Partial Fractions

Today we'll learn how to integrate functions of the form:

$$\frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials. Functions of this type are called *rational functions*. The technique for integrating functions of this type is called the method of *partial fractions*.

The method of partial fractions works by algebraically splitting $P(x)/Q(x)$ into pieces that are easier to integrate.

Example:

$$\int \left(\frac{1}{x-1} + \frac{3}{x+2} \right) dx = \ln|x-1| + 3\ln|x+2| + c$$

That was an easy integral. Next we'll see how the same integral might become difficult; if we add the two fractions together we get the same problem in a more challenging form:

$$\begin{aligned}
\frac{1}{x-1} + \frac{3}{x+2} &= \frac{1}{x-1} \cdot \frac{x+2}{x+2} + \frac{3}{x+2} \cdot \frac{x-1}{x-1} \\
&= \frac{(x+2)+3(x-1)}{(x-1)(x+2)} \\
&= \frac{4x-1}{x^2+x-2} \\
\int \frac{4x-1}{x^2+x-2}\, dx &= \ ?
\end{aligned}$$

This integral is algebraically the same as the one we just computed; that was easy, this one looks harder. The method of partial fractions involves algebraically manipulating integrals like the harder one to make them easier.

Introduction to the Cover-up Method

To integrate a rational function using the partial fractions method we must algebraically break it into parts. We're going to helped in this by a shortcut called the *cover-up method*.

We've disguised the easy integral:

$$\int \left(\frac{1}{x-1} + \frac{3}{x+2} \right) dx$$

as a harder one:

$$\int \frac{4x-1}{x^2+x-2}\, dx.$$

We'll use the method of partial fractions to unwind this disguise.

1. Our first step is to write down the integrand. Then we begin to undo the damage that we did by factoring the denominator (this can be a rather difficult step).
$$\frac{4x-1}{x^2+x-2} = \frac{4x-1}{(x-1)(x+2)}$$

2. Next we "set up" some unknowns, preparing to break the rational expression into pieces whose denominators are the factors we just found
$$\frac{4x-1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

3. The third step is to solve for the numerators; in this example they are A and B. Once we've completed this we've unwound the disguise.

The cover-up method makes step 3 more efficient and less clumsy.

We solve for A by multiplying by $x-1$.

$$\frac{4x-1}{x+2} = A + \frac{B}{x+2}(x-1)$$

Notice that we didn't try to clear the denominators completely; we just cleared one factor, which was all we needed to do to get A by itself.

If we plug in $x = 1$ we get:

$$\begin{aligned} \frac{4-1}{1+2} &= A + 0 \\ \frac{3}{3} &= A \\ A &= 1 \end{aligned}$$

This is not a surprise — we saw at the start of the lecture that:

$$\frac{4x-1}{x^2+x-2} = \frac{1}{x-1} + \frac{3}{x+2}.$$

Question: Why did you choose $x = 1$?

Answer: Because it works really fast — that's part of the cover-up method. Notice that if we'd set $x = 1$ in the original equation we'd have gotten a denominator of zero; that wouldn't have helped us at all.

What we did was multiply both sides by $x - 1$ and then immediately set $x = 1$; that's like mutiplying both sides by zero! It turns out to be OK though because the equation is true *except* when $x = 1$, so what we're really getting is the limit as x approaches 1.

Setting $x = 1$ helps us by canceling out the term with the variable B in it.

We're going to learn a quicker way to do this in a second, but first let's find the value of B. To isolate B we multiply both sides by $x + 2$:

$$\begin{aligned} \frac{4x-1}{(x-1)(x+2)} &= \frac{A}{x-1} + \frac{B}{x+2} \\ \frac{4x-1}{x-1} &= \frac{A}{x-1}(x+2) + B \end{aligned}$$

Then we plug in $x = -2$:

$$\begin{aligned} \frac{4(-2)-1}{(-2)-1} &= 0 + B \\ \frac{-9}{-3} &= 0 + B \\ B &= 3 \end{aligned}$$

We can now replace A and B by the values we've found to conclude that:

$$\frac{4x-1}{x^2+x-2} = \frac{1}{x-1} + \frac{3}{x+2}.$$

What we've seen here is why the cover-up method works. Next we'll see how we can make it work faster.

Question: Can we use this method if the exponents on x aren't whole numbers? Will it work on square roots?

Answer: This only works with polynomials, which have whole numbered powers of x. For example, x^2+x-2 has powers of 2, 1 and 0 in the denominator. It won't work with square roots.

The Cover-up Method

Here's how the cover-up method works in general:

1. Factor the denominator $Q(x)$,
2. Set-up (describe the target sum),
3. Cover-up (solve for unknown coefficients).

If we can go through these steps without writing writing down all the details of each one, the process will be much faster. It turns out that we can save a lot of time on step 3.

$$\overbrace{\frac{4x-1}{(x-1)(x+2)}}^{\text{Step 1:}} = \overbrace{\frac{A}{x-1}+\frac{B}{x+2}}^{\text{Step 2:}}$$

In Step 3 we cover up the $(x-1)$ in the denominator of $\frac{4x-1}{(x-1)(x+2)}$ and focus on the variable A (which is being divided by $x-1$). Plug the thing that makes $x-1=0$ (i.e. $x=1$) into the part of $\frac{4x-1}{(x-1)(x+2)}$ that's not covered up:

$$\begin{aligned} \frac{4x-1}{\cancel{(x-1)}(x+2)} &= \frac{A}{\cancel{x-1}} + \cancel{\frac{B}{x+2}} \\ \frac{4\cdot 1-1}{1+2} &= A \end{aligned}$$

This is equivalent to the algebra we did earlier, but much faster.

We do the same thing for B, plugging in the value $x=-2$ that makes the denominator associated with B equal to zero:

$$\begin{aligned} \frac{4x-1}{(x-1)\cancel{(x+2)}} &= \cancel{\frac{A}{x-1}} + \frac{B}{\cancel{x+2}} \\ \frac{4(-2)-1}{-2-1} &= B \end{aligned}$$

In general, the cover-up method works if $Q(x)$ has distinct linear factors and the degree of polynomial $P(x)$ is also strictly less than the degree of $Q(x)$. When we have several variables in our set-up, the cover-up method is much more convenient than the step-by-step algebraic solution.

Example: Suppose we're given the rational expression:

$$\frac{x^2+3x+8}{(x-1)(x-2)(x+5)}.$$

The denominator is already factored, so we can skip step 1. From step 2 we get:

$$\frac{x^2+3x+8}{(x-1)(x-2)(x+5)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x+5}$$

When using this method, there will always be one term in the final sum for each linear factor in the denominator.

Next we would find the value of each variable, A, B, and C, using the cover-up method.

$$\begin{aligned}
\frac{x^2+3x+8}{\cancel{(x-1)}(x-2)(x+5)} &= \frac{A}{\cancel{x-1}} + \cancel{\frac{B}{x-2} + \frac{C}{x+5}} \\
\frac{1^2+3\cdot 1+8}{(1-2)(1+5)} &= A \\
A &= -2
\end{aligned}$$

Repeated Factors

Now that we've seen the basic method of partial fractions we need to address possible complications. The first complication we'll consider is what to do if the factors in the denominator are not distinct — i.e. if some of the factors are repeated. In order for this technique to work, the degree of the numerator must still be less than the degree of the denominator.

Example: $\dfrac{x^2+2}{(x-1)^2(x+2)}$

Again, step 1 has already been done for us. In the set-up, step 2, we need to add a second term for the second factor of $(x-1)$.

$$\frac{x^2+2}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

In general, when you have $(x-a)^n$ in the denominator you get n corresponding terms in your sum; one for each of the powers $(x-a)^1, (x-a)^2, ..., (x-a)^n$. If the expression were:

$$\frac{x^2+2}{(x-1)^3(x+2)},$$

the setup would need to include another term with $(x-1)^3$ in the denominator.

Question: Why does it have to be squared?

Answer: This is a good question; we'll use an analogy to hint at the answer. The reasoning behind the $(x-1)^2$ is similar to the reasoning behind place value in the decimal expansion of a number. Similarly, we could expand the fraction $\frac{7}{16}$ as:

$$\frac{7}{16} = \frac{0}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4}.$$

Because of the 2^4 in the denominator of $\frac{7}{16}$ we need to use powers of 2 up to 2^4 in the denominator to represent $\frac{7}{16}$ in this way.

When you have repeated factors in the denominator the cover-up method still works, but it doesn't work quite as well. The cover-up method will work for the coefficients B and C but *not* for A.

We start by using the cover-up method to solve for C:

$$\begin{aligned}
\frac{x^2+2}{(x-1)^2\cancel{(x+2)}} &= \cancel{\frac{A}{x-1} + \frac{B}{(x-1)^2}} + \frac{C}{\cancel{x+2}} \\
\frac{(-2)^2+2}{((-2)-1)^2} &= C \\
\frac{6}{9} &= C \\
C &= \frac{2}{3}
\end{aligned}$$

The cover-up method will also work to find B:

$$\begin{aligned}
\frac{x^2+2}{\cancel{(x-1)^2}(x+2)} &= \cancel{\frac{A}{x-1}} + \frac{B}{\cancel{(x-1)^2}} + \cancel{\frac{C}{x+2}} \\
\frac{1^2+2}{1+2} &= B \\
B &= 1
\end{aligned}$$

Let's do that again the slow way to see why it worked:

$$\begin{aligned}
(x-1)^2\frac{x^2+2}{(x-1)^2(x+2)} &= \frac{A}{x-1}(x-1)^2 + \frac{B}{(x-1)^2}x-1)^2 + \frac{C}{x+2}(x-1)^2 \\
\frac{x^2+2}{(x+2)} &= A(x-1) + B + \frac{C}{x+2}(x-1)^2
\end{aligned}$$

When we set $x = 1$, every term with a multiple of $(x-1)$ in it becomes zero and we're left with the value of B.

We can't get everything to cancel so nicely to give us the value of A, which is divided only by a single power of $(x-1)$. If we multiply through by just $(x-1)$ we'll still have an $(x-1)$ in the denominator of the B term which would cause a division by 0. If we multiply through by $(x-1)^2$ the A term cancels completely.

So we have to find another strategy to solve for A. Let's try plugging in Professor Jerison's favorite number, $x = 0$. (Unfortunately, if you use $x = 0$ in solving for other terms in the decomposition you can't use it here. So far we've used $x = 1$ and $x = -2$ so it's ok to use $x = 0$.) Plugging in $x = 0$, $B = 1$ and $C = \frac{2}{3}$, we get:

$$\begin{aligned}
\frac{0^2+2}{(0-1)^2(0+2)} &= \frac{A}{0-1} + \frac{1}{(0-1)^2} + \frac{2/3}{0+2} \\
\frac{2}{2} &= \frac{A}{-1} + \frac{1}{1} + \frac{2/3}{2} \\
1 &= -A + 1 + \frac{1}{3} \\
A &= \frac{1}{3}
\end{aligned}$$

This is a lot of algebra, but if we're careful and thorough we get the right answer and our rational expression becomes easy to integrate.

Question: If $x = 0$ has already been used, what should we do?
Answer: Pick something else, like $x = 1$.

Question: If you had more powers would you have more variables?
Answer: Yes. As the degree of the denominator goes up, the number of variables goes up.

Question: How would you solve it if you had two unknowns?

Answer: When we plug in $x = 0$ (or whatever) we'll get an equation in however many unknowns are left. There are methods of solving systems of equations for those variables which we'll learn more about later.

Quadratic Factors

Our next example is one step more complicated. The degree of the numerator is still less than the degree of the denominator, but the denominator Q will have a quadratic factor:

$$\frac{x^2}{(x-1)(x^2+1)}.$$

Notice that there's nothing more you can do to factor the denominator.[1]

The set-up for this example looks like:

$$\frac{x^2}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}.$$

The difference is that the numerator corresponding to the factor (x^2+1) is linear, not constant; in general it will be a polynomial with degree one less than the degree of the numerator.

We can still use the cover-up method to solve for A:

$$\begin{aligned}
\frac{x^2}{\cancel{(x-1)}(x^2+1)} &= \frac{A}{\cancel{x-1}} + \cancel{\frac{Bx+C}{x^2+1}} \\
\frac{1^2}{1^2+1} &= A \\
A &= \frac{1}{2}
\end{aligned}$$

It turns out that the best way to solve for B and C is the "slow way"; to clear the denominators completely by multiplying both sides by $(x-1)(x^2+1)$.

$$\begin{aligned}
\frac{x^2}{(x-1)(x^2+1)}((x-1)(x^2+1)) &= \left(\frac{A}{x-1} + \frac{Bx+C}{x^2+1}\right)((x-1)(x^2+1)) \\
x^2 &= A(x^2+1) + (Bx+C)(x-1) \\
x^2 &= Ax^2 + A + Bx^2 + Cx - Bx - C \\
x^2 &= (A+B)x^2 + (C-B)x + (A-C)
\end{aligned}$$

The coefficient of x^2 on the left hand side is 1. On the right hand side, the coefficient of x^2 is $A+B$. We know that $A = \frac{1}{2}$, so we get:

$$\begin{aligned}
1 &= A + B \\
1 &= \frac{1}{2} + B \\
B &= \frac{1}{2}
\end{aligned}$$

[1] If you use complex numbers you can further factor the denominator and again use the cover-up method; in fact, this method was originally intended to be used with complex numbers. This technique was first used by Heaviside in his work with Laplace transforms and inversion of differential equations, but it also turns out to be very useful for integration.

The constant term on the left hand side is 0. On the right hand side the constant term is $A - C = \frac{1}{2} - C$. We conclude that:

$$C = \frac{1}{2}.$$

Question: Why didn't you use the coefficient of x^1?

Answer: It turns out that I didn't need it. From experience, I know that the highest and lowest degree terms of the product will be the easiest terms to work with. I avoid those when possible because they usually involve more arithmetic.

Question: Could you just set $x = 0$?

Answer: Absolutely. That's equivalent to equating the coefficients of the constant term. "Plugging in numbers" is an alternate method of solving these equations, but you still want to avoid plugging in the same number twice.

Question: What if we had $x^3 + 1$ in place of $x^2 + 1$?

Answer: If there's an $x^3 + 1$ in the denominator we haven't fully factored the denominator; we're back to step 1. This method of simplifying rational expressions doesn't work if we don't factor the denominator in the first step. Steps 1 and 2 would become:

$$\begin{aligned}
\frac{x^2}{(x-1)(x^3+1)} &= \frac{x^2}{(x-1)(x+1)(x^2-x+1)} \\
&= \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2-x+1}
\end{aligned}$$

Question: If the degree of the numerator equals the degree of the denominator, none of this works?

Answer: It definitely doesn't work. We're going to have to do something totally different to handle that case.

Remember that the point of all this work was to make it possible to compute an integral:

$$\begin{aligned}
\int \frac{x^2}{(x-1)(x^2+1)}\,dx &= \int \left(\frac{A}{x-1} + \frac{Bx+C}{x^2+1}\right) dx \\
&= \int \left(\frac{\frac{1}{2}}{x-1} + \frac{\frac{1}{2}x + \frac{1}{2}}{x^2+1}\right) dx \\
&= \int \frac{\frac{1}{2}}{x-1}\,dx + \int \frac{\frac{1}{2}x}{x^2+1}\,dx + \int \frac{\frac{1}{2}}{x^2+1}\,dx
\end{aligned}$$

We've split the integral of the rational expression into three simpler integrals. The first is easy to solve by substituting $u = x - 1$, the second can be solved by "advanced guessing" or by substituting $v = x^2 + 1$. The last one can be solved by trigonometric substitution or by remembering the antiderivative of $\frac{1}{x^2+1}$.

$$\int \frac{x^2}{(x-1)(x^2+1)}\,dx = \frac{1}{2}\ln|x-1| + \frac{1}{4}\ln(x^2+1) + \frac{1}{2}\tan^{-1}x + c.$$

Long Division

When you're integrating a rational expression $\frac{P(x)}{Q(x}$, what happens if the degree of P is greater than or equal to the degree of Q? We can think of this as an improper fraction, similar to fractions like $\frac{5}{4}$ and $\frac{8}{3}$.

Example:

$$\frac{x^3}{(x-1)(x+2)}$$

The numerator has degree 3 and the denominator has degree 2; our usual method is not going to work here.

The first step to simplifying this turns out to be to reverse our usual step 1; we don't want the denominator factored, we want it expanded, or multiplied out:

$$\frac{x^3}{(x-1)(x+2)} = \frac{x^3}{x^2+x-2}.$$

Our next step is to use long division to convert an improper fraction into a proper fraction:

$$\begin{array}{r|l}
 & \quad x-1 \\ \hline
x^2+x-2 & x^3 \\
 & -x^3 - x^2 + 2x \\ \hline
 & -x^2 + 2x \\
 & x^2 + x - 2 \\ \hline
 & 3x - 2
\end{array}$$

You may recall from grade school that $x-1$ is called the *quotient* and $3x-2$ is the *remainder*. We use this result to rewrite our rational expression as follows:

$$\frac{x^3}{(x-1)(x+2)} = \underbrace{x-1}_{\text{easy}} + \underbrace{\frac{3x-2}{x^2+x-2}}_{\text{use cover-up}}$$

We could now find the integral of this rational expression if we wished to.

Partial Fractions – Big Example

We've seen how to do partial fractions in several special cases; now we'll do a big example so that you can see how all these cases fit together.

Remember that partial fractions is a method for breaking up rational expressions into integrable pieces. The good news is, it always works. The bad news is that it can take a lot of time to make it work.

- **Step 0** Long Division:

  $$\frac{P(x)}{Q(x)} = \text{quotient} + \frac{R(x)}{Q(x)}.$$

  By completing this step you split your rational function into an easy to integrate quotient and a rational function for which the degree of the denominator is greater than the degree of the numerator.

- **Step 1** Factor the Denominator $Q(x)$.

  For example, suppose our remainder term looks like:

  $$\frac{R(x)}{(x+2)^4(x^2+2x+3)(x^2+4)^3}$$

  where the degree of $R(x)$ is less than 12. Polynomials can be extremely difficult to factor; we may need a machine to do this. This can be the hardest step in this method.

  If we expand the denominator in the example we get something like:
  $x^{12}+10x^{11}+55x^{10}+224x^9+716x^8+1856x^7+4000x^6+7168x^5+10624x^4+12800x^3+12032x^2+8192x+3072$
  Factoring this polynomial by hand would be unpleasant.

- **Step 2** Set-up:

  $$\frac{R(x)}{(x+2)^4(x^2+2x+3)(x^2+4)^3} = \begin{array}{l} \frac{A_1}{(x+2)} + \frac{A_2}{(x+2)^2} + \frac{A_3}{(x+2)^3} + \\ \frac{A_4}{(x+2)^4} + \frac{B_0x+C_0}{x^2+2x+3} + \\ \frac{B_1x+C_1}{(x^2+4)} + \frac{B_2x+C_2}{(x^2+4)^2} + \frac{B_3x+C_3}{(x^2+4)^3} \end{array}$$

  Note that repeated quadratic factors in the denominator are treated very much the same way as repeated linear factors. There's one term for each power of the repeated factor, and the degree of the numerator is the same in each term.

  There are 12 unknowns in this equation; that's not a coincidence. The degree of the denominator is 12. The numerator $R(x)$ can have at most 12 coefficients a_0, a_1, ..., a_{11}; i.e. the number of degrees of freedom of a polynomial of degree 11 is 12.

This is a very complicated system of equations: twelve equations for twelve unknowns. Machines handle this very well, but human beings have a little trouble.

- **Step 3** Cover-up.

  We can use the cover-up method to solve for A_4. That reduces the problem to eleven equations in eleven unknowns.

Question: I see that there are twelve unknowns, but isn't it just one big equation?

Answer: If you multiply both sides by $Q(x)$ it becomes a polynomial equation. On one side you have the known polynomial:

$$R(x) = a_{11}x^{11} + a_{10}x^{10} + ...$$

On the other side of the equation you'll have a polynomial whose coefficients are linear combinations of the unknowns:

$$A_1(x+2)^3(x^2+2x+3)(x^2+4)^3 + A_2(\cdots$$

When you set the coefficients on both sides equal to each other you get 12 equations in 12 unknowns.

Question: Should I write down all this stuff?

Answer: That's a good question! You'll notice that Professor Jerison didn't write it down. It's pages long. You're a human being, not a machine; don't try this at home.

Remember that once we've decomposed $\frac{P(x)}{Q(x)}$ into simpler fractions we still need to integrate it. The quotient and the fractions of the form $\dfrac{A_i}{(x+2)^i}$ are easy to integrate. However, we'll also need to compute something like:

$$\int \frac{x}{(x^2+4)^3}\,dx = -\frac{1}{4}(x^2+4)^{-2} + c$$

using advanced guessing or substitution of $u = x^2 + 4$. To calculate something like:

$$\int \frac{dx}{(x^2+4)^3}$$

we'd need to use the trig substitution $x = 2\tan u$, $dx = 2\sec^2 u\,du$

$$\begin{aligned}
\int \frac{dx}{(x^2+4)^3} &= \int \frac{2\sec^2 u\,du}{(4\sec^2 u)^3} \\
&= \frac{2}{64}\int \cos^4 u\,du \\
&= \frac{1}{32}\int \left(\frac{1+\cos(2u)}{2}\right)^2 du \\
&\ \ \vdots
\end{aligned}$$

When calculating:

$$\int \frac{dx}{x^2 + 2x + 3} = \int \frac{dx}{(x+1)^2 + 2},$$

we'll have to complete the square and then use a trig substitution to get something like:

$$\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x+2}{\sqrt{2}} \right) + c.$$

In addition, the integral of:

$$\int \frac{x}{x^2 + 2x + 3}\, dx$$

is an expression involving $\ln(x^2 + 2x + 3)$. In theory, we know how to do each of these twelve integrals. In practice it will take a long time.

There's no easier method. The method we use to compute these integrals is going to be at least as complicated as the results, and we've seen that the results can get very complicated. But this method always works, and there are computer programs that can do these calculations for us.

Introduction to Integration by Parts

Unlike the previous method, we already know everything we need to to understand integration by parts. Integration by parts is like the reverse of the product formula:

$$(uv)' = u'v + uv'$$

combined with the fundamental theorem of calculus.

To derive the formula for integration by parts we just rearrange and integrate the product formula:

$$\begin{aligned}
(uv)' &= u'v + uv' \\
uv' &= (uv)' - u'v \\
\int uv'\,dx &= \int (uv)'\,dx - \int u'v\,dx \\
\int uv'\,dx &= uv - \int u'v\,dx
\end{aligned}$$

The integration by parts formula is:

$$\int uv'\,dx = uv - \int u'v\,dx.$$

For definite integrals, it becomes:

$$\int_a^b uv'\,dx = uv\big|_a^b - \int_a^b u'v\,dx.$$

Example: $\int \ln x \, dx$

This looks intractable, but if we fit it into the form $\int uv' \, dx$, integration by parts makes the calculation relatively easy.

Here's the idea: if we let $u = \ln x$ then when we apply the formula for integration by parts we'll get an integral involving $u' = \frac{1}{x}$. The key element is that the derivative of $u = \ln x$ is easier to integrate than what we started with.

In order to fit the form $\int uv' \, dx$ we need a function v. If we choose $v = x$ then $v' = 1$ and:

$$\int \ln x \, dx = \int uv' \, dx.$$

The formula for integration by parts is:

$$\int uv' \, dx = uv - \int u'v \, dx.$$

So by plugging in $u = \ln x$ and $v = x$ we get:

$$\begin{aligned} \int \underbrace{\ln x}_{uv'} \, dx &= \underbrace{\ln x \cdot x}_{uv} - \int \underbrace{\frac{1}{x}}_{u'} \underbrace{x}_{v} \, dx \\ &= x \ln x - x + c \end{aligned}$$

Example: $\int (\ln x)^2 \, dx$

To finish learning the method of integration by parts we just need a lot of practice. To this end, we'll do two slightly more complicated examples.

To integrate:

$$\int (\ln x)^2 \, dx,$$

assign:

$$\begin{array}{rl} u = (\ln x)^2 & \quad u' = 2(\ln x)\dfrac{1}{x} \\ v = x & \quad v' = 1. \end{array}$$

When we differentiate u we get something simpler, which is a good start. Plugging u and v in to the formula for integration by parts we get:

$$\begin{aligned} \int \underbrace{(\ln x)^2}_{uv'} \, dx &= \underbrace{(\ln x)^2 \cdot x}_{uv} - \int \underbrace{2 \ln x \frac{1}{x}}_{u'} \underbrace{x}_{v} \, dx \\ &= x(\ln x)^2 - 2 \int \ln x \, dx. \end{aligned}$$

We haven't solved the problem, but we're back to the previous case; we recently computed that $\int \ln x \, dx = x \ln x - x + c$. So we have:

$$\int (\ln x)^2 \, dx = x(\ln x)^2 - 2 \underbrace{(x \ln x - x)}_{\int \ln x \, dx} + c.$$

As we'll see in the next example, this is typical. Integration by parts frequently involves replacing a "hard" integral by an easier one.

A Reduction Formula

When using a *reduction formula* to solve an integration problem, we apply some rule to rewrite the integral in terms of another integral which is a little bit simpler. We may have to rewrite that integral in terms of another integral, and so on for n steps, but we eventually reach an answer.

For example, to compute:

$$\int (\ln x)^n \, dx$$

we repeat the integration by parts from the previous example $n-1$ times, until we're just calculating $\int (\ln x) \, dx$.

For our first step we use:

$$\begin{aligned} u = (\ln x)^n & \qquad u' = n(\ln x)^{n-1}\frac{1}{x} \\ v = x & \qquad v' = 1. \end{aligned}$$

Then:

$$\begin{aligned} \int (\ln x)^n \, dx &= x(\ln x)^n - n\int (\ln x)^{n-1}\frac{1}{x} x \, dx \\ &= x(\ln x)^n - n\int (\ln x)^{n-1} \, dx \end{aligned}$$

So, if:

$$F_n(x) = \int (\ln x)^n \, dx$$

then we've just shown that:

$$F_n(x) = x(\ln x)^n - nF_{n-1}(x).$$

This is an example of a reduction formula; by applying the formula repeatedly we can write down what $F_n(x)$ is in terms of $F_1(x) = \int \ln x dx$ or $F_0(x) = \int 1 \, dx$.

We illustrate the use of a reduction formula by applying this one to the preceding two examples. We start by computing $F_0(x)$ and $F_1(x)$:

$$\begin{aligned} F_0(x) &= \int (\ln x)^0 \, dx = x + c \\ F_1(x) &= x(\ln x)^1 - 1F_0(x) \quad \text{(use reduction formula)} \\ &= x\ln x - x + c \quad \text{(Example 1)} \\ F_2(x) &= x(\ln x)^2 - 2F_1(x) \quad \text{(use reduction formula)} \\ &= x(\ln x)^2 - 2(x\ln x - x) + c \\ &= x(\ln x)^2 - 2x\ln x + 2x + c \quad \text{(Example 2.)} \end{aligned}$$

This is how reduction formulas work in general.

Another Reduction Formula: $\int x^n e^x \, dx$

To compute $\int x^n e^x \, dx$ we derive another reduction formula. We could replace e^x by $\cos x$ or $\sin x$ in this integral and the process would be very similar.

Again we'll use integration by parts to find a reduction formula. Here we choose

$$u = x^n$$

because

$$u' = nx^{n-1}$$

is a simpler (lower degree) function. If $u = x^n$ then we'll have to have

$$v' = e^x, \quad v = e^x.$$

(Note that the antiderivative of v is no more complicated than v' was — another indication that we've chosen correctly.)

On the other hand, if we used $u = e^x$, then $u' = e^x$ would not be any simpler.

Performing the integration by parts we get:

$$\int \underbrace{x^n e^x}_{uv'} \, dx = \underbrace{x^n e^x}_{uv} - \int \underbrace{x^{n-1} e^x}_{u'v} \, dx.$$

If:

$$G_n(x) = \int x^n e^x \, dx$$

then we get the reduction formula:

$$G_n(x) = x^n e^x - nG_{n-1}(x).$$

Let's illustrate this by computing a few integrals. First we directly compute:

$$G_0(x) = \int x^0 e^x \, dx = e^x + c.$$

Now we can use the reduction formula to conclude that:

$$\begin{aligned} G_1(x) &= xe^x - G_0(x) \\ &= xe^x - e^x + c. \end{aligned}$$

So $\int xe^x \, dx = xe^x - e^x + c$.

Question: How do you know when this method will work?

Answer: Good question! The answer is "only through experience and practice". To use this method on an integrand, we need one factor u of the integrand to get simpler when we differentiate and the other factor v not to get more complicated when we integrate.

We've seen how to use integration by parts to derive reduction formulas. We could also find these formulas by advanced guessing — guess what the formula should be and then check it. Either method is valid.

Volume of a Wine Glass: Horizontal Slices

Now we know all of the techniques of integration anyone knows. We'll celebrate by using our new techniques to answer an interesting question.

Find the volume of an exponential wine glass whose bowl is formed by rotating the portion of the graph of $y = e^x$ that joins $(0, 1)$ and $(1, e)$ about the y-axis.

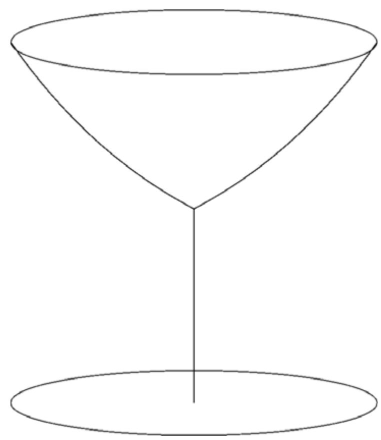

Figure 1: A wine glass formed by rotating the graph of $y = e^x$ about the y-axis.

There are two methods of solving this problem: horizontal and vertical slices. If we compute the volume using horizontal slices we'll be adding up the volumes of disks with height dy and radius $x = \ln y$. (See Figure 2.)

The volume will therefore be:

$$\begin{aligned}
\int_1^e \pi x^2\, dy &= \int_1^e \pi(\ln y)^2\, dy \\
&= \pi F_2(y)\big|_1^e \quad \text{(see previous example)} \\
&= \pi\left[y(\ln y)^2 - 2(y\ln y - y)\right]_1^e \\
&= \pi\left[(e(\ln e)^2 - 2(e\ln e - e)) - (1(\ln 1)^2 - 2(1\ln 1 - 1))\right] \\
&= \pi\left[(e(1)^2 - 2(e\cdot 1 - e)) - (1(0)^2 - 2(1\cdot 0 - 1))\right] \\
&= \pi\left[(e - 2(0)) - (-2(-1))\right] \\
&= \pi(e - 2).
\end{aligned}$$

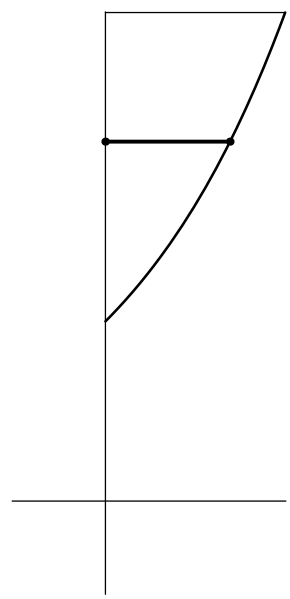

Figure 2: Rotating this horizontal slice about the y-axis forms a disk.

Volume of a Wine Glass: Vertical Slices

If we use vertical slices to compute the volume of our exponential wine glass, we'll be adding up volumes of shells with height $e - y$, radius x and thickness dx.

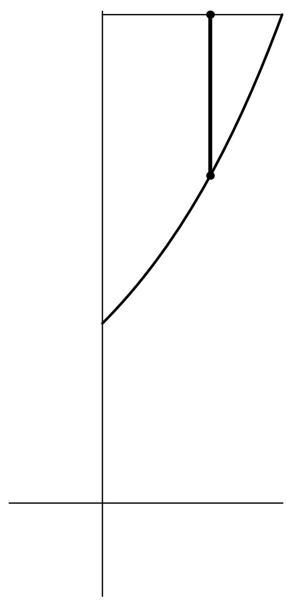

Figure 1: Rotating a slice of thickness dx about the y-axis produces a shell.

$$
\begin{aligned}
\text{Volume} &= \int_0^1 (e-y)2\pi x\,dx \\
&= \int_0^1 (e-e^x)2\pi x\,dx \\
&= \int_0^1 2\pi e x\,dx - \int_0^1 2\pi x e^x\,dx \\
&= \underbrace{\frac{2\pi e}{2}}_{\text{area of a triangle}} - 2\pi G_1(x)\Big|_0^1 \\
&= \pi e - 2\pi\left[x e^x - e^x\right]_0^1 \\
&= \pi e - 2\pi\left[(e-e)-(0-1)\right] \\
&= \pi e - 2\pi \\
&= \pi(e-2).
\end{aligned}
$$

Introduction to Arc Length

Now that we're done with techniques of integration, we'll return to doing some geometry; this will lead to some of the tools you'll need in multivariable calculus. Our first topic is *arc length*, which is calculated using another cumulative sum which will have an associated story and picture.

Suppose you have a roadway with mileage markers $s_0, s_1, s_2, ..., s_n$ along the road. The distance traveled along the road — the *arc length* — is described by this parameter s. If we look at the road as a graph, we can let a be the x coordinate of the first point s_0 on the curve or road and b be the x coordinate of the end point s_n of the curve, and x_i as the x-coordinate of s_i. This is reminiscent of what we did with Riemann sums.

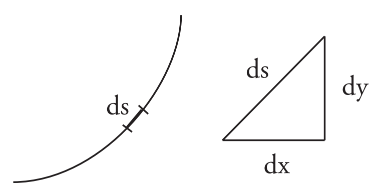

Figure 1: Straight line approximation of arc length.

We'll approximate the length s of the curve by summing the straight line distances between the points s_i. As n increases and the distance between the s_i decreases, the straight line distance from s_i to s_{i-1} will get closer and closer to the distance Δs along the curve. We can use the Pythagorean theorem to see that that distance equals $\sqrt{(\Delta x)^2 + (\Delta y)^2}$. In other words:

$$(\Delta s)^2 \approx \overbrace{(\Delta x)^2 + (\Delta y)^2}^{(\text{hypotenuse})^2}.$$

We apply the tools of calculus to this estimate; in the infinitesimal this is exactly correct:

$$(ds)^2 = (dx)^2 + (dy)^2.$$

In the future we'll omit the parentheses and write this as $ds^2 = dx^2 + dy^2$. These are squares of differentials; try not to mistake them for differentials of squares.

The next thing we do is take the square root:

$$ds = \sqrt{dx^2 + dy^2}.$$

This is the formula that Professor Jerison has memorized, but you can rewrite it in several useful ways. For instance, you can factor out the dx to get:

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

This is the form we'll be using today; when we add up all the infinitesimal values of ds we'll find that:

$$\begin{aligned}
\text{Arc Length} &= \text{distance along the curve from } s_0 \text{ to } s_n \\
&= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\
&= \int ds \\
&= \int_a^b \sqrt{1 + f'(x)^2}\, dx \quad (y = f(x))
\end{aligned}$$

Question: Is $f'(x)^2$ equal to $f''(x)$?

Answer: No. Suppose $f(x) = x^2$. Then $f'(x) = 2x$, $f'(x)^2 = 4x^2$ and $f''(x) = 2$.

Question: What are the limits of integration on $\int ds$, above?

Answer: If you're integrating with respect to s you'll start at s_0 and end at s_n. If you're integrating with respect to a different variable you'll have different limits of integration, as happens when we change variables. The values s_0 and s_n are mileage markers along the road; they're not the same as a and b. When we measure arc length, remember that we're measuring distance along a curved path.

Example: $y = mx$

This is a basic example that should help you get some perspective on this method.

$$\begin{aligned} y &= mx \\ y' &= m \\ ds &= \sqrt{1+(y')^2}\,dx \\ &= \sqrt{1+m^2}\,dx \end{aligned}$$

The length of the curve $y = mx$ on the interval $0 \le x \le 10$ is:

$$\int_0^{10} \sqrt{1+m^2}\,dx = 10\sqrt{1+m^2}$$

We've drawn a picture to confirm this; see Figure 1. We see that the

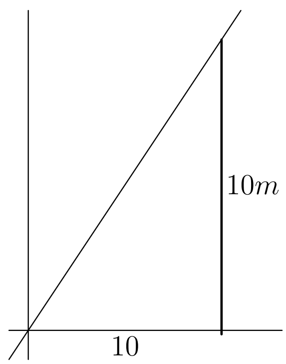

Figure 1: Arc Length of $y = mx$ over $0 \le x \le 10$.

arc whose length we're computing is the hypotenuse of a triangle, and the Pythagorean theorem tells us that its length is:

$$\sqrt{10^2 + (10m)^2} = 10\sqrt{1+m^2}.$$

You may be disdainful of how obvious this is. But if we can figure out these formulas for linear functions, we can use calculus to subdivide other functions into infinitesimal linear parts and then solve the problem for those functions. This is the main point of these integrals.

Example: Circular Arc

$$y = \sqrt{1-x^2}$$

describes the graph of a semicircle. We'll find the arc length of the piece of this semicircle above the interval $0 \le x \le a$. (See Figure 1.)

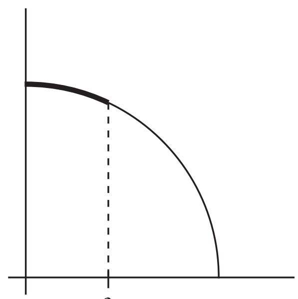

Figure 1: Arc length of $y = \sqrt{1-x^2}$ over $0 \le x \le a$.

We'll use the variable α to denote the arc length along the circle. We could calculate the exact value of α using trigonometry, but we'll first find it using calculus. We start by finding y'.

$$\begin{aligned} y' &= \frac{-x}{\sqrt{1-x^2}} \\ ds &= \sqrt{1+(y')^2}\,dx \\ &= \sqrt{1+\left(\frac{-x}{\sqrt{1-x^2}}\right)^2}\,dx \end{aligned}$$

Yuck. Let's simplify $1+\left(\frac{-x}{\sqrt{1-x^2}}\right)^2$ in a separate calculation:

$$\begin{aligned} 1+(y')^2 &= 1+\left(\frac{-x}{\sqrt{1-x^2}}\right)^2 \\ &= 1+\frac{x^2}{1-x^2} \\ &= \frac{1-x^2+x^2}{1-x^2} \\ 1+(y')^2 &= \frac{1}{1-x^2}. \end{aligned}$$

Plugging this in and using our formula for arc length, we get:

$$\alpha = \int_0^a \frac{dx}{\sqrt{1-x^2}}$$

$$
\begin{aligned}
&= \sin^{-1} x\Big|_0^a \\
\alpha &= \sin^{-1} a. \\
(\text{So } \sin\alpha &= a.)
\end{aligned}
$$

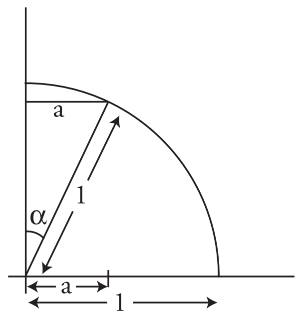

Figure 2: The arc length equals α.

This is a little deeper than it looks; we went a distance α along the arc of a circle that has radius 1 and ended up at a point whose x-coordinate was a.

Previously, if we had an angle whose measure was α radians, we'd say:

$$\sin\alpha = a.$$

You may have been told that radians measured the arc length along the curve of the circle, but this is the first time you've been able to derive it.

Remember that our first definition of the exponential function e^x involved the slope of its graph, but later we were able to define the natural log function as an integral. The same sort of thing is happening here; if you want to know what radians are you have to calculate this arc length. This gives you a new definition of the arcsine function, which gives you a new definition of the sine function, which leads to an improved definition and understanding of trig functions.

Example: Length of a Parabola

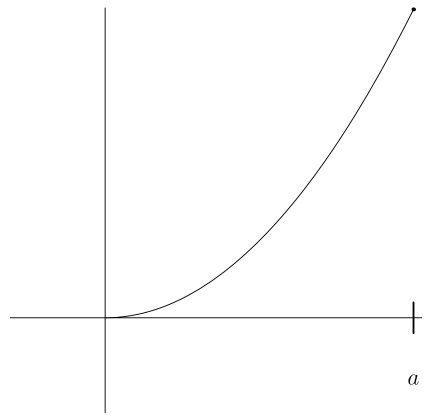

Figure 1: Arc length of $y = x^2$ over $0 \leq x \leq a$.

To find the arc length of a parabola we start with:

$$\begin{aligned} y &= x^2 \\ y' &= 2x \\ ds &= \sqrt{1 + (2x)^2}\, dx \\ &= \sqrt{1 + 4x^2}\, dx. \end{aligned}$$

So the arc length of the parabola over the interval $0 \leq x \leq a$ is:

$$\int_0^a \sqrt{1 + 4x^2}\, dx.$$

This is the answer to the question, but it would be more useful to us if we could write it in a simpler form. That's why we studied techniques of integration. To evaluate this integral we use the following trig substitution:

$$\begin{aligned} x &= \frac{1}{2} \tan u \\ dx &= \frac{1}{2} \sec^2 u \end{aligned}$$

When we do, we find that:

$$\int_0^a \sqrt{1 + 4x^2}\, dx = \left[\frac{1}{4} \ln(2x + \sqrt{1 + 4x^2}) + \frac{1}{2} x \sqrt{1 + 4x^2} \right]_0^a$$

(you may have seen parts of this calculation in a recitation video).

Introduction to Surface Area

We're going to move to three dimensions now to talk about surface area; we'll be doing a lot with surface area in multivariable calculus. If this starts to look too complicated, keep in mind that all we're doing is integrating infinitesimal pieces of simple, linear functions.

The only surface areas we'll compute in this class are surfaces of rotation. We'll start by rotating the parabola from our last example about the x-axis to get the trumpet shape shown in Figure 2. (Remember that we're only interested in the surface — the metal part of the trumpet — and not the interior.)

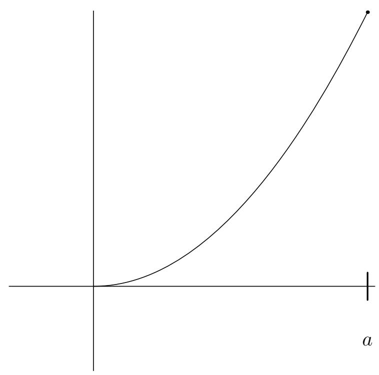

Figure 1: The parabola $y = x^2$.

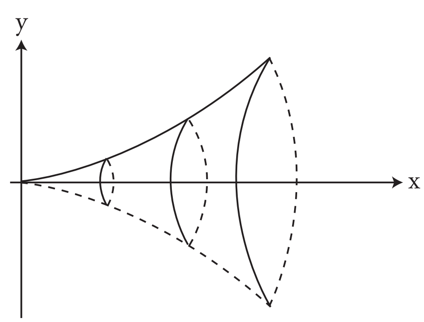

Figure 2: The parabola $y = x^2$ rotated about the x-axis.

We figure out the formula for surface area of a surface of rotation in much the same way we figured out the formula for volumes of revolution. Think of a small segment of arc length with length ds. If that segment is parallel to the x-axis, when you rotate it around the axis it sweeps out a shell shape. If the segment is tilted at an angle, then the surface swept out will have more area than a shell, proportional to the amount of tilt. The surface area swept out is proportional to the length ds of the segment.

In our example, the total surface area swept out by a small segment of arc will be:

$$dA = \underbrace{(2\pi y)}_{\text{circumference}} (ds).$$

You may also see S used for surface area (and s used for arc length):

$$dS = (2\pi y)(ds).$$

The surface area of our trumpet shape will then be:

$$\begin{aligned} \text{Surface area} &= \int_0^a \underbrace{2\pi x^2}_{2\pi y} \underbrace{\sqrt{1+4x^2}\,dx}_{ds \text{ from before}} \\ &\vdots \qquad\qquad \text{(substitute } x = \tfrac{1}{2}\tan u\text{)} \end{aligned}$$

The calculation for this integral is long; if we wanted to we could use a computer program to get an answer. Our goal is to be able to see that we could get an exact solution to this integral if we had to; i.e. that we could rewrite it as a product of trigonometric functions.

Surface Area of a Sphere

In this example we will complete the calculation of the area of a surface of rotation. If we're going to go to the effort to complete the integral, the answer should be a nice one; one we can remember. It turns out that calculating the surface area of a sphere gives us just such an answer.

We'll think of our sphere as a surface of revolution formed by revolving a half circle of radius a about the x-axis. We'll be integrating with respect to x, and we'll let the bounds on our integral be x_1 and x_2 with $-a \leq x_1 \leq x_2 \leq a$ as sketched in Figure 1.

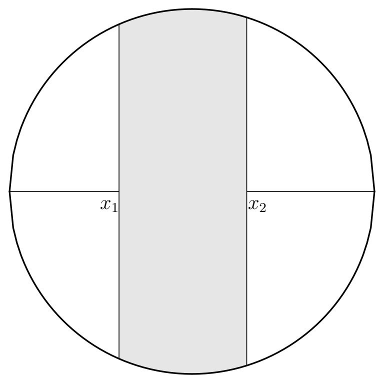

Figure 1: Part of the surface of a sphere.

Remember that in an earlier example we computed the length of an infinitesimal segment of a circular arc of radius 1:

$$ds = \sqrt{\frac{1}{1-x^2}}\,dx$$

In this example we let the radius equal a so that we can see how the surface area depends on the radius. Hence:

$$\begin{aligned} y &= \sqrt{a^2 - x^2} \\ y' &= \frac{-x}{\sqrt{a^2-x^2}} \\ ds &= \sqrt{1 + \frac{x^2}{a^2-x^2}}\,dx \\ &= \sqrt{\frac{a^2 - x^2 + x^2}{a^2-x^2}}\,dx \\ &= \sqrt{\frac{a^2}{a^2-x^2}}\,dx. \end{aligned}$$

The formula for the surface area indicated in Figure 1 is:

$$\text{Area} = \int_{x_1}^{x_2} 2\pi y\,ds$$

$$
\begin{aligned}
&= \int_{x_1}^{x_2} 2\pi \overbrace{\sqrt{a^2 - x^2}}^{y} \overbrace{\sqrt{\frac{a^2}{a^2 - x^2}}\, dx}^{ds} \\
&= \int_{x_1}^{x_2} 2\pi \sqrt{a^2 - x^2} \frac{a}{\sqrt{a^2 - x^2}}\, dx \\
&= \int_{x_1}^{x_2} 2\pi a\, dx \\
&= 2\pi a(x_2 - x_1).
\end{aligned}
$$

Special Cases

When possible, we should test our results by plugging in values to see if our answer is reasonable. Here, if we set $x_1 = 0$ and $x_2 = a$ we should get the surface area of a hemisphere of radius a:

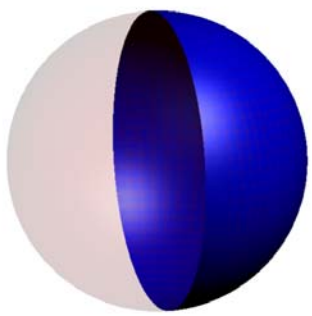

Figure 2: Right hemisphere.

$$
\begin{aligned}
2\pi a(x_2 - x_1) &= 2\pi a(a - 0) \\
&= 2\pi a^2
\end{aligned}
$$

We get the surface area of the whole sphere by letting $x_1 = -a$ and $x_2 = a$:

$$
\begin{aligned}
2\pi a(x_2 - x_1) &= 2\pi a(a - (-a)) \\
&= 4\pi a^2
\end{aligned}
$$

Question: Would it be possible to rotate around the y-axis?

Answer: Yes. If we rotate around the y axis and integrate with respect to x (calculating the surface area of a vertical slice, as we did here) we'd be adding up little strips of area. If we integrate with respect to y and find the surface area between two vertical positions y_1 and y_2 we'd get exactly the same calculation.

Question: Can you compute surface area using shells?

Answer: The short answer is "not quite". We use the word shell to describe something which has a thickness dx. Shells have volume, integrals which involve shells compute volumes, not surface areas.

To compute surface area you need to sum up the areas of small regions of your surface, but those small regions can have any shape whose area you can measure.