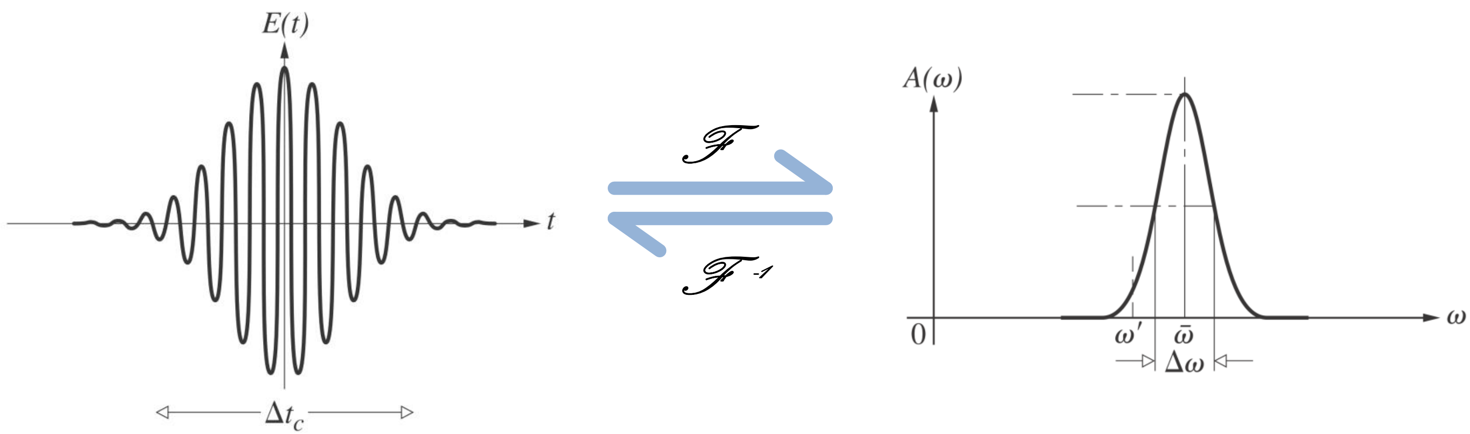
C2 Wave Motion

- Harmonic waves.
- Fourier Transform.
- **Propagation vector**, wavefront, **phase velocity**, frequency bandwidth.



§ 2.1 Mathematical description

① Harmonic vibrations

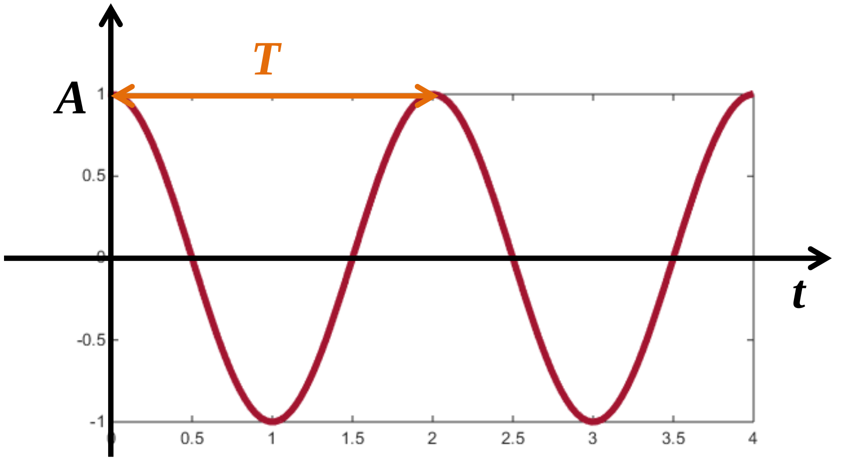
Vibration: A physical quantity changes periodically around its equilibrium position (or average value).

Harmonic vibration: The variation of the physical quantity of with time t is a periodic and it changes as a sine or cosine function in each period.

- Equation of motion:

$$U(t) = A\cos\left(\frac{2\pi}{T}t + \varphi_0\right)$$

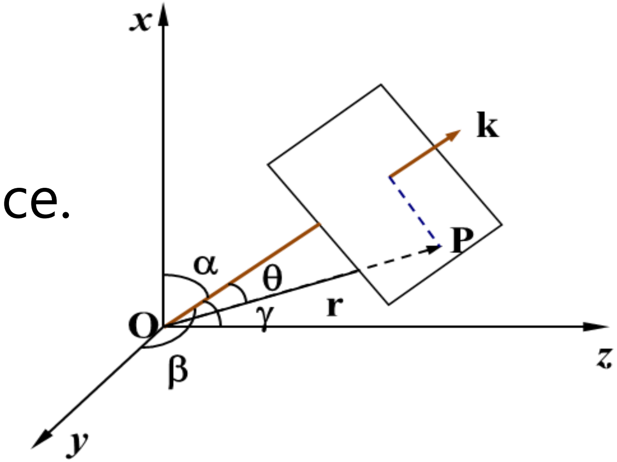
$$= A\cos(\omega t + \varphi_0)$$



A is amplitude, $\omega = \dfrac{2\pi}{T}$ is angular frequency.

$\varphi = (\omega t + \varphi_0)$ is the phase and φ_0 is the initial phase.

② Harmonic plane waves



- **Waves**: The spread of vibrations in space.
- If the wave is the spreading of harmonic vibration, it is called simple harmonic wave.
- Monochromatic parallel waves can be viewed as harmonic plane waves.
- A parallel light propagates in the k direction and consider the vibration of arbitrary point P in space. Set P(x, y, z), denoted by the vector $\mathbf{r}$. The velocity is v. The time it takes from point O to point P is t', then:

$$t' = r\cos\theta / v$$

∴ Vibration at P :

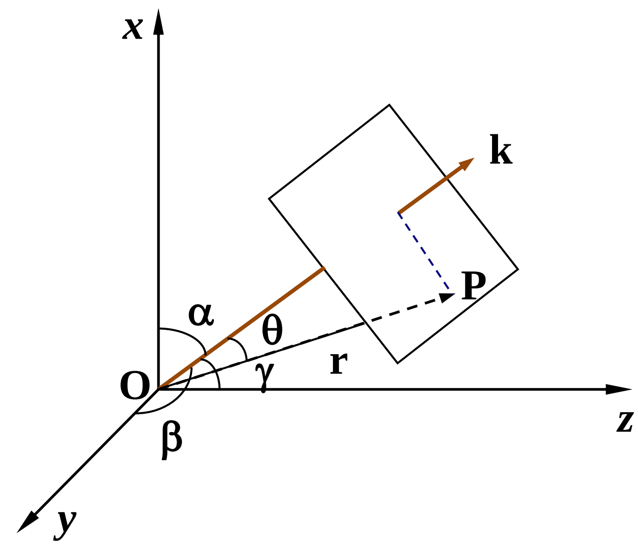
$$U(\mathbf{r},t) = A\cos(\omega t - \omega t') = A\cos\left(\omega t - (2\pi/\lambda) r\cos\theta\right) \qquad \omega = \frac{2\pi}{\lambda} v$$

- Let's define $\mathbf{k} = \frac{2\pi}{\lambda}\hat{k}$

$$U(\mathbf{r},t) = A\cos\left(\omega t - (2\pi/\lambda) r\cos\theta\right)$$
$$= A\cos\left(\omega t - \mathbf{k}\cdot\mathbf{r}\right)$$

- In Cartesian coordinate,

$$\mathbf{k}\cdot\mathbf{r} = k\left(x\cos\alpha + y\cos\beta + z\cos\gamma\right)$$
$$= k_x x + k_y y + k_z z$$



- λ is also called **spatial period** (wavelength). $1/\lambda$ is **spatial frequency**, i.e. the number of wavelengths per unit length in the propagation direction.
- $k = \frac{2\pi}{\lambda}$ is called angular spatial frequency or **propagation number**. $\mathbf{k}$ is the **wavevector**.

③ Wave function

- Wave function： the function that describes the wave. It is a function of $\mathbf{r}$ and t.

$$U(\mathbf{r},t) = A\cos(\omega t - \mathbf{k}\cdot\mathbf{r})$$

- **Temporal** period and **spatial** period are linked by v:

$$\lambda = vT$$

- Comparison：

Temporal period T

Temporal frequency $\nu = \frac{1}{T}$

Spatial period λ

Spatial frequency $1/\lambda$

$\omega = \frac{2\pi}{T}$ **Angular** temporal frequency

$k = \frac{2\pi}{\lambda}$ **Angular** spatial frequency（空间圆频率）

- The wave function can be further written as

$$U(\mathbf{r},t) = A(\mathbf{r})\cos\left[\omega t - \varphi(\mathbf{r})\right]$$

A function depends on space and time

Describe the spatial distribution of amplitude

Describe the temporal distribution of phase

Describe the spatial distribution of phase

Wavefront, Phase velocity

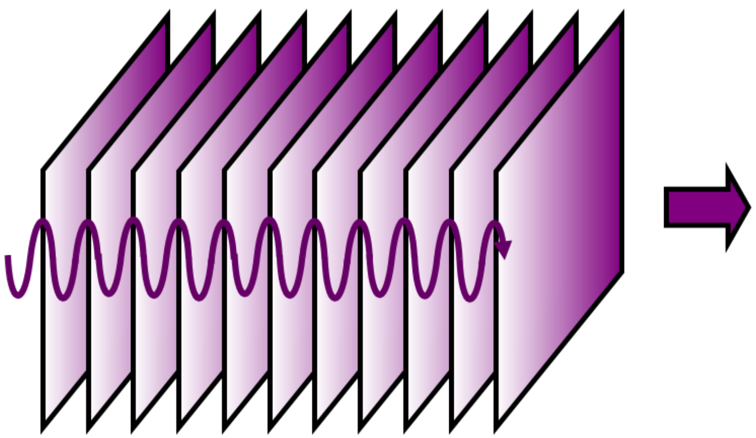
- At any instant a **wavefront** in three dimensions is a surface of constant phase.

$$(\omega t-\mathbf{k}\cdot\mathbf{r})=Const. \quad \omega \mathrm{d}t-\mathbf{k}\cdot\mathrm{d}\mathbf{r}=0 \quad \omega=\mathbf{k}\cdot\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}=\mathbf{k}\cdot v_{\mathrm{p}}$$

- If propagation direction is along the coordinate axis (e.g. *z* axis), the propagation velocity of wavefront, i.e., the **phase velocity**, is:

$$v_{\mathrm{p}}=\frac{\omega}{k}$$

- The wavefront of plane waves is a plane. -----> $(\omega t-kz)=Const.$



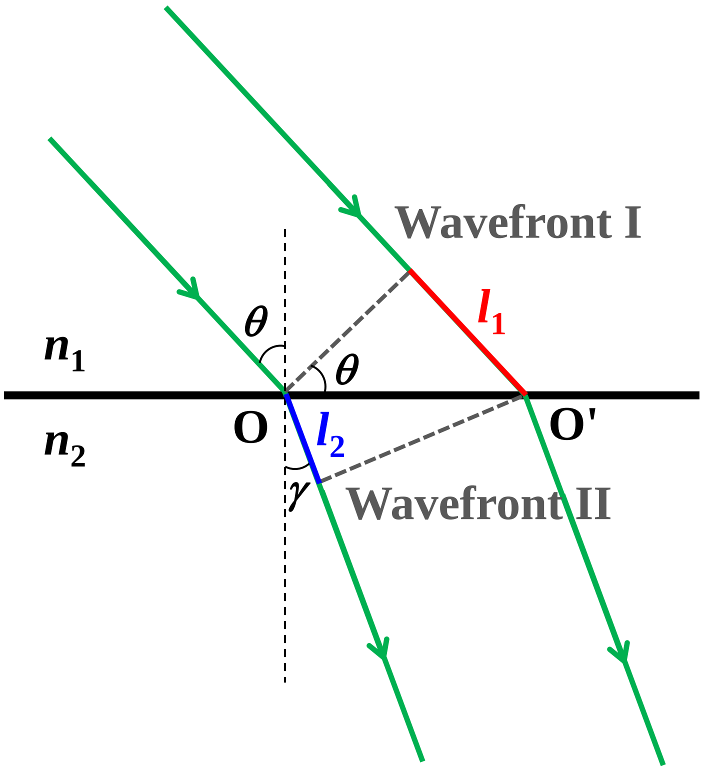
A wave's wavefronts sweep along the propagation direction at the speed of light.

Wavefront, Phase velocity

- The time that light travels from the wavefront I to II.

$$n_1 l_1 = n_1 \sin\theta OO'$$
$$= n_2 \sin\gamma OO' = n_2 l_2$$
$$= c\Delta t$$

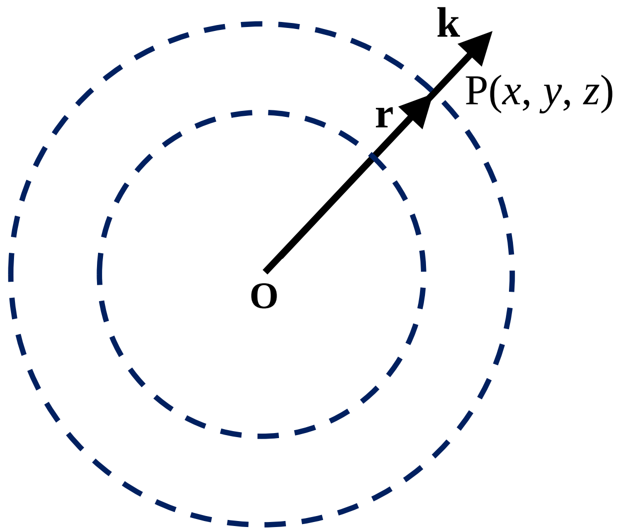
$$\Delta t = \frac{l_1}{v_1} = \frac{l_2}{v_2}$$



Wavefront, Phase velocity

④ Harmonic spherical wave

- In a homogeneous medium, the wavefront of a **point source** is spherical. >spherical waves



- Similar to the plane waves, the vibration at P(x, y, z):

$$U(\mathbf{r},t) = A(r)\cos(\omega t - \mathbf{k}\cdot\mathbf{r})$$

According to the energy conservation, it follows

$$A(r) = A_0/r.$$

Assuming r_0 to be a unit length, we have:

$$4\pi r_0^2 I_0 = 4\pi r^2 I$$

$$\Rightarrow r_0^2 A_0^2 = r^2 A^2$$

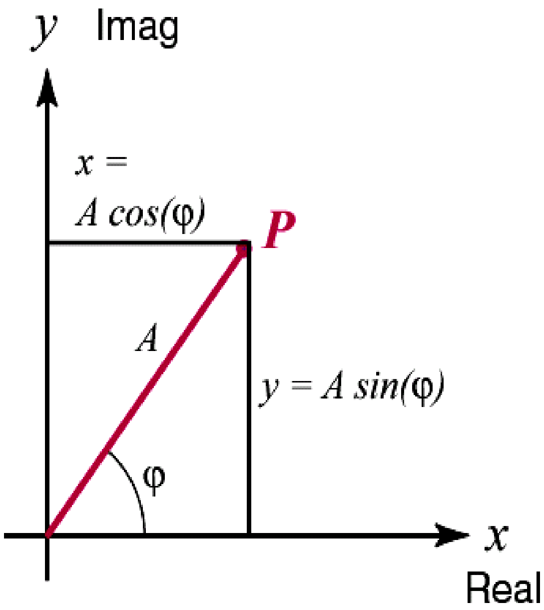
$$\Rightarrow A = A_0 r_0 / r$$

$$U(\mathbf{r},t) = (A_0/r)\cos(\omega t - \mathbf{k}\cdot\mathbf{r})$$

The complex representation

- In order to simplify the calculation, the cosine expression of the wave is often expressed in complex exponent.
- Since: $e^{\mathbf{i}\alpha} = \cos\alpha + \mathbf{i}\sin\alpha$

$$e^{-\mathbf{i}\alpha} = \cos\alpha - \mathbf{i}\sin\alpha$$



- For most waves, the wavefunction can be written as

$$U(\mathbf{r},t) = A(\mathbf{r})\cos\left[\omega t - \varphi(\mathbf{r})\right]$$

The complex form:

$$U(\mathbf{r},t) = A(\mathbf{r})e^{i\varphi(\mathbf{r})}e^{-i\omega t}$$

$\tilde{U}(\mathbf{r}) = A(\mathbf{r})e^{i\varphi(\mathbf{r})}$ is called the **complex amplitude**.

The complex representation

- Complex amplitude of plane waves:

$$\tilde{U}(\mathbf{r}) = A_0 e^{i\mathbf{k}\cdot\mathbf{r}} = A_0 e^{ik(x\cos\alpha + y\cos\beta + z\cos\gamma)}$$

- Complex amplitude of spherical waves:

$$\tilde{U}(\mathbf{r}) = \frac{A_0}{r} e^{i\mathbf{k}\cdot\mathbf{r}}$$

- In the complex representation, the optical **intensity** can be simply written as

$$I(\mathbf{r}) = \tilde{U}(\mathbf{r}) \cdot \tilde{U}^*(\mathbf{r})$$

The complex representation

- Consider a hamonic plane wave

$U(\mathbf{r},t) = A(\mathbf{r}) e^{i(kz+\varphi_0)} e^{-i\omega t}$ propagate in the **positive** z direction.

$\tilde{U}^*(\mathbf{r}) e^{-i\omega t}$ propagate in the **negative** z direction.

and $i(kz+\varphi_0)$ — Spatial phase factor

$-i\omega t$ — Temporal phase factor

The negative sign in the time factor enforces that the vibration phase at t > 0 to always lag behind the phase at t = 0.

$$\varphi = kz \uparrow - \omega t \uparrow = C$$

$t \uparrow$, the wavefront will move from z = 0 to z > 0.

Monochomatic light

⑤ Why we talk about harmonic waves

- In theory, monochromatic light contains a single frequency. An electric dipole without any damping can radiate light that is an ideal monochromatic light wave, a harmonic wave.

- The actual light source always comprises a band of frequencies. If the band is narrow, the light is **quasi-monochromatic**. The narrower the wavelength range, the better the monochromaticity.

- A lot of actual waves can be approximated as simple harmonics waves. For example, a laser light source has a wavelength range of about 10^{-8} Å.

Monochomatic light

- The **superpositon principle**. An intriguing property of waves, which is unlike the behavior of a stream of classical particles.

- Any non-harmonic wave can be viewed as the superposition of many harmonic waves.
- According to Fourier analysis, any complex waves can be decomposed into harmonic waves. That is why we study the harmonic waves.

§ 2.2 Fourier Transforms

- If the periodic function:

$$g(t)=g(t+T)$$

- Meet the Dirichlet（狄利克雷）conditions:

  （1）Single value;（2）A limited number of extreme points（极值点）and discontinuities in a period.

- Then the periodic function can be transformed into series

$$g(t)=\sum_{m=-\infty}^{\infty} c_m \underline{e^{i2m\pi\nu_0 t}}, \quad m=0,\pm1,\pm2,\cdots$$

$$c_m=\frac{1}{T}\int_{t=-T/2}^{T/2} g(t)e^{i2m\pi\nu_0 t}dt$$

Harmonic wave (eigenfunction)

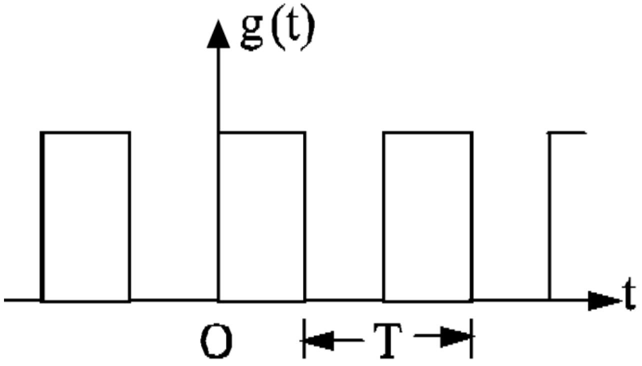
$\nu_0=1/T$ is fundamental frequency

The other $m\nu_0$ is called harmonic frequency.

Fourier series

E.g. 2.1 Determine the Fourier transform of the rectangle function $g(t)$.

$$g(t)=\begin{cases}1 & mT\le t\le mT+\dfrac{T}{2}\\ 0 & \text{other}\end{cases}$$



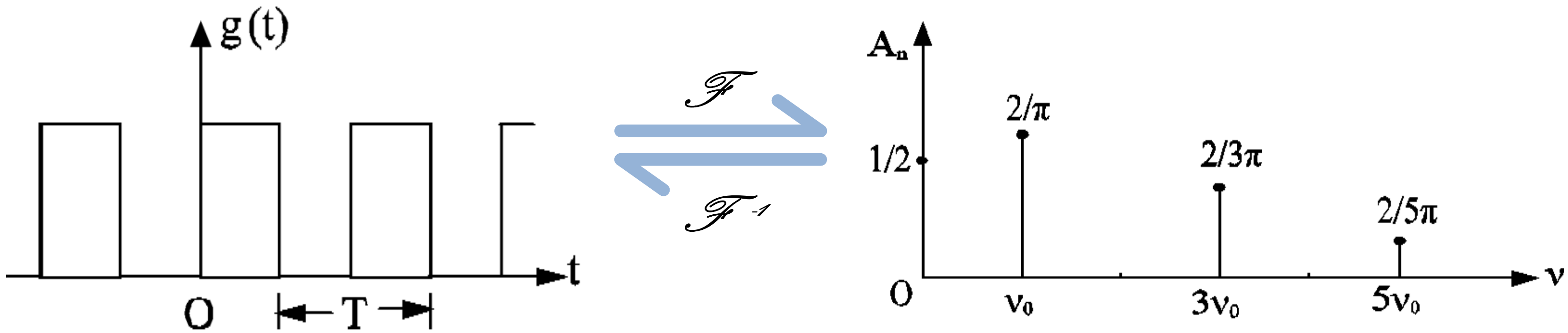
$$g(t)=\frac{a_0}{2}+\sum_{m=1}^{\infty}(a_m\cos 2\pi m\nu_0 t+b_m\sin 2\pi m\nu_0 t)$$

$$a_m=\frac{2}{T}\int_0^T g(t)\cos(m\frac{2\pi t}{T})dt=\begin{cases}1 & m=0\\ 0 & m\neq 0\end{cases}$$

$$b_m=\frac{2}{T}\int_0^T g(t)\sin(m\frac{2\pi t}{T})dt=\frac{1}{m\pi}\left[1-\cos(m\pi)\right]=\begin{cases}2/(m\pi) & m=1,3,5\\ 0 & m=2,4,6\end{cases}$$

Fourier series

$$g(t) = \frac{1}{2} + \frac{2}{\pi}\left(\sin \omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + \cdots\right)$$



- Time domain: Describes changes in waveform over time.
- Frequency domain: Describes the amplitude of each harmonic component.

Description of wave in the time domain and in the frequency domain are equivalent. So

① The spectrum of the periodic function is discontinuous (discrete).
② The spectrum only contains the frequencies that are integer multiples of the fundamental frequency.

Fourier integrals

- **Fourier transform of aperiodic function**
- For aperiodic function $g(t)$, if it meets Dirichlet condition, and it is absolutely integrable in infinite intervals（无穷区间绝对可积）.
- It can be viewed as a periodic function $g(t)$ with a period go to ∞. Then, the summation change into a integral.
- Fourier integrals:

$$g(t) = \int_{-\infty}^{\infty} G(\nu) e^{i2\pi\nu t} \mathrm{d}\nu$$

$$G(\nu) = \int_{-\infty}^{\infty} g(t) e^{-i2\pi\nu t} \mathrm{d}t$$

Fourier integrals

- The integral decomposes the function $g(t)$ into a linear combination of many eigenfunctions.
- Each eigenfunction takes the form of $e^{i2\pi\nu t}$
- $G(\nu)$ is the weight of each component. It is the **spectrum** of $g(t)$.
- $G(\nu)$ is the Fourier transforms of $g(t)$. $g(t)$ is the inverse Fourier transforms of $G(\nu)$.

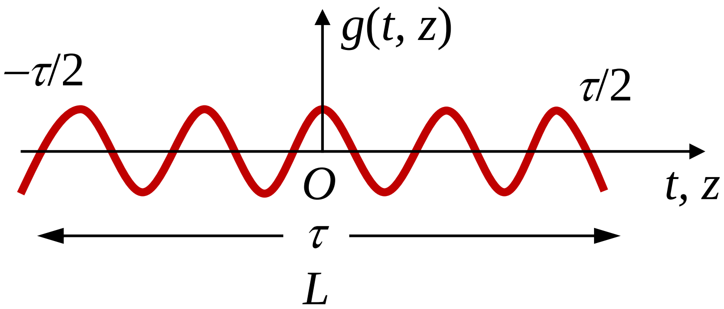
Two-dimensional Fourier transform in optics:

$$\begin{cases} g(x, y) = \iint G(f_x, f_y) e^{i2\pi(f_x x + f_y y)} \mathrm{d}f_x \mathrm{d}f_y \\ G(f_x, f_y) = \iint g(x, y) e^{-i2\pi(f_x x + f_y y)} \mathrm{d}x \mathrm{d}y \end{cases}$$

Fourier integrals

E.g. 2.2 Determine the spectrum of $g(t)$.

$$g(t)=\begin{cases} Ae^{i2\pi\nu_0 t}, & |t|\le \dfrac{\tau}{2} \\ 0, & |t|\ge \dfrac{\tau}{2} \end{cases}$$



$$G(\nu)=A\int_{-\infty}^{\infty} g(t)e^{-i2\pi\nu t}dt = A\int_{-\tau/2}^{\tau/2} e^{-i2\pi(\nu-\nu_0)t}dt$$

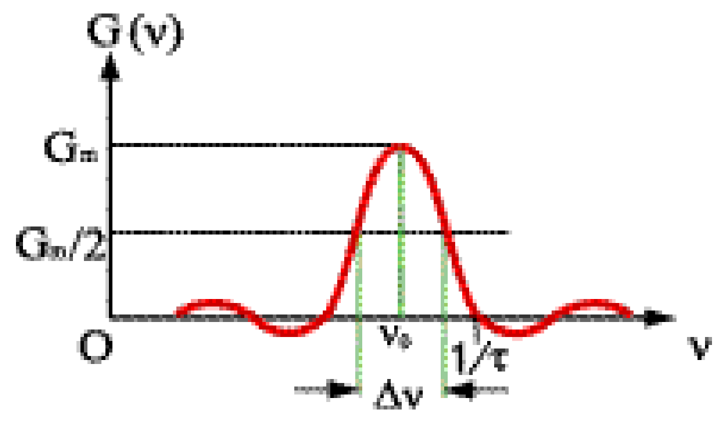
$$=\frac{A}{-i2\pi(\nu-\nu_0)}\cdot\left(e^{-i\pi(\nu-\nu_0)\tau}-e^{i\pi(\nu-\nu_0)\tau}\right)=A\tau\frac{\sin\left[\pi(\nu-\nu_0)\tau\right]}{\pi(\nu-\nu_0)\tau}$$

It consists of many harmonic waves with different frequencies and amplitudes.

An aperiodic function has a continuous spectrum.

Frequency Bandwidths

$$G(\nu) = A\tau \frac{\sin\left[\pi(\nu-\nu_0)\tau\right]}{\pi(\nu-\nu_0)\tau}$$



So, when $\begin{cases} \nu = \nu_0 \\ \nu = \nu_0 \pm 1/\tau \end{cases}$

We have $\begin{cases} G(\nu) = G_{\max}(\nu) = A\tau \\ G(\nu) = 0 \end{cases}$

- Half the width of the peak, **frequency bandwidth:**

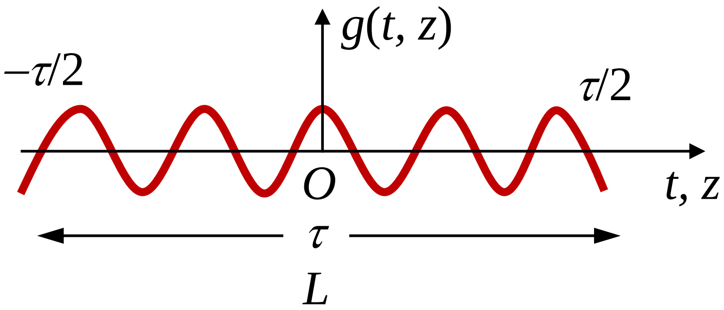
$$\Delta\nu = |\nu - \nu_0| = 1/\tau$$

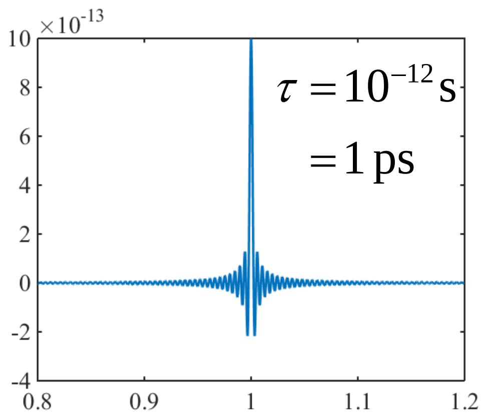
- Full-width at half maximum (**FWHM**)

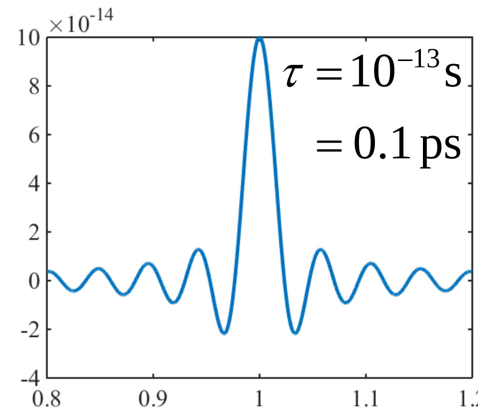
Length of wavetrain

- The relationship between wavetrain（波列）duration and spectrum width:

$$\Delta \nu = 1/\tau$$







- Spatial length of wavetrain:

$$L = c\tau = \frac{\lambda^2}{\Delta\lambda} \qquad \text{and,} \qquad \Delta \nu = \Delta\left(\frac{c}{\lambda}\right) = -\frac{c}{\lambda^2}\Delta\lambda$$

Homework

Read Section 7.4.1-7.4.4, and find the Fourier transform of a δ-function, a Gaussian function, a Lorentz peak, a periodic rectangle function (grating) etc.

Next week

Electromagnetic wave, Sections 3.1-3.3

Polarization, Sections 8.1, 8.13