The Second Fundamental Theorem of Calculus

We're going to start with a continuous function f and define a complicated function $G(x) = \int_a^x f(t)\,dt$. The variable x which is the input to function G is actually one of the limits of integration. The function f is being integrated with respect to a variable t, which ranges between a and x. The variable t is a dummy variable, and is the variable of integration. Don't get t and x mixed up, even if your textbook does.

Theorem: If f is continuous and $G(x) = \int_a^x f(t)\,dt$, then $G'(x) = f(x)$.

From the point of view of differential equations, $G(x)$ solves the differential equation

$$y' = f, \qquad y(a) = 0.$$

The second fundamental theorem of calculus tells us that we can always solve this equation (by using Riemann sums if necessary).

Using The Second Fundamental Theorem of Calculus

This is the quiz question which everybody gets wrong until they practice it.

Example: Evaluate $\frac{d}{dx}\int_1^x \frac{dt}{t^2}$.

This question challenges your ability to understand what the question means. It looks very complicated, but what it really is is an exercise in recopying!

By definition, we have a function of the form $G(x) = \int_a^x f(t)\,dt$ like the one in the second fundamental theorem of calculus. The second fundamental theorem then tells us that $G'(x) = f(x)$.

So if $G(x) = \int_1^x \frac{1}{t^2}\,dt$, then $\frac{d}{dx}G(x) = \frac{1}{x^2}$.

Despite the fact that this looks like a long, elaborate problem it is actually quite easy to solve.

Question: Why don't you integrate?

Answer: Because we don't need to! When you take the derivative of something and then the antiderivative, you get back to the original thing (plus some arbitrary constant). In this case, we're taking the antiderivative and then the derivative; again we get back to the same thing. There isn't even an arbitrary constant to worry about here because the derivative of that constant is zero.

Let's see what happens when we integrate. It's slower than just recopying, but we can check it to see that the theorem does work.

$$\int_1^x t^{-2}\,dt = -t^{-1}\Big|_1^x$$

(Remember that t is our variable of integration, not x.)

$$\int_1^x t^{-2}\,dt = -\frac{1}{x} - (-1).$$

Then we take the derivative to get $\frac{d}{dx}\int_1^x \frac{dt}{t^2}$:

$$\frac{d}{dx}\left(1 - \frac{1}{x}\right) = 0 - (-x^{-2}) = \frac{1}{x^2}.$$

Proof of the Second Fundamental Theorem of Calculus

Theorem: (The Second Fundamental Theorem of Calculus) If f is continuous and $F(x) = \int_a^x f(t)\,dt$, then $F'(x) = f(x)$.

Proof: Here we use the interpretation that $F(x)$ (formerly known as $G(x)$) equals the area under the curve between a and x. Our goal is to take the derivative of F and discover that it's equal to f.

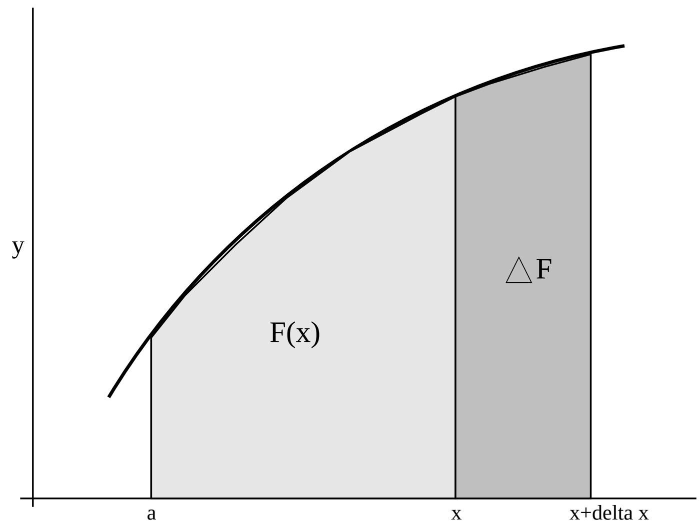

Figure 1: Graph of $f(x)$ with shaded area $F(x)$.

We graph the equation $y = f(x)$ and keep track of where a, x and $x + \Delta x$ are. This splits the area under the curve into pieces. The first piece is the area under the curve between a and x which is, by definition, $F(x)$. The second piece is a thin region; its area is ΔF, which is the change in the area under the curve as x increases by Δx.

We now approximate this thin region with area ΔF by a rectangle. Its base has width Δx and its height is close to $f(x)$ (because f is continuous). So

$$\Delta F \approx \Delta x f(x).$$

Divide both sides by Δx to get $\frac{\Delta F}{\Delta x} \approx f(x)$, then take the limit as Δx goes to zero to get the derivative:

$$F'(x) = \lim_{\Delta x \to 0} \frac{\Delta F}{\Delta x} = f(x).$$

Proof of the First Fundamental Theorem of Calculus

The first fundamental theorem says that the integral of the derivative is the function; or, more precisely, that it's the difference between two outputs of that function.

Theorem: (First Fundamental Theorem of Calculus) If f is continuous and $F' = f$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

Proof: By using Riemann sums, we will define an antiderivative G of f and then use $G(x)$ to calculate $F(b) - F(a)$.

We start with the fact that $F' = f$ and f is continuous. (It's not strictly necessary for f to be continuous, but without this assumption we can't use the second fundamental theorem in our proof.)

Next, we define $G(x) = \int_a^x f(t)\,dt$. (We know that this function exists because we can define it using Riemann sums.)

The second fundamental theorem of calculus tells us that:

$$G'(x) = f(x)$$

So $F'(x) = G'(x)$. Therefore,

$$(F - G)' = F' - G' = f - f = 0$$

Earlier, we used the mean value theorem to show that if two functions have the same derivative then they differ only by a constant, so $F - G =$ constant or

$$F(x) = G(x) + c.$$

This is an essential step in an essential proof; all of calculus is founded on the fact that if two functions have the same derivative, they differ by a constant.

Now we compute $F(b) - F(a)$ to see that it is equal to the definite integral:

$$\begin{aligned} F(b) - F(a) &= (G(b) + c) - (G(a) + c) \\ &= G(b) - G(a) \\ &= \int_a^b f(t)\,dt - \int_a^a f(t)\,dt \\ &= \int_a^b f(t)\,dt - 0 \\ F(b) - F(a) &= \int_a^b f(x)\,dx \end{aligned}$$

Antiderivative of $\dfrac{1}{x}$

We return to our theme of using $f = F'$ to understand F.

We recently found the antiderivative of t^{-2}. For the most part, it's easy to antidifferentiate x^n, except for the tricky case of $\frac{1}{x}$.

Example: Solve the differential equation $L'(x) = \frac{1}{x}$; $L(1) = 0$.

The second fundamental theorem of calculus tells us that the solution is:

$$L(x) = \int_1^x \frac{dt}{t} = \ln x.$$

We also know that this antiderivative must be the logarithm function, but it turns out that this way of looking at the function makes many calculations much easier.

One very interesting thing about this solution is that we started with a ratio of polynomials ($\frac{1}{x}$) and ended with a transcendental function. The natural log function can't be written in terms of our usual algebraic operations.

Area Under the Bell Curve

In addition to exotic but familiar functions like $\ln x$, we can also use definite integrals and Riemann sums to get truly *new* functions.

Example: The solution to $y' = e^{-x^2}$; $y(0) = 0$ is:

$$F(x) = \int_0^x e^{-t^2}\,dt$$

The graph of e^{-x^2} is known as the bell curve, and $F(x)$ describes the area under the curve. This function is extremely useful for computing probabilities.

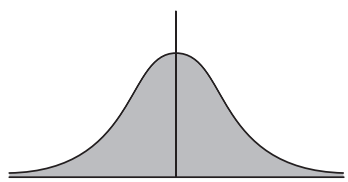

Figure 1: Graph of e^{-x^2}.

The exciting thing about $F(x)$ is that although we have a geometric definition and can compute it using Riemann sums, we can't describe it in terms of any function we've seen previously, including logarithmic and trigonometric functions. It's a completely new function. The problem of describing F is analogous to the problem of calculating the value of π — the area of a circle with radius 1. The number π is transcendental; it is not the root (zero) of an algebraic equation with rational coefficients.

Using definite integrals we can define a huge class of new functions, many of which are important tools in science and engineering.

Alternate Definition of Natural Log

The second fundamental theorem of calculus says that the derivative of an integral gives you the function back again:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

We saw a few examples of how to use this to solve differential equations. In particular, we can solve $y' = \frac{1}{x}$ to get

$$L(x) = \int_1^x \frac{dt}{t}.$$

We could use this formula to define the logarithm function and derive all of its properties.

The first property of the function L is that $L'(x) = \frac{1}{x}$. (We defined it that way.)

The other piece of information that we need in order to completely describe the function is its output at one value; knowing its derivative only tells us its value up to a constant.

$$L(1) = \int_1^1 \frac{dt}{t} = 0$$

This will be the case with all definite integrals; if we evaluate them at their starting place, we'll get 0.

Together these two properties uniquely describe $L(x)$.

Next we want to understand the properties of the function; we'll start by graphing it. We know that the derivative of $L(x)$ is $\frac{1}{x}$; when the function is given as an integral it's easy to compute its first derivative! The derivative of L "blows up" as x approaches 0. To avoid this problem we'll consider only positive values of x.

The second derivative of L is $-\frac{1}{x^2}$. From it we learn that the graph of $L(x)$ is concave down. We know that $L(1) = 0$ and $L'(1) = 1$, which gives us a point on the graph and its slope at that point. We also know that $L'(x) > 0$ when x is positive, so we know that the function is increasing — the graph rises as we move to the right.

Knowing that $L(x)$ is increasing when x is positive allows us to work backwards from this definition to the one we used previously. If we draw the line $y = 1$ it intersects the graph of $L(x)$ at some point (we could confirm this using a Riemann sum if we had to). We'll define the number e so that this point of intersection is $(e, 1)$. In other words, e is the unique value for which $L(e) = 1$. We know there's only one such value because the graph can never "dip down" to cross the line $y = 1$ again.

Since we know $L(x)$ is increasing, we know there are no critical points; the only other interesting thing is the ends. It turns out that the limit as x

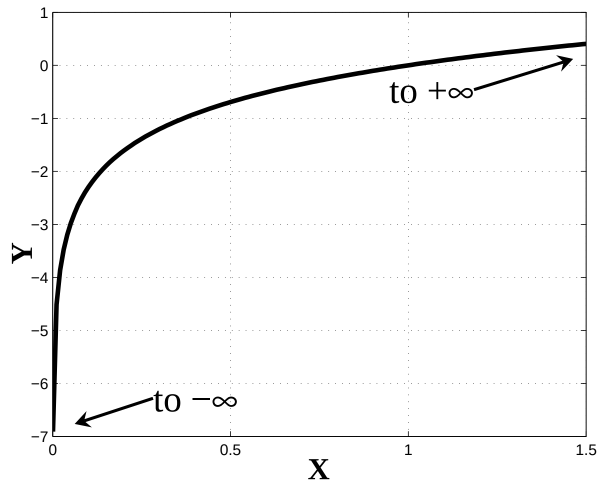

Figure 1: Graph of $y = \ln(x)$.

approaches 0 is minus infinity. As x approaches positive infinity, the limit is positive infinity (there's no horizontal asymptote). We won't get into those details today.

Instead, we'll look at one more qualitative feature of the graph; the fact that $L(x) < 0$ for $x < 1$. Why is this true?

- $L(1) = 0$ and L is increasing — if L increases to 0 as x goes toward 1 it must be negative before then.

- $L(x) = \int_1^x \frac{dt}{t} = -\int_x^1 \frac{dt}{t} < 0$ when $0 < x < 1$ because $\int_x^1 \frac{dt}{t}$ describes a positive area.

Log of a Product

Claim: $L(ab) = L(a) + L(b)$, where $L(x) = \int_1^x \frac{dt}{t}$ is an alternately defined natural log function.

To prove this, we just plug in the formula and see what happens. On the left hand side we have:

$$L(ab) = \int_1^{ab} \frac{dt}{t} = \int_1^{a} \frac{dt}{t} + \int_a^{ab} \frac{dt}{t}$$

By definition, $\int_1^a \frac{dt}{t} = L(a)$. If we could show that $\int_a^{ab} \frac{dt}{t} = L(b)$, we'd be done with the proof.

It turns out that we can prove this by using a change of variables. We start with $\int_a^{ab} \frac{dt}{t} = L(b)$, and substitute $t = au$ (so $dt = a\,du$). The limits of integration are from $u = 1$ to $u = b$. If we plug these into $\int_a^{ab} \frac{dt}{t}$, we get:

$$\int_a^{ab} \frac{dt}{t} = \int_{u=1}^{u=b} \frac{a\,du}{au} = \int_1^b \frac{du}{u} = L(b).$$

We can now conclude that:

$$L(ab) = L(a) + L(b)$$

The Area Under the Bell Curve

Our last example of a definite integral is:

$$F(x) = \int_0^x e^{-t^2}\,dt.$$

As we've already remarked, this is a new function that we can't express in terms of functions we already know. To begin to understand the properties of this function F we'll sketch its graph.

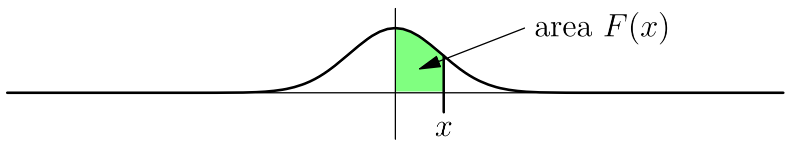

Figure 1: Area under e^{-t^2}.

The fundamental theorem tells us right away that:

$$F'(x) = e^{-x^2}.$$

It's easy to find the value of $F(0)$, because the "starting place" of the integral is 0.

$$F(0) = \int_0^0 e^{-t^2}\,dt = 0.$$

We can also compute the second derivative:

$$F''(x) = -2xe^{-x^2}$$

The first derivative is always positive, so $F(x)$ is always increasing. Because the sign of $-2xe^{-x^2}$ is just the sign of $-2x$, its graph will be concave down when $x > 0$ and concave up when $x < 0$. And finally, $F'(0) = e^{-0^2} = 1$. (We often define functions so that their graphs have slope 1 in convenient locations.)

By combining all this information we get a pretty good idea of what the graph of the function looks like, even though we cannot write down a strictly algebraic equation for $F(x)$.

We want to know as much as possible about this function, so we'll discuss a few more features before moving on.

First we show that $F(x)$ is odd. As mentioned in Figure 1, $F(x)$ is the area under the graph of e^{-t^2} between 0 and x. From the symmetry of the graph we see:

$$\int_{-x}^0 e^{-x^2}\,dx = \int_0^x e^{-x^2}\,dx,$$

so:

$$F(-x) = \int_0^{-x} e^{-x^2}\,dx = -\int_{-x}^0 e^{-x^2}\,dx = -\int_0^x e^{-x^2}\,dx = -F(x).$$

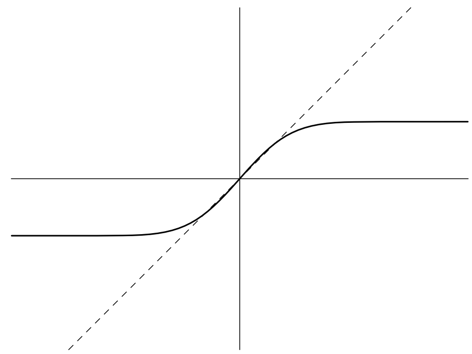

Figure 2: Graph of $F(x) = \int_0^x e^{-t^2}\,dt$.

We conclude that $F(x)$ is an odd function.

Because F is odd we know that the part of its graph to the right of the y-axis is exactly a rotation of the part of its graph to the left of the y-axis. If we know the shape of one branch we immediately know the shape of the other branch.

Our final step to understanding the graph is to figure out what happens at the ends. It turns out that the graph approaches a horizontal asymptote as x approaches positive infinite and, by symmetry, as x goes to negative infinity. On the right, the graph rises to a certain level; on the left it falls to the negative of that level.

What is that level? It's the area under the graph of $f(x) = e^{-t^2}$ between 0 and infinity: $\frac{\sqrt{\pi}}{2}$. It took several years for mathematicians to discover the exact value of this area.

$$\lim_{x\to\infty} F(x) = \frac{\sqrt{\pi}}{2}$$

$$\lim_{x\to-\infty} F(x) = -\frac{\sqrt{\pi}}{2}$$

Because 1 is a nicer number to work with than $\frac{\sqrt{\pi}}{2}$, people define the *error function* to be:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt = \frac{2}{\sqrt{\pi}} F(x).$$

This is a famous function, as is its cousin the standard normal distribution.

More New Functions from Old

Using definite integrals, we can define many transcendental or "new" functions which cannot be expressed in elementary terms.

Example: Fresnel Integrals

$$C(x) = \int_0^x \cos(t^2)\,dt$$

$$S(x) = \int_0^x \sin(t^2)\,dt$$

These are named after Augustin-Jean Fresnel and are used in optics.

Example: From Fourier Analysis

$$H(x) = \int_0^x \frac{\sin t}{t}\,dt$$

Example: Logarithmic Integral

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\ln t}$$

This function is significant because it is approximately equal to the number of primes less than x. If you can describe precisely how close the value of $\mathrm{Li}(x)$ is to the exact number of primes less than x you'll have proven the Riemann hypothesis; a task mathematicians have been working on for over a century.

Question: Are we supposed to understand this stuff?

Answer: That's a good question. We're going to see a lot more of the function $F(x) = \int_0^x e^{-t^2}\,dt$ because it's associated with the normal distribution. The functions described in this segment are simply examples of other transcendental functions that are important and described by definite integrals. For this class, the only thing that you'll need to do with such functions are things like understanding the derivative, the second derivative, and sketching their graphs.

Areas Between Curves

Suppose you have two curves, $y = f(x)$ above and $y = g(x)$ below. You want to find the area between the two curves bounded on the left by $x = a$ and on the right by $x = b$.

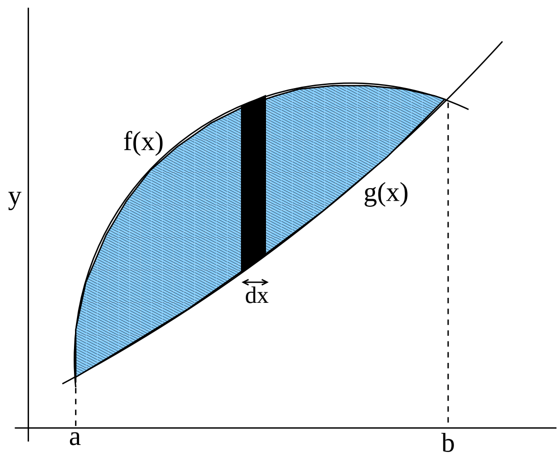

Figure 1: Finding the area between two intersecting functions.

As we did with Riemann sums, we can (approximately) chop this area up into thin rectangles. Each rectangle will have width dx and height $f(x) - g(x)$, so will have area

$$\underbrace{(f(x) - g(x))}_{\text{height}}\,\underbrace{dx}_{\text{base}}.$$

In order to get the whole area, sum the areas of all these rectangles:

$$A = \int_a^b (f(x) - g(x))\,dx.$$

There are two key steps to solving problems with integrals. The first is figuring out what to integrate. The function being integrated is called the *integrand*. The second is finding the *limits of integration* — in this case a and b. Once we have these we can compute the integral, either numerically or by finding an antiderivative. Without the integrand and limits of integration we can't find the value of the integral.

Example: Find the area between $x = y^2$ and $y = x - 2$

First, graph these functions. If skip this step you'll have a hard time figuring out what the boundaries of your area is, which makes it very difficult to compute the area!

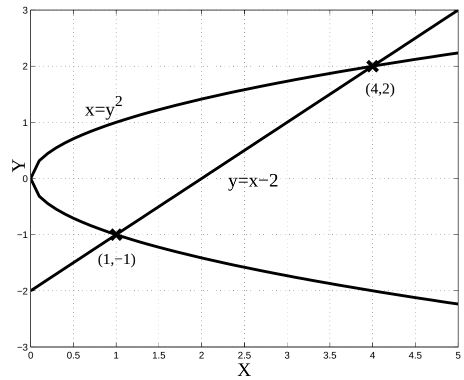

Figure 1: Finding the area between two intersecting graphs.

There are two ways of finding the area between these two curves: the hard way and the easy way.

Hard Way: slice it vertically

First, we'll try chopping the region up into vertical rectangles.

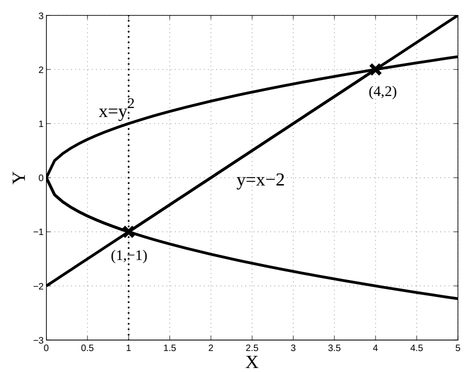

Figure 2: The area between $x = y^2$ and $y = x - 2$ split into two subregions.

If we slice the region between the two curves this way, we need to consider two different regions. Where $x > 1$, the region's lower bound is the straight line. For $x < 1$, however, the region's lower bound is the lower half of the sideways parabola. The left end of the region is at $x = 0$.

We must find the crossing points of the two curves; in other words, we find the values of x and y that satisfy both equations simultaneously.

$$x = y^2 \quad \text{and} \quad y = x - 2$$

so:

$$\begin{aligned} y - 2 &= y^2 \\ y^2 - y - 2 &= 0 \\ (y-2)(y+1) &= 0 \end{aligned}$$

We conclude that:

$$y = 2 \quad \text{or} \quad y = -1.$$

We can plug these values of y back in to either equation to find the associated x values:

$$\begin{aligned} y &= x - 2 \\ 2 &= x - 2 \\ 4 &= x. \end{aligned}$$

If we perform a similar equation with $y = -1$ we'll find that the two points of intersection are $(1, -1)$ and $(4, 2)$.

The equation of the upper half of the sideways parabola is $y = \sqrt{x}$ and that of the lower half is $y = -\sqrt{x}$. The equation of the lower right hand boundary of the region is just $y = x - 2$.

We find the area A between the two curves by integrating the difference between the top curve and the bottom curve in each region:

$$A = \underbrace{\int_0^1 (\overbrace{\sqrt{x}}^{\text{top}} - \overbrace{(-\sqrt{x})}^{\text{bottom-l}}\,))\,dx}_{\text{left}} + \underbrace{\int_1^4 (\overbrace{\sqrt{x}}^{\text{top}} - \overbrace{(x-2)}^{\text{bottom-r}}\,)\,dx}_{\text{right}}$$

The rest of this calculation is easy; just evaluate the integrals.

$$\begin{aligned} A &= 2\int_0^1 \sqrt{x}\,dx + \int_1^4 (-x + \sqrt{x} + 2)\,dx \\ &= 2\left[\frac{2}{3}x^{3/2}\right]_0^1 + \left[-\frac{1}{2}x^2 + \frac{2}{3}x^{3/2} + 2x\right]_1^4 \\ &= 2\left(\frac{2}{3} - 0\right) + \left(-\frac{4^2}{2} + \frac{2}{3}\cdot 4^{3/2} + 8\right) - \left(-\frac{1}{2} + \frac{2}{3} + 2\right) \\ &= \frac{4}{3} - 8 + \frac{16}{3} + 8 + \frac{1}{2} - \frac{2}{3} - 2 \\ A &= \frac{9}{2}. \end{aligned}$$

Easy Way: Slice it horizontally

There's a much quicker way to complete this area calculation; you should look for an easier way as soon as you notice the need to split the region into parts. The quicker way is similar in principle but reverses the roles of x and y; in this method we slice the area in question into horizontal rectangles.

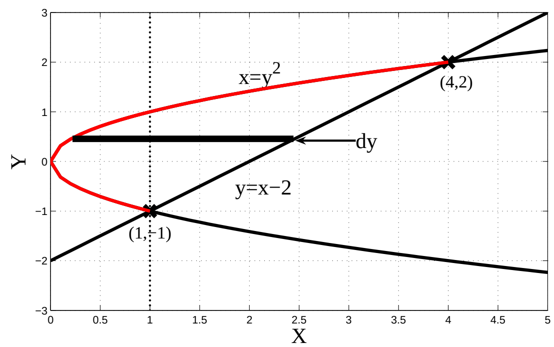

Figure 3: The area between $x = y^2$ and $y = x - 2$ and one horizontal rectangle.

The height of these rectangles is dy; we get their width by subtracting the x-coordinate of the edge on the left curve from the x-coordinate of the edge on the right curve. (If you get mixed up and subtract the right from the left you'll get a negative answer.) The left curve is the sideways parabola $x = y^2$. The right curve is the straight line $y = x - 2$ or $x = y + 2$.

The limits of integration come from the points of intersection we've already calculated. In this case we'll be adding the areas of rectangles going from the bottom to the top (rather than left to right), so from $y = -1$ to $y = 2$.

$$\begin{aligned}
A &= \int_{y=-1}^{y=2} \left[(y+2) - y^2\right] \, dy \\
&= \left[\frac{-y^3}{3} + 2y + \frac{y^2}{2}\right]_{-1}^{2} \\
&= \left(\frac{4}{2} + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right) \\
A &= \frac{9}{2}
\end{aligned}$$

You'll notice that if you plug the limits of integration into the integrand, you get 0. This makes sense; as y goes toward -1 and 2 the width of the rectangles approaches 0.

Volumes by Slicing

Suppose you have a loaf of bread and you want to find the volume of the loaf. One way to do this is to find the volume of each slice and then add up their volumes.

The volume of a slice of bread is its thickness dx times the area a of the face of the slice (the part you spread butter on). So $\Delta V \approx A\Delta x$. In the limit, $dV = A(x)\,dx$. (If your loaf of bread is not perfectly regular, the area of a face might change from slice to slice.) To get the entire volume, sum the volumes of all the slices:

$$V = \int A(x)\,dx$$

The Riemann sum approximating this volume looks like $\displaystyle\sum_{i=1}^{n} A_i\,\Delta x$ if the loaf has n slices.

Solids of Revolution

In theory we could take any three dimensional object and estimate its volume by slicing it into slabs and adding the volumes of the slabs. In practice we'll concentrate exclusively on *solids of revolution.* These are formed by taking an area — for example the arc over the x-axis shown in Figure 1 — and revolving it about an axis to see what volume it sweeps out. If you rotate that arc and its interior about the x-axis you get a shape like an American football or a rugby ball. (See Figure 2.)

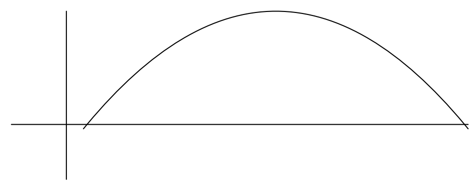

Figure 1: An arc over the x-axis.

Method of Disks

How would we slice up this ball to find its volume? We'll start with the two dimensional picture of an arc over the x-axis. Two dimensional figures are much easier to draw and understand than three dimensional ones; when possible you should avoid drawing three dimensional figures. In this picture we draw a thin rectangle whose base lies on the x-axis and whose height is the height of the arc. The width of this rectangle is dx.

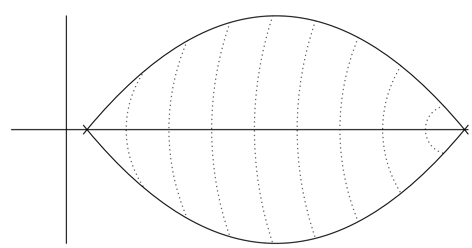

Figure 2: The solid formed by revolving an arc about the x-axis.

Next we need to visualize how this rectangle is related to the three dimensional volume of revolution. When we rotate the arc about the x-axis, the rectangle rotates also as if it were hinged. As it rotates it sweeps out a disk or coin shape. This corresponds to one "slice" of our solid ball, like a slice of bread. The method that we're describing for figuring out the volume of the ball is called the *method of disks* because we're slicing the ball into disks.

We add the volumes of these disks to find the volume of the ball. If the height of the arc over the x-axis is y, then the area $A(x)$ of a face of the disk or slice is πy^2 because the rectangle swept out a circular shape as it spun about the x-axis. The thickness of the disk is dx, so the volume of the disk is

$$dV = (\pi y^2)\, dx.$$

This will be the integrand in the formula for volume associated with the method of disks.

$$V = \int (\pi y^2)\, dx$$

Notice that we have both a y and a dx in our formula, yet we don't have a formula describing y in terms of x. That formula depends on the equation $y = f(x)$ of the arc over the x-axis, and will change depending on the situation.

In addition, we haven't specified any limits of integration yet; again those will depend on the situation.

Example: Volume of a Sphere

Let's practice the method of disks by finding the volume of a soccer ball. We'll find the volume of revolution formed by rotating a circle with center at $(a, 0)$ and radius a about the x-axis. This will sweep out a ball of radius a. Our goal is to find out the volume of this ball.

Remember that our integrand will be $dV = \pi y^2\, dx$. In order to use this we need a formula for y in terms of x, so we need the equation of the circle of radius a centered at $(a, 0)$:

$$(x-a)^2 + y^2 = a^2.$$

We need to describe y in terms of x:

$$\begin{aligned} (x-a)^2 + y^2 &= a^2 \\ y^2 &= a^2 - (x-a)^2 \\ &= a^2 - (x^2 - 2ax + a^2) \\ y^2 &= 2ax - x^2 \end{aligned}$$

We can stop here (without taking any square roots) because the value we need to plug into our formula is y^2. We get:

$$V = \int \pi(2ax - x^2)\, dx.$$

Of course we can't evaluate this integral without knowing the limits of integration. Luckily those limits are easy to find in this example. We're integrating with respect to x. The lowest value of x in our circle is $x = 0$ and the highest is $2a$, so x ranges between 0 and $2a$. Our formula for the volume of a ball becomes:

$$\begin{aligned} V &= \int_0^{2a} \pi(2ax - x^2)\, dx \\ &= \pi\left(ax^2 - \frac{x^3}{3}\right)\Big|_0^{2a} \\ &= \pi\left(4a^3 - \frac{8a^3}{3}\right) - 0 \\ &= \left(\frac{12}{3} - \frac{8}{3}\right)\pi a^3 \\ V &= \frac{4}{3}\pi a^3 \end{aligned}$$

One nice thing about this formula is that we've found more than just the volume of the ball. If we change the upper limit of integration we can also find the volume of a piece sliced off the ball.

$V(x) \quad = \quad$ Volume of a chopped off portion of the sphere with width x

$$
\begin{aligned}
&= \int_0^x \pi(2at - t^2)\,dt \\
V(x) &= \pi\left(ax^2 - \frac{x^3}{3}\right)
\end{aligned}
$$

If we plug in $x = a$ we should get half the volume of the sphere:

$$
\begin{aligned}
V(a) &= \text{Volume of a half sphere} \\
&= \pi\left(a^3 - \frac{a^2}{3}\right) \\
&= \pi\left(a^3 - \frac{a^2}{3}\right) \\
&= \pi\frac{2}{3}a^3 \\
&= \frac{1}{2}\left(\frac{4}{3}\pi a^3\right)
\end{aligned}
$$

This is a good way to check our work.

The formula $V(x) = \pi\left(ax^2 - \frac{x^3}{3}\right)$ turns out to be useful in predicting the behavior of particles in a fluid. When large spherical particles are being pushed around by small ones, will they tend to cluster together or to stick to the sides of the container? Finding the answer to this question involves adding the volumes of two slices off a sphere to find the volume of a lens shaped region.

Example: Volume of a Cauldron

In our next, Halloween themed, example we'll compute the volume of the region shown below.

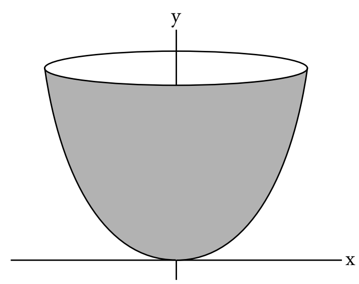

Figure 1: $y = x^2$ rotated around the y-axis.

We could use the method of disks to calculate this volume, but instead we will use the other standard method of finding volumes — the *method of shells*.

A cross section of the cauldron has boundaries $y = x^2$ and $y = a$. We revolve this cross section around the y-axis to get our cauldron. (Note that we can revolve shapes around the y-axis as well as the x-axis.) Our "shell" will be the result of revolving a thin rectangle with its base on $y = x^2$ and its top at $y = a$, as shown in Figure 2. This "shell" shape might also be described as a cylinder. Make your own by rolling a piece of piece of paper into a tube!

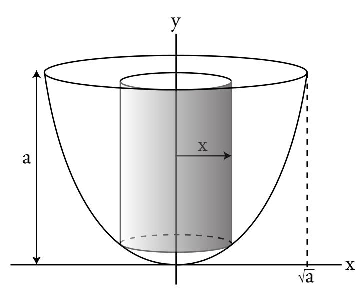

Figure 2: Cylindrical shell with radius x, thickness dx and height $a - y = a - x^2$.

We need to compute the volume of this shell. It's thickness is dx, and its height is $y_{top} - y_{bottom} = a - x^2$. Since the shell is very thin, we get a good approximation of its volume by "unrolling" it like a piece of paper and computing the volume of a rectangular slab with thickness dx, height $a - x^2$ and width equal to the circumference of the shell. To compute the circumference we multiply the radius r by 2π; our estimate of the shell's volume is then:

$$dV = (2\pi x)(a - x^2)\,dx = 2\pi(ax - x^3)\,dx$$

(We can roughly check our work by noting that we're multiplying three lengths here, so the units do match up on both sides of the equation.)

To compute the volume of the cauldron we'll integrate this; all that's left is to find the limits. The entire volume of the cauldron is swept out by the right side of the parabola as it spins about the y-axis, so our limits of integration start at 0 (not $-\sqrt{a}$). In other words, if we just rotate the right half of this it covers the left half; if we counted the volume swept out by the left half we would be counting the volume twice.

The upper limit of integration is at the farthest rightmost spot, where $y = a$ and $y = x^2$ simultaneously; in other words, at $x = \sqrt{a}$. Getting the correct limits is as important as getting the right integrand.

So the volume of the cauldron is:

$$\begin{aligned} V &= 2\pi \int_0^{\sqrt{a}} (ax - x^3)\, dx \\ &= 2\pi \left(a\frac{x^2}{2} - \frac{x^4}{4} \right) \Big|_0^{\sqrt{a}} \\ &= 2\pi \left(\frac{a^2}{2} - \frac{a^2}{4} \right) = 2\pi \left(\frac{a^2}{4} \right) \\ V &= \frac{\pi}{2} a^2. \end{aligned}$$

Question: How do you know whether the rectangle should be vertical or horizontal?

Answer: You can always set it up both ways. One way may be a difficult calculation and one way may be an easier calculation. In the example of $y = x^2$ and $y = x - 2$ the horizontal and the vertical calculations were quite different in character. One of them was really a mess, and one of them was a little easier. This is often the case, and once in a while, one of them is impossible and the other one is possible. By choosing your method carefully you can save yourself a lot of work.

Warning about units.

Previously, we calculated the volume of a parabolic "cauldron" to be $\frac{\pi}{2}a^2$. There's something fishy about this expression — it looks as if it has units of area, but it's describing a volume. In general, we must be very aware of what units we're using.

Suppose the height of the cauldron is $a = 100$cm. Then:

$$\begin{aligned} V &= \frac{\pi}{2}(100)^2\,\text{cm}^3 \\ &= \frac{\pi}{2}10^4\,\text{cm}^3 \\ &= \frac{\pi}{2}10 \sim 16\,\text{liters} \end{aligned}$$

Next, suppose that the height of the cauldron is $a = 1$m. Then:

$$\begin{aligned} V &= \frac{\pi}{2}(1)^2\,\text{m}^3 \\ &= \frac{\pi}{2}10^6\,\text{cm}^3 \\ &= \frac{\pi}{2}1000 \sim 1600\,\text{liters} \end{aligned}$$

But 100cm = 1m. Why are the answers different?

The problem is that we don't know the units in the equation $y = x^2$. If the units are centimeters, then 100cm $= 10^2$cm. If the units are meters then 1m $= 1^2$m. When we use centimeters as units, the cauldron is five times as tall as it is wide, so it looks like:

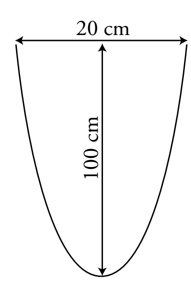

Figure 1: Cauldron cross section for units of centimeters.

When we interpret $y = x^2$ in meters, we find that the cauldron is twice as wide as it is tall, which seems more likely in the context of the problem.

This confusion about units arose because the equation $y = x^2$ is not scale-invariant.