第二章 分析力学(I)

(Analytical Mechanics)

基本概念---约束. 自由度. 广义坐标. 虚位移

平衡问题-----虚功原理

动力学

位形空间
- 拉格朗日方程
- 哈密顿原理

相空间
- 哈密顿正则方程
- 哈密顿原理
- 泊松括号

运动积分
- L判据. H判据. 泊松括号判据
- 时空对称性. 不可观测量和守恒定律

§1.基本概念(Basic Concepts)

牛顿力学两大困难

约束力未知　？　坐标不独立

一.约束

- 定义:物体运动过程中受到限制
- 约束方程: $f(\vec{r}.\dot{\vec{r}}.t) = 0$

约束分类:

- 几何约束: $f(\vec{r}, t) = 0$
- 微分约束: $f(\vec{r}, \dot{\vec{r}}, t) = 0$
- 完整约束与非完整约束:

$$\left.\begin{array}{l}\text{几何约束} \\ \text{可积分的微分约束}\end{array}\right\} \text{完整约束}$$

- 稳定约束与非稳定约束:

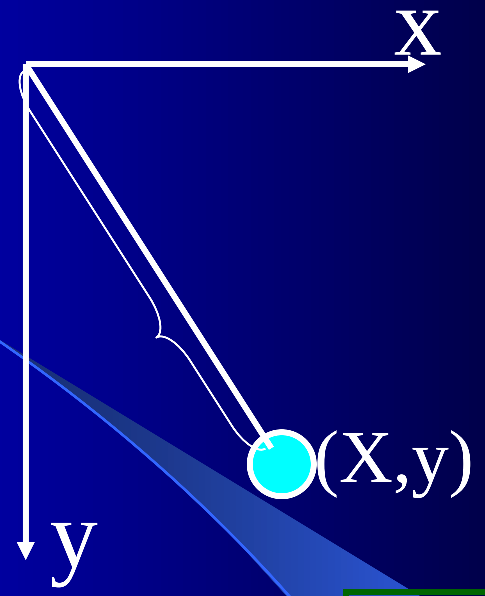
$f(\vec{r}) = 0$ $\qquad$ $f(\vec{r}, t) = 0$

- 可解约束与不可解约束

- 几何约束:

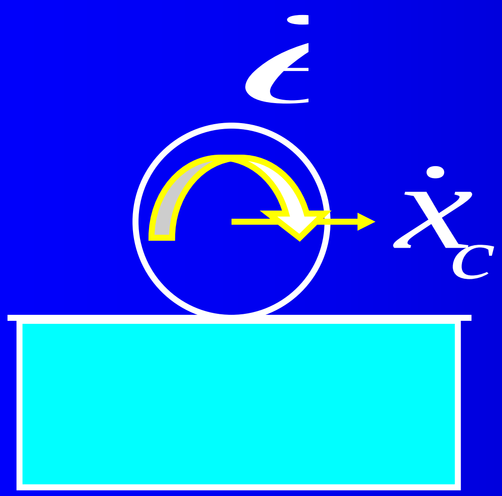
$$f(\vec{r}, t) = 0$$

$$x^2 + y^2 = l^2 \quad \ldots(1)$$



demonstration

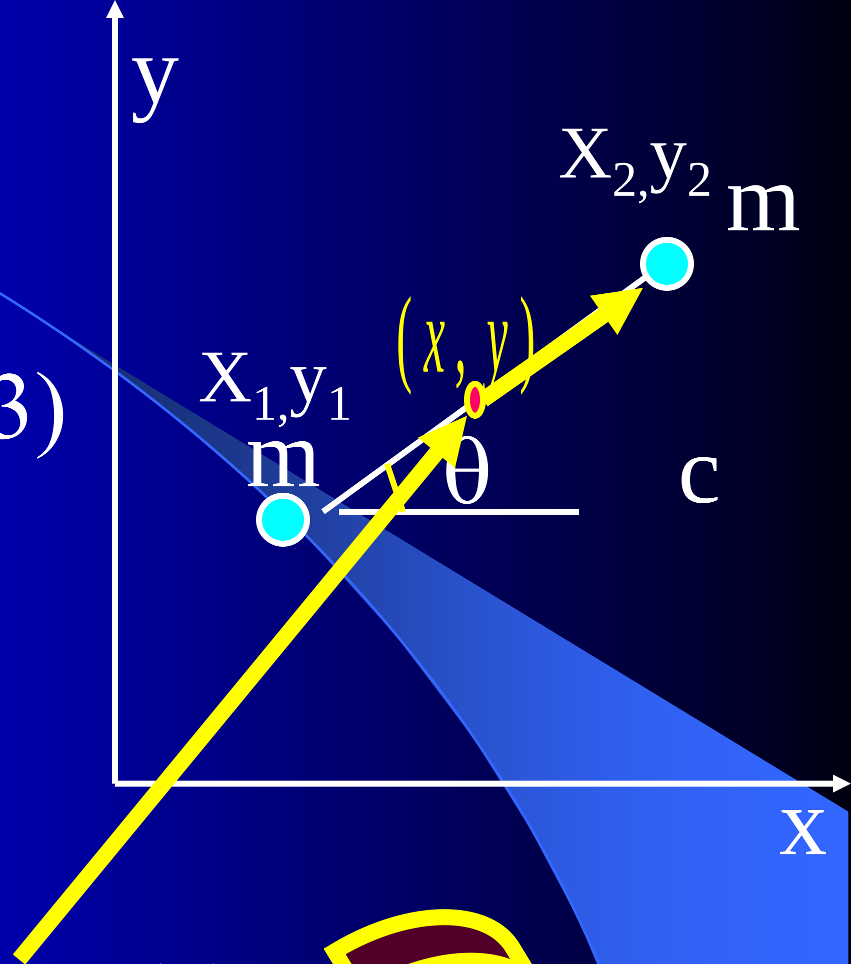
- 微分约束: $f(\vec{r}, \dot{\vec{r}}, t) = 0$



$$\dot{x}_c = a\dot{\theta} \quad \ldots(2)$$

Example:

$$\begin{cases}(x_2 - x_1)^2 + (y_2 - y_1)^2 = l^2 \\ z_1 = z_2 = 0\end{cases} \quad \ldots(3)$$



$$\begin{cases} x = \dfrac{mx_1 + mx_2}{m+m} = \frac{1}{2}(x_1 + x_2) \\ y = \dfrac{my_1 + my_2}{m+m} = \frac{1}{2}(y_1 + y_2)\end{cases}$$

$$\frac{\dot{y}}{\dot{x}} = \frac{\dot{y}_1 + \dot{y}_2}{\dot{x}_1 + \dot{x}_2} = tg\theta = \frac{y_2 - y_1}{x_2 - x_1} \quad \ldots(4)$$

$$(x_2 - x_1)dy_1 + (x_2 - x_1)dy_2 - (y_2 - y_1)dx_1 - (y_2 - y_1)dx_2 = 0$$

.....(4)

任一微分约束均可表示为

$$a_i dx_i + a_t dt = 0 \quad (i = 1,2,3\ldots\ldots N)$$

爱因斯坦求和约定

$$a_t = a_t(x_i, t) \qquad a_i = a_i(x_i, t)$$

如果: $\dfrac{\partial a_i}{\partial x_j} = \dfrac{\partial a_j}{\partial x_i}$ $\dfrac{\partial a_i}{\partial t} = \dfrac{\partial a_t}{\partial x_i}$ ⇒ 则微分方程可积

$$(x_2 - x_1)dy_1 + (x_2 - x_1)dy_2 - (y_2 - y_1)dx_1 - (y_2 - y_1)dx_2 = 0$$

$$\dot{x}_c = a\dot{\theta}$$

是否可积?

- 完整约束：
  - 几何约束和
  - 可积分的微分约束
- 非完整约束： 不可积分的微分约束
- 可解约束与不可解约束：

  用不等号表示约束 ⇒ 可解约束

  用等号表示约束 ⇒ 不可解约束

二. 自由度和描述度

系统有N个质点，受k个完整约束和m个非完整约束

定义自由度：

$$f = 3N-(k+m)$$

描述度：描述一个力学系统所需独立坐标数目：S

完整约系 ⇒ $f = S$

非完整约系 ⇒ $f < S$

三.广义坐标(Generalized coordinates)位形空间(Configurational Space)

完整约束系统的自由度为S(f),则可选S个独立参量来描述此系统 ⇒ 广义坐标

$$\boxed{q(q_1, q_2, q_3 \ldots\ldots q_s)}$$

描述系统既可用$\vec{r}_i$又可用q

它们间联系 ⇒ 变换方程

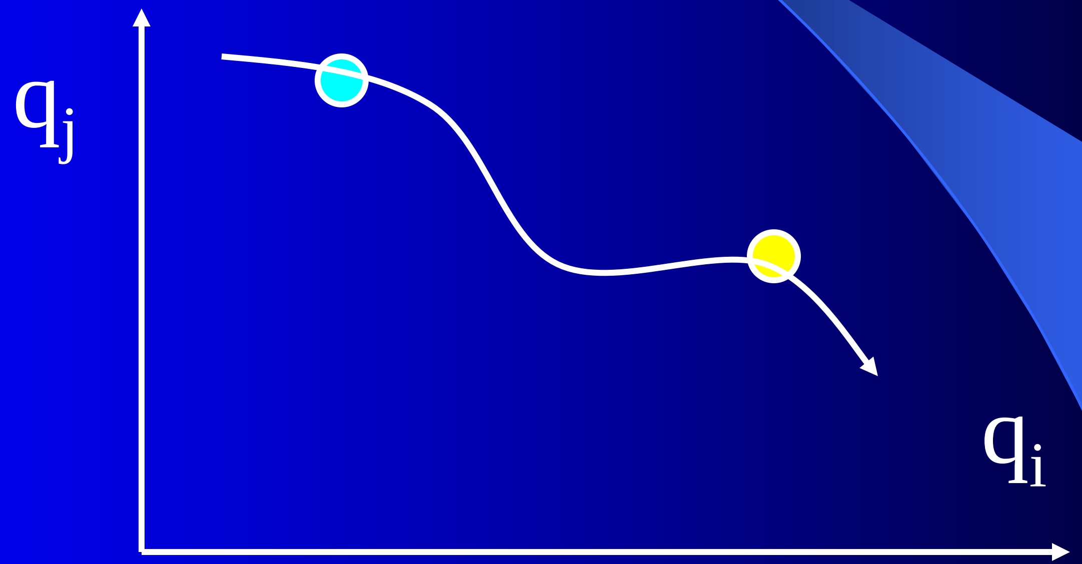
$$\vec{r}_i = \vec{r}_i(q, t)$$

Attention:

- 广义坐标数目由自由度确定
- “广义”二字的含义
- 对给定力学系统, 广义坐标选取不唯一
- 广义坐标正确与否的判断
  - 全部直角坐标能用广义坐标表示则对
  - 如果全部直角坐标不能用广义坐标表示则错
- 广义坐标克服了牛顿力学中坐标不不独立的困难

位形空间

由S个广义坐标张开成S维抽象空间



四.实位移 可能位移 虚位移(Real displacement, Possible displacement ,Virtual displacement)

实位移

设系统有N个质点，受k个几何约束

$$f_j(\vec{r}_1,\vec{r}_2,\ldots\vec{r}_i\ldots\vec{r}_N,t)=0 \qquad (j=1,2,3\ldots k)$$

(i=1 2 3…N)

$$\begin{cases} m\ddot{\vec{r}}_i=\vec{F}_i+\vec{N}_i \\ f_j(\vec{r}_1,\vec{r}_2,\ldots\vec{r}_i\ldots\vec{r}_N,t)=0 \\ (j=1,2,3\ldots k,\quad i=1,2,\ldots N) \\ dt\neq 0 \end{cases}$$

$\Rightarrow$ $d\vec{r}_i$ 称为实位移

特点： 唯一性 代表真实运动

实位移特点

- 唯一性
- 代表真实运动
- 既满足运动规律又满足约束方程
- $dt \neq 0$, ⇒ 需要时间

不考虑运动规律限制，只考虑约束限制条件下发生的位移 → **可能位移**

t时刻: $f_j(\vec{r}_i, t) = 0 \quad (j = 1.2.....k)$

t=dt 时刻: $f_j(\vec{r}_i + d\vec{r}_i, t + dt) = 0 \quad (i = 1.2....N)$

$$f_j(\vec{r}_i + d\vec{r}_i, t + dt) = 0$$

$$= f_j(\vec{r}_i, t) + \frac{\partial f_j}{\partial \vec{r}_i} d\vec{r}_i + \frac{\partial f_j}{\partial t} dt + \ldots = 0 \quad (i = 1.2....N)$$

$$\frac{\partial f_j}{\partial \vec{r}_i} d\vec{r}_i + \frac{\partial f_j}{\partial t} dt = 0 \quad (i = 1.2....N,\ j = 1.2....k)$$

Attention:

- 不考虑运动规律限制

$$\frac{\partial f_j}{\partial \vec r_i} d\vec r_i + \frac{\partial f_j}{\partial t} dt = 0$$

$$(i = 1.2....N,\ j = 1.2....k)$$

- 可能位移不唯一

可能位移的特点

可能位移产生的原因

约束变动引起 $\frac{\partial f_j}{\partial t}$

共性

$$\frac{\partial f_j}{\partial \vec r_i} \dot{\vec r}_i + \frac{\partial f_j}{\partial t} = 0$$

在约束面内各质点具有不同可能速度 $\dot{\vec r}_i$

个性

虚位移

可能位移

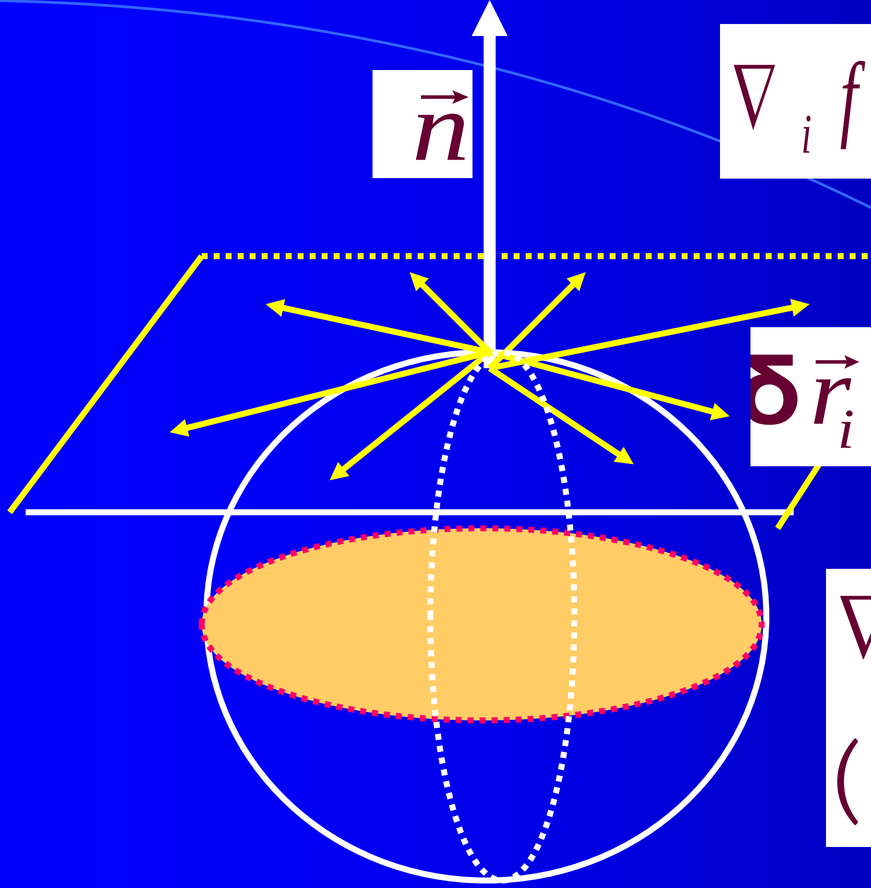
$$\frac{\partial f_j}{\partial \vec{r}_i} d\vec{r}_i + \frac{\partial f_j}{\partial t} dt = 0 \quad (i = 1.2\ldots.N\,,\ j = 1.2\ldots.k)$$

$$\nabla_i f_j \,\boldsymbol{\delta}\, \vec{r}_i + \frac{\partial f_j}{\partial t} \cdot \delta t = 0 \quad (j = 1.2\ldots k \quad i = 1.2\ldots.N),$$

$\delta t = 0, \nabla_i f_j \,\boldsymbol{\delta}\, \vec{r}_i = 0$ (等自变量的变分)

虚位移特点

- 不考虑运动规律限制
- 时间被冻结 ⟹ **δ t = 0 !!!**
- 约束被“凝固” ⟹ $\frac{\partial f_j}{\partial t} = 0$
- 满足约束条件
- 不唯一



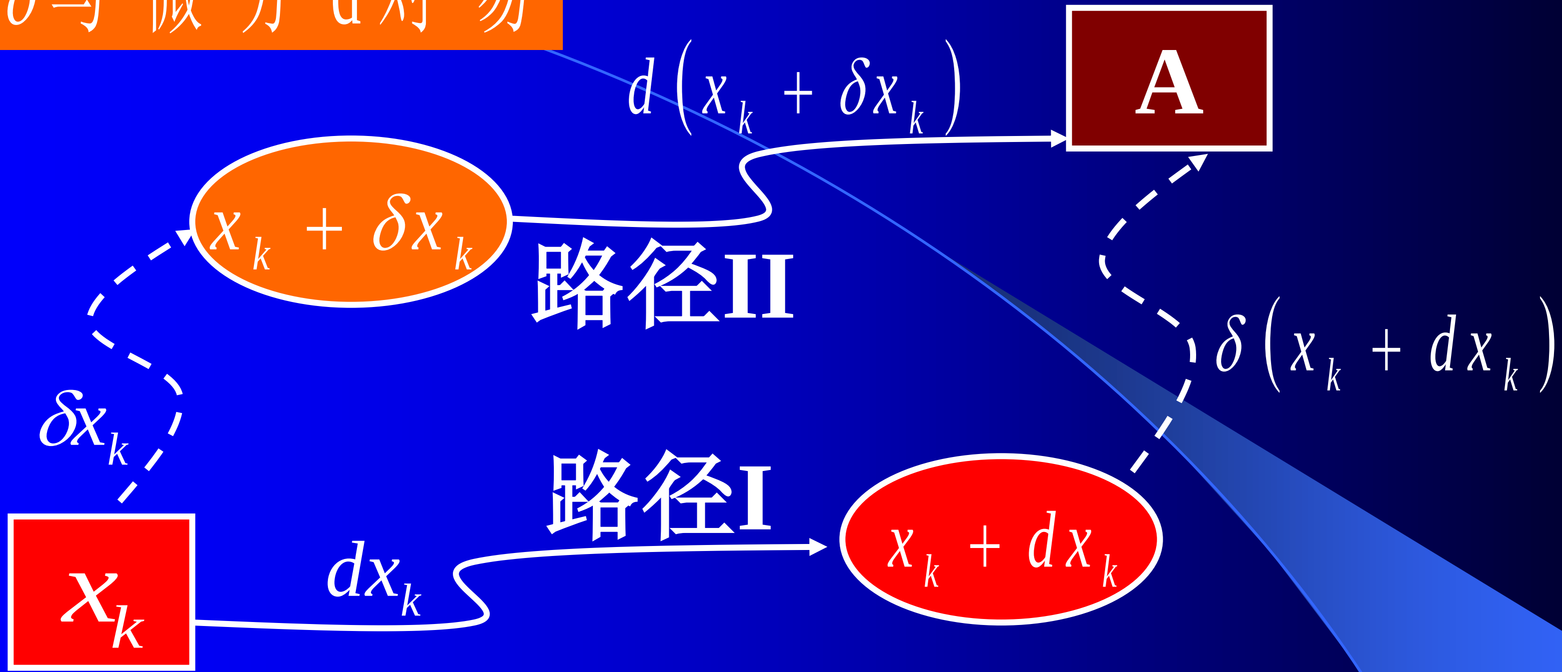
$\nabla_i f_j$：约束平面的法向 $\vec{n}$

$$\nabla_i f_j \, \delta\vec{r}_i = 0,$$
$$(i = 1,2,\ldots,N;\ j = 1,2,\ldots,k.)$$

虚位移不唯一；

稳定约束（不含t）,可能位移等于虚位移

变分 δ 与微分 d 对易



质点由位形x_k经两路径到达位形A，A是同一位形有：

$$x_k + dx_k + \delta(x_k + dx_k) = x_k + \delta x_k + d(x_k + \delta x_k)$$

整理得：$\delta(dx_k) = d(\delta x_k)$。

1. δ运算与运算d可交换；
2. δ运算规则与运算d规则相同；

变分δ与导数$\dfrac{\mathrm{d}}{\mathrm{d}t}$之间有条件对易性

$$\delta\left(\frac{\mathrm{d}x_k}{\mathrm{d}t}\right)=\frac{\delta(\mathrm{d}x_k)}{\mathrm{d}t}-\frac{\mathrm{d}x_k\delta(\mathrm{d}t)}{(\mathrm{d}t)^2}$$

$$=\frac{\mathrm{d}(\delta x_k)}{\mathrm{d}t}-\frac{\mathrm{d}x_k\mathrm{d}(\delta t)}{(\mathrm{d}t)^2}$$

$\delta t=0$

$$\delta\left(\frac{\mathrm{d}x_k}{\mathrm{d}t}\right)=\frac{\mathrm{d}(\delta x_k)}{\mathrm{d}t}$$

全变分Δ：

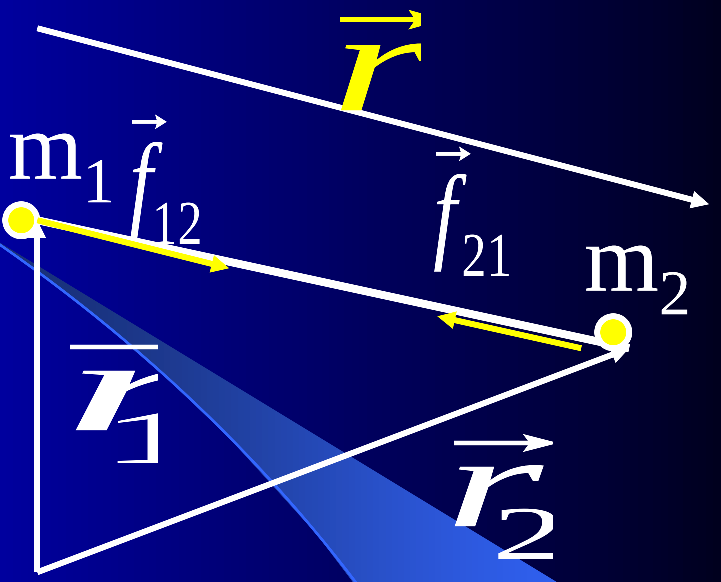
$$\Delta\left(\frac{\mathrm{d}x_k}{\mathrm{d}t}\right)=\frac{\mathrm{d}(\Delta x_k)}{\mathrm{d}t}-\frac{\mathrm{d}x_k}{\mathrm{d}t}\frac{\mathrm{d}(\Delta t)}{\mathrm{d}t},\quad(\Delta t\neq 0)$$

等时变分：变分δ与导数$\dfrac{\mathrm{d}}{\mathrm{d}t}$之间有条件（$\delta t=0$）对易：

$$\delta\left(\frac{\mathrm{d}x_k}{\mathrm{d}t}\right)=\frac{\mathrm{d}(\delta x_k)}{\mathrm{d}t},\quad(\delta t=0)$$

分析力学常用！

五. 理想约束



实例

$$\delta w = \vec f_{12}\cdot\delta\vec r_1 + \vec f_{21}\cdot\delta\vec r_2$$

$$= \vec f_{12}\cdot(\delta\vec r_1-\delta\vec r_2)$$

$$= \vec f_{12}\cdot\delta(\vec r_1-\vec r_2)$$

$\because \vec r_1+\vec r=\vec r_2$

$$= -\vec f_{12}\cdot\delta\vec r$$

$$\vec f_{12}= f\frac{\vec r}{r}$$

$$=-f\frac{\vec r}{r}\cdot\delta\vec r = -f\frac{1}{r}\cdot\boxed{\frac{1}{2}\delta\vec r^{2}}=0$$

非稳定约束 $f(\vec{r}\ \ t)=0$

对可能位移

$$df(\vec{r}\ \ t)=\frac{\partial f}{\partial \vec{r}}d\vec{r}+\frac{\partial f}{\partial t}dt=0$$

$$\lambda\frac{\partial f}{\partial \vec{r}}d\vec{r}=-\lambda\frac{\partial f}{\partial t}dt\neq 0$$

$$\because\ \vec{N}=\lambda\nabla f\ \Rightarrow\ \text{约 束 力}$$

$$dw=\lambda\frac{\partial f}{\partial \vec{r}}d\vec{r}=-\lambda\frac{\partial f}{\partial t}dt\neq 0$$

对可能位移所做元功≠0

对虚位移

$$\delta f(\vec{r}\ \ t)=\frac{\partial f}{\partial \vec{r}}\delta\vec{r}+\frac{\partial f}{\partial t}\delta t=0$$

$$\delta w=\vec{N}\bullet\delta\vec{r}=0$$

约束力在虚位移下的虚功=0

约束力“矢量”⊥虚位移“矢量”

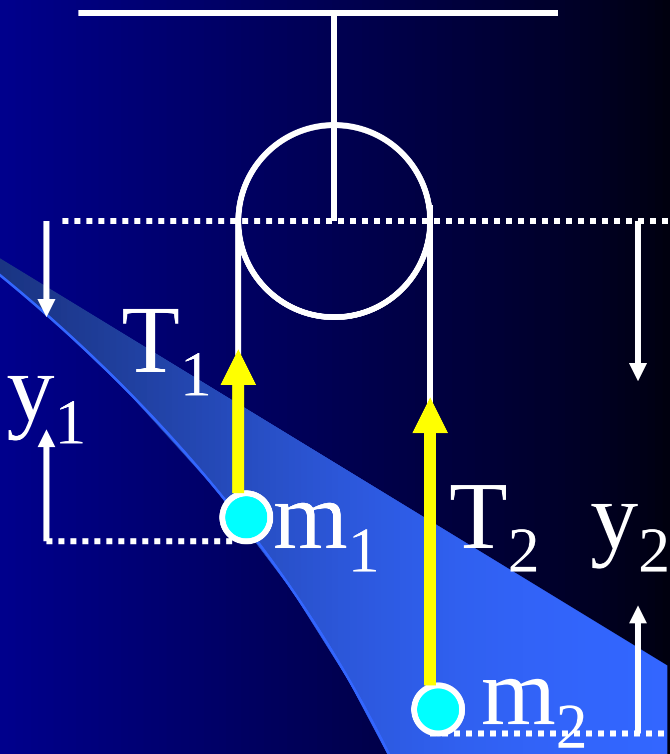
$$\delta w = T_1 \bullet \delta y_1 + T_2 \bullet \delta y_2$$

$$\left.\begin{aligned} T_1 = T_2 = T \\ y_2 = l - \pi a - y_1 \end{aligned}\right\}$$

$$\delta y_2 = -\delta y_1$$

$$\delta w = T \bullet \boxed{(\delta y_1 + \delta y_2)}$$

$$= 0$$



五. 理想约束

虚功: ⇒ 力在虚位移下所做的功

$$\delta w = \vec{F} \cdot \delta \vec{r}$$

理想约束: ⇒ 若作用在力学系统上所有的约束力在任意虚位移下所做的虚功之和为零

$$\delta W = \vec{N}_i \bullet \delta \vec{r}_i = 0 \quad (i = 1.2 \ldots . n)$$

n维约束力"矢量"⊥n维虚位移约束力"矢量"

§2.虚功原理(Principle of Virtual Work)

表述：完整的理想约束系统处于平衡的充要条件是

$\vec{F}_i$：主动力

$$\delta W = \vec{F}_i \bullet \delta \vec{r}_i = 0 \quad (i = 1.2......n)$$

证明：**必要性** ⇒ 系统处于平衡时

$$\vec{F}_i \bullet \delta \vec{r}_i = 0 \quad (i = 1.2.....n)$$

$$\vec{F}_i + \vec{N}_i = 0 \Rightarrow (\vec{F}_i + \vec{N}_i) \bullet \delta \vec{r}_i = 0$$

$$\because \quad \vec{N}_i \bullet \delta \vec{r}_i = 0$$

$$\boxed{\vec{F}_i \bullet \delta \vec{r}_i = 0}$$

充分性： **反证法**

系统不平衡，

假设 $\vec{F}_i \bullet \delta \vec{r}_i = 0$

k个质点

$$\vec{F}_j + \vec{N}_j \neq 0 \quad (j = 1, 2, 3, \ldots, k < n)$$

$$\sum_{j=1}^{k} (\vec{F}_j + \vec{N}_j) \neq 0$$

$$\sum_{j=1}^{k} (\vec{F}_j + \vec{N}_j) \bullet \delta \vec{r}_j \neq 0$$

$$\sum_{i=1}^{n} (\vec{F}_i + \vec{N}_i) \bullet \delta \vec{r}_i \neq 0$$

$$\sum_{i=1}^{n} \vec{N}_i \bullet \delta \vec{r}_i = 0$$

$$\sum_{i=1}^{n} \vec{F}_i \bullet \delta \vec{r}_i \neq 0$$

系统必平衡

$\delta W = \vec{F}_i \bullet \delta \vec{r}_i = 0$：

理想约束系统平衡所有主动力的虚功之和为0！

$$\vec{r}_i = \vec{r}_i(q, t)$$

$$\delta \vec{r}_i = \frac{\partial \vec{r}_i}{\partial q_\alpha} \delta q_\alpha$$

$$(\alpha = 1.2.3.....S)$$

$$\delta W = \vec{F}_i \bullet \delta \vec{r}_i = 0 \quad (i = 1.2......N)$$

$$\delta W = \vec{F}_i \bullet \frac{\partial \vec{r}_i}{\partial q_\alpha} \delta q_\alpha = 0 \quad (i = 1.2.....N \quad \alpha = 1.2....S)$$

定义：$Q_\alpha = \vec{F}_i \bullet \dfrac{\partial \vec{r}_i}{\partial q_\alpha} \quad (i = 1,2\ldots,3n)$ ⇒ 广义力

$$\delta W = \vec{F}_i \bullet \delta \vec{r}_i = 0 \quad (i = 1,2,\ldots,3n)$$

位形空间虚功原理：

$$\delta W = Q_\alpha \delta q_\alpha = 0 \qquad (\alpha = 1,2,\ldots,s)$$

$\because \delta q_\alpha \neq 0$, 独立 !!! ⇒ $\therefore\ Q_\alpha = 0 \qquad (\alpha = 1,2,\ldots,s)$

广义力的虚功？

$\delta W = Q_\alpha \bullet \delta q_\alpha = 0$：

理想约束系统平衡所有广义力的虚功均为0！

保守系：主动力是保守力

$$Q_\alpha = \vec{F}_i \bullet \frac{\partial \vec{r}_i}{\partial q_\alpha} = -\frac{\partial V}{\partial \vec{r}_i} \frac{\partial \vec{r}_i}{\partial q_\alpha} = -\frac{\partial V}{\partial q_\alpha} = 0 \qquad \begin{pmatrix} i = 1,2,\ldots,3n \\ \alpha = 1,2,\ldots,s \end{pmatrix}$$

Attention:

- 广义力的计算

$$Q_\alpha = \vec{F}_i \bullet \frac{\partial \vec{r}_i}{\partial q_\alpha} \quad (i = 1, 2 \ldots, N)$$

$$\delta W = Q_\alpha \delta q_\alpha = (\ )\delta q_1 + \ldots + (\ )\delta q_s$$

- 广义力的数目由自由度决定
- 广义力既可是力又可以是力矩，决定于广义坐标, 还可是其它物理量。

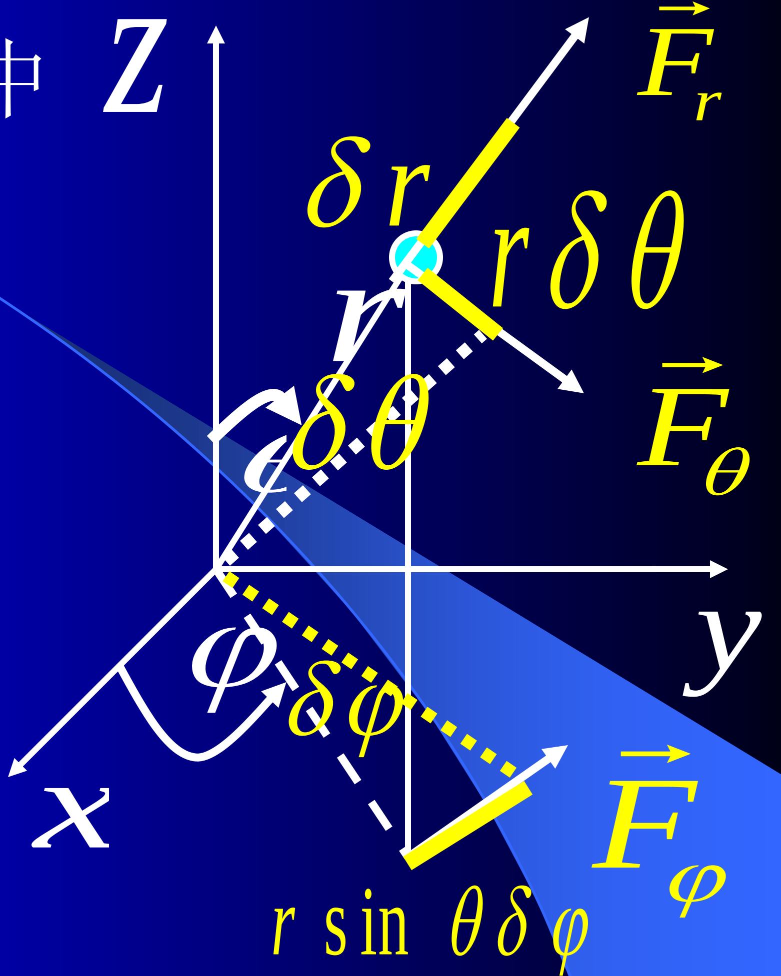
线量广义坐标：广义力即为力；
角量广义坐标：广义力即为力力矩。

- 不要将广义力和力混淆

已知:自由质点在球坐标系中

受力为$\vec{F}_r \quad \vec{F}_\theta \quad \vec{F}_\varphi$

求：广义力



解:

$$Q_r = F_r$$

$$Q_\theta = rF_\theta$$

$$Q_\varphi = r\sin\theta F_\varphi$$

$$\delta w = Q_r \delta r + \underline{Q_\theta} \delta\theta + Q_\varphi \delta\varphi$$

$$\delta w = F_r \delta r + \underline{rF_\theta} \delta\theta + r\sin\theta F_\varphi \delta\varphi$$

定义法求解：坐标（x,y,z）,广义坐标（r,θ,φ）

$$Q_\alpha = \vec{F}_i \bullet \frac{\partial \vec{r}_i}{\partial q_\alpha} = F_{ix} \bullet \frac{\partial x_i}{\partial q_\alpha} + F_{iy} \bullet \frac{\partial y_i}{\partial q_\alpha} + F_{iz} \bullet \frac{\partial z_i}{\partial q_\alpha} \quad (i = 1, 2, 3)$$

$$Q_r = F_x \bullet \frac{\partial x}{\partial r} + F_y \bullet \frac{\partial y}{\partial r} + F_z \bullet \frac{\partial z}{\partial r}$$

$$\cdots$$

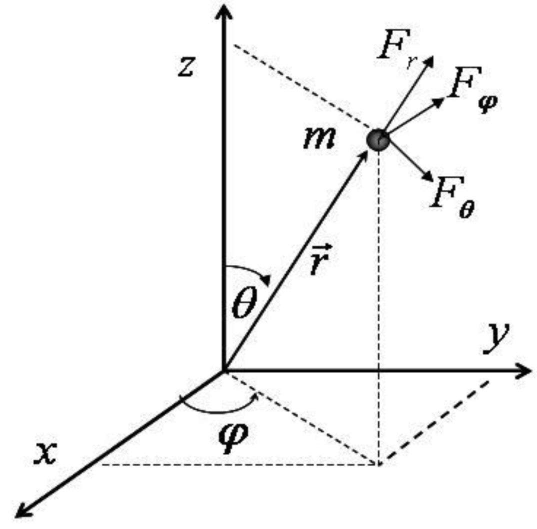
$$Q_\varphi = F_x \bullet \frac{\partial x}{\partial \varphi} + F_y \bullet \frac{\partial y}{\partial \varphi} + F_z \bullet \frac{\partial z}{\partial \varphi}$$

$$x = r \sin\theta \cos\varphi,$$
$$y = r \sin\theta \sin\varphi,$$
$$z = r \cos\theta,$$

$$Q_r = F_x \sin\theta \cos\varphi + F_y \sin\theta \sin\varphi + F_z \cos\theta,$$
$$Q_\theta = F_x r \cos\theta \cos\varphi + F_y r \cos\theta \sin\varphi - F_z \sin\theta,$$
$$Q_\varphi = -F_x r \sin\theta \sin\varphi + F_y \sin\theta \cos\varphi.$$

由于 $\vec{F}_r\ \ \vec{F}_\theta\ \ \vec{F}_\phi$ 坐标坐标系中投影为：

$$F_x = F_r \sin\theta\cos\varphi + F_\theta\cos\theta\cos\varphi - F_\varphi\sin\varphi,$$
$$F_y = F_r \sin\theta\sin\varphi + F_\theta\cos\theta\sin\varphi + F_\varphi\cos\varphi,$$
$$F_\varphi = F_r\cos\theta - F_\theta\sin\theta.$$



代入下式：

$$Q_r = F_x\sin\theta\cos\varphi + F_y\sin\theta\sin\varphi + F_z\cos\theta,$$
$$Q_\theta = F_x r\cos\theta\cos\varphi + F_y r\cos\theta\sin\varphi - F_z\sin\theta,$$
$$Q_\varphi = -F_x r\sin\theta\sin\varphi + F_y\sin\theta\cos\varphi.$$

得到： $Q_r = F_r$

$$Q_\theta = rF_\theta$$
$$Q_\phi = r\sin\theta F_\phi$$

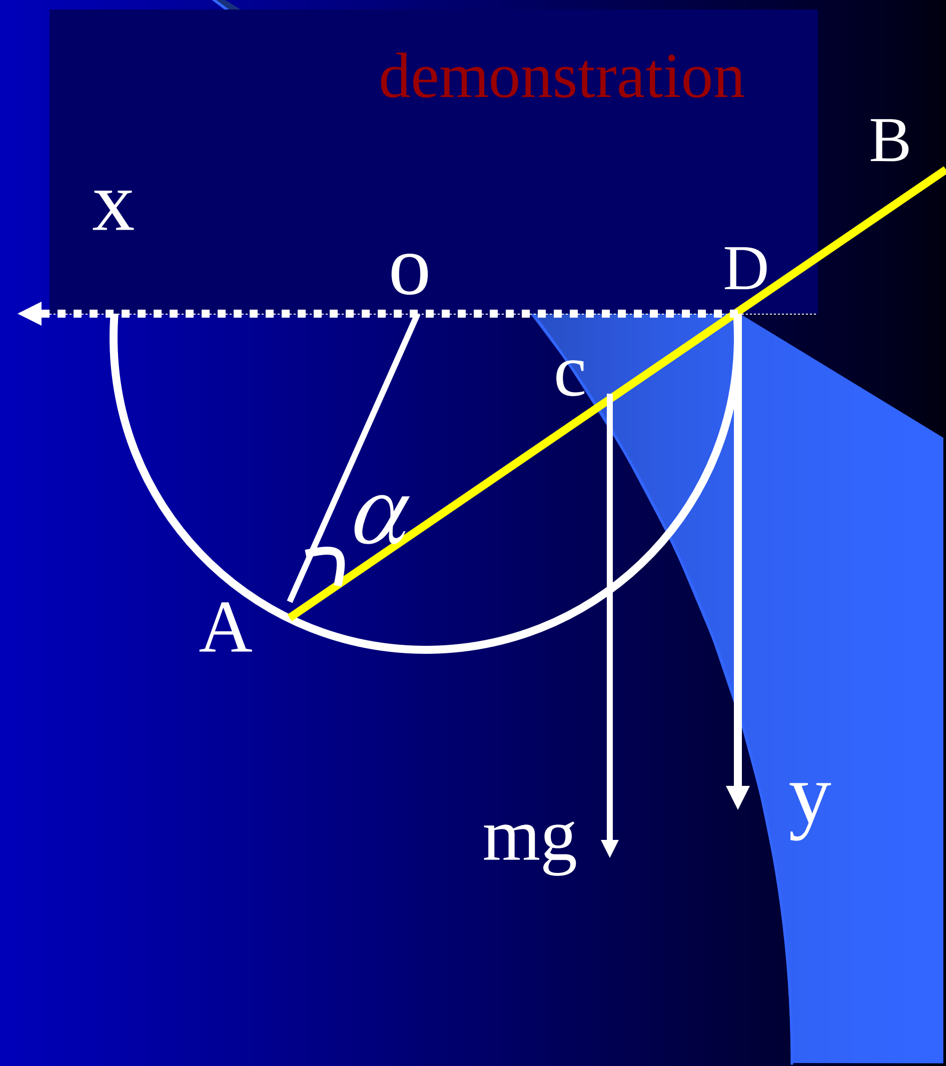
线量广义坐标：广义力即为力；
角量广义坐标：广义力即为力力矩。

虚功原理解题步骤

- 分析约束, 确定自由度, $\vec{r}_i$(坐标原点不动!)
- 选好广义坐标
- 写出主动力作用点的坐标并对其变分
- 代入虚功原理公式中求解

Attention:
- 静系中的平衡
- 只有广义坐标方可独立变化 $\delta q \neq 0$
- 只有正确写出 $\vec{r} = \vec{r}(q_\alpha, t)$
- 虚功原理中不出现约束力

例题1 半径为a的光滑半球形碗固定在水平面上。一匀质棒斜靠在碗缘，在碗内长度为c, 试用虚功原理求棒全长。



分析

坐标数 ⇨ 3

约束数目 ⇨ 2

自由度数目 ⇨ 1

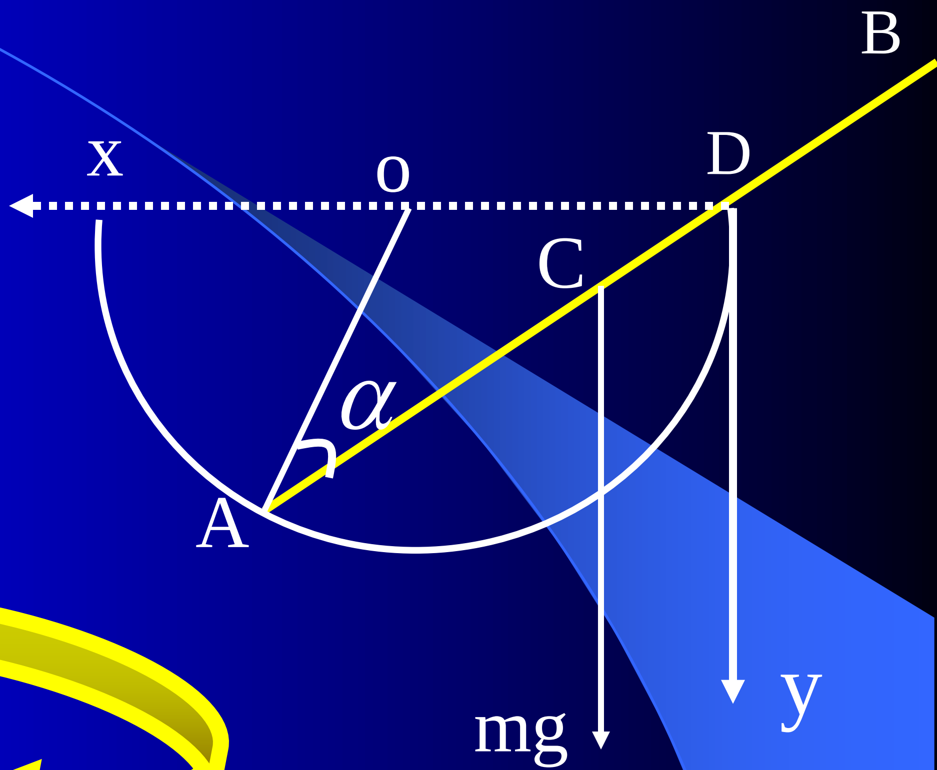
解：　取α为广义坐标，设杆长为

$$y_c = (AD - AC)\sin\alpha$$

$$= (c - \frac{l}{2})\sin\alpha$$

$$= (2a\cos\alpha - \frac{l}{2})\sin\alpha$$

$$\delta w = mg\delta y_c = 0$$

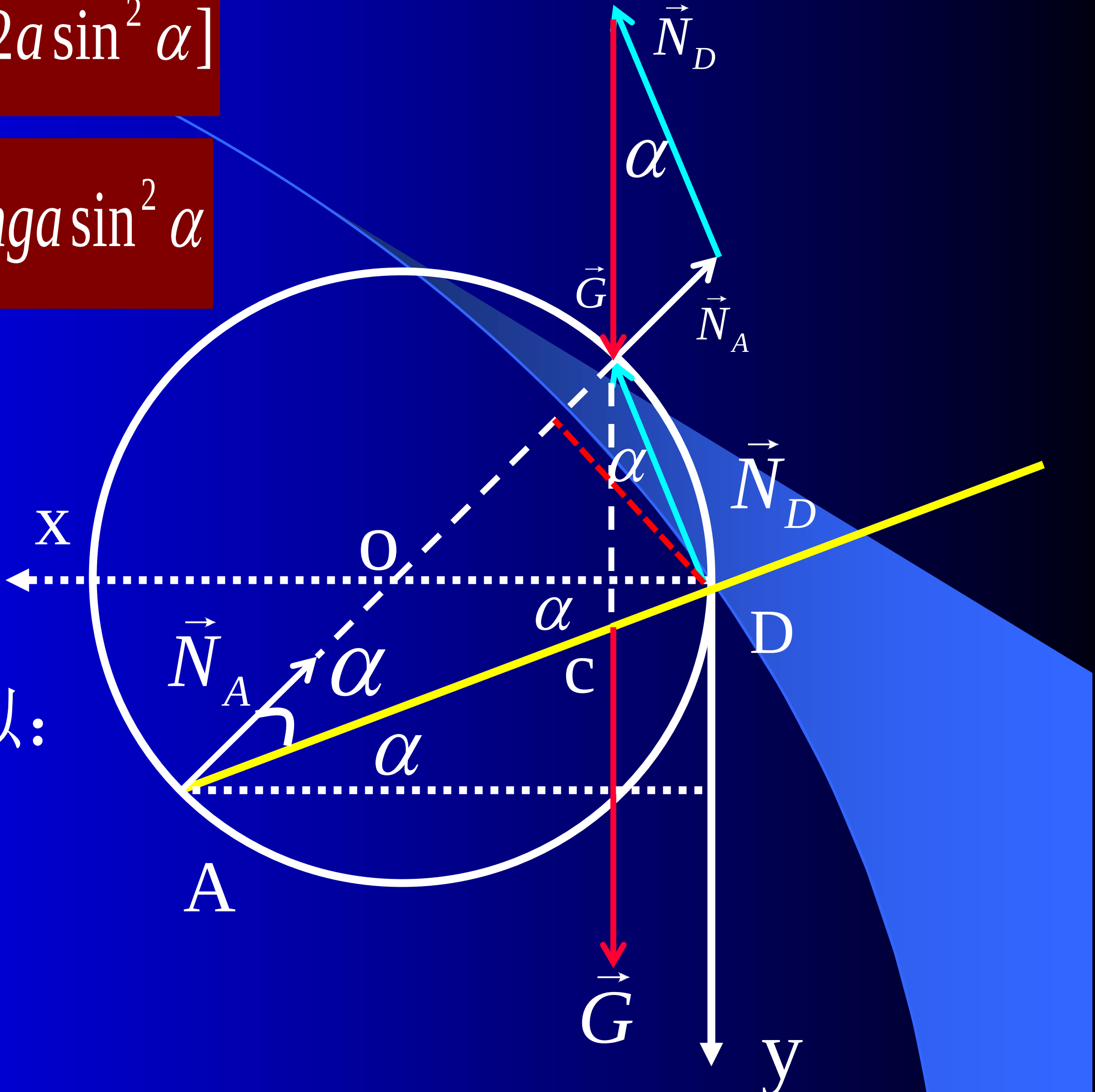


$$mg[(2a\cos\alpha - \frac{l}{2})\cos\alpha - 2a\sin^2\alpha]\delta\alpha = 0$$

$$Q_\alpha = mg[(2a\cos\alpha - \frac{l}{2})\cos\alpha - 2a\sin^2\alpha]$$

$$= mg(2a\cos\alpha - \frac{l}{2})\cos\alpha - 2mga\sin^2\alpha$$

$$= M_{\vec{G}} - M_{\vec{N}_A}$$



三力共点，三角形相似：

$$\frac{mg}{2a} = \frac{N_A}{\frac{2a\sin\alpha}{\cos\alpha}} = \frac{N_D}{\frac{l}{2}}$$

$$M_{N_A} = mg\frac{\sin\alpha}{\cos\alpha} \cdot 2a\sin\alpha\cos\alpha = 2mga\sin^2\alpha$$

$$mg\boxed{\left[\left(2a\cos\alpha-\frac{l}{2}\right)\cos\alpha-2a\sin^2\alpha\right]}\delta\alpha=0$$

$$=0$$

$$\because\ \delta\alpha\neq 0\quad !!!$$

$$\therefore l=\frac{4a(2\cos^2\alpha-1)}{\cos\alpha}$$

$$\because\ c=2a\cos\alpha$$

$$\therefore l=\frac{4(c^2-2a^2)}{c}$$

虚功原理中的功是“虚功”，质点没有真实运动；物理本质是？

$$Q_\alpha=-\frac{\partial V}{\partial q_\alpha}=0$$

系统总“势能”取极小值！

利用广义力解

$$y_c = (AD - AC)\sin\alpha = (2a\cos\alpha - \frac{l}{2})\sin\alpha$$

$$\boxed{Q_\alpha = \sum_i \vec{F}_i \bullet \frac{\partial \vec{r}_i}{\partial q_\alpha}} = mg\vec{j} \bullet \frac{\partial \vec{r}_c}{\partial \alpha} = 0$$

$$= mg[-2a\sin^2\alpha + 2a\cos^2\alpha - \frac{l}{2}\cos\alpha] = 0$$

$$mg[(2a\cos\alpha - \frac{l}{2})\cos\alpha - 2a\sin^2\alpha]\delta\alpha = 0$$

$$mg[(2a\cos\alpha - \frac{l}{2})\cos\alpha - 2a\sin^2\alpha] = 0$$

例二　长为l,质量为m的杆OA,OB光滑地连于A点,O点用光滑铰链固定.系统置于竖直面内.B端作用一水平恒力$\vec{F}$,试用虚功原理求两杆平衡位置.

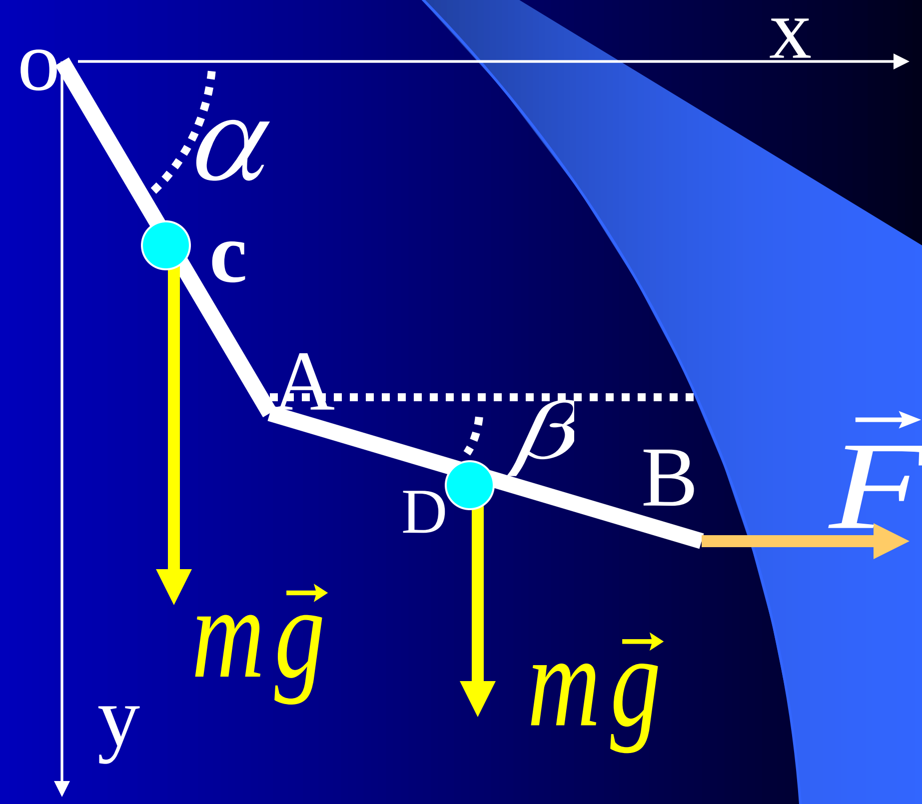
分析

坐标数 ⇒ 4

约束 ⇒ 2

自由度 ⇒ 2

广义坐标 ⇒ α , β



demonstration

解： 取如图所示α，β为广义坐标

$$y_c = \frac{l}{2}\sin\alpha \qquad y_D = l\sin\alpha + \frac{l}{2}\sin\beta$$

$$\delta y_c = \frac{l}{2}\cos\alpha\,\delta\alpha \qquad \delta y_D = l\cos\alpha\,\delta\alpha + \frac{l}{2}\cos\beta\,\delta\beta$$

$$x_B = l(\cos\alpha + \cos\beta) \qquad \delta x_B = -l(\sin\alpha\,\delta\alpha + \sin\beta\,\delta\beta)$$

$$\delta w = mg\,\delta y_C + mg\,\delta y_D + F\,\delta x_B$$

$$\delta w = mg\,\delta y_C + mg\,\delta y_D + F\,\delta x_B$$

$$\left\{\begin{aligned} \delta y_c &= \frac{l}{2}\cos\alpha\,\delta\alpha \\ \delta y_D &= l\cos\alpha\,\delta\alpha + \frac{l}{2}\cos\beta\,\delta\beta \\ \delta x_B &= -l(\sin\alpha\,\delta\alpha + \sin\beta\,\delta\beta) \end{aligned}\right.$$

$$\delta w = (\frac{3l}{2}mg\cos\alpha - Fl\sin\alpha)\delta\alpha + (\frac{l}{2}mg\cos\beta - Fl\sin\beta)\delta\beta = 0$$

$$\delta w = \left(\frac{3l}{2} mg\cos\alpha - Fl\sin\alpha\right)\delta\alpha + \left(\frac{l}{2} mg\cos\beta - Fl\sin\beta\right)\delta\beta = 0$$

$\because$ $\delta\alpha$ 和 $\delta\beta$ 可任意变化且 $\neq 0$!!!

$$\begin{cases} \left(\frac{3l}{2} mg\cos\alpha - Fl\sin\alpha\right) = 0 \\ \left(\frac{l}{2} mg\cos\beta - Fl\sin\beta\right) = 0 \end{cases}$$

$$\begin{cases} \left(\frac{3l}{2} mg\cos\alpha - Fl\sin\alpha\right) = 0 \\ \left(\frac{l}{2} mg\cos\beta - Fl\sin\beta\right) = 0 \end{cases}$$

$$\begin{cases} tg\alpha = \frac{3mg}{2F} \\ tg\beta = \frac{mg}{2F} \end{cases}$$

利用广义力解

$$\because\ \vec{F}_c = mg\,\vec{j} \qquad \vec{F}_d = mg\,\vec{j} \qquad \vec{F}_B = F\,\vec{i}$$

$$\vec{r}_c = \frac{l}{2}\sin\alpha\,\vec{j} + \frac{l}{2}\cos\alpha\,\vec{i}$$

$$\vec{r}_d = \left(l\cos\alpha + \frac{l}{2}\cos\beta\right)\vec{i} + \left(l\sin\alpha + \frac{l}{2}\sin\beta\right)\vec{j}$$

$$\vec{r}_B = \left(l\cos\alpha + l\cos\beta\right)\vec{i} + \left(l\sin\alpha + l\sin\beta\right)\vec{j}$$

$$\boxed{Q_\alpha = \sum_i \vec{F}_i \bullet \frac{\partial \vec{r}_i}{\partial q_\alpha}} = \vec{F}_c \bullet \frac{\partial \vec{r}_c}{\partial \alpha} + \vec{F}_D \bullet \frac{\partial \vec{r}_D}{\partial \alpha} + \vec{F}_B \bullet \frac{\partial \vec{r}_B}{\partial \alpha}$$

$$= \frac{3}{2}mgl\cos\alpha - Fl\sin\alpha = 0$$

$$tg\alpha = \frac{3mg}{2F}$$

$$\begin{cases} \vec{r}_c = \frac{l}{2}\sin\alpha\,\vec{j} + \frac{l}{2}\cos\alpha\,\vec{i} \\ \vec{r}_d = \left(l\cos\alpha + \frac{l}{2}\cos\beta\right)\vec{i} + \left(l\sin\alpha + \frac{l}{2}\sin\beta\right)\vec{j} \\ \vec{r}_B = \left(l\cos\alpha + l\cos\beta\right)\vec{i} + \left(l\sin\alpha + l\sin\beta\right)\vec{j} \end{cases}$$

$$Q_\beta = \sum_i \vec{F}_i \bullet \frac{\partial \vec{r}_i}{\partial q_\beta} = mg\frac{\partial y_c}{\partial \beta} + mg\frac{\partial y_d}{\partial \beta} + F\frac{\partial x_B}{\partial \beta}$$

$$\boxed{= \left(mg\frac{l}{2}\cos\beta - Fl\sin\beta\right)} = 0$$

$$tg\beta = \frac{mg}{2F}$$

例:长为l的四根轻杆光滑地连成菱形ABCD,AB和AD在E和F两点支于光滑钉子上,EF＝2a,BD间用一轻绳连接,C点系一重为W的重物,菱形的顶角为2α,试用虚功原理求平衡时绳中的张力.

分析

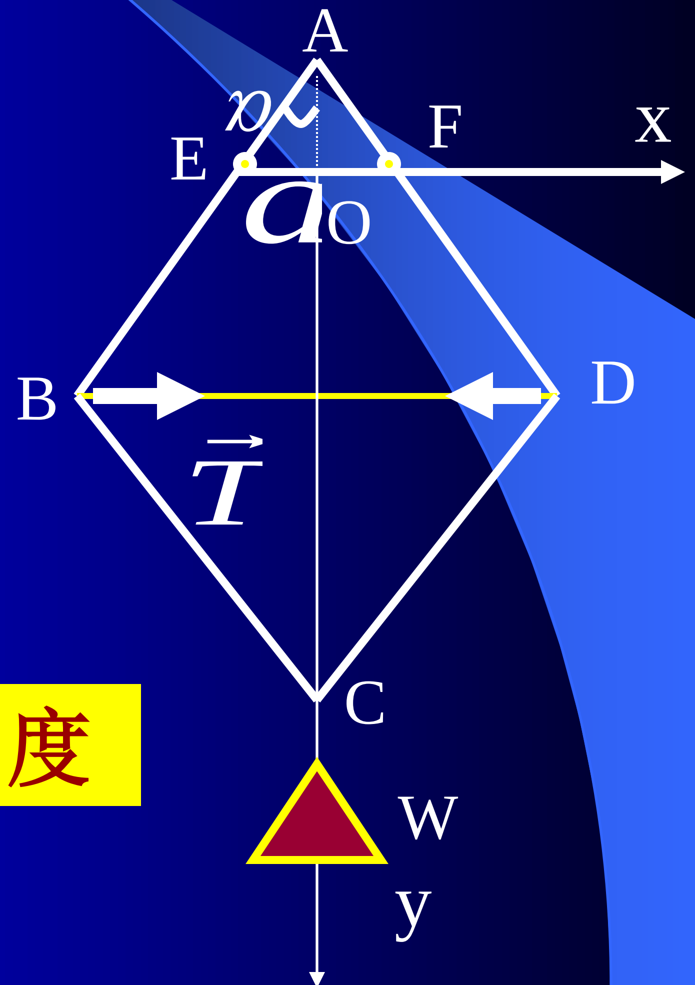
坐标数 ⇒ 6

约束 ⇒ 6

自由度 ⇒ 0 ?!!!

解除一个约束 ⇒ 一个自由度

广义坐标 ⇒ α



解: 取 α 为广义坐标

$$y_c = 2l\cos\alpha - a\,\mathrm{ctg}\,\alpha \qquad x_D = l\sin\alpha$$

$$\delta y_c = [-2l\sin\alpha + a\csc^2\alpha]\delta\alpha \qquad \delta x_D = l\cos\alpha\,\delta\alpha$$

$$\delta W = w\,\delta y_c - 2T\,\delta x_D = 0$$

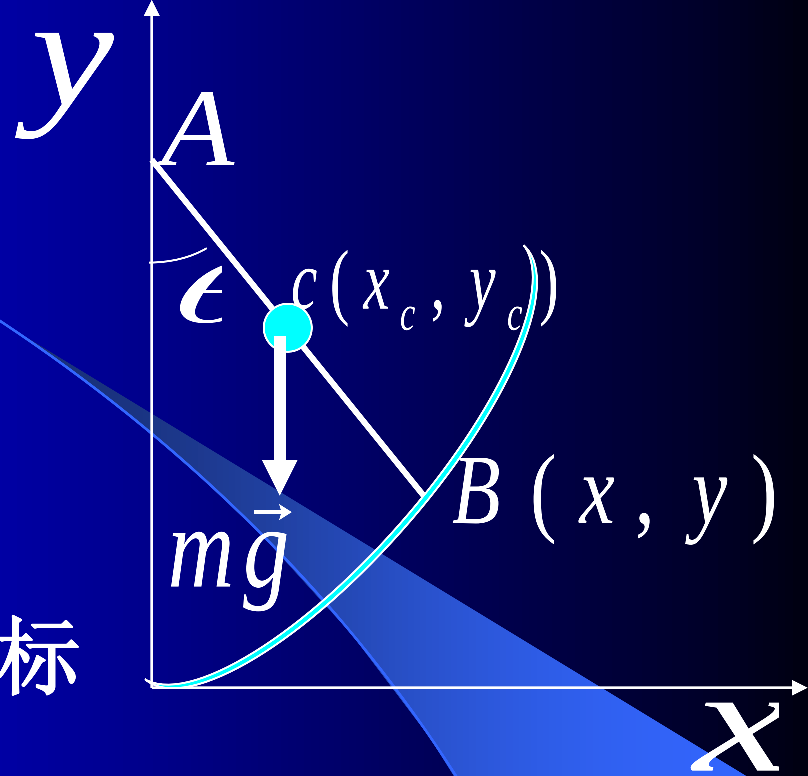
$$[w(a\csc^2\alpha - 2l\sin\alpha) - 2Tl\cos\alpha]\delta\alpha = 0$$

$\because$ $\delta\alpha$ 可任意变化且 $\neq 0$

$$T = \frac{w}{2l\cos\alpha}(a\csc^2\alpha - 2l\sin\alpha) = w\tan\alpha\left(\frac{a}{2l}\csc^3\alpha - 1\right)$$

例四

长为　　的匀质杆AB一端靠在光滑墙上，另一端靠在光滑固定曲面上，如果杆在与竖直墙间的夹角< $\frac{\pi}{2}$ 的任意位置均能平衡，试求曲面形状.



解：　取如图所示θ为广义坐标

$$y_c = y + \frac{l}{2}\cos\theta$$

$$\delta y_c = \delta y - \frac{l}{2}\sin\theta\,\delta\theta$$

$$= ???$$

$$\boxed{\delta y = \frac{dy}{d\theta}\delta\theta}$$

$$y = y(x(\theta))$$

由虚功原理有

$$\delta w = mg\delta y_c = 0 \qquad \delta y_c = 0$$

$$\delta y_c = (\frac{dy}{d\theta} - \frac{l}{2}\sin\theta)\delta\theta = 0$$

$$\because \ \delta\theta \neq 0 \qquad \therefore \frac{dy}{d\theta} - \frac{l}{2}\sin\theta = 0$$

$$\therefore \frac{dy}{dx}\frac{dx}{d\theta} - \frac{l}{2}\sin\theta = 0$$

$$\because \ x = l\sin\theta \qquad \therefore \frac{dx}{d\theta} = l\cos\theta$$

$$\therefore \frac{dy}{dx}\frac{dx}{d\theta}-\frac{l}{2}\sin\theta=0$$

$$\frac{dx}{d\theta}=l\cos\theta$$

$$\frac{dy}{dx}=\frac{l}{2}tg\theta=\frac{l}{2}\frac{x}{\sqrt{l^2-x^2}}$$

积分上式

$$y=-\frac{1}{2}\sqrt{l^2-x^2}+c$$

$$\because x=0, \quad y=0 \quad \therefore c=\frac{l}{2}$$

$$y=\frac{l}{2}-\frac{1}{2}\sqrt{l^2-x^2}$$

$$\frac{x^2}{l^2}+\frac{(y-\frac{l}{2})^2}{(\frac{l}{2})^2}=1$$

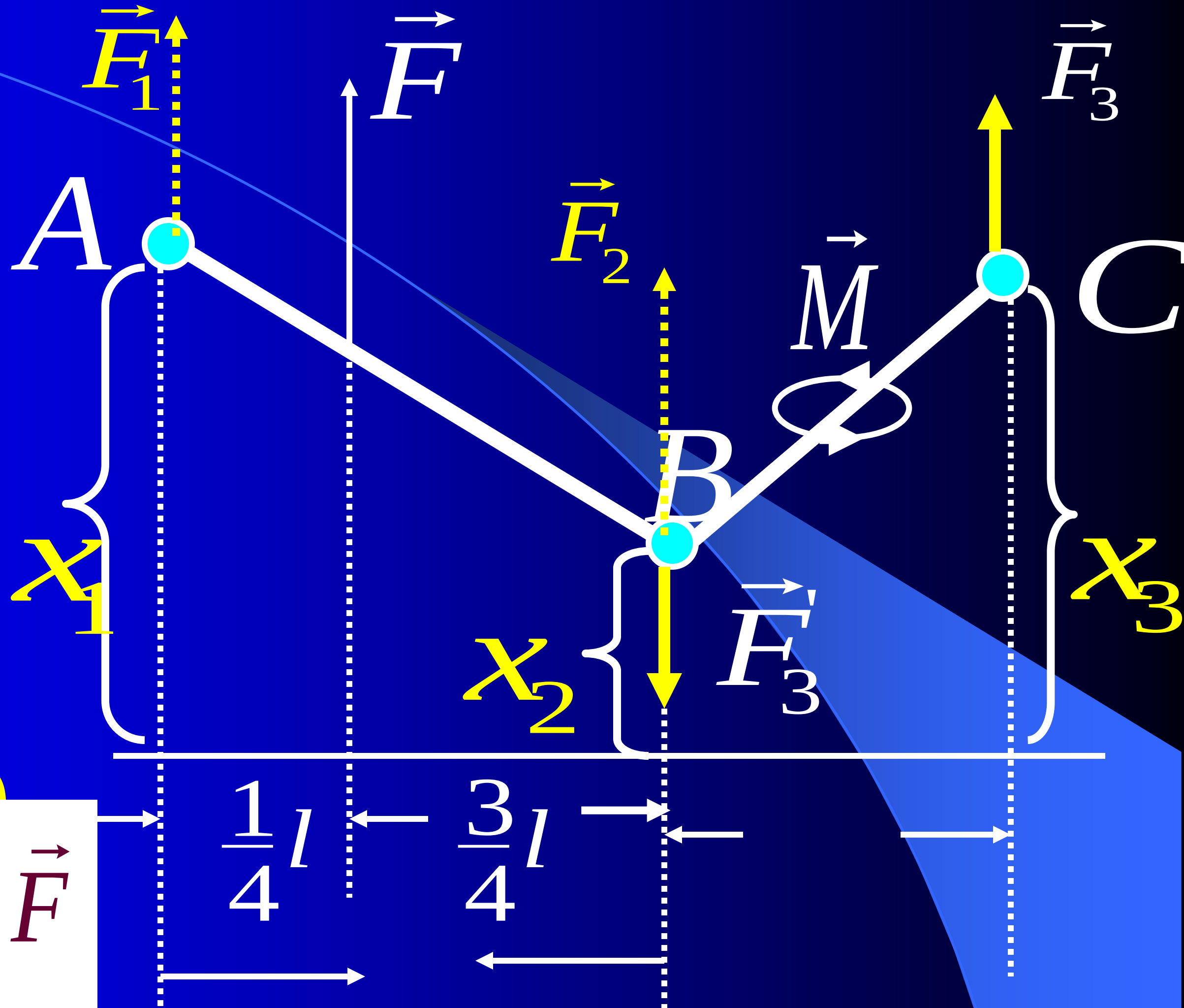
P162,10-3

取广义坐标为

$$q_1 = \frac{1}{3}(x_1 + x_2 + x_3)$$

$$q_2 = \frac{1}{2}(x_1 - x_3)$$

$$q_3 = \frac{1}{3}(x_1 - 2x_2 + x_3)$$



$$\vec{F}_1 = \frac{3}{4}\vec{F} \qquad \vec{F}_2 = \frac{1}{4}\vec{F}$$

$$\vec{F}_3 = \frac{\vec{M}}{l} \qquad \vec{F}_3' = -\frac{\vec{M}}{l}$$

$Q_3 = ?$

$$\because \ \delta W = Q_1\delta q_1 + Q_2\delta q_2 + Q_3\delta q_3 \qquad (1)$$

$$\because \ \delta W = \vec{F}_1\delta x_1 + \vec{F}_2\delta x_2 + \vec{F}_3\delta x_3 + \vec{F}_3^{\,'}\delta x_2 \qquad (2)$$

作用在系统上的广义力等效为:

$$Q_1 = F$$

$$Q_2 = \frac{3}{4}F - \frac{M}{l}$$

$$Q_3 = \frac{1}{8}F + \frac{3M}{2l}$$

$$\begin{cases} \delta x_1 = \delta q_1 + \delta q_2 + \frac{1}{2}\delta q_3 \\ \delta x_2 = \delta q_1 - \delta q_3 \\ \delta x_3 = \delta q_1 - \delta q_2 + \frac{1}{2}\delta q_3 \end{cases}$$

$$\because \ \delta W = F\delta q_1 + \left(\frac{3}{4}F - \frac{M}{l}\right)\delta q_2 + \left(\frac{1}{8}F + \frac{3M}{2l}\right)\delta q_3 \qquad (3)$$

§3.完整系的拉格朗日方程(Lagrange's Equation for Holonomic System)

一.达朗贝尔----拉格朗日方程

牛顿力学
- 平衡方程 ⇒ $\vec{F}_i + \vec{N}_i = 0 \quad or \quad \sum_{i=1}^{n} \vec{M}_i = 0$
- 动力学方程 ⇒ $m_i \ddot{\vec{r}}_i = \vec{F}_i + \vec{N}_i$

达朗贝尔原理 ⇒ $\vec{F}_i + \vec{N}_i - m_i \ddot{\vec{r}}_i = 0$

达朗贝尔--拉格朗日方程

$$(\vec{F}_i - m_i \ddot{\vec{r}}_i) \bullet \delta \vec{r}_i = 0 \quad (i = 1.2....N)$$

二. 完整系的拉格朗日方程

f(q,t): q_1, q_2, ⋮, q_α, q_s, t

两个重要公式

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial f}{\partial q_\alpha}\right) = \dfrac{\partial \dot{f}}{\partial q_\alpha} \\ \dfrac{\partial \dot{f}}{\partial \dot{q}_\alpha} = \dfrac{\partial f}{\partial q_\alpha} \end{cases}$$

证明:

$$\because \dot{f} = \frac{\partial f}{\partial q_\alpha}\dot{q}_\alpha + \frac{\partial f}{\partial t} \qquad (\alpha = 1.2\ldots.S)$$

$\dot{q}_\alpha = \dfrac{dq_\alpha}{dt} \Leftrightarrow$ 广 义 速 度

$\because \dfrac{\partial f}{\partial t}$ 和 $\dfrac{\partial f}{\partial q_\alpha}$ 与 $\dot{q}_\alpha$ 无关 !!!

$$\therefore \frac{\partial \dot{f}}{\partial \dot{q}_\alpha} = \frac{\partial f}{\partial q_\alpha}$$

$$\because \frac{d}{dt}\left(\frac{\partial f}{\partial q_\alpha}\right) = \frac{\partial}{\partial q_\beta}\left(\frac{\partial f}{\partial q_\alpha}\right)\left(\frac{dq_\beta}{dt}\right) + \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial q_\alpha}\right)\frac{dt}{dt}$$

(β=1 2 3…s)

$$= \frac{\partial}{\partial q_\alpha}\left[\frac{\partial f}{\partial q_\beta}\dot{q}_\beta + \frac{\partial f}{\partial t}\right]$$

=?

$$\Updownarrow$$

$$\dot{f}$$

$$\boxed{\therefore \frac{d}{dt}\left(\frac{\partial f}{\partial q_\alpha}\right) = \frac{\partial \dot{f}}{\partial q_\alpha}}$$

完整系的拉格朗日方程

$$(\vec{F}_i - m_i\ddot{\vec{r}}_i)\bullet\delta\vec{r}_i = 0 \quad (i = 1.2....N)$$

$$\because\ \vec{r} = \vec{r}(q_1, q_2, ...q_s, t)$$

$$\therefore\ \delta\vec{r}_i = \frac{\partial\vec{r}_i}{\partial q_\alpha}\delta q_\alpha \quad (\alpha = 1.2...S)$$

$$\vec{F}_i\bullet\delta\vec{r}_i = \vec{F}_i\bullet\frac{\partial\vec{r}_i}{\partial q_\alpha}\delta q_\alpha = Q_\alpha\delta q_\alpha \quad (i = 1.2...N)$$

$$-m_i\ddot{\vec{r}}_i\delta\vec{r}_i = -m_i\ddot{\vec{r}}\frac{\partial\vec{r}_i}{\partial q_\alpha}\delta q_\alpha \quad (i = 1.2...N)$$

$$= -[\frac{d}{dt}(m_i\dot{\vec{r}}_i\frac{\partial\vec{r}_i}{\partial q_\alpha}) - m_i\dot{\vec{r}}_i\frac{d}{dt}(\frac{\partial\vec{r}_i}{\partial q_\alpha})]\delta q_\alpha$$

$$-m_i\ddot{\vec{r}}_i\delta\vec{r}_i = -\left[\frac{d}{dt}\left(m_i\dot{\vec{r}}_i\frac{\partial \vec{r}_i}{\partial q_\alpha}\right) - m_i\dot{\vec{r}}_i\frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_\alpha}\right)\right]\delta q_\alpha$$

$$\frac{d}{dt}\left(\frac{\partial f}{\partial q_\alpha}\right) = \frac{\partial \dot{f}}{\partial q_\alpha}$$

$$\frac{\partial f}{\partial q_\alpha} = \frac{\partial \dot{f}}{\partial \dot{q}_\alpha}$$

$$-\left[\frac{d}{dt}\left(m_i\dot{\vec{r}}_i\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_\alpha}\right) - m_i\dot{\vec{r}}_i\frac{\partial \dot{\vec{r}}_i}{\partial q_\alpha}\right]\delta q_\alpha$$

$$= -\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha}\right]\delta q_\alpha$$

$$(\alpha = 1\ 2\ 3\ldots\ldots s)$$

$$(i = 1\ 2\ 3\ldots\ldots N)$$

$$T = \frac{1}{2}m_i\dot{\vec{r}}_i^{\,2}$$

$\Leftrightarrow$ 系 统 的 动 能

$$\begin{cases} -m_i\ddot{\vec{r}}_i\delta\vec{r}_i = -[\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_\alpha}) - \frac{\partial T}{\partial q_\alpha}]\delta q_\alpha \\ \vec{F}_i \bullet \delta\vec{r}_i = Q_\alpha \delta q_\alpha \quad (i = 1.2 \ldots N) \end{cases}$$

$$(\vec{F}_i - m_i\ddot{\vec{r}}_i)\bullet\delta\vec{r}_i = 0 \quad (i = 1.2\ldots.N)$$

$$\{[\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_\alpha}) - \frac{\partial T}{\partial q_\alpha}] - Q_\alpha\}\delta q_\alpha = 0$$

$\because$ δq_α 可 任 意 变 化 化 且 $\neq 0$!!!

$$\therefore \frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_\alpha}) - \frac{\partial T}{\partial q_\alpha} = Q_\alpha \quad (\alpha = 1.2\ldots S)$$

对保守系

$$Q_\alpha = \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_\alpha} = -\frac{\partial V}{\partial \vec{r}_i}\frac{\partial \vec{r}_i}{\partial q_\alpha} = -\frac{\partial V}{\partial q_\alpha}$$

$$T = T(q\ \dot{q}\ t) \qquad V = V(q)$$

$$\therefore \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = Q_\alpha = -\frac{\partial V}{\partial q_\alpha} \qquad (\alpha = 1.2\ldots S)$$

$$\therefore \frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}_\alpha}\right) - \frac{\partial (T-V)}{\partial q_\alpha} = 0 \qquad (\alpha = 1.2\ldots S)$$

L=T-V

拉格朗日函数

$$\therefore \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = 0$$

Summary:

- T= $T(q\ \dot{q}\ t)$ V=V(q)
- L=T-V $L = L(q\ \dot{q}\ t)$
- L 和 cL描述同一力学系统.c是任意常数
- L 和 $L+\dfrac{df}{dt}$ 描述同一力学系统 **规范不变性**

$f = f(q_1、q_2、q_3 \ldots\ldots q_s, t)$

拉格朗日函数不唯一!!!

- L可以给出力学系统的所有信息

L的标度特性

将空间尺度和时间尺度分别放大

$\vec{r}' = \alpha \vec{r} \qquad t' = \beta t$

相应的势能和动能的变化

$T' = \frac{\alpha^2}{\beta^2} T, \quad V' = \alpha^k V$

拉氏函数的改变

$L' = a^k L \qquad a$为常数

$$a^k = \frac{\alpha^2}{\beta^2} \qquad \frac{t'}{t} = \beta = a^{1-\frac{k}{2}} = [\frac{r'}{r}]^{1-\frac{k}{2}}$$

$$a^k = \frac{\alpha^2}{\beta^2} \quad \frac{t'}{t} = \beta = a^{1-\frac{k}{2}} = [\frac{r'}{r}]^{1-\frac{k}{2}}$$

Discussion:

$k = 2 \quad or \quad \beta = 1$ ⇒ 简谐振动的周期不变

$k = 1$ ⇒ 相当于重力场中自由落体

$k = -1$ ⇒ 相当于引力场的椭圆轨道开普勒第三定律

r视为椭圆轨道长轴

拉格朗日方称是牛顿运动方程在位形空间的投影

$$\boxed{\vec{F}_i = m_i\ddot{\vec{r}}_i = \frac{\mathrm{d}\,\overrightarrow{p_i}}{\mathrm{d}t}}$$

选 取 基 矢： $\vec{e}_\alpha \equiv \dfrac{\partial \vec{r}_i}{\partial q_\alpha}$

投影到位形空间：

$$\frac{\mathrm{d}\,\overrightarrow{p_i}}{\mathrm{d}t}\frac{\partial \vec{r}_i}{\partial q_\alpha} = F_i\frac{\partial \vec{r}_i}{\partial q_\alpha}$$

$$\frac{\mathrm{d}\,\overrightarrow{p_i}}{\mathrm{d}t}\frac{\partial \vec{r}_i}{\partial q_\alpha} = \frac{\mathrm{d}}{\mathrm{d}t}(\vec{p}_i\frac{\partial \vec{r}_i}{\partial q_\alpha}) - \vec{p}_i\frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial \vec{r}_i}{\partial q_\alpha}) = F_i\frac{\partial \vec{r}_i}{\partial q_\alpha} = Q_\alpha$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{\vec{r}}_i\frac{\partial \vec{r}_i}{\partial q_\alpha}) - m\dot{\vec{r}}\frac{\partial \dot{\vec{r}}_i}{\partial q_\alpha} = Q_\alpha \longrightarrow \frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial T}{\partial \dot{q}_\alpha}) - \frac{\partial T}{\partial q_\alpha} = Q_\alpha$$

对易与非对易关系

$$\delta \mathrm{d} = \mathrm{d}\delta,$$

$$\delta\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\delta, (\delta t = 0)$$

$$\frac{\partial}{\partial q_\alpha}\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial q_\alpha}$$

$$\frac{\partial}{\partial \dot{q}_\alpha}\frac{\mathrm{d}}{\mathrm{d}t} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_\alpha} = \frac{\partial}{\partial q_\alpha}$$

请证明！

拉格朗日方程：

非保守系：非保守力Q_α，又有保守力时：

保守系：

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right)-\frac{\partial L}{\partial q_\alpha}=Q_\alpha$$

特点

$T(q\ \dot{q}\ t)$ $V=V(q)$ $L=L(q\ \dot{q}\ t)$ $L=T-V$

$q,\dot{q}$ 为独立变量！

拉格朗日函数不唯一！ L和cL; L和$L+\frac{df}{dt}$

L可以给出力学系统的所有信息

拉氏方程是牛顿方程在位形空间的投影

拉氏方程是牛顿方程在位形空间的投影

主动力既有非保守力Q_α,又有保守力$-\dfrac{\partial V}{\partial q_\alpha}$时:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right)-\frac{\partial L}{\partial q_\alpha}=Q_\alpha, \qquad (\alpha=1,2,\ldots,s)$$

广义动量: $p_\alpha=\dfrac{\partial L}{\partial \dot{q}_\alpha}$, 拉格朗日力: $F_\alpha=Q_\alpha+\dfrac{\partial L}{\partial q_\alpha}$,

拉格朗日方程: $\dfrac{\mathrm{d}p_\alpha}{\mathrm{d}t}=F_\alpha$.

循环坐标q_k: $\dfrac{\partial T}{\partial q_k}=0$, 可遗坐标$q_l$: $\dfrac{\partial L}{\partial q_l}=0$

拉格朗日方程的应用

解题步骤

- 分析约束, 确定自由度
- 选好广义坐标
- 写出体系的动能和势能及拉格朗日函数
- 代入相应方程求解

非保守系:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)-\frac{\partial T}{\partial q_\alpha}=Q_\alpha \quad (\alpha=1.2\ldots S)$$

保守系:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right)-\frac{\partial L}{\partial q_\alpha}=0 \quad (\alpha=1.2\ldots S)$$

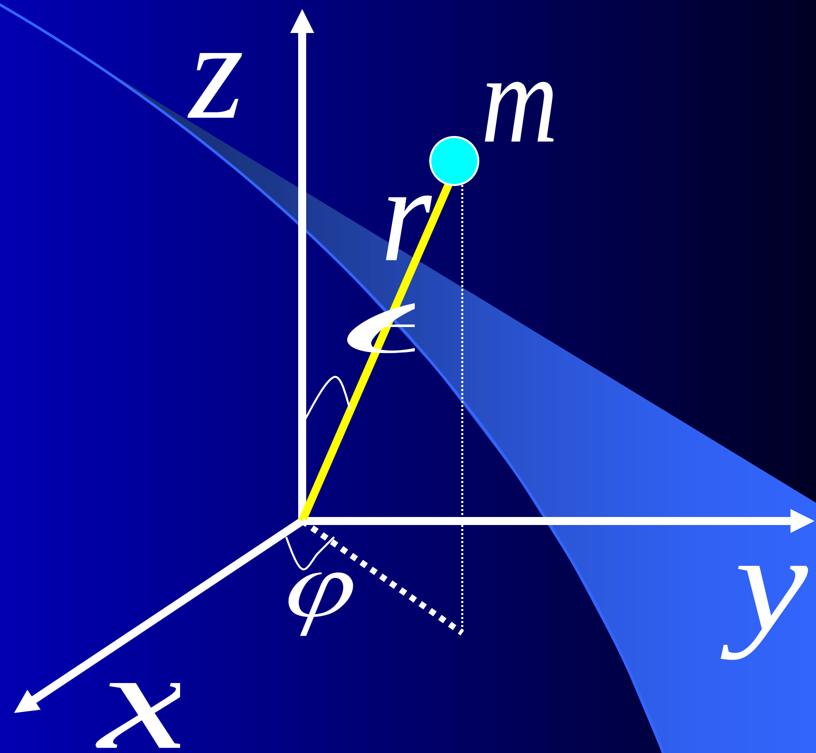
解题注意点

- 广义坐标选取至关重要
- 函数关系：$\begin{cases} T = T(q \; \dot{q} \; t) \\ V = V(q) \\ L = L(q \; \dot{q} \; t) \end{cases}$
- 动能形式柯尼西定理运用

$$T = T_c + T'$$

- T应是**绝对**动能

例1 用拉格朗日方程求自由质点在球坐标下广义力的表达式. 设其受力在r, θ, φ三个方向的分量分别为F_r, F_θ, F_φ



解: 广义力

↓

非保守系拉氏方程

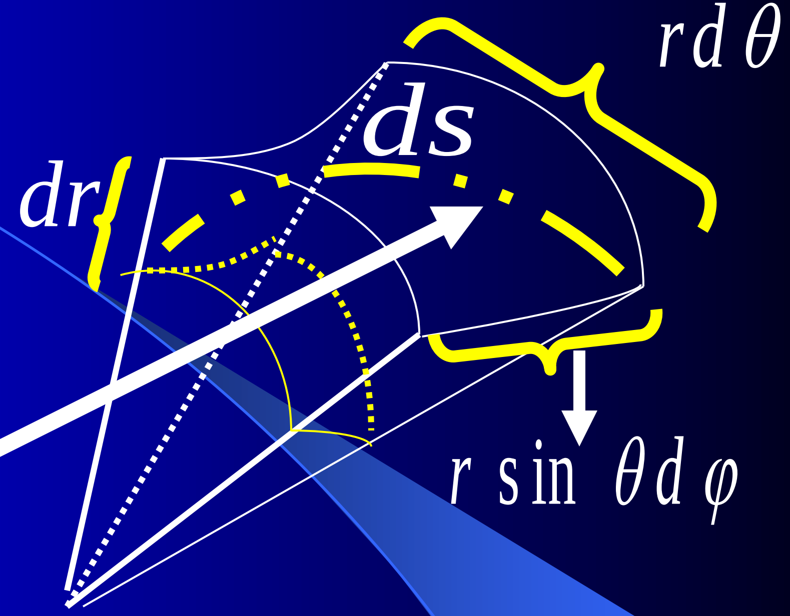
↓

必先求动能

$$x = r\sin\theta\cos\varphi$$

$$y = r\sin\theta\sin\varphi$$

$$z = r\cos\theta$$

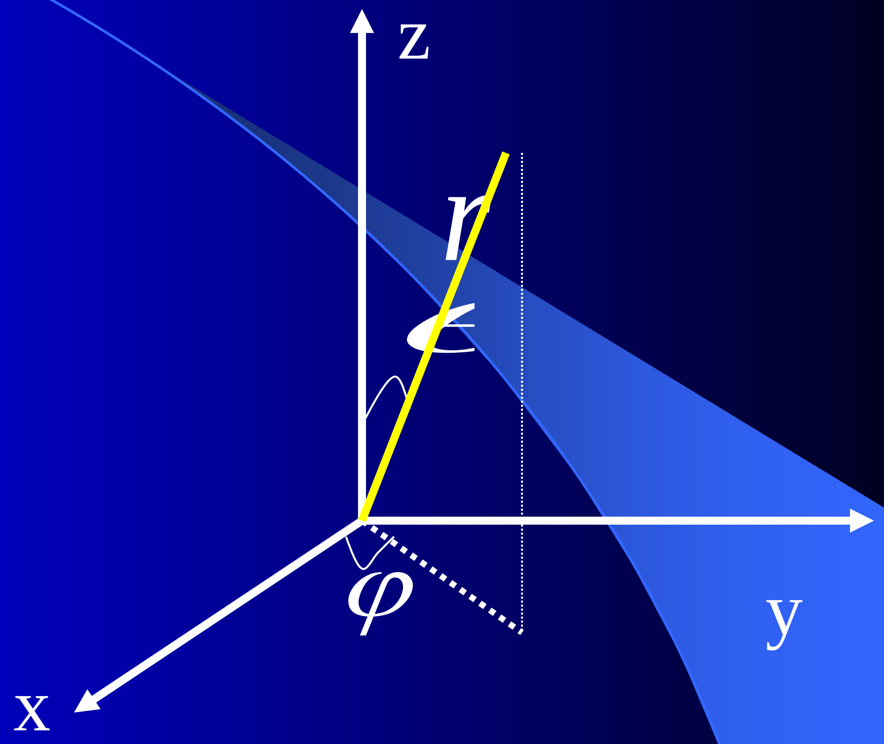


$$(ds)^2 = (dr)^2 + (rd\theta)^2 + (r\sin\theta d\varphi)^2$$

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2$$

$$r^2 = x^2 + y^2 + z^2$$

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$



$$T=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2+r^2\sin^2\theta\dot{\varphi}^2)$$

$$\begin{cases}\dfrac{\partial T}{\partial r}=mr\dot{\theta}^2+mr\sin^2\theta\dot{\varphi}^2\\[2ex]\dfrac{\partial T}{\partial \dot{r}}=m\dot{r}\end{cases}\Rightarrow\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right)=m\ddot{r}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right)-\frac{\partial T}{\partial r}=Q_r$$

$$Q_r=m(\ddot{r}-r\dot{\theta}^2-r\dot{\varphi}^2\sin^2\theta)$$

$$\because\ m(\ddot{r}-r\dot{\theta}^2-r\dot{\varphi}^2\sin^2\theta)=F_r\Rightarrow\boxed{Q_r=F_r}$$

$$T = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\,\dot{\varphi}^2)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta}$$

$$= m(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} - r^2\dot{\varphi}^2 \sin\theta\cos\theta) = Q_\theta$$

$$\because\ m(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\varphi}^2 \sin\theta\cos\theta) = F_\theta$$

$$Q_\theta = rF_\theta$$

$$T = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\dot{\varphi}^2)$$

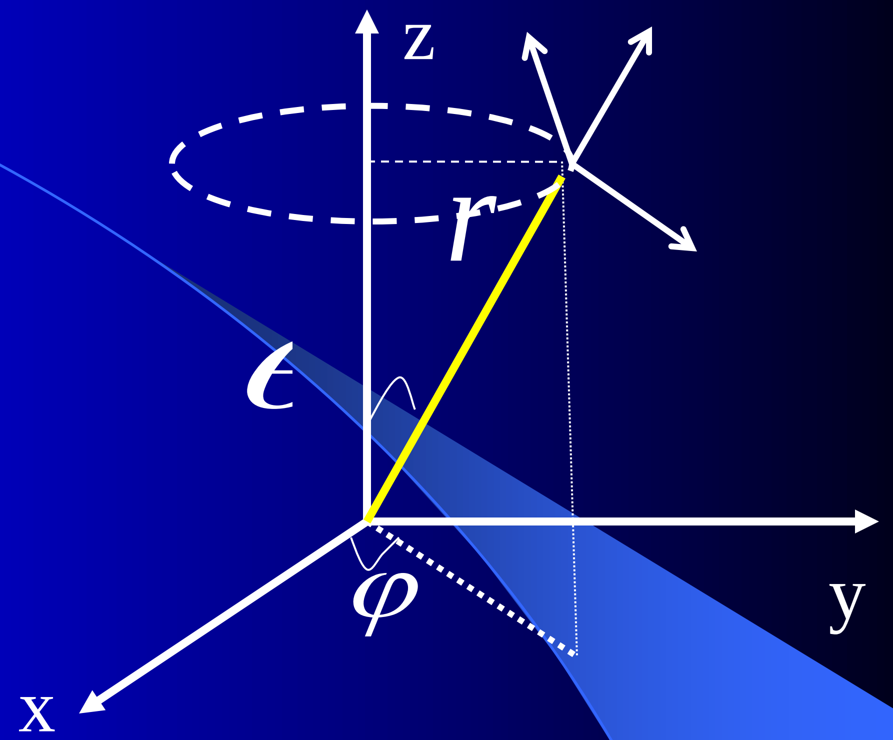
$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{\varphi}}) - \frac{\partial T}{\partial \varphi} = Q_\varphi$$

$$= r\sin\theta\{m(r\ddot{\varphi}\sin\theta + 2\dot{r}\dot{\varphi}\sin\theta + 2r\dot{\theta}\dot{\varphi}\cos\theta)\}$$

$$\because\ m(r\ddot{\varphi}\sin\theta + 2\dot{r}\dot{\varphi}\sin\theta + 2r\dot{\theta}\dot{\varphi}\cos\theta) = F_\varphi$$

$$Q_\varphi = rF_\varphi\sin\theta$$

$$
\begin{cases}
Q_\theta = rF_\theta \\
Q_\varphi = r\sin\theta F_\varphi \\
Q_r = F_r
\end{cases}
$$



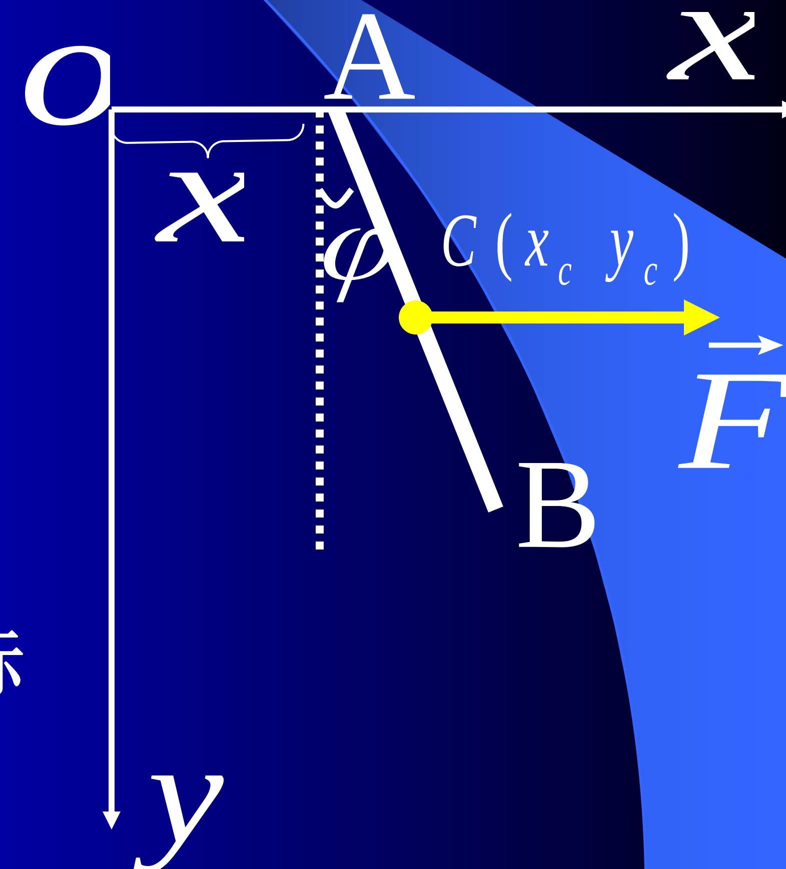
广义坐标　　广义力

$r \quad \rightarrow \quad$ 力F_r

$\theta, \varphi \rightarrow \quad$ 力矩

例2 质量为m，长为2a 的匀质棒AB，其A端可在光滑的水平导槽上滑动，而棒本身又可在竖直平面内绕A点摆动．C点受一水平恒力F作用，试用拉氏方程求其运动微分方程．

分析



坐标数: 3

约束数:

自由度: 2

取如图所示 x φ 为广义坐标

$$x_c = x + a\sin\varphi \qquad y_c = a\cos\varphi$$

$$\dot{x}_c = \dot{x} + a\dot{\varphi}\cos\varphi \qquad \dot{y}_c = -a\dot{\varphi}\sin\varphi$$

根据柯尼西定理

$$T = T_c + T'$$

$$T_c = \frac{1}{2}m(\dot{x}_c^2 + \dot{y}_c^2)$$

$$= \frac{1}{2}m(\dot{x}^2 + 2a\dot{x}\dot{\varphi}\cos\varphi + a^2\dot{\varphi}^2)$$

$$T' = \frac{1}{2}I_c\dot{\varphi}^2 = \frac{1}{2}\boxed{\frac{1}{12}m(2a)^2}\dot{\varphi}^2$$

$$T = T_c + T'$$
$$= \frac{1}{2} m(\dot{x}^2 + 2a\dot{\varphi}\dot{x}\cos\varphi + \frac{4}{3}a^2\dot{\varphi}^2)$$

$$\begin{cases} \dfrac{\partial T}{\partial \dot{x}} = m\dot{x} + ma\dot{\varphi}\cos\varphi & \dfrac{\partial T}{\partial x} = 0 \\ \dfrac{d}{dt}(\dfrac{\partial T}{\partial \dot{x}}) = m\ddot{x} + ma\ddot{\varphi}\cos\varphi - ma\dot{\varphi}^2 \sin\varphi \end{cases}$$

广义力 $Q_x = \sum_i \vec{F}_i \bullet \frac{\partial \vec{r}_i}{\partial x} = mg\frac{\partial y_c}{\partial x} + F\frac{\partial x_c}{\partial x}$

$$y_c = a\cos\varphi \qquad x_c = x + a\sin\varphi$$

$$Q_x = F$$

$$\begin{cases} m(\ddot{x} + a\ddot{\varphi}\cos\varphi - a\dot{\varphi}^2\sin\varphi) = F \\ m(\frac{4}{3}a^2\ddot{\varphi} + a\ddot{x}\cos\varphi) = Fa\cos\varphi - mga\sin\varphi \end{cases}$$

此题亦可用保存系拉格朗日方程

重力势能: $\Rightarrow$ $V_1 = -mgy_c = -mga\cos\varphi$

恒力势能: $\Rightarrow$ $V_2 = -Fx_c = -F(x + a\sin\varphi)$

$$V = V_1 + V_2 = -mga\cos\varphi - Fx - Fa\sin\varphi$$

$$L = T - V = \frac{1}{2}m(\dot{x}^2 + 2a\dot{\varphi}\dot{x}\cos\varphi + \frac{4}{3}a^2\dot{\varphi}^2) + mga\cos\varphi + Fx + Fa\sin\varphi$$

$$L = T - V = \frac{1}{2}m(\dot{x}^2 + 2a\dot{\varphi}\dot{x}\cos\varphi + \frac{4}{3}a^2\dot{\varphi}^2)$$
$$+ mga\cos\varphi + Fx + Fa\sin\varphi$$

$$\frac{\partial L}{\partial \dot{x}} = m(\dot{x} + a\dot{\varphi}\cos\varphi)$$

$$\frac{\partial L}{\partial x} = F$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m(\ddot{x} + a\ddot{\varphi}\cos\varphi - a\dot{\varphi}^2\sin\varphi)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$m(\ddot{x} + a\ddot{\varphi}\cos\varphi - a\dot{\varphi}^2\sin\varphi) = F$$

$$L = T - V = \frac{1}{2} m(\dot{x}^2 + 2a\dot{\varphi}\dot{x}\cos\varphi + \frac{4}{3}a^2\dot{\varphi}^2)$$

$$+ mga\cos\varphi + Fx + Fa\sin\varphi$$

$$\frac{\partial L}{\partial \varphi} = m[-a\dot{\varphi}\dot{x}\sin\varphi - mga\sin\varphi] + Fa\cos\varphi$$

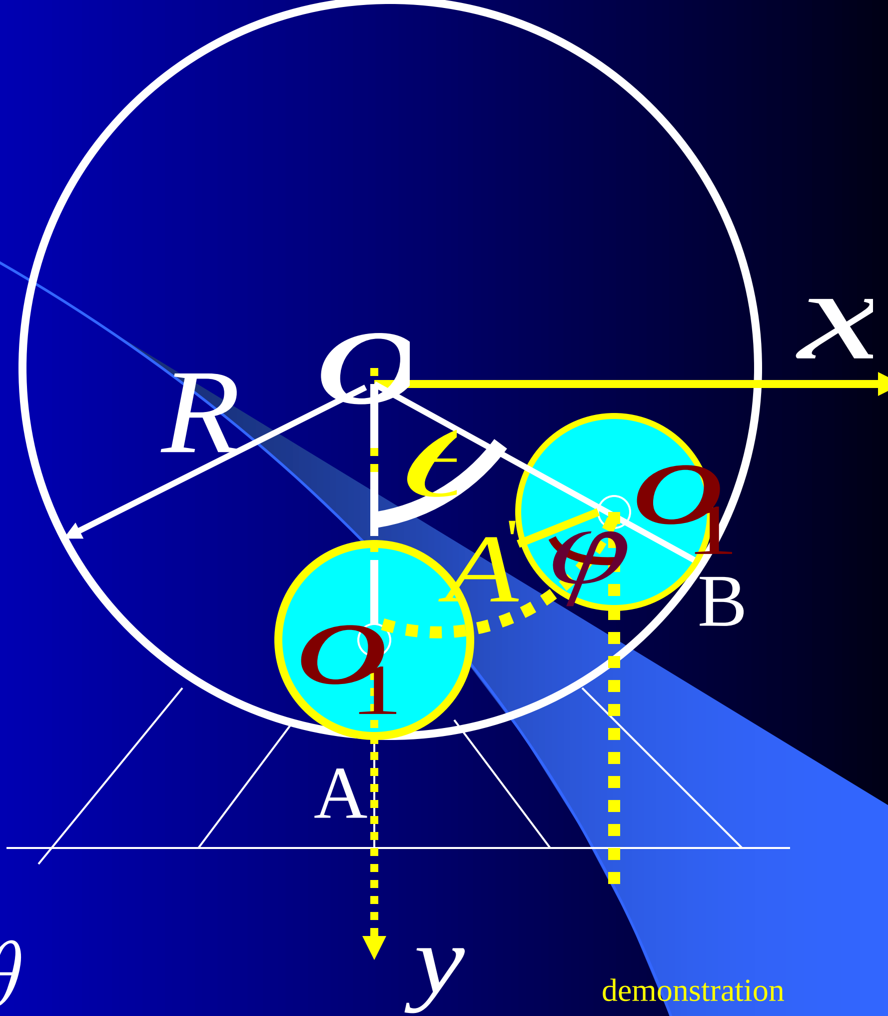
$$\frac{\partial L}{\partial \dot{\varphi}} = m[a\dot{x}\cos\varphi + \frac{4}{3}a^2\dot{\varphi}]$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} = m[a\ddot{x}\cos\varphi - a\dot{x}\dot{\varphi}\sin\varphi + \frac{4}{3}a^2\ddot{\varphi}]$$

$$m(\frac{4}{3}a^2\ddot{\varphi} + a\ddot{x}\cos\varphi) = Fa\cos\varphi - mga\sin\varphi$$

例3

一半径为r，质量为m的实心圆柱体在一半径为R的大圆柱体内表面作纯滚动，试用拉格朗日方程求其在平衡位置附近作微振动的周期.



demonstration

分析

$$\overset{\frown}{AB} = \overset{\frown}{A'B}$$

$$R\theta = r(\theta + \varphi)$$

$$\varphi = \frac{R-r}{r}\theta$$

坐标数 ⇒ 3

约束数 ⇒ 2 $\begin{cases} \varphi = \dfrac{R-r}{r}\theta \\ oo_1 = R - r \end{cases}$

自由度 ⇒

取θ为广义坐标

$$T = T_c + T'$$

$$T_c = \frac{1}{2} m V_c^2 = \frac{1}{2} m (R-r)^2 \dot{\theta}^2$$

$$T' = \frac{1}{2} I_c \dot{\varphi}^2 = \frac{1}{2}\frac{1}{2} m r^2 \left(\frac{R-r}{r}\right)^2 \dot{\theta}^2$$

$$\boxed{T = \frac{3}{4} m (R-r)^2 \dot{\theta}^2}$$

$$V = -mg(r-r)\cos\theta$$

$$L = T - V = \frac{3}{4} m (R-r)^2 \dot{\theta}^2 + mg(R-r)\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \Rightarrow \frac{3}{2} m (R-r)^2 \ddot{\theta} + mg(R-r)\sin\theta = 0$$

$$\sin\theta \approx \theta$$

$$\ddot{\theta} = -\frac{2g}{3(R-r)}\theta$$

$$\boxed{\omega = \sqrt{\frac{2g}{3(R-r)}}}$$

例五：

质量为m的相同三质点等距离系于长为2 的不可伸长的轻绳上，系统静止在光滑水平面上.若中间质点在某时刻获得与绳垂直且沿水平面的初速度 $\vec{v}_0$ ，试用拉格朗日方程求左右两质点相遇时的速率.

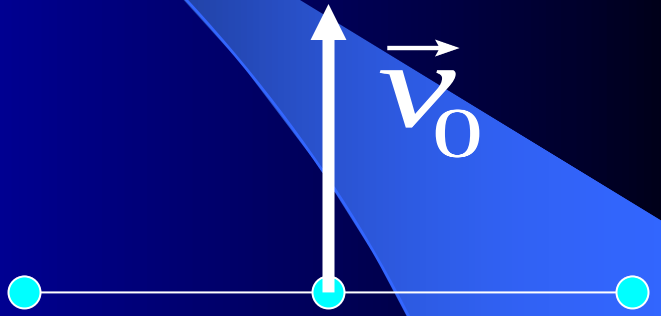
分析

坐标数 → 5

约束数 → 3

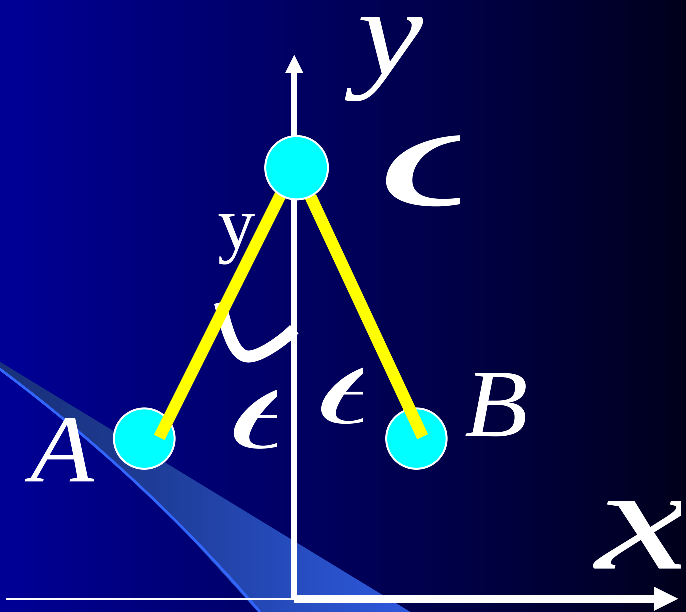
自由度数 → 2

取如图所示 (y,θ) 为广义坐标



demonstration

解:
$$\because x_B = l\sin\theta$$
$$\dot{x}_B = l\dot{\theta}\cos\theta$$
$$y_B = y - l\cos\theta$$
$$\dot{y}_B = \dot{y} + l\dot{\theta}\sin\theta$$



$$T = \frac{1}{2}m\dot{y}_c^2 + 2\times\frac{1}{2}m(\dot{x}_B^2 + \dot{y}_B^2)$$
$$= \frac{3}{2}m\dot{y}^2 + ml^2\dot{\theta}^2 + 2ml\dot{y}\dot{\theta}\sin\theta = L$$

$$L = \frac{3}{2} m \dot{y}^2 + m l^2 \dot{\theta}^2 + 2 m l \dot{y} \dot{\theta} \sin\theta$$

$$\begin{cases} \dfrac{\partial T}{\partial y} = 0 \\[2ex] \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{y}}\right) = 3m\ddot{y} + 2ml\ddot{\theta}\sin\theta + 2ml\dot{\theta}^2\cos\theta \end{cases}$$

$$\frac{\partial T}{\partial \dot{y}} = 3m\dot{y} + 2ml\dot{\theta}\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) - \frac{\partial T}{\partial y} = \boxed{3m\ddot{y} + 2ml\ddot{\theta}\sin\theta + 2ml\dot{\theta}^2\cos\theta = 0}$$

$$\boxed{3\ddot{y} + 2l\ddot{\theta}\sin\theta + 2l\dot{\theta}^2\cos\theta = 0 \qquad (1)}$$

$$L = \frac{3}{2} m \dot{y}^2 + m l^2 \dot{\theta}^2 + 2 m l \dot{y} \dot{\theta} \sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 2 m l^2 \ddot{\theta} + 2 m l \ddot{y} \sin\theta = 0$$

$$l \ddot{\theta} + \ddot{y} \sin\theta = 0 \qquad (2)$$

$$3\ddot{y} + 2 l \ddot{\theta} \sin\theta + 2 l \dot{\theta}^2 \cos\theta = 0 \qquad (1)$$

$$(3 - 2\sin^2\theta)\ddot{\theta} = 2\dot{\theta}^2 \sin\theta\cos\theta \qquad (3)$$

???

$$(3-2\sin^2\theta)\ddot{\theta} = 2\dot{\theta}^2 \sin\theta\cos\theta \qquad (3)$$

$$\because \ddot{\theta} = \frac{d\dot{\theta}}{dt} = \boxed{\frac{d\dot{\theta}}{d\theta}\frac{d\theta}{dt}} = \dot{\theta}\frac{d\dot{\theta}}{d\theta}$$

$$\dot{\theta}d\dot{\theta} = 2\dot{\theta}^2\frac{\sin\theta\cos\theta}{3-2\sin^2\theta}d\theta$$

$$\frac{d\dot{\theta}}{\dot{\theta}} = -\frac{1}{2}\frac{d(3-2\sin^2\theta)}{3-2\sin^2\theta}$$

$$\ln\dot{\theta} = \ln(3-2\sin^2\theta)^{-\frac{1}{2}} + c \qquad (4)$$

$$\ln\dot{\theta} = \ln(3-2\sin^2\theta)^{-\frac{1}{2}} + c \qquad (4)$$

$$\dot{\theta} = c_1(3-2\sin^2\theta)^{-\frac{1}{2}}$$

$$\because t=0, \quad \theta=\frac{\pi}{2}, \dot{y}_B = \dot{y}_A = 0, \quad \dot{y} = v_0$$

$$\dot{y}_B = \dot{y} + l\dot{\theta}\sin\theta \Rightarrow \dot{y} + l\dot{\theta}\sin\theta = v_0 + l\dot{\theta} = 0$$

$$c_1 = -\frac{v_0}{l} \qquad \dot{\theta}\Big|_{t=0} = -\frac{v_0}{l}$$

$$\dot{\theta} = -\frac{v_0}{l}(3-2\sin^2\theta)^{-\frac{1}{2}} \qquad (5)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) = \frac{d}{dt}(3m\dot{y} + 2ml\dot{\theta}\sin\theta) = 0$$

$$3\dot{y} + 2l\dot{\theta}\sin\theta = C_2$$

$$\because t = 0, \quad \theta = \frac{\pi}{2}, \dot{\theta} = -\frac{v_0}{l}, \quad \dot{y} = v_0$$

$$C_2 = v_0$$

$$\dot{y} = (3 - 2l\dot{\theta}\sin\theta)v_0 \big/ 3 \qquad (6)$$

$$\dot{\theta} = -\frac{v_0}{l}(3 - 2\sin^2\theta)^{-\frac{1}{2}} \qquad (5)$$

$$\dot{y} = {(3 - 2l\dot{\theta}\sin\theta)v_0} \Big/ 3 \qquad (6)$$

$$\begin{cases} \dot{x}_B = l\dot{\theta}\cos\theta \\ \dot{y}_B = \dot{y} + l\dot{\theta}\sin\theta \end{cases}$$

$$\dot{\theta} = -\frac{v_0}{l}(3 - 2\sin^2\theta)^{-\frac{1}{2}} \qquad (5)$$

相遇时$\theta = 0$

$$\dot{\theta}_{\theta=0} = -\frac{v_0}{\sqrt{3}l} \qquad (8)$$

$$\dot{y}_{\theta=0} = \frac{v_0}{3} \qquad (7)$$

$$[\dot{x}_B^2 + \dot{y}_B^2]^{\frac{1}{2}}_{\theta=0} = \frac{2}{3}v_0$$

利用守恒定律求解

$$\because \; L = \frac{3}{2} m\dot{y}^2 + ml^2\dot{\theta}^2 + 2ml\dot{y}\dot{\theta}\sin\theta$$

$$\therefore \frac{\partial L}{\partial y} = 0$$

y是循环坐标

$$\therefore p_y = \frac{\partial L}{\partial \dot{y}} = 3m\dot{y} + 2ml\dot{\theta}\sin\theta = C_1(const)$$

$$\because \frac{\partial L}{\partial t} = 0 \qquad \frac{\partial \vec{r}}{\partial t} = 0$$

完 整 、 保 守 、 稳 定

$$\therefore H = T + V = \frac{3}{2} m\dot{y}^2 + ml^2\dot{\theta}^2 + 2ml\dot{y}\dot{\theta}\sin\theta$$

$$\therefore\ p_y = \frac{\partial L}{\partial \dot{y}} = 3m\dot{y} + 2ml\dot{\theta}\sin\theta = C_1(const)$$

$$\therefore\ H = T + V = \frac{3}{2}m\dot{y}^2 + ml^2\dot{\theta}^2 + 2ml\dot{y}\dot{\theta}\sin\theta$$

$$= \frac{1}{2}mv_0^2 = E_0$$

$$\begin{cases} 3\dot{y}^2 + 2l^2\dot{\theta}^2 + 4l\dot{y}\dot{\theta}\sin\theta = v_0^2 \\ \because t = 0 \quad p_y = mv_0 \Rightarrow C_1 = mv_0 \\ 3\dot{y} + 2l\dot{\theta}\sin\theta = v_0 \end{cases}$$

$$\begin{cases} 3\dot{y}^2 + 2l^2\dot{\theta}^2 + 4l\dot{y}\dot{\theta}\sin\theta = v_0^2 \\ 3\dot{y} + 2l\dot{\theta}\sin\theta = v_0 \\ \because \quad \text{相遇时，} \theta = 0 \end{cases}$$

$$\begin{cases} \dot{y}\big|_{\theta=0} = \dfrac{1}{3}v_0 \\ \dot{\theta}\big|_{\theta=0} = \dfrac{1}{\sqrt{3}l}v_0 \end{cases}$$

例六

求一质量为m带电为q的带电粒子在电磁场 $\vec{B}$ 和 $\vec{E}$ 中运动时的拉格朗日函数

预备知识

$$\nabla \bullet \vec{A} + \frac{\partial \varphi}{\partial t} = 0 \Rightarrow \text{罗仑兹条件}$$

麦克斯韦方程组

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

电磁场矢势

解题思路

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = Q_\alpha$$

只要广义力满足：

$$Q_\alpha = \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_\alpha}\right) - \frac{\partial U}{\partial q_\alpha}，\text{广义势} U(q_\alpha, \dot{q}_\alpha)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = 0$$

$$Q_\alpha = \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_\alpha}\right) - \frac{\partial U}{\partial q_\alpha}$$

取直角坐标 $(x, y, z) \Rightarrow \vec{r}$ ⇒ 为广义坐标

广义速度 ⇒ $(\dot{x}, \dot{y}, \dot{z}) \Rightarrow \dot{\vec{r}} \Rightarrow \vec{V}$

广义力 $(Q_x, Q_y, Q_z) \Rightarrow (F_x, F_y, F_z) \Rightarrow \vec{F}(\vec{r}, \dot{\vec{r}})$

$$\because \vec{F} = q(\vec{E} + \vec{V} \times \vec{B}) \longleftrightarrow Q_\alpha = \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_\alpha}\right) - \frac{\partial U}{\partial q_\alpha}$$

$$\because \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Leftarrow \vec{B} = \nabla \times \vec{A}$$

$$\because \nabla \times \vec{E} = -\frac{\partial}{\partial t}(\nabla \times \vec{A}) \Rightarrow \because \nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0$$

$$\because \nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0 \Rightarrow \text{令 } \vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla \varphi$$

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla \varphi \qquad \varphi = \varphi(x, y, z, t) \Rightarrow$$

电磁场标势

$$= ??$$

$$\because \vec{A} = \vec{A}(x, y, z, t)$$

$$\therefore \frac{d\vec{A}}{dt} = \frac{\partial \vec{A}}{\partial x}\dot{x} + \frac{\partial \vec{A}}{\partial y}\dot{y} + \frac{\partial \vec{A}}{\partial z}\dot{z} + \frac{\partial \vec{A}}{\partial t}$$

$$= (\vec{V} \bullet \nabla)\vec{A} + \frac{\partial \vec{A}}{\partial t}$$

$$(\vec{V} \bullet \nabla) = (\dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k}) \bullet \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)$$

$$\because \frac{d\vec{A}}{dt} = (\vec{V} \bullet \nabla)\vec{A} + \frac{\partial \vec{A}}{\partial t}$$

$$\therefore \frac{\partial \vec{A}}{\partial t} = \frac{d\vec{A}}{dt} - (\vec{V} \bullet \nabla)\vec{A}$$

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla \varphi$$

$$\boxed{\vec{E} = -\frac{d\vec{A}}{dt} - \nabla \varphi + (\vec{V} \bullet \nabla)\vec{A}}$$

$$\because \vec{V} \times \vec{B} = \vec{V} \times (\nabla \times \vec{A})$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{c} \bullet \vec{a}) - \vec{c}(\vec{a} \bullet \vec{b})$$

$$\boxed{\vec{V} \times \vec{B} = \vec{V} \times (\nabla \times \vec{A}) = \nabla(\vec{V} \bullet \vec{A}) - (\vec{V} \bullet \nabla)A}$$

$$
\begin{cases}
\vec{E} = -\dfrac{d\vec{A}}{dt} - \nabla\varphi + (\vec{V}\bullet\nabla)\vec{A} \\
\vec{V}\times\vec{B} = \nabla(\vec{V}\bullet\vec{A}) - (\vec{V}\bullet\nabla)A
\end{cases}
$$

$$
\vec{F} = q(\vec{E} + \vec{V}\times\vec{B})
$$

$$
\vec{F} = q[-\nabla\varphi - \frac{d\vec{A}}{dt} + \nabla(\vec{V}\bullet\vec{A})]
$$

$$\vec{F} = q[-\nabla\varphi - \frac{d\vec{A}}{dt} + \nabla(\vec{V} \bullet \vec{A})]$$

$$\vec{A} = \vec{A}(x, y, z, t)$$

$$\varphi = \varphi(x, y, z, t)$$

$$\because\ q\vec{A} = \frac{\partial}{\partial\vec{V}}(q\vec{V}\cdot\vec{A} - q\varphi)$$

$$\vec{F} = -\nabla(q\varphi - q\vec{V} \bullet \vec{A}) - \frac{d}{dt}[\frac{\partial}{\partial\vec{V}}(q\varphi - q\vec{V} \bullet \vec{A})]$$

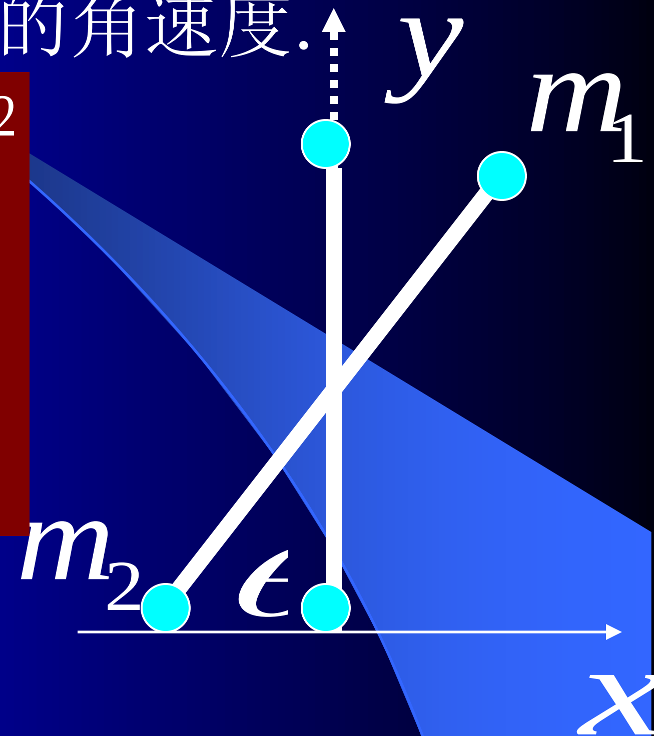
与速度相关广义势

$$U \equiv q\varphi - q\vec{V} \bullet \vec{A}$$

$$Q_\alpha = \frac{d}{dt}(\frac{\partial U}{\partial\dot{q}_\alpha}) - \frac{\partial U}{\partial q_\alpha}$$

$$L = T - U = \frac{1}{2}m\vec{V}^2 - q\varphi + q\vec{A} \bullet \vec{V}$$

例七 质量为m的相同二质点用一长为 的轻杆连接初始时直立静止在光滑水平面上, 以后任其倒下, 试用拉格朗日方程求杆落地时的角速度.



分析

$$(x_2 - x_1)^2 + y_1^2 = l^2$$

$$m_1 x_1 + m_2 x_2 = 0$$

$$x_1 = -x_2$$

坐标数 ⇒ 3

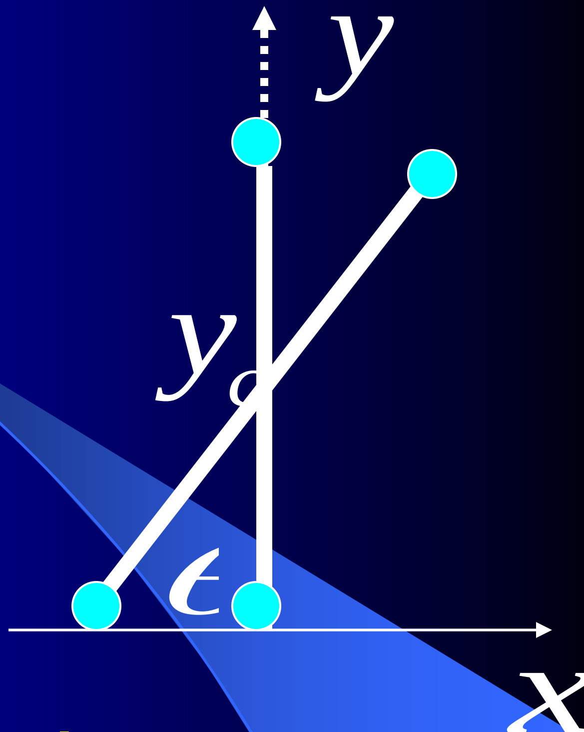
约束数 ⇒ 2

自由度数 ⇒

取如图所示 θ 为广义坐标

$$y_c = \frac{l}{2}\sin\theta$$

$$\dot{y}_c = \frac{l}{2}\dot{\theta}\cos\theta$$



根据柯尼西定理

$$T = \frac{1}{2}2m\dot{y}_c^2 + \frac{1}{2}I_c\dot{\theta}^2$$

$$I_c = 2m\left(\frac{l}{2}\right)^2 = \frac{1}{2}ml^2$$

$$T = \frac{1}{4}ml^2\dot{\theta}^2(1+\cos^2\theta)$$

$$V = mgl\sin\theta$$

$$L = T - V = \frac{1}{4}ml^2\dot{\theta}^2(1+\cos^2\theta) - mgl\sin\theta$$

$$L = T - V = \frac{1}{4} m l^2 \dot{\theta}^2 (1 + \cos^2\theta) - mgl \sin\theta$$

$$\frac{\partial L}{\partial \theta} = -\frac{1}{2} m l^2 \dot{\theta}^2 \cos\theta \sin\theta - mgl \cos\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2} m l^2 \dot{\theta} (1 + \cos^2\theta)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{1}{2} m l^2 \ddot{\theta} (1 + \cos^2\theta) - m l^2 \dot{\theta}^2 \sin\theta \cos\theta$$

拉格朗日方程

$$\frac{1}{2} m l^2 \ddot{\theta} (1 + \cos^2\theta) - \frac{1}{2} m l^2 \dot{\theta}^2 \sin\theta \cos\theta + mgl \cos\theta = 0$$

$$\frac{1}{2}ml^2\ddot{\theta}(1+\cos^2\theta)-\frac{1}{2}ml^2\dot{\theta}^2\sin\theta\cos\theta+mgl\cos\theta=0$$

$$L=T-V=\frac{1}{4}ml^2\dot{\theta}^2(1+\cos^2\theta)-mgl\sin\theta$$???

$\because \frac{\partial L}{\partial t}=0$ 约束稳定 $\quad \therefore H=T+V=E_0(const)$

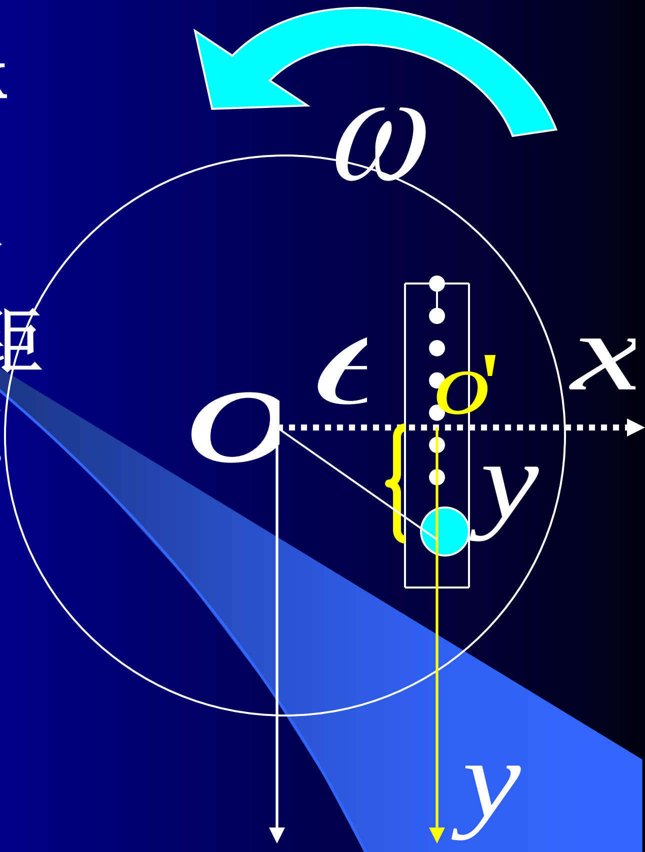
$$\frac{1}{4}ml^2\dot{\theta}^2(1+\cos^2\theta)+mgl\sin\theta=E_0$$

$$E_0=mgl$$

$$\boxed{\left.\dot{\theta}\right|_{\theta=0}=\sqrt{\frac{2g}{l}}}$$

落地瞬间N？可否跳起，起跳条件？

例八：质量为m的质点系在弹性 系数为k的弹簧上，弹簧系在以匀角速ω转动的水平转台上的光滑直槽内. 当弹簧处于原长时质点到转台中心距离最短，试用拉格朗日方程求质点作微振动的周期.



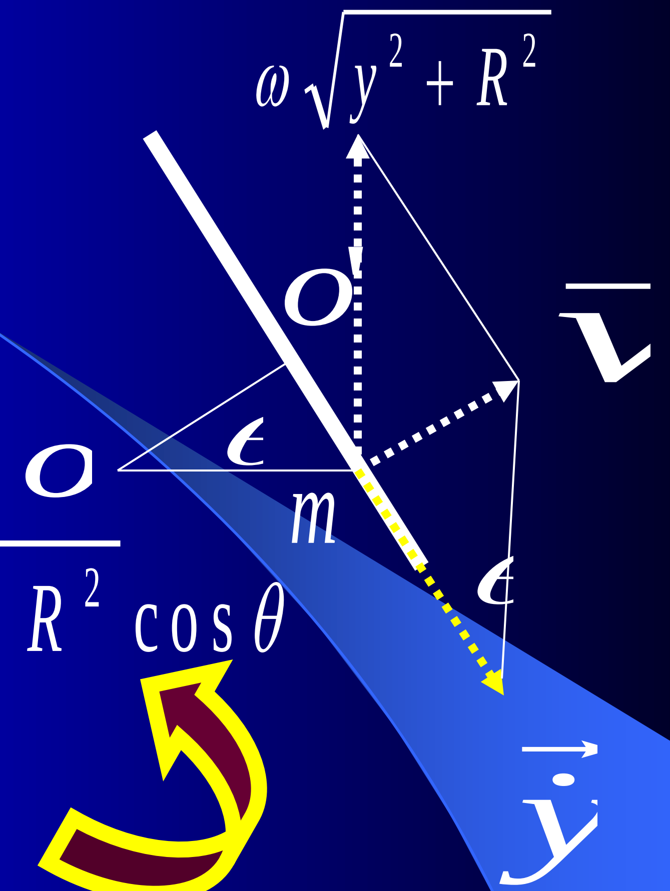
解：坐标数：２

约束数：

自由度数：

取如图所示弹簧伸长量y为广义坐标

$$R = OO'$$



根据余玄定理

$$v^2 = \dot{y}^2 + \omega^2(y^2+R^2) - 2\dot{y}\omega\sqrt{y^2+R^2}\cos\theta$$

$$\cos\theta = \frac{R}{\sqrt{y^2+R^2}}$$

$$v^2 = \dot{y}^2 + \omega^2(y^2+R^2) - 2\dot{y}\omega R \qquad V = \frac{1}{2}ky^2$$

$$L = T - V = \frac{m}{2}[\dot{y}^2 + \omega^2(y^2+R^2) - 2\dot{y}\omega R] - \frac{1}{2}ky^2$$

$$L = T - V = \frac{m}{2}[\dot{y}^2 + \omega^2(y^2 + R^2) - 2\dot{y}\omega R] - \frac{1}{2}ky^2$$

$$\frac{\partial L}{\partial y} = m\omega^2 y - ky \qquad \frac{\partial L}{\partial \dot{y}} = m\dot{y} - m\omega R$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = m\ddot{y}$$

$$m\ddot{y} - (m\omega^2 - k)y = 0$$

$$\ddot{y} = -\left(\frac{k}{m} - \omega^2\right)y$$

$$\Omega = \sqrt{\frac{k - m\omega^2}{m}} \qquad k > m\omega^2 \qquad T = 2\pi\sqrt{\frac{m}{k - m\omega^2}}$$

位形空间有“惯性力”吗？！