

Chapter 9: Part B

Quantum statistics of ideal gases

Zhi-Jie Tan
Wuhan University

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Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.1 Identical particles and symmetry requirements

Quantum state description:

$$\{s_1, s_2, \dots, s_N\}$$

Wave function:

$$\Psi = \Psi_{\{s_1, \dots, s_N\}}(Q_1, Q_2, \dots, Q_N)$$

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.1 Identical particles and symmetry requirements

“Classical limit”:

- 1, distinguishable;
- 2, any number of particles can be in the same state;
- 3, no symmetry requirement

Quantum mechanics:

- 1, exchange two particles does not obey a new state;
- 2, indistinguishable;
- 3, symmetry requirement ← fundamental postulate of quantum mechanics

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.1 Identical particles and symmetry requirements

A, Particles with integral spin (Bose-Einstein statistics)

$$\Psi(\dots Q_j \dots Q_i \dots) = \Psi(\dots Q_i \dots Q_j \dots)$$

- 1, exchange two particles does not obey a new state;
- 2, indistinguishable;
- 3, no restriction for number of particles in the same state

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.1 Identical particles and symmetry requirements

B, Particles with half-integral spin (Fermi-Dirac statistics)

$$\Psi(\dots Q_j \dots Q_i \dots) = -\Psi(\dots Q_i \dots Q_j \dots)$$

- 1, exchange two particles does not obey a new state;
- 2, indistinguishable;
- 3, restriction for number of particles in the same state??

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

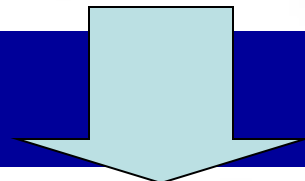
9.1 Identical particles and symmetry requirements

B, Particles with half-integral spin (Fermi-Dirac statistics)

$$\Psi(\dots Q_j \dots Q_i \dots) = -\Psi(\dots Q_i \dots Q_j \dots)$$

If two particles i and j are in the same state, then

$$\Psi(\dots Q_j \dots Q_i \dots) = \Psi(\dots Q_i \dots Q_j \dots)$$



$\Psi = 0$ when particles i and j are in the same state s

No state!!!

Pauli exclusion principle!!!

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.1 Identical particles and symmetry requirements

Illustration:

Two particles A and B can be
In three states: 1, 2, 3

Maxwell-Boltzmann statistics

<i>1</i>	<i>2</i>	<i>3</i>
<i>AB</i>	<i>...</i>	<i>...</i>
<i>...</i>	<i>AB</i>	<i>...</i>
<i>...</i>	<i>...</i>	<i>AB</i>
<i>A</i>	<i>B</i>	<i>...</i>
<i>B</i>	<i>A</i>	<i>...</i>
<i>A</i>	<i>...</i>	<i>B</i>
<i>B</i>	<i>...</i>	<i>A</i>
<i>...</i>	<i>A</i>	<i>B</i>
<i>...</i>	<i>B</i>	<i>A</i>

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.1 Identical particles and symmetry requirements

Illustration:

Two particles A and B can be
In three states: 1, 2, 3

Bose-Einstein statistics

<i>1</i>	<i>2</i>	<i>3</i>
<i>AA</i>	<i>...</i>	<i>...</i>
<i>...</i>	<i>AA</i>	<i>...</i>
<i>...</i>	<i>...</i>	<i>AA</i>
<i>A</i>	<i>A</i>	<i>...</i>
<i>A</i>	<i>...</i>	<i>A</i>
<i>...</i>	<i>A</i>	<i>A</i>

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.1 Identical particles and symmetry requirements

Illustration:

Two particles A and B can be

In three states: 1, 2, 3

Fermi-Dirac statistics

1	2	3
A	A	...
A	...	A
...	A	A

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.2 Formulation of the statistical problem

Energy:

$$E_R = n_1 \epsilon_1 + n_2 \epsilon_2 + n_3 \epsilon_3 + \dots = \sum_r n_r \epsilon_r$$

restriction:

$$\sum_r n_r = N$$

Partition function:

$$Z = \sum_R e^{-\beta E_R} = \sum_R e^{-\beta(n_1 \epsilon_1 + n_2 \epsilon_2 + \dots)}$$

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.2 Formulation of the statistical problem

Mean number
in s state:

$$\bar{n}_s = \frac{\sum_R n_s e^{-\beta(n_1 \epsilon_1 + n_2 \epsilon_2 + \dots)}}{\sum_R e^{-\beta(n_1 \epsilon_1 + n_2 \epsilon_2 + \dots)}}$$

$$= \frac{1}{Z} \sum_R \left(-\frac{1}{\beta} \frac{\partial}{\partial \epsilon_s} \right) e^{-\beta(n_1 \epsilon_1 + n_2 \epsilon_2 + \dots)}$$

$$= -\frac{1}{\beta Z} \frac{\partial Z}{\partial \epsilon_s}$$

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.2 Formulation of the statistical problem

Maxwell-Boltzmann statistics

$$n_r = 0, 1, 2, 3, \dots \quad \text{for each } r$$

distinguishable,

$$\sum_r n_r = N$$

Bose-Einstein statistics

$$n_r = 0, 1, 2, 3, \dots \quad \text{for each } r$$

indistinguishable,

$$\sum_r n_r = N$$

Fermi-Dirac statistics

$$n_r = 0, 1 \quad \text{for each } r$$

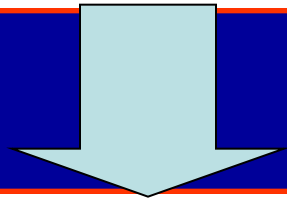
indistinguishable,

$$\sum_r n_r = N$$

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.3 Quantum distribution functions

$$\bar{n}_s = \frac{\sum_{n_1, n_2, \dots} n_s e^{-\beta(n_1 \epsilon_1 + n_2 \epsilon_2 + \dots + n_s \epsilon_s + \dots)}}{\sum_{n_1, n_2, \dots} e^{-\beta(n_1 \epsilon_1 + n_2 \epsilon_2 + \dots + n_s \epsilon_s + \dots)}}$$



$$\sum_r n_r = N$$

$$\bar{n}_s = \frac{\sum_{n_s} n_s e^{-\beta n_s \epsilon_s} \sum_{n_1, n_2, \dots}^{(s)} e^{-\beta(n_1 \epsilon_1 + n_2 \epsilon_2 + \dots)}}{\sum_{n_s} e^{-\beta n_s \epsilon_s} \sum_{n_1, n_2, \dots}^{(s)} e^{-\beta(n_1 \epsilon_1 + n_2 \epsilon_2 + \dots)}}$$

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.3 Quantum distribution functions

Photon statistics: BE statistics **without restricted N**

$$\bar{n}_s = \frac{\sum_{n_s} n_s e^{-\beta n_s \epsilon_s}}{\sum_{n_s} e^{-\beta n_s \epsilon_s}} \quad \Rightarrow \quad \bar{n}_s = \frac{\sum_{n_s} n_s e^{-\beta n_s \epsilon_s} \sum_{n_1, n_2, \dots}^{(s)} e^{-\beta(n_1 \epsilon_1 + n_2 \epsilon_2 + \dots)}}{\sum_{n_s} e^{-\beta n_s \epsilon_s} \sum_{n_1, n_2, \dots}^{(s)} e^{-\beta(n_1 \epsilon_1 + n_2 \epsilon_2 + \dots)}}$$

$$\begin{aligned} \bar{n}_s &= \frac{(-1/\beta)(\partial/\partial \epsilon_s) \sum e^{-\beta n_s \epsilon_s}}{\sum e^{-\beta n_s \epsilon_s}} \\ &= -\frac{1}{\beta} \frac{\partial}{\partial \epsilon_s} \ln (\sum e^{-\beta n_s \epsilon_s}) \end{aligned}$$

$$= \frac{1}{1 - e^{-\beta \epsilon_s}}$$

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.3 Quantum distribution functions

Photon statistics: BE statistics without restricted N

$$\bar{n}_\epsilon = \frac{1}{\beta} \frac{\partial}{\partial \epsilon_\epsilon} \ln (1 - e^{-\beta \epsilon_\epsilon}) = \frac{e^{-\beta \epsilon_\epsilon}}{1 - e^{-\beta \epsilon_\epsilon}}$$

$$\bar{n}_\epsilon = \frac{1}{e^{\beta \epsilon_\epsilon} - 1}$$

Plank distribution

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.3 Quantum distribution functions

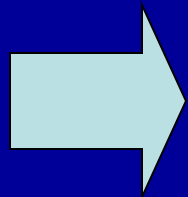
Fermi-Dirac statistics $n_r = 0 \text{ and } 1$

$$\sum_r n_r = N$$

define

$$Z_s(N) \equiv \sum_{n_1, n_2, \dots}^{(s)} e^{-\beta(n_1 \epsilon_1 + n_2 \epsilon_2 + \dots)}$$

$$n_s = 0 \quad 1$$



$$\bar{n}_s = \frac{0 + e^{-\beta \epsilon_s} Z_s(N-1)}{Z_s(N) + e^{-\beta \epsilon_s} Z_s(N-1)}$$

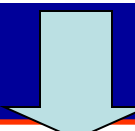
$$\bar{n}_s = \frac{1}{[Z_s(N)/Z_s(N-1)] e^{\beta \epsilon_s} + 1}$$

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

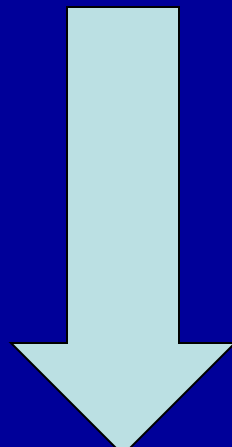
9.3 Quantum distribution functions

Fermi-Dirac statistics

define

$$\alpha = \frac{\partial \ln Z}{\partial N}$$


$$Z(N) / Z(N-1) = e^{\alpha}$$

$$\bar{n}_s = \frac{1}{[Z_s(N) / Z_s(N-1)] e^{\beta \epsilon_s} + 1}$$


$$\bar{n}_s = \frac{1}{e^{\alpha + \beta \epsilon_s} + 1}$$

Fermi-Dirac distribution

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.3 Quantum distribution functions

Fermi-Dirac statistics

define

$$\alpha = \frac{\partial \ln Z}{\partial N}$$



$$\alpha = - \frac{1}{kT} \frac{\partial F}{\partial N} =$$

$$= - \frac{\mu}{kT} = -\beta\mu$$

$$\bar{n}_s = \frac{1}{e^{\alpha + \beta \epsilon_s} + 1}$$

$$0 \leq \bar{n}_s \leq 1$$

$$\bar{n}_\epsilon \rightarrow 0$$

if ϵ_s becomes large enough

Chemical potential per particle

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.3 Quantum distribution functions

Bose-Einstein statistics

$$n_r = 0, 1, 2, 3, \dots$$

$$n_s = 0 \quad 1 \quad 2$$

$$\bar{n}_s = \frac{0 + e^{-\beta \epsilon_s} Z_s(N-1) + 2e^{-2\beta \epsilon_s} Z_s(N-2) + \dots}{Z_s(N) + e^{-\beta \epsilon_s} Z_s(N-1) + e^{-2\beta \epsilon_s} Z_s(N-2) + \dots}$$

$$Z(N)/Z(N-1) = e^\alpha$$

$$\bar{n}_s = \frac{Z_s(N)[0 + e^{-\beta \epsilon_s} e^{-\alpha} + 2e^{-2\beta \epsilon_s} e^{-2\alpha} + \dots]}{Z_s(N)[1 + e^{-\beta \epsilon_s} e^{-\alpha} + e^{-2\beta \epsilon_s} e^{-2\alpha} + \dots]}$$

$$\bar{n}_s = \frac{\sum_s n_s e^{-n_s(\alpha + \beta \epsilon_s)}}{\sum_s e^{-n_s(\alpha + \beta \epsilon_s)}}$$

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.3 Quantum distribution functions

Bose-Einstein statistics

The diagram illustrates the derivation of the Bose-Einstein distribution function through a series of steps:

- Step 1:** The grand partition function is given by
$$\bar{n}_s = \frac{\sum_s n_s e^{-n_s(\alpha + \beta \epsilon_s)}}{\sum_s e^{-n_s(\alpha + \beta \epsilon_s)}}$$
- Step 2:** A light blue arrow points to the expression for the average occupation number n_s as a derivative of the logarithm of the grand partition function:
$$n_s = -\frac{1}{\beta} \frac{\partial}{\partial \epsilon_s} \ln \left(\sum_s n_s e^{-n_s(\alpha + \beta \epsilon_s)} \right)$$
 The sum in the logarithm is circled in red.
- Step 3:** A red arrow points down to the derivative of the sum, resulting in:
$$\frac{1}{1 - e^{-(\alpha + \beta \epsilon_s)}}$$
- Step 4:** A light blue arrow points left to the final Bose-Einstein distribution function:
$$n_s = \frac{e^{-(\alpha + \beta \epsilon_s)}}{1 - e^{-(\alpha + \beta \epsilon_s)}}$$

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.3 Quantum distribution functions

Bose-Einstein statistics

$$\bar{n}_s = \frac{\sum_s n_s e^{-n_s(\alpha + \beta \epsilon_s)}}{\sum_s e^{-n_s(\alpha + \beta \epsilon_s)}}$$

$$\bar{n}_s = \frac{1}{e^{\alpha + \beta \epsilon_s} - 1}$$

α can be determined by

$$\sum_r \frac{1}{e^{\alpha + \beta \epsilon_r} - 1} = N$$

$$\alpha = -\beta \mu$$

$$\bar{n}_s = \frac{1}{e^{\beta(\epsilon_s - \mu)} - 1}$$

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.4 Maxwell-Boltzmann statistics: comparison

Partition function:

$$Z = \sum_R e^{-\beta(n_1 \epsilon_1 + n_2 \epsilon_2 + \dots)}$$

for given values of $\{n_1, n_2, \dots\}$

Possible way:

$$\frac{N!}{n_1! n_2! \dots}$$

$$Z = \sum_{n_1, n_2, \dots} \frac{N!}{n_1! n_2! \dots} e^{-\beta(n_1 \epsilon_1 + n_2 \epsilon_2 + \dots)}$$

$$\sum_r n_r = N$$

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

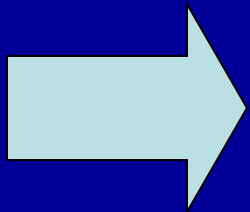
9.4 Maxwell-Boltzmann statistics

Partition function:

$$Z = \sum_{n_1, n_2, \dots} \frac{N!}{n_1! n_2! \dots} e^{-\beta(n_1 \epsilon_1 + n_2 \epsilon_2 + \dots)}$$

$$Z = \sum_{n_1, n_2, \dots} \frac{N!}{n_1! n_2! \dots} (e^{-\beta \epsilon_1})^{n_1} (e^{-\beta \epsilon_2})^{n_2} \dots$$

$$Z = (e^{-\beta \epsilon_1} + e^{-\beta \epsilon_2} + \dots)^N$$



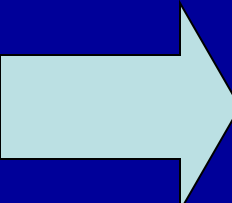
$$\ln Z = N \ln \left(\sum_r e^{-\beta \epsilon_r} \right)$$

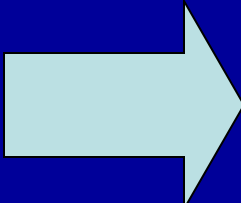
Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.4 Maxwell-Boltzmann statistics

Partition function:

$$\ln Z = N \ln \left(\sum_r e^{-\beta \epsilon_r} \right)$$


$$\bar{n}_s = - \frac{1}{\beta} \frac{\partial \ln Z}{\partial \epsilon_s} = - \frac{1}{\beta} N \frac{-\beta e^{-\beta \epsilon_s}}{\sum_r e^{-\beta \epsilon_r}}$$


$$\bar{n}_s = N \frac{e^{-\beta \epsilon_s}}{\sum_r e^{-\beta \epsilon_r}}$$

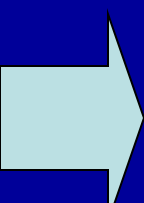
Maxwell-Boltzmann distribution

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

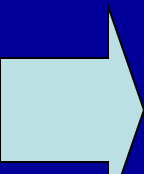
9.5 Photon statistics

Partition function:

$$Z = \sum_R e^{-\beta(n_1 \epsilon_1 + n_2 \epsilon_2 + \dots)}$$


$$Z = \sum_{n_1, n_2, \dots} e^{-\beta n_1 \epsilon_1} e^{-\beta n_2 \epsilon_2} e^{-\beta n_3 \epsilon_3} \dots$$

$$Z = \left(\sum_{n_1=0}^{\infty} e^{-\beta n_1 \epsilon_1} \right) \left(\sum_{n_2=0}^{\infty} e^{-\beta n_2 \epsilon_2} \right) \left(\sum_{n_3=0}^{\infty} e^{-\beta n_3 \epsilon_3} \right) \dots$$


$$Z = \left(\frac{1}{1 - e^{-\beta \epsilon_1}} \right) \left(\frac{1}{1 - e^{-\beta \epsilon_2}} \right) \left(\frac{1}{1 - e^{-\beta \epsilon_3}} \right) \dots$$

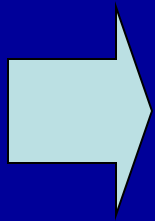

$$\ln Z = - \sum_r \ln (1 - e^{-\beta \epsilon_r})$$

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

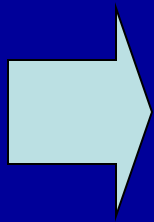
9.5 Photon statistics

Partition function:

$$\ln Z = - \sum_r \ln (1 - e^{-\beta \epsilon_r})$$



$$\bar{n}_r = - \frac{1}{\beta} \frac{\partial \ln Z}{\partial \epsilon_r} = \frac{e^{-\beta \epsilon_r}}{1 - e^{-\beta \epsilon_r}}$$



$$\bar{n}_r = \frac{1}{e^{\beta \epsilon_r} - 1}$$

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.6 Bose-Einstein statistics

Partition function:

$$Z = \sum_R e^{-\beta(n_1 \epsilon_1 + n_2 \epsilon_2 + \dots)}$$

$$n_r = 0, 1, 2, \dots$$

$$\sum_r n_r = N$$

Considering $Z(N')$.

$Z(N')$ increases rapidly with N' , but we are only interested in Z at $N'=N$.

Multiply $e^{-\alpha N'}$ to produce a function $Z(N')e^{-\alpha N'}$ with maximum at $N'=N$ by a proper choice of α .

A sum of all N' must select only terms of interest near N

$$\sum_{N'} Z(N') e^{-\alpha N'} = Z(N) e^{-\alpha N} \Delta^* N'$$

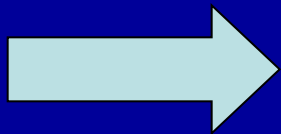
Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.6 Bose-Einstein statistics

Define Grand partition function

$$\sum_{N'} Z(N') e^{-\alpha N'} = Z(N) e^{-\alpha N} \Delta^* N'$$

$$\mathcal{Z} \equiv \sum_{N'} Z(N') e^{-\alpha N'}$$



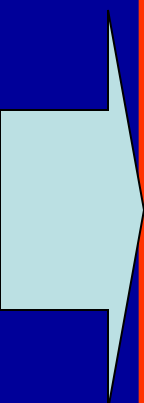
$$\ln Z(N) = \alpha N + \ln \mathcal{Z} \quad ?$$

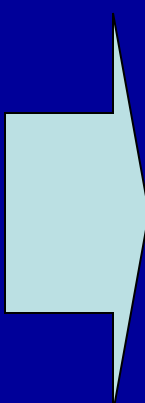
Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.6 Bose-Einstein statistics

Grand partition function

$$Z = \sum_R e^{-\beta(n_1 \epsilon_1 + n_2 \epsilon_2 + \dots)} e^{-\alpha(n_1 + n_2 + \dots)}$$


$$\begin{aligned} Z &= \sum_{n_1, n_2, \dots} e^{-(\alpha + \beta \epsilon_1)n_1 - (\alpha + \beta \epsilon_2)n_2 - \dots} \\ &= \left(\sum_{n_1=0}^{\infty} e^{-(\alpha + \beta \epsilon_1)n_1} \right) \left(\sum_{n_2=0}^{\infty} e^{-(\alpha + \beta \epsilon_2)n_2} \right) \dots \end{aligned}$$


$$\begin{aligned} Z &= \left(\frac{1}{1 - e^{-(\alpha + \beta \epsilon_1)}} \right) \left(\frac{1}{1 - e^{-(\alpha + \beta \epsilon_2)}} \right) \dots \\ \ln Z &= - \sum_r \ln (1 - e^{-\alpha - \beta \epsilon_r}) \end{aligned}$$

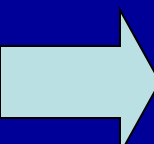
Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.6 Bose-Einstein statistics

Grand partition function

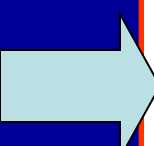
$$\ln Z(N) = \alpha N + \ln Z$$

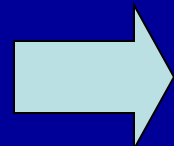
$$\ln Z = - \sum_r \ln (1 - e^{-\alpha - \beta \epsilon_r})$$


$$\ln Z = \alpha N - \sum_r \ln (1 - e^{-\alpha - \beta \epsilon_r})$$

Keep $N'=N$ by a proper choice of α

$$Z(N') e^{-\alpha N'}$$


$$\frac{\partial}{\partial N'} [\ln Z(N') - \alpha N'] = \frac{\partial \ln Z(N)}{\partial N} - \alpha = 0$$


$$\left[\alpha + \left(N + \frac{\partial \ln Z}{\partial \alpha} \right) \frac{\partial \alpha}{\partial N} \right] - \alpha = 0$$
$$N + \frac{\partial \ln Z}{\partial \alpha} = \frac{\partial \ln Z}{\partial \alpha} = 0$$

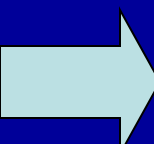
Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

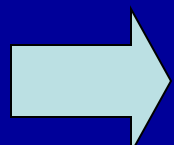
9.6 Bose-Einstein statistics

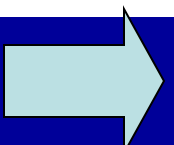
$$\ln Z = \alpha N - \sum_r \ln (1 - e^{-\alpha - \beta \epsilon_r})$$

$$N + \frac{\partial \ln Z}{\partial \alpha} = \frac{\partial \ln Z}{\partial \alpha} = 0$$

$$\ln Z = - \sum_r \ln (1 - e^{-\alpha - \beta \epsilon_r})$$


$$N - \sum_r \frac{e^{-\alpha - \beta \epsilon_r}}{1 - e^{-\alpha - \beta \epsilon_r}} = 0$$


$$\sum_r \frac{1}{e^{\alpha + \beta \epsilon_r} - 1} = N$$


$$\bar{n}_s = - \frac{1}{\beta} \frac{\partial \ln Z}{\partial \epsilon_s} = - \frac{1}{\beta} \left[- \frac{\beta e^{-\alpha - \beta \epsilon_s}}{1 - e^{-\alpha - \beta \epsilon_s}} + \frac{\partial \ln Z}{\partial \alpha} \frac{\partial \alpha}{\partial \epsilon_s} \right]$$

$$\bar{n}_s = \frac{1}{e^{\alpha + \beta \epsilon_s} - 1}$$

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.6 Bose-Einstein statistics

$$\bar{n}_s = \frac{1}{e^{\alpha + \beta \epsilon_s} - 1}$$



$$\sum_r \bar{n}_r = N$$

$$\frac{\partial \ln Z(N)}{\partial N} - \alpha = 0$$

$$\mu = \frac{\partial F}{\partial N} = -kT \frac{\partial \ln Z}{\partial N} = -kT \alpha$$

$$\alpha = -\beta \mu$$

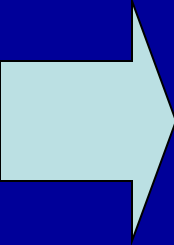
Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.7 Fermi-Dirac statistics

$$n_r = 0 \text{ and } 1 \quad \text{for each } r$$

Similar to the treatment in BE statistics

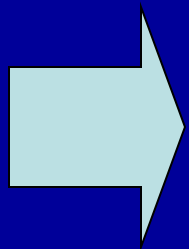
$$\begin{aligned} \mathcal{Z} &= \sum_{n_1, n_2, n_3, \dots} e^{-\beta(n_1 \epsilon_1 + n_2 \epsilon_2 + \dots) - \alpha(n_1 + n_2 + \dots)} \\ &= \left(\sum_{n_1=0}^1 e^{-(\alpha + \beta \epsilon_1) n_1} \right) \left(\sum_{n_2=0}^1 e^{-(\alpha + \beta \epsilon_2) n_2} \right) \dots \end{aligned}$$


$$\begin{aligned} \mathcal{Z} &= (1 + e^{-\alpha - \beta \epsilon_1})(1 + e^{-\alpha - \beta \epsilon_2}) \dots \\ \ln \mathcal{Z} &= \sum_r \ln (1 + e^{-\alpha - \beta \epsilon_r}) \end{aligned}$$

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.7 Fermi-Dirac statistics

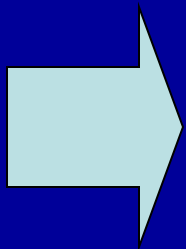
$$\ln Z = \sum_r \ln (1 + e^{-\alpha - \beta \epsilon_r})$$



$$\ln Z = \alpha N + \sum_r \ln (1 + e^{-\alpha - \beta \epsilon_r})$$

α is also determined by the condition

$$\frac{\partial \ln Z}{\partial \alpha} = N - \sum_r \frac{e^{-\alpha - \beta \epsilon_r}}{1 + e^{-\alpha - \beta \epsilon_r}} = 0$$

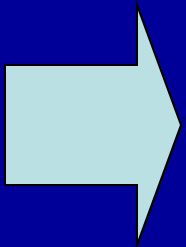


$$\sum_r \frac{1}{e^{\alpha + \beta \epsilon_r} + 1} = N$$

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.7 Fermi-Dirac statistics

$$\bar{n}_s = - \frac{1}{\beta} \frac{\partial \ln Z}{\partial \epsilon_s} = \frac{1}{\beta} \frac{\beta e^{-\alpha - \beta \epsilon_s}}{1 + e^{-\alpha - \beta \epsilon_s}}$$



$$\bar{n}_s = \frac{1}{e^{\alpha + \beta \epsilon_s} + 1}$$

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.8 Quantum statistics in the classic limit

BE and FD distributions:

$$\bar{n}_r = \frac{1}{e^{\alpha + \beta \epsilon_r} \pm 1}$$

Total particles:

$$\sum_r \bar{n}_r = \sum_r \frac{1}{e^{\alpha + \beta \epsilon_r} \pm 1} = N$$

Partition function:

$$\ln Z = \alpha N \pm \sum_r \ln (1 \pm e^{-\alpha - \beta \epsilon_r})$$

Limiting cases:

very low concentration

$$\bar{n}_r \ll 1$$

$$\exp(\alpha + \beta \epsilon_r) \gg 1$$

Very high T

$$\beta \rightarrow 0$$

$$\beta \epsilon_r \ll \alpha$$

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.8 Quantum statistics in the classic limit

Limiting cases:

very low concentration $\bar{n}_r \ll 1$ $\exp(\alpha + \beta\epsilon_r) \gg 1$

Very high T

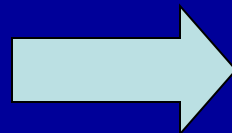
$$\beta \rightarrow 0 \quad \beta\epsilon_r \ll \alpha$$

Number of terms contribute
substantially to summation increases

Requires α must be large enough

To keep sum == N

$$\sum_r \bar{n}_r = \sum_r \frac{1}{e^{\alpha + \beta\epsilon_r} \pm 1} = N$$



$$\exp(\alpha + \beta\epsilon_r) \gg 1$$

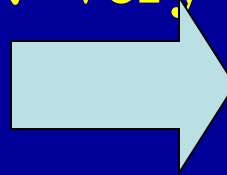
Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.8 Quantum statistics in the classic limit

Limiting cases:

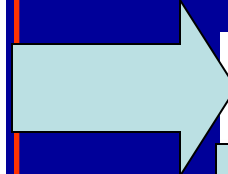
very low concentration, very high T

$$e^{\alpha + \beta \epsilon_r} \gg 1$$



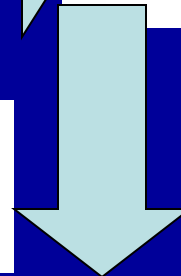
$$\bar{n}_r \ll 1$$

$$\bar{n}_r = \frac{1}{e^{\alpha + \beta \epsilon_r} \pm 1}$$



$$\bar{n}_r = e^{-\alpha - \beta \epsilon_r}$$

$$\sum_r e^{-\alpha - \beta \epsilon_r} = e^{-\alpha} \sum_r e^{-\beta \epsilon_r} = N$$



$$e^{-\alpha} = N \left(\sum_r e^{-\beta \epsilon_r} \right)^{-1}$$

Limiting cases:
low concentration
high T \rightarrow MB dis.

$$\bar{n}_r = N \frac{e^{-\beta \epsilon_r}}{\sum_r e^{-\beta \epsilon_r}}$$

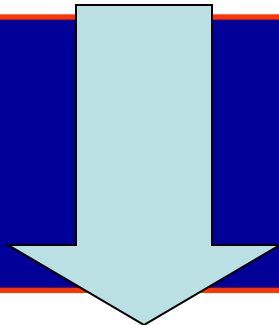
Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

$$\ln Z = \alpha N \pm \sum_r \ln (1 \pm e^{-\alpha - \beta \epsilon_r})$$

9.8 Quantum statistics in the classic limit

Partition function:

$$\ln Z = \alpha N \pm \sum_r (\pm e^{-\alpha - \beta \epsilon_r}) = \alpha N \pm N$$



$$\alpha = -\ln N \pm \ln \left(\sum_r e^{-\beta \epsilon_r} \right)$$

$$\ln Z = -N \ln N \pm N \pm N \ln \left(\sum_r e^{-\beta \epsilon_r} \right)$$

While MB gives:
????????????????

$$\ln Z = N \ln \left(\sum_r e^{-\beta \epsilon_r} \right)$$

Maxwell-Boltzmann, Bose-Einstein, Fermi-Dirac statistics

9.8 Quantum statistics in the classic limit

Partition function:

$$\ln Z = \ln Z_{\text{MB}} - (N \ln N - N)$$

$$\ln Z = \ln Z_{\text{MB}} - \ln N!$$

$$Z = \frac{Z_{\text{MB}}}{N!}$$

<<< distinguishable

Class-work

P 398 9.1

Homework

P 398 9.2-9.3

Homework

configuration			no. of states		
0	ϵ	3ϵ	MB	BE	FD
xx			1	1	--
	xx		1	1	--
		xx	1	1	--
x	x		2	1	1
x		x	2	1	1
	x	x	2	1	1

$$(a) \quad Z_{MB} = 1 + e^{-2\epsilon\beta} + e^{-6\epsilon\beta} + 2e^{-\epsilon\beta} + 2e^{-3\epsilon\beta} + 2e^{-4\epsilon\beta}$$

$$(b) \quad Z_{BE} = 1 + e^{-2\epsilon\beta} + e^{-6\epsilon\beta} + e^{-\epsilon\beta} + e^{-3\epsilon\beta} + e^{-4\epsilon\beta}$$

$$(c) \quad Z_{FD} = e^{-\epsilon\beta} + e^{-3\epsilon\beta} + e^{-4\epsilon\beta}$$