60\pi}\frac{p_0^2\omega^6 d^2}{c^3}$$