The Mean Value Theorem

The mean value theorem is a little theoretical, and will allow us to introduce the idea of integration in a few lectures. Integration is the subject of the second half of this course. We'll use the abbreviation "MVT" when discussing it.

Colloquially, the MVT theorem tells you that if you fly 3,000 kilometers in 6 hours, at some time during the flight you will be traveling at a speed of 500 kilometers per hour. (Because your average speed is 500 km/hr.)

The reason it's called the "mean value theorem" is because the word "mean" is the same as the word "average".

In math symbols, it says:

$$\boxed{\begin{array}{c}\dfrac{f(b)-f(a)}{b-a} = f'(c) \quad \text{(for some } c,\ a < c < b\text{)} \\ \text{Provided that } f \text{ is differentiable on } a < x < b, \text{ and continuous on } a \le x \le b.\end{array}}$$

Geometric Proof of MVT: Consider the graph of $f(x)$. Here, $\dfrac{f(b)-f(a)}{b-a}$ is the slope of a secant line joining the points $(a, f(a))$ and $(b, f(b))$, and $f'(c)$ is the slope of a tangent line. We need to show that somewhere between a and b there's a point on the graph $(c, f(c))$ whose tangent line has the same slope as that secant line.

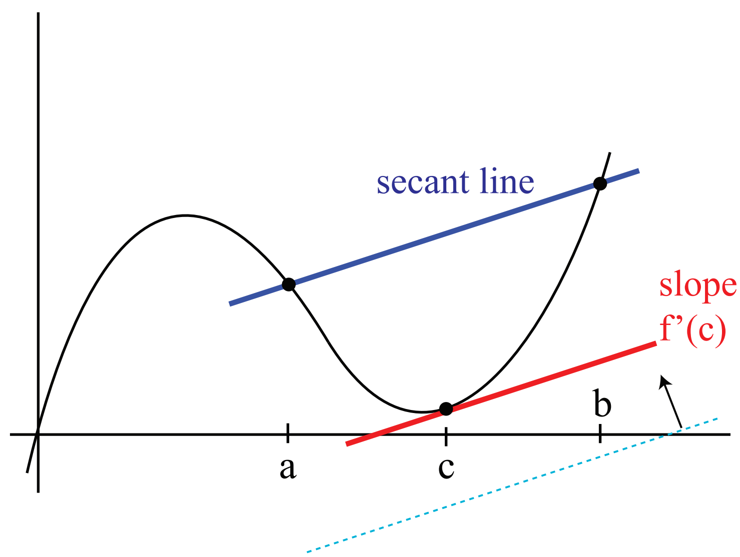

Figure 1: Illustration of the Mean Value Theorem.

Take (dotted) lines parallel to the secant line, as in Fig. 1 and shift them up from below the graph until one of them first touches the portion of the graph that lies between a and b. If it does not touch, start with a dotted line above the graph and move it down until it touches.

When reading a proof, you should always be thinking about why the hypotheses are necessary. Would the proof still work if the function were discontinuous or if it were not differentiable?

We need the hypotheses that f is continuous, because if you could sit still for six hours and then instantly teleport 3,000 km there would never be a time at which you were traveling 500 km/hour. The mean value theorem can't make any guarantees about discontinuous functions.

What if the function isn't differentiable? Suppose $f(x) = |x|$. Then the dotted line always touches the graph first at $x = 0$, no matter what its slope is (see Fig. 2). Even though f is differentiable everywhere except $x = 0$, the mean value theorem still doesn't work here; we need $f'(x)$ to exist at *all* x between a and b.

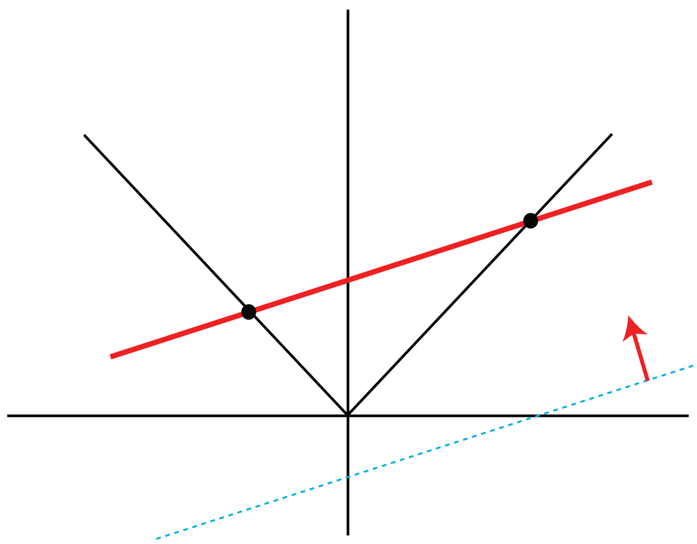

Figure 2: Graph of $y = |x|$ with secant line. (One bad point ruins the proof.)

Question: What if the line parallel to the secant line touches the graph in more than one point?

Answer: The more the merrier! The graph could wiggle a lot of times and the line could touch in ten places, or f could be constant and the line could touch every point on the graph at once. In mathematics, when we claim something is true for one point we don't necessarily mean that it isn't true for others.

The fact that this point exists is a touchy point; we can see why it ought to exist but we didn't really prove that it does. The formal proof has to do with the existence of tangent lines and uses more analysis then we can do in this class.

Mean Value Theorem: Consequences

The first thing we apply the MVT to is graphing, but we'll see later that this is significant in all the rest of calculus.

- If $f' > 0$ then f is increasing.
- If $f' < 0$ then f is decreasing.
- If $f' = 0$ then f is constant.

We told you that the first of these two are true, but we didn't prove them. We can now prove them using the MVT.

Proof: The mean value theorem tells us that:

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

for some c between a and b. For the purposes of this proof we'll assume that $b > a$. We write the equation for the MVT "backwards" because we want to use information about f' to get information about f.

We manipulate the equation to get:

$$\begin{aligned} f(b) - f(a) &= f'(c)(b - a) \\ f(b) &= f(a) + f'(c)(b - a) \end{aligned}$$

This new form of the MVT will let us check these three facts.

Since $a < b$, $b - a > 0$ and the sign of $f'(c)(b - a)$ is completely determined by the sign of $f'(c)(b - a)$.

- If $f'(c) > 0$ then $f(b) > f(a)$.
- If $f'(c) < 0$ then $f(b) < f(a)$.
- If $f'(c) = 0$ then $f(b) = f(a)$.

These facts may seem obvious, but they are not. The definition of the derivative is written in terms of infinitesimals. It's not a sure thing that these infinitesimals have anything to do with the large scale behavior of the function. Before, we were saying that the difference quotient was approximately equal to the derivative. Now we're saying that it's exactly equal to a derivative. (Although we don't know at what point that derivative should be taken.)

The Mean Value Theorem and Linear Approximation

What's the difference between the mean value theorem and the linear approximation?

The linear approximation to $f(x)$ near a has the formula:

$$f(x) \approx f(a) + f'(a)(x-a) \quad x \text{ near } a.$$

If we let $\Delta x = x - a$, we get:

$$\begin{aligned} f(x) &\approx f(a) + f'(a)(x-a) \\ f(x) - f(a) &\approx f'(a)\Delta x \\ \frac{\Delta f}{\Delta x} &\approx f'(a). \end{aligned}$$

Similarly the MVT says:

$$f(b) = f(a) + f'(c)(b-a) \quad \text{for some } c, a < c < b$$

If b is near a then we can write $b - a = \Delta x$ and rewrite the theorem as:

$$\frac{\Delta f}{\Delta x} = f'(c) \quad \text{for some } c, a < c < b.$$

The mean value theorem tells us that $\frac{\Delta f}{\Delta x}$ is *exactly* equal to $f'(c)$ for some c between a and b. We don't know precisely where c is; it depends on f, a, and b.

As Professor Jerison says in the video, this is telling us that the average change on the interval is between the maximum and minimum values $f'(x)$ reaches on the interval $[a, b]$ (because the derivative is continuous).

$$\min_{a \le x \le b} f'(x) \le \frac{f(b) - f(a)}{b - a} = f'(c) \le \max_{a \le x \le b} f'(x)$$

In other words, the average speed of your trip is somewhere between your minimum speed and your maximum speed.

Linear approximation, is based on the assumption that the average speed is approximately equal to the initial (or possibly final) speed. Figure 1 illustrates the approximation $1 + x \approx e^x$.

If the interval $[a, b]$ is short, $f'(x)$ won't vary much between a and b; the max and the min should be pretty close. The mean value theorem tells us absolutely that the slope of the secant line from $(a, f(a))$ to $(x, f(x))$ is no less than the minimum value and no more than the maximum value of f' on that interval, which assures us that the linear approximation does give us a reasonable approximation of the f.

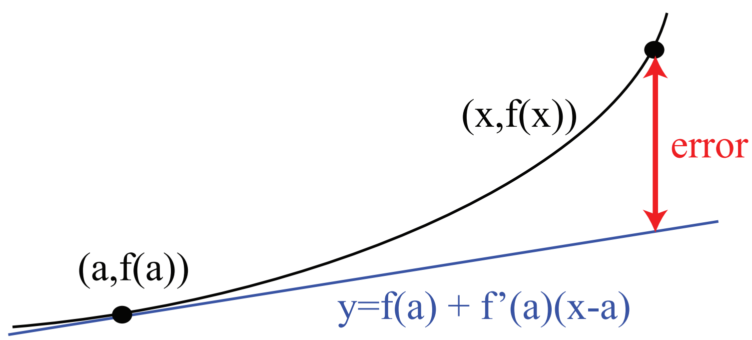

Figure 1: MVT vs. Linear Approximation.

The Mean Value Theorem and Inequalities

The mean value theorem tells us that if f and f' are continuous on $[a, b]$ then:

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

for some value c between a and b. Since f' is continuous, $f'(c)$ must lie between the minimum and maximum values of $f'(x)$ on $[a, b]$. In other words:

$$\min_{a \leq x \leq b} f'(x) \leq \frac{f(b) - f(a)}{b - a} = f'(c) \leq \max_{a \leq x \leq b} f'(x).$$

This is the form that the mean value theorem takes when it is used in problem solving (as opposed to mathematical proofs), and this is the form that you will need to know for the test.

In practice, you may even forget the mean value theorem and remember only these three inequalities:

- If $f'(c) > 0$ then $f(b) > f(a)$.
- If $f'(c) < 0$ then $f(b) < f(a)$.
- If $f'(c) = 0$ then $f(b) = f(a)$.

These can be used to prove mathematical inequalities. The following examples compare the function e^x to its linear and quadratic approximations and are the first steps toward a deeper understanding of the function.

Example: Show that $e^x > 1 + x$ for $x > 0$.

To prove this, we'll instead show that $f(x) = e^x - (1 + x)$ is always positive. We know that $f(0) = e^0 - (1 + 0) = 0$ and $f'(x) = e^x - 1$. When x is positive, $f'(x)$ is positive because $e^x > 1$.

We know that if $f'(x) > 0$ on an interval then $f(x)$ is increasing on that interval, so we can conclude that $f(x) > f(0)$ for $x > 0$. In other words,

$$e^x - (1 + x) > 0 \iff e^x > 1 + x.$$

Example: Show that $e^x > 1 + x + \frac{x^2}{2}$ for $x > 0$.

The value of $1 + x + \frac{x^2}{2}$ is slightly greater than that of $1 + x$, but it turns out that it's still less than the value of e^x. We let $g(x) = e^x - (1 + x + \frac{x^2}{2})$ and do the same thing we did before:

$$\begin{aligned} g(0) &= 1 - (1) = 0 \\ g'(x) &= e^x - (1 + x) \end{aligned}$$

We know $g'(x) > 0$ because we proved $f(x) > 0$ in the above example. Since $g'(x)$ is positive, g is increasing for $x > 0$, so $g(x) > g(0)$ when $x > 0$, so $e^x - (1 + x + \frac{x^2}{2}) > 0$ and $e^x > (1 + x + \frac{x^2}{2})$.

We can keep on going: $e^x > 1 + x + \frac{x^2}{2} + \frac{x^3}{3!}$ for $x > 0$. Eventually, it turns out that

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \quad \text{(an infinite sum)}$$

We will be discussing this when we get to Taylor series near the end of the course.

Differentials

Today we move on from differentiation to integration. For this we'll need a new notation for quantities called differentials.

Given a function $y = f(x)$, the *differential* of y is

$$\boxed{dy = f'(x)dx}$$

Because $y = f(x)$ we sometimes call this the differential of f. Both dy and $f'(x)dx$ are called *differentials*. You can think of

$$\frac{dy}{dx} = f'(x)$$

as a quotient of differentials. Get used to this idea; it comes up in many contexts, including this class and multivariable calculus.

This arises from the Leibniz interpretation of a derivative as a ratio of "infinitesimal" quantities; differentials are sort of like infinitely small quantities.

Working with differentials is much more effective than using the notation coined by Newton; good notation can help you think much faster. Leibniz's notation was adopted on the Continent and Newton dominated in Britain; as a result the British fell behind by one or two hundred years in the development of calculus.

Differentials and Linear Approximation

Linear approximation allows us to estimate the value of $f(x+\Delta x)$ based on the values of $f(x)$ and $f'(x)$. We replace the change in horizontal position Δx by the differential dx. Similarly, we replace the change in height Δy by dy. (See Figure 1.)

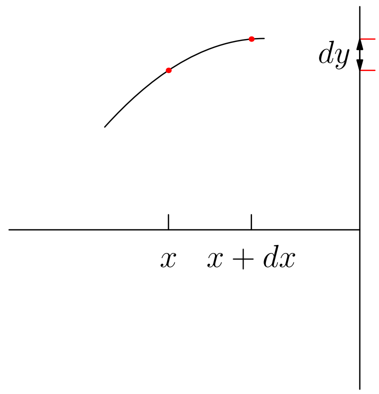

Figure 1: We use dx and dy in place of Δx and Δy.

Example: Find the approximate value of $(64.1)^{\frac{1}{3}}$.

Method 1 (using differentials)

We're going to use a linear approximation of the function $y = f(x) = x^{\frac{1}{3}}$. Our base point will be $x_0 = 64$ because it's easy to compute $y_0 = 64^{\frac{1}{3}} = 4$. By definition, $dy = f'(x)dx = \frac{1}{3}x^{-\frac{2}{3}}dx$.

$$\begin{aligned} dy &= \frac{1}{3}(64)^{-\frac{2}{3}}dx \\ &= \frac{1}{3}\frac{1}{16}dx \\ &= \frac{1}{48}dx \end{aligned}$$

We want to approximate $(64.1)^{\frac{1}{3}}$, so $x + dx = 64.1$ and $dx = 0.1 = \frac{1}{10}$. At the value $64.1 = x_0 + dx$, $f(x)$ is exactly equal to $y_0 + \Delta y$ (because this is how we defined Δy) and is approximately equal to $y_0 + dy$, where dy is is linear in dx as derived above.

In essence, the point $(x_0 + dx, y_0 + dy)$ is an infinitesimally small step away from (x_0, y_0) along the tangent line. Of course $\frac{1}{10}$ is not infinitesimally small, which is why this is an approximation rather than an exact value.

$$(64.1)^{\frac{1}{3}} \approx y + dy$$

$$\begin{aligned} &\approx \quad 4+\frac{1}{48}dx \\ &\approx \quad 4+\frac{1}{48}\frac{1}{10} \\ &\approx \quad 4.002 \end{aligned}$$

Method 2 (review)

When we compare this to our previous notation we discover that the calculations are the same; only the notation has changed.

The basic formula for linear approximation is:

$$f(x) = f(a) + f'(a)(x-a)$$

Here $a = 64$ and $f(x) = x^{\frac{1}{3}}$, so $f(a) = f(64) = 4$ and $f'(a) = \frac{1}{3}a^{-\frac{2}{3}} = \frac{1}{48}$

Our approximation then becomes:

$$\begin{aligned} f(x) &\approx f(a) + f'(a)(x-a) \\ x^{\frac{1}{3}} &\approx 4+\frac{1}{48}(x-64) \\ (64.1)^{\frac{1}{3}} &\approx 4+\frac{1}{48}\frac{1}{10} \\ (64.1)^{\frac{1}{3}} &\approx 4.002 \end{aligned}$$

We get the same answer as before, by doing a nearly identical calculation.

Introduction to Antiderivatives

This is a new notation and also a new concept. $G(x) = \int g(x)dx$ is the *antiderivative* of g. Other ways of saying this are:

$$G'(x) = g(x) \quad \text{or,} \quad dG = g(x)dx$$

There are a few things to notice about this definition. It includes a differential dx. It also includes the symbol $\int$, called an *integral sign*; the expression $\int g(x)dx$ is an *integral*. Another name for the antiderivative of g is the *indefinite integral* of g. (We'll learn what "indefinite" means in this context very shortly.)

If $G(x)$ is the antiderivative of $g(x)$ then $G'(x) = g(x)$. To find the antiderivative of a function g (to integrate g), we need to find a function whose derivative is g. In practice, finding antiderivatives is not as easy as finding derivatives, but we want to be able to integrate as many things as possible. We'll start with some examples.

Example: $\sin x$

We start with the integral of $g(x) = \sin x$. This is a function whose derivative is $\sin x$. What function has $\sin x$ as its derivative?

Student: $-\cos x$

Because the derivative of $-\cos x$ is $\sin x$, this is an antiderivative of $\sin x$. If:

$$\begin{aligned} G(x) &= -\cos x, \quad \text{then} \\ G'(x) &= \sin x \end{aligned}$$

On the other hand, if we had instead chosen $G(x) = -\cos x + 7$ we would still have had $G'(x) = \sin x$. Because the derivative of a constant is 0, we can add any constant to $G(x)$ and still have an antiderivative of $\sin x$. We write:

$$\int \sin x \; dx = -\cos x + c$$

and call this the *indefinite integral* of $\sin x$ because c can be any constant — it's an indefinite value. Whenever we take the antiderivative of something our answer is ambiguous up to a constant.

Antiderivative of x^a

What function has the derivative x^a? We know that the exponent decreases by one when we differentiate, so we guess x^{1+1}. This doesn't quite work:

$$d(x^{a+1}) = (a+1)x^a dx.$$

We have to divide both sides by the constant $(a+1)$ to get the correct answer.

$$\begin{aligned} d(\frac{x^{a+1}}{a+1}) &= x^a dx \\ \frac{x^{a+1}}{a+1} + c &= \int x^a dx \end{aligned}$$

But wait! Although it's true that $d(x^{a+1}) = (a+1)x^a dx$, it is not always true that $\int x^a dx = \frac{x^{a+1}}{a+1} + c$. When $a = -1$ the denominator is zero. However, we can still say that $\int x^a dx = \frac{x^{a+1}}{a+1} + c$ for $a \neq -1$.

What happens when $a = -1$? What is $\int \frac{1}{x} dx$?

So far we've used the formulas $\frac{d}{dx} \cos x = -\sin x$ and $\frac{d}{dx} x^{n+1} = (n+1)x^n$. An important part of integration is remembering formulas for derivatives and "reading them backward". In this case, the formula we need is $\frac{d}{dx} \ln x = \frac{1}{x}$. Using this, we get $\int \frac{1}{x} dx = \ln x + c$.

This formula is fine when $x > 0$, but $\ln x$ is not defined when x is negative. The more standard form of this equation is:

$$\int \frac{1}{x} dx = \ln |x| + c.$$

The absolute value doesn't change anything when $x \geq 0$, so we only need to check this formula when x is negative. In order to do so, we have to differentiate $\ln |x|$.

$$\begin{aligned} \frac{d}{dx} \ln |x| &= \frac{d}{dx} \ln(-x) \quad (|x| = -x \text{ when } x < 0) \\ &= \frac{1}{-x} \frac{d}{dx}(-x) \quad \text{(by the chain rule)} \\ &= -\frac{1}{-x} \\ &= \frac{1}{x} \end{aligned}$$

If we graph $\ln |x|$ we can see that this function does have slope $\frac{1}{x}$.

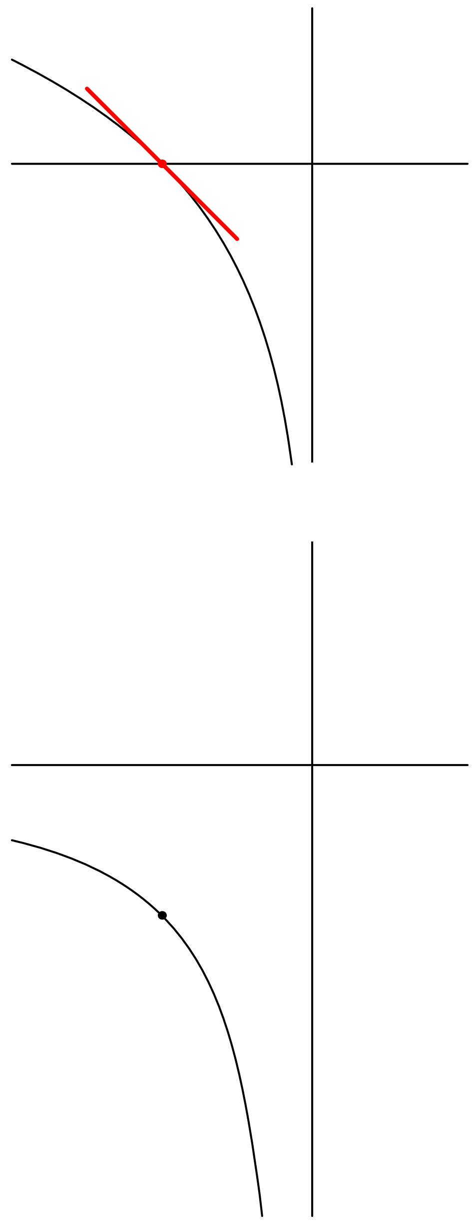

Figure 1: Graphs of $y = ln(-x)$ (above) and $y' = \dfrac{1}{x}$ (below).

Antiderivatives of $\sec^2 x$ and $\dfrac{1}{\sqrt{1-x^2}}$

Example: $\int \sec^2 x \, dx$

Searching for antiderivatives will help you remember the specific formulas for derivatives. In this case, you need to remember that $\frac{d}{dx} \tan x = \sec^2 x$.

$$\int \sec^2 x \, dx = \tan x + c$$

Example: $\displaystyle\int \frac{1}{\sqrt{1-x^2}} \, dx$

An alternate way to write this integral is $\int \frac{dx}{\sqrt{1-x^2}}$. This is consistent with the idea that dx is an infinitesimal quantity which can be treated like any other number.

We remember $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$ and conclude that:

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c.$$

Example: $\displaystyle\int \frac{dx}{1+x^2}$

$$\int \frac{dx}{1+x^2} = \tan^{-1} x + c$$

When looking for antiderivatives, you'll spend a lot of time thinking about derivatives. For a little while you may get the two mixed up and differentiate where you were meant to integrate, or vice-versa. With practice, this problem goes away.

Here is a list of the antiderivatives presented in this lecture:

1. $\displaystyle\int \sin x \, dx = -\cos x + c$ where c is any constant.
2. $\displaystyle\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$ for $n \neq -1$.
3. $\displaystyle\int \frac{dx}{x} = \ln|x| + c$ (This takes care of the exceptional case $n = -1$ in 2.)
4. $\displaystyle\int \sec^2 x \, dx = \tan x + c$.
5. $\displaystyle\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c$ (where $\sin^{-1} x$ denotes "inverse sine" or arcsin, and not $\dfrac{1}{\sin x}$.)
6. $\displaystyle\int \frac{dx}{1+x^2} = \tan^{-1}(x) + c$.

Antiderivatives are Unique up to a Constant

Theorem: If $F'(x) = f(x)$ and $G'(x) = f(x)$, then $F(x) = G(x) + c$.

In other words, once we've found one antiderivative of a function we know that any other antidervative we might find will only differ from it by some added constant.

Proof: If $F' = G'$ then $(F - G)' = F' - G' = f - f = 0$.

Recall that we proved as a corollary of the Mean Value Theorem that if a function's derivative is zero then it is constant. Hence $G(x) - F(x) = c$ (for some constant c). That is, $G(x) = F(x) + c$.

This is a very important fact. It's the basis for calculus; the reason why it makes sense to do calculus at all. This theorem tells us that if we know the rate of change of a function we can find out everything else about the function except this starting value c.

Substitution: $\int x^3(x^4+2)^5dx$

We want to compute $\int x^3(x^4+2)^5dx$.

We already have a formula for $\int x^n dx$, so we could expand $(x^4+2)^5$ and integrate the polynomial. That would be messy. Instead we'll use the method of substitution.

Finding the exact integral of a function is much harder than finding its derivative; occasionally it's impossible. In this unit we're only going to use one method, which means that whenever you see an integral, either you'll be able to divine immediately what the answer is or you'll use substitution. The method of substitution is tailor-made for differential notation.

To use the method of substitution, find the messiest function in your integral; in this case that will be $u = x^4+2$. The differential of u is $du = u'dx = 4x^3dx$. Luckily, we can substitute these two expressions into our original integral and simplify it considerably.

The original problem was to find $\int x^3(x^4+2)^5dx$. We can replace $(x^4+2)^5$ by u^5, and $x^3dx = \frac{1}{4}du$:

$$\begin{aligned}\int x^3(x^4+2)^5dx &= \int \underbrace{(x^4+2)^5}_{u^5}\underbrace{x^3dx}_{\frac{1}{4}du} \\ &= \int \frac{u^5du}{4} \\ &= \frac{1}{4}\cdot\frac{1}{6}u^6 = \frac{u^6}{24}\end{aligned}$$

This is not the answer to the question, because this answer is expressed in terms of u. The problem was posed in terms of the variable x. We change variables back to x by plugging in our definition of u:

$$\frac{u^6}{24} = \frac{(x^4+2)^6}{24}.$$

We conclude that:

$$\int x^3(x^4+2)^5dx = \frac{(x^4+2)^6}{24}.$$

Integration by "Advanced Guessing"

Example: $\int \frac{xdx}{\sqrt{1+x^2}}$

If we use the method of substitution, we start by setting u equal to the ugliest part of our integral:

$$u = 1 + x^2 \quad \text{and} \quad du = 2xdx.$$

The calculation looks like:

$$\begin{aligned}
\int \frac{xdx}{\sqrt{1+x^2}} &= \int \frac{\frac{1}{2}du}{\sqrt{u}} \\
&= \int \frac{u^{-\frac{1}{2}}}{2} du \\
&= 2\frac{u^{\frac{1}{2}}}{2} + c \\
&= u^{\frac{1}{2}} + c \\
&= (1+x^2)^{\frac{1}{2}} + c \\
&= \sqrt{1+x^2} + c
\end{aligned}$$

A better way to compute this is what we call "advanced guessing". Once you've done enough of these problems that you know what's going to happen, you can look at the $\sqrt{1+x^2}$ in the denominator and guess that the answer will involve $(1+x^2)^{1/2}$. Once you've made a guess, differentiate it and see if it works!

$$\begin{aligned}
\frac{d}{dx}(1+x^2)^{\frac{1}{2}} &= \frac{1}{2}(1+x^2)^{-\frac{1}{2}}(2x) \\
&= (1+x^2)^{-\frac{1}{2}}(x) \\
&= \frac{x}{\sqrt{1+x^2}}
\end{aligned}$$

As you can see, using this method we quickly confirm that:

$$\int \frac{xdx}{\sqrt{1+x^2}} = (1+x^2)^{1/2} + c.$$

This method is highly recommended, but it takes some getting used to.

Example: $\int e^{6x}dx$

We know that the derivative of e^x is e^x, so we guess e^{6x}. Then we check our guess using the chain rule:

$$\frac{d}{dx}(e)^{6x} = e^{6x}(6) = 6e^{6x}$$

This has a multiple of 6 that's not in the integral we're trying to compute, so we should divide our guess by 6 to get the correct answer:

$$\int e^{6x} dx = \frac{1}{6} e^{6x} + c.$$

We could also have used the substitution $u = 6x$. It would have worked, but it would have taken much longer.

More Examples of Integration

Example: $\int xe^{-x^2}\,dx$

For this we guess e^{-x^2}, hoping that the chain rule will somehow provide the missing factor of x in the integral. As usual, we take the derivative to check:

$$\frac{d}{dx}e^{-x^2} = (e^{-x^2})(-2x) = -2xe^{-x^2}$$

We're off by a factor of -2, so we divide our original guess by this constant to reach the conclusion that:

$$\int xe^{-x^2}\,dx = -\frac{1}{2}e^{-x^2} + c$$

Caution: If you solve integrals by guessing and don't check your answer by taking a derivative you're likely to make mistakes.

Example: $\int \sin x \cos x\, dx$

What's a good guess?

Student: $\sin^2 x$

Let's check it!

$$\frac{d}{dx}\sin^2 x = 2\sin x \cos x.$$

So:

$$\int \sin x \cos x\, dx = \frac{1}{2}\sin^2 x + c$$

An equally acceptable answer is:

$$\int \sin x \cos x\, dx = -\frac{1}{2}\cos^2 x + c$$

This seems like a contradiction; let's check our answer:

$$\frac{d}{dx}\cos^2 x = (2\cos x)(-\sin x) = -2\sin x \cos x$$

Both answers are correct! But we just proved that integrals are unique up to a constant. What's going on?

It turns out that the difference between the two answers *is* a constant:

$$\frac{1}{2}\sin^2 x - (-\frac{1}{2}\cos^2 x) = \frac{1}{2}(\sin^2 x + \cos^2 x) = \frac{1}{2}$$

So,

$$\frac{1}{2}\sin^2 x - \frac{1}{2} = \frac{1}{2}(\sin^2 x - 1) = \frac{1}{2}(-\cos^2 x) = -\frac{1}{2}\cos^2 x$$

The two answers are, in fact, equivalent. The constant c is shifted by $\frac{1}{2}$ from one answer to the other.

Example: $\displaystyle\int \frac{dx}{x \ln x}$

We will assume $x > 0$ so that $\ln x$ is defined. We don't quickly come up with a good guess, so we use the method of substitution (which is the only other method we know). The ugliest part of the integral is the natural log, so we choose:

$$u = \ln x.$$

One advantage of this choice is that taking the differential of $\ln x$ makes it simpler: $du = \dfrac{1}{x}\, dx$. Substitute these into the integral to get:

$$\begin{aligned}
\int \frac{dx}{x \ln x} &= \int \underbrace{\frac{1}{\ln x}}_{\frac{1}{u}} \underbrace{\frac{dx}{x}}_{du} \\
&= \int \frac{1}{u} du \\
&= \ln |u| + c \\
&= \ln |\ln(x)| + c
\end{aligned}$$

For this example, the method of substitution is better than guessing.

Introduction to Ordinary Differential Equations

MIT has an entire course on differential equations called 18.03. However, there is a technique using differentials that fits in well with what we've been doing with integration. We'll discuss that here.

The simplest type of differential equation looks like: $\dfrac{dy}{dx} = f(x)$. The solution to this equation is the antiderivative (integral) $y = \displaystyle\int f(x)\,dx$. We're going to assume for now that we can always solve this problem.

At the moment we know only the method of substitution (which includes "advanced guessing") for solving integration problems.

Example: $\left(\dfrac{d}{dx} + x\right) y = 0 \qquad \left(\text{or} \quad \dfrac{dy}{dx} + xy = 0\right)$

This is our first interesting example of a differential equation. The operation $\left(\dfrac{d}{dx} + x\right)$ is known in quantum mechanics as the *annihilation operator*. This is also the equation that governs the ground state of the harmonic oscillator, and it is relatively simple to solve.

The first step in solving it is to rewrite the equation by isolating $\frac{dy}{dx}$:

$$\begin{aligned}
\left(\frac{d}{dx} + x\right) y &= 0 \\
\frac{dy}{dx} + xy &= 0 \\
\frac{dy}{dx} &= -xy
\end{aligned}$$

The big difference between this example and the antiderivatives we've been studying is that here the rate of change depends upon both x and y. We don't yet have any strategies for solving this sort of equation.

However, it turns out that we can use differentials and Leibnitz' notation to solve this. The key step is to *separate variables* so that all the terms involving y are on one side of the equation and all terms involving x are on the other.

$$\frac{dy}{y} = -x\,dx$$

Because the problem is now set up in terms of differentials, as opposed to ratios of differentials (rates of change). Because of this, we can take advantage of Leibnitz' notation and integrate both sides of the equation. Notice that on the left the differential variable is y and on the right it is x.

$$\begin{aligned}
\int \frac{dy}{y} &= -\int x\,dx \\
\ln y + c_1 &= -\frac{x^2}{2} + c_2 \quad (\text{assume } y > 0)
\end{aligned}$$

$$\begin{aligned}
\ln y &= -\frac{x^2}{2} + c \quad \text{(we only need one constant } c = c_2 - c_1\text{)} \\
e^{\ln y} &= e^{c + -x^2/2} \\
y &= e^c e^{-x^2/2} \\
y &= Ae^{-x^2/2} \quad (A = e^c)
\end{aligned}$$

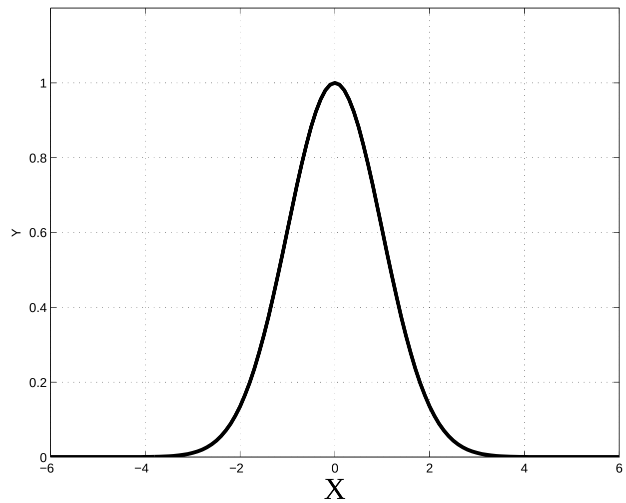

Figure 1: Graph of $y = e^{-\frac{x^2}{2}}$.

It turns out that the solution is $y = ae^{-x^2/2}$ for any constant multiple a, despite the fact that $e^c \neq 0$. We can check this solution using differentiation:

$$\begin{aligned}
y &= ae^{-x^2/2} \\
\frac{dy}{dx} &= \frac{d}{dx} ae^{-x^2/2} \\
&= ae^{-x^2/2} \cdot -2x/2 \quad \text{(chain rule)} \\
&= ae^{-x^2/2} \cdot -x \\
&= y \cdot -x \\
\frac{dy}{dx} &= -xy
\end{aligned}$$

This matches the equation for $\frac{dy}{dx}$ from our first step, so $y = ae^{-x^2/2}$ is a solution to our differential equation. We didn't make any assumptions about a in this calculation, so this is a solution no matter what value a has; $a = 0$ is possible along with all $a \neq 0$, depending on the initial conditions. For instance, if $y(0) = 1$, then $y = e^{-x^2/2}$. If $y(0) = a$, then $y = ae^{-x^2/2}$ (See Fig. 1).

This function is known as the normal distribution, which you may recognize from probability. In quantum mechanics it helps describe where a particle is.

The aim of differential equations is to solve them. Just as with algebraic equations. Usually, differential equations are telling you something about the balance between an acceleration and a velocity; for example, if you're doing calculations involving air resistance. Sometimes in applied problems, formulating a differential equation to describe a situation is very important. In order to see that you chose the right formulation you must confirm that your solution fits what actually happens in the real world.

Question: How can $y = ae^{-x^2/2}$ be our final solution when we don't know what a is?

Answer: We call $y = ae^{-x^2/2}$ the *general solution*; in other words, the whole family of solutions we get by choosing different values for a is the answer to the question.

Frequently we will be given more information than just $\left(\frac{d}{dx} + x\right) y = 0$. For example, we might know that when x is 0, y is 3. Given that extra piece of information, we can nail down exactly which function is the solution:

$$\begin{aligned} y &= ae^{-x^2/2} \\ 3 &= ae^{-(0^2)/2} \\ 3 &= ae^{0} \\ 3 &= a \end{aligned}$$

And so the final solution would be $y = 3e^{-x^2/2}$.

If we don't have the information you need to restrict our answer to a single function then the solution is not one function, it's a family of functions described by the different possible values of some parameter like a.

Question: Can you solve for x instead of y?

Answer: Sure! You would get the inverse function of the function that we're officially looking for but yes, it's legal. Sometimes we'll have to make do with just an implicit formula and sometimes we're stuck with x is a function of y. The way in which the solution is specified can be complicated; as you'll soon see, it's not necessarily the best thing to think y as a function of x.

Separation of Variables

We'll now look at the technique we used to solve the previous problem and discuss what other sorts of problems that method is useful for.

In general, the method of separation of variables applies to differential equations that can be written as:

$$\frac{dy}{dx} = f(x)g(y).$$

In our previous example, $f(x) = -x$ and $g(y) = y$.

The key step in the method is the separation of variables. It is possible because Leibniz designed his notation to make this work nicely, allowing us to treat differential calculations as if they were ordinary arithmetic.

$$\begin{aligned} \frac{dy}{dx} &= f(x)g(y) \\ \frac{dy}{g(y)} &= f(x)\,dx \quad \text{which we can write as} \\ h(y)\,dy &= f(x)\,dx \text{ where } h(y) = \frac{1}{g(y)}. \end{aligned}$$

Next we antidifferentiate both sides of the equation:

$$H(y) = \int h(y)\,dy; \quad F(x) = \int f(x)\,dx$$

In our example, $H(y) = \ln|y|$ and $F(x) = \frac{-x^2}{2}$.

These antiderivatives are equal, so we get:

$$H(y) = F(x) + c \quad \text{(Again, we only need one constant } c.\text{)}$$

This is what we call an *implicit equation*; in our example we had $\ln y = -x^2/2 + c$. It doesn't tell us exactly what y is but it does describe y implicitly. In order to solve for y explicitly we need the inverse function H^{-1}; in our example this inverse was the exponential function and the explicit equation was $y = Ae^{-x^2/2}$.

In practice, it's often easy to find the implicit equation and quite messy to perform the inverse operation. In that case we might leave the solution in implicit form.

Remark 1: In the example, we could have written $\ln|y| = -x^2/2 + c$ for $y \neq 0$. Then we would have gotten $|y| = Ae^{-x^2/2}$ or $y = \pm Ae^{-x^2/2}$, which is almost exactly the answer we did get, if $a = \pm A$ and $A > 0$.

Professor Jerison didn't bother with this because it makes the calculation more complicated. Once he had solved the problem for $y > 0$, he knew from previous experience what the answer would be and he could skip directly to that and check his work. (The exponential function comes up all the time, so you too will want to be completely comfortable dealing with it.)

This still leaves out the case $y = 0$. This is an extremely boring solution, but it is still a solution to this problem. You can verify that $y = 0$ (i.e. $a = 0$) is a solution to this problem by plugging 0 in for y in the equation $\left(\frac{d}{dx} + x\right) y = 0$ or $\frac{dy}{dx} = -xy$.

It's not so surprising that we missed that solution, because in the process of separating variables we divided by y. If you divide by something you may have problems when that thing equals zero, or miss that solution. To avoid these problems, take note of when you divide by something that might be zero and double check that case after you've finished your calculations.

Remark 2: We had:

$$\int h(y)\, dy = \int f(x)\, dx$$

which evaluated to:

$$H(y) = F(x) + c.$$

We could have written $H(y) + c_1 = F(x) + c_2$, but this is equivalent to $H(y) = F(x) + c_2 - c_1 = F(x) + c$. To save time and writing, we write down only one arbitrary constant when integrating both sides of an equation.

Remark 3: In our example, the additive constant c turned into a multiplicative constant A when we calculated $e^{-x^2/2+c}$. In general there will always be a free constant in the solution to a differential equation, but that constant will not always be additive.

Example: $\dfrac{dy}{dx} = f(x)$

We'll solve our first, very simple, example using the method of separation of variables. We start by multiplying both sides by dx:

$$dy = f(x)\,dx.$$

Then integrate both sides:

$$\int dy = \int f(x)\,dx$$

The antiderivative of dy is just y, so we get:

$$y = \int f(x)\,dx,$$

as we did before.

Differential Equations and Slope, Part 1

Suppose the tangent line to a curve at each point (x, y) on the curve is twice as steep as the ray from the origin to that point. Find a general equation for this curve. (See Fig. 1.)

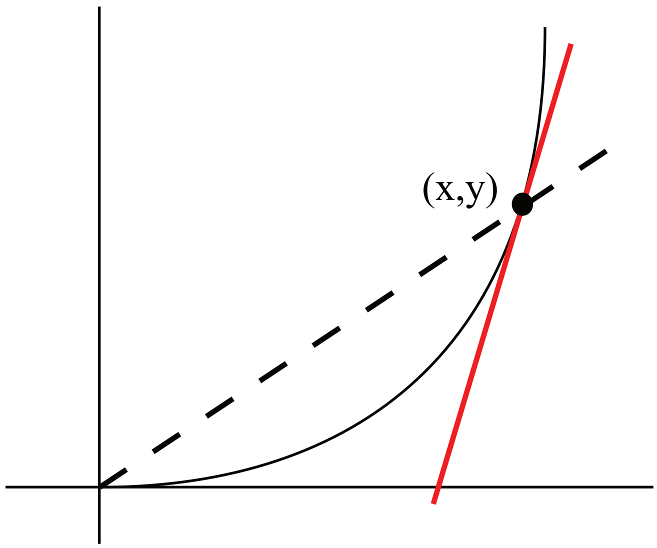

Figure 1: The slope of the tangent line (red) is twice the slope of the ray from the origin to the point (x, y).

This type of problem can be described very succinctly using differential equations. The slope of the tangent line is $\frac{dy}{dx}$. The slope of the ray from $(0, 0)$ to (x, y) is $\frac{y}{x}$. Since the slope of that ray is twice the slope of that ray, we get the differential equation:

$$\frac{dy}{dx} = 2\left(\frac{y}{x}\right).$$

We only have one method for solving differential equations; use it.

$$\begin{aligned}
\frac{dy}{dx} &= 2\frac{y}{x} \\
\frac{dy}{y} &= \frac{2\,dx}{x} \quad \text{(separate variables)} \\
\int \frac{dy}{y} &= \int \frac{2}{x}\,dx \quad \text{(integrate both sides)} \\
\ln|y| &= 2\ln|x| + c \quad \text{(antidifferentiate)} \\
e^{\ln|y|} &= e^{2\ln|x|+c} \quad \text{(apply an inverse function to isolate y)} \\
e^{\ln|y|} &= e^c e^{2\ln|x|} \quad \text{(exponentiate)}
\end{aligned}$$

$$
\begin{aligned}
e^{\ln|y|} &= e^c (e^{\ln|x|})^2 \\
|y| &= e^c x^2 \quad (e^{2\ln|x|} = x^2\ )
\end{aligned}
$$

There is an absolute value in this solution. When $y > 0$ we get $y = e^c x^2$. When $y < 0$ we get $y = -e^c x^2$. Based on prior experience we guess that the solution will be $y = ax^2$, where $a = \pm e^c$ or $a = 0$.

Because we divided by y in our calculations our solution doesn't include the case in which $a = 0$ and $y = 0x^2$. Graph the equation $y = 0$ and confirm that at each point on the graph the slope of the tangent line is twice the slope of the ray joining that point to the origin; this confirms that $y = 0x^2$ is a solution.

We conclude that the general solution to the problem is:

$$y = ax^2$$

where a could be positive, negative or zero. Some possible solutions include:

$$
\begin{aligned}
y &= x^2 \quad (a = 1) \\
y &= 2x^2 \quad (a = 2) \\
y &= -x^2 \quad (a = -1) \\
y &= 0x^2 = 0 \quad (a = 0) \\
y &= -2y^2 \quad (a = -2) \\
y &= 100x^2 \quad (a = 100)
\end{aligned}
$$

Some representatives of this family of curves are shown in black in Fig. 2.

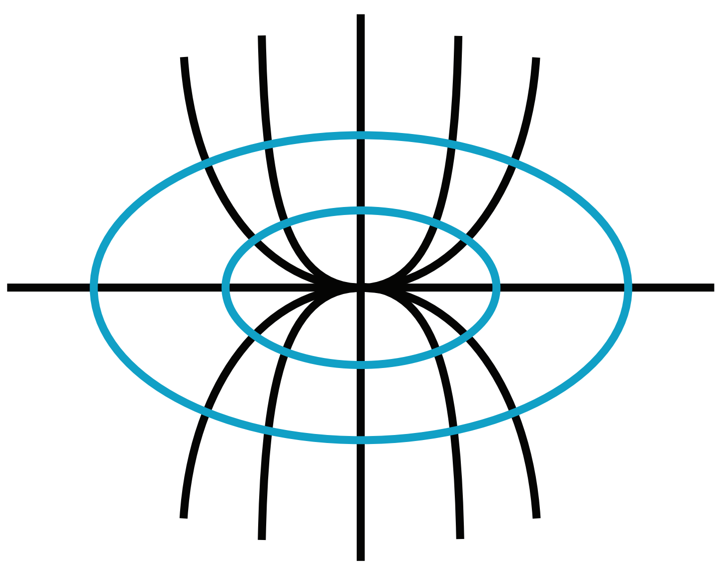

Figure 2: Parabolic curves, shown in black.

If we want to check our work, we can do so by taking the derivative:

$$\begin{aligned} y &= ax^2 \\ \frac{dy}{dx} &= 2ax \end{aligned}$$

Since $2ax = \frac{2ax^2}{x}$, we have $\frac{dy}{dx} = \frac{2y}{x}$. This works for $a > 0$, $a < 0$ and $a = 0$, so this solution is valid for all those values of a.

Warning: Notice that in the equation $\frac{dy}{dx} = \frac{2y}{x}$, $\frac{2y}{x}$ is undefined at $x = 0$. As you can see from Fig. 2, knowing the value of the function and its derivative at $x = 0$ doesn't tell us how the function will behave elsewhere. This is bad — for one thing, it contradicts our understanding of linear approximation.

What goes wrong is that the rate of change is not specified when $x = 0$. If you think carefully about what this function is doing, it could follow one branch when $x < 0$ and a completely different branch when $x > 0$. That's a very subtle point; you won't be asked to discuss this problem in your homework, but you should be aware that when x is equal to zero there's a problem for this differential equation.

Differential Equations and Slope, Part 2

Find the curves that are perpendicular to the parabolas $y = ax^2$ from the previous example.

We get a new differential equation from the one in the last example by using the fact that if a line has slope m, a line perpendicular to it will have slope $-\frac{1}{m}$. So:

$$\begin{aligned} \text{slope of curve} &= \frac{dy}{dx} \\ &= -\frac{1}{\text{slope of parabola}} \\ &= -\frac{1}{\frac{2y}{x}} \\ \frac{dy}{dx} &= \frac{-x}{2y} \end{aligned}$$

Separate variables:

$$2y\,dy = -x\,dx$$

Take the antiderivative:

$$\begin{aligned} \int 2y\,dy &= \int -x\,dx \\ y^2 &= -\frac{x^2}{2} + c \end{aligned}$$

So the general solution to this differential equation is:

$$y^2 + \frac{x^2}{2} = c.$$

This describes a family of ellipses. The y-semi-minor axis of these ellipses has length $\sqrt{c}$ and the x-semi-major axis has length $\sqrt{2c}$; the ratio of the x-semi-major axis to the y-semi-minor axis is $\sqrt{2}$ (see Fig. 1).

Unlike last time, this solution only works when $c > 0$. For some problems your constant parameter can be any real value; for some it can't.

Separation of variables leads to implicit formulas for y, but in this case you can solve for y.

$$y = \pm\sqrt{c - \frac{x^2}{2}}$$

Writing the solution in this form brings an important point to our attention — the equation of an ellipse does not describe a function! The explicit solution gives you functions that describe the top and bottom halves of the ellipses

The explicit solution also suggests that there's a problem when $y = 0$ and $x = \pm\sqrt{2c}$. Here the ellipse has a vertical tangent line; also the explicit solution isn't defined for $|x| > \sqrt{2c}$. This makes sense when we consider the fact that $\frac{dy}{dx} = \frac{-x}{2y}$. When $y = 0$ the slope of the tangent line to the curve should be infinite.

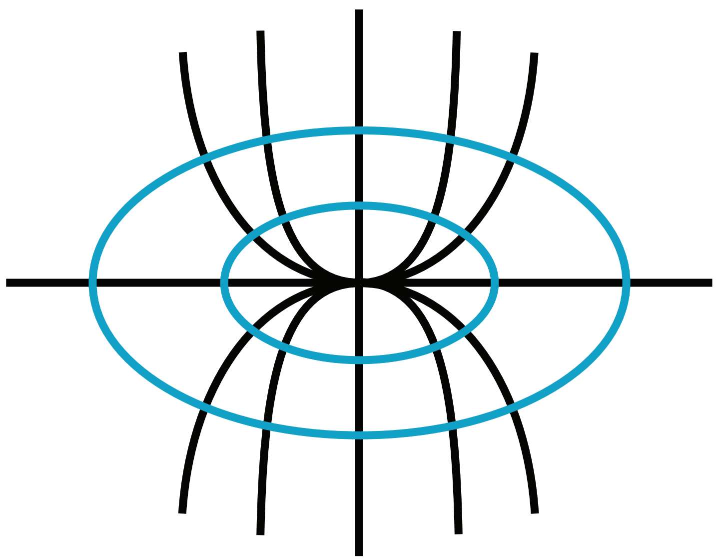

Figure 1: The curves perpendicular to the parabolas are ellipses.