Introduction to Series

Remember that we're dealing with infinity; we'll be studying *infinite* series. The most important and useful series is the geometric series. We'll start with a concrete example:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \cdots$$

We can visualize this infinite sum by marking its partial sums on the number line, as shown in Figure 1. The figure shows the results of adding 1, then $\frac{1}{2}$, then $\frac{1}{4}$, then $\frac{1}{8}$. Note that the value of each partial sum is midway between the value of the previous sum and 2. We say that the series converges to 2.

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \cdots = 2.$$

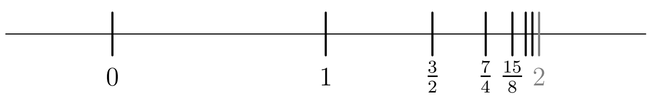


Figure 1: Adding a term gives a number half way between the previous value and 2.

Note that we never get to 2, we just get halfway there infinitely often. This is known as Zeno's paradox — if a rabbit gives a tortoise a head start in a race, the rabbit can never pass the tortoise because it must cross half the distance between itself and the tortoise infinitely many times. One resolution of this paradox requires understanding time as a continuum, which Zeno failed to do.

In general, a *geometric series* is a series of the form:

$$1 + a + a^2 + a^3 + \cdots = \frac{1}{1-a} \quad (|a| < 1).$$

We'll see how this formula is derived later.

Divergent Series

As with indefinite integrals, we're concerned about when infinite series converge. We're also interested in what goes wrong when a series diverges — when it fails to converge. Recall that when $|a| < 1$,

$$1 + a + a^2 + a^3 + \cdots = \frac{1}{1-a}.$$

How does this fail when $|a| \geq 1$?

One simple example of a divergent series is a geometric series with a equal to 1:

$$1 + 1 + 1 + 1 + \cdots = \frac{1}{1-1} = \frac{1}{0}.$$

This almost makes sense! Since the sum is infinite we conclude that the series diverge.

Now let's try $a = -1$. We get:

$$1 - 1 + 1 - 1 + \cdots$$

If we look at the partial sums of this sequence we see that they alternate between 1 and 0. If we plug $a = -1$ into our formula $\frac{1}{1-a}$ it predicts that the sum is $\frac{1}{2}$ which is halfway between 0 and 1 but is still wrong. Using the formula here is the equivalent of computing an indefinite integral without checking for singularities; it gives you an interesting but wrong result.

Because the partial sums of the series alternate between 0 and 1 without ever tending toward a single number, we say that this series is also divergent. The geometric series only converges when $|a| < 1$.

We'll look at one more case: $a = 2$. According to the formula,

$$1 + 2 + 2^2 + 2^3 + \cdots = \frac{1}{1-2} = -1.$$

This is clearly wrong. The sequence diverges; the left hand side is obviously infinite and the right hand side is -1. But number theorists actually have a way of making sense out of this, if they're willing to give up on the idea that 0 is less than 1.

Notation for Series

It's easier to understand and explain mathematics if we have good notation for what we're discussing. In this case, we define a *partial sum* to be:

$$S_N = \sum_{n=0}^{N} a_n.$$

Now we can do something similar to what we did for indefinite integrals and define:

$$S = \sum_{n=0}^{\infty} a_n = \lim_{N \to \infty} S_N.$$

Once again we have two choices. If the limit exists we say that the series converges. If the limit does not exist we say that the series diverges.

Examples of Series

Example: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

We'll now look at some other interesting series and will return to the (very important) geometric series later.

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ turns out to be very similar to the improper integral $\int_1^{\infty} \frac{dx}{x^2}$, which is convergent. The series is also convergent.

It's easy to calculate that $\int_1^{\infty} \frac{dx}{x^2} = 1$. It's hard to calculate that:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

This was computed by Euler in the early 1700's.

Example: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$

Notice that the series in these examples start with $n = 1$; we can't start with $n = 0$ because a_0 would be of the form $\frac{1}{0}$. That's ok — we can start these series at any positive value of n and it won't make a difference to whether they converge or diverge.

The series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is similar to the integral $\int_1^{\infty} \frac{dx}{x^3}$, which converges to $\frac{1}{2}$. The series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ was only recently proved to converge to an irrational number; there is no elementary way of describing its value.

We've been loosely comparing series to integrals; we can make this formal by using a Riemann sum with $\Delta x = 1$.

Comparison Tests

Integral Comparison

We used integral comparison when we applied Riemann sums to understanding $\sum_1^\infty \frac{1}{n}$ in terms of $\int_1^\infty \frac{dx}{x}$, and we've made several other comparisons between integrals and series in this lecture. Now we learn the general theory behind this technique.

Theorem: If $f(x)$ is decreasing and $f(x) > 0$ on the interval from 1 to infinity, then either the sum $\sum_1^\infty f(n)$ and the integral $\int_1^\infty f(x)\,dx$ both diverge or they both converge and:

$$\sum_{n=1}^{\infty} f(n) - \int_1^\infty f(x)\,dx \quad < \quad f(1).$$

For example, when $S_N = \sum_1^N \frac{1}{n}$ we showed that $|S_n - \ln N| < 1$.

Since it's very difficult to compute infinite sums and it's easy to compute indefinite integrals, this is an extremely useful theorem.

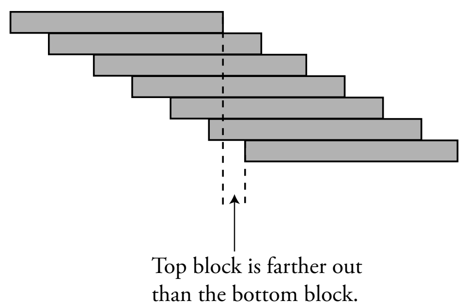
Limit Comparison

Theorem: If $f(n) \sim g(n)$ (i.e. if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$) and $g(n) > 0$ for all n, then either both $\sum_{n=1}^{\infty} f(n)$ and $\sum_{n=1}^{\infty} g(n)$ converge or both diverge.

This says that if f and g behave the same way in their tails, their convergence properties will be similar.

Preview of Stacking Blocks

Question: Is it possible to stack Professor Jerison's blocks so that no part of the top block is above the bottom block?



He answers this question in the next lecture.

Stacking Blocks

Is it possible to stack blocks as shown in Figure 1, so that no part of the bottom block is below the top block? In general, how much horizontal distance can there be between the top block and the bottom block?

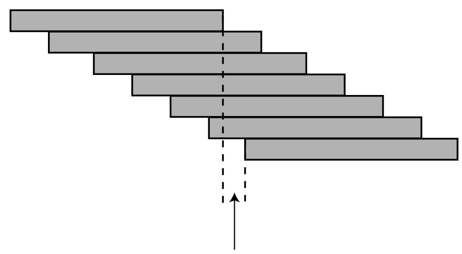


Figure 1: Stack of blocks.

This is a good kind of math question to ask. If there's a limit to how distant the top block can be from the bottom block it will be interesting to know what it is. It would also be interesting to discover that there's no limit to this distance. In the end, this becomes a question of whether the top block's position converges or diverges.

Professor Jerison has eight blocks; do you think he can achieve his goal with what he has?

In order to get the greatest possible horizontal distance, start at the top of the stack and work downward. The topmost block has to cover at least half of the block below it or else it will fall. Slide it so that its right end is at the midpoint of the block below it.

Next, slide the second block down as far to the left as you can without upsetting the tower. Then slide the block below it to the left as far as you can, and the one below that, and the one below that, and so on. In this way, Professor Jerison accomplished his goal using only 7 blocks.

How much further could we get if we had more blocks? Let's calculate it.

Calculation

To make the calculations simple, let's say that each block is 2 units long. Then if the left end of the topmost block is at position 0, then the left end of the block under it is at position 1.

In general the center of mass of the top n blocks must always be above the block supporting them.

- Let $C_1 =$ the x coordinate of the center of mass of the top block.

- Let C_2 = the x coordinate of the center of mass of the top two blocks.
- Put the left end of the next block below the center of mass of the previous ones.

Question: How do you know that this is the best way to stack them?

Answer: I can't answer that in general, but I can tell you that this is the best we can do if we start building from the top – we're using what computer scientists call "the greedy algorithm" and going as far as we can at each step.

If we tried using this algorithm starting from the bottom it wouldn't work. We'd stack the second block with its end on the midpoint of the first block and then be unable to gain any distance beyond that.

It seems possible that there is some other strategy that's better than using the greedy algorithm starting at the top. There isn't, but we're not going to prove that today.

According to our strategy we need to know C_N, the x coordinate of the center of mass of the top N blocks, in order to continue.

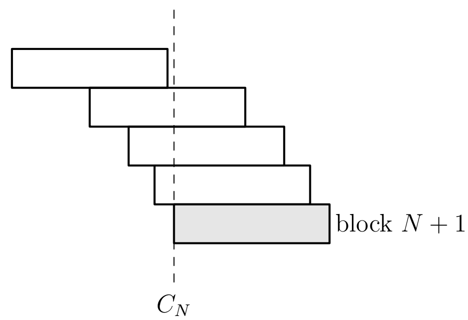


Figure 2: Adding a block.

If the center of mass of the top N blocks is on the line $x = C_N$, the center of mass of the $(N+1)^{st}$ block will have x coordinate $C_N + 1$. This shifts the center of mass of the stack to the right; the x coordinate of the new center of mass of the top $N+1$ blocks is given by the weighted average of the centers of mass of the stack:

$$\begin{aligned} C_{N+1} &= \frac{NC_N + 1(C_N+1)}{N+1} \\ &= \frac{(N+1)C_N + 1}{N+1} \\ C_{N+1} &= C_N + \frac{1}{N+1}. \end{aligned}$$

Adding the $(N+1)^{st}$ block added to the stack allows you to extend the stack $\frac{1}{N+1}$ units farther from its base.

So

$$\begin{aligned}
C_1 &= 1 \\
C_2 &= 1 + \frac{1}{2} \\
C_3 &= 1 + \frac{1}{2} + \frac{1}{3} \\
C_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \\
C_5 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} > 2.
\end{aligned}$$

It takes at least 5 blocks to extend the top block beyond the base.

$$C_N = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{N}$$

This sum C_N is the same as S_N from a previous lecture:

$$C_N = S_N = \sum_{n=1}^{N} \frac{1}{n}.$$

We know that:

$$\ln N < S_N < (\ln N) + 1.$$

Since $\ln N$ goes to infinity as N goes to infinity, $S_N = C_N$ must go to infinity as N does. If we have enough blocks we *can* extend our stack as far as we want.

In this example, the fact that $\sum_{n=1}^{N} \frac{1}{n}$ diverges means that it's possible to extend the stack as far to the left as we wish, provided we have enough blocks.

On the other hand, the inequality $S_N < (\ln N) + 1$ tells us that it will take a lot of blocks to extend the top of the stack very far.

How high would this stack of blocks be if it extended across the two lab tables at the front of the lecture hall? One lab table is 6.5 blocks, or 13 units, long. Two tables are 26 units long. There will be $26 - 2 = 24$ units of overhang in the stack. (We subtract 2 because the bottom block has no overhang and because the stack extends one unit past the center of mass of the top block.) Each block is approximately 3 centimeters tall.

If $\ln n = 24$ then $n = e^{24}$ and:

$$\text{Height} = 3cm \cdot e^{24} \approx 8 \times 10^8 m.$$

That height is roughly twice the distance to the moon.

If you want the stack to span this room ($\sim$ 30 ft.) it would have to be 10^{26} meters high. That's about the diameter of the observable universe.

We can learn one more thing from this experiment — if we look at the stack sideways we see that it follows the shape of the graph of $\ln x$. This experiment provides a concrete example of how slowly the function $\ln x$ increases.

We did not discover an important number that limited the reach of the stack, but we did discover that that reach is infinite — infinity is also an important number. We also discovered a property of that infinite value; that the rate of extension of the stack is very slow. Infinity doesn't have a single value; there are lots of different orders of infinity.

Power Series

Our last subject will be *power series*. We've seen one power series:

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \quad (|x| < 1).$$

This is our geometric series, with x in place of a. We'll now see why the sum should equal $\frac{1}{1-x}$.

Suppose that:

$$1 + x + x^2 + x^3 + \cdots = S$$

for some number S. Multiply both sides of this equation by x:

$$x + x^2 + x^3 + x^4 + \cdots = Sx.$$

Now subtract the two equations.

$$\begin{array}{ccccccccccc} 1 & + & x & + & x^2 & + & x^3 & + & \cdots & = & S \\ & & x & + & x^2 & + & x^3 & + & \cdots & = & Sx \\ \hline 1 & + & 0 & + & 0 & + & 0 & + & \cdots & = & S - Sx \end{array}$$

Lots of terms cancel! Continuing, we get:

$$\begin{aligned} 1 &= S - Sx \\ 1 &= S(1-x) \\ \frac{1}{1-x} &= S. \end{aligned}$$

There is a flaw in this reasoning — the argument only works if S exists. For example, if $x = 1$ this technique tells us that $\infty - \infty = \infty - \infty$. This is not a useful result.

This line of reasoning leads to a correct answer exactly when the series converges; in other words, when $|x| < 1$.

General Power Series

We now know exactly which geometric series converge and, if they do converge, what they converge to. Our next step is to extend this result to cover all power series. In general, a *power series* is a series of the form:

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_{n=0}^{\infty} a_n x^n.$$

As with geometric series, there's a very simple rule that tells us when power series converge. The series converges when $-R < x < R$ for some "magic number" R called the *radius of convergence*. The value of R depends on the values of the coefficients a_i. When $|x| > R$, the sum $\sum a_n x^n$ diverges. When $|x| = R$ the series might or might not converge — we won't deal with this case in this course.

What always happens is that for $|x| < R$, the values $|a_n x^n|$ tend to 0 exponentially fast. When $|x| > R$, the values $|a_n x^n|$ won't tend to 0 at all. For example, when $x = 1/2$ the geometric series looks like:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

We're adding together numbers that get smaller and smaller. When $x = 2$ the numbers in our summation get bigger and bigger:

$$1 + 2 + 4 + 8 + \cdots$$

Question: How can you tell when the numbers $a_n x^n$ are declining *exponentially* fast?

Answer: Any time the power series converges the numbers $a_n x^n$ decline exponentially fast. This is because of the power x^n in each term.

Question: How do you find R?

Answer: There's a long discussion of this question in many textbooks, but you won't need it. The radius of convergence will always be 1 or infinite or as obvious as it is for the geometric series.

Introduction to Taylor Series

Why are we looking at power series? If we reverse the equation for the geometric series:

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

we get a description of $\frac{1}{1-x}$ in terms of a series. In fact, we can represent all of the functions we've encountered in this course in terms of series.

The technique is similar to the use of a decimal expansion to represent $1/3$ or $\sqrt{2}$. When we describe a function like e^x or $\arctan x$ in terms of a series we can approximate and manipulate those functions as easily as we do polynomial functions.

Rules for convergent power series

What sorts of manipulations might we want to perform? Addition, multiplication, division, substitution (composition), integration and differentiation. The rules for manipulation of power series are essentially the same as those for manipulating polynomials!

$$f(x) + g(x), \quad f(x) \cdot g(x), \quad f(g(x)), \quad f(x)/g(x), \quad \frac{d}{dx} f(x), \quad \int f(x) dx$$

We can do all of these with power series; in this class integration and differentiation will be the most interesting manipulations.

We take the derivative of a power series just as we do for polynomials:

$$\frac{d}{dx}(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

Similarly, the formula for the integral of a power series is:

$$\int (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots) dx = c + a_0 x + \frac{a_1 x^2}{2} + \frac{a_2 x^3}{3} + \frac{a_3 x^4}{4} + \cdots$$

Here the arbitrary constant c takes the place of the constant term in the new series.

Question: Is that a series or a polynomial?

Answer: It's a polynomial if it ends; if it goes on infinitely far then it's a series.

Question: You can add up terms of x in a series?

Answer: When we introduced series we described them as infinite sums of *numbers*. At the start of this class we rewrote the geometric series using the *variable* x in place of the "constant value" a. When we plug in a value for x we get a sum of infinitely many numbers, so as long as we remember that x is a placeholder for a numerical value there's no problem.

In other words, what we're working with here are functions of x. These functions are defined for values of x inside the radius of convergence and undefined for values of x that are too large, just as the function $f(x) = \sqrt{x}$ is defined for positive values of x and undefined for $x < 0$.

Power series are infinite sums of powers of x, with coefficients. People also study and use series that are infinite sums of sines and cosines and lots of other series, but we're only going to study power series here.

Taylor's Formula

Taylor's formula describes how to get power series representations of functions. The function e^x doesn't look like a polynomial; we have to figure out what the values of a_i have to be in order to describe e^x as a series.

Taylor's formula says that given any function f for which the n^{th} derivative $f^{(n)}(x)$ exists for x near 0,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

We'll learn how to use it soon.

Why should this work? Suppose that:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

Then:

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots \quad \text{and}$$
$$f''(x) = 2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x^2 + \cdots \quad \text{and}$$
$$f^{(3)}(x) = 3 \cdot 2a_3 + 4 \cdot 3 \cdot 2a_4 x + \cdots$$

Evaluating each of these at 0 we see that: $f(0) = a_0$, $f'(0) = a_1$, $f''(0) = 2a_2$ and $f^{(3)}(0) = 3 \cdot 2a_3$. Solving for a_3 we get $a_3 = \dfrac{f^{(3)}(0)}{3 \cdot 2 \cdot 1}$ and in general:

$$a_n = \frac{f^{(n)}(0)}{n!},$$

where:

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 1.$$

We define $0! = 1$ because that makes our formulas work nicely.

Taylor's Formula

Recall that Taylor's formula says:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

Example: e^x

If $f(x) = e^x$ then $f'(x) = e^x$, $f''(x) = e^x$, and so on. This means that $f^{(n)}(0) = e^0 = 1$ for any n. Taylor's formula tells us that:

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n.$$

In particular, we know that:

$$\begin{aligned} e^1 &= \sum_{n=0}^{\infty} \frac{1}{n!}, \quad \text{or:} \\ e &= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots \end{aligned}$$

We finally have a straightforward way to compute the value of e.

Example: $\sin x$, $\cos x$

You may recall from the sessions on linear and quadratic approximation that

$$\sin x \approx x \quad \text{and}$$

$$\cos x \approx 1 - \frac{x^2}{2}.$$

We can use Taylor's formula to complete these formulas.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

You may feel that these are hard to memorize, especially since we skipped some steps in their derivation. Try to use their similarities to help you remember them. The exponential function has a power series in which all the $a_i = 1$. The power series expansion of sine has all the odd powers with alternating signs. The series for cosine has all the even powers with alternating signs. All three functions are part of the same family.

Review of Taylor's Series

Professor Jerison was away for this lecture, so Professor Haynes Miller took his place.

A *power series* or *Taylor's series* is a way of writing a function as a sum of integral powers of x:

$$f(x) = a_0 + a_x x + a_2 x^2 + \cdots$$

Polynomials are power series; they go on for a finite number of terms and then end, so that all of the a_j equal 0 after a certain point. Since polynomials are a special type of power series, it's not surprising that power series behave almost exactly like polynomials.

Given a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ there is a number R $(0 \leq R \leq \infty)$ for which, when $|x| < R$, the sum $\sum_{n=0}^{\infty} a_n x^n$ converges and when $|x| > R$ the sum diverges. R is called the *radius of convergence*.

For $|x| < R$, $f(x)$ has all its higher derivatives, and Taylor's formula tells us that $a_n = \dfrac{f^{(n)}(0)}{n!}$. So:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots$$

Whenever you write out a power series you should say what the radius of convergence is. The radius of convergence of this series is infinity; in other words, the series converges for any value of x.

Example: (Due to Leonhard Euler) e^x

We know that if $f(x) = e^x$ then $f^{(n)}(x) = e^x$ for all n, and so $f^{(n)}(0) = 1$. Applying Taylor's formula we see that:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots, \quad R = \infty.$$

Question: How many terms of the series do we need to write out?

Answer: Write out enough terms so that you can see what the pattern is.

Question: What functions can be written as power series?

Answer: Any function that has a reasonable expression can be written as a power series. This is not a very precise answer because the true answer is a little bit complicated. For now, it's enough that any of the functions that occur in calculus (like sines cosines, and tangents) all have power series expansions.

Taylor's Series of $\dfrac{1}{1+x}$

Our next example is the Taylor's series for $\frac{1}{1+x}$; this series was first described by Isaac Newton. Remember the formula for the geometric series:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \quad \text{if } |x| < 1.$$

If we replace x by $-x$ we get:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \quad R = 1.$$

You may recall that the graph of this function has an infinite discontinuity at $x = -1$; this gives us an idea of what R might be. If we try to replace x by -1 we get something of the form $\infty = \infty$; the radius of convergence of this series is 1.

Instead of deriving this from the formula for the geometric series we could also have computed it using Taylor's formula. Try it!

Question: If you put in -1 for x the series diverges. If you put in 1, it looks like it would converge.

Answer: The graph of $y = \frac{1}{1+x}$ looks smooth at $x = 1$, but there is still a problem. If the series converges for $|x| < |a|$ and then diverges for $x = a$ the radius of convergence is a; that's it.

What happens if we plug $x = 1$ into the series $1 - x + x^2 - x^3 + \cdots$? Let's look at the partial sums $S_N = \sum_{n=0}^{N} a_n x^n$.

$$\begin{aligned} S_0 &= 1 \\ S_1 &= 0 \\ S_2 &= 1 \\ S_3 &= 0 \\ S_4 &= 1 \\ &\vdots \end{aligned}$$

Even though these don't go off to infinity, they still don't converge.

Taylor's Series of $\sin x$

In order to use Taylor's formula to find the power series expansion of $\sin x$ we have to compute the derivatives of $\sin(x)$:

$$\begin{aligned}
\sin'(x) &= \cos(x) \\
\sin''(x) &= -\sin(x) \\
\sin'''(x) &= -\cos(x) \\
\sin^{(4)}(x) &= \sin(x).
\end{aligned}$$

Since $\sin^{(4)}(x) = \sin(x)$, this pattern will repeat.

Next we need to evaluate the function and its derivatives at 0:

$$\begin{aligned}
\sin(0) &= 0 \\
\sin'(0) &= 1 \\
\sin''(0) &= 0 \\
\sin'''(0) &= -1 \\
\sin^{(4)}(0) &= 0.
\end{aligned}$$

Again, the pattern repeats.

Taylor's formula now tells us that:

$$\begin{aligned}
\sin(x) &= 0 + 1x + 0x^2 + \frac{-1}{3!}x^3 + 0x^4 + \cdots \\
&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots
\end{aligned}$$

Notice that the signs alternate and the denominators get very big; factorials grow very fast.

The radius of convergence R is infinity; let's see why. The terms in this sum look like:

$$\frac{x^{2n+1}}{(2n+1)!} = \frac{x}{1} \cdot \frac{x}{2} \cdot \frac{x}{3} \cdots \frac{x}{(2n+1)}.$$

Suppose x is some fixed number. Then as n goes to infinity, the terms on the right in the product above will be very, very small numbers and there will be more and more of them as n increases.

In other words, the terms in the series will get smaller as n gets bigger; that's an indication that x may be inside the radius of convergence. But this would be true for any fixed value of x, so the radius of convergence is infinity.

Why do we care what the power series expansion of $\sin(x)$ is? If we use enough terms of the series we can get a good estimate of the value of $\sin(x)$ for any value of x.

This is very useful information about the function $\sin(x)$ but it doesn't tell the whole story. For example, it's hard to tell from the formula that $\sin(x)$ is periodic. The period of $\sin(x)$ is 2π; how is this series related to the number π?

Power series are very good for some things but can also hide some properties of functions.

Power Series Multiplication

Once you have one power series, there are ways to get new power series from it. One thing you can do is multiply — can we use power series to multiply x by $\sin(x)$?

We have a power series for $\sin(x)$:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Can we get one for x? Yes!

$$x = 0 + x + 0x^2 + 0x^3 + 0x^4 + \cdots$$

We can treat power series just like polynomials and multiply them together:

$$\begin{aligned} x \cdot \sin(x) &= x \cdot (x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots) \\ &= x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \cdots \end{aligned}$$

The radius of convergence will be the smaller of the two radii of convergence; in this case $R = \infty$.

Power series multiplication is just like polynomial multiplication. It can get tedious if you're dealing with a lot of terms, but this example was pretty simple.

Notice that we just multiplied two odd functions, $\sin x$ and x, and so their product is even. That's reflected in the fact that all the terms in the power series expansion of $x \sin(x)$ have even degree. For an odd function, like $\sin(x)$, all the terms in the power series have odd degree. In general, power series of even functions will have only even degree terms and power series of odd functions will consist of all odd degree terms.

Question: Why is the radius of convergence the smaller of the two radii?

Answer: If $|x|$ is larger than the smallest radius of convergence then one of your functions isn't defined at x, so you can't expect multiplication by that function to make sense.

Derivative of a Power Series

We can differentiate power series. For example, $\cos(x) = \sin'(x)$ so we can find a power series for $\cos(x)$ by differentiating the power series for $\sin(x)$ term by term — the same way we differentiate polynomials.

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\begin{aligned}
\cos(x) &= \sin'(x) \\
&= 1 - 3\frac{x^2}{3!} + 5\frac{x^4}{5!} - 7\frac{x^6}{7!} + \cdots \\
\cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots
\end{aligned}$$

Notice how $3\frac{x^2}{3!}$ became $\frac{x^2}{2!}$ when we canceled the 3's. This happens with each term of the power series.

The radius of convergence of the derivative of a power series is the same as the radius of convergence of the power series you started with. Here $R = 1$.

Of course, you could get this same formula using Taylor's formula and the derivatives of the cosine function.

Integral of a Power Series

We can multiply, add and differentiate power series. Can we integrate them? Yes; as you'd expect, integration of power series is very similar to integration of polynomials. We'll use integration to find a power series expansion for:

$$\ln(1+x) = \int_0^x \frac{dt}{1+t} \quad (x > -1).$$

We know that:

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$$

So:

$$\begin{aligned}
\ln(1+x) &= \int_0^x (1 - t + t^2 - t^3 + \cdots)dt \\
&= \left[t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots \right]_0^x \\
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots
\end{aligned}$$

Because we began with a power series whose radius of convergence was 1, the radius of convergence of the result will also be 1. This reflects the fact that $\ln(1+x)$ is undefined for $x \leq -1$.

Question: If you only use positive values of x is there still a radius of convergence?

Answer: Yes. If $x > 1$ then the numerators x, x^2, x^3, x^4 and so on are increasing exponentially. The denominators 1, 2, 3, 4 ... only grow linearly. So as n goes to infinity, $\frac{x^n}{n}$ will also go to infinity. If the terms of a series go to infinity then the series diverges.

Euler used this kind of power series expansion to calculate natural logarithms much more efficiently than was previously possible.

Substitution of Power Series

We can find the power series of e^{-t^2} by starting with the power series for e^x and making the substitution $x = -t^2$.

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad (R = \infty) \\
e^{-t^2} &= 1 + (-t^2) + \frac{(-t^2)^2}{2!} + \frac{(-t^2)^3}{3!} + \cdots \\
&= 1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \cdots
\end{aligned}$$

The signs of the terms alternate, the powers are all even, and the denominators are the factorials shown. The radius of convergence is infinity.

Power Series Expansion of the Error Function

Several times in this course we've seen the error function:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt.$$

(The factor of $\frac{2}{\sqrt{\pi}}$ guarantees that $\lim_{x\to\infty} \operatorname{erf}(x) = 1$.) This function is very important in probability theory, but we don't have a conventional algebraic description of it.

Because we can integrate power series and know that:

$$e^{-t^2} = 1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \cdots \quad (R = \infty),$$

we can now find a power series expansion for the error function.

$$\begin{aligned}
\operatorname{erf}(x) &= \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt \\
&= \frac{2}{\sqrt{\pi}} \int_0^x (1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \cdots)dt \\
&= \frac{2}{\sqrt{\pi}} \left[t - \frac{t^3}{3} + \frac{t^5}{5\cdot 2!} - \frac{t^7}{7\cdot 3!} + \cdots \right]_0^x \\
&= \frac{2}{\sqrt{\pi}} \left(x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \cdots \right)
\end{aligned}$$

To get this to look exactly like a power series we would distribute the factor of $\frac{2}{\sqrt{\pi}}$ across the sum, multiplying it by each term of the series. However, that's not strictly necessary.

This turns out to be a very good way to compute the value of the error function; your calculator probably uses this method.