

Different ways in which students handle chance encounters in the explorative setting of a dice game

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Abstract The purpose of this study is to investigate the ways in which Swedish seventh grade students (12 and 13 years old) handle chance encounters. Four groups of students working in pairs participated in the study. In the group discussions, which were tape-recorded and fully transcribed, the students were encouraged to explore strategies for winning a specifically designed dice game based on the sum of two dice. The dice game included four different set-ups of dice designed to bring to the fore different aspects of probability modelling and to offer the student the opportunity to encounter small differences in the mathematical structure of the sample space and of the probability distribution between the four different set-ups. The study describes strategies that the students use when confronted with these different set-ups, what their activities imply in terms of resources in handling random phenomena and what the dice game offers in terms of opportunities for learning probability. In order to explain such meaning-making processes the students' activities are viewed from a perspective that takes into consideration how the students' understanding varies with their interpretations of the situation they are confronted with, i.e., how they contextualize the different set-ups of the dice game. The results show how the students, during the course of the game, reorganize their interpretations of the mathematical content confronting them, and how a variation of guiding principles becomes the object of exploration. Approaches of extremes and a number model are described as a means for the students to identify and assign probabilities for the total of two dice.

Keywords Chance encounters · Probability thinking · Contextualization · Resources · Strategies · Dice game

1 Introduction

This study focuses on the ways in which students handle chance encounters. A particular study situation has been designed for dynamic purposes in terms of offering opportunities

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for exploration, application and evaluation of different aspects of the total of two dice. We will investigate strategies that come into play when Swedish seventh grade students (12 and 13 years old) encounter certain random phenomena and, in turn, to what extent their strategies reflect resources of a probability character. This means that a learner-oriented approach to analyzing students' chance encounters is adopted. In other words, the activities are viewed from the perspective of how the participants interpret the situation and chance encounters that emerge.

1.1 Previous research

Two research perspectives are prominent within the area of chance encounters: a psychological/cognitive perspective (Gilovich et al. 2002; Kahneman and Tversky 1972; Tversky and Kahneman 1973) focusing on cognitive patterns in order to identify students' misconceptions and judgemental heuristics, and a mathematics educational perspective focusing on learning from a mathematical point of view (Shaughnessy 1992). However, even if results from studies belonging to the former tradition are often reflected in mathematics education research the opposite is not always the case. Keeler and Steinhorst (2001) argue that this apparent dualism between perspectives is due to differences in research design and in particular how the focal task to be solved is constructed. Such differences in task construction may have an important impact on the learning outcomes (Cosmides and Tooby 1996). In the referred study the researchers chose a task originally used within a psychological/cognitive tradition (Casscells et al. 1978) with the purpose of replicating the results and in addition trying out whether a frequency version of the task would elicit Bayesian reasoning. Not only did the researchers succeed in reproducing the results of Casscells et al. they were also able to confirm their hypotheses concerning the importance of different task designs. By framing the single-event task in terms of frequencies the students were led to reflect on basic information never activated by the original task design. The result was that the proportion of students that reasoned soundly according to Bayes' rule increased. Even if it could be argued that the different task designs offered the students different kinds of information, another interpretation is possible: the frequency version offered the students the opportunity to reflect on data from a different point of view which in turn led to other associations and connections.

In Kahneman and Tversky (1972) subjects were asked which of the following two random distributions of 20 marbles was the most likely in Fig. 1.

Even though distribution II is more likely to appear randomly from a mathematical perspective, the results of Kahneman and Tversky show that a considerable percentage of the subjects (69%) chose distribution I. The authors argue that such a response is a matter of

Fig. 1 Two random distributions of 20 marbles

	I	II
Alan	4	4
Ben	4	4
Carl	5	4
Dan	4	4
Ed	3	4

representativeness. Accordingly, distribution I is judged as the most likely as it best represents irregularity, a feature that people tend to ascribe to randomness. Keren (1984) found reason to question if this really is the most proper way of interpreting responses which favour distribution I. Maybe the participants did not connect the distributions of marbles to specific children at all. What if they saw distribution I as including all of a group of outcomes in which three children get four marbles, one child gets three marbles and another child gets five marbles? This would imply that the children predict the likeliness of the two distributions by considering the number of ways that they bring about the outcomes of distribution II and I respectively. Keren (1984) moves on with a similar experiment from which he finds support for questioning representativeness as the only explanatory model. His findings point to the crucial importance of taking into consideration which sample space the students base their decisions on: “the knowledge of the sample space used by the students is crucial for understanding their responses” (p. 127).

1.1.1 Compound events – the sum of two dice

A single fair die has the equiprobable outcomes $\{1...6\}$, whereas for the sum of two fair dice, the 11 outcomes $\{2...12\}$ are not equally likely. Research has shown that students often find this transition difficult (Watson 2005). To understand compound events, such as the sums of two dice, requires one to be able to generate complete sets of outcomes for the events and use sample space symmetry and composition for making probability predictions (Polaki 2005).

In a study of factors affecting probabilistic judgments, Fischbein et al. (1991) discuss how probability predictions relate to available sample space. Regarding the sum of two dice they argue that some failures are due to the fact that there seems to be no natural intuition regarding the order of two dice. Even if the number of favourable representations is taken into account when predicting the chances of the totals of two dice, the proper responses may still not be given because one is not aware of all the possible configurations.

In Vidakovic et al. (1998), eighth grade students arrived at two different explanations for the behaviour of two dice. In one case they constructed a sample space of the distinctly different representations of the totals, not over the ordered pairs, resulting in a 21-element possibility space (e.g., (1, 2) and (2, 1) were not separated). In the alternative approach they took into consideration the order of the dice, resulting in a 36-element possibility space. Speiser and Walter (1998) reported similar results from their investigation of preservice teachers' reasoning in the setting of two dice.

If we focus on sample space information, equiprobability responses may be explained in several different ways. Regarding the total of two dice, Lecoutre (1992) argues that such responses mainly stem from a conceptualization of the random experiment as being only a matter of chance.

In Pratt (2000) 10 and 11-year-old students interacted within a computer-based microworld, called *Chance-maker*. Instead of focusing on the subjects' imperfections, Pratt investigated what students actually do in a study situation and, specifically, what different kinds of meanings they ascribed to specific randomly generated phenomena. In this respect he found it proper to introduce an approach in which actions are explained in terms of internal and external resources. External resources refer to components of the surroundings – tools, the task, the teacher and so on – as opposed to the internal resources, which are defined as individual decision models. Regarding equiprobability his results indicate that the four internal resources *unpredictability*, *irregularity*, *unsteerability* and *fairness* are mainly in play as decision models. As the students took notice of the frequencies of outcomes and

modified a gadget's working box to influence the appearance of its associated graphical display two new meanings for the behaviour of two dice emerged: the *Large Number Resource* – “the larger the number of trials, the more even the pie chart” (Pratt and Noss 2002, p. 471) and the *Distribution Resource* – “the more frequent an outcome occurs in the working box, the larger its sector in the pie chart” (Pratt and Noss 2002, p. 472).

Pratt's results emphasize that equiprobability responses in particular and probabilistic reasoning in general have to be explained in a broader perspective, a perspective in which the learners' activities are viewed from the different kinds of resources that come into play in a learning situation.

In contrast to Pratt, whose study was placed in a computer-based environment, the current study focuses on situations involving concrete objects. Furthermore, Pratt's distinction and connection between the internal and the external resources has been strongly influenced by the situated cognition perspective in that he takes into consideration how the setting may be thought of as providing meaning for knowledge: “...the setting provides meanings for knowledge in such a deep way that the knowledge seems to be embedded in the setting” (Pratt 2000, p. 604). The view of the current study however, is that there is nothing externally embedded that is transmitted to internal structures. Instead, according to constructivist principles, a perspective on learning is adopted in which interactions are viewed from the students' perspective, from the different ways the students interpret and experience chance encounters.

Summarizing the discussion so far, results indicate that students' responses in situations of uncertainty can be explained by how they process and bring to the fore information for their evaluation of the probability of an event. Perhaps it is not sensational that students' responses in a given situation are affected by what kinds of resources come into play, how resources become available and how they are made use of. However, fine-grained analyses of how such activities may be explained seem to be lacking. For instance, how do certain strategies come into play when students encounter compound random phenomena based on the total of two dice, and what in turn do their strategies imply in terms of probability resources? In the method section it is described how a specific dice game, based on the sum of two dice, was designed to challenge 12–13 year-old students' understanding of compound experiments.

2 Theoretical considerations

This paper adopts a constructivist approach to analyzing students' chance encounters, which means that the students' activities and articulated explanation models are viewed from a perspective which takes into account the students' interpretations of a given task, that is, what they think should be explained and dealt with. In order to explain the theoretical approach adopted in this study the chapter starts with an elaboration of the relation between the terms *task* and *problem*.

2.1 Task and problem

A task may be seen as a means for the teacher to communicate a learning object. It comprises what the teacher has said or has made known in other ways, as well as the learning aids the actual setting is equipped with. Halldén (1988) expresses this in short as follows: “the task is what is presented to the students by the teacher with the intention that

they are to do something and/or that they are to learn something” (p. 125). In order to understand what they are supposed to do and what they are expected to learn, the students have to interpret the task given. Such an interpretation constitutes the *problem* they see as being their obligation to solve. Furthermore it could be argued that it is exactly such individually formulated problems that are of crucial importance in an analysis of a learning activity “...when children give wrong answers it is not so often that they are wrong as they are answering another question...” (Bruner 1968, p. 4).

From a task-oriented perspective a learning situation may be looked upon as being equal to all students in the sense that it involves a set of objective learning material. Viewing the same situation from a problem-oriented perspective, i.e., from the individual students’ interpretations of the situation, the learning material leads to a variation in learning opportunities. Not taking this variability of interpretations into consideration, that is, adopting a normative approach when evaluating a study activity, would restrict the analysis to considering obstacles rather than possibilities. If it is the learners’ understanding of a study activity that determines what kind of explanations are relevant and what constitutes a sufficient explanation (Halldén 1988) it is more fruitful to base the analysis of students’ activities on their interpretations of that activity.

2.2 Contextualization

Learners’ conceptual understanding and theoretical models depend on how the learners experience and make sense of the learning material. Halldén (1999) has argued that such processes could be understood as processes of contextualization. Illustrating the principles of this line of reasoning he draws on a re-interpretation of one of Kahneman and Tversky’s earlier results. Kahneman and Tversky (1982) presented the participants in one of their studies with the following personal sketch:

Linda is 31 years old, single, outspoken and very bright. She majored in philosophy. As a student she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations (p. 496).

The participants were then asked to select which of the following two statements about Linda was the more probable: (1) Linda is a bank teller, or (2) Linda is a bank teller who is active in the feminist movement. From a mathematical point of view, in terms of the conjunction rule, the probability of the conjunction A and B cannot exceed the probability of either A or B alone. However, Kahneman and Tversky found that a considerable number of students chose the conjunction contained in statement (2), as being the more probable one. The authors concluded that the students had poor knowledge of probability theory, hence resorting to the representativeness heuristic. Halldén questions if we know whether the students who chose statement (2) in Kahneman and Tversky’s study understood the conjunction rule. Taking into account the context in which the students operate when trying to solve the task, Halldén (1999) argues that there are other ways to understand why the students do not always follow the norm. Presenting a group of 50 students of education with the same personal sketch used by Kahneman and Tversky, with the addition that he also asked them to give reasons for their choice, Halldén argues that the reasons why the students chose statement (2) was to be found, not in their understanding of the conceptual content of the task, but in their contextualizations of the situational framing of the task. Halldén reported that the students had placed the task within a setting different from the

expected statistical context and there they found the rules of probability inappropriate: "...students tend to perceive problem-solving tasks in the context of every-day life and consequently apply the kinds of problem-solving strategies that they use in every-day life" (p. 59).

One way of looking at the Linda example is to focus on how the learners' particular ways of viewing a situation activate and give priority to different aspects of the situation and the learning material: the individual perspective at hand determines what counts as a relevant and reasonable explanation and also what counts as empirical data. However, inversely, one can also say that aspects of the actual situation are allocated to different contexts in which the aspects in question make sense to the learner. Taking into consideration both of the mentioned explanation models one can talk of contextualization as a compound process, where the cognitive activity oscillates between interpretations and reflections about a coherent whole, the focal event and new information that comes into play (Halldén 1999). This implies that the learners' activities are viewed as being coherent in accordance with what the learner considers as being relevant.

Adopting such a student-oriented perspective implies a constructivist usage of the term context. In accordance with a constructivist view, context is considered to be a mental device referring to students' personal constructions shaped by individual interpretations. Research on contextualization has focused not only on the conceptual requirements of a given task but also on beliefs about the immediate setting as well as the appreciation of the discursive rules and social requirements. Accordingly, it is proper to introduce an explicit distinction between different kinds of contextual elements constituting different sub-contexts of a coherent whole. Slightly modified after Halldén (1999) and Wistedt and Brattström (2004) three types of contexts may be fruitful to refer to when analyzing students' contextualizations. First there is the *conceptual* context denoting personal constructions of concepts embedded in a study situation. Secondly, there is the *situational* context, which refers to aspects of interaction between the individual and the immediate surroundings in which learning occurs. This context may have a more prominent role in a dynamic learning environment than in a static one in that the former offers opportunities of feed-back in terms of exploration and applications (c.f. Tiberghien 1994). The third is the *cultural* context, referring to constructions of discursive rules, conventions and patterns of behaviour.

From a contextual point of view, handling tasks presented in a study situation and acquiring new concepts and strategies may be looked upon as a problem of finding relevant contexts by which new information can be interpreted and assimilated and new strategies be worked out (Caravita and Halldén 1994). In addition, some kind of meta-ability will be necessary to enable the learner to evaluate different approaches, different ways of looking at the world, or, in other words, an ability to differentiate between contextualizations. "Learning is then a process of decentering, in the Piagetian sense, rather than the acquisition of more embracing logical or conceptual systems replacing earlier less potent ones." (Caravita and Halldén 1994, p. 106).

2.3 Aim of the study

The study examines how Swedish students (12–13 years old) handle situations in which they encounter chance. More precisely, the aim of the study is to explain how various strategies come into play when students encounter certain random phenomena, and what in turn these strategies imply in terms of probability oriented resources. It will be described

how they contextualize a task, which is designed to provide opportunities for them to devise such strategies.

3 Method

3.1 Design objectives

What kind of tasks can serve the purpose of providing the information that is required to fulfil the aim of the study? In line with Wistedt et al. (1983, p. 15), the following global demands have been in focus to guide the design of the tasks used in the study:

- (1) They should concern phenomena that students are familiar with and may have intuitive models about,
- (2) they should also encourage variations in contextualizations and thereby promote discussion among students, and
- (3) they should at the same time incite reflection about mathematical aspects.

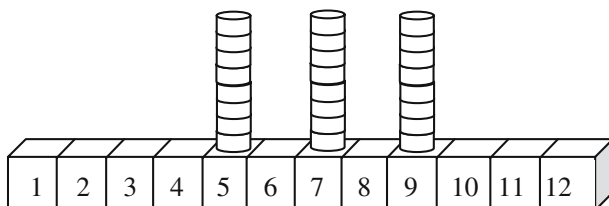
Game-situations offer good opportunities for learning (Speiser and Walter 1998; Vidakovic et al. 1998). In the current study a specific dice game was designed. Above all, dice games are a common feature in our childhood and hence the dice game situation was supposed to encourage the students to act intuitively. However, the intent was also to promote variation in interpretations and particularly to challenge the students to bring about certain probabilistic concerns. In Sweden, seventh grade students have not been formally introduced to probability theory.

3.1.1 The game

The game consisted of a play-board – numbered from 1 to 12 – a set of markers (Fig. 2) and especially designed dice.

Two teams competed in the game. The participants were instructed to distribute the markers among the 12 numbers on the play-board on the basis of the total of two dice. In order to capture diversity in the students' reasoning and to encourage them to take into consideration probabilistic aspects of the game the competitors were separated when working out their strategies. During the game the competitors took turns rolling the dice. If one team or both had at least one marker in the area marked with the sum of the dice, they were allowed to remove exactly one marker from this area regardless of which team rolled the dice. The team that first succeeded in removing all markers from the board won the game.

Fig. 2 The play-board of the game



3.1.2 An a priori analysis of the design

To stimulate reflection about mathematical aspects the design of the game was based on principles of variation (Marton et al. 2004). A design that accounts for variation offers the learner certain possibilities to discern, explore and compare features of the experienced phenomena with corresponding elements previously experienced.

A system of four specifically designed dice was arranged for the game. The overall guiding principle of the system was to challenge the students' ability to base probability predictions on sample space composition. The aim of the game, to reduce a certain number of markers, was also expected to encourage the students to adopt a probabilistic reasoning reflecting similarities with the inherent reasoning of the *distribution resource* (Pratt and Noss 2002). In the case of the current dice game, the more frequent an outcome is represented in a sample space, the more often it will appear in a series of throws.

Empirical evidence is understood to encourage students' probabilistic reasoning (Steinbring 1991). However, we conjecture that if students' notions of for instance equiprobability (Lecoutre 1992) are to be challenged, the part-part relation between the observed frequencies has to stand out noticeably. When rolling two ordinary numbered dice a number times this might not be the case as the probabilities between consecutive totals is too similar (e.g., the probability for 6 is $5/36$, whereas the probability for 7 is $6/36$, leading to a difference in probability of only $1/36$). Below will be presented a design in which the probabilities between consecutive totals differ by as much as $1/4$. To further encourage reflections on frequencies the students were asked to start the next discussion by reflecting on how they would place the markers if they played the game again with the same dice.

Now we shall examine more specifically the design of the four different dice set-ups that was arranged for the game. The dice were presented to the students in the following order:

- (1) The yellow setting – The faces were marked with one and two dots, distributed as (111 222) and (111 222).
- (2) The red setting – Included two different dice, each with a distribution of two outcomes among the faces as (222 444) and (333 555).
- (3) The blue setting – Similar to the yellow setting, with the difference that the dice now had four sides marked one and two sides marked two, that is (1111 22) and (1111 22).
- (4) The white setting – These dice were a mix of the red and the blue setting. The dice displayed the distribution (2222 44) and (3333 55).

The implicit probability distribution of the totals between setting one and two is consistent according to sample space structures. In Fig. 3 the 36-element possibility space is illustrated for the two first designs. In field A nine representations of sum three are combined through the ordered pairs (1,2). In field B, C and D we find nine pairs of (2,2), (1,1) and (2,1). The symmetry of the distributions of the single dice reduces the calculations of the 36-element possibility space to a sample space of four equally likely outcomes. Thus for the yellow setting, in which we have the three resulting sums 2, 3 and 4, the structure of the underlying sample space takes the format: $\Omega_Y = \{(1,1), (1,2), (2,1), (2,2)\}$. By sample space composition we ascribe the theoretical probabilities to the sums as $P(2)=P(4)=1/4$ and $P(3)=1/2$.

In the two first settings the students were asked to distribute 24 markers. There are several reasons for the choice of that number. First, from the results of earlier studies previously referred to there were reasons to assume that the students would have difficulty in accounting for the two different ways of rolling a three, which might lead them to equiprobability thinking. Distributing 24 markers offers the students the opportunity to

a

2	A	B
2		
2		
1	C	D
1		
1		
1		
	1 1 1	2 2 2

b

4	E	F
4		
4		
2	G	H
2		
2		
2		
	3 3 3	5 5 5

In Fig. 4 the 36-element sample space of the two last designs is presented. Similar to the yellow setting the sample space of the blue setting is formed as $\Omega_B = \{(1,1), (1,2), (2,1), (2,2)\}$. However, this time the four outcomes differ in the number of representations within the 36-element possibility space. From Fig. 4a we get 16 pairs of (1,1) in field K, eight pairs of (1,2) in field I, eight pairs of (2,1) in field L and four pairs of (2,2) in field J. Hence, we get 16 representations of the sum 2, 16 representations of the sum three and four representations of the sum four, which generate the theoretical probabilities $P(2) = P(3) = 4/9$ and $P(4) = 1/9$. Similar to the second setting the students in the fourth setting are

a

2	I	J
2	K	L
1		
1		
1		
1		

1 1 1 1 2 2

b

4	M	N
4	O	P
2		
2		
2		
2		

3 3 3 3 5 5

offered the opportunity to act on a design in which they can construct four distinct representations of each sum. In Fig. 4b there are 16 pairs of (2,3), eight of (2,5), eight of (4,3), and four of (4,5). This leads to the theoretical probabilities: $P(5)=P(7)=4/9$ and $P(9)=1/9$.

To provide for frequency data, i.e., generating sufficiently long series of throws, and to establish opportunities for the students to replicate exactly the proportions of all favourable outcomes when distributing markers in the game the two last rounds concerned the betting of 36 markers. Accordingly, the 36 markers in round three may be placed as: 16 markers on 5, 16 markers on 7 and four markers on 9.

Above the design objectives of the dice game have been outlined. Based on findings from previous research the design is assumed to mainly lead the students to sample space reasoning. However, we have to keep in mind that it is the ways in which the students handle the game that are in focus, how they interpret the situation and bring to the fore different kinds of resources and strategies for dealing with the underlying regularities of the game.

3.2 Data collection and data analysis

The study was carried out in the students' ordinary classroom. Eight seventh grade students (12 and 13 years old) were divided into four groups with two students in each group. The observation took about 70 min, of which 10 were spent on presenting the rules of the game.

The group discussions, which were tape-recorded and fully transcribed, were located in each corner of the room. When they had finished their discussions the teams entered the play-board, placed in the middle of the classroom, and started to play against one of the other teams. In order to capture the diversity of the students' reasoning they were asked to refrain from commenting on the game while at the play-board. However, spontaneous reactions during the play were videotaped.

The researcher was present during the session, taking the role of an active observer. The researcher introduced the game to the students and sometimes intervened during the small-group discussions asking the students to clarify their reasoning.

3.2.1 *Intentional analysis*

In the analysis of the data the different ways in which students handle chance encounters were explored by means of a theoretical model in which the students' activities are viewed from the perspective of personal constructions, that is, how the students contextualize chance encounters. Of course, we have no direct access to the learners' thinking, and thus, neither to their interpretative work. But when the students are involved in an activity, we are able to observe their behaviour and their articulated expressions. It is reasonable to assume that both behaviour and expressions rest on personal, rational considerations. Following the tradition of contextualization research a method of analysis, *intentional analysis*, was used in which the students' explicit performance is explained in terms of intentions (Halldén 1999; Ryve 2006; Scheja 2002; Wistedt and Brattström 2004). The principles of intentional analysis were introduced by von Wright (1971), to serve as a basis in the analysis of the meaning of human behaviour. This means that in order to understand a sequence of activities, these have to be regarded as intentional. Practically the researcher's way of reasoning when analyzing the data can be described in the following way: Our interpretation of the student's behaviour as a means to achieving a goal is the basis for deciding what act the student actually performs. On the basis of the student's activities and

articulated expressions we hypothesize a problem that the student is trying to solve and the ways in which the student perceives that problem. By systematically matching the hypothesis on these two levels of action we create a model of the student's contextualization described in terms of its conceptual, situational and cultural aspects. This matching procedure makes it possible to test and determine how well the model serves as an explanation of the observed behaviour.

4 Results and analysis

In this study the work of two children, Peter and Sabina (group C), will be the subject of investigation. By focusing on just one pair of students, we can describe in detail their contextualizations and how they differentiate between their contextualizations. However, the work of the other groups – Tom and Louise (group A), Mats and Jenny (group B) and Petra and Lars (group D) – will serve to support the argumentation.

In the group activities the participants' major intention is to win the game, that is, they perceive the activity in a context of competition. Trying to bring about winning strategies two overall intentions become prominent throughout the four rounds. In the two first rounds focus was on identifying the resulting sums. In the two last rounds the students changed their focus from attending to *the possible sums* to the *different number of representations* of the sums. Consequently, the analysis is divided into two parts.

4.1 The work in rounds 1 and 2

As the activities during the two first rounds are similar in several respects, the analysis will start with a presentation of excerpts from both of these rounds.

In the first round the dice were marked as (111 222) and (111 222). The students started the group discussion almost immediately by trying to work out which sums were obtainable. They started by rolling the dice a couple of times. Peter suggested that the sums could be all numbers up to four and Sabina restricted the sample space further:

- (1) S.: It can't be anything else. But it can't be smaller than two either...so we have to use /numbers/ between one and four. No it can't be one either.
- (2) P.: Between one and four?
- (3) S.: ...because it can't be one. We must use two, three and four.

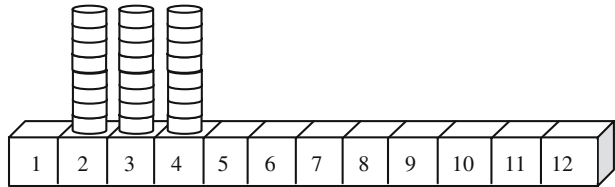
Even if their work does not seem to be based on any deeper systematic analysis, the students identify all possible totals. What can be seen so far is that their focus is basically on identifying the highest total, four, and the lowest total, two. Turning to the discussion of how they should distribute the 24 markers on the play-board Sabina almost immediately came up with a solution.

- (4) S.: How many on each? 24 divided by three...it will be eight, won't it?
- (5) P.: Yes, right hmm.

The strategy satisfies the group in that they did not find it necessary to explore the issue further. Sabina and Peter's set-up in the first round appears uniform as in Fig. 5.

In the first round Sabina and Peter beat group B, Mats and Jenny, who had placed their markers only on two and four with 12 markers on each. When asked about how they would

Fig. 5 The set-up in the first round, of group C



place their markers if they were to play again the students seemed encouraged by their victory and stated that they would try the same approach.

They then turned to reflect on the next pair of dice, marked as (222 444) and (333 555). Peter and Sabina approached this session similarly to the previous one, trying to identify the resulting totals.

- (6) S.: How high is it?
- (7) P.: It's five, from two to five.
- (8) S.: And on this one there is three to five. On this it is from three to five and on this from two to four.

Following an exploration of the design of each die the discussion continued with a mapping of the totals. During this activity they tried out various ways of rolling the dice.

- (9) S.: The highest it can be is...
- (10) P.: It's five!
- (11) S.: No, but with both the dice.
- (12) P.: All right, nine. Nine is the highest!
- (13) S.: And lowest? Lowest...five. Nine to five or five to nine?
- (14) P.: Yes!

Having identified the extremes, they turned to distributing the 24 markers.

- (15) S.: The same on each then?
- (16) P.: Yes, we can take that.
- (17) S.: 24...24 divided by five, it is not possible!
- (18) P.: Isn't it?
- (19) S.: No, we have to put more on some.
- (20) P.: Okay, but not on nine. On five, then it will be six.
- (21) S.: What?
- (22) P.: You said more on one number.
- (23) S.: No, not like that. It should be between five and nine. But how many on each? Then we have to have...24 divided by five, but it is not even, so we have to put only four on one instead. It is not even! ...then it will be only four there.
- (24) P.: 24 divided by three then?
- (25) S.: Yes but we shouldn't have that.
- (26) P.: It will be eight.
- (27) S.: Yes but we will still not take that.

Sabina started to draw their set-up and said:

- (28) S.: ...twenty, we should put in four more. Which shouldn't we put four on...or on which one should we have four?

The final question was not answered. However, the set-up came out as in Fig. 6.

4.1.1 Discerning impossible totals from possible ones

The excerpt above indicates that the group, in order to optimize the chances of winning the game, mainly interprets this as a question of differentiating possible sums from impossible ones. This was more or less the issue for all four groups during the two first rounds. However, regarding the second setting two different set-ups appeared. Groups B and D only placed markers on the possible totals 5, 7 and 9 while group A ended up with the same resulting sums, $\Omega_s = \{5, 6, 7, 8, 9\}$, like Peter and Sabina, who also included the impossible outcomes 6 and 8.

An extreme value approach Fischbein et al. (1991) argue that their participants did not have a clear definition of the terms possible, impossible and certain outcomes and thereby failed to make sound identifications of sample spaces. However, it could be argued for an alternative explanation where the reasons for involving impossible events is described as a problem of contextualization in terms of adopting an *approach of extremes*. Specifically, the problem is not that seventh grade students in Swedish schools cannot add two natural numbers and connect such an observation to the chances of winning the game. Instead, their ways of rationalizing the situation seems to make them unaware of the inappropriateness of how to place the markers. Investigating their interpretations further, the utterances [6]–[8] above indicate that Peter and Sabina activate situational elements in terms of the extremes of each die. Probably stimulated by everyday influences, in other words that ordinary dice are continuously numbered from 1 to 6, Sabina's summarizing in [8] indicates that she also assumes, mistakenly, that the dice in question are also continuously numbered between their extremes. Hence, it could be argued that the students base their work of identifying the possible totals of a particular pair of dice i.e., dice that are continuously numbered between two extremes. Perceiving the individual dice as continuously numbered, it makes sense for the students to assume that the possible totals include all totals between the highest and the lowest sum identified.

Uniform distributions In order to understand the ways in which students ascribe probabilities to the totals in terms of distributing markers, we have to take into account their interpretations of the situation as a whole. All four groups come up with set-ups reflecting equiprobability. In the work of Groups B and D, with $\Omega_s = \{5, 7, 9\}$, eight markers were placed on each total and group A distributed their markers in exactly the same way as Peter and Sabina. These uniformly directed responses could be explained in line with Lecoutre (1992), i.e., that the students consider the situation to be just a matter of chance. Since it seems as if the students just focus and base their judgments on the resulting totals, such responses appear to be reasonable. This means that an equiprobability response, based on the idea that everything is just a matter of chance, may be assumed since the students are apparently unaware of different representations of the resulting totals. Based on earlier

Fig. 6 The set-up in the second round, of group C



research findings there is reason to believe that the students would have difficulty in taking into account the order of the dice, i.e., the difference between the outcomes (1,2) and (2,1) in the first setting (Fischbein et al. 1991). However, since they still focus only on possible totals, none of the four groups in the second setting reflect on the distinctly different ways to represent seven ($5+2$ and $3+4$). In other words, as a result of their current contextualization, to identify possible totals, no further information is processed by the students, nor seems to be needed to be processed. If the markers are distributed according to a context in which only one representation of the total is available it is reasonable for the students to place their markers equally among the sums.

School cultural influences By reflecting on school cultural aspects we may be able to construct an alternative explanation to account for the students' equiprobability approach. Following the above line of reasoning, that the students have brought about three totals 2, 3 and 4 in the first round, the operation of division takes a prominent role particularly in Peter's reasoning. To support such a concern, there are reasons to believe that his behaviour has been determined by conceptions of norms and expectations relevant to school mathematics. His task interpretation is dominated by applications of tools that are relevant to the culture of education that is, that students know from experience that they are supposed to use all numbers appearing in a task and some computational device (cf. Säljö 1991). For Peter this means that he is conscious of three totals and 24 markers. He applies division to these numbers, which leads to his suggestion to distribute eight markers on each of the three totals.

4.2 The work in round 3 and 4

Before presenting the discussion between Sabina and Peter during the third set-up, we should observe how they reacted to what happened during the second set-up. Using the set-up shown in Fig. 6, they almost immediately understood that they would lose. Their competitors, members of group D, had placed their markers uniformly on the sums 5, 7 and 9. Before they began to discuss the third set-up Peter and Sabina were asked to consider how they would distribute their markers, if they were given the opportunity to play the second round again.

- (29) S.: We would not do the same, if we could play again.
- (30) P.: No!
- (31) S.: No markers on six and none on eight, 'cause it can't be those.

They were then asked to turn to the discussion of exploring strategies for winning the third game. Now they had to distribute 36 markers on the basis of the blue dice, marked as (1111 22) and (1111 22).

- (32) S.: There are more ones than twos, so it is twice as big a chance...that it will be two then it will be...
- (33) P.: Four.
- (34) S.: We put quite a lot on two.
- (35) P.: mm, and almost none on...
- (36) S.: ...four. Should we put more on three than on four?
- (37) P.: Well, but these have to have more.

(38) S.: Yes, but most of all on this one.

Even if it is a bit unclear which numbers they are referring to in the last two exchanges, we can observe that they are taking into consideration the differences in chance between the totals.

As in the first two rounds the different groups' activities during the remaining two rounds were similar. However, now the focus was shifted from the *resulting totals* towards an exploration of the *likelihood of each total* in terms of the number of favourable representations of the totals.

When the group prepared for the final set-up of markers, Sabina took on a prominent role. Unfortunately she did not clearly show which of the sums she was focusing on. However, reflecting on the final documentation it can be argued that she begins by placing a couple of markers on two and then adjusts the number of markers on four.

(39) S.: ...11, we have to have more on that. We can take 16 there, and then put eight there. I have put 24. Then we have 12 left. Should we have 12 here then?

Even though the last question is left unanswered, their strategy ends up as in Fig. 7.

During the third play the students notice that the piles of markers were reduced more or less fast during the game. When following up the third round Sabina and Peter concluded that they would place some extra markers on the numbers two and three as they needed fewer on four.

When confronted with the last pair of dice, the white setting designed as (2222 44) and (3333 55), they began by rolling and focusing on one die each. The discussion started thus:

(40) P.: Two and four.

(41) S.: Threes and fives.

(42) P.: 2 twos, no 3 twos...no 4 twos.

(43) S.: 4 twos, here there are 4 threes.

(44) P.: These were the same.

(45) S.: Now there are more threes and more twos. Consequently it can't be...

(46) P.: Now we put most on twos and...

(47) S.: It can't be two. The lowest it can be is five and the highest it can be is nine. It can't be six and eight, we learned that last time. It should be most on five, because there are most threes and twos.

Their approach of distributing these particular numbers of markers is to a great extent similar to how the group approached this problem during the previous round. They mainly take the design of the particular dice as their point of departure. However, probably

Fig. 7 The set-up in the third round, of group C

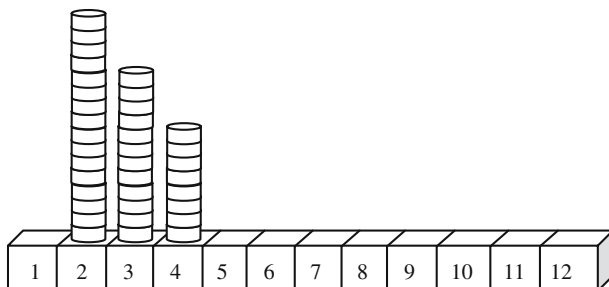
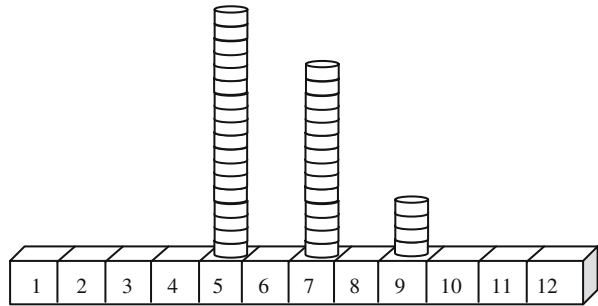


Fig. 8 The set-up in the fourth round, of group C



stimulated by the observed difficulty to receive fours in the third round the group chose to place a lower proportion of markers on the corresponding total in round four, i.e., on the number nine. In the documentation of their work we find that their final set-up is shaped by the notion that they should start out by placing markers on the total that is regarded as the most likely outcome, and to adjust the rest on the basis of that.

- (48) S.: 36, then we put... [counting]...16, 17 and 18 on five for example. And like 14 on that and four on that.

Based on their discussion they end up with the set-up for the last game as in Fig. 8.

4.2.1 Matching individual dice in the design of the totals

As can be seen from the excerpts from the third and fourth rounds, Peter and Sabina now give lower priority to figuring out possible and impossible sums. Utterances [32]–[38] above indicate that they now begin to reflect on how likely the possible totals are. However, issues of what counts as possible totals are not left without consideration. For instance, based on their second-round experience Sabina expresses in [47] the necessity to stop and confirm the actual sample space of the resulting sums in round four. Nevertheless, even if control of possible totals is activated as a component in their reasoning, this issue has a lower priority for Peter and Sabina in the last two rounds of the game.

With the help of fellow group members or the observer all groups were implicitly or explicitly aware of how the single dice of the third and the fourth round differed from the two previous settings. Let us therefore turn to a discussion of how they made use of this information in their later activities.

4.2.2 Proportionality thinking

In the current dice game situation the *distribution resource* (Pratt and Noss 2002) implies that the more frequently a total is represented in a sample space, the more often it will appear in a series of throws. With this in mind it has been suggested that the *equiprobability approach* prominent in the two first rounds resulted from sample space compositions in which the totals were represented only once. In the two closing rounds the students' ability to reason in line with the distribution resource and its implicit additional character, is strengthened. One could say however, that what is changed is the way in which the sample

space is interpreted and the corresponding sample space compositions are made for the purpose of modelling the implicit probability distribution of the totals.

The ways in which the students bring about the underlying sample spaces of the third and the fourth rounds can be said to be in accordance with what Lecoutre (1992) calls *the number model* and Pratt (2000) refers to as *the fairness resource*. Lecoutre identifies the number model in situations where you draw objects from an urn. Children applying the strategy argue that given an urn containing two orange candies and one lime candy you are more likely to get a pair of orange candies than one orange and one lime. The different proportions of candies in the urn heavily influence their reasoning. The basis of what Pratt calls the fairness approach is actually more related to the first two rounds described in this report, in that he refers to a discussion in which the children speak about individual fairly distributed dice. On the basis of his data Pratt draws the conclusion that children intuit that the total of two dice should be likely to appear equally often if the dice are individually fair. As argued in the analysis of the two first rounds, the participants in this study interpret the situation in such a way that the design of each die is given low, if any, consideration. Consequently, we found no reason to consider the activities in the two first rounds in terms of the fairness resource. However, as the children in the two last rounds go beyond their previous interpretations, aspects of Lecoutre's number model and Pratt's fairness resource become prominent in the children's reasoning process in the current study as well.

4.2.3 A number model approach

What will be argued is that the students perform some kind of matching procedure between the distribution of individual dice – the ratio between displayed values – and the number of representations of the totals, which can be spoken of in terms of a number model. From [32] a reasonable explanation could be to view this procedure as being situated in a context of addition. For instance, in the third round the proportions between placing markers on two and four seem to be arrived at by comparing the entire number of ones with the entire number of twos. Such a construction would make it plausible to conclude that there is twice as big a chance of twos occurring than fours. However, even if such a model gives echo to some extent to summarizing and combining procedures, it does not provide the students with a correct description of the proportions of the number of representations of the totals. Neither the two ordered ways of representing three in the third round, nor the distinct ways of representing seven in the fourth round are brought into focus and they are thus not available for reflection when judging how likely the different totals are. Actually, in none of the groups any explicit discussions emerge about how to arrive at the number of markers placed on 3 and 7 in the third and fourth round respectively. One interpretation of utterances [34]–[36] and [45]–[47], is that the students once again are heavily influenced by the notion of extremes. However, this time the extreme oriented strategy is composed both of aspects of extremes of borders and, following the above line of reasoning, of extremes in terms of the proportions of numbers displayed on the individual dice. Viewing the situation from such an extreme oriented point of view the sums in-between are given scant consideration. Nevertheless, the number of markers placed on three and seven indicates that the children again make use of continuity thinking in their strategy work.

In addition, we notice that in round four the students digress from the linear way of distributing markers, which they used in round three. It seems as if the students found reason to adjust their number model thinking on the basis of the observed frequencies during the third play. Thus, it appeared particularly difficult for the students to reduce the markers placed

on the sum four, the highest possible total in round three. On account of that, the students decreased the number of markers on the sum nine, the highest possible total in round four.

5 Discussion

In this study learning is considered as the ability to differentiate between parameters in a given situation as well as the ability to evaluate different approaches, different ways of looking at the world, that is, an ability to differentiate between contextualizations (Caravita and Halldén 1994). From such a perspective we may discuss some specific findings that indicate that children 12–13 years of age possess several important resources in dealing with random phenomena. We will also elaborate on some methodological concerns and on how certain aspects of the interaction during the dice game provided learning opportunities for the students.

5.1 Specific findings

In the analysis two overall contextualizations are described in detail, one prominent in the two first rounds and the other discernible in the two last settings. During the first two setups most of the students' activities were understandable if we ascribed to the students the efforts of trying to discern impossible totals from possible ones (cf. Fischbein et al. 1991). To accomplish that, several children used a strategy that took into consideration the extremes of possible totals. Identifying the lowest and highest possible total respectively, all possible totals were viewed as being constituted by these extremes and the numbers in-between. In addition, such a contextualization also reflects that the children to some extent make use of continuity thinking. However, as has been pointed out, the features of ordinary dice might also be a determinant factor. With ordinary dice, gaps in the sample space of the sums never appear.

In the analysis it has been argued that the focus on identifying resulting totals and in addition, on avoiding placing markers outside extremes, affected the students ways of assigning probabilities to different totals. As far as the participants are concerned the former problem at hand is dominant. Thus any further investigation of the situation has a low priority. Their interpretation seems to “hold them back”. To some extent this may be explained by relating their activity to the norms of education: the children believe that they have found what the teacher implied they should seek and therefore do not see the need to go any further in their exploratory activities. Anyway, since the students base the probabilities of the totals on a sample space composition in which each total is represented only once, it seems reasonable to conclude that they consider each total to be equally likely to appear in a series of throws and therefore distribute the same number of markers on the outcomes in question.

Fischbein et al. (1991) argue that conceptual constraints may explain why children fail to take into account the order of the dice. The results of the current study indicate that such negligence may, to some extent, also be a problem of contextualization. If the children solely focus on the resulting totals, and in particular on the extremes, they will not find it appropriate to examine different representations and combinations of these sums. In the analysis of the second round we gained further support for the interpretation that only the sums were taken into consideration by the students. In that case the order of the dice was never an issue of exploration. The students did not take into consideration the two distinctly

different ways of representing the total seven. They found one representation of seven and for the purpose of solving the current problem that was regarded as sufficient.

As far as the final two rounds are concerned the results indicate that the changes in the design of the dice played a prominent role in challenging the students to change their understanding of the game. The most obvious change was perhaps that all groups in the final two rounds complement their previous approach by restricting possible outcomes to be located within the extremes. Reflecting particularly on their experiences in round two, all groups became more cautious about how to place their markers during the latter rounds, with the result that none of the groups placed markers on impossible events in the third and the fourth rounds. However, the results also show that the students re-organize their contexts of interpretation in such a way that the chance of the totals becomes more focused. This shift stresses that the students' reasoning involves an interplay between empirical and theoretical ideas of probability. According to the distribution resource (Pratt and Noss 2002), the students realize that the totals will appear a different number of times in a series of throws since the chance of the various totals was observed to be different. In line with Vidakovic et al. (1998) and Speiser and Walter (1998), the present study also shows that learners are able to reflect on different sample spaces when modelling the chance of the totals. During round three and four the students' contextualizations concern the distribution of markers according to how they perceived the number of favourable representations of the different totals. Corresponding to the fairness resource (Pratt 2000), the number of representations of the totals was matched to the distributions displayed by the individual dice. Regarding this matching an approach of extremes also became prominent in the students' activities during the later rounds. However, in terms of a *number model*, the approach of extremes in this case refers to the perceived chance of the most likely and the less likely total.

Hence, just as Fischbein et al. (1991) and Keren (1984) conclude, the current study points to the fact that the sample space that is available to the children is of great importance for their responses. The students seem to understand intuitively that the number of favourable outcomes of an event affects the likeliness of that event. But, as their contextualizations do not encourage them to consider all representations of each total, their decision models are based on a limited amount of information.

5.2 Aspects of the learning situation

The overall guiding principle for the design of the game was to challenge the students' ability to base probability predictions on sample space composition. A specific task system based on the principles of variation (Marton et al. 2004) was designed to encourage reflection on the specifics of the subject matter. The fact that experimental evidence may promote probability reasoning (Steinbring 1991) was also taken into account when the dice were designed. Accordingly, to further the students' reflective processes the dice were designed to bring about distinct results in observed frequencies.

Above all, the game situation stimulated the students to adopt a competitive attitude to the game. To win the game was considered as essential. The task to bring about winning strategies encouraged the students to search for the underlying regularities of the game.

During the session all groups changed their focus from possible totals to the chance of the totals, in terms of their different number of representations. As this could be considered to be a desired progression of the students' activity the following question could be addressed: What significant aspects of the game served to support this learning route? First, the situation offered the students the possibility to reflect on concrete material. By turning,

twisting, and rolling the dice the participants became naturally stimulated to make observations. Second, the immediate feedback from the moment of play became of crucial importance in the process of evaluation. For instance, having placed markers on impossible events all teams realized almost immediately why one of the teams had to lose. Moreover, in the fourth round Peter and Sabina adjusted their *number model thinking* on the basis of how they experienced the difficulty getting the sum four in the third round. Third, the warped distribution of the single dice particularly challenged the students' equiprobability responses. In two of the groups this variation in design was spontaneously detected by the students and in the other two the observer chose to intervene to make the information available to the students. However, once familiar with the contrast in design the students were given the opportunity to compare and reflect on how certain aspects of the design should be dealt with.

From a teaching point of view however, we notice that a competitive attitude and confirmative feed-back does not always provide incitements for probabilistic explorations. Peter and Sabina won the first round. When they were asked how they would bet if they were to play with the first pair of dice again they did not express any need for re-consideration. Consequently they did not reflect at all on how the dice behaved during the first game and how they experienced the frequencies. One way to challenge such a direct re-using of a strategy could be achieved by letting the students play with the same couple of dice several times. In addition, in order to encourage further reflection on frequencies, the students could also be asked to document in certain frequency tables the outcomes that appear during a game.

However, having the competitors separated during their strategy-building seemed to encourage probabilistic explorations. The students were not able to base any decisions on how their opponents bet. Instead, they had to base their predictions on how they perceived that the dice would behave in the game.

Summarizing the potentiality of the dice game and how it was particularly designed to function as a learning material, it could be argued that on the one hand it stimulated the learners to come up with questions and suggestions about dealing with chance encounters and, on the other hand, brought into focus issues of evaluation and re-interpretation of previous encounters with the dice.

6 Concluding remarks

In this study the ways in which Swedish seventh grade students (12 and 13 years old) handle chance encounters have been scrutinized. In particular the focus has been on how certain strategies come into play when students are trying to devise winning strategies during a dice game, and what in turn these strategies imply in terms of abilities and opportunities for thinking and learning about probability. In order to explain such meaning-making processes, the ways in which students understand and explore random phenomena in a given situation has been described in terms of contextualizations. This means that we have taken into account how the students' understanding varies with their interpretations and constructions of conceptual, situational and cultural frames of references.

To gain a picture of the students' contextualization processes, their behaviour has been regarded as rational in terms of intentions: interpretations and conceptual constructions of the situation in question have been considered as being manifested in their goals for the activity, i.e., in the problems the students are trying to solve.

The report has shown how the model of contextualization may contribute to broadening the field of understanding chance encounters. Taking into account the students' interpretation of the situation in its entirety makes it possible to explain and understand how different kinds of resources evolve and interact when learners deal with situations involving elements of uncertainty.

We have described how the students change their interpretation of the game, a process which can be viewed as a process of decentering (Caravita and Halldén 1994). In the analysis two overall contextualizations have been singled out. During the two first rounds the students mainly try to discern impossible totals from possible ones and it has been argued that conceptual resources such as *continuity thinking* as well as *delimitations in extremes* were highlighted in that process. In the third and the fourth rounds the students turned their attention to the chances of the totals. Based on how likely the totals were perceived, the students showed evidence of the *distribution resource* (Pratt and Noss 2002) in that they reproduced the implicit probability distribution with the 36 markers. The students accounted for the probability distribution of the totals by means of a *number model*. This means that the chances of the total of two dice were matched to the proportions displayed by the distributions of the individual dice. What the number model implies is that we find good reason to extend the *fairness thinking* found in the uniform case (Pratt 2000) to count also when the individual dice are perceived as not being uniformly distributed. It also points to the fact that students who have not been taught probabilistic theory, spontaneously account for the number of favourable representations when predicting the chances of the totals of two dice.

However, the report also points to some issues of necessary interventions to be made if we wish to gain further information about the students' ability to deal with compound random events such as the total of two dice. Specifically, it would be interesting to scrutinize in detail activities in which students act in accordance with the number model, that is, to further investigate the students' conceptual repertoire underlying such combinatorial and proportional thinking. It would be very interesting for instance to ascertain how they make sense of the chance of numbers in-between the extremes. Particularly it seems as though the students' combinatorial capacity is constrained by symmetry thinking. Lecoutre (1992) discusses how geometrical concerns may promote combinatorial reasoning. The physical shape of ordinary dice only challenges the students to take into account the numbers displayed. What if the students were allowed to act on physical asymmetrical shaped dice. How would this challenge the students' combinatorial thinking? Using asymmetrical dice would probably also encourage the students to reflect on frequency data. With symmetrical dice the probability of the single outcomes are given. With asymmetrical dice samples may be needed in order to gain a picture of the behaviour of the dice.

Moreover, in further research it may be fruitful to encourage the participants to be more explicit about how they motivate their decisions. One idea of how to gain such information would be to have the students design their own dice for the game. Letting each team design and play with their own pair of dice will probably lead to further information about how they deal with issues of possible and impossible outcomes as well as how they assign probabilities to different totals.

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References

- Bruner, J. S. (1968). *Towards a theory of instruction*. New York: Norton.
- Caravita, S., & Halldén, O. (1994). Re-framing the problem of conceptual change. *Learning and Instruction*, 4, 89–111.
- Casscells, W., Schoenberger, A., & Graboyes, T. B. (1978). Interpretation by physicians of clinical laboratory results. *New England Journal of Medicine*, 299, 999–1001.
- Cosmides, L., & Tooby, J. (1996). Are humans good intuitive statisticians after all? Rethinking some conclusions from the literature on judgment under uncertainty. *Cognition*, 58, 1–73.
- Fischbein, E., Nello, M. S., & Marino, M. S. (1991). Factors affecting probabilistic judgements in children and adolescents. *Educational Studies in Mathematics*, 22, 523–549.
- Gilovich, T., Griffin, D., & Kahneman, D. (Eds.) (2002). *Heuristics and biases: The psychology of intuitive judgement*. Cambridge: Cambridge University Press.
- Halldén, O. (1988). Alternative frameworks and the concept of task. Cognitive constraints in students' interpretations of teachers' assignments. *Scandinavian Journal of Educational Research*, 32, 123–140.
- Halldén, O. (1999). Conceptual change and contextualisation. In W. Schnotz, M. Carretero & S. Vosniadou (Eds.), *New perspectives on conceptual change* (pp. 53–65). London: Elsevier.
- Kahneman, D., & Tversky, A. (1972). Subjective probability: A judgement of representativeness. *Cognitive Psychology*, 3, 430–454.
- Kahneman, D., & Tversky, A. (1982). On the study of statistical intuitions. In D. Kahneman, P. Slovic & A. Tversky (Eds.), *Judgment under uncertainty: Heuristics and biases* (pp. 493–508). Cambridge: Cambridge University Press.
- Keeler, C., & Steinhurst, K. (2001). A new approach to learning probability in the first statistics course. *Journal of Statistics Education*, 9, 1–23.
- Keren, G. (1984). On the importance of identifying the correct sample space. *Cognition*, 16, 121–128.
- Lecoutre, M. P. (1992). Cognitive models and problem spaces in 'Purely Random' situations. *Educational Studies in Mathematics*, 23, 557–568.
- Marton, F., Runesson, U., & Tsui, A. (2004). The space of learning. In F. Marton & A. Tsui (Eds.), *Classroom discourse and the space of learning*. Mahwah, NJ: Lawrence Erlbaum.
- Polaki, M. V. (2005). Dealing with compound events. In G. A. Jones (Ed.), *Exploring probability in school: Challenges for teaching and learning* (pp. 191–214). Berlin Heidelberg New York: Springer.
- Pratt, D. (2000). Making sense of the total of two dice. *Journal for Research in Mathematics Education*, 31 (5), 602–625.
- Pratt, D., & Noss, R. (2002). The micro-evolution of mathematical knowledge: The case of randomness. *Journal of the Learning Sciences*, 11(4), 453–488.
- Ryve, A. (2006). Making explicit the analysis of students' mathematical discourses – Revisiting a newly developed methodological framework. *Educational Studies in Mathematics*, 62, 191–210.
- Scheja, M. (2002). *Contextualising studies in higher education – First-year experiences of studying and learning in engineering*. PhD thesis, Department of Education, Stockholm University, Sweden.
- Shaughnessy, M. (1992). Research in probability and statistics: Reflections and directions. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 465–494). New York: Macmillan.
- Speiser, R., & Walter, C. (1998) Two dice, two sample spaces. In L. Pereira-Mendoza, L. Seu Kea, T. Wee Kee & W.-K. Wong (Eds.), *Proceedings of the Fifth International Conference on Teaching Statistics*, Vol. 1 (pp. 67–73). Voorburg, The Netherlands: International Statistical Institute Permanent Office.
- Steinbring, H. (1991). The theoretical nature of probability in the classroom. In R. Kapadia & M. Borovcnik (Eds.), *Chance encounters: Probability in education* (pp. 135–167). The Netherlands: Kluwer.
- Säljö, R. (1991). Learning and mediation – Fitting reality into a table. *Learning and Instruction*, 1, 261–272.
- Tiberghien, A. (1994). Modeling as a basis for analyzing teaching–learning situations. *Learning and Instruction*, 4, 71–87.
- Tversky, A., & Kahneman, D. (1973). Availability: A heuristic for judging frequency and probability. *Cognitive Psychology*, 5, 207–232.
- Vidakovic, D., Berenson, S., & Brandsma, J. (1998). Children's intuition of probabilistic concepts emerging from fair play. In L. Pereira-Mendoza, L. Seu Kea, T. Wee Kee & W.-K. Wong (Eds.), *Proceedings of the Fifth International Conference on Teaching Statistics*, Vol. 1 (pp. 67–73). Voorburg, The Netherlands: International Statistical Institute Permanent Office.
- von Wright, G. H. (1971). *Explanation and understanding*. London: Routledge and Kegan Paul.
- Watson, J. (2005). The probabilistic reasoning of middle school students. In G. A. Jones (Ed.), *Exploring probability in school: Challenges for teaching and learning* (pp. 145–169). Berlin Heidelberg New York: Springer.

- Wistedt, I., & Brattström, G. (2004). Mathematical induction in a cooperative setting. Merits and limitations of classroom communication among peers. In A. Chronaki & M. Christiansen (Eds.), *Challenging ways of viewing classroom communication*. Greenwich, UK: Information Age.
- Wistedt, I., Brattström, G., & Jacobsson, C. (1983). *Att använda barns informella kunskaper I matematikundervisningen* [Using children's informal knowledge in mathematics education]. Stockholms universitet: Pedagogiska institutionen.