## Review of Trigonometric Identities

The topic of this segment is the use of trigonometric substitutions in integration. We start by reviewing some basic facts about trigonometry.

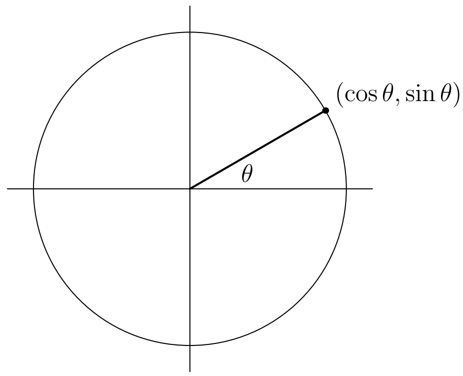


Figure 1: The unit circle.

Trigonometry is based on the circle of radius 1 centered at $(0, 0)$. A point on that circle at angle $\theta$ (see Figure **??**fig:l27g1) has coordinates $(\cos\theta, \sin\theta)$. Because the radius of the circle is 1, the Pythagorean theorem tells us right away that $\sin^2\theta + \cos^2\theta = 1$. (Remember that $\sin^2\theta$ means $(\sin\theta)^2$.) You may also remember some double angle formulas.

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \\ \cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ \sin(2\theta) &= 2\sin\theta\cos\theta \end{aligned}$$

From the double angle formula for $\cos(2\theta)$ we can derive the half angle formula:

$$\begin{aligned} \cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= \cos^2\theta - (1 - \cos^2\theta) \\ \cos(2\theta) &= 2\cos^2\theta - 1 \\ \Rightarrow \cos^2\theta &= \frac{1 + \cos(2\theta)}{2} \end{aligned}$$

This formula will allow us to rewrite powers like $\cos^2\theta$ in lower degree terms. A similar calculation shows that:

$$\sin^2\theta = \frac{1 - \cos(2\theta)}{2}.$$

## $\int \sin^n x \cos^m x \, dx, m = 1$

You already know something about integrating trigonometric functions; you can reverse anything you know about the derivative of a trigonometric function to get a fact about antiderivatives.

$$\begin{aligned} d\sin x = \cos x\, dx \quad &\Rightarrow \quad \int \cos x\, dx = \sin x + c \\ d\cos x = -\sin x\, dx \quad &\Rightarrow \quad \int \sin x\, dx = -\cos x + c \end{aligned}$$

Our plan is to use these two integration formulas and a few trig identities to derive more complicated formulas involving trig functions.

Our first topic is integrals of the form:

$$\int \sin^n x \cos^m x \, dx,$$

where $m$ and $n$ are non-negative integers. Integrals like this appear in Fourier series, among other places.

There are two cases to think about here. The easy case is the one in which at least one exponent is odd.

**Example:** $m = 1$

The trick in calculating $\int sin^n(x)\cos(x)\,dx$ is to make the substitution $u = \sin x$, so $du = \cos x\, dx$.

$$\begin{aligned} \int sin^n(x)\cos(x)\,dx &= \int u^n du \\ &= \frac{u^{n+1}}{n+1} + c \\ &= \frac{\sin^{n+1} x}{n+1} + c \end{aligned}$$

Although the answer $\frac{u^{n+1}}{n+1} + c$ looks nice, you need to reverse your substitution and plug in $\sin x$ for $u$ at the end because while you may know what $u$ is, your grader or employer does not.

What made this problem easy is that $\cos x$ is the derivative of $\sin x$.

**Example:** $\int \sin^3 x \cos^2 x dx$

One of the exponents is odd so this is an easy, but not as easy as the previous example. We turn this integral into one in which the odd exponent is 1 by using the trig identity $\sin^x x + \cos^2 x = 1$ to remove the largest even power in the term with the odd exponent.

In this case the odd exponent is on $\sin x$, so we use:

$$\sin^2 x = 1 - \cos^2 x.$$

$$\begin{aligned} \int \sin^3 x \cos^2 x dx &= \int (\sin^2 x \cdot \sin x) \cos^2 x dx \\ &= \int (1 - \cos^2 x) \cdot \sin x \cos^2 x dx \\ &= \int (\cos^2 - \cos^4 x) \sin x dx \end{aligned}$$

This is now very similar to our previous example. We use the substitution:

$$\begin{aligned} u &= \cos x \\ du &= -\sin x dx \end{aligned}$$

To get:

$$\begin{aligned} \int \sin^3 x \cos^2 x &= \int (u^2 - u^4 x) \cdot (-du) \\ &= \int \frac{-u^3}{3} + \frac{u^6}{5} + c \\ &= -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + c \end{aligned}$$

At this point it's a good idea to check your work by differentiating your answer and then applying trig identities to see that the result equals the original integrand.

**Example:** $\int \sin^3 x dx$

The integral $\int \sin^3 x dx$ is of the form $\int \sin^n x \cos^m x dx$ with one exponent odd, and the other exponent equal to zero, so it is in the easy case. We again use the trig identity $\sin^2 x + \cos^2 x = 1$ to remove the largest power of $\sin x$ that we can from the cube:

$$\int \sin^3 x dx = \int (1 - \cos^2 x) \sin x dx$$

Substitute $u = \cos x$ and $du = -\sin x dx$ to get:

$$\begin{aligned} \int \sin^3 x dx &= \int (1 - u^2)(-du) \\ &= -u + \frac{u^3}{3} + c \\ &= -\cos x + \frac{\cos^3 x}{3} + c \end{aligned}$$

In general, any time you have an odd power in an integral of the form $\int \sin^n x \cos^m x dx$ you can integrate it using the trig identity $\sin^2 x + \cos^2 x = 1$ and a substitution.

**Example:** $\int \cos^2 x \, dx$

What if we have to integrate $\int sin^n x \cos^m x \, dx$ when both exponents are even? This is a harder case; we'll use the half angle formulas to solve it.

$$\cos^2 \theta = \frac{1 + \cos(2\theta)}{2}$$
$$\sin^2 \theta = \frac{1 - \cos(2\theta)}{2}$$

These formulas help us by turning even powers of $\sin x$ and $\cos x$ into odd powers of $\cos(2x)$.

If we wanted to integrate:

$$\int \cos^2 x \, dx,$$

we could rewrite it as $\int (1 - \sin^2 x) \, dx$, but the new integral is at least as hard as the one we started with. Instead we use a half angle formula:

$$\begin{aligned} \int \cos^2 x \, dx &= \int \frac{1 + \cos(2x)}{2} \, dx \\ &= \frac{x}{2} + \frac{\sin 2x}{4} + c \end{aligned}$$

Notice that $\frac{x}{2}$ appears in the solution and is not a trigonometric function!

**Example:** $\int \sin^2 x \cos^2 x \, dx$

To integrate $\sin^2 x \cos^2 x$ we once again use the half angle formulas:

$$\cos^2 \theta = \frac{1 + \cos(2\theta)}{2}$$
$$\sin^2 \theta = \frac{1 - \cos(2\theta)}{2}$$

It's a good idea to do your trigonometric and algebraic manipulations of the integrand off to the side on your paper, so that you don't have to continuously copy over (and maybe forget) the integral sign and the $dx$.

**Side work:**

$$\begin{aligned} \sin^2 x \cos^2 x &= \left(\frac{1 - \cos(2x)}{2}\right)\left(\frac{1 + \cos(2x)}{2}\right) \\ &= \frac{1 - \cos^2(2x)}{4} \end{aligned}$$

We still have a square, so we're still not in the easy case. But this is an easier "hard" case, especially since we just computed $\int \cos^2 x \, dx$. We could use that, but instead let's continue to use half angle formulas until we reach an easy case:

$$\begin{aligned} \sin^2 x \cos^2 x &= \frac{1 - \cos^2(2x)}{4} \\ &= \frac{1}{4} - \frac{1 + \cos(4x)}{4 \cdot 2} \\ &= \frac{1}{8} - \frac{\cos(4x)}{8} \end{aligned}$$

Once we've done the side work we substitute back into the original integral to get:

$$\begin{aligned} \int \sin^2 x \cos^2 x \, dx &= \int \left(\frac{1}{8} - \frac{\cos(4x)}{8}\right) dx \\ &= \frac{x}{8} - \frac{\sin(4x)}{8 \cdot 4} + c \end{aligned}$$

We should now be able to calculate any integral of the form $\int \sin^n x \cos^m x \, dx$.

Here's an alternate method of doing the side work using the identity

$$\sin(2\theta) = 2 \sin\theta \cos\theta.$$

$$
\begin{aligned}
\sin^2 x \cos^2 x &= (\sin x \cos x)^2 \\
&= \left(\frac{1}{2}\sin(2x)\right)^2 \\
&= \frac{1}{4}\sin^2(2x) \\
&= \frac{1}{4}\left(\frac{1-\cos(4x)}{2}\right) \\
\sin^2 x \cos^2 x &= \frac{1}{8} - \frac{\cos(4x)}{8}
\end{aligned}
$$

Using the double angle formula for the sine function reduces the number of factors of $\sin x$ and $\cos x$, but not quite far enough; it leaves us with a factor of $\sin^2(2x)$. Next, the half angle formula for the sine function allows us to reduce this to a constant minus a multiple of the cosine function.

Note that we get the same expression we did before.

## Area of Part of a Circle

Given a circle of radius $a$, cut out a tab of height $b$. What is the area of this tab? (See Figure 1.)

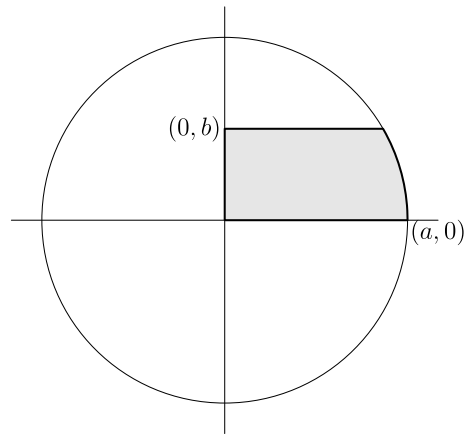


Figure 1: Tab cut out of a circle.

One way to compute the area would be split the area into vertical strips and integrate with respect to $x$:

$$\text{Area} = \int y\,dx.$$

This is awkward, because near the end the height of the region changes from a constant $y = b$ to the height of the circle $y = \sqrt{a^2 - x^2}$.

What if we integrate with respect to $y$? That seems to work better; there is a single simple expression for the length of each horizontal strip: $x = \sqrt{a^2 - y^2}$.

$$\begin{aligned} \text{Area} &= \int_0^b x\,dy \\ &= \int_0^b \sqrt{a^2 - y^2}\,dy \end{aligned}$$

We don't yet have a rule for integrating functions of this form. Considering that this integral arose from a question about a circle, it's not surprising that trigonometry will play a role in its solution.

When working with circles it often helps to use polar coordinates. In this case, note that the upper right hand corner of the region has polar coordinates $(a\cos\theta_0, a\sin\theta_0)$ where $\theta_0$ is the angle shown in Figure 2.

In general, $x = a\cos\theta$ and $y = a\sin\theta$. If we substitute $y = a\sin\theta$ into our integrand we get:

$$x = \sqrt{a^2 - y^2}$$

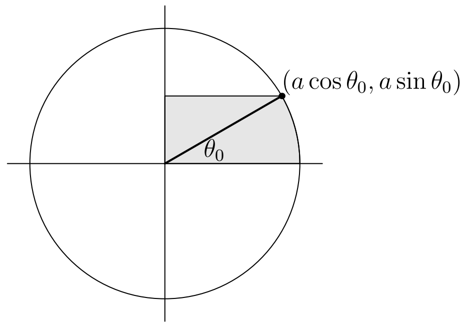


Figure 2: Polar coordinates of a point.

$$\begin{aligned}
&= \sqrt{a^2 - a^2\sin^2\theta} \\
&= a\sqrt{1-\sin^2\theta} \\
&= a\sqrt{\cos^2\theta} \\
x &= a\cos\theta
\end{aligned}$$

Changing to polar coordinates made our integrand look much nicer; we've gone from an integrand with a square root and no trig functions to an integrand with trig functions and no square root.

If we're going to use the substitution $y = a\sin\theta$ in our integral, we'll also need to replace $dy$ by something in polar coordinates.

$$\begin{aligned}
y &= a\sin\theta \\
dy &= a\cos\theta\, d\theta
\end{aligned}$$

Plugging in, we get:

$$\begin{aligned}
\int \sqrt{a^2-y^2}\, dy &= \int (a\cos\theta)(a\cos\theta\, d\theta) \\
&= a^2 \int \cos^2\theta\, d\theta
\end{aligned}$$

We computed the integral of $\cos^2 x$ earlier in the lecture:

$$\int \cos^2 x\, dx = \frac{x}{2} + \frac{\sin 2x}{4} + C.$$

Plugging this in, we get:

$$\int \sqrt{a^2-y^2} dy = a^2\left(\frac{\theta}{2} + \frac{\sin(2\theta)}{4}\right) + c.$$

Now we'd like to rewrite our solution in terms of the original variable $y$ so that we can plug in the limits of integration. In order to do this, it's helpful to rewrite $\sin(2\theta)$ using the double angle formula $\sin(2\theta) = 2\sin\theta\cos\theta$.

$$
\begin{aligned}
\int \sqrt{a^2 - y^2}\, dy &= a^2 \left( \frac{\theta}{2} + \frac{\sin(2\theta)}{4} \right) + c \\
&= a^2 \left( \frac{\theta}{2} + \frac{\sin\theta \cos\theta}{2} \right) + c \\
&= \left( \frac{a^2\theta}{2} + \frac{a \sin\theta a \cos\theta}{2} \right) + c.
\end{aligned}
$$

Next we solve $y = a\sin\theta$ for $y$ and plug in:

$$\theta = \arcsin\left(\frac{y}{a}\right).$$

Since $a\sin\theta = y$ and $a\cos\theta = x = \sqrt{a^2 - y^2}$, we get:

$$\int \sqrt{a^2 - y^2}\, dy = \left( \frac{a^2 \arcsin(y/a)}{2} + \frac{y\sqrt{a^2 - y^2}}{2} \right) + c.$$

We've used trigonometric substitution to find the indefinite integral of $\sqrt{a^2 - y^2}$. Whenever you see the square root of a quadratic in an integral you should think of trigonometry and $\sin^2\theta + \cos^2\theta$.

Our original problem asked us to compute the value of a definite integral; let's finish that.

$$
\begin{aligned}
\int_0^b \sqrt{a^2 - y^2}\, dy &= \left( \frac{a^2 \arcsin(y/a)}{2} + \frac{y\sqrt{a^2 - y^2}}{2} \right) \Bigg|_0^b \\
&= \left( \frac{a^2 \arcsin(b/a)}{2} + \frac{b\sqrt{a^2 - b^2}}{2} \right) - 0 \\
&= \frac{a^2 \arcsin(b/a)}{2} + \frac{b\sqrt{a^2 - b^2}}{2}
\end{aligned}
$$

Notice that $\theta_0 = \arcsin(b/a)$; we could rewrite this answer as:

$$\text{Area} = \frac{a^2\theta_0}{2} + \frac{b\sqrt{a^2 - b^2}}{2}$$

Does this make sense? The first term, $\frac{\theta_0}{2}a^2$, is exactly the area of the sector of the circle swept out by angle $\theta_0$. The second term, $\frac{1}{2}b\sqrt{a^2 - b^2}$, is the area of a triangle with base $b$ and height $\sqrt{a^2 - b^2}$. In other words, it's the area of the shaded triangle shown in Figure 2.

Using some basic geometry, we've checked that our answer to this complicated calculus problem is correct.

## Review of Trigonometric Identities

We've talked about trig integrals involving the sine and cosine functions. Now we'll look at trig functions like secant and tangent. Here's a quick review of their definitions:

$$\sec x = \frac{1}{\cos x} \qquad \tan x = \frac{\sin x}{\cos x} \tag{1}$$

$$\tag{2}$$

$$\csc x = \frac{1}{\sin x} \qquad \cot x = \frac{\cos x}{\sin x} \tag{3}$$

When you put a "co" in front of the name of the function, that exchanges the roles of sine and cosine in that function.

We have the following identities:

$$\begin{aligned} \sec^2 x &= 1 + \tan^2 x \\ \frac{d}{dx}\tan x &= \sec^2 x \\ \frac{d}{dx}\sec x &= \sec x \tan x \end{aligned}$$

We can verify these using familiar trig identities involving $\sin x$ and $\cos x$.

$$\begin{aligned} \sec^2 x &= \frac{1}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ \sec^2 x &= 1 + \tan^2 x \end{aligned}$$

This is the main trig identity behind what we'll do today.

$$\begin{aligned} \frac{d}{dx}\tan x &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\ &= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} \quad \text{(chain rule)} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} \\ \frac{d}{dx}\tan x &= \sec^2 x \end{aligned}$$

From this we get our first integral of the day:

$$\int \sec^2 x \, dx = \tan x + c.$$

$$
\begin{aligned}
\frac{d}{dx}\sec x &= \frac{d}{dx}\frac{1}{\cos x} \\
&= \frac{0-(-\sin x)}{\cos^2 x} \\
&= \frac{\sin x}{\cos^2 x} \\
\frac{d}{dx}\sec x &= \tan x \sec x
\end{aligned}
$$

Should we ever need an antiderivative of $\tan x \sec x$ we now have one.

## Integral of Tangent

How do we integrate one of these trig functions if we can't work backward from a derivative we already know?

$$\int \tan x \, dx \quad = \quad \int \frac{\sin x}{\cos x} \, dx$$

If you're working an integral like this and you see a trig function, it's good to look around and see if you can also find the derivative of that trig function. We make the substitution:

$$u = \cos x, \quad du = -\sin x \, dx$$

and rewrite our integral as:

$$\begin{aligned}
\int \tan x \, dx &= \int \frac{\sin x \, dx}{\cos x} \\
&= \int \frac{-du}{u} \\
&= -\ln|u| + c \\
\int \tan x \, dx &= -\ln|\cos(x)| + c
\end{aligned}$$

You'll find tables of formulas like this in the back of most textbooks. In addition, there is a certain amount of memorization that goes on in calculus; this is the kind of thing that you probably want to memorize.

## Integral of Secant

$$\int \sec x \, dx = ?$$

This calculation is not as straightforward as the one for the tangent function. What we need to do is add together the formulas for the derivatives of the secant and tangent functions.

$$\begin{aligned}
\frac{d}{dx}(\sec x + \tan x) &= \sec^2 x + \sec x \tan x \\
&= (\sec x)(\sec x + \tan x)
\end{aligned}$$

Notice that $\sec x + \tan x$ appears on both sides of the equation here. If we let $u = \sec x + \tan x$ and substitute, our equation becomes:

$$u' = u \cdot \sec x.$$

Which tells us that:

$$\sec x = \frac{u'}{u}.$$

We've seen this before; this is called the *logarithmic derivative*:

$$\frac{u'}{u} = \ln(u).$$

Putting this all together in order, we get:

$$\begin{aligned}
\sec x &= \frac{u'}{u} \quad (u = \sec x + \tan x) \\
&= \frac{d}{dx} \ln u \\
\sec x &= \frac{d}{dx} \ln(\sec x + \tan x).
\end{aligned}$$

Integrating both sides, we get:

$$\int \sec x \, dx = \ln(\sec x + \tan x) + c.$$

By taking the derivative of exactly the right function and looking at the results in the right way we got the formula we needed. You won't be expected to do this yourself in this class.

## Summary of Trig Integration

We now know the following facts about trig functions and calculus:

$$\sec x = \frac{1}{\cos x} \qquad \tan x = \frac{\sin x}{\cos x} \qquad \sin^2 x + \cos^2 x = 1$$

$$\csc x = \frac{1}{\sin x} \qquad \cot x = \frac{\cos x}{\sin x} \qquad \sec^2 x = 1 + \tan^2 x$$

$$\tfrac{d}{dx} \tan x = \sec^2 x \qquad \tfrac{d}{dx} \sec x = \sec x \tan x \qquad \textstyle\int \tan x \, dx = -\ln|\cos(x)| + c$$

$$\tfrac{d}{dx} \sin x = \cos x \qquad \tfrac{d}{dx} \cos x = -\sin x \qquad \textstyle\int \sec x \, dx = \ln(\sec x + \tan x) + c$$

We've also seen several useful integration techniques, including methods for integrating any function of the form $\sin^n x \cos^m x$. At this point we have the tools needed to integrate most trigonometric polynomials.

**Example:** $\displaystyle\int \sec^4 x \, dx$

We can get rid of some factors of $\sec x$ using the identity $\sec^2 x = 1 + \tan^2 x$. This is a particularly good idea because $\sec^2 x$ is the derivative of $\tan x$.

$$\begin{aligned} \int \sec^4 x \, dx &= \int (1 + \tan^2 x) \sec^2 x \, dx \\ &= \int (1 + \tan^2 x) \sec^2 x \, dx. \end{aligned}$$

Using the substitution $u = \tan x$, $du = \sec^2 x \, dx$, we get:

$$\begin{aligned} \int \sec^4 x \, dx &= \int (1 + u^2) du \\ &= u + \frac{u^3}{3} + c \\ \int \sec^4 x \, dx &= \tan x + \frac{\tan^3 x}{3} + c. \end{aligned}$$

**Example of Trig Substitution:** $\displaystyle\int \frac{dx}{x^2\sqrt{1+x^2}}$

$$\int \frac{dx}{x^2\sqrt{1+x^2}} = ?$$

This is an ugly integral. The square root is the ugliest part, so we'll try to rewrite it in such a way that we can get rid of the square. If we let $x = \tan\theta$ then the identity $\sec^2\theta = 1 + \tan^2\theta$ will allow this. We'll then have $dx = \sec^2\theta\, d\theta$:

$$\begin{aligned}
\int \frac{dx}{x^2\sqrt{1+x^2}} &= \int \frac{\sec^2\theta\, d\theta}{\tan^2\theta\sqrt{1+\tan^2\theta}} \\
&= \int \frac{\sec^2\theta\, d\theta}{\tan^2\theta\sqrt{\sec^2\theta}} \\
&= \int \frac{\sec^2\theta\, d\theta}{\tan^2\theta\sec\theta} \\
&= \int \frac{\sec\theta\, d\theta}{\tan^2\theta}
\end{aligned}$$

When faced with an assortment of different trig functions like this one, it's a good idea to rewrite everything in terms of $\sin\theta$ and $\cos\theta$:

$$\begin{aligned}
\int \frac{dx}{x^2\sqrt{1+x^2}} &= \int \frac{\frac{1}{\cos\theta}\, d\theta}{\frac{\sin^2\theta}{\cos^2\theta}} \\
&= \int \frac{\cos^2\theta\, d\theta}{\cos\theta\sin^2\theta} \\
&= \int \frac{\cos\theta\, d\theta}{\sin^2\theta}
\end{aligned}$$

The ugliest part of this integral is the $\sin^2\theta$ in the denominator. Since $\cos\theta\, d\theta$ is the derivative of $\sin\theta$, we make the substitution $u = \sin\theta$, $du = \cos\theta\, d\theta$:

$$\begin{aligned}
\int \frac{dx}{x^2\sqrt{1+x^2}} &= \int \frac{\cos\theta\, d\theta}{\sin^2\theta} \\
&= \int \frac{du}{u^2} \\
&= -\frac{1}{u} + c
\end{aligned}$$

Now we have to reverse our substitutions:

$$\begin{aligned}
\int \frac{dx}{x^2\sqrt{1+x^2}} &= -\frac{1}{\sin\theta} + c \\
&= -\csc\theta + c
\end{aligned}$$

It's not clear how to undo the substitution $x = \tan\theta$. Luckily there is a general method for undoing substitutions like this, which is to go back to thinking of trig functions as ratios of side lengths of a right triangle.

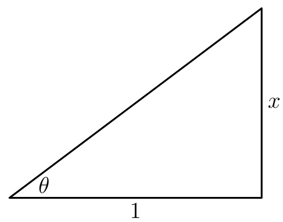


Figure 1: Undoing trig substitution.

We know $x = \tan\theta$ and we know that $\tan\theta$ equals the length of the leg opposite $\theta$ divided by the length of the leg adjacent to $\theta$. Figure 1 shows a right triangle with an angle $\theta$, an opposite leg of length $x$, and an adjacent leg of length 1.

The Pythagorean theorem tells us that the hypotenuse must have length $\sqrt{1+x^2}$. Now we can deduce that:

$$\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{\sqrt{1+x^2}}{x}.$$

Hence,

$$\int \frac{dx}{x^2\sqrt{1+x^2}} \quad = \quad -\frac{\sqrt{1+x^2}}{x} + c.$$

In the process of computing this integral we saw the following: trig substitution, rewriting trig functions in terms of sine and cosine, direct substitution, and undoing trig substitution.

What actually happened when we undid that trig substitution was that we computed $\csc(\arctan(x))$. In other words, we composed a trig function with the inverse of another trig function.

## Undoing Trig Substitution

Professor Miller plays a game in which students give him a trig function and an inverse trig function, and then he tries to compute their composition. As we've seen, this is sometimes the final step in integration by trig substitution.

$$\_\_(\text{arc}\_\_x) = ?$$

**Example:** $\tan(\operatorname{arccsc} x) = ?$

**Question:** Isn't $\tan(\operatorname{arccsc} x)$ acceptable as a final answer?

**Answer:** What does "acceptable" mean? The expression $-\csc(\arctan x)$ was a *correct* final answer, but $\dfrac{\sqrt{1+x^2}}{x}$ is a nicer, more insightful, and probably more useful answer.

To simplify $\tan(\operatorname{arccsc} x)$ we draw a triangle illustrating an angle whose cosecant is $x$; see Figure 1. We know that

$$x = \csc\theta = \frac{1}{\sin\theta} = \frac{\text{hyp}}{\text{opp}}$$

so we choose convenient values $x$ and 1 to be the lengths of the hypotenuse and opposite side.

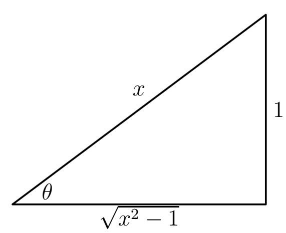


Figure 1: $\theta = \operatorname{arccsc} x$ so $x = \csc\theta$.

Once we've drawn our triangle we can compute that the length of the adjacent side must be $\sqrt{x^2-1}$, and so

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{1}{\sqrt{x^2-1}}.$$

Since $x = \csc\theta$, we have:

$$\tan(\operatorname{arccsc} x) = \tan\theta = \frac{1}{\sqrt{x^2-1}}.$$

Whenever you have to undo a trig substitution, this technique is likely to be useful.

## Summary of Trig Substitution

Here is a table of different trig substitutions and how they can be useful.

| If your integrand contains | Make substitution | To get |
|---|---|---|
| $\sqrt{a^2 - x^2}$ | $x = a\cos\theta$ or $x = a\sin\theta$ | $a\sin\theta$ or $a\cos\theta$ |
| $\sqrt{a^2 + x^2}$ | $x = a\tan\theta$ | $a\sec\theta$ |
| $\sqrt{x^2 - a^2}$ | $x = a\sec\theta$ | $a\tan\theta$ |

These are the three basic forms which are integrated using trig substitution. In general, you use trig substitution to replace the square root of a quadratic function by a trigonometric function. Once you've done this, integrate, then use what we've learned about right triangles and undoing trig substitution to get a final answer.

## Completing the Square

The last thing we need to discuss on the topic of trig substitution is completing the square. This is necessary when we want to integrate the square root of a quadratic whose form is not as simple as $ax^2 + c$.

**Example:** $\displaystyle\int \frac{dx}{\sqrt{x^2 + 4x}}$

The integrand contains the square root of a quadratic, but it doesn't match any of the forms in our table of trig substitutions. What we need to do is use substitution to rewrite it in a form that *is* in our table.

We'll start by trying to write the quadratic in the form:

$$(x + a)^2 + c.$$

$$\begin{aligned} x^2 + 4x &= (x + a)^2 + c \\ &= x^2 + 2xa + a^2 + c \end{aligned}$$

Since the coefficient on $x$ on the left must equal that on the right, it must be true that $2xa = 4x$; i.e. $a = 2$. Similarly, we get $a^2 + c = 0$ or $c = -4$. So:

$$x^2 + 4 = (x + 2)^2 - 4.$$

This process of eliminating the "middle term" using the square of a linear function is called *completing the square*. When we plug this into our integral, it becomes:

$$\int \frac{dx}{\sqrt{x^2 + 4x}} = \int \frac{dx}{\sqrt{(x + 2)^2 - 4}}$$

This is closer to the forms listed in our trig substitution table, but a direct substitution of $u = x + 2$ brings us even closer:

$$\int \frac{dx}{\sqrt{x^2 + 4x}} = \int \frac{du}{\sqrt{u^2 - 4}}$$

We now consult our table and perform the trig substitution $u = 2\sec\theta$, $du = 2\sec\theta\tan\theta\, d\theta$.

$$\begin{aligned} \int \frac{dx}{\sqrt{x^2 + 4x}} &= \int \frac{2\sec\theta\tan\theta\, d\theta}{\sqrt{4\sec^2\theta - 4}} \\ &= \int \frac{2\sec\theta\tan\theta\, d\theta}{2\sqrt{\tan^2\theta}} \\ &= \int \frac{\sec\theta\tan\theta\, d\theta}{\tan\theta} \\ &= \int \sec\theta\, d\theta \\ &= \ln(\sec\theta + \tan\theta) + c \end{aligned}$$

We've completed the integration, but we still need to reverse our two substitutions. We readily see that $\sec\theta = \frac{u}{2}$, and in the process of computing the integral we calculated that $\sqrt{u^2-4} = 2\tan\theta$, so $\tan\theta = \dfrac{\sqrt{u^2-4}}{2}$. Hence we can avoid drawing a triangle and say:

$$\int \frac{dx}{\sqrt{x^2+4x}} = \ln\left(\frac{u}{2} + \frac{\sqrt{u^2-4}}{2}\right) + c$$

Finally, we replace $u$ by $x+2$ to reverse the first substitution:

$$\begin{aligned}
\int \frac{dx}{\sqrt{x^2+4x}} &= \ln\left(\frac{x+2}{2} + \frac{\sqrt{(x+2)^2-4}}{2}\right) + c \\
&= \ln\left(\frac{x+2}{2} + \frac{\sqrt{x^2+4x}}{2}\right) + c
\end{aligned}$$