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Ashok K. Singh

Rohan J. Dalpatadu

Anthony F. Lucas

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Ashok K. Singh[1], Rohan J. Dalpatadu[2], Anthony F. Lucas[1]

Abstract

The probability distribution of the sum of two fair dice is used to calculate the house advantage of various bets in craps, and is readily available in probability and statistics books and gaming literature. The probability distribution of the sum of k dice (for $k \geq 3$) is derived in this paper using the method of moment generating function. A recursion formula for deriving the probability distribution of the sum of k dice (for $k \geq 3$) from the probability distribution of the sum of $k-1$ dice is also given. There are no gaming books or journal articles that demonstrate how a multi-reel slot game is developed. As an application of the probability distribution of the sum of k dice derived in this article, a slot game based on the sum of five dice is presented.

Introduction

There are many casino games that are based on rolling two or more fair dice. Craps, the most popular dice game, is played by rolling a pair of fair dice and wagers are taken on many outcomes, each with different odds, payoff amounts, and house advantages. The probability distribution of the sum of two faces of the dice, needed to compute the house advantages of various bets, is easy to calculate and is available in any elementary probability book or even on the web. Sic Bo or Chinese Chuck-a-luck is played with 3 fair dice, and the probability distribution of the sum of 3 faces, needed to analyze Sic Bo, can also be directly calculated. In this paper, we use the method of moment generating function (mgf) to derive the probability distribution of the sum of k dice, for k ³ 2. We also derive a recursive relationship between the probability distribution of the sum of j dice and the sum of j □1 dice, and show how the probability distribution of the sum of k dice can be obtained from this recursive relationship.

Finally, we use the probability distribution of the sum of 5 dice and develop an innovative slot game that is based on 5 dice. We use this example to illustrate how to compute the expected value and volatility index (VI) of a slot game.

The Moment Generating Function

The Moment Generating Function (mgf) of a random variable X is defined as (Scheaffer and Young, 2009) the expected value or weighted average of the function e^{tX}:

$$M_X(t) = E\left[e^{tX}\right] \text{ for some real number } t.$$

If the random variable (rv) is discrete (e.g., if it only takes integer values) then the mgf is computed by the formula:

$$M_X(t) = \sum_{\text{all } x} e^{tx} f(x), \text{ where } f(x) \text{ is the probability distribution of } X. \qquad (1)$$

Ashok K. Singh[1], Rohan J. Dalpatadu[2], Anthony F. Lucas[1]
[1]William F. Harrah College of Hotel Administration, University of Nevada, Las Vegas; e-mail: aksingh@unlv.nevada.edu
[2]Department of Mathematical Sciences, University of Nevada, Las Vegas; e-mail: dalpatad@unlv.nevada.edu

Whereas, if the rv X is continuous (*i.e.,* if it can take any value inside a real interval), then the mgf is computed by the formula:

$$M_X(t) = \int_{-\infty}^{+\infty} e^{tx} f(x)dx, \text{ where } f(x) \text{ is the probability density function of } X. \quad (2)$$

The mgf of a probability distribution is unique, *i.e.*, there exists a one-to-one correspondence between the probability distribution and its mgf. Suppose a discrete rv X has the probability distribution:

$$f(x_j) = P(X = x_j) = p_j, \; j = 1, \; 2, ..., \; k.$$

Then its mgf is:

$$M_X(t) = p_1 e^{tx_1} + p_2 e^{tx_2} + ... + p_k e^{tx_k}.$$

In other words, if the rv X has the mgf: $M_X(t) = p_1 e^{tx_1} + p_2 e^{tx_2} + ... + p_k e^{tx_k}$, then its probability distribution is: $f(x_j) = P(X = x_j) = p_j, \; j = 1, \; 2, ..., \; k.$ (3)

The mgf, as the name suggests, is used to compute the moments (such as the mean and variance) in one of two ways:

(a) The k-th moment $E\left[X^k\right]$ can be obtained by finding the *k-th* derivative of the mgf with respect to t at the origin $t = 0$:

$$E\left[X^k\right] = \left.\frac{d^k M_X(t)}{dt^k}\right|_{t=0}$$

(b) The k-th moment $E\left[X^k\right]$ can be obtained by expanding the mgf as a Taylor series and finding the coefficient of the term $\frac{t^k}{k!}$:

$$M_X(t) = 1 + tE\left[X\right] + \frac{t^2}{2!}E\left[X^2\right] + \frac{t^3}{3!}E\left[X^3\right] + ... + \frac{t^k}{k!}E\left[X^k\right] + ...$$

$$E\left[X^k\right] = \text{ coefficient of } \frac{t^k}{k!} \text{ in the series expansion.}$$

Once the moments $E\left[X\right]$ and $E\left[X^2\right]$ have been calculated, the variance of the rv can be computed by the well-known formula:

$$Var(X) = E\left[X^2\right] - \left\{E\left[X\right]\right\}^2.$$

$$M_X(t) = 1 + tE[X] + \frac{t^2}{2!}E[X^2] + \frac{t^3}{3!}E[X^3] + ... + \frac{t^k}{k!}E[X^k] + ...$$

$$E[X^k] = \text{ coefficient of } \frac{t^k}{k!} \text{ in the series expansion.}$$

Once the moments $E[X]$ and $E[X^2]$ have been calculated, the variance of the rv can be computed by the well-known formula:

$$Var(X) = E[X^2] - \{E[X]\}^2.$$

The Probability Distribution of the Sum of *k* Dice

Consider the experiment of rolling *k* fair dice, and let X_i represents the number that comes up when *i-th* fair die is rolled, $i = 1, 2, \cdots, k$. In this paper, we derive the probability distribution of the sum *X*. The probability distribution of each X_i is given by:

$$f(x) = \begin{cases} \frac{1}{6} & x = 1, 2, \cdots, 6 \\ 0 & \text{otherwise} \end{cases}$$

and its moment generating function (*mgf*) is:

$$M_{X_i}(t) = E(e^{tX_i}) = \frac{1}{6}\left(e^t + e^{2t} + e^{3t} + \cdots e^{6t}\right).$$

Since the random variables $X_1, X_2, ..., X_k$ are independent, the *mgf* of the sum *S* is:

$$M_S(t) = E\left[e^{tX}\right] = E\left[e^{t(X_1 + \cdots + X_k)}\right] = \prod_{i=1}^{k} E\left[e^{tX_i}\right]$$
$$= \prod_{i=1}^{k}\left[\frac{1}{6}\left(e^t + e^{2t} + \cdots + e^{6t}\right)\right].$$

$$M_S(t) = \frac{1}{6^k}\left(e^t + e^{2t} + \cdots + e^{6t}\right)^k. \quad (4)$$

We can now expand the right hand side of the expression in (4), and obtain the probability distribution of *X* by using result (3). This is illustrated for the cases where the value of *k* is: 2, 3, 4, and 5.

Sum of 2 fair dice $(k=2)$

$$M_x(t)=\frac{1}{6^2}\left(e^t+e^{2t}+\cdots+e^{6t}\right)^2$$
$$=\frac{1}{36}\left(e^{2t}+2e^{3t}+3e^{4t}+4e^{5t}+5e^{6t}+6e^{7t}+5e^{8t}+4e^{9t}+3e^{10t}+2e^{11t}+e^{12t}\right).$$

Which, from result (3), is the *mgf* of the following probability distribution.

Table 1: Probability distribution of the sum of 2 fair dice

x	2	3	4	5	6	7	8	9	10	11	12
$f(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

This is the probability distribution of the sum of two fair dice.

The mgf $M_X(t)$ of the sum S for $k = 3, 4, 5$ is similarly obtained from equation (4):

Sum of 3 fair dice $(k=3)$

$$M_X(t)=\frac{1}{6^3}\left(e^t+e^{2t}+\cdots+e^{6t}\right)^3$$
$$=\frac{1}{216}\begin{pmatrix} e^{3t}+3e^{4t}+6e^{5t}+10e^{6t}+15e^{7t}+21e^{8t}+25e^{9t}+27e^{10t} \\ +27e^{11t}+25e^{12t}+21e^{13t}+15e^{14t}+10e^{15t}+6e^{16t}+3e^{17t}+e^{18t} \end{pmatrix} \quad (5)$$

Sum of 4 fair dice $(k=4)$

$$M_X(t)=\frac{1}{6^4}\left(e^t+e^{2t}+e^{3t}+e^{4t}+e^{5t}+e^{6t}\right)^4$$
$$=\frac{1}{1296}\begin{pmatrix} e^{4t}+4e^{5t}+10e^{6t}+20e^{7t}+35e^{8t}+56e^{9t}+80e^{10t}+104e^{11t}+ \\ 125e^{12t}+140e^{13t}+146e^{14t}+140e^{15t}+125e^{16t}+104e^{17t}+ \\ 80e^{18t}+56e^{19t}+35e^{20t}+20e^{21t}+10e^{22t}+4e^{23t}+e^{24t} \end{pmatrix} \quad (6)$$

Sum of 5 fair dice $(k=5)$

$$M_X(t)=\frac{1}{6^5}\left(e^t+e^{2t}+e^{3t}+e^{4t}+e^{5t}+e^{6t}\right)^5$$
$$=\frac{1}{7776}\begin{pmatrix} e^{5t}+5e^{6t}+15e^{7t}+35e^{8t}+70e^{9t}+126e^{10t}+205e^{11t}+ \\ 305e^{12t}+420e^{13t}+540e^{14t}+651e^{15t}+735e^{16t}+780e^{17t}+ \\ 780e^{18t}+735e^{19t}+651e^{20t}+540e^{21t}+420e^{22t}+305e^{23t}+ \\ 205e^{24t}+126e^{25t}+70e^{26t}+35e^{27t}+15e^{28t}+5e^{29t}+e^{30t} \end{pmatrix} \quad (7)$$

The probability distributions of the sum S for $k=3,\ 4,$ and 5 are easily obtained from the above expressions for mgf, and result (3). Table 2 shows the probability distributions of the sum of k dice for $k=3,\ 4,$ and 5.

Table 2: Probability distribution of the sum of k dice, $k = 3, 4, 5$.

$k=3$		$k=4$		$k=5$	
Sum x	$f(x)$	Sum x	$f(x)$	Sum x	$f(x)$
3	1/216	4	1/1296	5	1/7776
4	3/216	5	4/1296	6	5/7776
5	6/216	6	10/1296	7	15/7776
6	10/216	7	20/1296	8	35/7776
7	15/216	8	35/1296	9	70/7776
8	21/216	9	56/1296	10	126/7776
9	25/216	10	80/1296	11	205/7776
10	27/216	11	104/1296	12	305/7776
11	27/216	12	125/1296	13	420/7776
12	25/216	13	140/1296	14	540/7776
13	21/216	14	146/1296	15	651/7776
14	15/216	15	140/1296	16	735/7776
15	10/216	16	125/1296	17	780/7776
16	6/216	17	104/1296	18	780/7776
17	3/216	18	80/1296	19	735/7776
18	1/216	19	56/1296	20	651/7776
		20	35/1296	21	540/7776
		21	20/1296	22	420/7776
		22	10/1296	23	205/7776
		23	4/1296	24	205/7776
		24	1/1296	25	126/7776
				26	70/7776
				27	35/7776
				28	15/7776
				29	5/7776
				30	1/7776

A Recursion Formula for the Probability Distribution of the Sum of *k* Dice

In this section we derive a recursion formula for the probability distribution of the sum of j dice, using the probability distribution of the sum of $j-1$ dice.

Let X_j represent the number that comes up when *j-th* fair die is rolled, $j=1,\ 2,\cdots,\ k$. The probability distribution of each X_j is given by:

$$f(x)=\begin{cases}\dfrac{1}{6} & x=1,\ 2,\cdots,\ 6\\ 0 & \text{otherwise}\end{cases}$$

Let $S_j = X_1 + X_2 + \cdots X_j =$ sum of j dice, with probability distribution $f_j(x) = P(S_j = x)$.

The random variable S_j can take any integer value $j, j+1, \ldots, 6j$. Then the event $\{S_j = m\}$ can be expressed as:

$$\{S_j = m\} = \{S_{j-1} = m-1, x_j = 1\} \cup \{S_{j-1} = m-2, x_j = 2\} \cup \\ \cdots \cup \{S_{j-1} = m-6, x_j = 6\}, \qquad j \le m \le 6j.$$

Hence $f_j(m) = f_{j-1}(m-1)f(1) + f_{j-1}(m-2)f(2) + \cdots + f_{j-1}(m-6)f(6)$.

This gives us the recursion formula:

$$f_j(m) = \frac{1}{6}\left[f_{j-1}(m-1) + f_{j-1}(m-2) + \cdots + f_{j-1}(m-6)\right],$$

where $j=1,\ 2,\cdots,\ k$ and $j \le m \le 6j$.

We now illustrate how to use the above recursion formula to derive the probability distribution $f_k(x)$ of the sum S_k of *k* fair dice; we assume that the probability distribution of S_2 has already been computed.

Three fair dice $(k=3)$

$$f_3(3) = f_2(2)f(1) = \frac{1}{36} \times \frac{1}{6} = \frac{1}{216}$$

$$f_3(4) = f_2(3)f(1) + f_2(2)f(2) = \frac{(2+1)}{36} \times \frac{1}{6} = \frac{3}{216}$$

$$f_3(5) = f_2(4)f(1) + f_2(3)f(2) + f_2(4)f(3) = \frac{(3+2+1)}{36} \times \frac{1}{6} = \frac{6}{216}$$

Proceeding in this manner, we can obtain $f_3(m)$ for $m = 6,\ 7, \ldots,\ 18$

Four fair dice $(k=4)$

$$f_4(4)=f_3(3)f(1)=\frac{1}{216}\times\frac{1}{16}=\frac{1}{1296}$$

$$f_4(5)=f_3(4)f(1)+f_3(3)f(2)=\frac{(3+1)}{216}\times\frac{1}{6}=\frac{4}{1296}$$

$$f_4(6)=f_3(5)f(1)+f_3(4)f(2)+f_3(3)f(3)=\frac{(6+3+1)}{216}\times\frac{1}{6}=\frac{10}{1296}$$

Similarly $f_4(m)$ can be computed for $m=7,\ 8,\cdots,\ 24$.

A Slot Game Based on Dice

In the US casinos, slot machines are more popular than table games, occupying at least 80% of the floor space, and generating 70% - 75% of the gaming revenue (Hannum and Cabot, 2005). A par sheet of a slot game shows all possible outcomes from the slot game with their probabilities, the Expected Value (EV) or Payback Percentage of the game, and typically the 95% Volatility Index (VI) which equals 1.96s, where s is the population standard deviation of the slot game. The mathematical details of these calculations, however, are not available in the gaming literature. In this paper, we develop a slot game based on five fair dice, and demonstrate how the EV, sd, and VI of a slot game are calculated.

In the US casinos, slot machines are more popular than table games, occupying at least 80% of the floor space, and generating 70% - 75% of the gaming revenue.

In this section, we use the probability distribution of the sum of 5 fair dice to develop a new slot game.

1) The game is played on a 3 x 5 matrix. For each of the 15 cells, a fair die is rolled independently.
2) The sum of all 5 dice on a pay-line determines the payoff.

Table 3 shows the winning combinations and their payoff amounts. The 4-th column labeled EV is obtained by multiplying the probability column P(x) by the Pays column. The total of the 4-th column (0.943951) is the expected value (EV) or payback % of this game.

Table 3: Winning combinations and payoff amounts of the slot game

Sum x	P(x)	Pays	EV	
5	0.000129	300	0.03858	
6	0.000643	60	0.03858	
7	0.001929	20	0.03858	
8	0.004501	10	0.04501	
9	0.009002	5	0.04501	
10	0.016204	3.596322*	0.058274	FS
11	0.026363	2.277778**	0.060049	BONUS
12	0.039223	2	0.078447	
14	0.069444	1	0.069444	
21	0.069444	1	0.069444	
23	0.039223	2	0.078447	
24	0.026363	2.277778**	0.060049	BONUS

25	0.016204	3.596322*	0.058274	FS
26	0.009002	5	0.04501	
27	0.004501	10	0.04501	
28	0.001929	20	0.03858	
29	0.000643	60	0.03858	
30	0.000129	300	0.03858	
		Total	0.943951	

* Since average number of free spins awarded to the player entering the Free Spin game is 3.5, and the EV of the Free Spin game is 1.027521, the average number of credits won during Free Spin game is 3.5 x 1.027521 = 3.596322.

** This is the average number of credits won during the Bonus Game is 2.277778, as shown in Step 4 below.

In this section, we use the probability distribution of the sum of 5 fair dice to develop a new slot game.

A lower bound for the variance of the slot game is obtained by using the average payoffs of the Free Spin and Bonus games in Table 3. This lower bound for the variance is 30.40433, and hence the lower bound for thr standard deviation of the slot game is 5.514013.

3) If the sum of 5 dice is 10 or 25, the FREE SPIN FEATURE starts: A fair die is rolled, and if *r* comes up on the die, the player is awarded *r* Free Spin Games (r = 1, 2, …, 6). The average number of Free Spin games awarded to the player who enters the Free Spin game is 3.5.

Table 4 shows the winning combinations and their payoff amounts of the free spin game. The total of the 4-th column (1.027521) is the EV of the free spin game.

Table 4: Winning combinations and payoff amounts of the Free Spin game

Sum x	P(x)	Pays	EV
5	0.000129	300	0.03858
6	0.000643	60	0.03858
7	0.001929	20	0.03858
8	0.004501	10	0.04501
9	0.009002	5	0.04501
10	0.016204	5	0.081019
11	0.026363	3	0.07909
12	0.039223	2	0.078447
14	0.069444	1	0.069444
21	0.069444	1	0.069444
23	0.039223	2	0.078447
24	0.026363	3	0.07909

25	0.016204	5	0.081019
26	0.009002	5	0.04501
27	0.004501	10	0.04501
28	0.001929	20	0.03858
29	0.000643	60	0.03858
30	0.000129	300	0.03858
		Total	1.027521

4) If the sum of 5 dice is 11 or 24, the player enters the BONUS GAME. In the BONUS GAME, 2 fair dice are rolled and the sum on the 2 dice determines the bonus award won, as shown in Table 5.

Table 5: Payoffs won by player in Bonus Game

sum s	P(s)	Pays	EV
2	0.027778	6	0.166667
3	0.055556	5	0.277778
4	0.083333	3	0.25
5	0.111111	2	0.222222
6	0.138889	1	0.138889
7	0.166667	1	0.166667
8	0.138889	1	0.138889
9	0.111111	2	0.222222
10	0.083333	3	0.25
11	0.055556	5	0.277778
12	0.027778	6	0.166667
		Total	2.277778

NOTE: The volatility index of a slot game at 95% confidence is given by

$$VI = 1.96 \times \sigma = 1.96 \times 5.51 = 10.8$$

where 1.96 is the value from the standard normal probability table corresponding to 95% confidence, and σ is the standard deviation (sd) of the slot game. The VI is used to calculate the 95% confidence interval for payback% of a slot machine by the formula

$$Payback\% \pm \frac{VI}{\sqrt{n}} \times 100$$

Since the awards won during Free Spin game and Bonus game are random, computing the exact sd of the slot game is quite complicated. Typically, game developers use the average awards won during Free Spin game and Bonus game (see Table 3) to compute the EV as well as the sd of a slot game; it should be noted that this substitution of average values does give the correct value of the EV (0.943951), but the sd (5.51) thus obtained is an underestimate.

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