Introduction to Linear Approximation

We're starting a new unit: applications of differentiation.

We're going to do two applications today. The first is linear approximations, which are encompassed by the single formula:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0),$$

but it will take at least half an hour to explain how this formula is used.

Linear Approximation to $\ln x$ at $x = 1$

If you have a curve $y = f(x)$, it is approximately the same as its tangent line $y = f(x_0) + f'(x_0)(x - x_0)$.

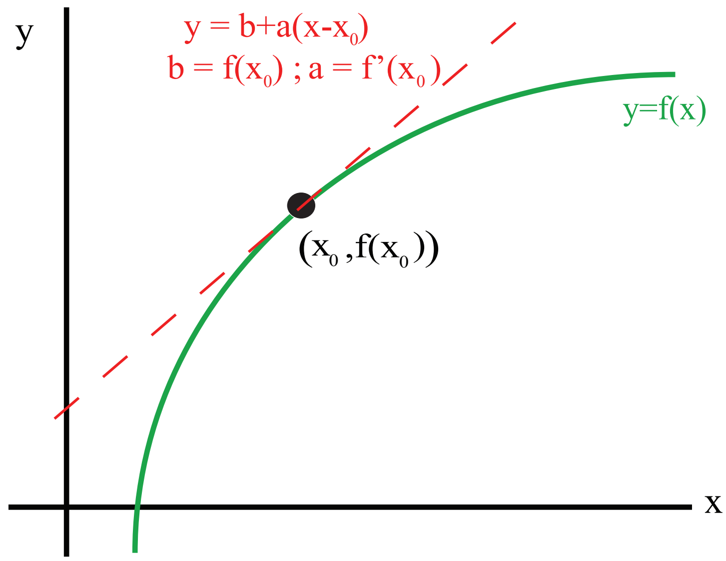

Figure 1: Tangent as a linear approximation to a curve

The tangent line approximates $f(x)$. It gives a good approximation near the tangent point x_0. As you move away from x_0, however, the approximation grows less accurate.

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

Example 1 Let $f(x) = \ln x$. Then $f'(x) = \frac{1}{x}$. We'll use the base point $x_0 = 1$ because we can easily evaluate $\ln 1 = 0$. Note also that $f'(x_0) = \frac{1}{1} = 1$.

Then the formula for linear approximation tells us that:

$$\begin{aligned} f(x) &\approx f(x_0) + f'(x_0)(x - x_0) \\ \ln x &\approx \ln(1) + 1(x - 1) \\ \ln x &\approx 0 + (x - 1) \\ \ln x &\approx (x - 1) \end{aligned}$$

Graph the curve $y = \ln x$ and the line $y = x - 1$. You'll see that the two graphs are very close together when $x = x_0 = 1$. You'll also see that they're only near each other when x is near 1.

The point of linear approximation is that the curve (in this case $y = \ln x$) is approximately the same as the tangent line ($y = x - 1$) when x is close to the base point x_0.

Linear Approximation and the Definition of the Derivative

Another way to understand the formula for linear approximation involves the definition of the derivative:

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$$

Look at this backward:

$$\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = f'(x_0)$$

We can interpret this to mean that:

$$\frac{\Delta f}{\Delta x} \approx f'(x_0) \quad \text{when } \Delta x \approx 0.$$

In other words, the average rate of change $\frac{\Delta f}{\Delta x}$ is nearly the same as the infinitesimal rate of change $f'(x_0)$.

We can see that this is the same as our original formula

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

if we multiply both sides by Δx and remind ourselves what Δx and Δf are abbreviations for:

$$\begin{aligned}
\frac{\Delta f}{\Delta x} &\approx f'(x_0) \\
\Delta x \cdot \frac{\Delta f}{\Delta x} &\approx f'(x_0) \cdot \Delta x \\
\Delta f &\approx f'(x_0) \cdot \Delta x \\
f(x) - f(x_0) &\approx f'(x_0)(x - x_0) \\
f(x) &\approx f(x_0) + f'(x_0)(x - x_0)
\end{aligned}$$

So we have two different ways of writing a formula for linear approximation. When you're solving linear approximation problems, try to choose the most appropriate formula for the problem you're working on.

Approximations at 0 for Sine, Cosine and Exponential Functions

Here is a list of several linear approximations which you may want to memorize. Half the work of memorizing a linear approximation is memorizing the derivative of a function at a base point, so memorizing these formulas should improve your knowledge of derivatives.

To make things as simple as possible, we always use base point $x_0 = 0$ and assume that $x \approx 0$. Then our general formula becomes:

$$f(x) \approx f(0) + f'(0)x.$$

Remember that when x is not near zero, this approximation probably won't work.

(Later we'll discuss exactly how close x has to be to zero; this is partly a matter of intuition and is very important in applications.)

We want to find linear approximations for the functions $\sin x$, $\cos x$ and e^x when x is near 0. We'll start by building a table of values of $f'(x)$, $f(0)$, and $f'(0)$; from these we can "read off" the linear approximations.

$f(x)$	$f'(x)$	$f(0)$	$f'(0)$
$\sin x$	$\cos x$	0	1
$\cos x$	$-\sin x$	1	0
e^x	e^x	1	1

We can now plug the values for $f(0)$ and $f'(0)$ into our formula $f(x) \approx f(0) + f'(0)x$ to get linear approximations for these functions:

1. $\sin x \approx x \quad (\text{if } x \approx 0) \quad$ (see part (a) of Fig. 1)
2. $\cos x \approx 1 \quad (\text{if } x \approx 0) \quad$ (see part (b) of Fig. 1)
3. $e^x \approx 1 + x \quad (\text{if } x \approx 0)$

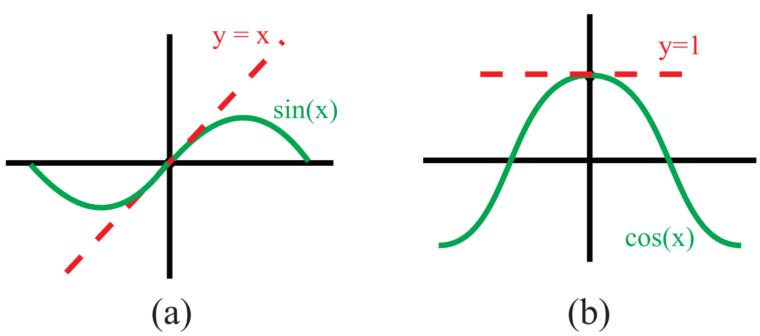

Figure 1: Linear approximations to sine and cosine at $x = 0$.

Approximations at 0 for $\ln(1+x)$ and $(1+x)^r$

Next, we compute two linear approximations that are slightly more challenging.

Here's the table of values:

$f(x)$	$f'(x)$	$f(0)$	$f'(0)$
$\ln(1+x)$	$\frac{1}{1+x}$	0	1
$(1+x)^r$	$r(1+x)^{r-1}$	1	r

And here are the linear approximations we get from the table:

1. $\ln(1+x) \approx x \quad (\text{if } x \approx 0)$
2. $(1+x)^r \approx 1 + rx \quad (\text{if } x \approx 0)$

Remember that we computed the linear approximation to $\ln x$ at $x_0 = 1$. Since our base point wasn't 0 we couldn't include that here. Because $\ln x \to -\infty$ as $x \to 0$, a linear approximation of $\ln x$ near $x_0 = 0$ is useless to us. Instead we have a linear approximation of the function $\ln(1+x)$ near our default base point $x_0 = 0$, which works out to nearly the same thing as a linear approximation of $\ln x$ near $x_0 = 1$.

Similarly, we found a linear approximation to $(1 + x)^r$; not to x^r. For some values of r, x^r is not well behaved when $x = 0$. If we really need an approximation of x^r we can get one by a change of variables.

For example, in a previous example we computed that $\ln u \approx u - 1$ for $u \approx 0$ (we've just replaced x by u.) Now we change variables by setting $u = 1 + x$. If we plug in $1 + x$ everywhere we had a u we get:

$$\ln(1 + x) \approx (1 + x) - 1 = x,$$

which is exactly the formula we have above.

If you've memorized $\ln(1+x) \approx x$ for $x \approx 0$ you can quickly find an approximation for $\ln u$ for $u \approx 1$ through the change of variables $x = u - 1$.

Curves are Hard, Lines are Easy

How are linear approximations used? We'll start with an example, then discuss.

Suppose I want to know the value of $\ln(1.1)$. If I've memorized the formula $\ln(1+x) = x$ for $x \approx 0$ I know right away that $\ln(1.1) \approx 0.1$ just by plugging in $x = 0.1$. (This works because 0.1 is "close enough" to zero.)

So what? The value of $\ln(1.1)$ is hard to compute; the value of 0.1 is easy. We used linear approximation to make a "hard" value "easy" to understand.

In general, $f(x)$ is hard to compute and $f(x_0) + f'(x_0)(x - x_0)$ is easy to compute (even though $f(x_0) + f'(x_0)(x - x_0)$ looks uglier). Look at the list below; evaluating the expressions on the left is hard and evaluating the ones on the right is easy.

$$\begin{aligned} \sin x &\approx x \\ \cos x &\approx 1 \\ e^x &\approx 1 + x \\ \ln(1+x) &\approx x \\ (1+x)^r &\approx 1 + rx \end{aligned}$$

That's the main advantage of linear approximation: it lets you work with expressions that are much easier to compute, which lets you make faster progress in solving problems.

Linear Approximation of a Complicated Exponential

Example 3: Find the linear approximation of $f(x) = \dfrac{e^{-3x}}{\sqrt{1+x}}$ near $x = 0$. We could calculate $f'(x)$ and find $f'(0)$. But instead, we will do this by algebraically combining the linear approximations we already have.

From our list of linear approximations we have:

$$\begin{aligned} e^x &\approx 1+x \\ (1+x)^r &\approx 1+rx \end{aligned}$$

So $\frac{1}{\sqrt{(1+x)}} = (1+x)^{-1/2} \approx 1 - \frac{x}{2}$.

We rewrite $\dfrac{e^{-3x}}{\sqrt{1+x}}$ as $e^{-3x} \cdot (1+x)^{-1/2}$ so that all we'll have to do is multiply two linear equations to get our approximation.

$$e^{-3x} \approx 1 + (-3x)$$

$$\frac{1}{\sqrt{1+x}} = (1+x)^{-1/2} \approx 1 - \frac{1}{2}x$$

Put these two approximations together to get:

$$\begin{aligned} \frac{e^{-3x}}{\sqrt{1+x}} &\approx (1-3x)(1-\frac{1}{2}x) \\ &\approx 1 - \frac{1}{2}x - 3x + \frac{3}{2}x^2 \\ &\approx 1 - \frac{1}{2}x - \frac{6}{2}x + \frac{3}{2}x^2 \end{aligned}$$

Remember that x is a number close to 0, so x^2 is a number very close to 0. In fact, when working with linear approximations we assume that x^2 is so small that we can safely ignore it. Our approximation then becomes:

$$\frac{e^{-3x}}{\sqrt{1+x}} \approx 1 - \frac{7}{2}x$$

When you're allowed to ignore all the higher degree terms — x^2, x^3 and so on — this sort of algebra becomes very simple. Why are we allowed to do that here and not in our algebra classes? For one thing, we're not calculating exact values only approximations. We know that the graph of the tangent line probably isn't the same as the graph of the function; we've already decided that finding an exact value is too hard. For another thing, these approximations are only valid when x is close to 0. If x is $1/100$ then x^2 is $1/10000$. The $\frac{3}{2}x^2$ we discarded has a value of 0.00015 when $x = .01$. When x is "small enough", any terms involving x^2 are so small that they don't significantly affect the final estimate.

Question: Can we use the original formula?

Earlier, we found that:

$$f(x) = \frac{e^{-3x}}{\sqrt{1+x}} \approx 1 - \frac{7}{2}x.$$

Could we use a different method to get a linear approximation of the function $f(x)$?

Yes. We could calculate f' and use the formula for linear approximation to find:

$$f(x) \approx f(0) + f'(0)x.$$

This must also be a linear approximation to $\frac{e^{-3x}}{\sqrt{1+x}}$.

We can easily find that $f(0) = 1$. Computing $f'(x)$ by the product rule is an annoying, somewhat long computation. Because of what we've just done we know that $f'(0)$ must equal $-\frac{7}{2}$. We used linear approximation as a shortcut to avoid computing $f'(0)$ directly.

When we study quadratic approximation we'll quickly see that combining approximations for complicated functions is far superior to differentiating them twice.

Question: If we find the linear approximation by differentiating, do we have to throw away an x^2 term?

Answer: No. But remember that when x is close to 0 throwing away an x^2 term has very little influence on our final value. Throwing away the x^2 was an easy way to simplify our expression; it's not something we should be trying to avoid here. (Linear approximation just captures the linear features of the function; we are not concerning ourselves with higher order terms here.)

Applications of Linear Approximation

In this unit we're trying to learn about applications of the derivative to real problems. Here is one such example that involves math as well as physics.

Example 4: Planet Quirk

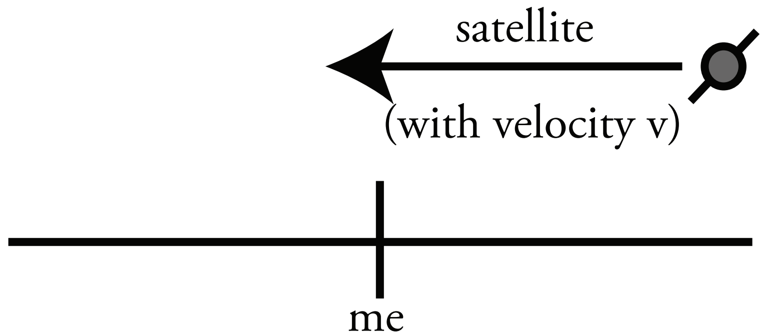

Figure 1: Illustration of Example 4: a satellite with velocity v speeding past "me" on planet Quirk.

Let's say I am on Planet Quirk, and that a satellite is whizzing overhead with a velocity v. The satellite has a clock on it that reports a time T. I have a watch that reports a time T_m. We want to calculate the time dilation (a concept from special relativity) that describes the difference between T and T_m.

We borrow the following equation from special relativity:

$$T_m = \frac{T}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Again, T_m is the time I measure on my wristwatch, and T is the time measured on board the satellite. How different are T_m and T?

To avoid dividing by a square root, we'll once again use linear approximation:

$$\frac{T}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{T}{\sqrt{1-u}} \approx T\left(1 + \frac{1}{2}u\right),$$

where $u = \frac{v^2}{c^2}$. (If you're wondering why we got $(1 + \frac{1}{2}u)$ and not $(1 - \frac{1}{2}u)$, try applying the change of variables $v = -u$ before approximating.)

And so, we find that:

$$T_m = \frac{T}{\sqrt{1 - \frac{v^2}{c^2}}} \approx T\left(1 + \frac{1}{2}\left(\frac{v^2}{c^2}\right)\right)$$

How does this affect you in the real world? GPS transmitters are mounted on satellites, the satellites are moving, and you might be moving too. According to special relativity, there will be a difference between the time on your GPS device and the time on the satellite. This time difference will affect the GPS device's estimate of your position.

The engineers who set up the GPS satellite system knew this and had to decide if they needed to take this into account when designing the system. It turns out that GPS satellites move at about $v = 4$ kilometers per second (km/s) and $c = 3 \times 10^5$ km/sec, and so $\frac{v^2}{c^2} \approx 10^{-10}$ and $T_m \approx T(1.00000000005)$. There's hardly any difference between the times measured on the ground and in the satellite.

Because $\frac{v^2}{c^2}$ is very close to zero, our linear approximation should be quite close to the actual value of T_m. Another good reason for using linear approximation here is that if the answer is "the difference is too small to matter", the person doing the calculation has no use for a more precise answer which may be more difficult to calculate.

Nevertheless, engineers used this very approximation (along with several other such approximations) to calibrate the radio transmitters on GPS satellites. The satellites transmit at a slightly offset frequency.

Relative Error

We continue with our example of time dilation in GPS satellite operation. We started with the following formula from special relativity:

$$T_m = \frac{T}{\sqrt{1-\frac{v^2}{c^2}}}$$

and used a linear approximation to find that:

$$T_m \approx T\left(1+\frac{1}{2}\left(\frac{v^2}{c^2}\right)\right).$$

This formula describes the difference due to time dilation between clocks on the ground and on the satellite. Algebraically, the difference is $\Delta T = T_m - T$; it turns out that there's a very simple relation between T, ΔT, v and c:

$$\begin{aligned} T_m &\approx T\left(1+\frac{1}{2}\left(\frac{v^2}{c^2}\right)\right) \\ T_m &\approx T+T\frac{1}{2}\left(\frac{v^2}{c^2}\right) \\ T_m - T &\approx T\frac{1}{2}\left(\frac{v^2}{c^2}\right) \\ \Delta T &\approx T\frac{1}{2}\left(\frac{v^2}{c^2}\right) \\ \frac{\Delta T}{T} &\approx \frac{1}{2}\left(\frac{v^2}{c^2}\right) \end{aligned}$$

In other words, the relative or percent error $\frac{\Delta T}{T}$ caused by time dilation is proportional to the ratio $\dfrac{v^2}{c^2}$, which relates the speed of the satellite to the speed of light.

As in the example of falling stock prices, this value $\frac{\Delta T}{T}$ gives us an idea of the relative size of the error introduced by time dilation.

The Formula for Quadratic Approximation

Quadratic approximation is an extension of linear approximation – we're adding one more term, which is related to the second derivative. The formula for the quadratic approximation of a function $f(x)$ for values of x near x_0 is:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 \quad (x \approx x_0)$$

Compare this to our old formula for the linear approximation of f:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) \quad (x \approx x_0).$$

We got from the linear approximation to the quadratic one by adding one more term that is related to the second derivative:

$$f(x) \approx \underbrace{f(x_0) + f'(x_0)(x - x_0)}_{\text{Linear Part}} + \underbrace{\frac{f''(x_0)}{2}(x - x_0)^2}_{\text{Quadratic Part}} \quad (x \approx x_0)$$

These are more complicated and so are only used when higher accuracy is needed.

We'd like to develop a catalog of quadratic approximations similar to our catalog of linear approximations. Let's start by looking at the quadratic version of our estimate of $\ln(1.1)$. The formula for the quadratic approximation turns out to be:

$$\ln(1 + x) \approx x - \frac{x^2}{2},$$

and so $\ln(1.1) = \ln(1 + \frac{1}{10}) \approx \frac{1}{10} - \frac{1}{2}(\frac{1}{10})^2 = 0.095$. This is not the value 0.1 that we got from the linear approximation, but it's pretty close (and slightly more accurate).

Explaining the Formula by Example

As we saw last time, quadratic approximations are a little more complicated than linear approximation. Use these when the linear approximation is not enough. For example, most modeling in economics uses quadratic approximation. When using approximation you sacrifice some accuracy for the ability to perform complex calculations; using approximations more precise than quadratic approximations can make your calculations too unwieldy to be useful.

The basic formula for quadratic approximation with base point $x_0 = 0$ is:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 \quad (x \approx x_0)$$

This works for values of x near 0; it is just the formula for linear approximation with one new term.

Where did that extra term come from, and what is the $\frac{1}{2}$ doing in front of the x^2? We'll explain this in terms of what happens if the graph of our function is a parabola. If the graph of a function is a parabola, that function *is* a quadratic function. Ideally, the quadratic approximation of a quadratic function should be identical to the original function.

For instance, consider:

$$f(x) = a + bx + cx^2; \qquad f'(x) = b + 2cx; \qquad f''(x) = 2c.$$

Set the base point $x_0 = 0$. Now we try to recover the values of a, b and c from our information about the derivatives of $f(x)$.

$$\begin{array}{rcll} f(0) & = & a + b \cdot 0 + c \cdot 0^2 & \implies \quad a = f(0) \\ f'(0) & = & b + 2c \cdot 0 = b & \implies \quad b = f'(0) \\ f''(0) & = & 2c & \implies \quad c = \dfrac{f''(0)}{2} \end{array}$$

This tells us what the coefficients of the quadratic approximation formula must be in order for the quadratic approximation of a quadratic function to equal that function. To confirm this, we see that applying the formula:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 \quad (x \approx x_0)$$

to our quadratic function $f(x) = a+bx+cx^2$ yields the quadratic approximation:

$$f(x) \approx a + bx + \frac{2c}{2}x^2.$$

Another way to think about this is that in the linear approximation of a function, the first derivative of the approximation is the same as the first derivative of the function. In a quadratic approximation, the first and second derivatives of the approximation are the same as the first and second derivatives of the function. If the $\frac{1}{2}$ weren't there, this wouldn't be true.

Quadratic Approximation at 0 for Several Examples

We'll save the derivation of the formula:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 \quad (x \approx x_0)$$

for later; right now we're going to find formulas for quadratic approximations of the functions for which we have a library of linear approximations.

Basic Quadratic Approximations:

In order to find quadratic approximations we need to compute second derivatives of the functions we're interested in:

$f(x)$	$f'(x)$	$f''(x)$	$f(0)$	$f'(0)$	$f''(0)$
$\sin x$	$\cos x$	$-\sin x$	0	1	0
$\cos x$	$-\sin x$	$-\cos x$	1	0	-1
e^x	e^x	3^x	1	1	1
$\ln(1+x)$	$\frac{1}{1+x}$	$\frac{-1}{(1+x)^2}$	0	1	-1
$(1+x)^r$	$r(1+x)^{r-1}$	$r(r-1)(1+x)^{r-2}$	1	r	$r(r-1)$.

Plugging the values for $f(0)$, $f'(0)$ and $f''(0)$ in to the quadratic approximation we get:

1. $\sin x \approx x \quad (\text{if } x \approx 0)$
2. $\cos x \approx 1 - \frac{x^2}{2} \quad (\text{if } x \approx 0)$
3. $e^x \approx 1 + x + \frac{1}{2}x^2 \quad (\text{if } x \approx 0)$
4. $\ln(1+x) \approx x - \frac{1}{2}x^2 \quad (\text{if } x \approx 0)$
5. $(1+x)^r \approx 1 + rx + \frac{r(r-1)}{2}x^2 \quad (\text{if } x \approx 0)$

We've computed some formulas; now let's think about their meaning.

Geometric significance (of the quadratic term)

A quadratic approximation gives a best-fit parabola to a function. For example, let's consider $f(x) = \cos(x)$ (see Figure 1).

The linear approximation of $\cos x$ near $x_0 = 0$ approximates the graph of the cosine function by the straight horizontal line $y = 1$. This doesn't seem like a very good approximation.

The quadratic approximation to the graph of $\cos(x)$ is a parabola that opens downward; this is much closer to the shape of the graph at $x_0 = 0$ than the line

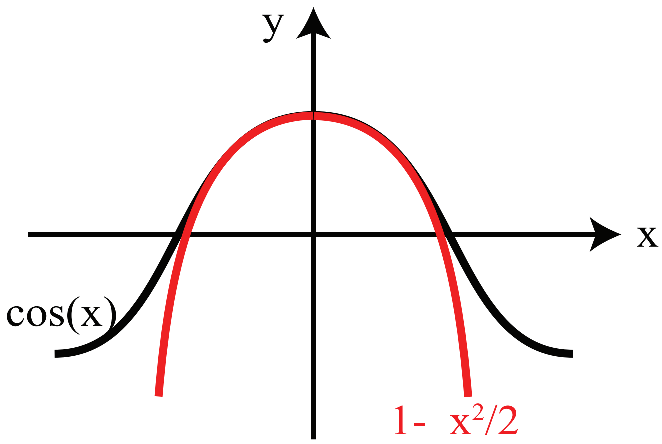

Figure 1: Quadratic approximation to $\cos(x)$.

$y = 1$. To find the equation of this quadratic approximation we set $x_0 = 0$ and perform the following calculations:

$$\begin{aligned} f(x) = \cos(x) \quad &\Longrightarrow \quad f(0) = \cos(0) = 1 \\ f'(x) = -\sin(x) \quad &\Longrightarrow \quad f'(0) = -\sin(0) = 0 \\ f''(x) = -\cos(x) \quad &\Longrightarrow \quad f''(0) = -\cos(0) = -1. \end{aligned}$$

We conclude that:

$$\cos(x) \approx 1 + 0 \cdot x - \frac{1}{2}x^2 = 1 - \frac{1}{2}x^2.$$

This is the closest (or "best fit") parabola to the graph of $\cos(x)$ when x is near 0.

Quadratic Approximation

Last class we derived a list of quadratic approximations for values of x near 0.

Using the formula:

$$f(x) \approx f(0) + f'(0)x + \frac{f''(0)}{2}x^2 \quad (x \approx 0)$$

We can (and eventually will) calculate the following approximations:

- $\sin x \approx x \quad$ (if $x \approx 0$)
- $\cos x \approx 1 - \frac{x^2}{2} \quad$ (if $x \approx 0$)
- $e^x \approx 1 + x + \frac{1}{2}x^2 \quad$ (if $x \approx 0$)
- $\ln(1 + x) \approx x - \frac{1}{2}x^2 \quad$ (if $x \approx 0$)
- $(1 + x)^r \approx 1 + rx + \frac{r(r-1)}{2}x^2 \quad$ (if $x \approx 0$)

Eventually you will recognize and remember all of these formulas, but it may take time and practice.

We have not derived the final two approximations on this list; we'll use them in several examples then describe their derivation.

Approximation of $\ln e$

Here's an example of the power of linear approximation, and of what quadratic approximation can do for us that linear approximation cannot.

Recall that when we discussed exponential and logarithmic functions we said that:

$$a_k = \left(1 + \frac{1}{k}\right)^k$$

tends to e as k goes to infinity. We did that by taking the logarithm of both sides:

$$\ln a_k = k \ln\left(1 + \frac{1}{k}\right),$$

and then analyzing the limit on the right hand side. That was a fairly difficult calculation which is made much easier by linear approximation. Since the linear approximation of $\ln(1+x)$ is just x,

$$\ln a_k = k \ln\left(1 + \frac{1}{k}\right) \approx k(1/k) = 1.$$

Can we conclude that $\ln a_k = 1$ so that $a_k = e$? (We used an approximation "$\approx$" in the above, not "$=$".) The linear approximation only works when x is near 0, but as k goes to infinity $\frac{1}{k}$ is indeed near 0. So as k approaches infinity, the linear approximation gets closer and closer to the exact value of the function, and $\ln a_k$ approaches 1. Linear approximation is often used in this way to evaluate limits.

Now if we want to find the *rate* of convergence — if we want to find out how fast the value of $k \ln\left(1 + \frac{1}{k}\right)$ approaches 1 — we need to look at the size of $\ln a_k - 1$ for large values of k. To do this you'll use quadratic approximation; the formula for the quadratic approximation of the natural log function was given above:

$$\ln(1+x) \approx x - \frac{1}{2}x^2 \quad \text{(for } x \text{ near } 0\text{)}.$$

You need the next higher order term to get a more detailed understanding of $\lim_{k\to\infty} \ln a_k$; this question will be on the problem set.

Question: How do we know when to use a linear approximation and when to use a quadratic one?

Answer: This is a very good question. For now the questions you get will specify whether to use a linear or a quadratic approximation. As time goes on you should try to get a feel for when you can get away with a linear approximation. In general, only use a quadratic approximation if a linear approximation won't work, because quadratic approximations are much more complicated (as you'll see in the next example).

In real life when you're faced with a problem like this — for example, determining the effects of gravity on an orbiting satellite — nobody is going to tell you anything. You won't even be told whether a linear approximation is relevant; you're on your own.

Example: $\dfrac{e^{-3x}}{\sqrt{1+x}}$

Last lecture we computed the linear approximation for x near 0 of

$$\frac{e^{-3x}}{\sqrt{1+x}} = e^{-3x}(1+x)^{-1/2}.$$

This lecture we'll compute a quadratic approximation for this function when x is near 0.

To do this we need to use the quadratic approximations for e^{-3x} and $(1+x)^{-1/2}$. We'll use the following two approximation formulas:

$$\begin{aligned} e^x &\approx 1 + x + \frac{1}{2}x^2 \\ (1+x)^r &\approx 1 + rx + \frac{r(r-1)}{2}x^2 \end{aligned}$$

substituting $x = 3x$ into the first and $r = -\frac{1}{2}$ into the second.

$$e^{-3x}(1+x)^{-1/2} \approx \left(1 + (-3x) + \frac{1}{2}(-3x)^2\right)\left(1 + \left(-\frac{1}{2}\right)x + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}x^2\right)$$

This looks awful! But we can ignore any terms of higher degree than x^2 and avoid doing those multiplications when we apply the distributive law, so it's not as bad as it looks.

$$e^{-3x}(1+x)^{-1/2} \approx 1 - 3x - \frac{1}{2}x + \frac{3}{2}x^2 + \frac{9}{2}x^2 + \frac{3}{8}x^2$$

Now we combine like terms:

$$e^{-3x}(1+x)^{-1/2} \approx 1 - \frac{7}{2}x + \frac{51}{8}x^2$$

Remember that this approximation is only valid for $x \approx 0$, and notice that the first two terms are exactly the linear approximation we got last time.

As you can see, calculations with quadratic approximations are much more involved than those with just linear approximations.

Question: Why do we get to drop all the higher order terms?

Answer: Because in the situation in which we're going to apply this, x is a very small number like $\frac{1}{100}$. That means that $x^2 \approx \frac{1}{10000}$ and $x^3 \approx \frac{1}{1000000}$. We don't need an exact answer so we can safely ignore anything as small as a millionth, which is what our x^3 terms represent.

Approximating $\ln(1+x)$ and $(1+x)^r$

- $\sin x \approx x \quad (\text{if } x \approx 0)$
- $\cos x \approx 1 - \frac{x^2}{2} \quad (\text{if } x \approx 0)$
- $e^x \approx 1 + x + \frac{1}{2}x^2 \quad (\text{if } x \approx 0)$
- $\ln(1+x) \approx x - \frac{1}{2}x^2 \quad (\text{if } x \approx 0)$
- $(1+x)^r \approx 1 + rx + \frac{r(r-1)}{2}x^2 \quad (\text{if } x \approx 0)$

Now that we've seen a couple of examples of quadratic approximation, we'll derive the last two formulas in our library, shown above. The general formula for a quadratic approximation is:

$$f(x) \approx f(0) + f'(0)x + \frac{f''(0)}{2}x^2 \quad (x \approx 0)$$

As usual, we chose the base point $x_0 = 0$. Shown below are the first and second derivatives of the functions we're interested in and their values at $x_0 = 0$. Combining this with the general formula yields the quadratic approximations listed above.

$f(x)$	$f'(x)$	$f''(x)$	$f(0)$	$f'(0)$	$f''(0)$
$\sin x$	$\cos x$	$-\sin x$	0	1	0
$\cos x$	$-\sin x$	$-\cos x$	1	0	-1
e^x	e^x	3^x	1	1	1
$\ln(1+x)$	$\frac{1}{1+x}$	$\frac{-1}{(1+x)^2}$	0	1	-1
$(1+x)^r$	$r(1+x)^{r-1}$	$r(r-1)(1+x)^{r-2}$	1	r	$r(r-1)$

We can approximate most common functions using algebraic combinations of the functions in this library.

Introduction to Curve Sketching

Goal: To draw the graph of f using information about whether f' and f'' are positive or negative. We want the graph to be qualitatively correct, but not necessarily to scale.

WARNING: Don't abandon your precalculus skills and common sense — you already know a lot about graphing functions; calculus just fills in the gaps.

The first principle is that if f' is positive, then f is increasing. In other words, if the tangent line is pointing up then the function is going up too. Similarly, if the derivative of f is negative then f is decreasing.

The second principle is just a second order effect of the same type. If f'' is positive, that means that f' is increasing. This is just the first principle applied to the second derivative; f'' is the derivative of f'.

Figure 1 shows the graph of a function for which the second derivative is positive. You can see from the tangent lines sketched on the graph that their slopes increase from negative on the left to positive on the right. We say that curves with this shape are *concave up*.

Similarly, if $f'' < 0$ then f' is decreasing and the graph of f is *concave down*.

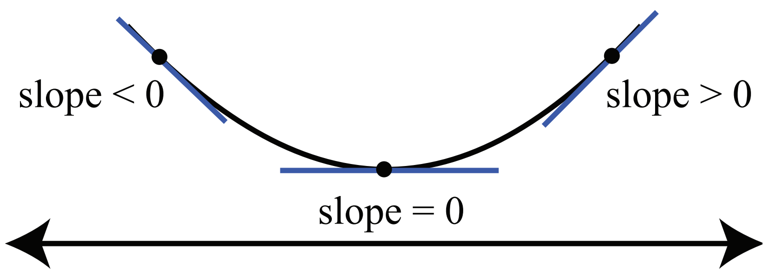

Figure 1: f is concave-up. The slope increases from negative to positive as x increases.

Curve Sketching Example 1

Example 1: Sketch the graph of $f(x) = 3x - x^3$.

First we note that:

$$f'(x) = 3 - 3x^2 = 3(1 - x)(1 + x)$$

We can see that if $-1 < x < 1$, both $(1 - x)$ and $(1 + x)$ are positive, so $f'(x)$ must also be positive, so f is increasing (by our first principle.)

When $x > 1$, $f'(x) < 0$ and f is decreasing. When $x < -1$, then $(1 - x)$ is positive and $(1 + x)$ is negative, so $f'(x) = 3(1 - x)(1 + x)$ is negative and f is decreasing.

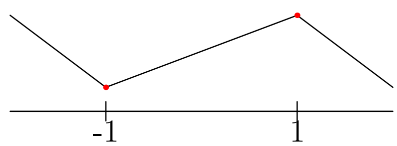

Figure 1: Turning points of $f(x) = 3x - x^3$.

We get a rough schematic of the graph of the function by drawing a number line at the bottom of our page as shown in Figure 1. Above the interval from negative infinity to -1, draw a diagonal line slanting down and to the right — this is one of the intervals on which f is decreasing. Above the interval from -1 to 1 draw a line slanting up and to the right — f is increasing. Finally, draw a line slanting down and to the right above the interval from 1 to positive infinity. The resulting zigzag gives us an idea of what an accurately drawn graph might look like.

We immediately notice two important features of the function lie at the points where the graph changes direction. These places where the derivative changes sign are turning points in the graph.

Definition: If $f'(x_0) = 0$, we call x_0 a *critical point* and $y_0 = f(x_0)$ is a *critical value* of f.

Our next step in understanding the graph of $f(x)$ is to plot the critical points and values of f. The critical points are $x = 1$ and $x = -1$; these are the values of x for which $(x - 1)$ and $(x + 1)$ are zero. The critical values are $f(1) = 3 \cdot 1 - 1^3 = 2$ and $f(-1) = 3 \cdot (-1) - (-1)^3 = -2$.

We plot the two points $(-1, -2)$ and $(1, 2)$, which we know are on the graph of f. Because we know where f' is positive and where it is negative, we also know that the graph decreases to $(-1, -2)$ and then starts to increase and that it increases toward $(1, 2)$ and then starts to decrease. In other words, the graph is shaped like a smile near $(-1, -2)$ and like a frown near $(1, 2)$. (We could also learn this from the second derivative.)

If we notice that $f(0) = 0$, we can now guess what the entire graph might look like but we still don't know for sure how far it decreases or increases to the left and right.

We can also notice that because all the powers of x are odd, the function f is odd; $f(-x) = 3(-x) - (-x)^3 = -3x + x^3 = -f(x)$. This means that if we can graph the function accurately for $x > 0$ we can reflect the graph across the y-axis to get a graph of the entire function.

In general, you should use precalculus skills to get information like $f(0) = 0$ or "f is odd" as much as you can.

The final detail we need to worry about is what happens at the "ends" of the graph. This topic is often neglected, but if you're using a graphing calculator or computer this can be the hardest part of understanding the graph of a function.

We want to know what happens as $x \to \infty$ and as $x \to -\infty$. As $x \to \infty$, the value of $-x^3$ grows very rapidly while the value of $3x$ is much smaller. So:

$$f(x) \approx -x^3 \to -\infty \text{ as } x \to \infty.$$

Similarly,

$$f(x) \approx x^3 \to \infty \text{ as } x \to -\infty.$$

We now know how far $f(x)$ increases or decreases as $x \to \pm\infty$ — it goes off toward infinity rather than, say, hugging the x-axis.

We now know a lot about the shape of the graph, but we can learn a little bit more about it by looking at the second derivative $f''(x) = -6x$. From this we learn that $f''(x) < 0$ when $x > 0$ and $f''(x) > 0$ when $x < 0$, so the graph is concave down to the right of the y-axis and concave up on the left. The value $x = 0$ is of interest not only because $f(0) = 0$ but also because it is an *inflection point* — a value x_0 for which $f''(x_0) = 0$.

We can now combine everything we've learned to get something like the graph shown in Figure 2.

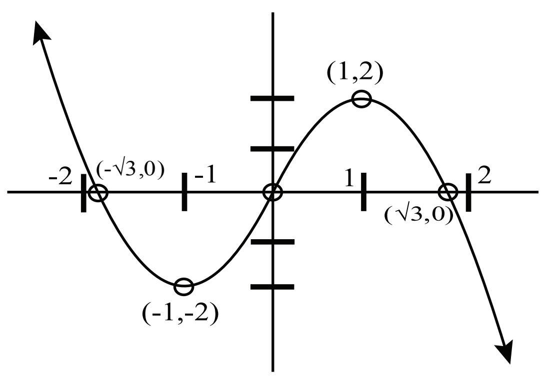

Figure 2: Sketch of the function $y = 3x - x^3$.

Question: What if the graph had a sharp point like the one in the schematic?

Answer: Points like that aren't called critical points, but they are very important. We'll talk about them later.

Graph Sketching Example

Example: Sketch the graph of $f(x) = \frac{x+1}{x+2}$.

What does the derivative of this function tell us about its graph? To save time, I'll tell you that

$$f'(x) = \frac{1}{(x+2)^2}.$$

The special thing about this example is that $f'(x) \neq 0$, i.e. there are no values of x_0 for which $f(x_0) = 0$ and there are no critical points. If you've been trained that the first step in sketching graphs is to find out where the derivative is zero, you might be tempted to give up at this point. But we're not going to give up — can you think of anything you'd like to try here?

Student: Find values of x where f is undefined.

That's a sophisticated way of putting the point that I want to make, which is that we should go back to our precalculus skills and just plot points.

It turns out that the most important point to plot is the one that's not there: the one with x coordinate -2. This is what was just suggested — this is a point where the function is not defined.

If we try to compute $f(-2)$ we find that we can't because that makes the denominator 0. Instead we'll look at the limit as x approaches -2 from the left and from the right. We'll write a^+ for "a number x that is a little big bigger than a" to make these limits easier to talk about.

$$\lim_{x \to -2^+} f(x) = f(-2^+) = \frac{-2+1}{(-2)^+ + 2} = -10^+ = -\infty.$$

We see that the limit comes down to dividing -1 by a very small positive number; the result of this division is a very large negative number. From the other direction we get:

$$\lim_{x \to -2^-} f(x) = f(-2^-) = \frac{-2+1}{(-2)^- + 2} = -10^- = +\infty$$

This tells us what the graph of the function is doing near $x = 2$. We could evaluate the function at any point; this was the most interesting point.

Our next step is to consider the "ends" of the graph, where $x \to +\infty$ and $x \to -\infty$. This tells us what happens to the left and right of the "screen" of our graph, just as the previous calculation told us what was happening above and below.

A simple algebraic trick makes it easy to see what will happen to the value of $f(x)$ as $x \to \pm\infty$:

$$\begin{aligned} f(x) &= \frac{x+1}{x+2} \\ &= \frac{x+1}{x+2}\frac{1/x}{1/x} \\ f(x) &= \frac{1+\frac{1}{x}}{1+\frac{2}{x}} \end{aligned}$$

$$\lim_{x\to\pm\infty} f(x) \quad = \quad \frac{1}{1} = 1$$

As x gets huge, the values $\frac{1}{x}$ and $\frac{2}{x}$ approach 0. We could abbreviate this conclusion as $f(\pm\infty) = 1$.

To draw the graph we use asymptotes; there's a horizontal asymptote at $y = 1$ and a vertical one at $x = -2$. We use dotted lines to sketch these asymptotes so that we won't confuse them with the sketch of our graph as in Figure 1.

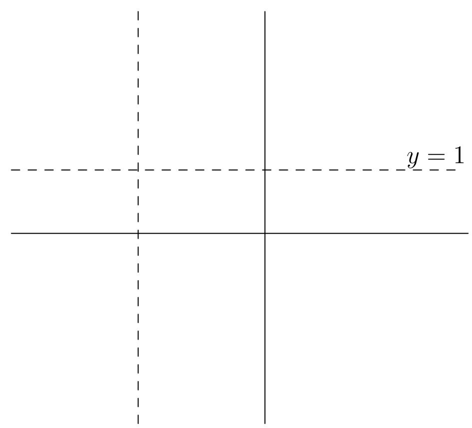

Figure 1: Axes and asymptotes.

The fact that $\lim_{x\to -2^+} f(x) = -\infty$ tells us that the graph plunges down as it approaches $x = -2$ from the right. As x approaches -2 from the left, the graph goes up toward positive infinity on the other side of the vertical asymptote. As x approaches negative infinity, the graph gets closer and closer to the horizontal asymptote $y = 1$, as it does when $x \to +\infty$.

We can almost finish this graph now, but there's one outstanding question. How do we know whether the graph crosses the line $y = 1$ or not?

Student: It can't dip below the line because there are no critical points.

That's exactly right. Because f' is never 0 the graph of f can't double back on itself, because the graph can't have any horizontal tangent lines.

We've practically got the entire graph now. We could improve it by adding details like the exact x- and y-intercepts, but we already understand the qualitative behavior of this function.

Let's look at the sign of the first derivative of the function to double check our graphing. To compute the derivative we can use the following algebraic trick:

$$f(x) = \frac{x+1}{x+2} = \frac{(x+2)-1}{x+2} = 1 - \frac{1}{x+2} = 1 + (x+2)^{-1}.$$

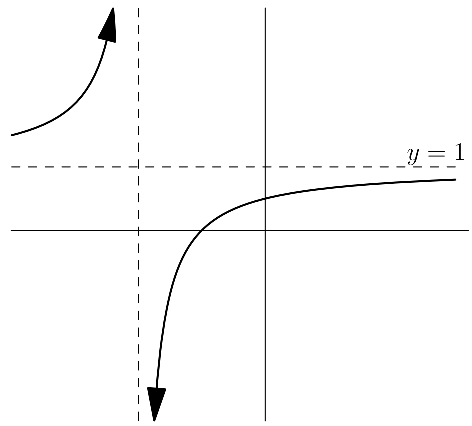

Figure 2: Sketch of $y = \dfrac{x+1}{x+2}$.

So $f'(x) = 0 - (x+2)^{-2} = \frac{1}{(x+2)^2}$ for $x \neq -2$. (This calculation also shows us that the graph of $f(x)$ is a hyperbola similar to the graph of $\frac{1}{x}$.)

We can see from our graph that $f(x)$ is increasing from negative infinity to -2 and from -2 to positive infinity, so $f'(x)$ must be positive. The derivative we calculated is always positive, so we've successfully checked that result.

Despite the fact that we know that f is increasing on $(-\infty, 2)$ and $(2, \infty)$, we shouldn't think that f is increasing for all x. There's a break at $x = -2$, which is the most important feature of the function.

The second derivative of $f(x)$ is

$$f''(x) = \frac{-2}{(x+2)^3} \quad (x \neq -2).$$

This is negative when $(x+2)^3$ is positive, so $f''(x) < 0$ when $x > -2$ and the graph is concave down on the interval $(-2, \infty)$. The second derivative is positive when $(x+2)^3$ is negative, so the graph is concave up on the interval $(-\infty, -2)$. Again, this agrees with the graph we have drawn. It also confirms that the graph doesn't "wiggle".

Question: Are we defining "increasing" to mean that the first derivative is positive?

Answer: No. Whenever f' is positive the function is increasing, but it's possible for the function to be increasing even when f' is 0.

General Strategy for Curve Sketching

1. (Precalc skill) Plot

   a discontinuities of f (especially infinite ones)

   b endpoints (or $x \to \pm\infty$)

   c easy points (optional)

2. Find the critical points — usually where the slope changes from positive to negative, or vice versa.

   a Solve $f'(x) = 0$

   b Plot critical points and values, but only if it's relatively easy to do so.

3. Decide whether $f'(x) < 0$ or $f'(x) > 0$ on each interval between critical points and discontinuities. (This just double checks steps 1 and 2.)

4. Decide whether $f''(x) < 0$ or $f''(x) > 0$ on each interval between critical points and discontinuities. This tells us whether the graph is concave up or concave down. Inflection points occur when $f''(x_0) = 0$. (If you can, skip this step.)

5. Combine this information to draw the graph.

Detailed Example of Curve Sketching

Example Sketch the graph of $f(x) = \dfrac{x}{\ln x}$. (Note: this function is only defined for $x > 0$)

1. Plot

   a The function is discontinuous at $x = 1$, because $\ln 1 = 0$.

   $$f(1^+) = \frac{1}{\ln 1^+} = \frac{1}{0^+} = \infty$$

   $$f(1^-) = \frac{1}{\ln 1^-} = \frac{1}{0^-} = -\infty$$

   b endpoints (or $x \to \pm\infty$)

   $$f(0^+) = \frac{0^+}{\ln 0^+} = \frac{0^+}{-\infty} = 0$$

   The situation is a little more complicated at the other end; we'll get a feel for what happens by plugging in $x = 10^{10}$.

   $$f(10^{10}) = \frac{10^{10}}{\ln 10^{10}} = \frac{10^{10}}{10 \ln 10} = \frac{10^9}{\ln 10} >> 1$$

   We conclude that $f(\infty) = \infty$.

We can now start sketching our graph. The point $(0, 0)$ is one endpoint of the graph. There's a vertical asymptote at $x = 1$, and the graph is descending before and after the asymptote. Finally, we know that $f(x)$ increases to positive infinity as x does. We already have a pretty good idea of what to expect from this graph!

2. Find the critical points

   $$\begin{aligned} f'(x) &= \frac{1 \cdot \ln x - x\left(\frac{1}{x}\right)}{(\ln x)^2} \\ &= \frac{(\ln x) - 1}{(\ln x)^2} \end{aligned}$$

   a $f'(x) = 0$ when $\ln x = 1$, so when $x = e$. This is our only critical point.

   b $f(e) = \frac{e}{\ln e} = e$ is our critical value. The point (e, e) is a critical point on our graph; we can label it with the letter c. (It's ok if our graph is not to scale; we'll do the best we can.)

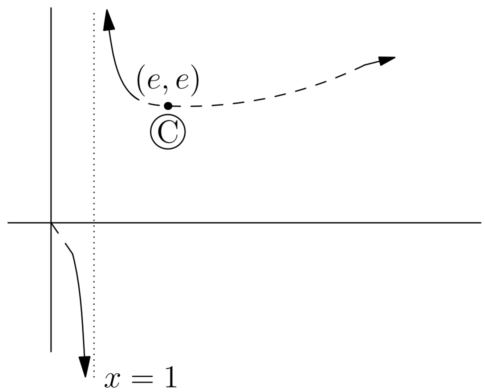

Figure 1: Sketch using starting point, asymptote, critical point and endpoints.

We now know the qualitative behavior of the graph. We know exactly where f is increasing and decreasing because the graph can only change direction at critical points and discontinuities; we've identified all of those. The rest is more or less decoration.

3. Double check using the sign of f'.

   We already know:

$$\begin{array}{ccc} f & \text{is decreasing on} & 0 < x < 1 \\ f & \text{is decreasing on} & 1 < x < e \\ f & \text{is increasing on} & e < x < \infty \end{array}$$

   We now double check this.

$$f'(x) = \frac{(\ln x) - 1}{(\ln x)^2}$$

   When x is between 0 and 1, $f'(x)$ equals a negative number divided by a positive number so is negative.

   When x is between 1 and e, $f'(x)$ again equals a negative number divided by a positive number so is negative.

   When x is between e and ∞, $f'(x)$ equals a positive number divided by a positive number so is positive.

   This confirms what we learned in steps 1 and 2.

   Sometimes steps 1 and 2 will be harder; then you might need to do this step first to get a feel for what the graph looks like.

   There's one more piece of information we can get from the first derivative $f'(x) = \frac{(\ln x)-1}{(\ln x)^2}$. It's possible for the denominator to be infinite; this is another situation in which the derivative is zero. So $f'(0^+) = 0$ and $x = 0^+$ is another critical point with critical value $-\infty$.

An easier way to see this is to rewrite $f'(x)$ as:

$$\frac{1}{\ln x} - \frac{1}{(\ln x)^2}$$

and note that:

$$f'(0^+) = \frac{1}{\ln 0^+} - \frac{1}{(\ln 0^+)^2} = \frac{1}{-\infty} - \frac{1}{(\infty)^2} = 0 - 0 = 0.$$

4. Use $f''(x)$ to find out whether the graph is concave up or concave down.

$$f'(x) = (\ln x)^{-1} - (\ln x)^{-2}$$

So

$$\begin{aligned} f''(x) &= -(\ln x)^{-2}\frac{1}{x} - -2(\ln x)^{-3}\frac{1}{x} \\ &= \frac{-(\ln x)^{-2} + 2(\ln x)^{-3}}{x}\frac{(\ln x)^3}{(\ln x)^3} \\ &= \frac{-\ln x + 2}{x(\ln x)^3} \\ f''(x) &= \frac{2 - \ln x}{x(\ln x)^3} \end{aligned}$$

We need to figure out where this is positive or negative. There are two places where the sign might change – when $2 - \ln x$ changes sign or when $(\ln x)^3$ changes sign. (Remember x will always be positive.)

The value of $2 - \ln x$ is positive when $\ln x < 2$ (when $x < e^2$) and negative when $x > e^2$. The denominator is positive when $x > 1$ and negative when $x < 1$. Combining these, we get:

$$\begin{aligned} 0 < x < 1 &\implies f''(x) < 0 \text{ (concave down)} \\ 1 < x < e^2 &\implies f''(x) > 0 \text{ (concave up)} \\ e^2 < x < \infty &\implies f''(x) < 0 \text{ (concave down)} \end{aligned}$$

This means that there's a "wiggle" at the point $\left(e^2, \frac{e^2}{2}\right)$ on the graph. The value of $f(x)$ is still increasing and the graph continues to rise, but the graph is rising less and less steeply as the values of $f'(x)$ decrease.

5. Combine this information to draw the graph.

We've been doing this as we go. If you're working a homework problem, at this point you might copy your graph to a clean sheet of paper.

This is probably as detailed a graph as we'll ever draw. In fact, one advantage of our next topic is that it will reduce the need to be this detailed.

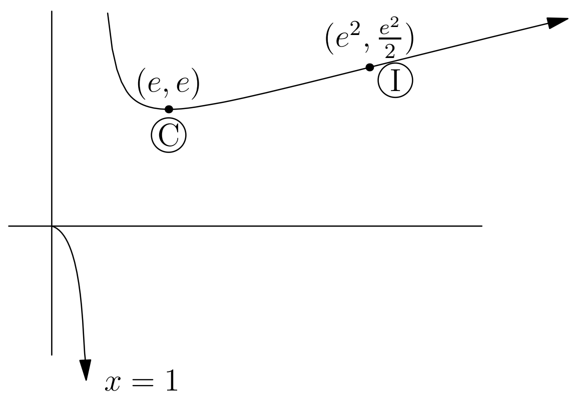

Figure 2: Final sketch.