Delta Hedging Simulation for European Call Options

Computational Finance - Lab Assignment 1

1 Introduction

This document provides a self-contained derivation of the delta hedging simulation for a European call option. We begin with the geometric Brownian motion model for stock prices, derive the Euler discretization scheme for simulation, present the Black-Scholes pricing formula and delta, and finally derive the discrete-time self-financing hedging strategy used in our numerical experiments.

2 The Stock Price Model

2.1 Geometric Brownian Motion

We assume that the stock price S_t follows a geometric Brownian motion (GBM) under the real-world probability measure $\mathbb{P}$. The dynamics are given by the stochastic differential equation

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t \tag{1}$$

where μ is the drift (expected return), $\sigma > 0$ is the volatility, and W_t is a standard Brownian motion under $\mathbb{P}$.

For derivative pricing and hedging purposes, we work under the risk-neutral measure $\mathbb{Q}$. Under this measure, the discounted stock price is a martingale, and the dynamics become

$$dS_t = rS_t \, dt + \sigma S_t \, dW_t^{\mathbb{Q}} \tag{2}$$

where r is the risk-free interest rate and $W_t^{\mathbb{Q}}$ is a standard Brownian motion under $\mathbb{Q}$. The key observation is that the drift μ has been replaced by r, while the volatility σ remains unchanged.

2.2 Solution to the GBM Equation

To solve equation (2), we apply Itô's lemma to $\ln S_t$. Let $f(S) = \ln S$. Then

$$df = \frac{\partial f}{\partial S} dS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(dS)^2 = \frac{1}{S} dS - \frac{1}{2S^2}\sigma^2 S^2 \, dt. \tag{3}$$

Substituting the dynamics of S_t yields

$$d(\ln S_t) = \left(r - \frac{1}{2}\sigma^2\right) dt + \sigma \, dW_t^{\mathbb{Q}}. \tag{4}$$

Integrating from time t to time T gives

$$\ln S_T - \ln S_t = \left(r - \frac{1}{2}\sigma^2\right)(T - t) + \sigma(W_T^{\mathbb{Q}} - W_t^{\mathbb{Q}}). \tag{5}$$

Since the increment $W_T^{\mathbb{Q}} - W_t^{\mathbb{Q}}$ is normally distributed with mean zero and variance $T - t$, we can write $W_T^{\mathbb{Q}} - W_t^{\mathbb{Q}} = \sqrt{T-t}\,Z$ where $Z \sim \mathcal{N}(0,1)$. Therefore, the exact solution is

$$S_T = S_t \exp\left[\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\sqrt{T-t}\,Z\right]. \tag{6}$$

This shows that S_T is log-normally distributed given S_t.

3 The Euler Discretization Scheme

3.1 Motivation for Discretization

While equation (6) provides the exact solution for GBM, numerical simulation requires discretizing the continuous-time process. The Euler-Maruyama scheme is the simplest and most widely used method for this purpose.

Consider a partition of the time interval $[0, T]$ into N equally spaced points

$$0 = t_0 < t_1 < t_2 < \cdots < t_N = T \tag{7}$$

with time step $\Delta t = T/N$. Our goal is to approximate $S_{t_{i+1}}$ given S_{t_i}.

3.2 The Euler-Maruyama Scheme

For a general stochastic differential equation of the form

$$dX_t = \mu(t, X_t)\,dt + \sigma(t, X_t)\,dW_t, \tag{8}$$

the Euler-Maruyama discretization approximates the solution by

$$X_{t_{i+1}} \approx X_{t_i} + \mu(t_i, X_{t_i})\Delta t + \sigma(t_i, X_{t_i})\sqrt{\Delta t}\,Z_i \tag{9}$$

where $Z_i \sim \mathcal{N}(0,1)$ are independent standard normal random variables.

Applying this to the GBM equation (2) with $\mu(t, S) = rS$ and $\sigma(t, S) = \sigma S$ yields

$$S_{t_{i+1}} = S_{t_i} + rS_{t_i}\Delta t + \sigma S_{t_i}\sqrt{\Delta t}\,Z_i. \tag{10}$$

This is sometimes called the arithmetic Euler scheme.

3.3 The Geometric Euler Scheme

A numerically more stable approach is to apply the Euler scheme to the log-price process. Since

$$d(\ln S_t) = \left(r - \frac{1}{2}\sigma^2\right)dt + \sigma\,dW_t, \tag{11}$$

the Euler discretization gives

$$\ln S_{t_{i+1}} = \ln S_{t_i} + \left(r - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t}\,Z_i. \tag{12}$$

Exponentiating both sides yields

$$S_{t_{i+1}} = S_{t_i} \exp\left[\left(r - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t}\,Z_i\right]. \tag{13}$$

This geometric form ensures that stock prices remain strictly positive and corresponds exactly to the analytical solution (6) over each time step. For GBM, this scheme is therefore exact in distribution and is preferred in practice.

Remark 1. The Euler-Maruyama scheme has strong convergence of order $\mathcal{O}(\sqrt{\Delta t})$ and weak convergence of order $\mathcal{O}(\Delta t)$. For hedging simulations where we care about the distribution of terminal payoffs, weak convergence is the relevant measure.

4 European Call Option Pricing

4.1 The Black-Scholes Framework

Consider a European call option with strike price K and maturity T written on a non-dividend paying stock. Under the assumptions of the Black-Scholes model, the price $C(S, t)$ of this option satisfies the partial differential equation

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - rC = 0 \tag{14}$$

with terminal condition $C(S, T) = \max(S - K, 0)$.

Equivalently, by risk-neutral pricing, the option value is the discounted expected payoff under the measure $\mathbb{Q}$:

$$C(S_t, t) = e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}\left[\max(S_T - K, 0)\right]. \tag{15}$$

4.2 The Black-Scholes Formula

The solution to equation (14) is the celebrated Black-Scholes formula

$$C(S, t) = S\,\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) \tag{16}$$

where $\Phi(\cdot)$ denotes the cumulative distribution function of the standard normal distribution, and

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}, \tag{17}$$

$$d_2 = d_1 - \sigma\sqrt{T - t} = \frac{\ln(S/K) + (r - \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}. \tag{18}$$

The term $\Phi(d_2)$ represents the risk-neutral probability that the option expires in the money, while $\Phi(d_1)$ is related to the expected stock price conditional on the option being exercised.

4.3 The Option Delta

The delta of an option measures its sensitivity to changes in the underlying stock price. For a European call option, the delta is

$$\Delta = \frac{\partial C}{\partial S} = \Phi(d_1). \tag{19}$$

Since $0 < \Phi(d_1) < 1$, the delta of a call option is always between zero and one. For at-the-money options, $\Delta \approx 0.5$. As the option moves deeper in the money, $\Delta \to 1$, and as it moves out of the money, $\Delta \to 0$.

The delta plays a central role in hedging because it tells us how many shares of stock we need to hold to neutralize the first-order exposure to stock price movements.

5 The Delta Hedging Strategy

5.1 Replication and Self-Financing Portfolios

The fundamental insight of Black-Scholes is that the payoff of an option can be replicated by dynamically trading in the underlying stock and a risk-free bond. Consider a portfolio consisting of x_t units of the cash account (with value $B_t = e^{rt}$) and y_t shares of stock. The portfolio value is

$$\Pi_t = x_t B_t + y_t S_t. \tag{20}$$

A portfolio is called self-financing if changes in its value arise solely from changes in asset prices, with no external cash flows. Mathematically, this means

$$d\Pi_t = x_t\, dB_t + y_t\, dS_t = r x_t B_t\, dt + y_t(rS_t\, dt + \sigma S_t\, dW_t). \tag{21}$$

By matching the stochastic terms in the option price dynamics (derived via Itô's lemma) with those of the replicating portfolio, one finds that the hedge ratio must be

$$y_t = \frac{\partial C}{\partial S} = \Delta_t. \tag{22}$$

This is the delta hedging strategy: at each instant, hold Δ_t shares of stock to eliminate exposure to small movements in S_t.

5.2 Discrete-Time Hedging

In practice, continuous rebalancing is impossible. Instead, we rebalance at discrete times $t_0, t_1, \ldots, t_N$. Let Π_{t_i} denote the portfolio value at time t_i and $\delta_{t_i} = \Delta(S_{t_i}, t_i)$ the Black-Scholes delta computed at that time.

At time t_i, the portfolio consists of δ_{t_i} shares of stock and a cash position of $\Pi_{t_i} - \delta_{t_i} S_{t_i}$. Over the interval $[t_i, t_{i+1}]$, the cash earns interest at rate r and the stock position changes in value. For a non-dividend paying stock, the self-financing condition implies

$$\Pi_{t_{i+1}} = \Pi_{t_i} + \underbrace{(\Pi_{t_i} - \delta_{t_i} S_{t_i}) \cdot r\Delta t}_{\text{interest on cash}} + \underbrace{\delta_{t_i}(S_{t_{i+1}} - S_{t_i})}_{\text{stock gain/loss}}. \tag{23}$$

This formula has a clear interpretation. The first term is the portfolio value at the start of the period. The second term is the interest earned on the cash position (which may be negative if we have borrowed money to buy shares). The third term is the profit or loss from holding δ_{t_i} shares as the stock price moves from S_{t_i} to $S_{t_{i+1}}$.

Remark 2. If the stock pays a continuous dividend yield q, the update formula becomes

$$\Pi_{t_{i+1}} = \Pi_{t_i} + (\Pi_{t_i} - \delta_{t_i} S_{t_i}) r\Delta t + \delta_{t_i}(S_{t_{i+1}} - S_{t_i} + q S_{t_i} \Delta t) \tag{24}$$

where the additional term $\delta_{t_i} q S_{t_i} \Delta t$ represents dividends received on the stock holding.

5.3 Initialization and Terminal P&L

For a short position in a call option, we initialize the replicating portfolio with the option premium received at time zero:

$$\Pi_0 = C(S_0, 0). \tag{25}$$

This is the fair value of the option computed using the Black-Scholes formula (16).

At maturity $T = t_N$, the profit and loss of the hedged position is the difference between the replicating portfolio value and the option payoff we must deliver:

$$\text{P\&L} = \Pi_T - \max(S_T - K, 0). \tag{26}$$

If the hedge were perfect (continuous rebalancing with the true volatility), we would have $\text{P\&L} = 0$ almost surely. In practice, discrete hedging introduces replication error, and the P&L is a random variable whose distribution depends on the hedging frequency and any mismatch between the volatility used for computing delta and the realized volatility of the stock.

6 The Complete Simulation Algorithm

We now summarize the complete algorithm for simulating the delta hedging strategy.

6.1 Parameters

The simulation requires the following inputs: the initial stock price S_0, the strike price K, the maturity T, the risk-free rate r, the volatility σ_{sim} used for simulating stock prices, the volatility σ_{hedge} used for computing the Black-Scholes delta, and the number of rebalancing periods N.

6.2 Algorithm

The simulation proceeds as follows.

First, compute the initial option price $C_0 = C(S_0, K, T, r, \sigma_{\text{hedge}})$ using the Black-Scholes formula and initialize the portfolio $\Pi_0 = C_0$.

Second, compute the initial delta $\delta_0 = \Phi(d_1)$ where d_1 is evaluated at S_0 with time to maturity T.

Third, for each time step $i = 0, 1, \ldots, N-1$, perform the following operations. Generate a standard normal random variable Z_i. Update the stock price using the geometric Euler scheme

$$S_{t_{i+1}} = S_{t_i} \exp\left[\left(r - \frac{1}{2}\sigma_{\text{sim}}^2\right)\Delta t + \sigma_{\text{sim}}\sqrt{\Delta t}\, Z_i\right]. \tag{27}$$

Update the portfolio value using the self-financing condition

$$\Pi_{t_{i+1}} = \Pi_{t_i} + (\Pi_{t_i} - \delta_{t_i} S_{t_i}) r \Delta t + \delta_{t_i}(S_{t_{i+1}} - S_{t_i}). \tag{28}$$

Compute the new delta $\delta_{t_{i+1}} = \Phi(d_1)$ where d_1 is now evaluated at $S_{t_{i+1}}$ with time to maturity $T - t_{i+1}$.

Fourth, compute the final P&L as $\text{P\&L} = \Pi_T - \max(S_T - K, 0)$.

6.3 Monte Carlo Estimation

To analyze the hedging performance, we repeat the above simulation for M independent paths and collect the P&L values $\{\text{P\&L}^{(1)}, \ldots, \text{P\&L}^{(M)}\}$. We then compute summary statistics including the sample mean

$$\overline{\text{P\&L}} = \frac{1}{M}\sum_{j=1}^{M} \text{P\&L}^{(j)} \tag{29}$$

and the sample variance

$$\widehat{\mathrm{Var}}(\mathrm{P\&L}) = \frac{1}{M-1} \sum_{j=1}^{M} \left(\mathrm{P\&L}^{(j)} - \overline{\mathrm{P\&L}} \right)^2. \tag{30}$$

7 Theoretical Results on Hedging Performance

7.1 Continuous-Time P&L Formula

In the limit of continuous hedging, it can be shown that the P&L of a delta-hedged short option position satisfies

$$\mathrm{P\&L} = \int_0^T \frac{1}{2} S_t^2 \frac{\partial^2 C}{\partial S^2} \left(\sigma_{\mathrm{hedge}}^2 - \sigma_t^2\right) dt \tag{31}$$

where σ_t is the instantaneous realized volatility at time t and $\Gamma_t = \partial^2 C / \partial S^2$ is the option gamma.

This formula reveals that the P&L depends on the gamma-weighted difference between the hedging volatility and the realized volatility. When $\sigma_{\mathrm{hedge}} = \sigma_t$ for all t, the integrand vanishes and the expected P&L is zero. When $\sigma_{\mathrm{hedge}} > \sigma_t$, the option seller profits on average because they sold volatility at a higher level than was subsequently realized.

7.2 Effect of Hedging Frequency

For discrete hedging with N rebalancing periods, the variance of the hedging error scales approximately as

$$\mathrm{Var}(\mathrm{P\&L}) \propto \frac{1}{N}. \tag{32}$$

This means that doubling the hedging frequency roughly halves the variance. Daily hedging ($N \approx 252$) will therefore have much smaller hedging error variance than weekly hedging ($N \approx 52$).

The intuition is that between rebalancing points, the delta changes but our hedge does not. The gamma of the option quantifies this sensitivity of delta to stock price movements. More frequent hedging keeps the portfolio closer to delta-neutral, reducing the accumulated hedging slippage.

8 Application to the Assignment

For the assignment, we consider a European call option with the following parameters: initial stock price $S_0 = 100$ EUR, strike price $K = 99$ EUR, maturity $T = 1$ year, risk-free rate $r = 6\%$, and volatility $\sigma = 20\%$.

In Task 1 (Matching Volatility), we set $\sigma_{\mathrm{sim}} = \sigma_{\mathrm{hedge}} = 20\%$ and vary the hedging frequency from daily to weekly. The expected P&L should be approximately zero, and we expect the variance to decrease as the hedging frequency increases.

In Task 2 (Mismatched Volatility), we introduce a discrepancy between σ_{sim} and σ_{hedge}. According to equation (31), when $\sigma_{\mathrm{hedge}} > \sigma_{\mathrm{sim}}$, the option seller should profit on average, and vice versa.