电动力学-第四次作业

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Problem3.7

Answer:

For this question, we want to solve the electric field above the plane(z>0), and the Dirichlet Boundary condition is:

$$\begin{cases} V = 0, where\, z = 0 \\ V = 0, where\, \sqrt{x^2+y^2+z^2} \to \infty \end{cases} \tag{1.1}$$

If we remove the grounded conductor, and place a charge -2q at (0,0,-d), and a charge +q at (0,0,-3d), it's easy to prove that it has the same Dirichlet boundary condition with the origin question. So, the force exert on +q is:

$$F = \frac{1}{4\pi\epsilon_0}(-\frac{2q^2}{4d^2} - \frac{2q^2}{16d^2} + \frac{q^2}{36d^2}) = -\frac{1}{4\ pi\epsilon_0}\frac{43q^2}{72d^2} \tag{1.2}$$

Problem3.15

Answer:

(a).

According to the symmetry of the pipe, it's easy to know that the potential inside the pipe is only the function of x and y.

And we can determined the Dirichlet boundary condition:

$$\begin{cases} V(x=0,y) = 0 \\ V(x=b,y) = V_0(y) \\ V(x,y=0) = 0 \\ V(x,y=b) = 0 \end{cases} \tag{2.1}$$

Assume that the V(x,y) could be writen in following form:

$$V(x,y) = \mathcal{X}(x)\mathcal{Y}(y) \tag{2.2}$$

So, the laplace's equation turn to:

$$\mathcal{Y}\frac{\partial^2 \mathcal{X}}{\partial x^2} + \mathcal{X}\frac{\partial^2 \mathcal{Y}}{\partial y^2} = 0 \tag{2.3}$$

Which equal to:

$$\frac{1}{\mathcal{X}}\frac{\partial^2 \mathcal{X}}{\partial x^2} + \frac{1}{\mathcal{Y}}\frac{\partial^2 \mathcal{Y}}{\partial y^2} == 0 \tag{2.4}$$

The two terms in the left side of the equation is only the function of x and of y. If we want (2.4) always correct, the following equation must be correct.

$$\begin{cases} \dfrac{\partial^2 \mathcal{X}}{\partial x^2} = k^2\mathcal{X} \\ \dfrac{\partial^2 \mathcal{Y}}{\partial y^2} = -k^2\mathcal{Y} \end{cases} \tag{2.5}$$

Solve (2.5) we get:

$$V(x,y) = (C_1e^{kx} + C_2e^{-kx})(C_3sin(ky) + C_4Cos(ky)) \tag{2.6}$$

By consider the boundary condition, we can determined the coefficient:

$$\begin{aligned} V(x,y) &= C_1(e^{\frac{n\pi x}{a}} - e^{-\frac{n\pi x}{a}})(C_4sin(\frac{n\pi y}{a})) \\ &= 2C_1C_4sinh(\frac{n\pi x}{a})sin(\frac{n\pi y}{a}) \end{aligned} \tag{2.7}$$

(2.7) could be writen in a sum form:

$$V(x,y) = \sum_{n=1}^{\infty} C_nsinh(\frac{n\pi x}{a})sin(\frac{n\pi y}{a}) \tag{2.8}$$

And solve C_n with $V_0(y)$:

$$C_n = \frac{2}{a\,sinh(n\pi b/a)}\int_0^a V_0(y)sin(\frac{n\pi y}{a})dy \tag{2.9}$$

(b)

$$C_n = \frac{2V_0}{a\,sinh(n\pi b/a)}\int_0^a sin(\frac{n\pi y}{a})dy \tag{2.10}$$

If n is even:

$$C_n = 0 \tag{2.11}$$

If n in odd:

$$C_n = \frac{4V_0}{n\pi\, sinh(n\pi b/a)} \tag{2.12}$$

The potential is:

$$V(x,y) = \frac{4V_0}{\pi} \sum_{n=1,3,5} \frac{sinh(n\pi x/a) sin(n\pi y/a)}{n\, sinh(n\pi b/a)} \tag{2.13}$$

Problem3.19

Answer:

$$V_0(\theta) = k\, cos(3\theta) \tag{3.1}$$

Express (3.1) in the combination of Legendre polynomials:

$$V_0(\theta) = k[\alpha P_3(cos\theta) + \beta P_1(cos\theta)] \tag{3.2}$$

Determined the coefficient α and β:

$$cos(3\theta) = 4cos^3\theta - 3cos\theta = \alpha[\frac{1}{2}(5cos^3\theta - 3cos\theta)] + \beta cos\theta \tag{3.3}$$

So:

$$\begin{cases} \alpha = \frac{8}{5} \\ \beta = -\frac{3}{5} \end{cases} \tag{3.4}$$

Therefore:

$$V_0(\theta) = \frac{k}{5}[8P_3(cos\theta) - 3P_1(cos\theta)] \tag{3.5}$$

For the inside space:

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l \tau^l P_l(cos\theta) \qquad r \le R \tag{3.6}$$

A_l is determined by:

$$\begin{aligned}
A_l =& \frac{(2l+1)}{2R^l}\int_0^\pi V_0(\theta)P_l(cos\theta)sin\theta d\theta \\
=& \frac{(2l+1)}{2R^l}\frac{k}{5}\{8\int_0^\pi P_3(cos\theta)P_l(cos\theta)sin\theta d\theta - 3\int_0^\pi P_1(cos\theta)P_l(cos\theta)sin\theta d\theta\} \\
=& \frac{(2l+1)}{2R^l}\frac{k}{5}\{8\frac{2}{(2l+1)}\delta_{l3} - 3\frac{2}{(2l+1)\delta_{l1}}\} \\
=& \frac{k}{5R^l}[8\delta_{l3} - 3\delta_{l1}]
\end{aligned} \tag{3.7}$$

So:

$$V(r,\theta) = \frac{k}{5}[8(\frac{r}{R})^3 P_3(cos\theta) * 3(\frac{r}{R})P_1(cos\theta)] \tag{3.8}$$

Then, look outside:

$$V(r,\theta) = \sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}P_l(cos\theta) \qquad r \geq R \tag{3.9}$$

And we can determined B_l:

$$B_l = \frac{k}{5}\frac{1}{R^{l+1}}[8\delta_{l3} - 3\delta_{l1}] \tag{3.10}$$

So:

$$V(r,\theta) = \frac{k}{5}[8(\frac{R}{r})^4 P_3(cos\theta) - 3(\frac{R}{r})^2 P_1(cos\theta)] \tag{3.11}$$

For the charge density:

$$\sigma(\theta) = \epsilon_0\sum_{l=0}^{\infty}(2l+1)A_l R^(l-1)P_l(cos\theta) = \frac{\epsilon_0 k}{5R}[-9P_1(cos\theta) + 56P_3(cos\theta)] \tag{3.12}$$

Problem3.45

Answer:

(a)

$$\frac{1}{2}\sum_{i,j=1}^{3}\hat{r_i}\hat{r_j}Q_{ij} = \frac{1}{2}\int 3\sum_{i=1}^{3}\hat{r_i}r'_j\sum_{j=1}^{3}\hat{r_j}r'_i - (r')^2\sum_{i,j}\hat{r_i}\hat{r_j}\delta_{ij}\rho dr' \tag{4.1}$$

So, the potential of the quadrupole is:

$$V_{quad} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\int\frac{1}{2}(r'^2 cos\theta' - r')\rho d\tau' = \frac{1}{4\pi\epsilon_0}\int r'^2 P_2(cos\theta')\rho d\tau' \tag{4.2}$$

(b)

Since $x^2 = y^2 = (a/2)^2$ for all four charges:

$$Q_{xx} = Q_{yy} = 0 \tag{4.3}$$

And z=0:

$$Q_{zz} = Q_(xz) = Q_(yz) = Q_(zx) = Q_(zy) = 0 \tag{4.4}$$

For $Q_{xy} and Q_{yx}$:

$$Q_{xy} = Q_{yx} = 3a^2 q \tag{4.5}$$