

Derivation of the Hydra Effect from Spatial Structure

Team 13: Evolutionary Suicide in Stochastic Cellular Automata

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1 Derivation

Consider a lattice \mathcal{L} of size $L \times L$ with periodic boundary conditions. Each cell $i \in \mathcal{L}$ has state $s_i \in \{0, 1, 2\}$ representing empty, prey, and predator respectively. Let \mathcal{N}_i denote the Moore neighborhood of cell i with $|\mathcal{N}_i| = k = 8$ neighbors.

Definition 1 (CA Transition Rules). *For asynchronous (random-sequential) updates, each cell is selected uniformly at random and updated according to:*

$$P(s_i : 0 \rightarrow 1) = P_b \cdot \mathbf{1}[\exists j \in \mathcal{N}_i : s_j = 1] \cdot \frac{1}{k} \sum_{j \in \mathcal{N}_i} \mathbf{1}[s_j = 1] \quad (1)$$

$$P(s_i : 1 \rightarrow 0) = P_d \quad (\text{natural death}) \quad (2)$$

$$P(s_i : 1 \rightarrow 2) = P_c \cdot \mathbf{1}[\exists j \in \mathcal{N}_i : s_j = 2] \cdot \frac{1}{k} \sum_{j \in \mathcal{N}_i} \mathbf{1}[s_j = 2] \quad (3)$$

$$P(s_i : 2 \rightarrow 0) = P_{dc} \quad (\text{predator death}) \quad (4)$$

where P_b : prey birth, P_d denotes: prey death, P_c : predator death (consumption), and P_{dc} : predator death.

The mean-field model assumes well-mixed populations with no spatial correlations

Definition 2 (Mean-Field Equations).

$$\frac{dR}{dt} = R(b - d_r - cC - eR) \quad (5)$$

$$\frac{dC}{dt} = C(aR - d_c - qC) \quad (6)$$

where $R, C \in [0, 1]$ are population densities, and parameters are b : prey birth rate, d_r : prey death rate, c : consumption rate (predation pressure), e : prey intraspecific competition, a : predator conversion efficiency ($= c \times$ conversion factor), d_c : predator death rate, q : predator intraspecific competition.

The mean-field assumption is that spatial correlations vanish:

$$\langle s_i s_j \rangle = \langle s_i \rangle \langle s_j \rangle \quad \forall i, j \quad (7)$$

Hence, the probability of finding a neighbor in state X equals the global density:

$$P(\text{neighbor is } X) = \rho_X = \frac{1}{L^2} \sum_{i \in L} \mathbf{1}[s_i = X] \quad (8)$$

In the parent-centric formulation, a prey cell attempts reproduction with probability P_b by selecting a random neighbor. The target is required to be empty. The per-prey birth rate is given by:

$$\lambda_{\text{birth}} = P_b \times P(\text{random neighbor is empty}) \quad (9)$$

Under the mean-field assumption:

$$P(\text{neighbor empty}) = 1 - \rho_R - \rho_C \quad (10)$$

Let $N_R = L^2 \times \rho_R$ denote the total prey population. Then we have

$$\left. \frac{dN_R}{dt} \right|_{\text{birth}} = N_R \cdot P_b \cdot (1 - \rho_R - \rho_C) \quad (11)$$

Converting to density

$$\left. \frac{d\rho_R}{dt} \right|_{\text{birth}} = \rho_R \cdot P_b \cdot (1 - \rho_R - \rho_C) \quad (12)$$

Furthermore, each prey dies with probability P_d per time step

$$\left. \frac{d\rho_R}{dt} \right|_{\text{death}} = -\rho_R \cdot P_d \quad (13)$$

Each predator attempts to consume a prey neighbor with probability P_c . The predator consumption rate can be defined as

$$\lambda_{\text{consume}} = P_c \times P(\text{random neighbor is prey}) = P_c \cdot \rho_R \quad (14)$$

So, the total predation rate is given by

$$\left. \frac{d\rho_R}{dt} \right|_{\text{predation}} = -\rho_C \cdot P_c \cdot \rho_R \quad (15)$$

Summing the above equations, we obtain

$$\begin{aligned} \frac{d\rho_R}{dt} &= \rho_R \cdot P_b \cdot (1 - \rho_R - \rho_C) - \rho_R \cdot P_d - \rho_C \cdot P_c \cdot \rho_R \\ &= \rho_R [P_b - P_d - P_b \rho_R - P_b \rho_C - P_c \rho_C] \\ &= \rho_R [(P_b - P_d) - P_b \rho_R - (P_b + P_c) \rho_C] \end{aligned} \quad (16)$$

Comparing this result with the standard form of the equation, we propose the following justification for the prey competition parameter e :

Proposition 1. *The CA exhibits density dependent growth without an explicit competition term. The factor $(1 - \rho_R - \rho_C)$ acts a logistic carrying capacity*

$$\lim_{\rho_R + \rho_C \rightarrow 1} \left. \frac{d\rho_R}{dt} \right|_{\text{birth}} = 0 \quad (17)$$

This is the exclusion principle. Each cell can hold at most one individual.

Following similar logic for the predator dynamics, we obtain the following relations for birth and death shown below:

$$\left. \frac{d\rho_C}{dt} \right|_{\text{birth}} = \rho_C \cdot P_c \cdot \rho_R \quad (18)$$

$$\left. \frac{d\rho_C}{dt} \right|_{\text{death}} = -\rho_C \cdot P_{dc} \quad (19)$$

Hence

$$\frac{d\rho_C}{dt} = \rho_C [P_c \rho_R - P_{dc}] \quad (20)$$

which maps to

$$\frac{dC}{dt} = C(aR - d_c) \quad (21)$$

2 Equilibria

Setting $dR/dt = 0$ and $dC/dt = 0$ in the full model (5)–(6) we obtain the trivial equilibrium $(R^*, C^*) = (0, 0)$. For $C = 0$ and $dR/dt = 0$:

$$R_{\text{prey}}^* = \frac{b - d_r}{e} = \frac{r}{e} \quad (22)$$

where $r = b - d_r$ is the net prey growth rate. This exists when $r > 0$. From $dC/dt = 0$, we have

$$R^* = \frac{d_c + qC^*}{a} \quad (23)$$

the coexistence equilibrium. Substituting into $dR/dt = 0$ and solving:

$$R^* = \frac{rq + d_c c}{ca + eq} \quad (24)$$

$$C^* = \frac{ra - d_c e}{ca + eq} \quad (25)$$

Coexistence requires $R^* > 0$ and $C^* > 0$, which holds when:

$$\frac{d_c}{a} < R_{\text{prey}}^* = \frac{r}{e} \quad \Leftrightarrow \quad r > \frac{d_c e}{a} \quad (26)$$

The above results motivate the following proposition.

Proposition 2 (No Hydra Effect in Mean-Field). *In the mean-field model with $e, q > 0$, the equilibrium prey density is a monotonically decreasing function of prey death rate:*

$$\frac{\partial R^*}{\partial d_r} < 0 \quad (27)$$

Proof. From equation (24) with $r = b - d_r$:

$$R^* = \frac{(b - d_r)q + d_c c}{ca + eq} \quad (28)$$

Taking the derivative:

$$\frac{\partial R^*}{\partial d_r} = \frac{-q}{ca + eq} < 0 \quad (29)$$

since $q, c, a, e > 0$. \square

3 Spatial Correlation

The mean-field assumption $\langle s_i s_j \rangle = \langle s_i \rangle \langle s_j \rangle$ fails in spatial systems. Define the pair correlation:

$$C_{XY} = \langle \mathbf{1}[s_i = X] \cdot \mathbf{1}[s_j = Y] \rangle_{j \in \mathcal{N}_i} - \rho_X \rho_Y \quad (30)$$

A pair approximation modifies the prey equation to:

$$\frac{d\rho_R}{dt} = \rho_R \cdot P_b \cdot q_{R|0} - \rho_R \cdot P_d - \rho_C \cdot P_c \cdot q_{R|C} \quad (31)$$

where $q_{R|0}$ = probability a neighbor of empty cell is prey (birth targets) and $q_{R|C}$ = probability a neighbor of predator is prey (predation targets). We propose the following condition for the hydra effect.

Proposition 3 (Spatial Hydra Condition). *The Hydra effect ($\partial \rho_R^* / \partial d_r > 0$) occurs when the reduction in effective predation from spatial fragmentation exceeds the direct mortality cost:*

$$\left| \frac{\partial}{\partial d_r} (\rho_C \cdot P_c \cdot q_{R|C}) \right| > P_d \cdot \left| \frac{\partial \rho_R}{\partial d_r} \right|_{\text{direct}} \quad (32)$$

Remark 1. Proposition 3 provides a heuristic condition; a rigorous derivation would require pair approximation with explicit calculation of $\frac{\partial q_{R|C}}{\partial d_r}$. We validate this condition numerically via PCF measurements.

4 Hypotheses

In particular, we propose the following hypotheses:

- Mean-field baseline: For well-mixed initial conditions $\partial\rho_R^*/\partial d_r < 0$.
- Hydra regime: For clustered spatial configurations, $\exists d_r^*$ such that $\partial\rho_R^*/\partial d_r > 0$ for $d_r < d_r^*$.
- Cluster size scaling: In the Hydra regime, mean prey cluster size should decrease with d_r while global density increases.
- Percolation effect: Near the Hydra-to-extinction transition, prey clusters should exhibit power-law size distributions with exponent $\tau \approx 2.05$