Delta Hedging with Real SPX Option Data

A Mathematical Guide to Part 3 of Lab Assignment 1

Computational Finance

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1 Overview and Motivation

In Part 2 of this assignment, we performed delta-hedging simulations in a *controlled* Black–Scholes world, where the data-generating process was known. Part 3 brings us into the real world: we hedge real SPX options observed over February 2023, where the true data-generating process is unknown and models are necessarily misspecified.

The central question is: *given that the Black–Scholes model is wrong, how can we choose a hedge ratio δ_t that minimises the daily hedging error?*

2 Notation and Setup

We establish a unified notation that reconciles the lecture notes with the lab assignment.

2.1 Lecture notes notation

In the lecture notes, the option price is denoted $P(t, S_t)$ and the underlying satisfies

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t^{\mathbb{P}}, \tag{1}$$

under the physical measure $\mathbb{P}$. The delta hedge is $\Delta_t = \frac{\partial P}{\partial S}(t, S_t)$ and the one-step P&L of a short-option, long-Δ portfolio over a small interval δt is (eq. 102 in lecture notes):

$$\text{PnL} = -\underbrace{\left[\frac{\partial P}{\partial t} - rP + (r-q)S_t \frac{\partial P}{\partial S}\right]}_{A(t,S_t)} \delta t \; - \underbrace{\tfrac{1}{2} S_t^2 \frac{\partial^2 P}{\partial S^2}}_{\text{dollar gamma } \Gamma^{\$}} (\delta S)^2. \tag{2}$$

2.2 Lab assignment notation

The assignment uses a discrete, data-driven framework. Let t index trading days.

Symbol	Definition	Lecture equivalent
S_t	SPX close on day t	S_t
ΔS_t	$S_{t+1} - S_t$	δS
V_t	Option midquote $\frac{\text{bid}_t + \text{ask}_t}{2}$	$P(t, S_t)$
ΔV_t	$V_{t+1} - V_t$	$P(t+\delta t, S_{t+\delta t}) - P(t, S_t)$
δ_t	Hedge ratio (delta) used on day t	$\Delta_t = \frac{\partial P}{\partial S}$
$\varepsilon_t(\delta)$	$\Delta V_t - \delta_t \, \Delta S_t$	Hedging residual
σ^{mkt}	Market-implied volatility	σ_{imp}
τ_t	$(T-t)/365$, time to expiry in years	$T - t$
$r_t(\tau)$	Continuously-compounded zero rate	r
K	Strike price (in index points)	K
D_t	Days to expiry $T - t$	—

Key Insight

The lecture notes use infinitesimal $\delta t \to 0$; the lab uses discrete daily steps $\Delta t = 1$ day. Everything that was exact in continuous time becomes an *approximation* with nonzero residuals.

3 The Hedging Residual: Connecting Theory to Data

3.1 From continuous to discrete

In continuous time (lecture eq. 97–102), setting $\Delta_t = \frac{\partial P}{\partial S}$ eliminates the first-order exposure to δS. The remaining P&L is second-order:

$$\mathrm{PnL} = -A(t, S_t)\,\delta t - B(t, S_t)\left(\frac{\delta S}{S_t}\right)^2, \tag{3}$$

where $A(t, S_t) = \frac{\partial P}{\partial t} - rP + (r-q)S_t\frac{\partial P}{\partial S}$ is the theta-like term and $B(t, S_t) = \frac{1}{2}S_t^2\frac{\partial^2 P}{\partial S^2}$ is the dollar gamma.

In the discrete lab setting, we cannot observe A or B directly. Instead, we observe the *total* change ΔV_t and the stock move ΔS_t, and we define the hedging residual:

$$\boxed{\varepsilon_t(\delta) = \Delta V_t - \delta_t\,\Delta S_t.} \tag{4}$$

This residual captures *everything* that the linear hedge $\delta_t\,\Delta S_t$ failed to explain: gamma effects, theta, discrete rebalancing error, model misspecification, and bid–ask noise.

3.2 What makes ε_t large?

Expanding ΔV_t using a Taylor-like decomposition (analogous to eq. (2)):

$$\Delta V_t \approx \underbrace{\frac{\partial P}{\partial S}\Delta S_t}_{\text{delta component}} + \underbrace{\frac{\partial P}{\partial t}\Delta t}_{\text{theta}} + \underbrace{\frac{1}{2}\frac{\partial^2 P}{\partial S^2}(\Delta S_t)^2}_{\text{gamma}} + \text{higher order.} \tag{5}$$

Substituting into (4) with $\delta_t = \frac{\partial P}{\partial S}$:

$$\varepsilon_t \approx \Theta_t\,\Delta t + \tfrac{1}{2}\Gamma_t\,(\Delta S_t)^2 + \text{noise}, \tag{6}$$

where $\Theta_t = \frac{\partial P}{\partial t}$ and $\Gamma_t = \frac{\partial^2 P}{\partial S^2}$.

Under Black–Scholes, Θ and Γ are related through the PDE:

$$\Theta + \tfrac{1}{2}\sigma^2 S^2 \Gamma + (r-q)S\,\Delta - rP = 0.$$

If the model were exact *and* we could hedge continuously, the theta and gamma terms would cancel on average. In discrete time with daily rebalancing, they don't cancel exactly, leaving a nonzero ε_t.

4 Scoring: SSE and Gain

The assignment measures hedging quality through:

$$\mathrm{SSE}(\delta) = \sum_{t\in\mathcal{I}} \varepsilon_t(\delta)^2, \tag{7}$$

where $\mathcal{I}$ is the filtered observation set (after removing extreme deltas and short maturities). Lower SSE means better hedging.

Given two strategies $\delta^{(A)}$ and $\delta^{(B)}$, the **Gain** metric is:

$$\mathrm{Gain}(A \text{ vs } B) = 1 - \frac{\mathrm{SSE}(\delta^{(A)})}{\mathrm{SSE}(\delta^{(B)})}. \tag{8}$$

A positive gain means strategy A beats B. The baseline B is the standard Black–Scholes delta δ_t^{BS}.

The bucketed SSE (across 9 moneyness $\times$ 7 maturity $=$ 63 cells) ensures that improvements are measured uniformly across the option surface, not dominated by one region.

5 Why We Need Treasury Data and the Risk-Free Rate

The Black–Scholes delta for a European call is:

$$\delta_t^{\mathrm{BS}} = N(d_+), \qquad d_+ = \frac{\ln(S_t/K) + \big(r_t(\tau) + \frac{1}{2}\sigma^2\big)\tau}{\sigma\sqrt{\tau}}, \tag{9}$$

where $N(\cdot)$ is the standard normal CDF. The rate $r_t(\tau)$ appears in two critical places:

(i) **In d_+**: it shifts the forward moneyness. Underestimating r makes d_+ too small, biasing delta downward for calls.

(ii) **In the forward price**: $F_t(\tau) = S_t\, e^{(r_t(\tau)-q)\tau}$. The forward determines where ATM sits on the strike axis.

(iii) **In discounting**: the discount factor $P_t(\tau) = e^{-r_t(\tau)\,\tau}$ is used for present-valuing cash flows.

Since each option has a different time-to-expiry τ, we need a *term structure* $r_t(\tau)$ — not a single scalar. This is why we fit the Nelson–Siegel–Svensson (NSS) yield curve to Treasury par yields and evaluate it at each option's τ.

5.1 The conversion to continuous compounding

Treasury par yields are quoted with semiannual compounding. To convert to the continuously-compounded rate required by Black–Scholes:

$$r_t(\tau) = 2\ln\!\left(1 + \frac{\hat{y}_t(\tau)}{2}\right), \tag{10}$$

where $\hat{y}_t(\tau)$ is the NSS-fitted yield. For short-dated SPX options (a few weeks to months), $r_t(\tau) \approx \hat{y}_t(\tau)$ since the compounding adjustment is small. But for consistency and to avoid systematic bias, the conversion is necessary.

6 The Filters and Why They Matter

The assignment mandates removing observations with:

$$\delta_t^{\mathrm{BS}} \le 0.05, \qquad \delta_t^{\mathrm{BS}} \ge 0.95, \qquad D_t \le 14. \tag{11}$$

Deep OTM ($\delta \le 0.05$): These options have tiny midquotes (often a few cents). The hedging residual ε_t is dominated by bid–ask noise rather than genuine model error. Including them inflates SSE with information-free variance.

Deep ITM ($\delta \ge 0.95$): The option behaves like the stock ($\delta \approx 1$) regardless of model choice. Every strategy gives $\delta \approx 1$, so no strategy can differentiate itself. These observations are uninformative for comparison.

Short maturity ($D_t \le 14$): Near expiry, gamma Γ_t grows as $\sim 1/\sqrt{\tau}$ for ATM options. From eq. (6), the gamma term $\frac{1}{2}\Gamma_t(\Delta S_t)^2$ becomes extremely large, making daily rebalancing hopelessly inadequate. This is a discretization artefact, not a model problem.

Key Insight

The competitive region is $0.05 < \delta < 0.95$ and $D_t > 14$. This is where your choice of δ_t genuinely matters. Focus your optimisation efforts here.

7 The Volatility Misspecification Effect

This is the theoretical backbone of the competitive advantage. From Exercise 2 (eq. 6 in the assignment), the cumulative P&L of a delta hedge using implied vol σ_{imp} when the true (realised) vol is σ_t is:

$$\text{PnL}_{\text{total}} = \int_0^T e^{-rt}\,\frac{1}{2}S_t^2\,\Gamma_t^{\sigma_{\text{imp}}}\left(\sigma_{\text{imp}}^2 - \sigma_t^2\right)dt, \tag{12}$$

where $\Gamma_t^{\sigma_{\text{imp}}}$ is the gamma computed under σ_{imp}.

7.1 Interpretation

Eq. (12) says the P&L depends on the mismatch $(\sigma_{\text{imp}}^2 - \sigma_t^2)$ *weighted by dollar gamma*. For the P&L to vanish on average:

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T e^{-rt}\,\frac{1}{2}S_t^2\,\Gamma_t^{\sigma_{\text{imp}}}\left(\sigma_{\text{imp}}^2 - \sigma_t^2\right)dt\right] = 0. \tag{13}$$

This means the "correct" volatility for hedging is not the simple average of σ_t, but a **gamma-weighted average**:

$$\sigma_{\text{imp}}^2 = \frac{\mathbb{E}\left[\int_0^T e^{-rt}\,\frac{1}{2}S_t^2\,\Gamma_t\,\sigma_t^2\,dt\right]}{\mathbb{E}\left[\int_0^T e^{-rt}\,\frac{1}{2}S_t^2\,\Gamma_t\,dt\right]}. \tag{14}$$

Key Insight

Since Γ_t is largest for ATM options, the break-even vol is disproportionately influenced by realised vol around the times when the option is near ATM. Deep OTM/ITM periods contribute less.

7.2 Discrete-time analogue

In our daily framework, eq. (12) becomes approximately:

$$\varepsilon_t(\delta^\sigma) \approx \frac{1}{2}\Gamma_t^\sigma\,S_t^2\left[\left(\frac{\Delta S_t}{S_t}\right)^2 - \sigma^2\Delta t\right] + \text{lower order terms}. \tag{15}$$

The residual is small when $\sigma^2\Delta t \approx (\Delta S_t/S_t)^2$, i.e., when the volatility parameter matches the realised daily squared return. This motivates using a vol closer to realised vol rather than implied vol.

8 Strategies for Competitive Edge

The baseline strategy uses the Black–Scholes delta with the market-implied volatility σ^{mkt} (or the vendor-provided delta from WRDS). Here are mathematically motivated ways to improve.

8.1 Strategy 1: Adjusted Volatility Delta

From the misspecification analysis, the optimal hedging vol is the gamma-weighted realised vol, not the implied vol. In practice, for SPX options there exists a persistent **variance risk premium**: implied vol is systematically higher than realised vol.

Define an adjusted delta:

$$\delta_t^{\text{adj}} = N\Big(d_+(\alpha\,\sigma^{\text{mkt}})\Big), \qquad \alpha \in (0,1], \tag{16}$$

where $\alpha < 1$ scales down the implied vol toward realised vol. You can find the optimal α by minimising SSE over a grid:

$$\alpha^* = \arg\min_{\alpha} \ \mathrm{SSE}\left(\delta^{\alpha\,\sigma^{\mathrm{mkt}}}\right). \tag{17}$$

Warning

This is an in-sample optimisation. The optimal α is specific to this dataset (Feb 2023). Report it transparently and discuss overfitting risk.

8.2 Strategy 2: Minimum Variance (MV) Delta

Instead of using a model to compute δ, we can estimate it from the data directly. For each option i, the hedging residual is $\varepsilon_{i,t} = \Delta V_{i,t} - \delta_{i,t}\,\Delta S_t$. If we treat $\delta_{i,t}$ as a regression coefficient:

$$\Delta V_{i,t} = \delta_i\,\Delta S_t + \varepsilon_{i,t}, \tag{18}$$

then the OLS estimate $\hat{\delta}_i = \frac{\sum_t \Delta V_{i,t}\,\Delta S_t}{\sum_t (\Delta S_t)^2}$ minimises the sum of squared residuals by construction. However, this is purely in-sample and cannot be used directly for out-of-sample hedging. It does, however, provide a *target* that tells you what the ideal delta would have been. You can then study the systematic difference $\hat{\delta}_i - \delta_i^{\mathrm{BS}}$ across moneyness/maturity to guide adjustments.

8.3 Strategy 3: Smile-Aware (Sticky-Strike) Delta

The standard BS delta assumes the volatility surface is flat. In reality, the implied volatility smile $\sigma^{\mathrm{mkt}}(K,\tau)$ varies with strike. Two common assumptions lead to different deltas:

- **Sticky-strike**: The implied vol of a fixed (K,T) contract doesn't change when S moves. This gives the standard BS delta $N(d_+)$.
- **Sticky-delta (or sticky-moneyness)**: The implied vol at a fixed moneyness level K/S stays constant. This adds a correction:

$$\delta_t^{\mathrm{smile}} = N(d_+) + \underbrace{\frac{\partial C}{\partial \sigma}\cdot\frac{\partial \sigma}{\partial S}}_{\text{vega} \times \text{smile slope}} \approx N(d_+) + \nu_t \cdot \frac{\partial \sigma^{\mathrm{mkt}}}{\partial K}\cdot\left(-\frac{K}{S_t}\right), \tag{19}$$

  where $\nu_t = S_t N'(d_+)\sqrt{\tau}$ is the Black–Scholes vega. The correction is typically negative for SPX (the skew is downward-sloping: $\frac{\partial \sigma}{\partial K} < 0$), which means the smile-adjusted delta is *larger* than the BS delta.

Estimating $\frac{\partial \sigma^{\mathrm{mkt}}}{\partial K}$ at each (t,K,τ) from the option panel requires care (finite differences across neighbouring strikes on the same day and expiry).

8.4 Strategy 4: Per-Bucket Volatility Scaling

Since the competition is scored on bucketed SSE (9 moneyness $\times$ 7 maturity), you can allow the adjustment factor to vary by bucket:

$$\delta_t^{(i,j)} = N\left(d_+\left(\alpha_{i,j}\,\sigma_t^{\mathrm{mkt}}\right)\right), \tag{20}$$

where (i,j) denotes the moneyness–maturity bucket. This gives you 63 free parameters but risks overfitting. A middle ground is to fit a smooth surface $\alpha(\delta, D_t)$ with a small number of parameters (e.g., linear in δ and $\log D_t$).

8.5 Strategy 5: Realised Volatility Blending

Compute the recent realised volatility from the SPX close series (e.g., over a rolling window of 5–10 days):

$$\hat{\sigma}_t^{\text{real}} = \sqrt{\frac{252}{n} \sum_{k=0}^{n-1} \left(\ln \frac{S_{t-k}}{S_{t-k-1}} \right)^2}, \tag{21}$$

and blend it with implied vol:

$$\sigma_t^{\text{blend}} = w\, \sigma_t^{\text{mkt}} + (1-w)\, \hat{\sigma}_t^{\text{real}}, \qquad w \in [0,1]. \tag{22}$$

Then compute delta using σ_t^{blend}. The rationale from eq. (14): the optimal hedging vol lies between implied and realised. The weight w can be optimised over the dataset.

9 What to Watch Out For

1. **Overfitting**: Any in-sample SSE minimisation (grid search over α, per-bucket $\alpha_{i,j}$) will overfit to this specific month. Prefer theoretically motivated, low-parameter strategies. Report both the optimised SSE and discuss robustness.
2. **Data alignment**: Ensure ΔV_t and ΔS_t correspond to the *same* day transition $t \to t+1$. A one-day misalignment will produce nonsensical residuals.
3. **Strike rescaling**: WRDS stores strikes as $K \times 1000$. If you forget to divide, d_+ will be wildly wrong and your deltas will be meaningless.
4. **Consecutive trading days**: When computing ΔV_t, ensure both V_t and V_{t+1} come from consecutive trading days. If a contract doesn't trade on both days, ΔV_t spans a gap and the residual is inflated.
5. **The gamma effect near ATM**: ATM options have the largest gamma, so their residuals ε_t are naturally larger. This is not a model failure — it's the physics of options. The bucketing ensures your score isn't dominated by these points alone.
6. **Implied vol vs. delta from WRDS**: The WRDS dataset provides both σ^{mkt} and a vendor delta δ^{vendor}. Using the vendor delta directly as the baseline is the simplest approach. If you recompute delta yourself (to try a different vol), make sure your BS formula, rate convention, and day-count are all consistent.
7. **Dividends**: The assignment says set $q = 0$. For SPX in Feb 2023 this is a reasonable approximation, but be aware that ignoring dividends introduces a small systematic bias in the forward price.

10 Summary: The Pipeline

Step	Description
1	Load and clean the WRDS option panel (calls only, rescale strikes, midquote, drop missing IV)
2	Construct ΔV_t by tracking each contract across consecutive trading days
3	Merge SPX close S_t and compute ΔS_t (with one extra day beyond Feb 28)
4a–e	Fit NSS yield curve → extract $r_t(\tau)$ for each option observation
5a	Compute baseline $\delta_t^{\mathrm{BS}} = N(d_+)$ using σ^{mkt} and $r_t(\tau)$
5b	Compute residuals $\varepsilon_t = \Delta V_t - \delta_t^{\mathrm{BS}} \Delta S_t$ and baseline SSE
5c	Apply filters: $\delta \in (0.05, 0.95)$ and $D_t > 14$
5d	Study misspecification: recompute $\delta_t(\sigma)$ for σ on a grid, observe SSE curve
5e	Bucket into 9×7 moneyness–maturity grid
6	Try improved strategies (vol scaling, smile-aware delta, blending) → report Gain

Competitive Edge

Bottom line: The variance risk premium means $\sigma_{\mathrm{imp}} > \sigma_{\mathrm{real}}$ on average for SPX. The BS delta computed with implied vol is therefore systematically "too high" (it uses a vol that overstates future moves). Hedging with a slightly lower vol — closer to the gamma-weighted realised vol from eq. (14) — should reduce SSE. The simplest competitive strategy is to find $\alpha^* \in (0.85, 0.99)$ such that $\delta_t = N(d_+(\alpha^* \sigma^{\mathrm{mkt}}))$ minimises SSE. Combine this with smile-slope corrections for additional edge.