附录 B　统计物理学中常用的积分公式

B1. 高 斯 积 分

$$\int_0^\infty x^{2n} e^{-\lambda x^2} dx = \frac{1 \cdot 3 \cdot \cdots \cdot (2n-1)}{2^{n+1}} \sqrt{\frac{\pi}{\lambda^{2n+1}}},$$

$$\int_0^\infty x^{2n+1} e^{-\lambda x^2} dx = \frac{n!}{2\lambda^{n+1}}.$$

特例：

$$\int_0^\infty e^{-\lambda x^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{\lambda}}, \quad \int_0^\infty x e^{-\lambda x^2} dx = \frac{1}{2\lambda},$$

$$\int_0^\infty x^2 e^{-\lambda x^2} dx = \frac{1}{4} \sqrt{\frac{\pi}{\lambda^3}}, \quad \int_0^\infty x^3 e^{-\lambda x^2} dx = \frac{1}{2\lambda^2},$$

$$\int_0^\infty x^4 e^{-\lambda x^2} dx = \frac{3}{8} \sqrt{\frac{\pi}{\lambda^5}}, \quad \int_0^\infty x^5 e^{-\lambda x^2} dx = \frac{1}{\lambda^3},$$

后面的积分可以从前面的积分对 λ 求微商得到.

B2. 某些包含玻色分布函数的积分

$$\int_0^\infty \frac{x^{\nu-1}\mathrm{d}x}{\mathrm{e}^x - 1} = \int_0^\infty x^{\nu-1} \sum_{\lambda=1}^\infty (\mathrm{e}^{-x})^\lambda \mathrm{d}x$$

$$= \Gamma(\nu) \sum_{\lambda=1}^\infty \frac{1}{\lambda^\nu} = \Gamma(\nu)\zeta(\nu) \quad (\nu > 1),$$

$\Gamma(\nu)$为 Γ 函数，$\zeta(\nu)$为黎曼(Riemann)ζ **函数. 对某些特殊的** ν，$\zeta(\nu)$**的值为**：

$$\zeta\left(\frac{3}{2}\right) \approx 2.612, \quad \zeta(2) = \frac{\pi^2}{6} \approx 1.645, \quad \zeta\left(\frac{5}{2}\right) \approx 1.341;$$

$$\zeta(3) \approx 1.202, \quad \zeta(4) = \frac{\pi^4}{90} \approx 1.082, \quad \zeta(5) \approx 1.037.$$

于是得

$$\int_0^\infty \frac{x^{\frac{1}{2}}\mathrm{d}x}{\mathrm{e}^x - 1} \approx 2.315,$$

$$\int_0^\infty \frac{x\mathrm{d}x}{\mathrm{e}^x - 1} = \frac{\pi^2}{6} \approx 1.645,$$

$$\int_0^\infty \frac{x^{\frac{3}{2}}\mathrm{d}x}{\mathrm{e}^x - 1} \approx 1.783,$$

$$\int_0^\infty \frac{x^2\mathrm{d}x}{\mathrm{e}^x - 1} \approx 2.404,$$

$$\int_0^\infty \frac{x^3\mathrm{d}x}{\mathrm{e}^x - 1} = \frac{\pi^4}{15} \approx 6.494,$$

$$\int_0^\infty \frac{x^4\mathrm{d}x}{\mathrm{e}^x - 1} \approx 24.889,$$

$$\cdots$$

参看 R. K. Pathria, Statistical Mechanics, 2nd edition, Butterworth-Heinemann, 1996, p. 506.

B3. 某些包含费米分布函数的积分

$$\int_0^\infty \frac{x^{\nu-1}}{e^x+1} = \int_0^\infty x^{\nu-1}\sum_{\lambda=1}^{\infty}(-1)^{\lambda+1}e^{-\lambda x}dx = \Gamma(\nu)\sum_{\lambda=1}^{\infty}(-1)^{\lambda+1}\frac{1}{\lambda^\nu}$$
$$= (1-2^{1-\nu})\Gamma(\nu)\zeta(\nu) \quad (\nu>0),$$

于是得

$$\int_0^\infty \frac{x^{\frac{1}{2}}dx}{e^x+1} \approx 0.678,$$
$$\int_0^\infty \frac{xdx}{e^x+1} = \frac{\pi^2}{12} \approx 0.823,$$
$$\int_0^\infty \frac{x^{\frac{3}{2}}dx}{e^x+1} \approx 1.152,$$
$$\int_0^\infty \frac{x^2dx}{e^x+1} \approx 1.803,$$
$$\int_0^\infty \frac{x^3dx}{e^x+1} = \frac{7\pi^4}{120} \approx 5.682,$$
$$\int_0^\infty \frac{x^4dx}{e^x+1} \approx 23.333,$$

…

参看 L. D. Landau and E. M. Lifshitz, Statistiacl Physics, 3rd edition, Part 1, Pergamon Press, 1986, p. 170.

附录 C　误 差 函 数

定义误差函数

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \mathrm{e}^{-y^2}\mathrm{d}y.$$

级数展开：

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\left(x - \frac{x^3}{1! \times 3} + \frac{x^5}{2! \times 5} - \frac{x^7}{3! \times 7} + \cdots\right).$$

渐近展开（**当** x **很大时**）：

$$\mathrm{erf}(x) = 1 - \frac{\mathrm{e}^{-x^2}}{x\sqrt{\pi}}\left(1 - \frac{1}{2x^2} + \frac{1 \times 3}{(2x^2)^2} - \frac{1 \times 3 \times 5}{(2x^2)^3} + \cdots\right).$$

3.3.3 二维正态分布

下面引入的二维正态分布也是一种重要的分布.

定义 3.3.3. 若二维随机变量 (X,Y) 的联合概率密度为

$$f(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_1)^2}{\sigma_1^2}-2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2}+\frac{(y-\mu_2)^2}{\sigma_2^2}\right]\right\} \tag{3.3.13}$$

其中 μ_1,μ_2 为实数, $\sigma_1>0, \sigma_2>0$, $|\rho|<1$, 则称 (X,Y) 服从参数为 $\mu_1,\mu_2,\sigma_1,\sigma_2,\rho$ 的 **二维正态分布 (two-dimensional normal distribution)**, 也称 (X,Y) 是 **二维正态随机变量 (two-dimensional normal random variable)**, 记作 $(X,Y)\sim N(\mu_1,\mu_2,\sigma_1^2,\sigma_2^2,\rho)$. 称上述 $f(x,y)$ 为**二维正态概率密度**.

二维正态概率密度 $f(x,y)$ 在三维空间的图形, 类似一个椭圆切面的钟倒扣在 Oxy 平面上, 其中心在 (μ_1, μ_2) 处. 见图 3.5.

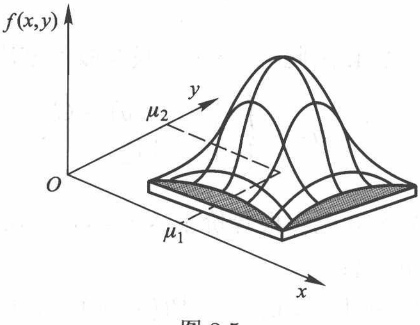

图 3.5

下面求二维正态随机变量的边缘分布. 为此, 令

$$\frac{x-\mu_1}{\sigma_1}=u,\quad \frac{y-\mu_2}{\sigma_2}=v$$

由 (3.3.7) 式知

$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f(x,y)\,\mathrm{d}y \\
&= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_1)^2}{\sigma_1^2}-2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2}+\frac{(y-\mu_2)^2}{\sigma_2^2}\right]}\mathrm{d}y \\
&= \frac{1}{2\pi\sigma_1\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{1}{2(1-\rho^2)}[u^2-2\rho uv+v^2]}\mathrm{d}v \\
&= \frac{1}{\sqrt{2\pi}\sigma_1}\mathrm{e}^{-\frac{u^2}{2}} \cdot \frac{1}{\sqrt{2\pi(1-\rho^2)}} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{\rho^2u^2-2\rho uv+v^2}{2(1-\rho^2)}}\mathrm{d}v \\
&= \frac{1}{\sqrt{2\pi}\sigma_1}\mathrm{e}^{-\frac{u^2}{2}} \cdot \frac{1}{\sqrt{2\pi(1-\rho^2)}} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{(v-\rho u)^2}{2(1-\rho^2)}}\mathrm{d}v \\
&= \frac{1}{\sqrt{2\pi}\sigma_1}\mathrm{e}^{-\frac{u^2}{2}} \\
&= \frac{1}{\sqrt{2\pi}\sigma_1}\mathrm{e}^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}} \qquad (3.3.14)
\end{aligned}
$$

由此知 $f_X(x)$ 是正态分布 $N(\mu_1,\,\sigma_1^2)$ 的概率密度. 由对称性还可得

$$
f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}\sigma_2}\mathrm{e}^{-\frac{(y-\mu_2)^2}{2\sigma_2^2}} \qquad (3.3.15)
$$

因而二维正态分布 $N(\mu_1,\mu_2,\sigma_1^2,\sigma_2^2,\rho)$ 的两个边缘分布均为一维正态分布, 分别为 $N(\mu_1,\sigma_1^2)$ 和 $N(\mu_2,\sigma_2^2)$, 它们与参数 ρ 无关.

如果 $\rho_1 \neq \rho_2$, 则两个二维正态分布:

$$
N(\mu_1,\mu_2,\sigma_1^2,\sigma_2^2,\rho_1), \quad N(\mu_1,\mu_2,\sigma_1^2,\sigma_2^2,\rho_2)
$$

是不同的, 但是由 (3.3.14) 和 (3.3.15) 知它们有完全相同的两个边缘分布. 对这个现象的解释是: 边缘分布只考虑了单个分量各自的情况, 而未涉及 X,Y 之间的联系, X,Y 之间的联系这个信息是包含在 (X,Y) 的联合分布之内的. 在下一章将指出, 参数 ρ 正好刻画了 X 和 Y 之间关系的密切程度. 这一事实再一次说明边缘分布不能完全决定它们的联合分布. 还值得一提的是, 两个边缘分布都是一维正态分布的二维随机变量, 它们的联合分布不仅是不唯一确定的, 还可以不是一个二维正态分布. 下面就是一个这样的例子.

例 3.3.6. 设

$$
f(x,y) = \frac{1}{2\pi}\mathrm{e}^{-\frac{x^2+y^2}{2}}(1+\sin x\sin y), \quad (x,y)\in\mathbf{R}^2
$$

则易见此 $f(x,y)$ 满足 (3.3.2) 和 (3.3.3), 且它的两个边缘分布均为标准正态分布 $N(0,1)$, 但显然 $f(x,y)$ 不是二维正态分布.

常 用 积 分 公 式

（一）含有$ax+b$的积分$(a\neq 0)$

1. $\int\frac{\mathrm{d}x}{ax+b}=\frac{1}{a}\ln|ax+b|+C$

2. $\int(ax+b)^{\mu}\mathrm{d}x=\frac{1}{a(\mu+1)}(ax+b)^{\mu+1}+C$ （$\mu\neq -1$）

3. $\int\frac{x}{ax+b}\mathrm{d}x=\frac{1}{a^2}(ax+b-b\ln|ax+b|)+C$

4. $\int\frac{x^2}{ax+b}\mathrm{d}x=\frac{1}{a^3}\left[\frac{1}{2}(ax+b)^2-2b(ax+b)+b^2\ln|ax+b|\right]+C$

5. $\int\frac{\mathrm{d}x}{x(ax+b)}=-\frac{1}{b}\ln\left|\frac{ax+b}{x}\right|+C$

6. $\int\frac{\mathrm{d}x}{x^2(ax+b)}=-\frac{1}{bx}+\frac{a}{b^2}\ln\left|\frac{ax+b}{x}\right|+C$

7. $\int\frac{x}{(ax+b)^2}\mathrm{d}x=\frac{1}{a^2}(\ln|ax+b|+\frac{b}{ax+b})+C$

8. $\int\frac{x^2}{(ax+b)^2}\mathrm{d}x=\frac{1}{a^3}(ax+b-2b\ln|ax+b|-\frac{b^2}{ax+b})+C$

9. $\int\frac{\mathrm{d}x}{x(ax+b)^2}=\frac{1}{b(ax+b)}-\frac{1}{b^2}\ln\left|\frac{ax+b}{x}\right|+C$

（二）含有$\sqrt{ax+b}$ 的积分

10. $\int\sqrt{ax+b}\,\mathrm{d}x=\frac{2}{3a}\sqrt{(ax+b)^3}+C$

11. $\int x\sqrt{ax+b}\,\mathrm{d}x=\frac{2}{15a^2}(3ax-2b)\sqrt{(ax+b)^3}+C$

12. $\int x^2\sqrt{ax+b}\,\mathrm{d}x=\frac{2}{105a^3}(15a^2x^2-12abx+8b^2)\sqrt{(ax+b)^3}+C$

13. $\int\frac{x}{\sqrt{ax+b}}\mathrm{d}x=\frac{2}{3a^2}(ax-2b)\sqrt{ax+b}+C$

14. $\int \frac{x^2}{\sqrt{ax+b}}\mathrm{d}x = \frac{2}{15a^3}(3a^2x^2 - 4abx + 8b^2)\sqrt{ax+b} + C$

15. $\int \frac{\mathrm{d}x}{x\sqrt{ax+b}} = \begin{cases} \frac{1}{\sqrt{b}}\ln\left|\frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right| + C & (b>0) \\ \frac{2}{\sqrt{-b}}\arctan\sqrt{\frac{ax+b}{-b}} + C & (b<0) \end{cases}$

16. $\int \frac{\mathrm{d}x}{x^2\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{bx} - \frac{a}{2b}\int \frac{\mathrm{d}x}{x\sqrt{ax+b}}$

17. $\int \frac{\sqrt{ax+b}}{x}\mathrm{d}x = 2\sqrt{ax+b} + b\int \frac{\mathrm{d}x}{x\sqrt{ax+b}}$

18. $\int \frac{\sqrt{ax+b}}{x^2}\mathrm{d}x = -\frac{\sqrt{ax+b}}{x} + \frac{a}{2}\int \frac{\mathrm{d}x}{x\sqrt{ax+b}}$

（三）含有 $x^2 \pm a^2$ 的积分

19. $\int \frac{\mathrm{d}x}{x^2+a^2} = \frac{1}{a}\arctan\frac{x}{a} + C$

20. $\int \frac{\mathrm{d}x}{(x^2+a^2)^n} = \frac{x}{2(n-1)a^2(x^2+a^2)^{n-1}} + \frac{2n-3}{2(n-1)a^2}\int \frac{\mathrm{d}x}{(x^2+a^2)^{n-1}}$

21. $\int \frac{\mathrm{d}x}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$

（四）含有 $ax^2 + b(a>0)$ 的积分

22. $\int \frac{\mathrm{d}x}{ax^2+b} = \begin{cases} \frac{1}{\sqrt{ab}}\arctan\sqrt{\frac{a}{b}}\,x + C & (b>0) \\ \frac{1}{2\sqrt{-ab}}\ln\left|\frac{\sqrt{a}\,x-\sqrt{-b}}{\sqrt{a}\,x+\sqrt{-b}}\right| + C & (b<0) \end{cases}$

23. $\int \frac{x}{ax^2+b}\mathrm{d}x = \frac{1}{2a}\ln\left|ax^2+b\right| + C$

24. $\int \frac{x^2}{ax^2+b}\mathrm{d}x = \frac{x}{a} - \frac{b}{a}\int \frac{\mathrm{d}x}{ax^2+b}$

25. $\int \frac{\mathrm{d}x}{x(ax^2+b)} = \frac{1}{2b}\ln\frac{x^2}{|ax^2+b|} + C$

26. $\int \frac{\mathrm{d}x}{x^2(ax^2+b)} = -\frac{1}{bx} - \frac{a}{b}\int \frac{\mathrm{d}x}{ax^2+b}$

27. $\int \frac{\mathrm{d}x}{x^3(ax^2+b)} = \frac{a}{2b^2}\ln\frac{|ax^2+b|}{x^2} - \frac{1}{2bx^2} + C$

28. $\int \frac{\mathrm{d}x}{(ax^2+b)^2} = \frac{x}{2b(ax^2+b)} + \frac{1}{2b}\int \frac{\mathrm{d}x}{ax^2+b}$

（五）含有 $ax^2+bx+c\ (a>0)$ 的积分

29. $\int \frac{\mathrm{d}x}{ax^2+bx+c} = \begin{cases} \frac{2}{\sqrt{4ac-b^2}}\arctan\frac{2ax+b}{\sqrt{4ac-b^2}} + C & (b^2<4ac) \\ \frac{1}{\sqrt{b^2-4ac}}\ln\left|\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right| + C & (b^2>4ac) \end{cases}$

30. $\int \frac{x}{ax^2+bx+c}\mathrm{d}x = \frac{1}{2a}\ln|ax^2+bx+c| - \frac{b}{2a}\int \frac{\mathrm{d}x}{ax^2+bx+c}$

（六）含有 $\sqrt{x^2+a^2}\ (a>0)$ 的积分

31. $\int \frac{\mathrm{d}x}{\sqrt{x^2+a^2}} = \operatorname{arsh}\frac{x}{a} + C_1 = \ln(x+\sqrt{x^2+a^2}) + C$

32. $\int \frac{\mathrm{d}x}{\sqrt{(x^2+a^2)^3}} = \frac{x}{a^2\sqrt{x^2+a^2}} + C$

33. $\int \frac{x}{\sqrt{x^2+a^2}}\mathrm{d}x = \sqrt{x^2+a^2} + C$

34. $\int \frac{x}{\sqrt{(x^2+a^2)^3}}\mathrm{d}x = -\frac{1}{\sqrt{x^2+a^2}} + C$

35. $\int \frac{x^2}{\sqrt{x^2+a^2}}\mathrm{d}x = \frac{x}{2}\sqrt{x^2+a^2} - \frac{a^2}{2}\ln(x+\sqrt{x^2+a^2})+C$

36. $\int \frac{x^2}{\sqrt{(x^2+a^2)^3}}\mathrm{d}x = -\frac{x}{\sqrt{x^2+a^2}} + \ln(x+\sqrt{x^2+a^2})+C$

37. $\int \frac{\mathrm{d}x}{x\sqrt{x^2+a^2}} = \frac{1}{a}\ln\frac{\sqrt{x^2+a^2}-a}{|x|}+C$

38. $\int \frac{\mathrm{d}x}{x^2\sqrt{x^2+a^2}} = -\frac{\sqrt{x^2+a^2}}{a^2 x}+C$

39. $\int \sqrt{x^2+a^2}\,\mathrm{d}x = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln(x+\sqrt{x^2+a^2})+C$

40. $\int \sqrt{(x^2+a^2)^3}\,\mathrm{d}x = \frac{x}{8}(2x^2+5a^2)\sqrt{x^2+a^2} + \frac{3}{8}a^4\ln(x+\sqrt{x^2+a^2})+C$

41. $\int x\sqrt{x^2+a^2}\,\mathrm{d}x = \frac{1}{3}\sqrt{(x^2+a^2)^3}+C$

42. $\int x^2\sqrt{x^2+a^2}\,\mathrm{d}x = \frac{x}{8}(2x^2+a^2)\sqrt{x^2+a^2} - \frac{a^4}{8}\ln(x+\sqrt{x^2+a^2})+C$

43. $\int \frac{\sqrt{x^2+a^2}}{x}\mathrm{d}x = \sqrt{x^2+a^2} + a\ln\frac{\sqrt{x^2+a^2}-a}{|x|}+C$

44. $\int \frac{\sqrt{x^2+a^2}}{x^2}\mathrm{d}x = -\frac{\sqrt{x^2+a^2}}{x} + \ln(x+\sqrt{x^2+a^2})+C$

（七）含有$\sqrt{x^2-a^2}$ $(a>0)$的积分

45. $\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}} = \frac{x}{|x|}\operatorname{arch}\frac{|x|}{a}+C_1 = \ln\left|x+\sqrt{x^2-a^2}\right|+C$

46. $\int \frac{\mathrm{d}x}{\sqrt{(x^2-a^2)^3}} = -\frac{x}{a^2\sqrt{x^2-a^2}}+C$

47. $\int \frac{x}{\sqrt{x^2-a^2}}\mathrm{d}x = \sqrt{x^2-a^2}+C$

48. $\int \frac{x}{\sqrt{(x^2-a^2)^3}}\mathrm{d}x = -\frac{1}{\sqrt{x^2-a^2}}+C$

49. $\int \frac{x^2}{\sqrt{x^2-a^2}}\mathrm{d}x = \frac{x}{2}\sqrt{x^2-a^2}+\frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right|+C$

50. $\int \frac{x^2}{\sqrt{(x^2-a^2)^3}}\mathrm{d}x = -\frac{x}{\sqrt{x^2-a^2}}+\ln\left|x+\sqrt{x^2-a^2}\right|+C$

51. $\int \frac{\mathrm{d}x}{x\sqrt{x^2-a^2}} = \frac{1}{a}\arccos\frac{a}{|x|}+C$

52. $\int \frac{\mathrm{d}x}{x^2\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{a^2x}+C$

53. $\int \sqrt{x^2-a^2}\mathrm{d}x = \frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right|+C$

54. $\int \sqrt{(x^2-a^2)^3}\mathrm{d}x = \frac{x}{8}(2x^2-5a^2)\sqrt{x^2-a^2}+\frac{3}{8}a^4\ln\left|x+\sqrt{x^2-a^2}\right|+C$

55. $\int x\sqrt{x^2-a^2}\mathrm{d}x = \frac{1}{3}\sqrt{(x^2-a^2)^3}+C$

56. $\int x^2\sqrt{x^2-a^2}\mathrm{d}x = \frac{x}{8}(2x^2-a^2)\sqrt{x^2-a^2}-\frac{a^4}{8}\ln\left|x+\sqrt{x^2-a^2}\right|+C$

57. $\int \frac{\sqrt{x^2-a^2}}{x}\mathrm{d}x = \sqrt{x^2-a^2}-a\arccos\frac{a}{|x|}+C$

58. $\int \frac{\sqrt{x^2-a^2}}{x^2}\mathrm{d}x = -\frac{\sqrt{x^2-a^2}}{x}+\ln\left|x+\sqrt{x^2-a^2}\right|+C$

（八）含有$\sqrt{a^2-x^2}$ $(a>0)$的积分

59. $\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a}+C$

60. $\int \frac{\mathrm{d}x}{\sqrt{(a^2-x^2)^3}} = \frac{x}{a^2\sqrt{a^2-x^2}}+C$

61. $\int \frac{x}{\sqrt{a^2-x^2}}\mathrm{d}x=-\sqrt{a^2-x^2}+C$

62. $\int \frac{x}{\sqrt{(a^2-x^2)^3}}\mathrm{d}x=\frac{1}{\sqrt{a^2-x^2}}+C$

63. $\int \frac{x^2}{\sqrt{a^2-x^2}}\mathrm{d}x=-\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\arcsin\frac{x}{a}+C$

64. $\int \frac{x^2}{\sqrt{(a^2-x^2)^3}}\mathrm{d}x=\frac{x}{\sqrt{a^2-x^2}}-\arcsin\frac{x}{a}+C$

65. $\int \frac{\mathrm{d}x}{x\sqrt{a^2-x^2}}=\frac{1}{a}\ln\frac{a-\sqrt{a^2-x^2}}{|x|}+C$

66. $\int \frac{\mathrm{d}x}{x^2\sqrt{a^2-x^2}}=-\frac{\sqrt{a^2-x^2}}{a^2x}+C$

67. $\int \sqrt{a^2-x^2}\mathrm{d}x=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\arcsin\frac{x}{a}+C$

68. $\int \sqrt{(a^2-x^2)^3}\mathrm{d}x=\frac{x}{8}(5a^2-2x^2)\sqrt{a^2-x^2}+\frac{3}{8}a^4\arcsin\frac{x}{a}+C$

69. $\int x\sqrt{a^2-x^2}\mathrm{d}x=-\frac{1}{3}\sqrt{(a^2-x^2)^3}+C$

70. $\int x^2\sqrt{a^2-x^2}\mathrm{d}x=\frac{x}{8}(2x^2-a^2)\sqrt{a^2-x^2}+\frac{a^4}{8}\arcsin\frac{x}{a}+C$

71. $\int \frac{\sqrt{a^2-x^2}}{x}\mathrm{d}x=\sqrt{a^2-x^2}+a\ln\frac{a-\sqrt{a^2-x^2}}{|x|}+C$

72. $\int \frac{\sqrt{a^2-x^2}}{x^2}\mathrm{d}x=-\frac{\sqrt{a^2-x^2}}{x}-\arcsin\frac{x}{a}+C$

（九）含有 $\sqrt{\pm ax^2+bx+c}$ $(a>0)$ 的积分

73. $\int \frac{\mathrm{d}x}{\sqrt{ax^2+bx+c}}=\frac{1}{\sqrt{a}}\ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right|+C$

74. $\int\sqrt{ax^2+bx+c}\mathrm{d}x=\frac{2ax+b}{4a}\sqrt{ax^2+bx+c}$

$$+\frac{4ac-b^2}{8\sqrt{a^3}}\ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right|+C$$

75. $\int\frac{x}{\sqrt{ax^2+bx+c}}\mathrm{d}x=\frac{1}{a}\sqrt{ax^2+bx+c}$

$$-\frac{b}{2\sqrt{a^3}}\ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right|+C$$

76. $\int\frac{\mathrm{d}x}{\sqrt{c+bx-ax^2}}=-\frac{1}{\sqrt{a}}\arcsin\frac{2ax-b}{\sqrt{b^2+4ac}}+C$

77. $\int\sqrt{c+bx-ax^2}\mathrm{d}x=\frac{2ax-b}{4a}\sqrt{c+bx-ax^2}+\frac{b^2+4ac}{8\sqrt{a^3}}\arcsin\frac{2ax-b}{\sqrt{b^2+4ac}}+C$

78. $\int\frac{x}{\sqrt{c+bx-ax^2}}\mathrm{d}x=-\frac{1}{a}\sqrt{c+bx-ax^2}+\frac{b}{2\sqrt{a^3}}\arcsin\frac{2ax-b}{\sqrt{b^2+4ac}}+C$

（十）含有$\sqrt{\pm\frac{x-a}{x-b}}$或$\sqrt{(x-a)(b-x)}$的积分

79. $\int\sqrt{\frac{x-a}{x-b}}\mathrm{d}x=(x-b)\sqrt{\frac{x-a}{x-b}}+(b-a)\ln(\sqrt{|x-a|}+\sqrt{|x-b|})+C$

80. $\int\sqrt{\frac{x-a}{b-x}}\mathrm{d}x=(x-b)\sqrt{\frac{x-a}{b-x}}+(b-a)\arcsin\sqrt{\frac{x-a}{b-x}}+C$

81. $\int\frac{\mathrm{d}x}{\sqrt{(x-a)(b-x)}}=2\arcsin\sqrt{\frac{x-a}{b-x}}+C\qquad(a<b)$

82. $\int\sqrt{(x-a)(b-x)}\mathrm{d}x=\frac{2x-a-b}{4}\sqrt{(x-a)(b-x)}+\frac{(b-a)^2}{4}\arcsin\sqrt{\frac{x-a}{b-x}}+C$

$(a<b)$

（十一）含有三角函数的积分

83. $\int\sin x\mathrm{d}x=-\cos x+C$

84. $\int \cos x\mathrm{d}x = \sin x + C$

85. $\int \tan x\mathrm{d}x = -\ln|\cos x| + C$

86. $\int \cot x\mathrm{d}x = \ln|\sin x| + C$

87. $\int \sec x\mathrm{d}x = \ln\left|\tan(\frac{\pi}{4}+\frac{x}{2})\right| + C = \ln|\sec x + \tan x| + C$

88. $\int \csc x\mathrm{d}x = \ln\left|\tan\frac{x}{2}\right| + C = \ln|\csc x - \cot x| + C$

89. $\int \sec^2 x\mathrm{d}x = \tan x + C$

90. $\int \csc^2 x\mathrm{d}x = -\cot x + C$

91. $\int \sec x \tan x\mathrm{d}x = \sec x + C$

92. $\int \csc x \cot x\mathrm{d}x = -\csc x + C$

93. $\int \sin^2 x\mathrm{d}x = \frac{x}{2} - \frac{1}{4}\sin 2x + C$

94. $\int \cos^2 x\mathrm{d}x = \frac{x}{2} + \frac{1}{4}\sin 2x + C$

95. $\int \sin^n x\mathrm{d}x = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\mathrm{d}x$

96. $\int \cos^n x\mathrm{d}x = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\mathrm{d}x$

97. $\int \frac{\mathrm{d}x}{\sin^n x} = -\frac{1}{n-1}\cdot\frac{\cos x}{\sin^{n-1} x} + \frac{n-2}{n-1}\int \frac{\mathrm{d}x}{\sin^{n-2} x}$

98. $\int \frac{\mathrm{d}x}{\cos^n x} = \frac{1}{n-1}\cdot\frac{\sin x}{\cos^{n-1} x} + \frac{n-2}{n-1}\int \frac{\mathrm{d}x}{\cos^{n-2} x}$

99. $$\begin{aligned}\int \cos^m x\sin^n x\mathrm{d}x &= \frac{1}{m+n}\cos^{m-1} x\sin^{n+1} x + \frac{m-1}{m+n}\int \cos^{m-2} x\sin^n x\mathrm{d}x \\ &= -\frac{1}{m+n}\cos^{m+1} x\sin^{n-1} x + \frac{n-1}{m+n}\int \cos^m x\sin^{n-2} x\mathrm{d}x\end{aligned}$$

100. $\int \sin ax\cos bx\mathrm{d}x = -\frac{1}{2(a+b)}\cos(a+b)x - \frac{1}{2(a-b)}\cos(a-b)x + C$

101. $\int \sin ax \sin bx \mathrm{d}x = -\frac{1}{2(a+b)}\sin(a+b)x + \frac{1}{2(a-b)}\sin(a-b)x + C$

102. $\int \cos ax \cos bx \mathrm{d}x = \frac{1}{2(a+b)}\sin(a+b)x + \frac{1}{2(a-b)}\sin(a-b)x + C$

103. $\int \frac{\mathrm{d}x}{a+b\sin x} = \frac{2}{\sqrt{a^2-b^2}}\arctan\frac{a\tan\frac{x}{2}+b}{\sqrt{a^2-b^2}} + C \qquad (a^2>b^2)$

104. $\int \frac{\mathrm{d}x}{a+b\sin x} = \frac{1}{\sqrt{b^2-a^2}}\ln\left|\frac{a\tan\frac{x}{2}+b-\sqrt{b^2-a^2}}{a\tan\frac{x}{2}+b+\sqrt{b^2-a^2}}\right| + C \qquad (a^2<b^2)$

105. $\int \frac{\mathrm{d}x}{a+b\cos x} = \frac{2}{a+b}\sqrt{\frac{a+b}{a-b}}\arctan(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}) + C \qquad (a^2>b^2)$

106. $\int \frac{\mathrm{d}x}{a+b\cos x} = \frac{1}{a+b}\sqrt{\frac{a+b}{b-a}}\ln\left|\frac{\tan\frac{x}{2}+\sqrt{\frac{a+b}{b-a}}}{\tan\frac{x}{2}-\sqrt{\frac{a+b}{b-a}}}\right| + C \qquad (a^2<b^2)$

107. $\int \frac{\mathrm{d}x}{a^2\cos^2 x + b^2\sin^2 x} = \frac{1}{ab}\arctan(\frac{b}{a}\tan x) + C$

108. $\int \frac{\mathrm{d}x}{a^2\cos^2 x - b^2\sin^2 x} = \frac{1}{2ab}\ln\left|\frac{b\tan x + a}{b\tan x - a}\right| + C$

109. $\int x\sin ax \mathrm{d}x = \frac{1}{a^2}\sin ax - \frac{1}{a}x\cos ax + C$

110. $\int x^2\sin ax \mathrm{d}x = -\frac{1}{a}x^2\cos ax + \frac{2}{a^2}x\sin ax + \frac{2}{a^3}\cos ax + C$

111. $\int x\cos ax \mathrm{d}x = \frac{1}{a^2}\cos ax + \frac{1}{a}x\sin ax + C$

112. $\int x^2\cos ax \mathrm{d}x = \frac{1}{a}x^2\sin ax + \frac{2}{a^2}x\cos ax - \frac{2}{a^3}\sin ax + C$

（十二）含有反三角函数的积分（其中$a>0$）

113. $\int \arcsin\frac{x}{a}\mathrm{d}x = x\arcsin\frac{x}{a} + \sqrt{a^2-x^2} + C$

114. $\int x\arcsin\frac{x}{a}\mathrm{d}x=(\frac{x^2}{2}-\frac{a^2}{4})\arcsin\frac{x}{a}+\frac{x}{4}\sqrt{a^2-x^2}+C$

115. $\int x^2\arcsin\frac{x}{a}\mathrm{d}x=\frac{x^3}{3}\arcsin\frac{x}{a}+\frac{1}{9}(x^2+2a^2)\sqrt{a^2-x^2}+C$

116. $\int \arccos\frac{x}{a}\mathrm{d}x=x\arccos\frac{x}{a}-\sqrt{a^2-x^2}+C$

117. $\int x\arccos\frac{x}{a}\mathrm{d}x=(\frac{x^2}{2}-\frac{a^2}{4})\arccos\frac{x}{a}-\frac{x}{4}\sqrt{a^2-x^2}+C$

118. $\int x^2\arccos\frac{x}{a}\mathrm{d}x=\frac{x^3}{3}\arccos\frac{x}{a}-\frac{1}{9}(x^2+2a^2)\sqrt{a^2-x^2}+C$

119. $\int \arctan\frac{x}{a}\mathrm{d}x=x\arctan\frac{x}{a}-\frac{a}{2}\ln(a^2+x^2)+C$

120. $\int x\arctan\frac{x}{a}\mathrm{d}x=\frac{1}{2}(a^2+x^2)\arctan\frac{x}{a}-\frac{a}{2}x+C$

121. $\int x^2\arctan\frac{x}{a}\mathrm{d}x=\frac{x^3}{3}\arctan\frac{x}{a}-\frac{a}{6}x^2+\frac{a^3}{6}\ln(a^2+x^2)+C$

（十三）含有指数函数的积分

122. $\int a^x\mathrm{d}x=\frac{1}{\ln a}a^x+C$

123. $\int \mathrm{e}^{ax}\mathrm{d}x=\frac{1}{a}\mathrm{e}^{ax}+C$

124. $\int x\mathrm{e}^{ax}\mathrm{d}x=\frac{1}{a^2}(ax-1)\mathrm{e}^{ax}+C$

125. $\int x^n\mathrm{e}^{ax}\mathrm{d}x=\frac{1}{a}x^n\mathrm{e}^{ax}-\frac{n}{a}\int x^{n-1}\mathrm{e}^{ax}\mathrm{d}x$

126. $\int xa^x\mathrm{d}x=\frac{x}{\ln a}a^x-\frac{1}{(\ln a)^2}a^x+C$

127. $\int x^n a^x\mathrm{d}x=\frac{1}{\ln a}x^n a^x-\frac{n}{\ln a}\int x^{n-1}a^x\mathrm{d}x$

128. $\int \mathrm{e}^{ax}\sin bx\mathrm{d}x=\frac{1}{a^2+b^2}\mathrm{e}^{ax}(a\sin bx-b\cos bx)+C$

129. $\int \mathrm{e}^{ax}\cos bx\mathrm{d}x=\frac{1}{a^2+b^2}\mathrm{e}^{ax}(b\sin bx+a\cos bx)+C$

130. $\int e^{ax}\sin^n bx\mathrm{d}x=\frac{1}{a^2+b^2n^2}e^{ax}\sin^{n-1}bx(a\sin bx-nb\cos bx)$

$$+\frac{n(n-1)b^2}{a^2+b^2n^2}\int e^{ax}\sin^{n-2}bx\mathrm{d}x$$

131. $\int e^{ax}\cos^n bx\mathrm{d}x=\frac{1}{a^2+b^2n^2}e^{ax}\cos^{n-1}bx(a\cos bx+nb\sin bx)$

$$+\frac{n(n-1)b^2}{a^2+b^2n^2}\int e^{ax}\cos^{n-2}bx\mathrm{d}x$$

（十四）含有对数函数的积分

132. $\int \ln x\mathrm{d}x=x\ln x-x+C$

133. $\int \frac{\mathrm{d}x}{x\ln x}=\ln|\ln x|+C$

134. $\int x^n\ln x\mathrm{d}x=\frac{1}{n+1}x^{n+1}(\ln x-\frac{1}{n+1})+C$

135. $\int (\ln x)^n\mathrm{d}x=x(\ln x)^n-n\int(\ln x)^{n-1}\mathrm{d}x$

136. $\int x^m(\ln x)^n\mathrm{d}x=\frac{1}{m+1}x^{m+1}(\ln x)^n-\frac{n}{m+1}\int x^m(\ln x)^{n-1}\mathrm{d}x$

（十五）含有双曲函数的积分

137. $\int \mathrm{sh}x\mathrm{d}x=\mathrm{ch}x+C$

138. $\int \mathrm{ch}x\mathrm{d}x=\mathrm{sh}x+C$

139. $\int \mathrm{th}x\mathrm{d}x=\ln\mathrm{ch}x+C$

140. $\int \mathrm{sh}^2x\mathrm{d}x=-\frac{x}{2}+\frac{1}{4}\mathrm{sh}2x+C$

141. $\int \mathrm{ch}^2x\mathrm{d}x=\frac{x}{2}+\frac{1}{4}\mathrm{sh}2x+C$

（十六）定积分

142. $\int_{-\pi}^{\pi}\cos nx\mathrm{d}x=\int_{-\pi}^{\pi}\sin nx\mathrm{d}x=0$

143. $\int_{-\pi}^{\pi}\cos mx\sin nx\mathrm{d}x=0$

144. $\int_{-\pi}^{\pi}\cos mx\cos nx\mathrm{d}x=\begin{cases}0, & m\neq n\\ \pi, & m=n\end{cases}$

145． $\int_{-\pi}^{\pi} \sin mx \sin nx \mathrm{d}x = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$

146． $\int_{0}^{\pi} \sin mx \sin nx \mathrm{d}x = \int_{0}^{\pi} \cos mx \cos nx \mathrm{d}x = \begin{cases} 0, & m \neq n \\ \dfrac{\pi}{2}, & m = n \end{cases}$

147． $I_n = \int_{0}^{\frac{\pi}{2}} \sin^n x \mathrm{d}x = \int_{0}^{\frac{\pi}{2}} \cos^n x \mathrm{d}x$

$I_n = \dfrac{n-1}{n} I_{n-2}$

$I_n = \dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} \cdot \cdots \cdot \dfrac{4}{5} \cdot \dfrac{2}{3}$ （n 为大于 1 的正奇数），$I_1 = 1$

$I_n = \dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} \cdot \cdots \cdot \dfrac{3}{4} \cdot \dfrac{1}{2} \cdot \dfrac{\pi}{2}$ （n 为正偶数），$I_0 = \dfrac{\pi}{2}$