Chapter 12

Electrodynamics and Relativity

- 12.1 The Special Theory of Relativity
- 12.2 Relativistic Mechanics
- 12.3 Relativistic Electrodynamics

The Special Theory of Relativity

Einstein's Postulates

Classical mechanics obeys the principle of relativity: the same laws apply in any **inertial reference frame**. "inertial" means that the system is at rest or moving with constant velocity.

Does it also apply to the laws of electrodynamics?

At first glance, the answer would seem to be no. After all, a charge in motion produces a magnetic field, whereas a charge at rest does not.

There is an extraordinary coincidence that gives us pause. As the loop rides through the magnetic field, a motional emf is established; according to the flux rule ($Eq. 7.13$),

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

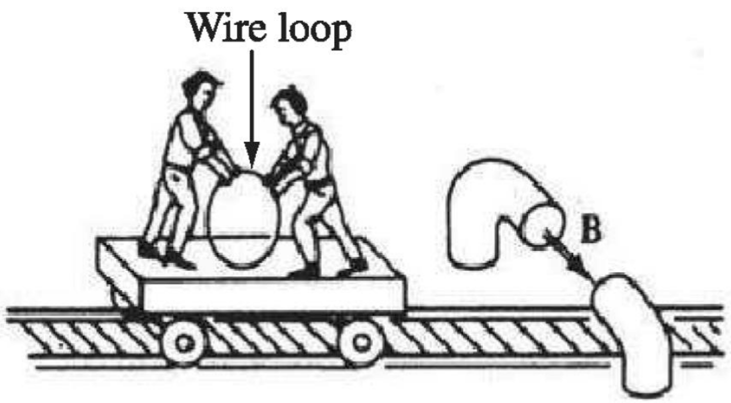


FIGURE 12.1

This emf, remember, is due to the magnetic force on charges in the wire loop, which are moving along with the train. On the other hand, if someone on the train naively applied the laws of electrodynamics in that system, no *magnetic* force, because the loop is at rest. But as the magnet flies by, a changing magnetic field induces an electric field, by Faraday's law.

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

- Faraday's law and the flux rule predict exactly the same emf , even though their physical interpretation of the process is completely wrong.
- Einstein could not believe this was a mere coincidence he wrote on the first page of his 1905 paper introducing the special theory of relativity (see details in the book).
- To Einstein's predecessors, the equality of the two emfs was just a lucky accident; They thought of electric and magnetic fields as strains in an invisible jellylike medium called **ether**. The speed of the charge was to be measured with respect to the ether-only then would the laws of electrodynamics be valid.
- How do we know the *ground* observer isn't moving relative to the ether, too?
- Earth rotates, the solar system circulates around the galaxy, the galaxy itself is moving at a high speed through the cosmos. All told, we should be traveling at well over 50 km/s with respect to the ether, unless we just happen to find ourselves in a tailwind of precisely the right strength, or the earth has some sort of “windshield”, it becomes a matter of crucial importance to *find* the ether frame, **experimentally**.

Michelson and Morley, using an optical interferometer of fantastic precision.
Two essential points:
1. compare the speed of light in different directions
2. discovered was that this speed is exactly the same in all directions

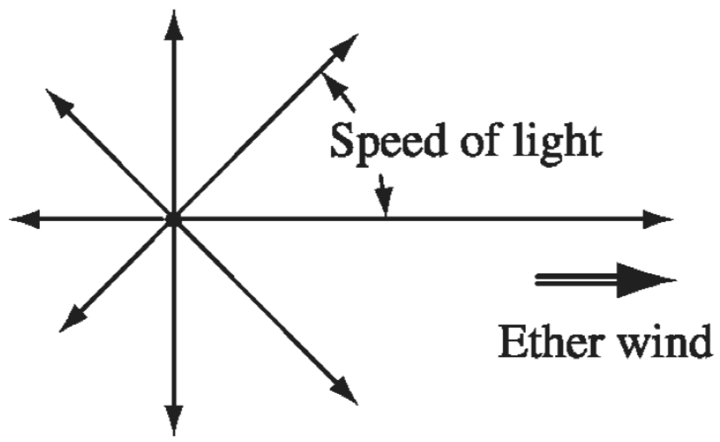


FIGURE 12.2

All other waves (water waves, sound waves, waves on a string) travel at a prescribed speed relative to the propagating medium(the stuff that does the waving), and if this medium is in motion with respect to the observer, the net speed is always greater "downstream" than "upstream." Over the next 20 years, a series of improbable schemes were concocted in an effort to explain why this does not occur with light. Michelson and Morley themselves interpreted their experiment as confirmation of the "ether drag" hypothesis, which held that the earth somehow pulls the ether along with it. But this was found to be inconsistent with other observations, notably the aberration of starlight.

At any rate, it was not until Einstein that anyone took the Michelson-Morley result at face value, and suggested that the speed of light is a universal constant, the same in all directions, regardless of the motion of the observer or the source. Einstein proposed his two famous postulates:

1. **The principle of relativity.**
   The laws of physics apply in all inertial reference systems.

2. **The universal speed of light.**
   The speed of light in vacuum is the same for all inertial observers, regardless of the motion of the source.

The special theory of relativity derives from these two postulates.

Galileo's velocity addition rule $v_{AC} = v_{AB} + v_{BC}$

Einstein's velocity addition rule

$$\boxed{v_{AC} = \frac{v_{AB} + v_{BC}}{1 + (v_{AB} v_{BC}/c^2)}.}$$

(will derive later)

if $v_{AB} = c$,then automatically $v_{AC} = c$:

$$v_{AC} = \frac{c + v_{BC}}{1 + (c v_{BC}/c^2)} = c$$

But how can Galileo's rule, which seems to rely on nothing but common sense, possibly be wrong? And if it is wrong, what does this do to all of classical physics? The answer is that special relativity compels us to alter our notions of space and time themselves, and therefore also of such derived quantities as velocity, momentum, and energy.

The Geometry of Relativity

In this section I present a series of gedanken (thought) experiments that serve to introduce the three most striking geometrical consequences of Einstein's postulates: time dilation, Lorentz contraction, and the relativity of simultaneity.

(*i*)The relativity of simultaneity.

Two events that are simultaneous in one inertial system are not, in general, simultaneous in another.

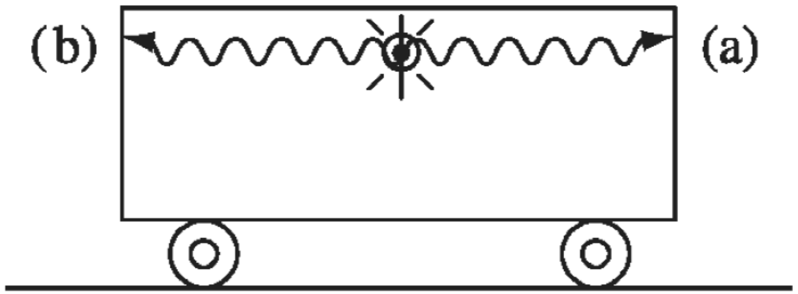


FIGURE 12.4

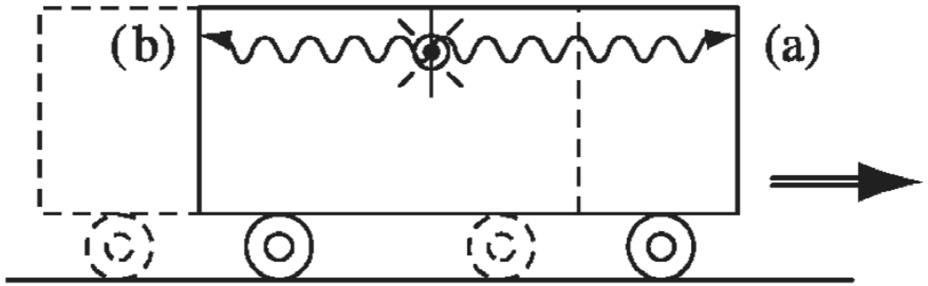


FIGURE 12.5

(*ii*) **Time dilation.**

Now let's consider a light ray that leaves the bulb and strikes the floor of the car directly below.

Question: How long does it take the light to make this trip?

$$\Delta\bar{t} = \frac{h}{c}$$

an overbar denotes measurements made on the train

$$\Delta t = \frac{\sqrt{h^2 + (v\Delta t)^2}}{c}$$

$$\Delta t = \frac{h}{c}\frac{1}{\sqrt{1 - v^2/c^2}}$$

$$\boxed{\Delta\bar{t} = \sqrt{1 - v^2/c^2}\,\Delta t.}$$

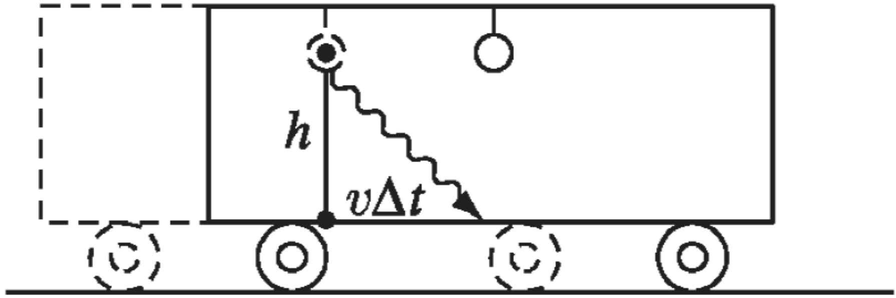


FIGURE 12.7

In fact, the interval recorded on the train clock, $\Delta\vec{t}$, is shorter by the factor

$$\boxed{\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}}.}$$

Conclusion:

Moving clocks run slow. This is called **time dilation.**

Of all Einstein's predictions, none has received more spectacular and persuasive confirmation than time dilation. Most elementary particles are unstable :
The lifetime of a neutron is 15 min; of a muon, 2×10^{-6}s; and of a neutral pion,$9 \times 10^{-17} s$.
But these are lifetimes of particles at rest.

It may strike you that time dilation is inconsistent with the principle of relativity. For if the ground observer says the train clock runs slow, the train observer can with equal justice claim that the ground clock runs slow.

Who's right?

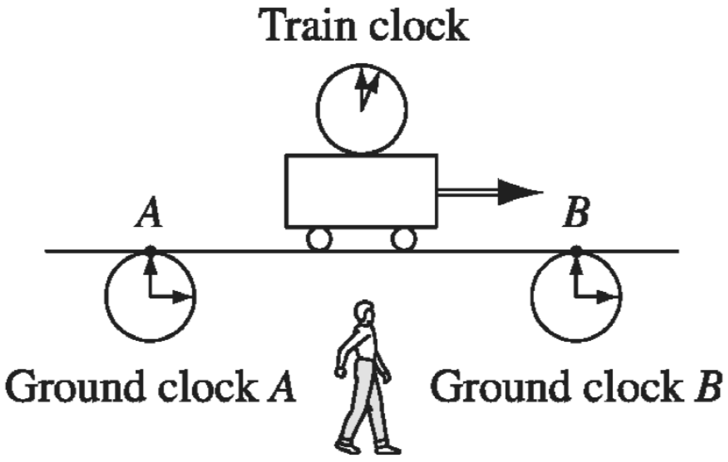


FIGURE 12.8

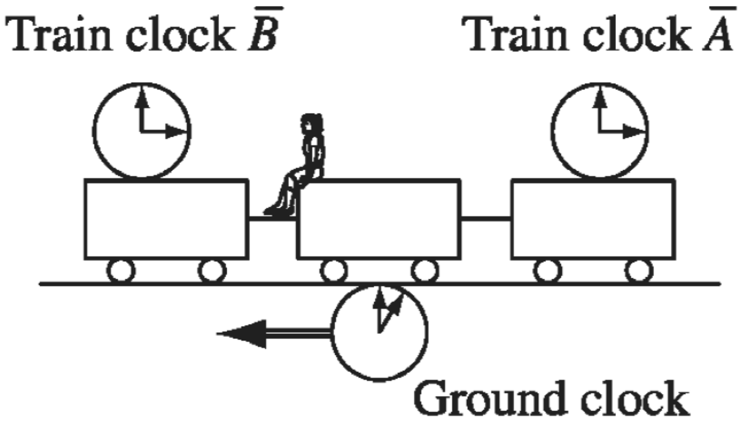


FIGURE 12.9

They're both right! Clocks that are properly synchronized in one system will not be synchronized when observed from another system.

Example 12.2.

The twin paradox. On her 21st birthday, an astronaut takes off in a rocket ship at a speed of $\frac{12}{13}$c. After 5 years have elapsed on her watch, she turns around and heads back at the same speed to rejoin her twin brother, who stayed at home.
Question : How old is each twin at their reunion?

Solution

The traveling twin has aged 10 years (5 years out, 5 years back);

$$\gamma = \frac{1}{\sqrt{1-(12/13)^2}} = \frac{13}{5}$$

The time elapsed on earthbound clocks is $\frac{13}{5} \times 10 = 26$

The so-called twin paradox arises when you try to tell this story from the point of view of the traveling twin. She sees the earth fly off, turn around after 5 years, and return. From her point of view, it would seem, she's at rest, whereas her brother is in motion, and hence it is he who should be younger at the reunion.

The two twins are not equivalent. The traveling twin experiences ***acceleration*** when she turns around to head home, but her brother does not. To put it in fancier language, the traveling twin is not in an inertial system: she's in one inertial system on the way out and a completely different inertial system on the way **back.**

The *traveling twin cannot claim to be a stationary observer* because you can't undergo acceleration and remain stationary.

(iii) **Lorentz contraction.**

For the third gedanken experiment you must imagine that we have set up a lamp at one end of a boxcar and a mirror at the other, so that a light signal can be sent down and back ($Fig.$ 12.10).

Question: How long does the signal take to complete the round trip? To an observer on the train, the answer is:

$$\Delta\bar{t} = 2\frac{\Delta\bar{x}}{c}$$

where $\Delta\bar{x}$ is the length of the car, To an observer on the ground

$$\Delta t_1 = \frac{\Delta x + v\Delta t_1}{c}, \quad \Delta t_2 = \frac{\Delta x - v\Delta t_2}{c}$$

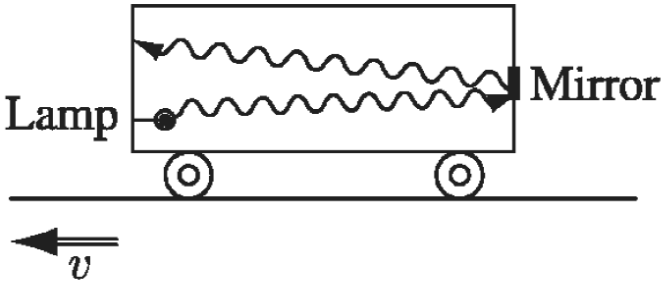


FIGURE 12.10

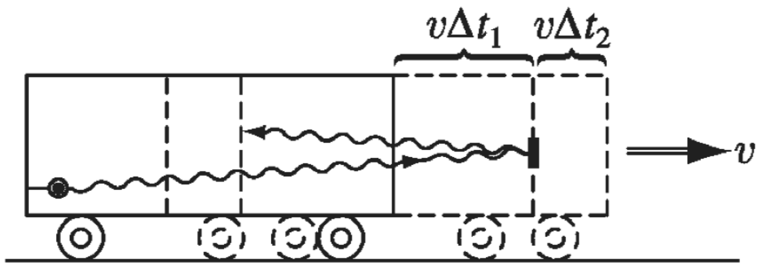


FIGURE 12.11

$$\Delta t_1 = \frac{\Delta x}{c - v}, \quad \Delta t_2 = \frac{\Delta x}{c + v}$$

the round-trip time is

$$\Delta t = \Delta t_1 + \Delta t_2 = 2\frac{\Delta x}{c}\frac{1}{(1 - v^2/c^2)}$$

Meanwhile, $$\Delta \bar{t} = \sqrt{1 - v^2/c^2}\,\Delta t$$

$$\boxed{\Delta \bar{x} = \frac{1}{\sqrt{1 - v^2/c^2}}\Delta x.}$$

Conclusion:
Moving objects are shortened.
We call this **Lorentz contraction.**

Notice that the same factor, $\gamma \equiv \dfrac{1}{\sqrt{1 - v^2/c^2}}$

Example 12.3

The barn and ladder paradox. Unlike time dilation, there is no direct experimental confirmation of Lorentz contraction, simply because it's too difficult to get an object of measurable size going anywhere near the speed of light. The following parable illustrates how bizarre the world would be if the speed of light were more accessible.

There once was a farmer who had a ladder too long to store in his barn. Will the ladder fit inside the barn?

Solution

When you say "the ladder is in the barn," you mean that all parts of it are inside at one instant of time, but in view of the relativity of simultaneity, that's a condition that depends on the observer. There are really two relevant events here:

a. Back end of ladder makes it in the door.
b. Front end of ladder hits far wall of barn

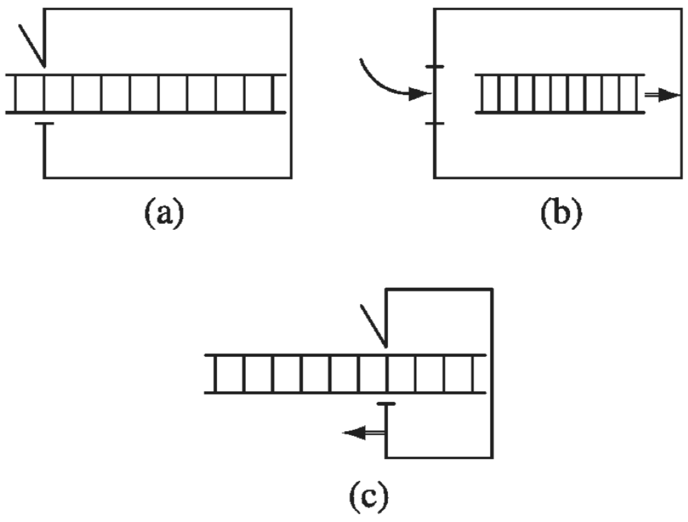


FIGURE 12.13

One final comment on Lorentz contraction. A moving object is shortened only along the direction of its motion:

Dimensions perpendicular to the velocity are not contracted

A lovely gedanken experiment suggested by Taylor and Wheeler[7].
Question: Does the passenger's red line lie above or below our blue one?
(see answers in the textbook)

[7]E. F. Taylor and J. A. Wheeler, Spacetime Physics 2nd ed. (San Francisco: W. H. Freeman, 1992).

The Lorentz Transformations

$$x = d + vt \qquad (12.10)$$

Before Einstein,

$$d = \bar{x} \qquad (12.11)$$

Thus,

Galilean transformations

$$\left.\begin{aligned} &\text{(i)} \quad \bar{x} = x - vt, \\ &\text{(ii)} \quad \bar{y} = y, \\ &\text{(iii)} \quad \bar{z} = z, \\ &\text{(iv)} \quad \bar{t} = t. \end{aligned}\right\} \qquad (12.12)$$

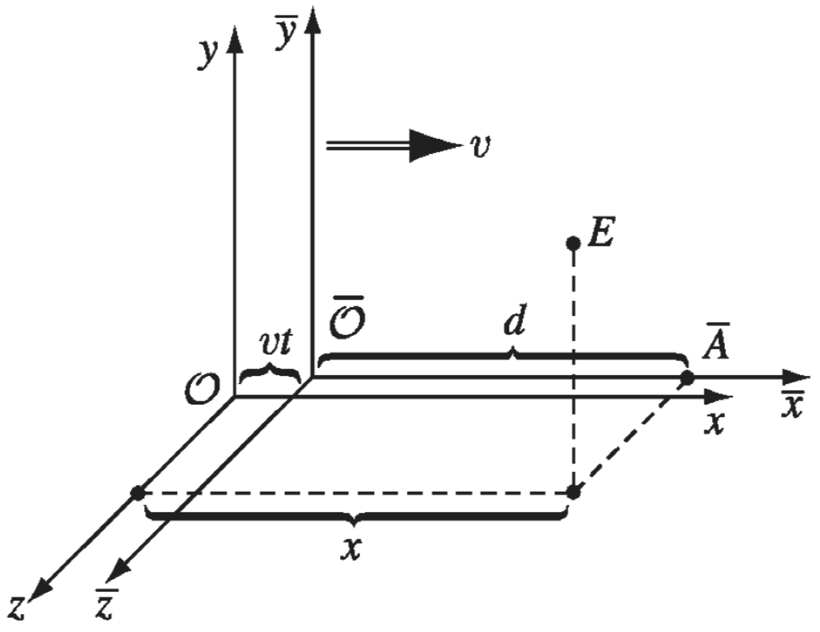


FIGURE 12.16

$$d = \frac{1}{\gamma}\bar{x} \qquad \bar{x} = \gamma(x - vt) \qquad (12.14)$$

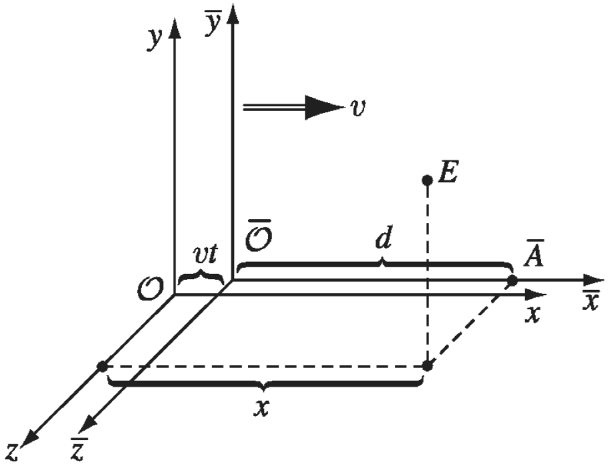


FIGURE 12.16

$$\bar{x} = \bar{d} - v\bar{t}$$

$$\bar{d} = \frac{1}{\gamma}x \qquad x = \gamma(\bar{x} + v\bar{t}) \qquad (12.17)$$

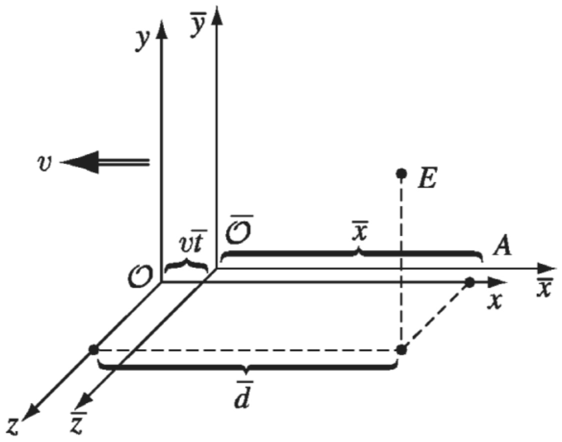


FIGURE 12.17

substitute $\bar{x}$ from $Eq. 12.14$, and solve for $\bar{t}$, we complete the relativistic "dictionary":

$$\boxed{\begin{aligned} &\text{(i)}\quad \bar{x} = \gamma(x - vt), \\ &\text{(ii)}\quad \bar{y} = y, \\ &\text{(iii)}\quad \bar{z} = z, \\ &\text{(iv)}\quad \bar{t} = \gamma\left(t - \frac{v}{c^2}x\right). \end{aligned}}$$

These are the famous **Lorentz transformations**

$$\left.\begin{aligned} &\text{(i}'\text{)}\quad x = \gamma(\bar{x} + v\bar{t}), \\ &\text{(ii}'\text{)}\quad y = \bar{y}, \\ &\text{(iii}'\text{)}\quad z = \bar{z}, \\ &\text{(iv}'\text{)}\quad t = \gamma\left(\bar{t} + \frac{v}{c^2}\bar{x}\right). \end{aligned}\right\} \qquad (12.19)$$

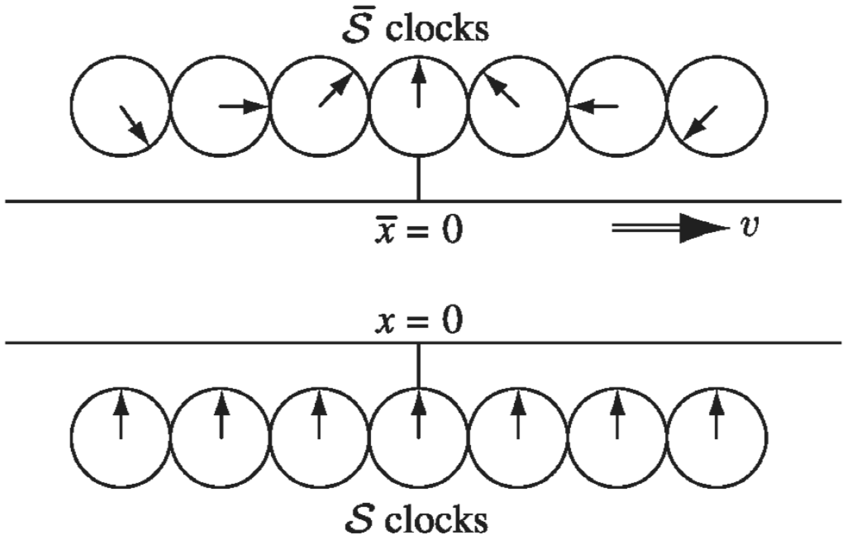
The reverse dictionary, which carries you from S back to S, can be obtained algebraically by solving (i) and (iv) for x and t, or, more simply, by switching the sign of v:

Example 12.4

Simultaneity, synchronization, and time dilation. Suppose event A occurs at $x_A = 0$, $t_A = 0$, and event B occurs at $x_B = b$, $t_B = 0$. The two events are simultaneous in S (they both take place at $t = 0$). But they are not simultaneous in $\bar{S}$, for the Lorentz transformations give $\bar{x}_A = 0$, $\bar{t}_A = 0$ and $\bar{x}_B = \gamma b$, $\bar{t}_B = -\gamma\left({}^{v}/_{c^2}\right)b$. According to the $\bar{S}$ clocks, then, B occurred before A. This is nothing new, of course-just the relativity of simultaneity.

At time $t = 0$

$$\bar{t} = -\gamma \frac{v}{c^2} x$$



Those to the left of the origin (negative x) are ahead, and those to the right are behind,

Finally, suppose S focuses his attention on a single clock at rest in the S frame. (iv') gives $\Delta t = \gamma \Delta \bar{t}$,

$$\Delta \bar{t} = \frac{1}{\gamma} \Delta t$$

That's the old time dilation formula

Example 12.5

Lorentz contraction. Imagine a stick at rest in $\bar{S}$ (hence moving to the right at speed v in S). Its rest length (that is, its length as measured in $\bar{S}$) is $\Delta\bar{x} = \overline{x_r} - \overline{x_l}$, where the subscripts denote the right and left ends of the stick. If an observer in S were to measure the stick, he would subtract the positions of the two ends at one instant of his time t: $\Delta x = x_r - x_l$ (for $t_l = t_r$). According to (i), then,

$$\Delta x = \frac{1}{\gamma}\Delta\bar{x}$$

This is the old Lorentz contraction formula.

Example 12.6

Einstein's velocity addition rule. Suppose a particle moves a distance dx (in S) in a time dt. Its velocity u is then

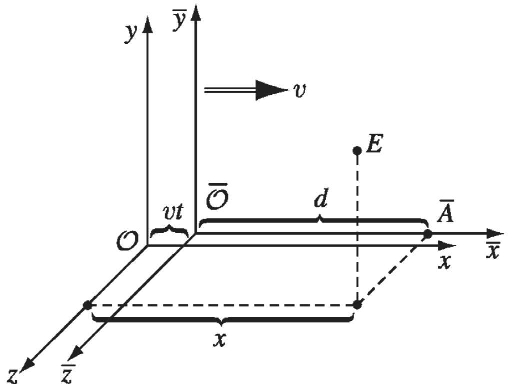


FIGURE 12.16

$$u = \frac{dx}{dt}$$

In $\bar{S}$, it has moved a distance

$$d\bar{x} = \gamma(dx - v\,dt)$$

as we see from (i), in a time given by (iv):

$$d\bar{t} = \gamma\left(dt - \frac{v}{c^2}dx\right)$$

The velocity in $\bar{S}$ is therefore

$$\bar{u} = \frac{d\bar{x}}{d\bar{t}} = \frac{\gamma(dx - v\,dt)}{\gamma\left(dt - v/c^2 dx\right)} = \frac{(dx/dt - v)}{1 - v/c^2 dx/dt} = \frac{u - v}{1 - uv/c^2}$$

(i) $\bar{x} = \gamma(x - vt)$,

(ii) $\bar{y} = y$,

(iii) $\bar{z} = z$,

(iv) $\bar{t} = \gamma\left(t - \frac{v}{c^2}x\right)$.

This is Einstein's velocity addition rule $u = v_{AB}$, $\bar{u} = v_{AC}$,

$$v_{AC} = \frac{v_{AB} + v_{BC}}{1 + (v_{AB} v_{BC}/c^2)}$$

The Structure of Spacetime

(i) **Four-vectors.** The Lorentz transformations take on a simpler appearance when expressed in terms of the quantities

$$x^0 \equiv ct, \quad \beta \equiv \frac{v}{c}$$

Using x^0(instead of t) and β (instead of v) amounts to changing the unit of time from the second to the meter—1 meter of x^0corresponds to the time it takes light to travel 1 meter (in vacuum). number the x, y, z coordinates, so that

$$x^1 = x, \quad x^2 = y, \quad x^3 = z \qquad (12.22)$$

then the Lorentz transformations read

$$\left.\begin{aligned}
\bar{x}^0 &= \gamma(x^0 - \beta x^1),\\
\bar{x}^1 &= \gamma(x^1 - \beta x^0),\\
\bar{x}^2 &= x^2,\\
\bar{x}^3 &= x^3.
\end{aligned}\right\} \qquad (12.23)$$

Or, in matrix form:

$$\begin{pmatrix} \bar{x}^0 \\ \bar{x}^1 \\ \bar{x}^2 \\ \bar{x}^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

this can be distilled into a single equation:

$$\bar{x}^{\mu} = \sum_{\nu=0}^{3} (\Lambda^{\mu}_{\nu}) x^{\nu}$$

where Λ (lambda) is the Lorentz transformation matrix (the superscript μ labels the row, the subscript ν labels the column).

We now define a **4-vector** as any set of four components that transform in the same manner as (x^0, x^1, x^2, x^3) under Lorentz transformations:

$$\bar{a}^{\mu} = \sum_{\nu=0}^{3} \Lambda^{\mu}_{\nu} a^{\nu} \qquad (12.26)$$

For the particular case of a transformation along the x axis,

$$\left.\begin{aligned}\bar{a}^0 &= \gamma(a^0 - \beta a^1),\\ \bar{a}^1 &= \gamma(a^1 - \beta a^0),\\ \bar{a}^2 &= a^2,\\ \bar{a}^3 &= a^3.\end{aligned}\right\}$$

There is a **4-vector** analog to the dot product ($A \cdot B \equiv A_x B_x + A_y B_y + A_z B_z$), but it's not just the sum of the products of the components; rather, the zeroth components have a minus sign:

$$-a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3$$

This is the four-dimensional scalar product; it has the same value in all inertial systems:

$$-\bar{a}^0\bar{b}^0 + \bar{a}^1\bar{b}^1 + \bar{a}^2\bar{b}^2 + \bar{a}^3\bar{b}^3 = -a^0b^0 + a^1b^1 + a^2b^2 + a^3b^3$$

just as the ordinary dot product is invariant (unchanged) under rotations, this combination is invariant under Lorentz transformations.

To keep track of the minus sign, it is convenient to introduce the covariant (协变) vector a_μ, which differs from the contravariant (逆变) a_μ only in the sign of the zeroth component:

$$a_\mu = (a_0, a_1, a_2, a_3) \equiv (-a^0, a^1, a^2, a^3)$$

upper indices designate contravariant vectors; lower indices are for covariant vectors. $(a_0 = -a^0)$;$(a_1 = a^1, a_2 = a^2, a_3 = a^3)$

Formally,

$$a_\mu = \sum_{\nu=0}^{3} g_{\mu\nu} a^\nu \quad \text{where} \quad g_{\mu\nu} \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

or, more compactly still,

$$a^\mu b_\mu$$

Summation is implied whenever a Greek index is repeated in a product. This is called the **Einstein summation convention**

$$a_\mu b^\mu = a^\mu b_\mu = -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3$$

(ii) The invariant interval. Suppose event A occurs at $(x_A^0, x_A^1, x_A^2, x_A^3)$ and event B at $(x_B^0, x_B^1, x_B^2, x_B^3)$. The difference,

$$\Delta x^\mu \equiv x_A^\mu - x_B^\mu$$

is the displacement 4-vector. The scalar product of Δx^μ with itself is called the invariant interval between two events:

$$I \equiv (\Delta x)^\mu (\Delta x)_\mu = -(\Delta x^0)^2 + (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2 = -c^2 t^2 + d^2$$

where t is the time difference between the two events and d is their spatial separation. When you transform to a moving system, the time between A and B is altered ($\bar{t} \neq t$), and so is the spatial separation ($\bar{d} \neq d$), but the interval $\boldsymbol{I}$ remains the same.

Depending on the two events in question, the interval can be positive ,negative ,or zero.

$$\text{If} \quad a^\mu a_\mu > 0, \quad a^\mu \text{ is called } \textbf{spacelike}.$$

$$\text{If} \quad a^\mu a_\mu < 0, \quad a^\mu \text{ is called } \textbf{timelike}.$$

$$\text{If} \quad a^\mu a_\mu = 0, \quad a^\mu \text{ is called } \textbf{lightlike}.$$

If the displacement between two events is timelike $(I < 0)$, there exists an inertial system (accessible by Lorentz transformation) in which they occur at the same point. For if I hop on a train going from (A) to (B) at the speed $v = {}^{d}/_{t}$, leaving event A when it occurs, I shall be just in time to pass B when it occurs; in the train system, A and B take place at the same point. You cannot do this for a spacelike interval, of course, because v would have to be greater than c, and no observer can exceed the speed of light (γ would be imaginary and the Lorentz transformations would be nonsense). On the other hand, if the displacement is spacelike $(I > 0)$, then there exists a system in which the two events occur at the same time.

(iii) Space-time diagrams. A particle at rest is represented by a vertical line; a photon, traveling at the speed of light, is described by a 45° line; and a rocket going at some intermediate speed follows a line of slope ${}^{c}/_{v}={}^{1}/_{\beta}$. We call such plots **Minkowski diagrams.**

The trajectory of a particle on a Minkowski diagram is called a **world line.**

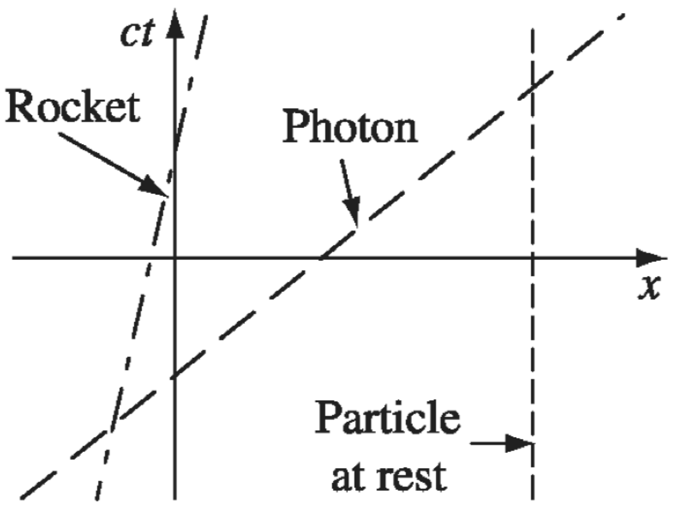


FIGURE 12.21

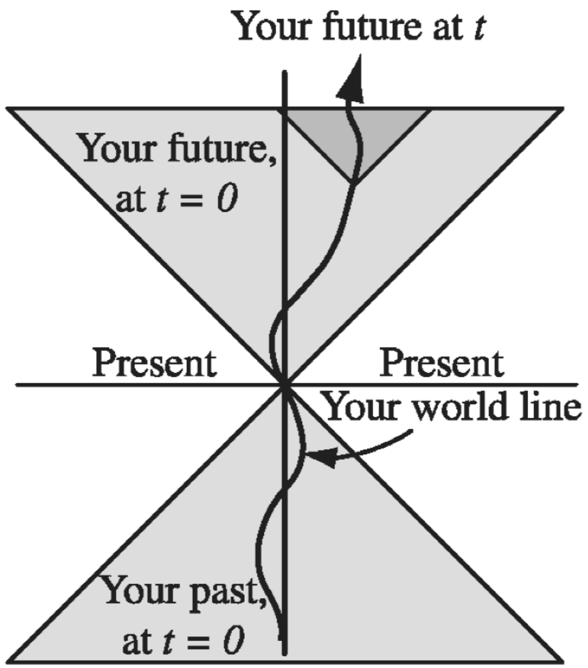


FIGURE 12.22

If we include a y axis coming out of the page, the “wedges” become cone——**forward light cone** and the **backward light cone.**

the slope of the line connecting two events on a space-time diagram tells you at a glance whether the displacement between them is timelike (slope greater than 1), spacelike (slope less than 1), or lightlike (slope 1).

Under Lorentz transformations, however, it is the interval $I = \left(x^2 - c^2 t^2\right)$ that is preserved, and the locus of all points with a given value of I is a hyperbola (双曲线), or, if we include the y axis, a hyperboloid (双曲面) of revolution. When the displacement is time like, it's a "hyperboloid of two sheets" ($Fig. 12.24a$); when the displacement is spacelike, it's a "hyperboloid of one sheet" ($Fig. 12.24b$). When you perform a Lorentz transformation (that is, when you go into a moving inertial system), the coordinates (x, t) of a given event will change to $(\bar{x}, \bar{t})$, but these new coordinates will lie on the same hyperbola as (x, t).

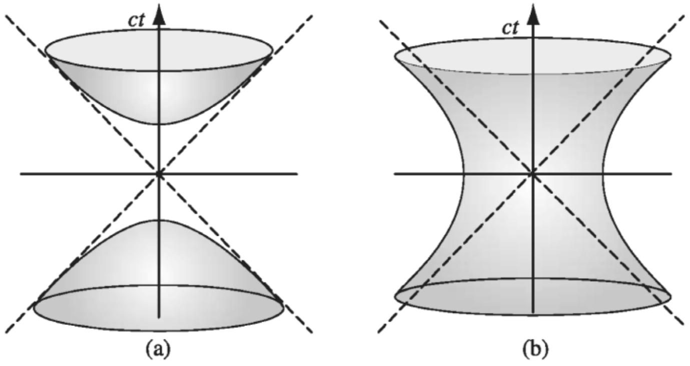


FIGURE 12.24

If the displacement 4-vector between two events is timelike, their ordering is absolute (they can occur at the same point for certain observer); if the interval is space like, their ordering depends on the inertial system from which they are observed. In terms of the space-time diagram, an event on the upper sheet of a timelike hyperboloid definitely occurred after $(0, 0)$, and one on the lower sheet certainly occurred before; but an event on a spacelike hyperboloid occurred at positive t, or negative t, depending on your reference frame.

Causality: If it were *always* possible to reverse the order of two events, then we could never say "A caused B ," since a rival observer would retort that B preceded A .

Conclusion: The displacement between causally related events is always timelike, and their temporal ordering is the same for all inertial observers.

Relativistic Mechanics

Proper Time and Proper Velocity

The time τ your watch registers (or, more generally, the time associated with the moving object) is called **proper time**. (The word suggests a mistranslation of the French "pro pre", meaning "own.") proper time is invariant, whereas "ordinary" time depends on the particular reference frame you have in mind.

Now, imagine you're on a flight, and the pilot announces that the plane's velocity is $\frac{4}{5}$c. What precisely does he mean by "velocity"?

$$\mathbf{u} = \frac{d\mathbf{l}}{dt}$$

both dl and dt are to be measured by the ground observer

proper velocity $$\boldsymbol{\eta} \equiv \frac{d\mathbf{l}}{d\tau}$$

eta

u the **ordinary velocity**

$$\boldsymbol{\eta} = \frac{1}{\sqrt{1 - u^2/c^2}}\mathbf{u}$$

From a theoretical standpoint, however, proper velocity has an enormous advantage over ordinary velocity: it transforms simply, when you go from one inertial system to another. In fact, η (eta) is the spatial part of a 4-vector,

$$\eta^\mu \equiv \frac{dx^\mu}{d\tau}$$

proper velocity

whose zeroth component is

$$\eta^0 = \frac{dx^0}{d\tau} = c\frac{dt}{d\tau} = \frac{c}{\sqrt{1 - u^2/c^2}}$$

the numerator, dx^μ, is a displacement 4-vector, while the denominator, $d\tau$ is invariant.

when you go from system S to system $\overline{S}$, moving at speed v along the common $x\bar{x}$ axis,

$$\left.\begin{aligned} \bar{\eta}^0 &= \gamma(\eta^0 - \beta\eta^1), \\ \bar{\eta}^1 &= \gamma(\eta^1 - \beta\eta^0), \\ \bar{\eta}^2 &= \eta^2, \\ \bar{\eta}^3 &= \eta^3. \end{aligned}\right\} \qquad (12.43)$$

More generally, $\bar{\eta}^\mu = \Lambda^\mu_\nu \eta^\nu$

η^μ is called the proper velocity 4-vector, or simply the 4-velocity.

By contrast, the transformation rule for ordinary velocities is quite cumbersome

$$
\left.
\begin{aligned}
\bar{u}_x &= \frac{d\bar{x}}{d\bar{t}} = \frac{u_x - v}{(1 - v u_x/c^2)}, \\
\bar{u}_y &= \frac{d\bar{y}}{d\bar{t}} = \frac{u_y}{\gamma(1 - v u_x/c^2)}, \\
\bar{u}_z &= \frac{d\bar{z}}{d\bar{t}} = \frac{u_z}{\gamma(1 - v u_x/c^2)}.
\end{aligned}
\right\}
\qquad (12.45)
$$

The reason for the added complexity is plain:
we're obliged to transform both the numerator dl and the denominator dt, whereas for proper velocity, the denominator $d\tau$ is invariant.

Relativistic Energy and Momentum

In classical mechanics, momentum is mass times velocity. In the relativistic domain, should we use *ordinary* velocity or *proper* velocity?
It is essential that we use *proper* velocity, for the law of conservation of momentum would be inconsistent with the principle of relativity if we were to define momentum as m**u**

Relativistic Momentum

$$\boxed{\mathbf{p} \equiv m\boldsymbol{\eta} = \frac{m\mathbf{u}}{\sqrt{1-u^2/c^2}};}$$

Relativistic momentum is the spatial part of a 4-vector,

$$p^\mu \equiv m\eta^\mu$$

and it is natural to ask what the temporal component

$$p^0 = m\eta^0 = \frac{mc}{\sqrt{1 - u^2/c^2}}$$

represents. Einstein identified $p^0 c$ as relativistic energy:

$$\boxed{E \equiv \frac{mc^2}{\sqrt{1 - u^2/c^2}};}$$

p^μ is called the energy-momentum 4-vector (or the momentum 4-vector, for short)

the relativistic energy is nonzero even when the object is stationary;
we call this rest energy:

$$E_{\text{rest}} \equiv mc^2$$

The remainder, which is attributable to the motion, is kinetic energy

$$E_{\text{kin}} \equiv E - mc^2 = mc^2\left(\frac{1}{\sqrt{1-u^2/c^2}} - 1\right)$$

In every closed system, the total relativistic energy and momentum are conserved.

Note: the distinction between an invariant quantity (same value in all inertial systems) and a conserved quantity (same value before and after some process). Mass is invariant but not conserved; energy is conserved but not invariant; electric charge is both conserved and invariant; velocity is neither conserved nor invariant.

The scalar product of p^μ with itself is

$$p^\mu p_\mu = -(p^0)^2 + (\mathbf{p}\cdot\mathbf{p}) = -m^2c^2$$

In terms of the relativistic energy and momentum

$$\boxed{E^2 - p^2c^2 = m^2c^4.}$$

$$\boxed{E \equiv \frac{mc^2}{\sqrt{1-u^2/c^2}};}$$

Example 12.7

Two lumps of clay, each of (rest) mass m, collide head-on at $\frac{3}{5}c$ $(Fig. 12.26)$. They stick together.
Question: what is the mass (M) of the composite lump?

Solution

The energy of each lump prior to the collision is

$$\frac{mc^2}{\sqrt{1-(3/5)^2}} = \frac{5}{4}mc^2$$

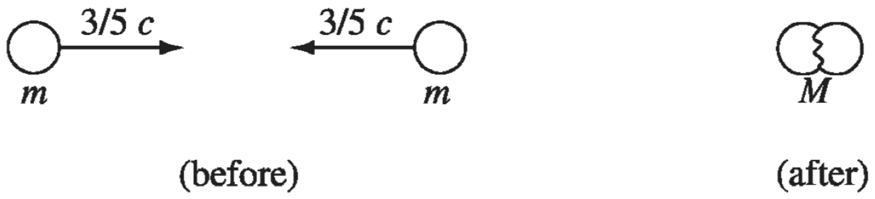


FIGURE 12.26

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and the energy of the composite lump after the collision is Mc^2 (since it's at rest).
So conservation of energy says

$$\frac{5}{4}mc^2 + \frac{5}{4}mc^2 = Mc^2$$

and hence

$$M = \frac{5}{2}m$$

Note: we call the process elastic if kinetic energy is conserved; the next one is elastic.

Example 12.9

Compton scattering. A photon of energy E_0 "bounces" off an electron, initially at rest. Find the energy E of the outgoing photon, as a function of the scattering angle θ (see $Fig.$ 12.28).

Solution

Conservation of momentum in the "vertical" direction gives $p_e \sin\phi = P_p \sin\theta$, or, since $P_p = {}^{E}/_{c}$,

$$\sin\phi = \frac{E}{p_e c}\sin\theta$$

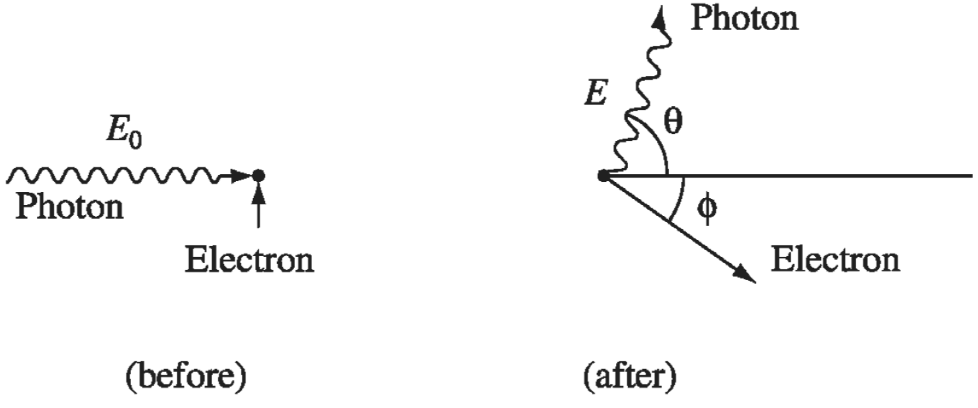

FIGURE 12.28

Conservation of momentum in the "horizontal" direction gives

$$\frac{E_0}{c} = p_p\cos\theta + p_e\cos\phi = \frac{E}{c}\cos\theta + p_e\sqrt{1-\left(\frac{E}{p_e c}\sin\theta\right)^2}$$

or $$p_e^2c^2 = (E_0 - E\cos\theta)^2 + E^2\sin^2\theta = E_0^2 - 2E_0E\cos\theta + E^2$$

Finally, conservation of energy says that

$$\begin{aligned} E_0 + mc^2 &= E + E_e = E + \sqrt{m^2c^4 + p_e^2c^2} \\ &= E + \sqrt{m^2c^4 + E_0^2 - 2E_0E\cos\theta + E^2} \end{aligned}$$

Solving for E, we find that

$$E = \frac{1}{(1-\cos\theta)/mc^2 + (1/E_0)}$$

$$E = h\nu = \frac{hc}{\lambda}$$

The quantity $({}^{h}/_{mc})$ is called the Compton wavelength of the electron

$$\lambda = \lambda_0 + \frac{h}{mc}(1-\cos\theta)$$

Relativistic Dynamics

Newton's second law,

$$\boxed{\mathbf{F} = \frac{d\mathbf{p}}{dt},}$$

retains its validity in relativistic mechanics, provided we use the relativistic momentum.

Example 12.10

Motion under a constant force. A particle of mass m is subject to a constant force $\boldsymbol{F}$. If it starts from rest at the origin, at time $t = 0$, find its position (x), as a function of time.

Solution

$$\frac{dp}{dt} = F \quad \Rightarrow \quad p = Ft + \text{constant}$$

but since $p = 0$ at $t = 0$, the constant must be zero, and hence

$$p = \frac{mu}{\sqrt{1 - u^2/c^2}} = Ft$$

Solving for u, we obtain

$$u = \frac{(F/m)t}{\sqrt{1 + (Ft/mc)^2}}$$

$$x(t) = \frac{F}{m} \int_0^t \frac{t'}{\sqrt{1 + (Ft'/mc)^2}} \, dt'$$

$$= \frac{mc^2}{F} \sqrt{1 + (Ft'/mc)^2} \Big|_0^t = \frac{mc^2}{F} \left[\sqrt{1 + (Ft/mc)^2} - 1 \right]$$

In place of the classical parabola, $x(\mathrm{t}) = (F/_{2m})t^2$, the graph is a hyperbola, for this reason, motion under a constant force is often called **hyperbolic motion.**

Work, as always, is the line integral of the force:

$$W \equiv \int \mathbf{F} \cdot d\mathbf{l}$$

The work-energy theorem

$$W = \int \frac{d\mathbf{p}}{dt} \cdot d\mathbf{l} = \int \frac{d\mathbf{p}}{dt} \cdot \frac{d\mathbf{l}}{dt}\, dt = \int \frac{d\mathbf{p}}{dt} \cdot \mathbf{u}\, dt$$

while

$$\frac{d\mathbf{p}}{dt} \cdot \mathbf{u} = \frac{d}{dt}\left(\frac{m\mathbf{u}}{\sqrt{1 - u^2/c^2}}\right) \cdot \mathbf{u}$$

$$= \frac{m\mathbf{u}}{(1 - u^2/c^2)^{3/2}} \cdot \frac{d\mathbf{u}}{dt} = \frac{d}{dt}\left(\frac{mc^2}{\sqrt{1 - u^2/c^2}}\right) = \frac{dE}{dt}$$

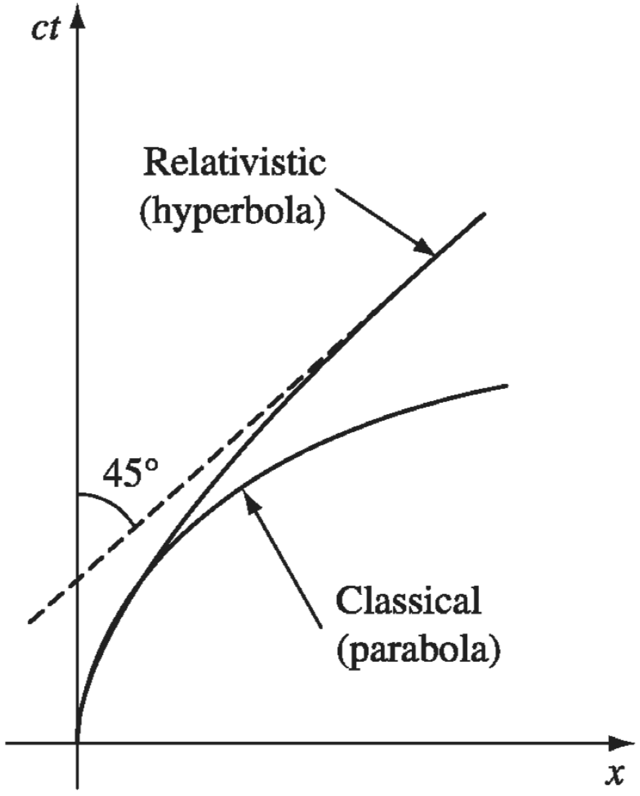

FIGURE 12.30

$$W = \int \frac{dE}{dt} dt = E_{\text{final}} - E_{\text{initial}}$$

Unlike the first two, Newton's third law does not, in general, extend to the relativistic domain. Indeed, if the two objects in question are separated in space, the third law is incompatible with the relativity of simultaneity.
Only in the case of contact interactions, where the two forces are applied at the same physical point (and in the trivial case where the forces are constant) can the third law be retained.

Because **F** is the derivative of momentum with respect to ordinary time, it shares the ugly behavior of (ordinary) velocity, when you go from one inertial system to another: both the numerator and the denominator must be transformed. Thus,

$$\bar{F}_y = \frac{d\bar{p}_y}{d\bar{t}} = \frac{dp_y}{\gamma\, dt - \frac{\gamma\beta}{c} dx} = \frac{dp_y/dt}{\gamma\left(1 - \frac{\beta}{c}\frac{dx}{dt}\right)} = \frac{F_y}{\gamma(1 - \beta u_x/c)}$$

and similarly for the z component

$$\bar{F}_z = \frac{F_z}{\gamma(1 - \beta u_x/c)}$$

The x component is even worse

$$\bar{F}_x = \frac{d\bar{p}_x}{d\bar{t}} = \frac{\gamma\, dp_x - \gamma\beta\, dp^0}{\gamma\, dt - \dfrac{\gamma\beta}{c}\, dx} = \frac{\dfrac{dp_x}{dt} - \beta\dfrac{dp^0}{dt}}{1 - \dfrac{\beta}{c}\dfrac{dx}{dt}} = \frac{F_x - \dfrac{\beta}{c}\left(\dfrac{dE}{dt}\right)}{1 - \beta u_x/c}$$

Use ${}^{dE}/_{dt}$ in $Eq. 12.63$:

$$\bar{F}_x = \frac{F_x - \beta(\mathbf{u}\cdot\mathbf{F})/c}{1 - \beta u_x/c}$$

In one special case these equations are reasonably tractable: If the particle is (instantaneously) at rest in S, so that u = 0, then

$$\bar{\mathbf{F}}_\perp = \frac{1}{\gamma}\mathbf{F}_\perp, \quad \bar{F}_\parallel = F_\parallel$$

The "proper" force is the derivative of momentum with respect to proper time :

$$K^{\mu} \equiv \frac{dp^{\mu}}{d\tau}$$

which is called the **Minkowski force.**

The spatial components of K^{μ} are related to the "ordinary" force by

$$\mathbf{K} = \left(\frac{dt}{d\tau}\right)\frac{d\mathbf{p}}{dt} = \frac{1}{\sqrt{1-u^2/c^2}}\mathbf{F}$$

while the zeroth component

$$K^0 = \frac{dp^0}{d\tau} = \frac{1}{c}\frac{dE}{d\tau}$$

However, since we are interested in the particle's trajectory as a function of *ordinary* time, the ordinary force is often more useful.

the (proper) rate at which the energy of the particle increases-in other words, the (proper) power delivered to the particle

The Lorentz force, as it turns out, is an ordinary force - will explain later why this is so.

$$\mathbf{F} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

Magnetism as a Relativistic Phenomenon

Unlike Newtonian mechanics, classical electrodynamics is already consistent with special relativity. Maxwell's equations and the Lorentz force law can be applied legitimately in any inertial system. Of course what observer interprets as an electrical process another may regard as magnetic, but the actual particle motions they predict will be identical.

See an example in right figure, a moving charge q and two currents, and there is no electric force on q, because no net charge in the conductor, the total current is.

$$I = 2\lambda v.$$

However, from the point of view of system $\bar{S}$, which is moving to the right with u, the velocity of positive and negative charges are different

$$v_{\pm} = \frac{v \mp u}{1 \mp vu/c^2}$$

in this frame, therefore, the wire carries a net negative charge!

$$\lambda_{\pm} = \pm(\gamma_{\pm})\lambda_0$$

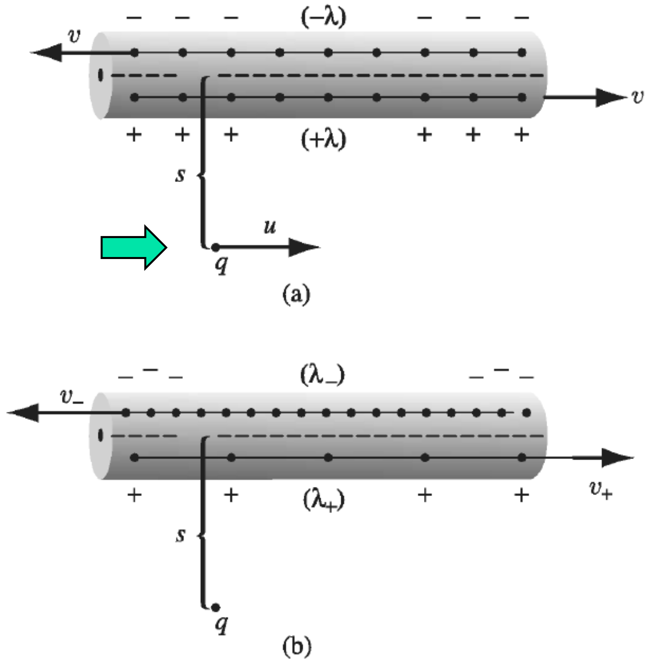

FIGURE 12.34

$$\gamma_\pm = \frac{1}{\sqrt{1 - v_\pm^2/c^2}},$$

$$\lambda_{\text{tot}} = \lambda_+ + \lambda_- = \lambda_0(\gamma_+ - \gamma_-) = \frac{-2\lambda u v}{c^2\sqrt{1 - u^2/c^2}}.$$

Conclusion:

As a result of unequal Lorentz contraction of the positive and negative lines, a current-carrying wire that is electrically neutral in one inertial system will be charged in another.

Now, a line charge λ_{tot} sets up an electric field

$$E = \frac{\lambda_{\text{tot}}}{2\pi\epsilon_0 s},$$

so there is an electrical force on q in $\bar{S}$:

$$\bar{F} = qE = -\frac{\lambda v}{\pi\epsilon_0 c^2 s}\frac{qu}{\sqrt{1 - u^2/c^2}}.$$

the force in S is given by $Eq. 12.67$

$$F=\sqrt{1-u^2/c^2}\,\bar{F}=-\frac{\lambda v}{\pi\epsilon_0 c^2}\frac{qu}{s}. \qquad (12.85)$$

The charge is attracted toward the wire by a force that is purely electric in $\bar{S}$ (where the wire is charged and q is at rest), but distinctly nonelectrical in S (where the wire is neutral). Then electrostatics and relativity imply the existence of another force.

This "other force" is, of course, magnetic. In fact, we can cast $Eq. 12.85$ into more familiar form by using $c^2=\frac{1}{\epsilon_0\mu_0}$ and expressing λv in terms of the current ($Eq. 12.76$):

$$F=-qu\left(\frac{\mu_0 I}{2\pi s}\right)$$

Magnetic field of a long, straight wire

How the Fields Transform

We have learned that observer's electric field is another's magnetic field, and it would be nice to know the general transformation rules for electromagnetic fields. Given the fields in S, what are the fields in $\bar{S}$?

The complete set of transformation rules (see details on section 12.3.2):

$$\boxed{\begin{array}{lll} \bar{E}_x = E_x, & \bar{E}_y = \gamma(E_y - vB_z), & \bar{E}_z = \gamma(E_z + vB_y), \\ \bar{B}_x = B_x, & \bar{B}_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right), & \bar{B}_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right). \end{array}}$$

(12.109)

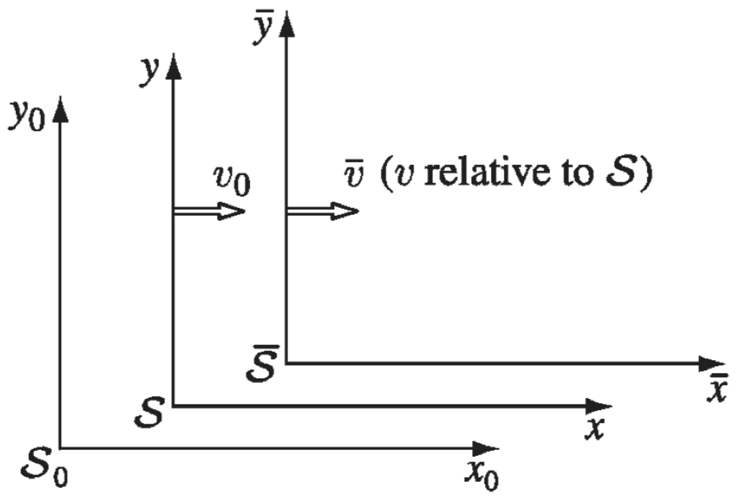

FIGURE 12.38

The Field Tensor

$As\ Eq.$ 12.109 indicates, **E** and **B** certainly do not transform like the spatial parts of the two 4-vectors. A 4-vector transforms by the rule

$$\bar{a}^{\mu} = \Lambda^{\mu}_{\nu} a^{\nu}$$

where Λ (lambda) is the Lorentz transformation matrix. If $\bar{S}$ is moving in the x direction at speed v, Λ has the form

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (12.114)$$

and Λ^{μ}_{ν} is the entry in row μ, column ν. A (second-rank) tensor is an object with two indices, which transforms with two factors of Λ (one for each index):

$$\bar{t}^{\mu\nu} = \Lambda^{\mu}_{\lambda} \Lambda^{\nu}_{\sigma} t^{\lambda\sigma}$$

$$t^{\mu\nu} = \begin{Bmatrix} t^{00} & t^{01} & t^{02} & t^{03} \\ t^{10} & t^{11} & t^{12} & t^{13} \\ t^{20} & t^{21} & t^{22} & t^{23} \\ t^{30} & t^{31} & t^{32} & t^{33} \end{Bmatrix}$$

$$\bar{t}^{\mu\nu} = \Lambda^{\mu}_{\lambda}\Lambda^{\nu}_{\sigma}t^{\lambda\sigma}$$

However, the 16 elements need not all be different

$$t^{\mu\nu} = t^{\nu\mu} \quad \textbf{(symmetric tensor)}$$

$$t^{\mu\nu} = -t^{\nu\mu} \quad \textbf{(antisymmetric tensor)}$$

$$t^{\mu\nu} = \begin{Bmatrix} 0 & t^{01} & t^{02} & t^{03} \\ -t^{01} & 0 & t^{12} & t^{13} \\ -t^{02} & -t^{12} & 0 & t^{23} \\ -t^{03} & -t^{13} & -t^{23} & 0 \end{Bmatrix}$$

Starting with $\bar{t}^{01}$, we have

$$\bar{t}^{01} = \Lambda^0_\lambda \Lambda^1_\sigma t^{\lambda\sigma} \qquad \bar{t}^{01} = \Lambda^0_0 \Lambda^1_0 t^{00} + \Lambda^0_0 \Lambda^1_1 t^{01} + \Lambda^0_1 \Lambda^1_0 t^{10} + \Lambda^0_1 \Lambda^1_1 t^{11}$$

the complete set of transformation rules

$$\left.\begin{array}{lll} \bar{t}^{01} = t^{01}, & \bar{t}^{02} = \gamma(t^{02} - \beta t^{12}), & \bar{t}^{03} = \gamma(t^{03} + \beta t^{31}), \\ \bar{t}^{23} = t^{23}, & \bar{t}^{31} = \gamma(t^{31} + \beta t^{03}), & \bar{t}^{12} = \gamma(t^{12} - \beta t^{02}). \end{array}\right\}$$

we can construct the field tensor $F^{\mu\nu}$ by direct comparison

$$\boxed{\begin{array}{lll} \bar{E}_x = E_x, & \bar{E}_y = \gamma(E_y - vB_z), & \bar{E}_z = \gamma(E_z + vB_y), \\ \bar{B}_x = B_x, & \bar{B}_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right), & \bar{B}_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right). \end{array}}$$

$$F^{01} \equiv \frac{E_x}{c}, \quad F^{02} \equiv \frac{E_y}{c}, \quad F^{03} \equiv \frac{E_z}{c}, \quad F^{12} \equiv B_z, \quad F^{31} \equiv B_y, \quad F^{23} \equiv B_x. \qquad (12.118)$$

Written as an array

$$F^{\mu\nu} = \begin{Bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{Bmatrix}$$

there was a different way of imbedding **E** and **B** in an antisymmetric tensor: Instead of comparing the first line of $Eq.$ 12.109 with the first line of $Eq.$ 12.118, and the second with the second, we could relate the first line of $Eq.$ 12.109 to the second line of $Eq.$ 12.118, and vice versa. This leads to dual tensor, $G^{\mu\nu}$:

$$G^{\mu\nu} = \begin{Bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{Bmatrix}.$$

$G^{\mu\nu}$ can be obtained directly from $F^{\mu\nu}$ by the substitution ${}^{\boldsymbol{E}}/_{\boldsymbol{c}} \rightarrow \boldsymbol{B}$, $\boldsymbol{B} \rightarrow -{}^{\boldsymbol{E}}/_{\boldsymbol{c}}$

Electrodynamics in Tensor Notation

Now that we know how to represent the fields in relativistic notation, it is time to reformulate the laws of electrodynamics (Maxwell's equations and the Lorentz force law) in that language.

To begin with, we must determine how the source of the field, ρ and $\boldsymbol{J}$, transform.

$$\rho = \frac{Q}{V} \qquad \mathbf{J} = \rho \mathbf{u}$$

We would like to express them in terms of proper charge density, i.e., the density in the rest system of the charge:

$$\rho_0 = \frac{Q}{V_0}$$

Because one dimension is Lorentz-contracted:

$$V = \sqrt{1 - u^2/c^2}\, V_0$$

We arrive

$$\rho = \rho_0 \frac{1}{\sqrt{1 - u^2/c^2}} \qquad \mathbf{J} = \rho_0 \frac{\mathbf{u}}{\sqrt{1 - u^2/c^2}}$$

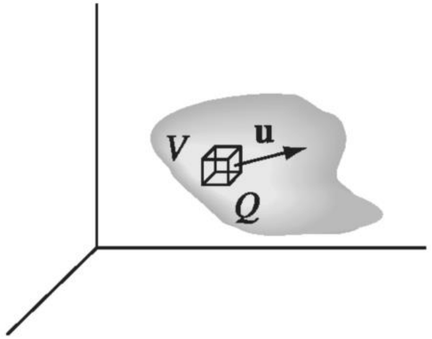

FIGURE 12.43

Electrodynamics in Tensor Notation

$$\rho = \rho_0 \frac{1}{\sqrt{1 - u^2/c^2}} \qquad \mathbf{J} = \rho_0 \frac{\mathbf{u}}{\sqrt{1 - u^2/c^2}}$$

Evidently charge density and current density go together to make a 4-vector:

$$J^\mu = \rho_0 \eta^\mu$$

current density 4-vector:

$$\boxed{J^\mu = (c\rho, J_x, J_y, J_z).}$$

The continuity equation $\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$

$$\boxed{J^{\mu} = (c\rho, J_x, J_y, J_z).}$$

$$\nabla \cdot \mathbf{J} = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} = \sum_{i=1}^{3} \frac{\partial J^i}{\partial x^i} \qquad \frac{\partial \rho}{\partial t} = \frac{1}{c}\frac{\partial J^0}{\partial t} = \frac{\partial J^0}{\partial x^0}$$

Thus, bringing ${\partial \rho}/{\partial t}$ at over to the left side (in the continuity equation), we have:

$$\boxed{\frac{\partial J^{\mu}}{\partial x^{\mu}} = 0,}$$

Incidentally, ${\partial J^{\mu}}/{\partial x^{\mu}}$ is the four-dimensional divergence of J^{μ}, so the continuity equation states that the current density 4-vector is divergenceless.

Now Maxwell's equations can be written as

$$\boxed{\frac{\partial F^{\mu\nu}}{\partial x^{\nu}} = \mu_0 J^{\mu}, \quad \frac{\partial G^{\mu\nu}}{\partial x^{\nu}} = 0,}$$

$$\boxed{\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu, \quad \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0,}$$

Each of these stands for four equations--one for every value of μ. If $\mu = 0$, the first equation reads

$$\frac{\partial F^{0\nu}}{\partial x^\nu} = \frac{\partial F^{00}}{\partial x^0} + \frac{\partial F^{01}}{\partial x^1} + \frac{\partial F^{02}}{\partial x^2} + \frac{\partial F^{03}}{\partial x^3}$$

$$= \frac{1}{c}\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) = \frac{1}{c}(\nabla \cdot \mathbf{E})$$

$$= \mu_0 J^0 = \mu_0 c \rho,$$

or $$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho$$ Gauss's law

If μ=1, we have
$$\frac{\partial F^{1\nu}}{\partial x^\nu} = \frac{\partial F^{10}}{\partial x^0} + \frac{\partial F^{11}}{\partial x^1} + \frac{\partial F^{12}}{\partial x^2} + \frac{\partial F^{13}}{\partial x^3}$$

$$= -\frac{1}{c^2}\frac{\partial E_x}{\partial t} + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \left(-\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{B}\right)_x$$

$$= \mu_0 J^1 = \mu_0 J_x.$$

Combining this with the corresponding results for μ=2 and μ=3 gives

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

which is Ampere's law with Maxwell's correction.

Meanwhile, the second equation in 12.127, with μ=0, becomes

$$\frac{\partial G^{0\nu}}{\partial x^{\nu}} = \frac{\partial G^{00}}{\partial x^{0}} + \frac{\partial G^{01}}{\partial x^{1}} + \frac{\partial G^{02}}{\partial x^{2}} + \frac{\partial G^{03}}{\partial x^{3}}$$

$$= \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = \nabla \cdot \mathbf{B} = 0$$

(the third of Maxwell's equations), whereas μ=1 yields

$$\frac{\partial G^{1\nu}}{\partial x^{\nu}} = \frac{\partial G^{10}}{\partial x^{0}} + \frac{\partial G^{11}}{\partial x^{1}} + \frac{\partial G^{12}}{\partial x^{2}} + \frac{\partial G^{13}}{\partial x^{3}}$$

$$= -\frac{1}{c}\frac{\partial B_x}{\partial t} - \frac{1}{c}\frac{\partial E_z}{\partial y} + \frac{1}{c}\frac{\partial E_y}{\partial z} = -\frac{1}{c}\left(\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E}\right)_x = 0.$$

So, combining this with the corresponding results for μ=2 and μ=3,

Faraday's law $$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

In terms of $F^{\mu\nu}$ and the proper velocity η^μ, the Minkowski force on a charge q is given by

$$\boxed{K^\mu = q\eta_\nu F^{\mu\nu}.} \tag{12.128}$$

if $\mu = 1$, $$K^1 = q\eta_\nu F^{1\nu} = q(-\eta^0 F^{10} + \eta^1 F^{11} + \eta^2 F^{12} + \eta^3 F^{13})$$

$$= q\left[\frac{-c}{\sqrt{1-u^2/c^2}}\left(\frac{-E_x}{c}\right) + \frac{u_y}{\sqrt{1-u^2/c^2}}(B_z) + \frac{u_z}{\sqrt{1-u^2/c^2}}(-B_y)\right]$$

$$= \frac{q}{\sqrt{1-u^2/c^2}}[\mathbf{E} + (\mathbf{u} \times \mathbf{B})]_x,$$

with a similar formula for $\mu = 2$ and $\mu = 3$. Thus,

$$\mathbf{K} = \frac{q}{\sqrt{1 - u^2/c^2}} [\mathbf{E} + (\mathbf{u} \times \mathbf{B})],$$

Therefore, referring back to $Eq.$ 12.69

$$\mathbf{K} = \left(\frac{dt}{d\tau}\right) \frac{d\mathbf{p}}{dt} = \frac{1}{\sqrt{1 - u^2/c^2}} \mathbf{F} \qquad (12.69)$$

$$\mathbf{F} = q[\mathbf{E} + (\mathbf{u} \times \mathbf{B})]$$

which is the Lorentz force law. $Eq.$ 12.128, then, represents the Lorentz force law in relativistic notation.

$$\boxed{K^\mu = q\eta_\nu F^{\mu\nu}.} \qquad (12.128)$$

Relativistic Potentials

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A} \tag{12.131}$$

V and $\boldsymbol{A}$ together constitute a 4-vector:

$$\boxed{A^{\mu} = (V/c, A_x, A_y, A_z).}$$

In terms of this 4-vector potential, the field tensor can be written

$$\boxed{F^{\mu\nu} = \frac{\partial A^{\nu}}{\partial x_{\mu}} - \frac{\partial A^{\mu}}{\partial x_{\nu}}.}$$

To check that Eq. 12.133 is equivalent to Eq. 12.131, let's evaluate a few terms explicitly.
For $\mu = 0, \nu = 1,$

$$F^{01} = \frac{\partial A^1}{\partial x_0} - \frac{\partial A^0}{\partial x_1} = -\frac{\partial A_x}{\partial(ct)} - \frac{1}{c}\frac{\partial V}{\partial x}$$

$$= -\frac{1}{c}\left(\frac{\partial \mathbf{A}}{\partial t} + \nabla V\right)_x = \frac{E_x}{c}.$$

That (and its companions with $\nu = 2$ and $\nu = 3$) is the first equation in $Eq.$ 12.131. For $\mu = 1, \nu = 1$, we get

$$F^{12} = \frac{\partial A^2}{\partial x_1} - \frac{\partial A^1}{\partial x_2} = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = (\nabla \times \mathbf{A})_z = B_z$$

which (together with the corresponding results for F^{23} and F^{31}) is the second equation in $Eq.$ 12.131

The potential formulation automatically takes care of the homogeneous Maxwell equation ($\partial G^{\mu\nu}/\partial x^{\nu}$).
As for the inhomogeneous equation ($\partial F^{\mu\nu}/\partial x^{\nu}$), that becomes

$$\frac{\partial}{\partial x_\mu}\left(\frac{\partial A^\nu}{\partial x^\nu}\right) - \frac{\partial}{\partial x_\nu}\left(\frac{\partial A^\mu}{\partial x^\nu}\right) = \mu_0 J^\mu .$$

you could add to A^μ the gradient of any scalar function λ:

$$A^\mu \longrightarrow A^{\mu\prime} = A^\mu + \frac{\partial\lambda}{\partial x_\mu}$$

without changing $F^{\mu\nu}$. This is precisely the gauge invariance.
In particular, the Lorenz gauge condition

$$\nabla \cdot \mathbf{A} = -\frac{1}{c^2}\frac{\partial V}{\partial t}$$

becomes, in relativistic notation

$$\frac{\partial A^{\mu}}{\partial x^{\mu}} = 0.$$

In the Lorenz gauge, therefore, $Eq.$ 12.134 reduces to

$$\boxed{\square^2 A^{\mu} = -\mu_0 J^{\mu},} \qquad (12.137)$$

where $\boxdot^2$ is the d' Alembertian

$$\square^2 \equiv \frac{\partial}{\partial x_{\nu}} \frac{\partial}{\partial x^{\nu}} = \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}$$

$Equation$ 12.137 combines our previous results into a single 4-vector equation-it represents the most elegant formulation of Maxwell's equations.