Example: $\lim_{x\to 0} \dfrac{\sin 5x}{\sin 2x}$

This is similar to an example we saw earlier in the course. Here $f(x) = \sin 5x$, $g(x) = \sin 2x$, and $a = 0$. Since $f(a) = g(a) = \sin 0 = 0$, we can apply l'Hôpital's rule and find this limit:

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin 5x}{\sin 2x} &= \lim_{x\to 0} \frac{5\cos 5x}{2\cos 2x} \qquad \text{(l'Hop)} \\
&= \lim_{x\to 0} \frac{5\cos(5\cdot 0)}{2\cos(2\cdot 0)} \\
&= \frac{5}{2}.
\end{aligned}$$

Repeating L'Hôpital's Rule

This example illustrates the superiority of Version 1 of l'Hôpital's rule; it works even if $g'(a) = 0$.

In this case, $f(x) = \cos x - 1$, $g(x) = x^2$, and $a = 0$. We're trying to find:

$$\lim_{x \to 0} \frac{\cos x - 1}{x^2}.$$

We can easily verify that $f(a) = g(a) = 0$.

We apply l'Hôpital's rule:

$$\lim_{x \to 0} \frac{\cos x - 1}{x^2} = \lim_{x \to 0} \frac{-\sin x}{2x}.$$

Notice that $\dfrac{-\sin x}{2x}$ is undefined when $x = 0$; it's of the type $\frac{0}{0}$. That's OK; this version of l'Hôpital's rule still works when $g'(x) = 0$; it works as long as $\lim_{x \to a} \frac{f'(a)}{g'(a)}$ is defined.

We need to find $\displaystyle\lim_{x \to 0} \frac{-\sin x}{2x}$. We can do that by applying l'Hôpital's rule!

$$\lim_{x \to 0} \frac{-\sin x}{2x} = \lim_{x \to 0} \frac{-\cos x}{2}.$$

All together, the calculation looks like:

$$\begin{aligned}
\lim_{x \to 0} \frac{\cos x - 1}{x^2} &= \lim_{x \to 0} \frac{-\sin x}{2x} && \text{(l'Hop)} \\
&= \lim_{x \to 0} \frac{-\cos x}{2} && \text{(l'Hop)} \\
&= \frac{-\cos 0}{2} \\
&= -\frac{1}{2}.
\end{aligned}$$

Notice that we only know that the hypotheses of l'Hôpital's rule hold for our problem when we reach the end and get the limit. The theorem says that l'Hôpital's rule only works if the limit exists, and we find out that the limit exists by using l'Hôpital's rule to calculate it. This is logically somewhat subtle, but in practice the theorem works very well.

Question: Why does the limit have to exist? Isn't it just the derivative that has to exist?

Answer: No, we need the derivative of the numerator, the derivative of the denominator *and* the limit to exist.

Surprisingly enough, we don't need $f'(a)$ to exist; we're working with limits as x approaches a, so what happens when $x = a$ doesn't necessarily matter to us. Once again we're using limits to get close to something that's not defined at the exact value $x = a$.

If any one of the three limits $\lim_{x\to a} f'(x)$, $\lim_{x\to a} g'(x)$, and $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ does not exist then we can't apply l'Hôpital's rule. This is because the theorem we used to prove the rule works might not be true if these limits are undefined.

To get a little ahead of ourselves, notice that in:

$$\lim_{x\to\infty} xe^{-x}$$

the limit $\lim_{x\to\infty} x$ is undefined. Nevertheless, l'Hôpital's rule will apply here.

Comparison With Approximation

We just used l'Hôpital's rule to show that:

$$\lim_{x \to 0} \frac{\cos x - 1}{x^2} = -\frac{1}{2}.$$

Let's compare that to what we get using the method of approximations, replacing the functions involved with their linear or quadratic approximations.

In the example of $\lim_{x \to 0} \frac{\sin 5x}{\sin 2x}$, we would use the linear approximation $\sin u \approx u$ for u near 0 to get:

$$\frac{\sin 5x}{\sin 2x} \approx \frac{5x}{2x} = \frac{5}{2}$$

for x near 0. Since the approximation $\sin u \approx u$ becomes exact as u approaches 0, we could then conclude that:

$$\lim_{x \to 0} \frac{\sin 5x}{\sin 2x} = \frac{5}{2}.$$

To use this method to find:

$$\lim_{x \to 0} \frac{\cos x - 1}{x^2},$$

we would approximate:

$$\frac{\cos x - 1}{x^2}$$

by:

$$\frac{(1 - x^2/2) - 1}{x^2} = \frac{-x^2/2}{x^2} = -\frac{1}{2}.$$

Again the approximation becomes exact as x approaches 0, so:

$$\lim_{x \to 0} \frac{\cos x - 1}{x^2} = -\frac{1}{2}.$$

The method of approximation gives us the same result as l'Hôpital's rule, as it should. Both methods are valid and they both involve about the same amount of work. The approximation we used for the cosine function is related to the second derivative of $\cos x$, and we had to find the second derivative of $\cos x$ when we applied l'Hôpital's rule.

Extensions of L'Hôpital's Rule

Our first version of l'Hôpital's rule told us that:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

provided that $f(a) = g(a) = 0$ and $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists.

We've seen that we can get nearly equivalent results by replacing $f(x)$ and $g(x)$ by linear or quadratic approximations. L'Hôpital's rule is superior to the method of approximation because it works better in some situations.

It turns out that l'Hôpital's rule works even under the following conditions:

- $a = \pm\infty$
- $f(a), g(a) = \pm\infty$
- $\lim_{x \to a} \frac{f'(a)}{g'(a)} = \pm\infty$

In other words, l'Hôpital's rule works not just in the $\frac{0}{0}$ case but also when you're taking a limit of the form $\frac{\infty}{\infty}$. It will give us the right answer if $\frac{f'(a)}{g'(a)}$ approaches $-\infty$, ∞ or some finite number. It fails if $\frac{f'(a)}{g'(a)}$ oscillates wildly, but we don't encounter those conditions in this class; l'Hôpital's rule handles everything we could expect it to handle, and it's easy to use.

Rate of Growth of $\ln x$

This expression is in indeterminate form but looks like it might be the wrong type. This isn't a fraction, so we have to think about how to apply l'Hôpital's rule.

In the expression, the factor x is approaching 0 while the factor $\ln x$ is approaching negative infinity.

$$\lim_{x\to 0^+} \underbrace{x}_{\to 0} \underbrace{\ln x}_{\to -\infty}$$

We're multiplying a number that's getting smaller and smaller by one that's getting larger and larger; the result could be really large or really small, depending on rates of growth.

The first step in finding the limit is to rewrite the expression as a ratio, rather than as a product. We'll choose to write it as:

$$x \ln x = \frac{\ln x}{1/x}.$$

This is an expression of the type $\frac{-\infty}{\infty}$, which is one of the forms we can apply l'Hôpital's rule to. Let's do that:

$$\begin{aligned}
\lim_{x\to 0^+} x \ln x &= \lim_{x\to 0^+} \frac{\ln x}{1/x} \\
&= \lim_{x\to 0^+} \frac{1/x}{-1/x^2} \qquad \text{(l'Hop)} \\
&= \lim_{x\to 0^+} -x \\
&= 0.
\end{aligned}$$

We conclude that x goes to 0 faster than $\ln x$ goes to negative infinity, and so the limit of the product is 0. You might not have been able to guess this in advance.

Rate of Growth of e^{px}

When we looked at $\lim_{x\to 0^+} x \ln x$ we found that the value of the limit was 0, so x shrinks to 0 faster than $\ln x$ grows to negative infinity. The next two examples illustrate similar rate properties, which will be important when we study improper integrals and elsewhere.

Example: $\displaystyle\lim_{x\to\infty} xe^{-px}, \quad (p > 0)$

The expression xe^{-px} is a product, not a ratio, so we need to rewrite it before we use l'Hôpital's rule. We choose to rewrite it as $\dfrac{x}{e^{px}}$. This is of the form $\frac{\infty}{\infty}$, so we can use l'Hôpital's rule to calculate:

$$\begin{aligned}
\lim_{x\to\infty} xe^{-px} &= \lim_{x\to\infty} \frac{x}{e^{px}} \\
&= \lim_{x\to\infty} \frac{1}{pe^{px}} \qquad \text{(l'Hop)} \\
&= \frac{1}{\infty} \\
&= 0.
\end{aligned}$$

We conclude that when $p > 0$, x grows more slowly than e^{px} as x goes to infinity.

Example: $\displaystyle\lim_{x\to\infty} \frac{e^{px}}{x^{100}} \quad (p > 0)$

This example doesn't give us much more information, but it's good practice. The value of this limit gives us information about the relative rates of growth of e^{px} and x^{100}.

The expression $\displaystyle\lim_{x\to\infty} \frac{e^{px}}{x^{100}}$ is of the form $\frac{\infty}{\infty}$, so we can use l'Hôpital's rule again. In fact, there are two ways we could use l'Hôpital's rule. The slow way looks like:

$$\begin{aligned}
\lim_{x\to\infty} \frac{e^{px}}{x^{100}} &= \lim_{x\to\infty} \frac{pe^{px}}{100x^{99}} \qquad \text{(l'Hop)} \\
&= \lim_{x\to\infty} \frac{p^2 e^{px}}{100 \cdot 99 x^{98}} \qquad \text{(l'Hop)} \\
&= \lim_{x\to\infty} \frac{p^3 e^{px}}{100 \cdot 99 \cdot 98 x^{97}} \qquad \text{(l'Hop)} \\
&\vdots
\end{aligned}$$

We could apply l'Hôpital's rule 100 times and we'd eventually get an answer.

The clever way is to rewrite the expression as follows:

$$\lim_{x\to\infty} \frac{e^{px}}{x^{100}} = \left(\lim_{x\to\infty} \frac{e^{px/100}}{x}\right)^{100}$$

$$
\begin{aligned}
&= \left(\lim_{x\to\infty} \frac{\frac{p}{100}e^{px/100}}{1}\right)^{100} && \text{(l'Hop)} \\
&= \left(\lim_{x\to\infty} \frac{p \cdot e^{px/100}}{100}\right)^{100} \\
&= \infty
\end{aligned}
$$

In this example $\lim_{x\to a} \frac{f'(a)}{g'(a)} = \infty$, another possible outcome of l'Hôpital's rule. We conclude that e^{px} grows faster than x^{100} when p is positive. In fact, e^{px} grows faster than *any* polynomial in x; exponential functions grow faster than powers of x.

Comparing Growth of $ln(x)$ and $x^{\frac{1}{3}}$

We have one more item on our original list of limits to cover; again we'll look at a slight variation on the original problem. We're going to find:

$$\lim_{x \to \infty} \frac{\ln x}{x^{1/3}}.$$

This limit is of the form $\frac{\infty}{\infty}$, so we apply l'Hôpital's rule to find:

$$\begin{aligned} \lim_{x \to \infty} \frac{\ln x}{x^{1/3}} &= \lim_{x \to \infty} \frac{1/x}{\frac{1}{3}x^{-2/3}} \qquad \text{(l'Hop)} \\ &= \lim_{x \to \infty} 3x^{-1/3} \\ &= 0 \end{aligned}$$

We conclude that $\ln x$ grows more slowly as x approaches infinity than $x^{1/3}$ or any positive power of x. In other words, $\ln x$ increases very slowly.

Question: When we discussed extensions of l'Hôpital's rule, we learned that we're allowed to change some hypotheses. How many hypotheses can we change at once?

Answer: We can make any or all of the three changes listed. However, $\dfrac{f(a)}{g(a)}$ must always be of the form $\frac{\infty}{\infty}$, $-\frac{\infty}{\infty}$, or $\frac{0}{0}$.

The Indeterminate Form 0^0

We next consider the limit:

$$\lim_{x \to 0^+} x^x.$$

Can we compute this?

There are many different indeterminate forms; x^x is one of the simpler examples. In this case, because x is a moving exponent, we can use a trick to evaluate the limit.

Since we have a moving exponent, we will use base e. We rewrite our original expression as follows:

$$x^x = e^{x \ln x}.$$

Now we can focus our attention on the exponent:

$$\begin{aligned}
\lim_{x \to 0^+} x \ln x &= \lim_{x \to 0^+} \frac{\ln x}{1/x} \\
&= \lim_{x \to 0^+} \frac{1/x}{-1/x^2} \qquad \text{(l'Hop)} \\
&= \lim_{x \to 0^+} -x \\
&= 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lim_{x \to 0^+} x^x &= \lim_{x \to 0^+} e^{x \ln x} \\
&= e^0 \\
&= 1.
\end{aligned}$$

This was relatively easy to calculate because we have so many powerful tools to work with.

Example: $\lim_{x\to 0} \frac{\sin x}{x^2}$

If we apply l'Hôpital's rule to this problem we get:

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin x}{x^2} &= \lim_{x\to 0} \frac{\cos x}{2x} \qquad \text{(l'Hop)} \\
&= \lim_{x\to 0} \frac{-\sin x}{2} \qquad \text{(l'Hop)} \\
&= 0.
\end{aligned}$$

If we instead apply the linear approximation method and plug in $\sin x \approx x$, we get:

$$\begin{aligned}
\frac{\sin x}{x^2} &\approx \frac{x}{x^2} \\
&\approx \frac{1}{x}.
\end{aligned}$$

We then conclude that:

$$\begin{aligned}
\lim_{x\to 0^+} \frac{\sin x}{x^2} &= \infty \\
\lim_{x\to 0^-} \frac{\sin x}{x^2} &= -\infty.
\end{aligned}$$

There's something fishy going on here. What's wrong?

Student: L'Hôpital's rule wasn't applied correctly the second time.

That's correct; $\lim_{x\to 0} \frac{\cos x}{2x}$ is of the form $\frac{1}{0}$, not $\frac{0}{0}$ or some other indeterminate form.

This is where you have to be careful when using l'Hôpital's rule. You have to verify that you have an indeterminate form like $\frac{0}{0}$ or $\frac{\infty}{\infty}$ before applying the rule. The moral of the story is: **Look before you l'Hôp**.

Also, don't use l'Hospital's rule as a crutch. If we want to evaluate:

$$\lim_{x\to\infty} \frac{x^5 - 2x^4 + 1}{x^4 + 2}$$

we can apply l'Hôpital's rule four times, or we could divide the numerator and denominator by x^5 to conclude:

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^5 - 2x^4 + 1}{x^4 + 2} &= \lim_{x\to\infty} \frac{1 - 2/x + 1/x^5}{1/x + 2/x^5} \\
&= \frac{1}{0} \\
&= \infty.
\end{aligned}$$

After enough practice with rates of growth, we can calculate this limit almost instantly:

$$\lim_{x\to\infty} \frac{x^5 - 2x^4 + 1}{x^4 + 2} \sim \lim_{x\to\infty} \frac{x^5}{x^4} = \infty.$$

l'Hôpital's Rule, Continued

In keeping with the spirit of "dealing with infinity" we look at an application of l'Hôpital's rule to a limit of the form $\frac{\infty}{\infty}$. In other words, as x approaches a we have:

- $f(x) \to \infty$
- $g(x) \to \infty$
- $\dfrac{f'(x)}{g'(x)} \to L$

and so we can conclude that:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = L.$$

(Recall that a and L may be infinite.)

Rates of Growth

We apply this to "rates of growth"; the study of how rapidly functions increase. We know that the functions $\ln x$ and x^2 both go to infinity as x goes to infinity, and that x^2 increases much more rapidly than $\ln x$. We can formalize this idea as follows:

If $f(x) > 0$ and $g(x) > 0$ as x approaches infinity, then

$$f(x) << g(x) \text{ as } x \to \infty \quad \text{means} \quad \lim_{x \to \infty} \frac{f(x)}{g(x)} = 0.$$

(Read $f(x) << g(x)$ as "$f(x)$ is a lot less than $g(x)$".) In our example, $f(x) = \ln x$ and $g(x) = x^2$. If we use l'Hôpital's rule to evaluate $\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{\ln x}{x^2}$ we get:

$$\begin{aligned}
\lim_{x \to \infty} \frac{\ln x}{x^2} &= \lim_{x \to \infty} \frac{\frac{1}{x}}{2x} \\
&= \lim_{x \to \infty} \frac{1}{2x^2} \\
&= 0.
\end{aligned}$$

We conclude that $\ln x << x^2$ as $x \to \infty$.

If $p > 0$ then:

$$\ln x << x^p << e^x << e^{x^2} \text{ as } x \to \infty.$$

Rates of Decay

"Rates of decay" are rates at which functions tend to 0 as x goes to infinity. Again our new notation comes in handy; if $p > 0$ then:

$$\frac{1}{\ln x} >> \frac{1}{x^p} >> e^{-x} >> e^{-x^2} \text{ as } x \to \infty.$$

Introduction to Improper Integrals

An *improper integral* of a function $f(x) > 0$ is:

$$\int_a^\infty f(x)\,dx = \lim_{N\to\infty}\int_a^N f(x)\,dx.$$

We say the improper integral *converges* if this limit exists and *diverges* otherwise.

Geometrically then the improper integral represents the total area under a curve stretching to infinity. If the integral $\int_a^\infty f(x)\,dx$ converges the total area under the curve is finite; otherwise it's infinite. (See Figure 1.)

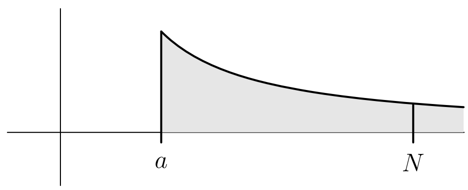

Figure 1: Infinite area under a curve.

How can an area that extends to infinity be finite? Obviously the area between a and N (i.e. $\int_a^N f(x)\,dx$) is finite. As N goes to infinity this quantity will either grow without bound or it will converge to some finite value. Our next step is to look at examples of each of these possibilities.

Example: $\int_0^\infty e^{-kx}\,dx \quad (k > 0)$

This is the most fundamental, by far, of the improper integrals. We start by calculating $\int_0^N e^{-kx}\,dx$:

$$\begin{aligned}\int_0^N e^{-kx}\,dx &= -\frac{1}{k}e^{-kx}\Big|_0^N \\ &= -\frac{1}{k}e^{-kN} - -\frac{1}{k}e^0 \\ &= -\frac{1}{k}e^{-kN} + \frac{1}{k}\end{aligned}$$

As N goes to infinity the $\frac{1}{k}$ does not change, but $-\frac{1}{k}e^{-kN}$ gets closer and closer to zero. (This is only true if k is positive!) So

$$\int_0^\infty e^{-kx}\,dx = \lim_{N\to\infty}\int_0^N e^{-kx}\,dx = \frac{1}{k}.$$

We can abbreviate this calculation as follows:

$$\begin{aligned}\int_0^\infty e^{-kx}\,dx &= -\frac{1}{k}e^{-kx}\Big|_0^\infty \\ &= -\frac{1}{k}e^{-\infty} - -\frac{1}{k}e^0 \quad \text{(using the fact that } k > 0\text{)} \\ &= -0 + \frac{1}{k} \\ &= \frac{1}{k}\end{aligned}$$

Question: What if the limit is infinity?

Answer: Good question. There's a difference between the limit existing and the limit being inifinte. Where our definition of improper integral says "if this limit exists" it means "exists and is finite"; if infinite limits are allowed they're mentioned explicitly, as in l'Hôpital's rule.

There is another part of this subject which we will not study here. If f changes sign (e.g. $f(x) = \frac{\sin x}{x}$) there can be some cancellation in the integral as f oscillates. Sometimes the limit exists, but the total area enclosed above and below the x-axis is infinite. In order to avoid this possibility we require that $f(x) > 0$.

Physical Interpretation

The number of radioactive particles in some radioactive substance that decay in time $0 \le t \le T$ is given (on average) by:

$$\int_0^T Ae^{-kt}\,dt.$$

If we let T go to infinity we get:

$$\int_0^{\infty} Ae^{-kt}\,dt = \frac{A}{k} = \text{ total number of particles.}$$

The notion that T goes to infinity is an idealization; we're not actually going to wait forever for the substance to decay. However, it's useful for us to write down and use this quantity even if it's not physically realistic. One reason to use this value as opposed to those for finite time intervals is that $\frac{A}{k}$ is much easier to work with than $-\frac{A}{k}e^{-kT} + \frac{A}{k}$.

Example: $\int_{-\infty}^{\infty} e^{-x^2}\,dx$

Another famous improper integral is:

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

This is a key number in probability. It is used to understand standard deviation, predict outcomes of elections, calculate insurance rates, and estimate income from lotteries.

This number was first calculated numerically around the end of the the seventeenth century by de Moivre, who was selling his services to various royalty who were running lotteries. Although he didn't know the exact value of $\sqrt{\pi}$, he approximated the integral well enough to predict how much money their lotteries would make.

Example: $\int_1^\infty \frac{dx}{x}$

Our next examples of indefinite integrals come close to the dividing line between infinite integral values and finite ones. We're exploring this boundary because we often want to know how large a variable value we must account for before we can ignore the rest.

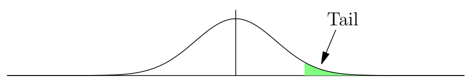

Figure 1: The tail of e^{-x^2}.

For example, when calculating probabilities the area of the "tail" of the normal curve gives the probability of an extreme value of x arising. If the area of the tail is negligible you don't have to compute it, but if it's not negligible (if it's a "fat tail") and you don't take it into account you can get a nasty surprise — like the recent mortgage scandal. The job of a mathematician is to know what finite regions require careful calculation and what areas are too small to affect the final outcome.

How "fat" does a tail have to be before its area becomes infinite and overwhelms the area of the central body? The borderline cases are x^p, where p is negative. Let's start by looking at the case for which $p = 1$; in other words:

$$\int_1^\infty \frac{dx}{x}.$$

As usual we start by computing the integral from 1 to N and then let N go to infinity.

$$\begin{aligned} \int_1^N \frac{dx}{x} &= \ln x\Big|_1^N \\ &= \ln N - \ln 1 \\ &= \ln N - 0 \end{aligned}$$

As N goes to infinity so does $\int_1^N \frac{dx}{x}$, so we conclude that:

$$\int_1^\infty \frac{dx}{x} \quad \text{diverges.}$$

Example: $\displaystyle\int_1^\infty \frac{dx}{x^p}$

We know that $\int_1^\infty \frac{dx}{x}$ diverges. Next we'll find $\int_1^\infty \frac{dx}{x^p}$ for any value of p; we'll see that $p = 1$ is a borderline when we do this calculation.

$$\begin{aligned}
\int_1^\infty \frac{dx}{x^p} &= \int_1^\infty x^{-p}dx \\
&= \left.\frac{x^{-p+1}}{-p+1}\right|_1^\infty \\
&= \frac{\infty^{-p+1}}{-p+1} - \frac{1^{-p+1}}{-p+1} \\
&= \frac{\infty^{-p+1}}{-p+1} + \frac{1}{p-1}
\end{aligned}$$

Remember that the ∞ in this expression is shorthand for "a number approaching infinity".

When we think about raising a very large number to the $p+1$ power we see that there are two cases that split exactly at $p = 1$. When $p = 1$, the exponent is zero and so is the denominator; the expression doesn't make any sense. For all other values of p the expression makes sense and the value of the integral depends on whether $-p+1$ is positive or negative.

$$\frac{\infty^{-p+1}}{-p+1} \quad \text{is infinite when } -p+1 > 0$$

and

$$\frac{\infty^{-p+1}}{-p+1} \quad \text{is zero when } -p+1 < 0.$$

Check this yourself — this is the sort of problem that will be on the exam.

Conclusion: Combining this with our previous example we see that:

$$\int_1^\infty \frac{dx}{x^p} \quad \text{diverges if } p \leq 1$$

and

$$\int_1^\infty \frac{dx}{x^p} \quad \text{converges to } \tfrac{1}{p-1} \text{ if } p > 1.$$

Notice that when $p = 1$ our formula for the antiderivative is wrong; the antiderivative is $\ln x$ and not $\frac{x^{-p+1}}{-p+1}$. We really needed to do three separate calculations to compute the value of this integral: one for $p < 1$, one for $p = 1$ and one for $p > 1$.

Introduction to Limit Comparison

Recall that we're trying to find out whether the tail of a function is fat or thin — whether we can safely ignore the values of the function after a certain point or whether the area under the graph of the function is infinite. If you can't directly compute the area of the tail you can use *limit comparison* to compare your function to some function whose values you are able to compute.

If $f(x) \sim g(x)$ as x goes to infinity then $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ either both converge or both diverge.

(Remember that "$f \sim g$ as x goes to infinity" means that $\frac{f(x)}{g(x)} \to 1$ as $x \to \infty$.)

Example: $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x^2+10}}$

Since $\sqrt{x^2+10} \sim \sqrt{x^2} = x$, we can compare this integral to $\int_1^\infty \frac{dx}{x}$. (Because $1/x$ is singular at $x = 0$ we'll start by ignoring the finite value $\int_0^1 \frac{dx}{\sqrt{x^2+10}}$ — if it turns out that the integral converges we can add it later.)

$$\int_1^\infty \frac{dx}{\sqrt{x^2+10}} \sim \int_1^\infty \frac{dx}{x}$$

We know that $\int_1^\infty \frac{dx}{x}$ diverges, so limit comparison tells us that $\int_0^\infty \frac{dx}{\sqrt{x^2+10}}$ diverges as well.

Question: Why did we switch from 0 to 1?

Answer: Because $\int_0^1 \frac{dx}{x}$ is infinite for unrelated reasons, and we didn't want that to affect our results. The interval between 0 and 1 is not the part of the function that we care about. What we're really worried about is what the area of the tail of the function is. We could just as well have done the comparison:

$$\int_{100}^\infty \frac{dx}{\sqrt{x^2+10}} \sim \int_{100}^\infty \frac{dx}{x}$$

which leads us to the same conclusion — the area of the tail of the graph of $\frac{1}{\sqrt{x^2+10}}$ is infinite.

Example: $\int_{10}^{\infty} \frac{dx}{\sqrt{x^3+3}}$

We could have used a trig substitution to compute $\int_0^{\infty} \frac{dx}{\sqrt{x^2+10}}$ in the previous example. We can use the limit comparison method to determine whether an integral is finite even if we're unable to find an antiderivative.

For instance, we can't evaluate $\int_{10}^{\infty} \frac{dx}{\sqrt{x^3+3}}$. But because:

$$\frac{1}{\sqrt{x^3+3}} \cong \frac{1}{\sqrt{x^3}} = \frac{1}{x^{3/2}}$$

we know that:

$$\int_{10}^{\infty} \frac{dx}{\sqrt{x^3+3}} \cong \int_{10}^{\infty} \frac{dx}{x^{3/2}}$$

and so we know that the integral converges to some finite value.

Example: $\int_{-\infty}^{\infty} e^{-x^2}\,dx$

We've been told that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$. We can't compute the exact value of this integral, but *can* use a simple comparison to check that the value is finite.

We start by using the fact that this is an even function, symmetric about the y-axis, to rewrite the integral as:

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \int_{0}^{\infty} e^{-x^2}\,dx.$$

The function $f(x) = e^{-x^2}$ goes to zero so quickly that we can't find a function $g(x)$ that's comparable to $f(x)$ for a limit comparison, so we'll have to use an ordinary comparison to determine whether this improper integral converges.

Because $x^2 \geq x$ when $x \geq 1$, we know that $-x^2 \leq -x$ and $e^{-x^2} \leq e^{-x}$ for $x \geq 1$. To show that $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ converges we split the integral again between $x > 1$ and $x < 1$. We compare integrals using our understanding that increasing the integrand increases the value of the integral:

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-x^2}\,dx &= 2\int_{0}^{\infty} e^{-x^2}\,dx \\
&= 2\int_{0}^{1} e^{-x^2}\,dx + 2\int_{1}^{\infty} e^{-x^2}\,dx \\
&\leq 2\int_{0}^{1} e^{-x^2}\,dx + 2\int_{1}^{\infty} e^{-x}\,dx \quad \text{(larger integrand)}
\end{aligned}$$

Since $\int_0^1 e^{-x^2}\,dx$ is finite and $\int_1^{\infty} e^{-x}\,dx$ converges, we conclude that $\int_0^{\infty} e^{-x^2}\,dx$ converges.

Ordinary comparison is a good tool for proving the convergence of integrals whose integrands decay very rapidly.

Indefinite Integrals over Singularities

When computing $\int_0^\infty \frac{dx}{\sqrt{x^2+10}}$ we had to take an extra step to avoid the integral $\int_0^1 \frac{dx}{x}$. We'll now go back and discuss integration near singular points.

Integrals like $\int_0^1 \frac{dx}{x}$ are known as *indefinite integrals of the second type*. Examples include:

$$\int_0^1 \frac{dx}{\sqrt{x}}, \quad \int_0^1 \frac{dx}{x}, \quad \text{and} \quad \int_0^1 \frac{dx}{x^2}.$$

These integrals turn out to be fairly straightforward to calculate:

$$\begin{aligned} \int_0^1 \frac{dx}{\sqrt{x}} &= \int_0^1 x^{-1/2} dx \\ &= \left. \frac{1}{1/2} x^{1/2} \right|_0^1 \\ &= \left. 2x^{1/2} \right|_0^1 \\ &= 2 \cdot 1^{1/2} - 2 \cdot 0^{1/2} \\ &= 2. \end{aligned}$$

$$\begin{aligned} \int_0^1 \frac{dx}{x} &= \ln x|_0^1 \\ &= \ln 1 - \ln 0 \quad \text{(diverges.)} \end{aligned}$$

$$\begin{aligned} \int_0^1 \frac{dx}{x^2} &= -x^{-1}\Big|_0^1 \\ &= -\frac{1}{1} - \left(-\frac{1}{0}\right) \quad \text{(diverges.)} \end{aligned}$$

However, you can get into trouble if you're not careful. Consider the following calculation:

$$\begin{aligned} \int_{-1}^1 \frac{dx}{x^2} &= -x^{-1}\Big|_{-1}^1 \\ &= -(1^{-1}) - (-(-1)^{-1}) \\ &= -1 - 1 \\ &= -2. \end{aligned}$$

This is ridiculous! As we see from Figure 1, $\frac{1}{x^2}$ is always positive. The area under the graph of $y = \frac{1}{x^2}$ between -1 and 1 is clearly greater than 2; in particular it cannot be a negative number.

In fact, the area under the graph of $y = \frac{1}{x^2}$ between -1 and 1 is infinite, not -2. The calculation above is nonsense.

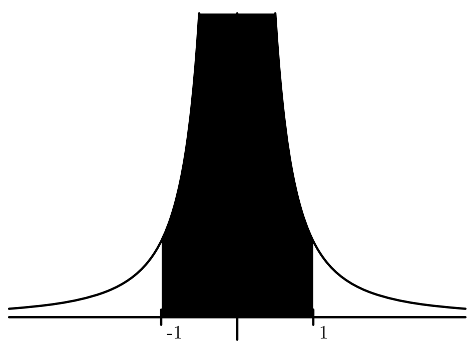

Figure 1: Graph of $y = \frac{1}{x^2}$.

Improper Integrals of the Second Kind, Continued

We'll continue our discussion of integrals of functions which have singularities at finite values; for example, $f(x) = \frac{1}{x}$. If $f(x)$ has a singularity at 0 we define

$$\int_0^1 f(x)\,dx = \lim_{a\to 0^+} \int_a^1 f(x)\,dx.$$

As before, we say the integral *converges* if this limit exists and *diverges* if not.

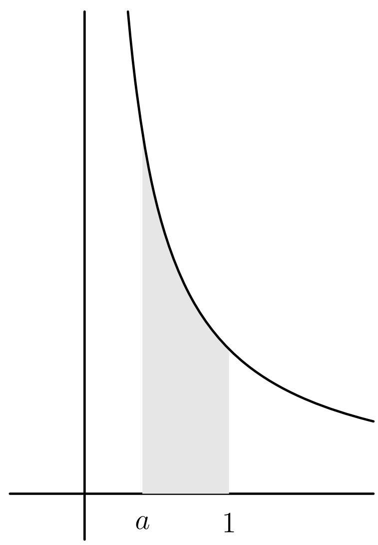

Figure 1: Area under the graph of $y = \frac{1}{x}$.

We treat this infinite vertical "tail" the same way we treated horizontal tails. Figure 1 shows a function whose value goes to positive infinity as x goes to zero from the right hand side. We don't know whether the area under its graph between 0 and 1 is going to be infinite or finite, so we cut it off at some point a where we know it will be finite. Then we let a go to zero from above ($a \to 0^+$) and see whether the area under the curve between a and 1 goes to infinity or to some finite limit.

Example: $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$

$$\begin{aligned}
\int_0^1 \frac{dx}{\sqrt{x}} &= \int_0^1 x^{-1/2}\,dx \\
&= \frac{1}{1/2}x^{1/2}\Big|_0^1 \\
&= 2x^{1/2}\Big|_0^1
\end{aligned}$$

$$
\begin{aligned}
&= 2 \cdot 1^{1/2} - 2 \cdot 0^{1/2} \\
&= 2.
\end{aligned}
$$

This is a convergent integral.

Example: $\displaystyle\int_0^1 \frac{dx}{x}$

$$
\begin{aligned}
\int_0^1 \frac{dx}{x} &= \ln x\big|_0^1 \\
&= \ln 1 - \ln 0^+ \\
&= 0 - (-\infty) \\
&= +\infty.
\end{aligned}
$$

This integral diverges.

In general:

$$
\begin{aligned}
\int_0^1 \frac{dx}{x^p} &= \left.\frac{x^{-p+1}}{-p+1}\right|_0^1 \quad (\text{for } p \neq 1) \\
&= \frac{1^{-p+1}}{-p+1} - \frac{0^{-p+1}}{-p+1} \\
&= \begin{cases} \dfrac{1}{1-p} & \text{if } p < 1 \\[2ex] \text{diverges} & \text{if } p \geq 1. \end{cases}
\end{aligned}
$$