Average Value

You already know how to take the average of a finite set of numbers:

$$\frac{a_1 + a_2}{2} \text{ or } \frac{a_1 + a_2 + a_3}{3}$$

If we want to find the average value of a function $y = f(x)$ on an interval, we can average several values of that function:

$$\text{Average} \approx \frac{y_1 + y_2 + ... + y_n}{n}.$$

As was mentioned previously, if we let the number of values n approach infinity we get:

$$\text{Continuous Average} = \frac{1}{b-a}\int_a^b f(x)\,dx = \text{Ave}(f).$$

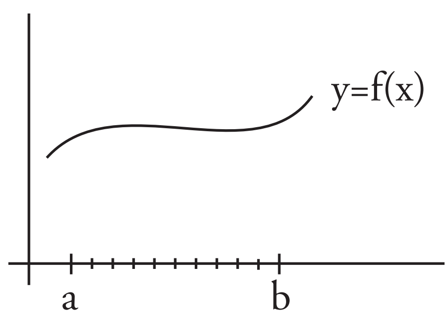

Figure 1: $a \leq x \leq b$.

Why does this describe the average value of $f(x)$? Imagine that you have $n+1$ equally spaced points $a = x_0 < x_1 < x_2 < ... < x_n = b$. The distance between each pair of points is $\Delta x = \frac{b-a}{n}$. Let $y_0 = f(x_0)$, $y_1 = f(x_1), ...y_n = f(x_n)$.

Then the Riemann sum approximating the area under the curve is:

$$(y_1 + y_2 + \cdots + y_n)\Delta x.$$

As n approaches infinity this approaches the area under the curve, which is:

$$\int_a^b f(x)\,dx.$$

$$\begin{aligned}
\frac{1}{b-a}\int_a^b f(x)\,dx &\approx \frac{1}{b-a}(y_1+y_2+\cdots+y_n)\Delta x \\
&= \frac{1}{b-a}(y_1+y_2+\cdots+y_n)\frac{b-a}{n} \\
&= \frac{y_1+y_2+\cdots+y_n}{n},
\end{aligned}$$

so:

$$\frac{1}{b-a}\int_a^b f(x)\,dx \approx \frac{y_1+y_2+\cdots+y_n}{n}.$$

The only difference between the average value and the integral (area under the curve) is that we're dividing by the length of the interval.

Example: Find the average value of $f(x) = c$ on the interval $[a, b]$, where a, b and c are arbitrary constants.

$$\begin{aligned}
\frac{1}{b-a}\int_a^b c\,dx &= \frac{1}{b-a}\cdot(\text{Area of a } (b-a) \text{ by } c \text{ rectangle}) \\
&= \frac{1}{b-a}\cdot(b-a)\cdot c \\
&= c
\end{aligned}$$

If the value of $f(x)$ is always c, then the average value of $f(x)$ had better be c. This confirms that our formula for the average value of a function works, and in particular it confirms that $\frac{1}{b-a}$ is the correct normalizing factor. In this case our Riemann sum becomes:

$$\begin{aligned}
\frac{y_1+y_2+\cdots+y_n}{n} &= \frac{\overbrace{c+c+\cdots+c}^{n\text{times}}}{n} \\
&= \frac{nc}{n} \\
&= c
\end{aligned}$$

and we see why we needed the n in the denominator.

Average Height

Find the average height of a point on a unit semicircle.

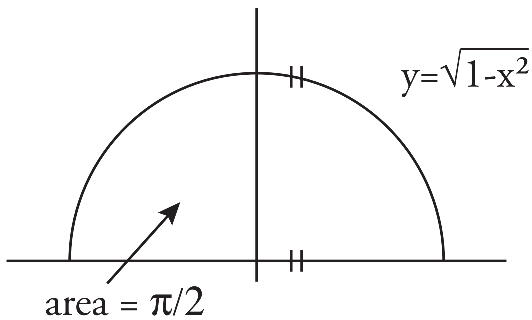

Figure 1: The unit semicircle and an interval dx.

Here $f(x) = \sqrt{1-x^2}$ for $-1 \le x \le 1$, so $a = -1$ and $b = 1$. The average value of $f(x)$ is:

$$
\begin{aligned}
\mathrm{Avg}(f) &= \frac{1}{b-a}\int_a^b f(x)\,dx \\
&= \frac{1}{2}\int_{-1}^{1}\sqrt{1-x^2}\,dx \\
&= \frac{1}{2}(\text{Area of a unit semicircle}) \\
&= \frac{1}{2}\left(\frac{\pi}{2}\right) \\
&= \frac{\pi}{4}.
\end{aligned}
$$

(We will eventually learn how to find the antiderivative of $\sqrt{1-x^2}$ in the unit on techniques of integration.)

Average with Respect to Arc length

Find the average height of a point on a unit semicircle *with respect to the arc length* θ.

When taking averages, it's extremely important to specify the variable with respect to which the average is taking place. The answer may be different depending on the variable!

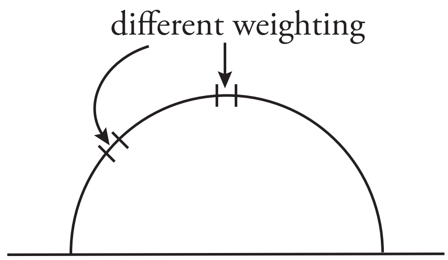

Figure 1: Equal arc lengths correspond to different distances on the x-axis.

As you can see from Figure 1, equal distances along the arc of the semicircle overshadow different lengths on the x-axis. Taking the average with respect to θ will weight the lower parts of the semicircle more heavily than the higher ones. We expect the average with respect to arc length be less than $\frac{\pi}{4}$.

Then average is still given by $\frac{1}{b-a}\int_a^b f(\theta)d\theta$. This time, $a = 0$ and $b = \pi$. The integrand is $y = \sin\theta$, which is the height of the semicircle in terms of θ. So our average height is:

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(\theta)\,d\theta &= \frac{1}{\pi}\int_0^\pi \sin\theta\,d\theta \\ &= \frac{1}{\pi}(-\cos\theta)\Big|_0^\pi \\ &= \frac{1}{\pi}\left(-\cos\pi - (-\cos 0)\right) \\ &= \frac{2}{\pi} \end{aligned}$$

Let's see if we can check our work. The average height with respect to arc length is $\frac{2}{\pi}$. The average height with respect to horizontal distance is $\frac{\pi}{4}$. Points on the semicircle with lower height "count for more" in the computation with respect to arc length, so we'd expect a lower average. This does turn out to be the case because:

$$\frac{2}{\pi} < \frac{\pi}{4} \quad \text{if and only if} \quad 8 < \pi^2.$$

Since $\pi^2 > 9$ we have reason to believe we've found the right answer

Question: How do we interpret the result of the average with respect to arc length?

Answer: One way of thinking of it anticipates our next subject, which is probability. Suppose you picked a point at random along the base of the semicircle (with equal likelihood between -1 and 1) and checked the height above that point. The expected value of that height is given by the first calculation: $\frac{\pi}{4}$.

The second calculation tells you the expected value of the height of a point picked at random on the semicircle, if you were equally likely to pick any point on the semicircle.

Those two average heights are different because distance along the semicircle is different from distance along the x-axis.

Question: Shouldn't the average with respect to arc length have a *bigger* value because the arc length is *longer*?

Answer: Averages never work that way; when we multiply by $\frac{1}{b-a}$ we are dividing by the total length.

However, the average of a constant is that same constant regardless of what variable we use because we compensate by dividing by $b - a$. The difference between an integral and an average is that we're dividing by that total.

Weighted Averages

A *weighted average* is calculated by dividing the weighted total value of a fraction by the total of the weighting function:

$$\frac{\int_a^b f(x)w(x)\,dx}{\int_a^b w(x)\,dx}.$$

Multiplying by $w(x)$ makes some values of $f(x)$ contribute more to the total than other values, depending on the value of x and $w(x)$. Dividing by the integral of $w(x)$ is analogous to dividing by the length or by the number of values.

First we check that this makes sense by confirming that the weighted average of a constant is that same constant:

$$\frac{\int_a^b cw(x)\,dx}{\int_a^b w(x)\,dx} = \frac{c\int_a^b w(x)\,dx}{\int_a^b w(x)\,dx} = c.$$

We see that we were correct to put $\int_a^b w(x)\,dx$ in the denominator.

Now pretend you have a stock which you bought for \$10 one year. Six months later you brought some more for \$20, and then you bought some more for \$30. What's the average price of your stock?

It depends on how many shares you bought. If you bought w_1 shares the first time, w_2 shares the second time and w_3 shares the third time, the total amount that you spent is

$$10w_1 + 20w_2 + 30w_3.$$

The average price per share is the total price divided by the total number of shares:

$$\frac{10w_1 + 20w_2 + 30w_3}{w_1 + w_2 + w_3}$$

This is the discrete analog of the continuous average

$$\frac{\int_a^b f(x)w(x)\,dx}{\int_a^b w(x)\,dx}.$$

The function f is the function describing the price of a share and the weights are the amounts (relative importance) of the different purchases.

Question: You can't factor out the $f(x)$, can you?

Answer: When we found the weighted average of a constant, we factored out c. In

$$\frac{\int_a^b f(x)w(x)\,dx}{\int_a^b w(x)\,dx}$$

we cannot factor out $f(x)$. If the weighted average is interesting you have to do two different integrals to calculate it. It's only when $f(x)$ is constant that you can factor it out (in which case, the calculation is not very interesting at all).

Boiling Cauldron: Introduction

Now let's fill the cauldron from our example with water and light a fire under it to get the water to boil (at 100°C). Let's say it's a cold day: the temperature of the air outside the cauldron is 0^oC. How much energy does it take to boil this water, i.e. to raise the water's temperature from 0°C to 100°C?

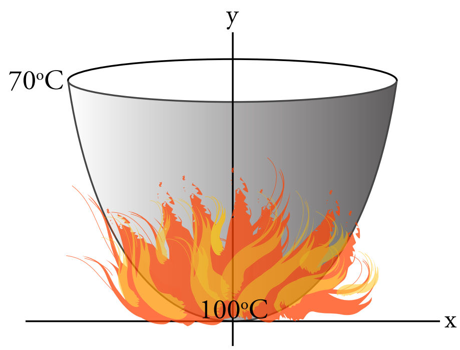

Figure 1: The boiling cauldron.

The temperature of the water is not the same at each level in the kettle. At the bottom of the kettle, where you're heating it up, it's at its highest temperature — 100 degrees Celsius. At the top, it's going to be, say, 70 degrees Celsius. The temperature is varying in height; if it varies linearly the temperature at height y will be $100 - \frac{30}{a}y$ degrees Celsius.

The total amount of heat you need to add is going to be temperature times volume, and some places will get more heat than others. The base of the cauldron will be hottest, but it also has the least volume. The cauldron is widest at the top, so we have more water to heat at that level.

At each horizontal level, the temperature is constant, so we'll use horizontal rectangles in this calculation. We'll revolve these short horizontal rectangles about the y-axis to get disks, calculate the amount of heat needed for each disk, then integrate that value with respect to dy.

Boiling Cauldron, Continued

Last class we asked how much energy it would take to boil water in a cauldron whose shape is found by rotating a parabola with height 1 meter and width 2 meters about the y-axis. The approximately 1600 liters of water would start at a uniform $T = 0$ degrees Celsius. We assumed that the final temperature would be given by the formula

$$T = 100 - 30y \text{degrees Celsius.}$$

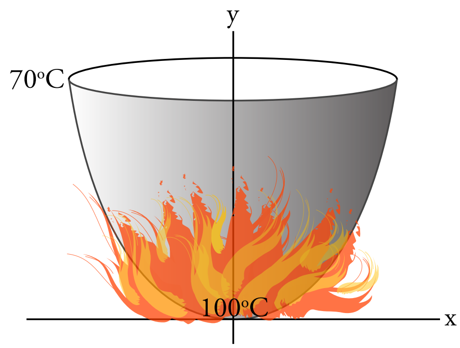

Figure 1: The boiling cauldron.

It's important to realize T is constant on each horizontal level and so the method of disks is the best way to add up the energy needed. If we set up the integral with shells, T would vary from top to bottom of the shell and we'd have to solve a second calculus problem to figure out how much heat was needed for that shell.

The equation of the parabola is $y = x^2$, so each disk will have radius $x = \sqrt{y}$ and height dy. To calculate the amount of energy needed we'll have to add up

$$\text{energy} = \text{degrees} \cdot \text{volume}.$$

So our answer will be:

$$\int_0^1 \underbrace{(100 - 30y)}_{\text{temperature}} \underbrace{\pi x^2\, dy.}_{\text{disk volume}}$$

$$\begin{aligned} \int_0^1 (100 - 30y)\pi y\, dy &= \int_0^1 (100y - 30y^2)\pi\, dy \\ &= \left[50\pi y^2 - 10\pi y^3\right]_0^1 \\ &= 50\pi - 10\pi \\ &= 40\pi \text{ deg} \cdot \text{m}^3 \end{aligned}$$

Our answer has units of degrees Celsius times cubic meters. Let's translate that into something more familiar.

$$\begin{aligned}
40\pi \deg \cdot \mathrm{m}^3 \left(\frac{1\mathrm{cal}}{\deg \cdot \mathrm{cm}^3}\right)\left(\frac{100\mathrm{cm}}{\mathrm{m}}\right)^3 &= 40\pi \cdot 10^6 \text{ calories} \\
&= 40\pi \cdot 1000 \,\mathrm{Kcal} \\
&\approx 125,000\pi \,\mathrm{Kcal}
\end{aligned}$$

One candy bar has about 250 Kcal, so it takes the energy of about 500 candy bars to heat the cauldron.

Question: What does this integral give us?

Answer: We must heat each milliliter of water in the cauldron up to some temperature between 70 and 100 degrees Celsius. For each milliliter there's some amount of energy needed to do that — water lower down in the pot needs to be heated to a higher temperature, and so will require more energy. A calorie is the amount of energy needed to raise one cubic centimeter of water by one degree Celsius. This integral adds up the energy needed to heat every single drop of water in the cauldron to exactly the right temperature.

Before we go on, let's compute the average final temperature. (The average initial temperature is 0 because the temperature is initially constant.) The formula for a weighted average is

$$\frac{\int_a^b f(x)w(x)\,dx}{\int_a^b w(x)\,dx}.$$

In this example, our function $f(x)$ is the temperature T and our weight $w(y) = \pi y$ corresponds to the volume of water in a horizontal disk; the denominator is the total volume of water in the cauldron. The limits of integration are still 0 and 1.

$$\begin{aligned}
\frac{\int_0^1 T\pi y\,dy}{\int_0^1 \pi y\,dy} &= \frac{40\pi}{\pi/2} \\
&= 80 \text{ degrees}
\end{aligned}$$

The value of the weight function $w(y)$ is different at different heights. This makes sense; there's more water at the top of the cauldron than at the bottom. If you tried to take the average the ordinary way you would get:

$$\frac{T_{\max} + T_{\min}}{2} = \frac{100 + 70}{2} = 85 \text{ degrees}.$$

This estimate is higher than the weighted average because it doesn't take into account that there is more water at the top of the cauldron (70 degrees) than at the bottom (100 degrees).

Probability Example

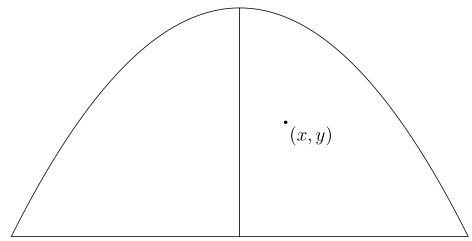

Figure 1: Choose a point at random.

Probability, volumes and weighted averages are three of the most important applications of integration. We'll analyze the probability experiment of picking a point "at random" in the region bounded below by $y = 0$ and above by $y = 1 - x^2$. Inside this parabolic region, the probability of picking a point in a given location is proportional to the area of the location.

What is the chance that $x > 1/2$? In other words, for a point picked at random, what is the probability that $x > 1/2$? Or, what is $P(x > 1/2)$?

$$\begin{aligned} \text{Probability} &= \frac{\text{Part}}{\text{Whole}} \\ &= \frac{\text{Target Area}}{\text{Entire Area}} \\ &= \frac{\text{Success}}{\text{All Possibilities}} \end{aligned}$$

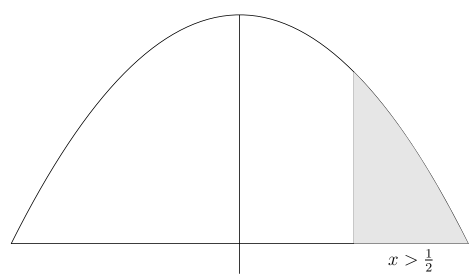

Figure 2: What is the probability that $x > \frac{1}{2}$?

The probability will just be the ratio of the two areas:

$$\frac{\int_{1/2}^{1} (1 - x^2)\, dx}{\int_{-1}^{1} (1 - x^2)\, dx}.$$

If we like, we can think of this as a weighted average with $w(x) = 1 - x^2$, $a = -1$, $b = 1$ and:

$$f(x) = \begin{cases} 0 & \text{when } x < 1/2 \\ 1 & \text{when } x \geq 1/2. \end{cases}$$

$$\begin{aligned}
P(x > 1/2) &= \frac{\int_{1/2}^{1} (1 - x^2)\, dx}{\int_{-1}^{1} (1 - x^2)\, dx} \\
&= \frac{(x - \frac{x^3}{3})\Big|_{1/2}^{1}}{(x - \frac{x^3}{3})\Big|_{-1}^{1}} \\
&= \frac{\left(\frac{2}{3} - \frac{11}{24}\right)}{\left(\frac{2}{3} - \left(-\frac{2}{3}\right)\right)} \\
&= \frac{5}{24} \div \frac{4}{3} \\
&= \frac{5}{32}.
\end{aligned}$$

Probability Summary

If $a \leq x_1 < x_2 \leq b$ and we pick x at random between a and b, then:

$$P(x_1 < x < x_2) = \frac{\int_{x_1}^{x_2} w(x)\,dx}{\int_a^b w(x)\,dx} = \frac{\text{Part}}{\text{Whole}}.$$

In our previous example, the weighting function described the height of a curve above the x-axis.

Our next probability problem will be more realistic. Suppose you're throwing darts at a dart board and your little brother is standing next to the dart board. How likely are you to hit your little brother?

Errata: Heat is Energy

While computing the energy needed to boil the witch's cauldron last class, Professor Jerison said that we were computing *energy* and not *heat*. Energy, heat and work are all different names for the same thing, despite the fact that heat is measured in calories and work in foot-pounds. The only difference between the quantities is the units we use to describe them.

Example: Boy Near a Dart Board

Suppose a seven year old child is throwing darts at a dartboard while her little brother is standing nearby. What is the probability that the brother gets hit by a dart?

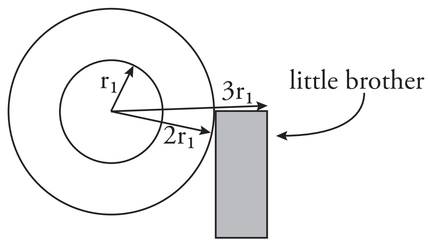

Figure 1: A younger brother stands by a dart board.

In order to turn this into a math problem we must make some assumptions. First, we'll assume something about her aim:

$$\text{Number of hits} = ce^{-r^2}$$

This says that the locations she hits are normally distributed (like a bell curve) about the center of the dart board. She's more likely to hit the dart board, but there's some chance she'll hit the wall next to it, or her brother.

As usual, we'll compute the probability by finding the ratio of the part to the whole. For the final calculation, the "part" will be where the brother is, and the "whole" is all the possible places the dart could hit.

We start by considering a thin ring or annulus of the dart board. The inside of the ring will have radius r_1 and the outside will have radius r_2.

If we graph $y = e^{-r^2}$ we get a "side view" of our probability distribution that looks like Figure 3. The height of the graph indicates that the darts are most likely to land near the center of the dartboard and less likely to land further out. (It turns out that the c in ce^{-r^2} will cancel, so we'll forget about it for now. The value of c depends on the number of darts thrown.)

To calculate the probability of a dart landing in the annulus described by r_1 and r_2 we look at the area between r_1 and r_2, above the r-axis, and below $y = e^{-r^2}$. We'll calculate the volume of revolution of this area, revolving about the center of the dart board (which corresponds to the y-axis in our graph). This volume of revolution will have a ring or washer shape.

Since we've written probability as a function of radius, we'll use the method of shells; the probability of a hit is constant at a given radius. To find the probability of hitting inside the annulus, we use the limits of integration r_1 and

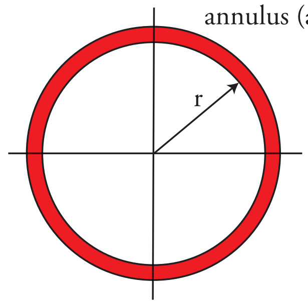

Figure 2: A annulus or ring about the center of the dartboard.

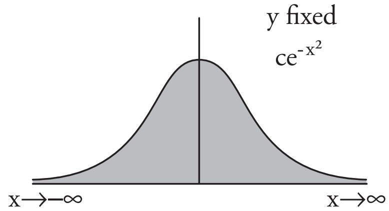

Figure 3: Graph of ce^{-x^2}.

r_2, circumference $2\pi r$, and height e^{-r^2}. We're integrating with respect to r so the thickness of a shell is dr.

$$\int_{r_1}^{r_2} (2\pi r)e^{-r^2}\, dr$$

Question: Why not just find the area under the curve between r_1 and r_2, then revolve that?

Answer: Because a shell with radius r_1 may have significantly less volume than one with radius r_2. We can't simply multiply that area by 2π to get an estimate of the volume of the ring.

Question: So r_1 and r_2 could be anything?

Answer: Yes. We'll do this in general first, and then later we can plug in values for r_1 and r_2.

Question: Why do we have to compute the volume?

Answer: We're computing the number of hits per unit area. The height of the graph corresponds to the number of hits, so we're computing area times height equals volume.

Question: Why is this a realistic calculation?

Answer: The function e^{-r^2} turns out to be the most accurate one for describing where the darts are likely to hit; this model was used to analyze where the V-2 rockets from Germany hit London during World War II.

We're constructing a mathematical model of dart throwing. The probability function e^{-r^2} models the fact that the kid is aiming for the center of the dartboard, but that due to inaccuracies in her throwing she won't always hit the bullseye. The assumption that the model describes the situation accurately does need to be justified, but for now Professor Jerison is asking us to accept it as-is.

Now we'll calculate the probability of a dart landing in the annulus between radii r_1 and r_2:

$$\begin{aligned} \int_{r_1}^{r_2} (2\pi r) e^{-r^2}\, dr &= -\pi e^{-r^2}\Big|_{r_1}^{r_2} \\ &= -\pi e^{-r_2^2} - (-\pi e^{-r_1^2}) \\ &= \pi(e^{-r_1^2} - e^{-r_2^2}). \end{aligned}$$

Note that the calculation doesn't change much if we instead use ce^{-r^2} as the probability function. We'll put it back now:

$$\text{Part} = c\pi(e^{-r_1^2} - e^{-r_2^2})$$

To calculate the volume of the "Whole", we let r range from zero to infinity. This is slightly artificial because when you're playing darts you don't hit the floor or the ceiling. But for the same reason, pretending the wall goes on forever doesn't have much effect on our final answer. The easiest value to calculate has the upper limit of integration equal to infinity; using this approximation will make the numbers come out more cleanly and because the e^{-r^2} is close to zero when r is large our answer will still be pretty accurate.

$$Whole = c\pi(e^{-0^2} - e^{-\infty^2}) = c\pi(1-0) = c\pi$$

The probability of hitting the annulus is then:

$$\begin{aligned} P(r_1 < r < r_2) &= \frac{\text{Part}}{\text{Whole}} \\ &= \frac{c\pi(e^{-r_1^2} - e^{-r_2^2})}{c\pi} \\ P(r_1 < r < r_2) &= e^{-r_1^2} - e^{-r_2^2}. \end{aligned}$$

(Notice that the c did, in fact, cancel.)

For this to be a realistic probability function, it should be true that

$$P(0 < r < \infty) = 1.$$

Since $e^0 = 1$ and $e^{-\infty^2} = 0$, this is in fact true. Another realistic assumption is that the probability of the child hitting the target is about 1/2. In other words, if a is the radius of the target,

$$P(0 \le r \le a) = \frac{1}{2}.$$

To figure out the probability of hitting the little brother, we have to define a "part" corresponding to the little brother. Let's suppose he's not standing too close — maybe he's standing a distance of $2a$ away from the dart board. Volumes of revolution are much easier to compute than volumes of preschoolers, so we'll assume that the parts of the little brother that are most likely to be hit lie on an arc of an annulus. We'll say that he's between $2a$ and $3a$ units away from the dartboard, and that he fills the angular arc between 3 o'clock and 5 o'clock to the right of the board, which is 1/6 of a circle.

So the probability of the little brother getting hit is:

$$\frac{1}{6}P(2a < r < 3a).$$

In order to get a number out of this we need to use the fact that the probability of the girl hitting the dart board is 1/2 or 50%.

$$\begin{aligned} P(0 < r < a) &= 1/2 \\ e^{0^2} - e^{-a^2} &= 1/2 \\ 1 - e^{-a^2} &= 1/2 \\ 1/2 &= e^{-a^2}. \end{aligned}$$

We could calculate a from this, but it turns out we don't need to. We now know all we need to calculate the probability of the little brother being hit. We start by calculating the probability of hitting the entire annulus between $2a$ and $3a$.

$$\begin{aligned} P(2a < r < 3a) &= e^{-(2a)^2} - e^{-(3a)^2} \\ &= e^{-2^2a^2} - e^{-3^2a^2} \\ &= e^{-4a^2} - e^{-9a^2} \\ &= (e^{-a^2})^4 - (e^{-a^2})^9 \\ &= \left(\frac{1}{2}\right)^4 - \left(\frac{1}{2}\right)^9 \\ &= \frac{1}{16} - \frac{1}{512} \\ &= \frac{1}{16} - \frac{1}{512} \\ &\approx \frac{1}{16}. \end{aligned}$$

(Getting the value $\frac{1}{512}$ for $r = 3a$ reassures us that the probably of getting hit does indeed decrease sharply as you get further from the dart board.)

So the probability that the girl hits her little brother is:

$$\begin{aligned}
\frac{1}{6}P(2a < r < 3a) &\approx \frac{1}{6}\frac{1}{16} \\
&\approx \frac{1}{100} \\
&= 1\%.
\end{aligned}$$

In this problem, our weight function was:

$$w(r) = 2\pi c r e^{-r^2}.$$

This is different from the ce^{-r^2} that we started out with. That function is one dimensional; to take into account that we're describing the probability for a whole ring around the center of the dart board we include a factor of $2\pi r$.

If you graph this function you'll see that as r goes to 0 the value of $w(r)$ also goes to zero. This reflects the fact that it's hard to hit the center of the target because the center is very small. It's more likely that a dart will hit a ring around the center, because those rings have greater area than the bull's eye.

Introduction to Numerical Integration

Many functions don't have easy to describe antiderivatives, so many integrals must be (approximately) calculated by computer or calculator. These calculations also take the form of (simpler) weighted averages. There are many different techniques for computing numerical estimates of definite integrals. We'll go over three of these techniques.

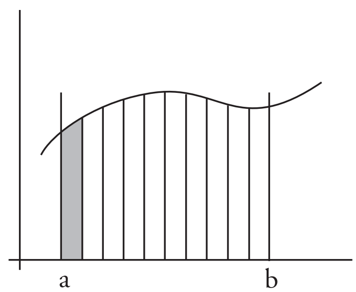

Figure 1: The area under the curve is divided into n regions of equal width.

- **Riemann Sums**

  Riemann sums are a very inefficient way to estimate the area under a continuous curve.

- **Trapezoidal Rule**

  Much more reasonable than Riemann sums, but still lousy.

- **Simpson's Rule**

  Slightly trickier, clever, and pretty good.

Review of Riemann Sums

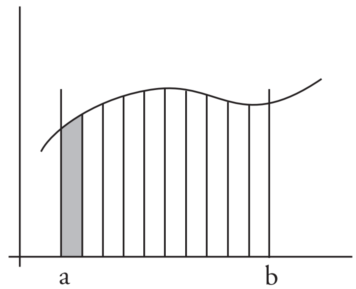

Figure 1: The area under the curve is divided into n regions of equal width.

As was mentioned at the start of this unit, Riemann sums approximate the area between the x-axis and a curve over the interval $[a, b]$ by a sum of areas of rectangles. Each rectangle has width $x_i - x_{i-1} = \Delta x$; there are n rectangles whose sides have x-coordinates $a = x_o < x_1 < x_2 ... < x_n = b$. The heights of the rectangles are $y_o = f(x_o)$, $y_1 = f(x_1), ..., y_{n-1} = f(x_{n-1})$ (if the left edge of each rectangle is exactly as high as the graph).

Our goal is to "average" or add these y-values to get an approximation to

$$\int_a^b f(x)\, dx.$$

The formula for the (left) Riemann sum is:

$$(y_0 + y_1 + \ldots + y_{n-1})\Delta x.$$

If we let the right hand side of each rectangle be as high as the graph, using right endpoints instead of the left endpoints, we get the right Riemann sum:

$$(y_1 + y_2 + \ldots + y_n)\Delta x.$$

Trapezoidal Rule

$$\text{Area} \approx \Delta x \left(\frac{y_o}{2} + y_1 + y_2 + ... + y_{n-1} + \frac{y_n}{2} \right).$$

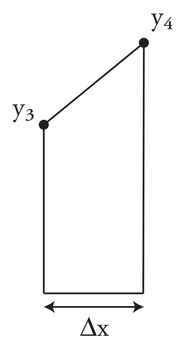

Figure 1: Approximation by areas of trapezoids.

The trapezoidal rule divides up the area under the graph into trapezoids (using segments of secant lines), rather than rectangles (using horizontal segments). As you can see from Figure 1, these diagonal lines come much closer to the curve than the tops of the rectangles used in the Riemann sum.

Remember that the area of a trapezoid is the area of the base times its average height. When applying the trapezoidal rule, the base of a trapezoid has length Δx and its sides have heights y_{i-1} and y_i; trapezoid i has area $\Delta x \frac{y_{i-1}+y+i}{2}$.

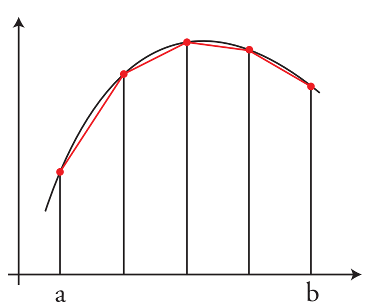

Figure 2: Area $= \left(\frac{y_3+y_4}{2}\right) \Delta x$.

When we add up the areas of all the trapezoids under the curve, we get:

$$\text{Area} \quad = \quad \Delta x \left\{ \frac{y_o + y_1}{2} + \frac{y_1 + y_2}{2} + \frac{y_2 + y_3}{2} + ... + \frac{y_{n-1} + y_n}{2} \right\}$$

$$= \quad \Delta x \left(\frac{y_o}{2} + y_1 + y_2 + ... + y_{n-1} + \frac{y_n}{2} \right).$$

Notice that the trapezoidal rule is the average of the left Riemann sum and the right Riemann sum; it gives a more symmetric treatment of the endpoints a and b than a Riemann sum does.

This looks good and in fact it is much better than a Riemann sum; however, it's still not very efficient.

Simpson's Rule

This approach often yields much more accurate results than the trapezoidal rule does. Again we divide the area under the curve into n equal parts, but for this rule n must be an even number because we're estimating the areas of regions of width $2\Delta x$.

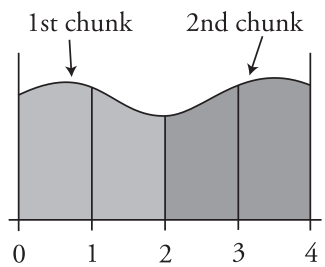

Figure 1: Simpson's rule for n intervals (n must be even!)

When computing Riemann sums, we approximated the height of the graph by a constant function. Using the trapezoidal rule we used a linear approximation to the graph. With Simpson's rule we match quadratics (i.e. parabolas), instead of straight or slanted lines, to the graph. When Δx is small this approximates the curve very closely, and we get a fantastic numerical approximation of the definite integral.

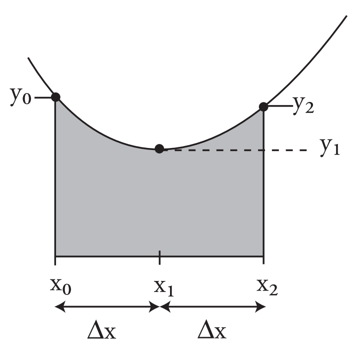

Figure 2: Using a parabolic approximation of the curve.

The derivation of the formula for Simpson's Rule is left as an exercise, but the area of this region is essentially the base times some average height of the graph:

$$\text{Area} = (\text{base})(\text{average height}) = (2\Delta x)\left(\frac{y_o + 4y_1 + y_2}{6}\right).$$

This emphasizes the middle more than the sides, which is consistent with the equations for parabolic approximation.

Simpson's rule gives you the following estimate for the area under the curve:

$$\text{Area} = (2\Delta x)\left(\frac{1}{6}\right)\left[(y_o + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + \cdots + (\cdots y_n)\right].$$

We can combine terms here by exploiting the following pattern in the coefficients:

1	4	1				
		1	4	1		
				1	4	1
1	4	2	4	1	4	1

To get the final form of Simpson's rule:

$$\int_a^b f(x)dx \approx \frac{\Delta x}{3}(y_o + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n).$$

Trapezoid Rule Approximation of $\int_1^2 \frac{dx}{x}$

Continuing our discussion of numerical integration, we'll look at:

$$\int_1^2 \frac{dx}{x}$$

Of course we already know:

$$\begin{aligned}\int_1^2 \frac{dx}{x} &= \ln x\Big|_1^2 \\ &= \ln 2 - \ln 1 \\ &= \ln 2\end{aligned}$$

We can use a calculator to find that this value is approximately 0.693147.

Numerical methods allow us to estimate integrals with accuracy about equal to our accuracy in estimating the integrand. We can approximate the value of $1/x$ pretty well, so we can get a pretty accurate estimate of the value of $\ln 2$. To make life even easier for ourselves, we'll do a very simple case of the approximation — we'll only use two intervals.

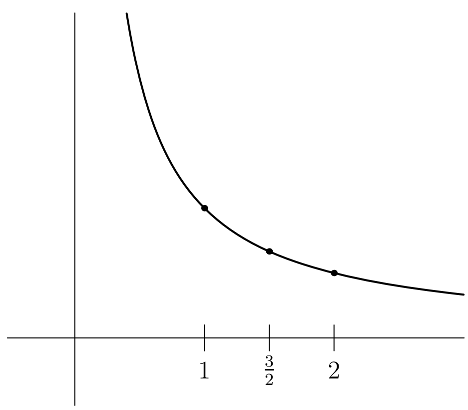

Figure 1: Two intervals; three points on the hyperbola.

We can't expect to get a very good approximation of $\ln 2$ using only two intervals. With only two intervals, we're making estimates of the area under a hyperbola based on only three points (see Figure 1).

Trapezoidal Rule

The trapezoidal rule gives us the following formula for the area under the curve:

$$\text{Area} \approx \Delta x \left(\frac{1}{2} y_0 + y_1 + \frac{1}{2} y_2 \right).$$

In this case $\Delta x = \frac{b-a}{n} = \frac{1}{2}$ because $b = 2$, $a = 1$ and $n = 2$. By evaluating $y_i = f(x_i) = \dfrac{1}{1 + \frac{1}{2}i}$ and plugging these values in, we get:

$$\text{Area} \approx \frac{1}{2}\left(\frac{1}{2} \cdot 1 + \frac{2}{3} + \frac{1}{2} \cdot \frac{1}{2}\right)$$

You wouldn't have to perform this addition on an exam, but if there are only two terms you should add them together. When we do put these numbers into a calculator, we find that the trapezoidal rule gives an estimate of about 0.708 for the area of this region; that's pretty close under the circumstances!

Simpson's Rule Approximation of $\int_1^2 \frac{dx}{x}$

If we use Simpson's rule with two intervals to estimate the value of $\int_1^2 \frac{dx}{x}$, the result is surprisingly accurate:

$$\frac{\Delta x}{3}(y_0 + 4y_1 + y_2) = \frac{1}{6}\left(1 + 4\frac{2}{3} + \frac{1}{2}\right) \approx 0.69444...$$

This is impressively close to our calculator's estimate of 0.693147. (Our calculator uses more than two intervals!)

The accuracy of Simpson's rule is proportional to Δx^4. In this example, that means that if we'd used 10 intervals the error would be about 10^{-4}; we'd have four digit accuracy. The calculation isn't even particularly difficult — it's possible to do it by hand. The other rules for numerical approximation aren't as accurate.

The reason Simpson's rule is more accurate is that it's matching a parabola to the curve, rather than a straight line. Simpson's rule gives the exact area beneath the graphs of functions of degree two or less (parabolas and straight lines), while the other methods are only exact for functions whose graphs are linear.

Oddly, Simpson's rule also gives the exact area under the curve for cubic curves, which explains why the error is comparable to the fourth power of Δx.

However, Simpson's rule does have problems if the derivative is unbounded (e.g. near 0 for the function $f(x) = \frac{1}{x}$) or if the graph is not "smooth" enough.

Numerical Integration Study Tips

How can we remember the formulas for the trapezoidal rule and Simpson's rule to use on the exam? How can we know if we've remembered the formulas correctly?

Instead of memorizing the entire formula, you might memorize a very simple case:

$$f(x) = 1$$

and reconstitute the complete formula from that. The formula for $f(x) = 1$ is easy to check — if the area under the curve doesn't come out to $b - a$, you're doing it wrong.

Consider the formula for the trapezoid rule:

$$\text{Area} \approx \Delta x \left(\frac{1}{2} y_0 + y_1 + \cdots + y_{n-1} + \frac{1}{2} y_n \right).$$

If the function is $f(x) = 1$, then $y_i = 1$ for each value of i:

$$\begin{aligned}
\Delta x \left(\frac{1}{2} + \overbrace{1 + 1 + \cdots + 1}^{(n-1) \text{ of these}} + \frac{1}{2} \right) &= \Delta x \left(\frac{1}{2} + (n-1) + \frac{1}{2} \right) \\
&= \Delta x \cdot n \\
&= \frac{b-a}{n} \cdot n \\
&= b - a \\
&= \int_a^b 1 \, dx.
\end{aligned}$$

You can do the same thing with Simpson's rule.

Area Under the Bell Curve

Today, we'll complete the calculation first mentioned during the discussion of the fundamental theorem of calculus. We've said that:

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

Now we'll prove it. The proof relies on a very clever trick which we would be unlikely to come up with ourselves. We study the proof because the result is very important and because it's related to adding up slices, as we've been doing in this unit.

In our example with the little brother and the dart board, we found the volume of revolution created by rotating the curve e^{-r^2} around the vertical axis.

The calculation went by shells:

$$V = \int_0^\infty 2\pi r\ e^{-r^2}\,dr$$

where $2\pi r$ was the circumference of the shell, e^{-r^2} was the height, and dr was the thickness.

$$\begin{aligned} V &= \int_0^\infty 2\pi r\, e^{-r^2}\,dr \\ &= -\pi\ e^{-r^2}\Big|_0^\infty \\ V &= \pi \end{aligned}$$

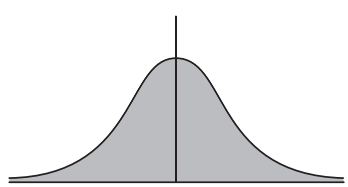

Figure 1: $Q =$ area under e^{-t^2}.

What we need to find now is not a volume but an area:

$$Q = \int_{-\infty}^\infty e^{-t^2}\,dt$$

This is the area under the bell curve shown in Figure 1.

The trick is to compute V in a different way — by slices. Amazingly, we'll discover that $V = Q^2$, which will tell us the value of Q:

$$\begin{aligned} Q^2 &= V \\ Q^2 &= \pi \\ Q &= \sqrt{\pi}. \end{aligned}$$

So now we need to check that $V = Q^2$.

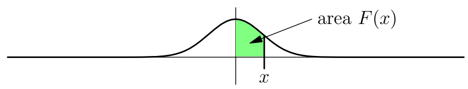

Figure 2: $F(x) = \int_0^x e^{-t^2 dt}\, dt$.

Remember that we have already discussed:

$$F(x) = \int_0^x e^{-t^2}\, dt.$$

We were interested in the limit of $F(x)$ as x approached infinity:

$$F(\infty) = \int_0^\infty e^{-t^2}\, dt.$$

That's the area under half of the bell curve, so:

$$\begin{aligned} Q &= 2F(\infty) \\ F(\infty) &= \frac{\sqrt{\pi}}{2} \end{aligned}$$

By calculating the value of Q we're reassuring ourselves that this is true.

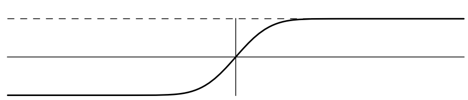

Figure 3: $\lim_{x\to\infty} F(x) = \frac{\sqrt{\pi}}{2}$

We need to prove $V = Q^2$ by using slices; two slices of the surface e^{-r^2} are shown in Figure 4. This is not particularly easy to visualize. It may help to imagine slicing an anthill, a pile of gravel, or some other bump that has circular symmetry.

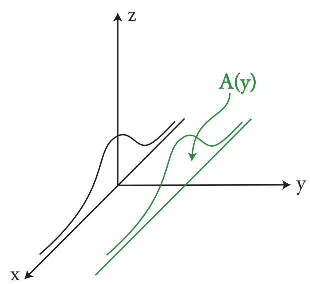

Figure 4: Three dimensional slices of the volume of rotation of e^{-r^2}.

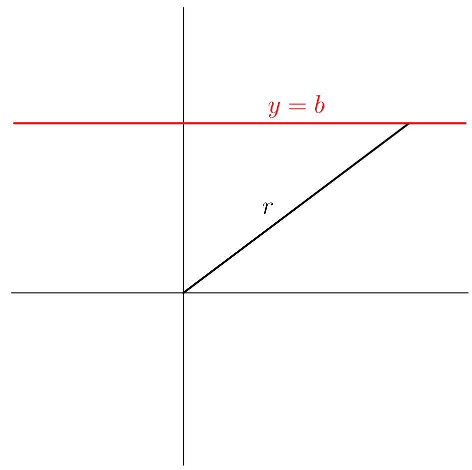

Figure 5: Top view of a slice of the surface of revolution of e^{-r^2}.

The formula for volume by slices is:

$$V = \int_{-\infty}^{\infty} A(y)\,dy.$$

We're going to fix $y = b$ and calculate $A(b)$.

Figure 5 shows a top view of the slice whose area we're calculating. The height of the surface at a point r units away from $(0,0)$ is given by:

$$\text{height} = e^{-r^2}.$$

In terms of b and x, $r^2 = b^2 + x^2$, so

$$\begin{aligned} \text{height} &= e^{-(b^2+x^2)} \\ &= e^{-b^2}e^{-x^2} \\ &= ce^{-x^2} \end{aligned}$$

where c is a constant equal to e^{-b^2}.

The area under the curve is:

$$\begin{aligned} A(b) &= \int_{\infty}^{\infty} e^{-b^2} e^{-x^2}\, dx \\ &= e^{-b^2} \int_{\infty}^{\infty} e^{-x^2}\, dx \\ A(b) &= e^{-b^2} Q \end{aligned}$$

We've calculated the area of a slice of our "bump" in terms of the value Q that we're looking for.

Remember that the volume of that bump is given by:

$$\begin{aligned} V &= \int_{-\infty}^{\infty} A(y)\, dy \\ &= \int_{-\infty}^{\infty} e^{-y^2} Q\, dy \\ &= Q \int_{-\infty}^{\infty} e^{-y^2}\, dy \quad (Q \text{ is a constant}) \\ &= Q^2 \quad (\text{by definition, } Q = \textstyle\int_{-\infty}^{\infty} e^{-t^2}\, dt). \end{aligned}$$

As desired, we've proven that $V = Q^2$ and so we can conclude that $Q = \sqrt{\pi}$.

Question: Doesn't x change as y changes?

Answer: Good question. That's the way x and y have been used in this whole course. But x and y can be different variables; they won't always depend on each other.

In this example, we're fixing $y = b$. The value of y doesn't change. The value of x does change; x varies from negative infinity to positive infinity. In this example y is not a function of x, and that's ok but it's not what we're used to.