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Larry Feldman & Fred Morgan

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Larry Feldman and Fred Morgan
Indiana University of Pennsylvania

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Key Words: Expected value; Optimization; Probability.

Abstract

The dice game HOG has been used successfully in a variety of educational situations as an activity that not only introduces students to concepts in probability, statistics, and simulation but also fosters student interest in these concepts. This article presents several areas in the statistics curriculum where important concepts can be dealt with in a hands-on way. These areas include probability as decision making, experimental versus theoretical probability, expected value, and optimization.

This article explains the rules for HOG, gives examples of students' understanding, develops the probability theory, and identifies a "best" strategy for playing the game. This "best" strategy is developed in the context of fair six-sided dice and then generalized to fair *s*-sided dice.

1. Introduction

The dice game HOG was developed by the Mathematical Sciences Education Board (1994) and is described in Bohan and Schultz (1996). The game has been played with a wide variety of audiences including graduate students in mathematics, Masters of Business Administration (MBA) students, students in undergraduate introductory statistics courses, and elementary and secondary school students. The authors have also played HOG numerous times in Statistics Education through Quantitative Literacy (SEQuaL) workshops for K-12 teachers at sites throughout Pennsylvania. The rules for the game are as follows:

1. Players take turns rolling dice. Each player may choose any number of dice from one up to the total number of dice available. (We recommend that at least ten dice be available for each player or team.)
2. The number of dice a player chooses to roll can vary from turn to turn.
3. The player's score for a turn is zero if at least one of the dice comes up with the value one. Otherwise, the player's score for the turn is the sum of the faces showing on the dice. (Rolling a one sets only the score for that turn to zero, not the total cumulative score for the player.)
4. A cumulative running total of the scores is kept for each player.
5. The first player to reach or exceed a predetermined score (100 works well) wins the game. If more than one player reaches the predetermined score on the same turn then the player with the highest point total wins the game.

Table 1 displays a sample of the first four turns in a hypothetical two-person game.

Table 1. Sample of the first 4 turns for a two-person game.

Person A rolls	Person A's score for that roll	Person A's cumulative score	Person B rolls	Person B's score for that roll	Person B's cumulative score
2, 4, 5	11	11	3, 6	9	9
1, 4, 5, 5, 6	0	11	2, 3, 5	10	19
3, 4	7	19	1, 2, 2, 2, 4, 6, 6	0	19
2, 4, 4, 5, 6	21	39	3, 4	7	26

We have found that the dice game HOG provides an activity with accompanying assessments that reflect accepted standards and recommendations in the mathematics education community, such as those described in Mathematical Sciences Education Board (1994).

1.1 Probability as Decision Making

There exists an obvious trade-off in deciding how many dice to roll. The more dice the player rolls, the less the likelihood of a non-zero score. Yet if a one does not appear on any of the dice rolled, more dice will lead to a greater expected score. The number of dice a player chooses to roll may also depend on how far the player is behind or ahead in the cumulative score, or how close the player and any opponents are to winning. These and other factors provide an interesting activity for students with respect to decision making in the face of uncertainty.

1.2 Experimental versus Theoretical Probability

Too often in statistics classes students are provided with results of experiments with little motivation. When playing the game HOG students are excited to try out different strategies. After allowing the students time to play against one another using fair six-sided dice, many students are curious to see if a best strategy exists. To this end, we suggest that each student or small group of students be assigned a fixed number of dice to be rolled. For example, one group must always roll exactly one die while the next group must always roll exactly two dice and so on. A fixed number of turns is assigned to the groups (ten turns works well if using actual dice and we recommend about thirty turns if the dice rolls are being simulated using technology).

After playing the game with fixed numbers of dice and comparing the results, students get the feeling that perhaps somewhere around six dice is the optimum number of dice to roll. However, they see that on any single turn different numbers of dice may produce the best score. This helps students to see that the results of experiments may vary and has motivated our students to investigate the “real” (theoretical) answer, which, as will be shown, is to roll either five or six dice.

For example, one of the authors had his “Probability and Statistics for Elementary and Middle School Teachers” students play the game. After they played the game to cumulative scores of 100 in groups of three to four students, each group was asked to list their top three guesses for the best number of dice to use. The results are given in Table 2.

Table 2. Students’ guesses for the best number of dice after playing once.

Group	Guess of best number	Guess of second best number	Guess of third best number
A	3	4	2
B	3	2	4

C	4	3	2
D	5	4	3
E	3	2	4
F	4	3	2

For the most part, the students did not believe numbers above four would be the best, probably since they tended to exaggerate the effect of getting a zero for any single turn.

The groups were then assigned a fixed number of dice to roll ten times with the goal of finding a mean score per roll. Their findings are in Table 3.

Table 3. Mean values for ten rolls using the same number of dice each time.

Number of dice	Mean score per roll
1	3.6
2	5.8
3	5.8
4	8.3
5	8.7
6	8.9

After obtaining these results the students were asked once again to list their top three guesses for the best number of dice to use. Their guesses are summarized in Table 4. Note that these guesses are closer to the theoretical best number of dice, which is either five or six dice.

Table 4. Students' guesses for the best number of dice after finding mean scores.

Group	Guess of best number	Guess of second best number	Guess of third best number
A	4	5	3
B	3	5	4
C	5	6	4
D	6	5	4
E	5	4	3
F	5	4	6

Within each group, each person then chose a number to stick with for each roll in a new game of HOG. The winner for two of the groups was the person who chose to roll five dice. For another two groups the winner was the person who chose to roll six dice, and for another group the winner rolled ten dice.

1.3 Expected Value

Sometimes the concept of expected value is presented as a formula that students do not intuitively understand. For many students a visual approach to expected value can be very meaningful. The possible outcomes and the expected score after rolling just one die are displayed in Table 5.

Table 5. Equally likely scores, rolling one die.

Possible scores					
0	2	3	4	5	6

Mean of all scores = 3 1/3, Mean of non-zero scores = 4.0

The expected value for rolling one die can be computed by evaluating $(1/6 \times 0) + (1/6 \times 2) + (1/6 \times 3) + (1/6 \times 4) + (1/6 \times 5) + (1/6 \times 6) = 20/6 = 3\ 1/3$. If you roll one die ten times, your expected score will be approximately 33. Since the mean of the non-zero values (2, 3, 4, 5, and 6) is 4.0, the concept of conditional expectation can be introduced by calculating the expected value for one die as $(1/6 \times 0) + (5/6 \times 4.0) = 20/6 = 3\ 1/3$.

Table 6 shows that when rolling two dice the expected value is 200/36 or approximately 5.6, since the sum of all 36 of the possible outcomes is 200 and each outcome is equally likely.

Table 6. Equally likely scores, rolling two dice.

	Die 2					
Die 1	1	2	3	4	5	6
1	0	0	0	0	0	0
2	0	4	5	6	7	8
3	0	5	6	7	8	9
4	0	6	7	8	9	10
5	0	7	8	9	10	11
6	0	8	9	10	11	12

Mean of all scores = 5 5/9. Mean of non-zero scores = 8

The proportion of non-zero area in this table can be represented by 5/6 of the possible outcomes for Die 1 times 5/6 of the possible outcomes for Die 2, or 25/36. The mean of the non-zero values will be four times the number of dice ($4 \times 2 = 8$ for two dice). Using conditional expectation, the expected value is determined by $(5/6 \times 5/6 \times 8) = 200/36$. For the general case, there will be five non-zero values out of six for each die used, so the probability of scoring positive points with a roll is $(5/6)^n$, where n denotes the number of dice rolled. Given that zero 1's are rolled, the expected score is $4n$, since 4 is the mean of the non-zero scores 2, 3, 4, 5, and 6. The expected score is then the product of the probability of a positive score with the conditional expectation given a positive score, namely $(4n) \times (5/6)^n$. These values for rolling 1 through 12 dice are displayed in Table 7.

Table 7. Expected scores for rolling 1-12 fair six-sided dice.

Number of dice	Probability of a 1	Score for a 1	Probability of no 1's	Expected score given no 1's	Expected score
1	0.167	0	0.833	4	3.333
2	0.306	0	0.694	8	5.556
3	0.421	0	0.579	12	6.944
4	0.518	0	0.482	16	7.716
5	0.598	0	0.402	20	8.038
6	0.665	0	0.335	24	8.038
7	0.721	0	0.279	28	7.814
8	0.767	0	0.233	32	7.442
9	0.806	0	0.194	36	6.977
10	0.838	0	0.162	40	6.460
11	0.865	0	0.135	44	5.922
12	0.888	0	0.112	48	5.384

The graph of the expected score function for rolling up to 40 fair six-sided dice is shown in Figure 1.

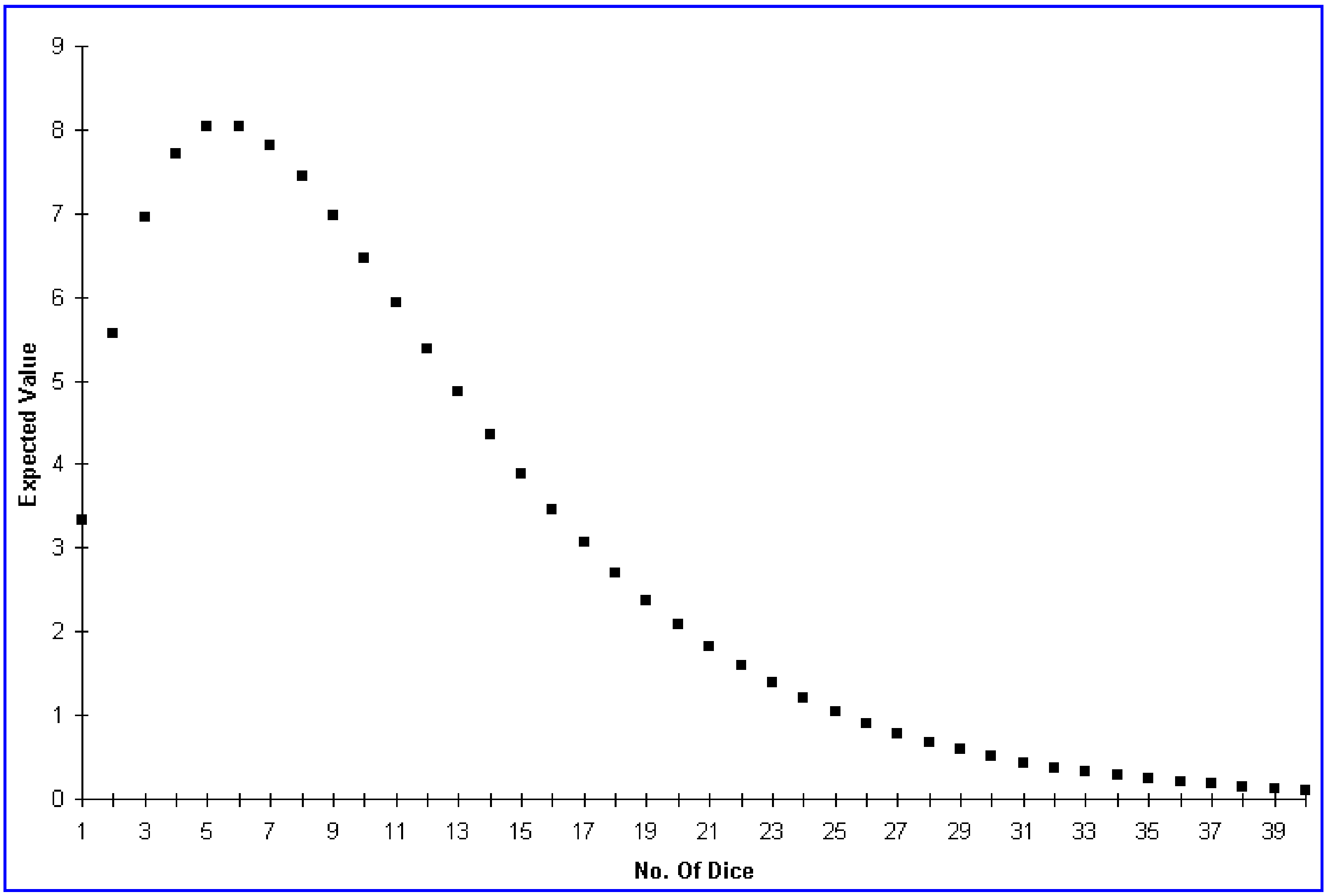

Figure 1. Expected scores for rolling from 1 to 40 fair six-sided dice.

These results demonstrate that an optimal strategy is to use either five or six dice for each roll. Of course, a player still might want to change the number of dice to roll if the player is far behind or far ahead in the cumulative score.

2. Extension to Fair *s*-sided Dice

One of the authors was using this activity with a class of MBA students when one student asked what an optimal strategy would be for ten-sided dice. This led to the authors' search for the optimal strategy for a fair *s*-sided dice. The results below demonstrate that the optimal strategy is to use *s*-1 or *s* dice when rolling fair *s*-sided dice.

Let s = the number of sides for the dice being used, k = the number of dice rolled, and p = the probability of no 1 occurring $= (s-1)/s$.

Let X = the sum of the dice $= \sum_{i=1}^{k} X_i$ (where X_i = the number of dots on the top face of the ith die), and

$$Y = \begin{cases} 0 \text{ if at least one of the dice is a 1} \\ 1 \text{ if none of the dice are 1s} \end{cases}.$$

Then U = score per turn = XY.

The expected score per turn is given by

$$\begin{aligned}
E(U) &= E(XY) = E_Y[E(XY)|Y] \\
&= \underset{(\text{if } Y=0)}{0 \cdot (1-p^k)} \quad + \underset{(\text{if } Y=1)}{\sum_{i=1}^{k} E(X_i) p^k} \\
&= k\left(\frac{2+3+\ldots+s}{s-1}\right)p^k \\
&= k\left(\frac{s(s+1)/2-1}{(s-1)}\right)p^k \qquad \text{(since } 1+2+\ldots+n = n(n+1)/2) \\
&= k\left(\frac{s(s+1)-2}{2(s-1)}\right)p^k \\
&= k\left(\frac{(s+2)(s-1)}{2(s-1)}\right)p^k \\
&= k\left(\frac{s+2}{2}\right)p^k \qquad (1)
\end{aligned}$$

The value of k that maximizes the expected score is also the value that maximizes the natural logarithm of the expected score. Thus,

$$\frac{d}{dk}[\ln E(U)] = \frac{d}{dk}\left[\ln k + \ln\frac{s+2}{2} + k\ln p\right] = \frac{1}{k} + \ln p = 0$$

implies

$$k = -\frac{1}{\ln(p)} = \frac{1}{\ln\left(\frac{s}{s-1}\right)} \qquad (2)$$

Since k must be an integer, more analysis is required to determine which integer on either side of Equation (2) achieves the maximum.

Since

$$\ln\left(\frac{s}{s-1}\right) = \ln(s) - \ln(s-1)$$
$$= \int_{s-1}^{s} \frac{1}{x}\,dx$$
$$\geq \frac{1}{s} \qquad \text{(see Figure 2),}$$

it follows that
$$s \geq \frac{1}{\ln\left(\frac{s}{s-1}\right)}$$

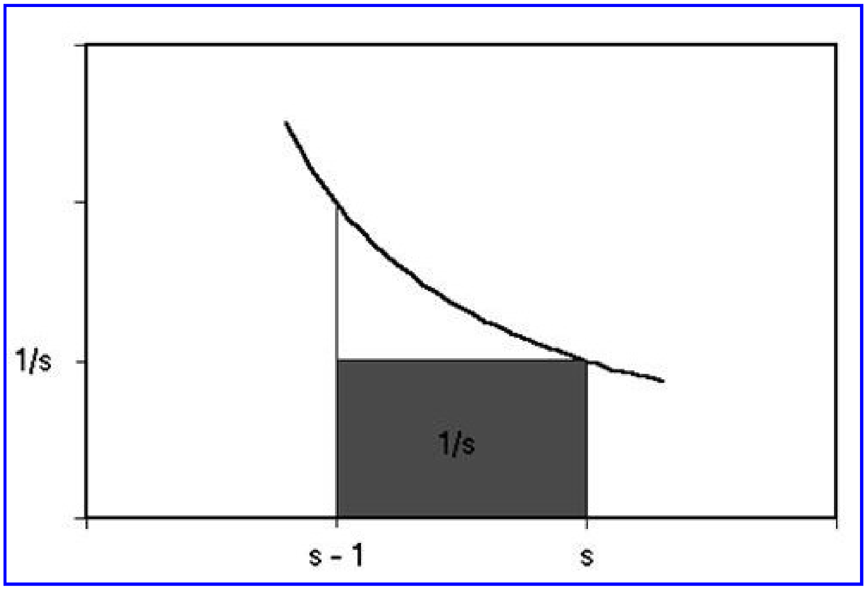

Figure 2. Graph of $Y = 1 / X$.

Similarly,
$$s-1 \leq \frac{1}{\ln\left(\frac{s}{s-1}\right)}.$$

Therefore,
$$s-1 \leq \frac{1}{\ln\left(\frac{s}{s-1}\right)} \leq s$$
.

Using the first derivative test for test values s-1 and s, it is easy to show that the value for k in Equation (2) above maximizes the expected score. This number k is between s-1 and s, which are consecutive integers. Therefore, substituting $k = s$ and $k = s$-1 into Equation (1), we find:
$$E(U)\,|_{k=s-1} = E(U)\,|_{k=s} = \frac{(s-1)^s}{s^{s-1}}\left(\frac{s+2}{2}\right).$$

The number of dice to be rolled, k, that will maximize the expected score per turn is s-1 or s. For standard six-sided dice this means the number of dice to roll to maximize the expected score is five or six.

The result extends to "two-sided" dice, or coin flips. If heads is assigned the value 1, and tails is assigned the value 2, the expected value of the score is maximized by flipping either one or two coins. Recall that the score for a turn is zero if at least one "1" (heads) appears. With one coin, the expected score is $(1/2 \times 0) + (1/2 \times 2) = 1$. With two coins, the expected score is $(3/4 \times 0) + (1/4 \times 4) = 1$. The expected score decreases with three coins to $(7/8 \times 0) + (1/8 \times 6) = 3/4$.

As a different extension, it can be shown that the variance of a player's score per turn is:

$$\mathrm{Var}(U) = \mathrm{Var}(XY) = \mathrm{E_Y}[\mathrm{Var}(XY|Y)] + \mathrm{Var_Y}[\mathrm{E}(XY|Y)]$$
$$= k\left(\frac{(s-1)^2-1}{12} + k\left(\frac{s+2}{2}\right)^2 (1-p^k)\right)p^k$$

3. Summary

The dice game HOG can be played and analyzed by elementary school children up through graduate statistics students. This activity is typical of the kinds of activities that teachers and their students seem to enjoy the most. It is a fun activity that has “hidden” deep probabilistic, statistical, and mathematical concepts. These embedded concepts can be adapted for students at a wide variety of levels.

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Larry Feldman
Mathematics Department
Indiana University of Pennsylvania
Indiana, PA 15701
USA
lmfeldmn@iup.edu

Fred Morgan
Mathematics Department
Indiana University of Pennsylvania
Indiana, Pennsylvania 15701
USA
fwmorgan@iup.edu
