Chapter 1

Mathematical Preliminaries

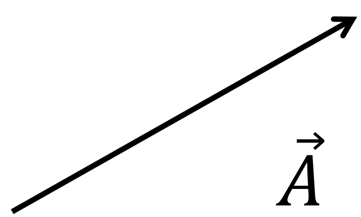
- 1.1 Vectors
- 1.2 Derivatives
- 1.3 Integrals
- 1.4 Generalized Functions
- 1.5 Orthogonal Transformations
- 1.6 Cartesian Tensors
- 1.7 The Helmholtz Theorem
- 1.8 Lagrange Multipliers

1.1 Vectors

Scalars: quantities have **magnitude** but no direction. Mass, charge, density and temperature

Vectors: quantities have both **magnitude** and **direction**. Velocity, acceleration, force and momentum

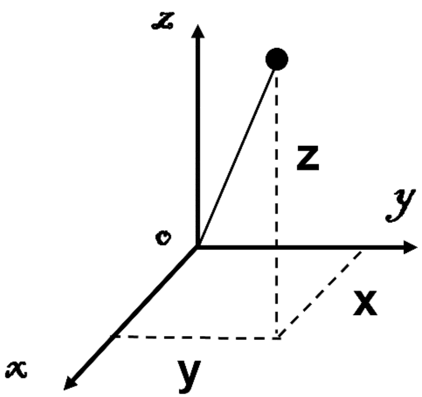
$$\vec{F} = m\vec{a}$$



$\vec{A}$

1.1 Vectors

1.1.1 Cartesian coordinates



$$V = V_x\hat{\mathbf{x}} + V_y\hat{\mathbf{y}} + V_z\hat{\mathbf{z}}$$

Gradient: $$\nabla V = \frac{\partial V}{\partial x}\hat{\mathbf{x}} + \frac{\partial V}{\partial y}\hat{\mathbf{y}} + \frac{\partial V}{\partial z}\hat{\mathbf{z}}$$

Divergence: $$\nabla \cdot \mathbf{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

Curl: $$\nabla \times \mathbf{V} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

$$= \hat{\mathbf{x}}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

1.1 Vectors

1.1.1 Cartesian coordinates

The gradient operator:

$$\boldsymbol{\nabla} = \frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} + \frac{\partial}{\partial z}\hat{\mathbf{z}}$$

Product rules:

$$\boldsymbol{\nabla}(fg) = f\boldsymbol{\nabla}g + g\boldsymbol{\nabla}f$$

$$\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B}) + \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{A}) + (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} + (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A}$$

$$\boldsymbol{\nabla}\cdot(f\mathbf{A}) = f(\boldsymbol{\nabla}\cdot\mathbf{A}) + \mathbf{A}\cdot(\boldsymbol{\nabla}f)$$

$$\boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) - \mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B})$$

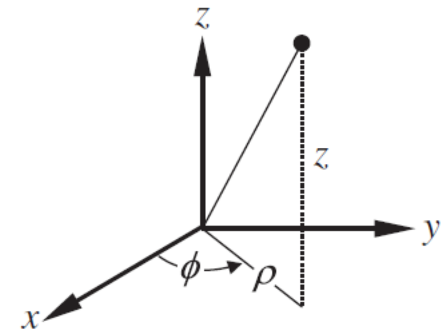
$$\boldsymbol{\nabla}\times(f\mathbf{A}) = f(\boldsymbol{\nabla}\times\mathbf{A}) - \mathbf{A}\times(\boldsymbol{\nabla}f)$$

$$\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} - (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} + \mathbf{A}(\boldsymbol{\nabla}\cdot\mathbf{B}) - \mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{A})$$

1.1 Vectors

1.1.2 Cylindrical coordinates

$$V = V_\rho \hat{\boldsymbol{\rho}} + V_\varphi \hat{\boldsymbol{\phi}} + V_z \hat{\mathbf{z}}$$



$$x = \rho cos\phi \quad y = \rho sin\phi \quad \mathrm{z} = \mathrm{z}$$

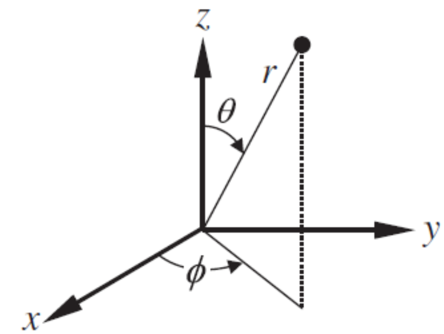
$$\hat{\rho} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi \qquad \hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$$

The gradient operator is:

$$\nabla = \frac{\partial}{\partial\rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial}{\partial z}\hat{\mathbf{z}}$$

1.1 Vectors

1.1.3 Spherical coordinates



$$V = V_r\hat{\boldsymbol{r}} + V_\theta\hat{\boldsymbol{\theta}} + V_\phi\hat{\boldsymbol{\phi}}$$

$$x = r sin\theta cos\phi \qquad y = r sin\theta sin\phi \qquad z = r cos\theta$$

$$\hat{\mathbf{r}} = \hat{\mathbf{x}}\sin\theta\cos\phi + \hat{\mathbf{y}}\sin\theta\sin\phi + \hat{\mathbf{z}}\cos\theta$$

$$\hat{\boldsymbol{\theta}} = \hat{\mathbf{x}}\cos\theta\cos\phi + \hat{\mathbf{y}}\cos\theta\sin\phi - \hat{\mathbf{z}}\sin\theta \qquad \hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$$

The gradient operator is:

$$\nabla = \frac{\partial}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r sin\theta}\frac{\partial}{\partial\phi}\hat{\phi}$$

1.1.4 Einstein summation convention

$$V = \sum_{k=1}^{3} V_k \hat{\mathbf{e}}_k = V_k \hat{\mathbf{e}}_k$$

In a Cartesian basis, the gradient and divergence are:

$$\nabla\varphi = \hat{\mathbf{e}}_k \nabla_k \varphi = \hat{\mathbf{e}}_k \partial_k \varphi = \hat{\mathbf{e}}_k \frac{\partial\varphi}{\partial r_k}$$

$$\nabla \cdot D = \nabla_k D_k = \partial_k D_k = \frac{\partial D_k}{\partial r_k}$$

1.1 Vectors

1.1.5 Kronecker and Levi-Cività permutation Symbols

$$\delta_{ij} = \begin{cases} 1 & i = j, \\ 0 & i \neq j, \end{cases} \qquad \epsilon_{ijk} = \begin{cases} 1 & ijk = xyz \quad yzx \quad zxy, \\ -1 & ijk = xzy \quad yxz \quad zyx, \\ 0 & \text{otherwise.} \end{cases}$$

Some useful identities:

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij} \qquad \delta_{kk} = 3$$

$$\partial_k r_j = \delta_{jk} \qquad V_k \delta_{kj} = V_j$$

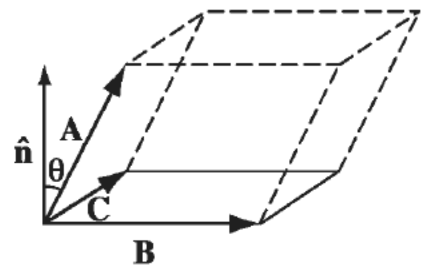
$$[\mathbf{V} \times \mathbf{F}]_i = \epsilon_{ijk} V_j F_k \qquad [\nabla \times \mathbf{A}]_i = \epsilon_{ijk} \partial_j A_k$$

$$\delta_{ij} \epsilon_{ijk} = 0 \qquad \epsilon_{ijk} \epsilon_{ijk} = 6.$$

$$\boxed{\epsilon_{ijk} \epsilon_{ist} = \delta_{js} \delta_{kt} - \delta_{jt} \delta_{ks}}$$

1.1 Vectors

Triple products:



$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=\mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})=\mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$$

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

BAC-CAB rule

Use the delta symbol and permutation symbol to prove the vector identities. For example:

the vector triple product: $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

the double-curl identity: $\nabla\times(\mathbf{A}\times\mathbf{B})=(\mathbf{B}\cdot\nabla)\mathbf{A}-(\mathbf{A}\cdot\nabla)\mathbf{B}+\mathbf{A}(\nabla\cdot\mathbf{B})-\mathbf{B}(\nabla\cdot\mathbf{A})$

1.2 Derivatives

The position vector is $\mathbf{r} = r\hat{\boldsymbol{r}}$ with $\mathrm{r} = \sqrt{x^2 + y^2 + z^2}$. If $f(r)$ is a scalar function with $f'(r) = df/dr$,

$$\nabla r = \hat{\mathbf{r}} \qquad \nabla \times \mathbf{r} = 0$$

$$\nabla f = f'\hat{\mathbf{r}} \qquad \nabla^2 f = \frac{(r^2 f')'}{r^2}$$

$$\nabla \cdot (f\mathbf{r}) = \frac{(r^3 f)'}{r^2} \qquad \nabla \times (f\mathbf{r}) = 0$$

$$\nabla = \frac{\partial}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{rsin\theta}\frac{\partial}{\partial \phi}\hat{\phi}$$

1.2 Derivatives

The Convective Derivative:

Let $\phi(\mathbf{r}, t)$ be a scalar function, an observer repeatedly samples ϕ along a trajectory in $\mathbf{r}(t)$ with velocity $\mathbf{v}(t) = \dot{\mathbf{r}}(t)$, the time rate of change of ϕ as the *convective derivative* is:

$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial t} + \frac{dx}{dt}\frac{\partial \phi}{\partial x} + \frac{dy}{dt}\frac{\partial \phi}{\partial y} + \frac{dz}{dt}\frac{\partial \phi}{\partial z} = \frac{\partial \phi}{\partial t} + (\boldsymbol{v} \cdot \nabla)\phi$$

The partial derivative $\partial\phi/\partial t$ gives the time rate of the change of ϕ at a fixed point in space $\mathbf{r}$.

1.3 Integrals

Jacobian Determinant: relates volume elements when changing variables in an integral.

Suppose **x** and **y** are *N*-dimensional space vectors in two different coordinates, the volume elements $d^N x$ and $d^N y$ are related by:

$$d^N x = |\mathbf{J}(\mathbf{x},\mathbf{y})|\, d^N y = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_N} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial x_N}{\partial y_1} & \frac{\partial x_N}{\partial y_2} & \cdots & \frac{\partial x_N}{\partial y_N} \end{vmatrix} d^N y$$

An example: Spherical-Cartesian transformation

$$\begin{aligned} x &= r\sin\theta\cos\varphi \\ y &= r\sin\theta\sin\varphi \\ z &= r\cos\theta \end{aligned}$$

1.3 Integrals

$\mathbf{F}(\mathbf{r})$ be a vector function defined in a volume *V* enclosed by a surface *S* with an outward normal $\hat{\mathbf{n}}$. If $d\mathbf{S} = dS\hat{\mathbf{n}}$, the divergence theorem is

$$\boxed{\int_V d^3r\, \nabla \cdot \mathbf{F} = \int_S d\mathbf{S} \cdot \mathbf{F}}$$

Gauss's theorem:

That is:

$$\int_V d^3r\nabla \rightarrow \oint_S d\vec{S}$$

Special choices for **F** gives:

$$\int_V d^3r\, \nabla \psi = \int_S d\mathbf{S}\, \psi$$

$$\int_V d^3r\, \nabla \times \mathbf{A} = \int_S d\mathbf{S} \times \mathbf{A}$$

1.3 Integrals

Substituting $\mathbf{F}(\mathbf{r}) = \phi(\mathbf{r})\nabla\psi(\mathbf{r})$ into the Gauss's theorem gives the **Green's first identity**:

$$\int_V d^3r\,[\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi] = \int_S d\mathbf{S}\cdot\phi\nabla\psi$$

Rewriting the equation with the role of ϕ and ψ exchanged gives the **Green's second identity**:

$$\int_V d^3r\,[\phi\nabla^2\psi - \psi\nabla^2\phi] = \int_S d\mathbf{S}\cdot[\phi\nabla\psi - \psi\nabla\phi]$$

$$\int_V d^3r\,\nabla\cdot\mathbf{F} = \int_S d\mathbf{S}\cdot\mathbf{F}$$

1.3 Integrals

The **Green's second identity**:

$$\int_V d^3r\,[\phi\nabla^2\psi - \psi\nabla^2\phi] = \int_S d\mathbf{S}\cdot[\phi\nabla\psi - \psi\nabla\phi]$$

We define $\psi = 1/R \equiv 1/|\mathbf{x}-\mathbf{x}'|$, **x** is the observation point and **x'** is the integration variable, ϕ is the scalar potential with $\nabla^2\phi = -\rho/\varepsilon_0$. The $\nabla^2(1/R) = -4\pi\delta(\mathbf{x}-\mathbf{x}')$. Then, the Green's second identity becomes:

$$\int_V \left[-4\pi\phi(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}') + \frac{1}{\varepsilon_0 R}\rho(\mathbf{x}')\right] d^3x' = \oint_S \left[\phi\frac{\partial}{\partial n'}\left(\frac{1}{R}\right) - \frac{1}{R}\frac{\partial\phi}{\partial n'}\right] dS$$

If the point **x** lies within the volume *V*, we get:

$$\Rightarrow\quad \phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \left[\frac{\rho(\mathbf{x}')}{R}\right] d^3x' + \frac{1}{4\pi}\oint_S \left[\phi\frac{\partial}{\partial n'}\left(\frac{1}{R}\right) - \frac{1}{R}\frac{\partial\phi}{\partial n'}\right] dS$$

1.3 Integrals

A vector function $\mathbf{F}(\mathbf{r})$ defined on an open surface *S* bounded by a closed curve *C*, if *dl* is a line element of *C*:

$$\boxed{\int_S d\mathbf{S}\cdot\nabla\times\mathbf{F} = \oint_C d\boldsymbol{\ell}\cdot\mathbf{F}}$$

Stokes' theorem:

The curve *C* is traversed in the direction given by the right-hand rule when the thumb points in the direction of *d* **S**.

That is:

$$\int_S d\mathbf{S}\times\nabla \rightarrow \oint_C d\boldsymbol{l}$$

$$\mathbf{F} = \mathbf{c}\psi \implies \int_S d\mathbf{S}\times\nabla\psi = \oint_C d\boldsymbol{\ell}\,\psi$$

$$\mathbf{F} = \mathbf{A}\times\mathbf{c} \implies \int_S (d\mathbf{S}\times\nabla)\times\mathbf{F} = \oint_C d\boldsymbol{l}\times\mathbf{F}$$

1.3 Integrals

The time derivative of a flux integral:

Leibniz' rule for the time derivative of a one-dimensional integral is:

$$\frac{d}{dt}\int_{x_1(t)}^{x_2(t)} dx\, b(x,t) = b(x_2,t)\frac{dx_2}{dt} - b(x_1,t)\frac{dx_1}{dt} + \int_{x_1(t)}^{x_2(t)} dx\,\frac{\partial b}{\partial t}$$

The Leibniz' rule for a two dimensions: the time derivative of the magnetic flux integral is

$$\frac{d}{dt}\int_{S(t)} d\mathbf{S}\cdot\mathbf{B} = \int_{S(t)} d\mathbf{S}\cdot\left[\boldsymbol{\upsilon}(\nabla\cdot\mathbf{B}) - \nabla\times(\boldsymbol{\upsilon}\times\mathbf{B}) + \frac{\partial\mathbf{B}}{\partial t}\right]$$

The individual area elements of the surface *S(t)* move with velocity *v*(**r**,t).

1.4 Generalized Functions

1.41 The Delta Function in One Dimension:

The $\delta(x)$ is defined by its "filtering" action on a smooth test function f(x):

$$\int_{-\infty}^{\infty} dx f(x)\delta(x-x') = f(x')$$

An informal definition:

$$\delta(x) = 0 \quad \text{for} \quad x \neq 0 \qquad \text{but} \qquad \int_{-\infty}^{\infty} dx\, \delta(x) = 1$$

If the variables x has dimensions of length, the integrals of the $\delta(x)$ make sense only if it has dimensions of inverse length. The integration ranges need only be large enough to include the point where the argument of the delta function vanishes.

1.4 Generalized Functions

1.41 The Delta Function in One Dimension:

The delta function can be understood as the **limit** of a sequence of functions which become more and more highly peaked at the point where its argument vanishes:

$$\delta(x) = \lim_{m\to\infty} \frac{\sin mx}{\pi x}$$

$$\delta(x) = \lim_{m\to\infty} \frac{m}{\sqrt{\pi}} \exp(-m^2 x^2)$$

$$\delta(x) = \lim_{\epsilon\to 0} \frac{\epsilon/\pi}{x^2 + \epsilon^2}$$

1.4 Generalized Functions

1.41 The Delta Function in One Dimension:

The delta function identities:

$$\delta(ax) = \frac{1}{|a|}\delta(x), \qquad a \neq 0$$

$$\int_{-\infty}^{\infty} dx f(x) \frac{d}{dx}\delta(x-x') = -\left.\frac{df}{dx}\right|_{x=x'}$$

$$\delta[g(x)] = \sum_n \frac{1}{|g'(x_n)|}\delta(x-x_n) \qquad \text{where} \qquad g(x_n)=0, \quad g'(x_n)\neq 0$$

$$\delta(x-x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ik(x-x')}$$

1.4 Generalized Functions

1.41 The Delta Function in One Dimension:

$$\delta(x-x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{ik(x-x')}$$

Above formula may be read as a statement of the completeness of plane waves labeled with the continuous index k:

$$\psi_k(x) = \frac{1}{\sqrt{2\pi}} e^{-ikx}$$

The general result for a complete set of normalized basis function $\psi_n(x)$ labeled with the discrete index n is :

$$\delta(x-x') = \sum_{n=1}^{\infty} \psi_n^*(x)\psi_n(x')$$

1.4 Generalized Functions

1.42 The Principal Value Integral and Plemelj Formula:

The Cauchy principal value is a generalized function defined by its action under an integral with an arbitrary function f(x), as:

$$\mathcal{P}\int_{-\infty}^{\infty} dx\,\frac{f(x)}{x-x_0} = \lim_{\epsilon\to 0}\left[\int_{-\infty}^{x_0-\epsilon} dx\,\frac{f(x)}{x-x_0} + \int_{x_0+\epsilon}^{\infty} dx\,\frac{f(x)}{x-x_0}\right]$$

An important application where the principal value plays a role is the *Plemelj formula*:

$$\lim_{\epsilon\to 0}\frac{1}{x-x_0\pm i\epsilon} = \mathcal{P}\frac{1}{x-x_0} \mp i\pi\delta(x-x_0)$$

This expression is symbolic in the sense that it gains meaning when we multiply every term by an arbitrary function $f(x)$ and integrate over x from $-\infty$ to ∞.

1.4 Generalized Functions

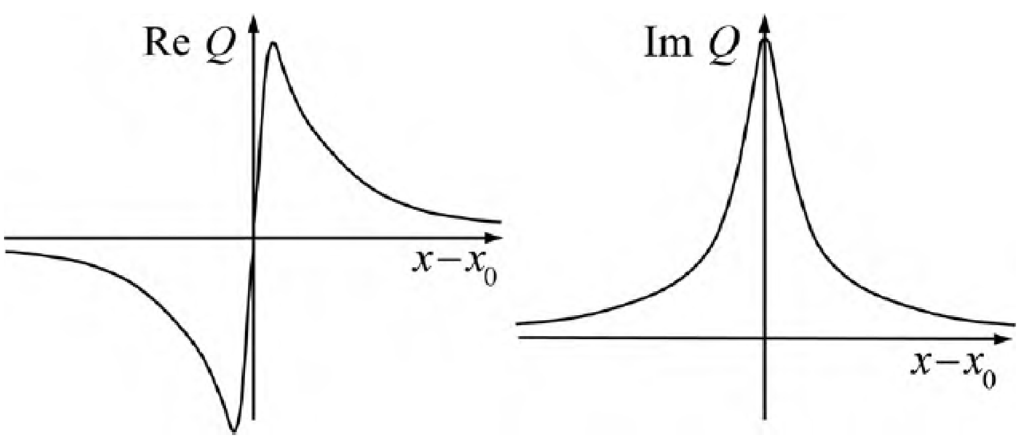
1.42 The Principal Value Integral and Plemelj Formula:

An important application where the principal value plays a role is the *Plemelj formula*:

$$\lim_{\epsilon \to 0} \frac{1}{x - x_0 \pm i\epsilon} = \mathcal{P}\frac{1}{x - x_0} \mp i\pi\delta(x - x_0)$$

The correctness of the Plemelj formula can be appreciated from the identity:

$$\frac{1}{x - x_0 \pm i\epsilon} = \frac{x - x_0}{(x - x_0)^2 + \epsilon^2} \mp i\frac{\epsilon}{(x - x_0)^2 + \epsilon^2}$$



Laplace's equation in Spherical Coordinates:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}=0.$$

Assuming the problem has azimuthal symmetry, V is independent of ϕ:

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)=0.$$

Since the first term depends only on r, and the second only on θ, it follows that each must be a constant:

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=l(l+1),\quad \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=-l(l+1).$$

$$R(r)=Ar^l+\frac{B}{r^{l+1}},$$

$$\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -l(l+1)\sin\theta\,\Theta,$$

Assume $\zeta = cos\theta$

$$\frac{d}{d\zeta}\left[(1-\zeta^2)\frac{d\Theta}{d\zeta}\right] + l(l+1)\Theta = 0$$

Legendre Function

The solutions are **Legendre polynomials** in the variable $\cos\theta$:

$$\Theta(\theta) = P_l(\cos\theta).$$

$P_l(x)$ is most conveniently defined by the **Rodrigues formula:**

$$P_l(x) \equiv \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l.$$

$$\boxed{\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= (3x^2-1)/2 \\ P_3(x) &= (5x^3-3x)/2 \\ P_4(x) &= (35x^4-30x^2+3)/8 \\ P_5(x) &= (63x^5-70x^3+15x)/8 \end{aligned}}$$

The general solution for the Laplace's equation:

$$V(r,\theta) = \left(Ar^l + \frac{B}{r^{l+1}}\right)P_l(\cos\theta).$$

Laplace's equation in Spherical Coordinates:

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0$$

What happened if the problem has azimuthal variations? The potential Φ will dependent on ϕ.

$$\Phi = \frac{U(r)}{r} P(\theta)Q(\phi)$$

$$r^2 \sin^2\theta\left[\frac{1}{U}\frac{d^2U}{dr^2} + \frac{1}{Pr^2 \sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)\right] + \frac{1}{Q}\frac{d^2Q}{d\phi^2} = 0$$

The ϕ dependence of the equation has now been isolated in the last term. Consequently that term must be a constant which we call $(-m^2)$:

$$\frac{1}{Q}\frac{d^2Q}{d\phi^2} = -m^2$$

This has solutions

$$Q = e^{\pm im\phi}$$

By similar considerations we find separate equations for $P(\theta)$ and $U(r)$:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+\left[l(l+1)-\frac{m^2}{\sin^2\theta}\right]P=0$$

$$\frac{d^2U}{dr^2}-\frac{l(l+1)}{r^2}U=0$$

From the form of the radial equation it is apparent that a single power of r (rather than a power series) will satisfy it. The solution is found to be:

$$U=Ar^{l+1}+Br^{-l} \tag{3.8}$$

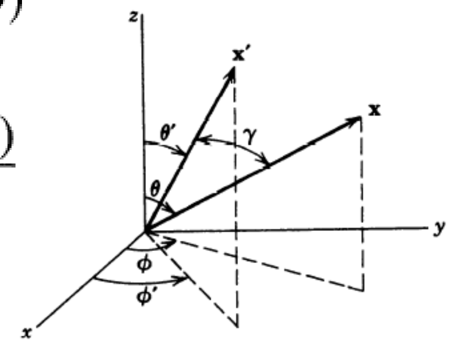
The θ equation for $P(\theta)$ is customarily expressed in terms of $x=\cos\theta$, instead of θ itself. Then it takes the form:

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right]+\left[l(l+1)-\frac{m^2}{1-x^2}\right]P=0$$

This equation is called the generalized Legendre equation, and its solutions are the associated Legendre functions.

$$Y_{lm}(\theta,\phi)=\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\,P_l^m(\cos\theta)\,e^{im\phi}$$

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi)$$



$$\Phi(\mathbf{x}) = \frac{1}{\epsilon_0} \sum_{l,m} \frac{1}{2l+1} \left[\int Y_{lm}^*(\theta', \phi') r'^l \rho(\mathbf{x}') \, d^3x' \right] \frac{Y_{lm}(\theta, \phi)}{r^{l+1}}$$

$$q_{lm} = \int Y_{lm}^*(\theta', \phi') r'^l \rho(\mathbf{x}') \, d^3x'$$

These coefficients are called *multipole moments*. To see the physical interpretation of them we exhibit the first few explicitly in terms of Cartesian coordinates:

$$q_{00} = \frac{1}{\sqrt{4\pi}} \int \rho(\mathbf{x}') \, d^3x' = \frac{1}{\sqrt{4\pi}} q$$

$$\left.\begin{aligned} q_{11} &= -\sqrt{\frac{3}{8\pi}} \int (x' - iy')\rho(\mathbf{x}') \, d^3x' = -\sqrt{\frac{3}{8\pi}} (p_x - ip_y) \\ q_{10} &= \sqrt{\frac{3}{4\pi}} \int z'\rho(\mathbf{x}') \, d^3x' = \sqrt{\frac{3}{4\pi}} p_z \end{aligned}\right\}$$

$$\left.\begin{aligned} q_{22} &= \frac{1}{4}\sqrt{\frac{15}{2\pi}} \int (x' - iy')^2 \rho(\mathbf{x}') \, d^3x' = \frac{1}{12}\sqrt{\frac{15}{2\pi}} (Q_{11} - 2iQ_{12} - Q_{22}) \\ q_{21} &= -\sqrt{\frac{15}{8\pi}} \int z'(x' - iy')\rho(\mathbf{x}') \, d^3x' = -\frac{1}{3}\sqrt{\frac{15}{8\pi}} (Q_{13} - iQ_{23}) \\ q_{20} &= \frac{1}{2}\sqrt{\frac{5}{4\pi}} \int (3z'^2 - r'^2)\rho(\mathbf{x}') \, d^3x' = \frac{1}{2}\sqrt{\frac{5}{4\pi}} Q_{33} \end{aligned}\right\}$$

Dipole moment

$$\mathbf{p} = \int \mathbf{x}'\rho(\mathbf{x}') \, d^3x'$$

and Q_{ij} is the traceless quadrupole moment tensor:

$$Q_{ij} = \int (3x_i'x_j' - r'^2\delta_{ij})\rho(\mathbf{x}') \, d^3x'$$

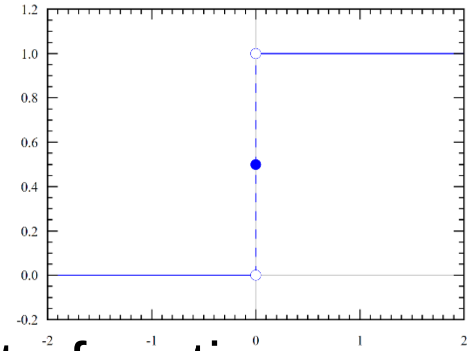
l	m	$\Phi(\varphi)$	$\Theta(\theta)$		極坐標中的表達式	直角坐標中的表達式
0	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{1}{\sqrt{2}}$		$\frac{1}{2\sqrt{\pi}}$	$\frac{1}{2\sqrt{\pi}}$
1	0	$\frac{1}{\sqrt{2\pi}}$	$\sqrt{\frac{3}{2}}\cos\theta$		$\frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta$	$\frac{1}{2}\sqrt{\frac{3}{\pi}}\frac{z}{r}$
1	+1	$\frac{1}{\sqrt{2\pi}}\exp(i\varphi)$	$\frac{\sqrt{3}}{2}\sin\theta$		$\frac{1}{2}\sqrt{\frac{3}{\pi}}\sin\theta\cos\varphi$	$\frac{1}{2}\sqrt{\frac{3}{\pi}}\frac{x}{r}$
1	-1	$\frac{1}{\sqrt{2\pi}}\exp(-i\varphi)$	$\frac{\sqrt{3}}{2}\sin\theta$		$\frac{1}{2}\sqrt{\frac{3}{\pi}}\sin\theta\sin\varphi$	$\frac{1}{2}\sqrt{\frac{3}{\pi}}\frac{y}{r}$
2	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{1}{2}\sqrt{\frac{5}{2}}(3\cos^2\theta-1)$		$\frac{1}{4}\sqrt{\frac{5}{\pi}}(3\cos^2\theta-1)$	$\frac{1}{4}\sqrt{\frac{5}{\pi}}\frac{2z^2-x^2-y^2}{r^2}$
2	+1	$\frac{1}{\sqrt{2\pi}}\exp(i\varphi)$	$\frac{\sqrt{15}}{2}\sin\theta\cos\theta$		$\frac{1}{2}\sqrt{\frac{15}{\pi}}\sin\theta\cos\theta\cos\varphi$	$\frac{1}{2}\sqrt{\frac{15}{\pi}}\frac{zx}{r^2}$
2	-1	$\frac{1}{\sqrt{2\pi}}\exp(-i\varphi)$	$\frac{\sqrt{15}}{2}\sin\theta\cos\theta$		$\frac{1}{2}\sqrt{\frac{15}{\pi}}\sin\theta\cos\theta\sin\varphi$	$\frac{1}{2}\sqrt{\frac{15}{\pi}}\frac{yz}{r^2}$
2	+2	$\frac{1}{\sqrt{2\pi}}\exp(2i\varphi)$	$\frac{\sqrt{15}}{4}\sin^2\theta$		$\frac{1}{4}\sqrt{\frac{15}{\pi}}\sin^2\theta\cos 2\varphi$	$\frac{1}{4}\sqrt{\frac{15}{\pi}}\frac{x^2-y^2}{r^2}$
2	-2	$\frac{1}{\sqrt{2\pi}}\exp(-2i\varphi)$	$\frac{\sqrt{15}}{4}\sin^2\theta$		$\frac{1}{4}\sqrt{\frac{15}{\pi}}\sin^2\theta\sin 2\varphi$	$\frac{1}{2}\sqrt{\frac{15}{\pi}}\frac{xy}{r^2}$

1.4 Generalized Functions

1.43 The Step Function:

The Heaviside step function $\Theta(x)$ is defined by:

$$\Theta(x) = \begin{cases} 0 & x < 0, \\ 1 & x > 0. \end{cases}$$



The delta function is the derivative of the theta function:

$$\frac{d\Theta(x)}{dx} = \delta(x)$$

A useful representation is:

$$\Theta(x) = \lim_{\epsilon \to 0} \frac{i}{2\pi} \int_{-\infty}^{\infty} dk \frac{1}{k + i\epsilon} e^{-ikx}$$

1.44 The Sign Function:

The sign function sgn(x) is defined by:

$$\mathrm{sgn}(x) = \frac{d}{dx}|x| = \begin{cases} -1 & x < 0, \\ 1 & x > 0. \end{cases}$$

A convenient representation is:

$$\mathrm{sgn}(x) = -1 + 2\int_{-\infty}^{x} dy\,\delta(y).$$

1.4 Generalized Functions

1.45 The Delta Function in Three Dimensions:

The three-dimensional delta function is defined by using an integral over a volume *V* and an arbitrary “test” function *f(***r***)*:

$$\int_V d^3r\, f(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}') = \begin{cases} f(\mathbf{r}') & \mathbf{r}' \in V, \\ 0 & \mathbf{r}' \notin V. \end{cases}$$

An a less formal definition:

$$\delta(\mathbf{r}) = 0 \quad \text{for} \quad \mathbf{r} \neq 0 \qquad \text{but} \qquad \int_V d^3r\, \delta(\mathbf{r}) = \begin{cases} 1 & \mathbf{r} = 0 \in V, \\ 0 & \mathbf{r} = 0 \notin V. \end{cases}$$

The $\delta(\mathbf{r})$ has dimensions of inverse volume.

1.4 Generalized Functions

1.45 The Delta Function in Three Dimensions:

In Cartesian coordinate:

$$\delta(\mathbf{r}) = \delta(x)\delta(y)\delta(z).$$

In Cylindrical coordinate:

$$\delta(\mathbf{r}) = \frac{\delta(\rho)\delta(\phi)\delta(z)}{\rho}$$

In Spherical coordinate:

$$\delta(\mathbf{r}) = \frac{\delta(r)\delta(\theta)\delta(\phi)}{r^2 sin\theta}$$

$$\boxed{\delta(\mathbf{x} - \mathbf{x}_0) = \frac{1}{|\mathbf{J}(\mathbf{x}, \mathbf{y})|}\delta(\mathbf{y} - \mathbf{y}_0).}$$

1.4 Generalized Functions

1.45 The Delta Function in Three Dimensions:

Some useful Delta Function Identities:

$$\delta(\mathbf{r}-\mathbf{r}') = \frac{1}{(2\pi)^3}\int d^3k\, e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}$$

$$\int d^3r\, f(\mathbf{r})\delta[g(\mathbf{r})] = \int_S dS \frac{f(\mathbf{r}_S)}{|\nabla g(\mathbf{r}_S)|} \qquad \text{where } g(\mathbf{r}_S) = 0 \text{ defines } S$$

$$\boxed{\nabla^2 \frac{1}{|\mathbf{r}-\mathbf{r}'|} = -4\pi\delta(\mathbf{r}-\mathbf{r}')}$$

$$\frac{\partial}{\partial r_k}\frac{\partial}{\partial r_m}\frac{1}{r} = \frac{3r_k r_m - r^2\delta_{km}}{r^5} - \frac{4\pi}{3}\delta_{ij}\delta(\mathbf{r}).$$

1.4 Generalized Functions

$$\boxed{\nabla^2 \frac{1}{r} = -4\pi\delta(\mathbf{r})}$$

Laplacian:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}.$$

In spherical coordinates:

$$\nabla^2 \frac{1}{r} = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\frac{1}{r} = 0 \quad \text{when} \quad r \neq 0.$$

For r=0, we integrate $\nabla^2(1/r)$ over a tiny spherical volume V centered at the origin.

$$\int_V d^3r\, \nabla^2 \frac{1}{r} = \int_V d^3r\, \nabla \cdot \nabla \frac{1}{r} = \int_S d\mathbf{S} \cdot \left(-\frac{\hat{\mathbf{r}}}{r^2}\right)$$

Gauss's theorem

$$\int_V d^3r\, \nabla \cdot \mathbf{F} = \int_S d\mathbf{S} \cdot \mathbf{F}$$

Since $d\boldsymbol{S} = r^2 sin\theta d\theta d\phi \mathbf{r}$ and $\nabla(1/r) = -\mathbf{r}/r^2$

$$\int_S d\mathbf{S} \cdot \left(-\frac{\hat{\mathbf{r}}}{r^2}\right) = -\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta = -4\pi.$$

1.5 Orthogonal Transformations

Let $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$ and $(\hat{\mathbf{e}}'_1, \hat{\mathbf{e}}'_2, \hat{\mathbf{e}}'_3)$ be two sets of orthogonal Cartesian unit vectors. Each is a complete basis for vectors in three dimensions, so

$$\hat{\mathbf{e}}'_i = A_{ij}\hat{\mathbf{e}}_j .$$

The set of scalars A_{ij} are called *direction cosines*. Using the unit vector properties, gives:

$$\delta_{ij} = \hat{\mathbf{e}}'_i \cdot \hat{\mathbf{e}}'_j = A_{ik}A_{jk} .$$

It says that the transpose of the matrix **A**, $\mathbf{A}^{\mathbf{T}}$, is identical to the inverse of the matrix **A**, $\mathbf{A}^{-1}$. This is the definition of a matrix **A** that describes an ***orthogonal transformation.***

$$\mathbf{A}\mathbf{A}^{\mathsf{T}} = \mathbf{A}\mathbf{A}^{-1} = 1 .$$

There are two classes of orthogonal coordinate transformations. These follow from the determinant

$$\det\left[\mathbf{A}\mathbf{A}^{\mathsf{T}}\right] = \det\mathbf{A}\,\det\mathbf{A}^{\mathsf{T}} = (\det\mathbf{A})^2 = 1 .$$

1.5 Orthogonal Transformations

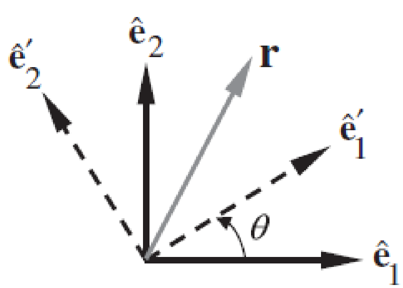
A **rotation** has det $\mathbf{A} = 1$. Figure 1.5(a) shows an example where

$$\mathbf{A} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

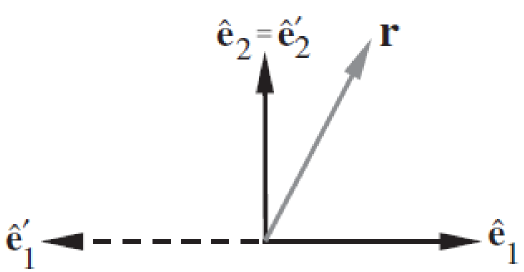
A **reflection** has det $\mathbf{A} = -1$. Figure 1.5(b) shows an example where

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The **inversion** transformation is represented by $A_{ij} = -\delta_{ij}$ so det $\mathbf{A} = -1$ like a reflection. However, a sequence of reflections can have det $\mathbf{A} = 1$ like a rotation.



(a) Rotation



(b) Reflection

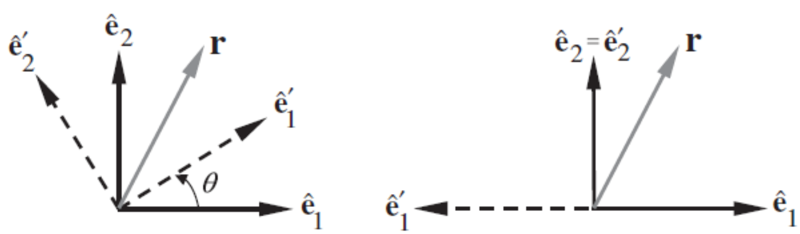
1.5 Orthogonal Transformations

1.5.1 Passive Point of View

The position vector **r** can be decomposed as:

$$\mathbf{r} = r_i \hat{\mathbf{e}}_i = r'_j \hat{\mathbf{e}}'_j = (\mathbf{r})'.$$

$$r'_i = A_{ij} r_j \qquad r_j = A_{kj} r'_k.$$

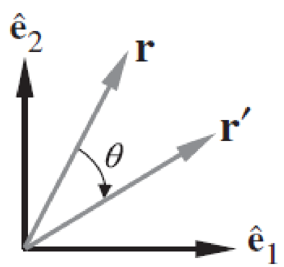


This description is called the *passive* point of view. The position vector **r** is a spectator fixed in space while the coordinate system transforms. The matrix **A** is regarded as an operator that transforms the components of **r** in the $\{\hat{\mathbf{e}}\}$ basis to the components of **r** in the $\{\hat{\mathbf{e}}'\}$ basis. The matrix form of the transformation connects components of the *same* vector in *different* coordinate systems:

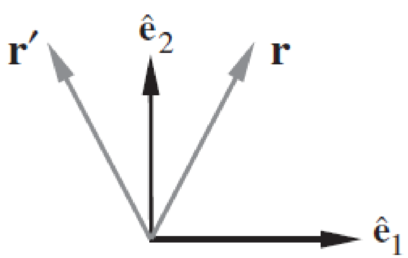
1.5.2 Active Point of View

The *active* point of view is an alternative (and equivalent) way to think about an orthogonal transformation. Here, the matrix $\mathbf{A}$ is regarded as an operator which transforms $\mathbf{r}$ to a new vector $\mathbf{r}'$ with no change in the underlying coordinate system. The matrix form of the transformation connects the components of *different* vectors in the *same* coordinate system:

$$\mathbf{r}' = \mathbf{A}\mathbf{r}.$$



(a) Rotation



(b) Reflection

1.6 Cartesian Tensors

Tensors are mathematical objects defined by their behavior under orthogonal coordinate transformations. Physical quantities are classified as *rotational tensors* of various ranks depending on how they transform under rotations. In this section, we adopt the passive point of view

A *tensor of rank 0* is a one-component quantity where the result of a rotational transformation from the passive point of view is

$$f'(\mathbf{r}) = f(\mathbf{r}).$$

An ordinary **scalar** is a tensor of rank 0. A *tensor of rank 1* is an object whose three components transform under rotation

$$V_i'(\mathbf{r}) = A_{ij} V_j(\mathbf{r}).$$

a vector is characterized by the preservation of its length under a change of coordinates:

$$V_i' V_i' = A_{ij} V_j A_{ik} V_k = V_k V_k.$$

1.6 Cartesian Tensors

A *tensor of rank 0* is a one-component quantity where the result of a rotational transformation from the passive point of view is

$$f'(\mathbf{r}) = f(\mathbf{r}).$$

An ordinary **scalar** is a tensor of rank 0. A *tensor of rank 1* is an object whose three components transform under rotation

$$V_i'(\mathbf{r}) = A_{ij} V_j(\mathbf{r}).$$

A *tensor of rank 2* is a nine-component quantity whose components transform under rotation by the rule

$$T_{ij}'(\mathbf{r}) = A_{ik} A_{jm} T_{km}(\mathbf{r}).$$

A **dyadic** is a tensor of rank 2 composed of a linear combination of pairs of juxtaposed (not multiplied) vectors. The examples we will encounter in this book all have the form

$$\mathbf{T} \equiv \hat{\mathbf{e}}_i T_{ij} \hat{\mathbf{e}}_j .$$

1.6 Cartesian Tensors

Inversion, Reflection and Pseudotensors

We study here the transformation properties of rotational tensors under a general orthogonal transformation **A**. The cross product $\mathbf{m} = \mathbf{p} \times \mathbf{w}$ in component form:

$$m_i = \epsilon_{ijk} p_j w_k.$$

The component transforms under rotation as:

$$m'_i = \epsilon_{ijk} p'_j w'_k = \epsilon_{ijk} A_{js} p_s A_{k\ell} w_\ell = \epsilon_{pjk} \delta_{ip} A_{js} A_{k\ell} p_s w_\ell.$$

Using $\delta_{ij} = A_{ik} A_{jk}$ to eliminate δ_{ip}, gives:

$$m'_i = \epsilon_{pjk} A_{iq} A_{pq} A_{js} A_{k\ell} p_s w_\ell = (\epsilon_{pjk} A_{pq} A_{js} A_{k\ell}) A_{iq} p_s w_\ell.$$

$$\Rightarrow \quad m'_i = (\det \mathbf{A}) A_{iq} \epsilon_{qs\ell} p_s w_\ell.$$

$$\boxed{\epsilon_{\ell mn} \det \mathbf{C} = \epsilon_{ijk} C_{\ell i} C_{mj} C_{nk}.}$$

$$\Rightarrow \quad m'_i = (\det \mathbf{A}) A_{iq} m_q.$$

1.6 Cartesian Tensors

$$\boxed{m'_i = (\det \mathbf{A}) A_{iq} m_q.}$$

The **m** transforms like an ordinary vector under rotations when det **A=1**. An extra minus sign occurs when **A** corresponds to a reflection or an inversion where det **A=-1**.

Pseudovector (or *axial vector*): any rotational vector where an explicit determinant factor appears in its transformation rule.

Polar vector : any rotational vector where no determinant factor appears in the transformation rule.

The nature of the vector produced by the cross product of two other vectors is summarized by:

$$\text{axial vector} \times \text{polar vector} = \text{polar vector}$$

$$\text{polar vector} \times \text{polar vector} = \text{axial vector}$$

$$\text{axial vector} \times \text{axial vector} = \text{axial vector}.$$

1.6 Cartesian Tensors

Inversion and Reflection

- The position vector **r** is an ordinary polar vector, the transformation law $r_i' = A_{ij}r_j$ does not include the determinant factor in $m_i' = (\det A)A_{iq}m_q$. Therefore, if the orthogonal transformation **A** corresponds to inversion, it gives:

$$\mathbf{r} \to \mathbf{r}' = -\mathbf{r}.$$

By definition, any other polar vector **P** behaves the same way under inversion:

$$\mathbf{P} \to \mathbf{P}' = -\mathbf{P} \qquad \text{(inversion of a polar vector).}$$

- For an axial vector **Q**:

$$\mathbf{Q} \to \mathbf{Q}' = \mathbf{Q} \qquad \text{(inversion of an axial vector).}$$

1.6 Cartesian Tensors

Inversion and Reflection

- The operation of mirror reflection through the x-y plane inverts the z-component of the position vector

$$(x, y, z) \rightarrow (x', y', z') = (x, y, -z).$$

A polar vector **P** behaves the same way under inversion:

$$(P_x, P_y, P_z) \rightarrow (P'_x, P'_y, P'_z) = (P_x, P_y, -P_z) \qquad \text{(reflection of a polar vector).}$$

- For an axial vector **Q**:

$$(Q_x, Q_y, Q_z) \rightarrow (Q'_x, Q'_y, Q'_z) = (-Q_x, -Q_y, Q_z) \qquad \text{(reflection of an axial vector).}$$

1.7 The Helmholtz Theorem

Statement:

An arbitrary vector field $\mathbf{C}(\mathbf{r})$ can always be decomposed into the sum of two vector fields: one with zero divergence and one with zero curl. Specifically,

$$\mathbf{C} = \mathbf{C}_{\perp} + \mathbf{C}_{\parallel},$$

where

$$\nabla \cdot \mathbf{C}_{\perp} = 0 \qquad \text{and} \qquad \nabla \times \mathbf{C}_{\parallel} = 0.$$

An explicit representation of special interest is

$$\mathbf{C}(\mathbf{r}) = \nabla \times \mathbf{F}(\mathbf{r}) - \nabla\Omega(\mathbf{r}).$$

When the following integrals over all space converge, $\Omega(\mathbf{r})$ and $\mathbf{F}(\mathbf{r})$ are uniquely given by

$$\Omega(\mathbf{r}) = \frac{1}{4\pi}\int d^3r' \, \frac{\nabla' \cdot \mathbf{C}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

and

$$\mathbf{F}(\mathbf{r}) = \frac{1}{4\pi}\int d^3r' \, \frac{\nabla' \times \mathbf{C}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}.$$

This result is valid for both static and time-dependent vector fields.

1.7 The Helmholtz Theorem

$$\nabla^2 \frac{1}{|\mathbf{r}-\mathbf{r}'|} = -4\pi\,\delta(\mathbf{r}-\mathbf{r}')$$

Proof:
$$\mathbf{C}(\mathbf{r}) = \int d^3r'\mathbf{C}(\mathbf{r}')\,\delta(\mathbf{r}-\mathbf{r}') = -\frac{1}{4\pi}\int d^3r'\,\mathbf{C}(\mathbf{r}')\,\nabla^2\frac{1}{|\mathbf{r}-\mathbf{r}'|}.$$

Exchanging $\mathbf{C}(\mathbf{r}')$ and ∇^2 in the last term and using the double-curl identity

$$\mathbf{C}(\mathbf{r}) = -\frac{1}{4\pi}\nabla\int d^3r'\,\nabla\cdot\left[\frac{\mathbf{C}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right] + \frac{1}{4\pi}\nabla\times\int d^3r'\,\nabla\times\left[\frac{\mathbf{C}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right].$$

Now, since $\nabla f(|\mathbf{r}-\mathbf{r}'|) = -\nabla' f(|\mathbf{r}-\mathbf{r}'|)$, we deduce that

$$\nabla'\cdot\left[\frac{\mathbf{C}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right] = \frac{\nabla'\cdot\mathbf{C}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} - \mathbf{C}(\mathbf{r}')\cdot\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}.$$

Moving $\mathbf{C}(\mathbf{r}')$ to the right of ∇ in the last term and rearranging gives

$$\nabla\cdot\left[\frac{\mathbf{C}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right] = \frac{\nabla'\cdot\mathbf{C}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} - \nabla'\cdot\frac{\mathbf{C}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}.$$

In exactly the same way,

$$\nabla\times\left[\frac{\mathbf{C}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right] = \frac{\nabla'\times\mathbf{C}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} - \nabla'\times\frac{\mathbf{C}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}.$$

1.7 The Helmholtz Theorem

$$\mathbf{C}(\mathbf{r}) = -\frac{1}{4\pi}\nabla\int d^3r'\,\frac{\nabla'\cdot\mathbf{C}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} + \frac{1}{4\pi}\nabla\times\int d^3r'\,\frac{\nabla'\times\mathbf{C}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$+\frac{1}{4\pi}\nabla\int d^3r'\,\nabla'\cdot\left[\frac{\mathbf{C}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right] - \frac{1}{4\pi}\nabla\times\int d^3r'\,\nabla'\times\left[\frac{\mathbf{C}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right].$$

The last two integrals are zero when the first two integrals converge. Using the divergence theorem to transform the volume integrals in the last two terms into the surface integrals:

$$\int d\mathbf{S}'\cdot\frac{\mathbf{C}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \qquad \text{and} \qquad \int d\mathbf{S}'\times\frac{\mathbf{C}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}.$$

The surface of integration for both integrals lies at infinity. Therefore, both integrals vanish if **C(r)** goes to zero faster that 1/r as $r\rightarrow\infty$. The same condition guarantees that the integrals in the first two terms converge. The final result is exactly the representation of **C(r)** given in the statement of the theorem.

1.7 The Helmholtz Theorem

Uniqueness:

Suppose that $\nabla \cdot \mathbf{C_1} = \nabla \cdot \mathbf{C_2}$ and $\nabla \times \mathbf{C_1} = \nabla \times \mathbf{C_2}$. Then, if **W=C$_1$-C$_2$**, we have $\nabla \cdot \mathbf{W} = 0$, $\nabla \times \mathbf{W} = 0$. The double-curl identity tell us that $\nabla^2 \boldsymbol{W} = 0$ also. With this information, Green's first identity with $\phi = \psi = \mathrm{W}$ (W is any Cartesian component of **W**) takes the form:

$$\int_V d^3r\, |\nabla W|^2 = \int_S d\mathbf{S} \cdot W \nabla W.$$

The surface integral on the right side goes to zero when **C(r)** behaves at infinity as indicated above. Therefore, $\nabla W = 0$ or W=const. But W→∞ at infinity so W=0 or **C1**=**C2** as required.

1.8 Lagrange Multipliers

Suppose we wish to minimize (or maximize) a function of two variables f(x,y). The rule of calculus is to set the total differential equal to zero:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0.$$

But *dx* and *dy* are arbitrary, so:

$$\frac{\partial f}{\partial x} = 0 \qquad \text{and} \qquad \frac{\partial f}{\partial y} = 0.$$

Suppose the two variables are constrained by the equation:

$$g(x, y) = const.$$

$$\Rightarrow \quad dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy = 0.$$

$$\frac{\partial f/\partial x}{\partial g/\partial x} = \frac{\partial f/\partial y}{\partial g/\partial y} = \lambda.$$

1.8 Lagrange Multipliers

$$\frac{\partial f/\partial x}{\partial g/\partial x} = \frac{\partial f/\partial y}{\partial g/\partial y} = \lambda.$$

The constant λ appears because the two ratios cannot be equal for all values of *x* and *y*. In other words,

$$\frac{\partial f}{\partial x} - \lambda\frac{\partial g}{\partial x} = 0 \qquad \text{and} \qquad \frac{\partial f}{\partial y} - \lambda\frac{\partial g}{\partial y} = 0.$$

These are the equations that extremize, **without constraint**, the function:

$$F(x, y) = f(x, y) - \lambda g(x, y).$$

The Lagrange constant λ is not determined by this procedure.