Introduction to Maxima and Minima

Suppose you have a function like the one in Figure 1. Find the maximum value of the function. Then find the minimum.

To find the maximum value the function could output, we look at the graph and find the highest point. To find the lowest possible value we find the lowest point on the graph.

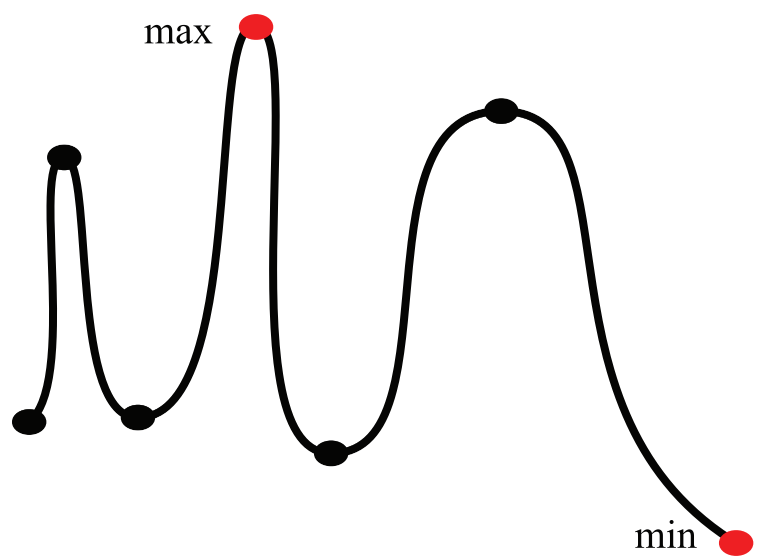

Figure 1: Search for max and min among critical points and endpoints

If you have a sketch, it is very easy to find max and the min. The problem is that the sketch is a lot of work. We don't want to do all that work every single time we need to find a maximum and minimum. Our goal is to use shortcuts; all we need to know is whether the graph is up or down.

Key to Finding Maxima and Minima: We only need to look at critical points *and* endpoints *and* points of discontinuity.

Looking back at our graph of $f(x) = \frac{x}{\ln x}$ we see five places to look for maxima and minima: $x = 0^+$, $x = 1^-$, $x = 1^+$, $x = e$ and $x = \infty$. Of these five points, only one is a critical point; we can't allow ourselves to forget about endpoints and points of discontinuity.

Maximum Area of Two Squares

Consider a wire of length 1, cut into two pieces. Bend each piece into a square. We want to figure out where to cut the wire in order to enclose as much area in the two squares as possible.

In all of these problems you start with a “bunch of words” — a story problem. The two main tasks in starting the problem are to draw a diagram and to pick variables.

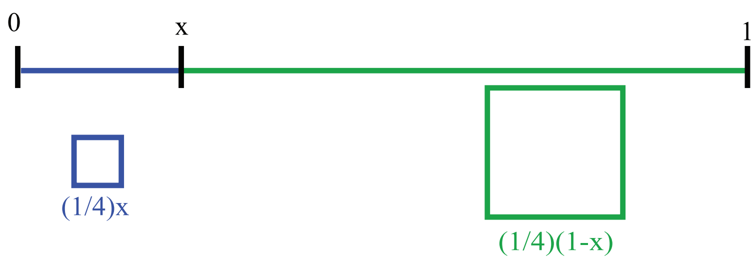

Figure 1: Two pieces of wire enclose two squares.

If we cut the wire so that one piece has length x, the other piece will have length $1-x$. We know that we're bending these pieces of wire into squares, so we can add those squares to our diagram. The first square will have sides of length $\frac{x}{4}$ and the second square will have sides of length $\frac{1-x}{4}$.

We want to find a maximum area, so we'll need formulas for the areas of these squares. The first square's area is $\frac{x^2}{16}$ and the second square has area $\left(\frac{1-x}{4}\right)^2$. The total area is then

$$\begin{aligned} A &= \left(\frac{x}{4}\right)^2 + \left(\frac{1-x}{4}\right)^2 \\ &= \frac{x^2}{16} + \frac{(1-x)^2}{16} \end{aligned}$$

Most calculus students' instinct at this point is to find the critical points — the values x_0 for which $A'(x_0) = 0$. We can do that now if you like:

$$\begin{aligned} A' &= \frac{2x}{16} + \frac{2(1-x)}{16}(-1) \\ &= \frac{x}{8} - \frac{1}{8} + \frac{x}{8} \\ A' &= \frac{2x-1}{8} \\ A' &= 0 \Longrightarrow 2x - 1 = 0 \Longrightarrow x = \frac{1}{2} \end{aligned}$$

So $x = \frac{1}{2}$ is a critical point with critical value:

$$A\left(\frac{1}{2}\right) = \left(\frac{\frac{1}{2}}{4}\right)^2 + \left(\frac{\frac{1}{2}}{4}\right)^2 = \frac{1}{32}$$

We're not done yet, though. We still need to check the endpoints! The length x of the first piece of wire has to satisfy $0 < x < 1$, so we should check the limits as x approaches the endpoints. In this case we can find those limits just by plugging in values. At $x = 0$,

$$A(0^+) = 0^2 + \left(\frac{1-0}{4}\right)^2 = \frac{1}{16}.$$

At $x = 1$,

$$A(1^-) = \left(\frac{1}{4}\right)^2 + 0^2 = \frac{1}{16}.$$

If we try to graph the function with the information we have now, we see that it starts at the point $\left(0, \frac{1}{16}\right)$, dips down to the point $\left(\frac{1}{2}, \frac{1}{32}\right)$, then goes back up to $\left(1, \frac{1}{16}\right)$. (See Fig. **??**.)

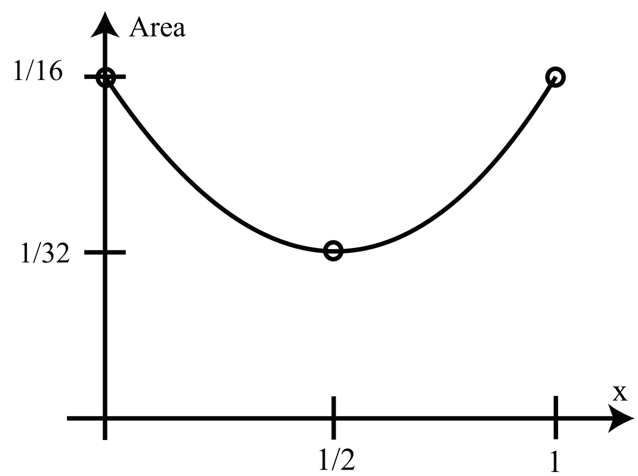

Figure 2: Graph of the area function.

When we found the critical point we did not find the maximum enclosed area — the *minimum* area was achieved at $x = \frac{1}{2}$. The maximum area is not achieved in $0 < x < 1$, but it is achieved at $x = 0$ or 1. The maximum corresponds to using the whole length of wire for one square.

Moral: If you don't pay attention to what the function looks like you may find the worst answer, rather than the best one.

We conclude that the least area enclosed by the two squares is $\frac{1}{32}$, when $x = \frac{1}{2}$; i.e when the two squares are equal. The greatest area enclosed is $\frac{1}{16}$ when $x = 0$ or $x = 1$ and there is only one square.

Commonly asked questions:

What is the minimum? It's the minimum value $\frac{1}{32}$.

Where is the minimum? It's at the critical point $x = \frac{1}{2}$.

These are two different questions. Be sure to answer the correct one; you may get so involved in doing the calculus to find $\frac{1}{2}$ that you forget to find the minimum value $\frac{1}{32}$. Both the critical point and the critical value are important; together they form the point on the graph $(\frac{1}{2}, \frac{1}{32})$ where it turns around.

There are many more and less precise ways to ask these questions. You'll have to do your best to understand what the questions (and answers) mean from the context of the question or example.

Question: Since the goal was to enclose as much area as possible, why did we find the minimum area?

Answer: The reason is that when we go about our procedure of looking for the least or the most, we'll automatically find both. We won't know which one is which until we compare values. It's actually to your advantage to figure out both the maximum and minimum whenever you answer such a question; otherwise you won't understand the behavior of the function very well.

Question: Could we also use the second derivative test here?

Answer: Yes, and we're going to see an example of the second derivative test soon. We could also look at the equation of $A(x)$ and notice that the graph must be a parabola that opens upward.

Max/Min Example 2

This is an example of a minimization problem with a constraint.

Example: Find the box (without a top) with least surface area for a fixed volume.

Again, we start by drawing a diagram and choosing variables. (This time we'll have four variable names.) I'll make things simpler by telling you in advance that the best box has a square bottom — knowing that the length of the box equals its width will let us use one fewer variable. It's often true that the correct answer is something symmetric, but you weren't expected to know this in advance.

Draw a picture of an open-topped box whose length and width equals x and whose height is y. The volume of the box is $V = x^2y$ and its surface area is the area of the base plus the area of each of the four sides: $A = x^2 + 4(xy)$.

The big difference between this problem and the last is that we have the *constraint* that the box must have a certain volume — this determines the relationship between x and y.

$$y = \frac{V}{x^2}$$

We can use this to rewrite the formula for A in terms of a single variable:

$$\begin{aligned}
A(x) &= x^2 + 4x\frac{V}{x^2} \\
A(x) &= x^2 + \frac{4V}{x}
\end{aligned}$$

Now that we have a formula for the value we're trying to minimize — the surface area — we'll follow the same procedure as before. Namely, we'll look for the critical points then check the endpoints and any discontinuities.

To find the critical points we take the derivative of $A(x)$ and set it equal to zero.

$$\begin{aligned}
A(x) &= x^2 + \frac{4V}{x} \\
A'(x) &= 2x - \frac{4V}{x^2}
\end{aligned}$$

$$\begin{aligned}
2x - \frac{4V}{x^2} &= 0 \\
2x &= \frac{4V}{x^2} \\
x &= \frac{2V}{x^2} \\
x^3 &= 2V \\
x &= 2^{\frac{1}{3}}V^{\frac{1}{3}} \quad \text{(critical point)}
\end{aligned}$$

We are not done; we don't even know whether this critical point gives us the most surface area or the least.

Let's check the ends: what are they? What's the smallest possible value of x?

Student: $x > 0$

This is a good answer! We can't build a box with a negative side length.

What's the largest possible value of x? It's true that:

$$x < \sqrt{\frac{V}{y}}$$

but since we said $y = \frac{V}{x^2}$ we can't use this as a limit.

Student: x is less than infinity?

That's right. $0 < x < \infty$.

This is an important realization; if there's no obvious limit on a variable the upper limit is infinity. And infinity is a very important endpoint, and is usually an easy endpoint to check.

We have to consider the possibility that if we shrink the sides all the way to $x = 0$ we'll get a better box. It would be very strange — maybe infinitely tall — but it might have the least surface area; we'll have to see.

$$A(0^+) = \left(x^2 + \frac{4V}{x}\right)\bigg|_{x=0^+}$$

Here the term $\frac{4V}{x}$ is going to infinity as x approaches 0 from the positive side; a box whose surface area approaches infinity is a bad box.

At the other end we get:

$$A(\infty) = \left(x^2 + \frac{4V}{x}\right)\bigg|_{x=\infty}$$

which is also infinite.

We can now draw a schematic for our graph. The surface area goes to infinity as x goes to 0 (from the positive side) and as x goes to infinity. There's only one critical point — only one place where the graph can turn around — so that point must be where the surface area reaches its minimum.

An alternative to checking ends is the *second derivative test*. The second derivative test is not recommended, and for most of the problems in this class it will be hard to use the second derivative test. However, this is a simple enough problem that we are able to use it.

$$\begin{aligned} A'(x) &= 2x - \frac{4V}{x^2} \\ A''(x) &= 2 + \frac{8V}{x^3} \end{aligned}$$

The second derivative is always positive (because x is always positive), so the graph of $A(x)$ is always concave up. This tells us that the critical point is at the lowest point of a "smile shape" and so must correspond to a minimum value of $A(x)$.

Question: Is this the answer to the question, or do we have to calculate y and A and so on?

Answer: What answer is appropriate depends on what the question is. We know that $x = 2^{\frac{1}{3}}V^{\frac{1}{3}}$. If we're going to build to build the box we'll also need to know the y value, which is the height of the box.

$$\begin{aligned} y &= \frac{V}{x^2} \\ &= \frac{V}{(2^{\frac{1}{3}}V^{\frac{1}{3}})^2} \\ y &= 2^{-\frac{2}{3}}V^{\frac{1}{3}} \end{aligned}$$

We could figure out the value of A; this would be appropriate if we wanted to know how much money it was going to cost to build this box.

$$\begin{aligned} A(x) &= x^2 + \frac{4V}{x} \\ &= (2^{\frac{1}{3}}V^{\frac{1}{3}})^2 + \frac{4V}{2^{\frac{1}{3}}V^{\frac{1}{3}}} \\ &= 2^{\frac{2}{3}}V^{\frac{2}{3}} + 4V \cdot 2^{-\frac{1}{3}}V^{-\frac{1}{3}} \\ &= 2^{\frac{2}{3}}V^{\frac{2}{3}} + (2 \cdot 2)2^{-\frac{1}{3}}V^{\frac{2}{3}} \\ &= (2^{\frac{2}{3}} + 2 \cdot 2^{\frac{2}{3}})V^{\frac{2}{3}} \\ &= 3 \cdot 2^{\frac{2}{3}}V^{\frac{2}{3}} \end{aligned}$$

So one possible answer to this problem is that the box with minimum surface area has:

$$\begin{aligned} \text{length} &= 2^{\frac{1}{3}}V^{\frac{1}{3}} \\ \text{height} &= 2^{-\frac{2}{3}}V^{\frac{1}{3}} \text{ and} \\ \text{surface area} &= 3 \cdot 2^{\frac{2}{3}}V^{\frac{2}{3}} \end{aligned}$$

However, there are much more meaningful ways of answering this question. If we use *dimensionless variables* it won't make a difference whether the sides of the box are measured in inches or kilometers and we may learn more about the problem. One famous dimensionless quantity is π, the ratio of the circumference of a circle to its diameter.

For example, $\frac{A}{V^{\frac{2}{3}}} = 3 \cdot 2^{\frac{1}{3}}$ is a dimensionless quantity, because if A is measured in $\in^2$ and V is measured in $\in^3$ the dimensions will cancel. If you increase the volume, you'll increase the surface area by the $\frac{2}{3}$ power of the volume.

The other dimensionless quantity is:

$$\frac{x}{y} = \frac{2^{\frac{1}{3}} V^{\frac{1}{3}}}{2^{-\frac{2}{3}} V^{\frac{1}{3}}} = 2$$

This is the best answer to the question; it tells us that the box with minimum surface area has a base that is twice as wide as its height. This is the optimal shape of the box.

Question: Could we have gotten the answer if we weren't told that the bottom was square?

Answer: Yes. You can do it all in one step if you use multivariable calculus and include variables for both the length and width of the box. Or you can do it in two steps using ideas from this class by first figuring out what rectangle has the smallest perimeter for a fixed area.

Question: Why did you divide x by y?

Answer: We were looking for dimensionless quantities. The lengths x and y are measured in the same units, and x/y gives you the proportions of the box. Another word for what we're interested in here is *proportions*. These are universal, independent of the volume V. The proportions would be the same for any box, at any scale; this is why they provide the nicest answer.

Implicit Differentiation and Min/Max

Example: Find the box (without a top) with least surface area for a fixed volume.

Another way to solve this problem is by using implicit differentiation. As before, this method has some advantages and some disadvantages.

We start the same way:

$$V = x^2 y, \quad A = x^2 + 4xy$$

The goal is to find the minimum value of A while holding V constant.

Next, we just differentiate:

$$\frac{d}{dx}V = 2xy + x^2\frac{dy}{dx} \Longrightarrow 0 = 2xy + x^2 y'$$

So $y' = -\frac{2y}{x}$.

$$\frac{dA}{dx} = 2x + 4y + 4xy'$$

And when we plug in $y' = -\frac{2y}{x}$ we get:

$$\begin{aligned}\frac{dA}{dx} &= 2x + 4y + 4x\left(-\frac{2y}{x}\right)\\ &= 2x + 4y - 8y\\ \frac{dA}{dx} &= 2x - 4y\end{aligned}$$

To find the critical points, we set $\frac{dA}{dx}$ equal to zero and get $0 = 2x - 4y$ or

$$\frac{x}{y} = 2.$$

This method gets to the answer faster and gets the nicer answer — the scale invariant proportions.

The disadvantage is that we did not check whether this critical point is a maximum, minimum, or neither.

Question: How would we check it?

Answer: By looking at the values of $A(0^+)$ and $A(\infty)$ or perhaps by using your intuition — would a very tall box with a tiny base have more or less surface area than a box that's the lower half of a cube? What about a very short box with a wide base?

Introduction to Related Rates

Next we'll look at the subject of related rates, which will give us another opportunity to practice working with several different variables and equations.

Example: Police are 30 feet from the side of the road. Their radar sees your car approaching at 80 feet per second when your car is 50 feet away from the radar gun. The speed limit is 65 miles per hour (which translates to 95 feet per second). Are you speeding?

How do you set up a problem like this? First, draw a diagram of the setup (as in Fig. 1):

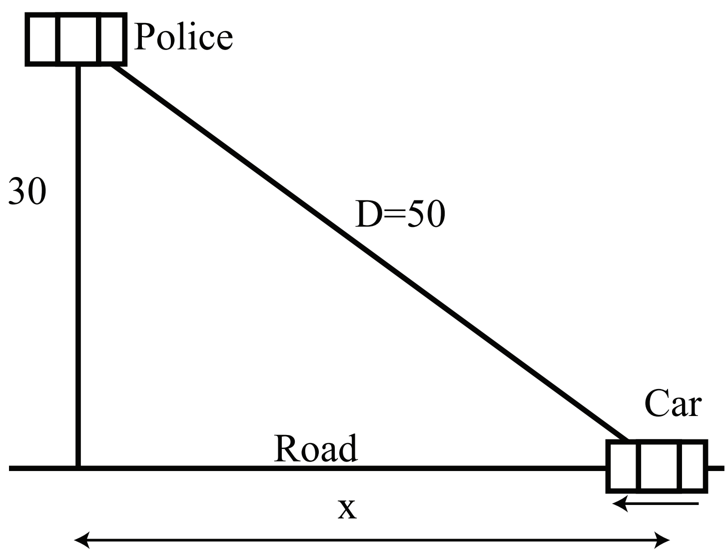

Figure 1: Illustration of example 1: triangle with the police, the car, the road, and labeled variables.

We see that this is a question about a right triangle — the line from the police car to your car is the hypotenuse, the road is one leg of the triangle, and the other leg is a segment from the police car to the road. The Pythagorean Theorem tells us that you're 40 feet away from the right angled vertex of the triangle.

How do we name the variables needed to turn this into a calculus problem? The important thing to figure out is which variables are changing.

We'll choose variable t to stand for time in seconds. The distance along the road from your car to the police car is an important value which we will call x. (We know that $x = 40$ to begin with, but that value will change as the car moves.) Once we have those two variables, we can rewrite the question as "Is $\frac{dx}{dt}$ bigger than 95 feet per second?"

There's another value that is changing, which is the length of the hypotenuse, or the distance between the police car and your car. We'll call this D. (Because we know something about speed limit enforcement, we can correctly assume that the police car is not moving and so the length of the third side of the triangle is not variable.) The rate at which D is changing is exactly what the police are measuring with their radar gun:

$$\frac{dD}{dt} = D' = -80.$$

The derivative must be negative because the value of D is decreasing.

Introduction to Related Rates

We're continuing with a related rates problem from last class.

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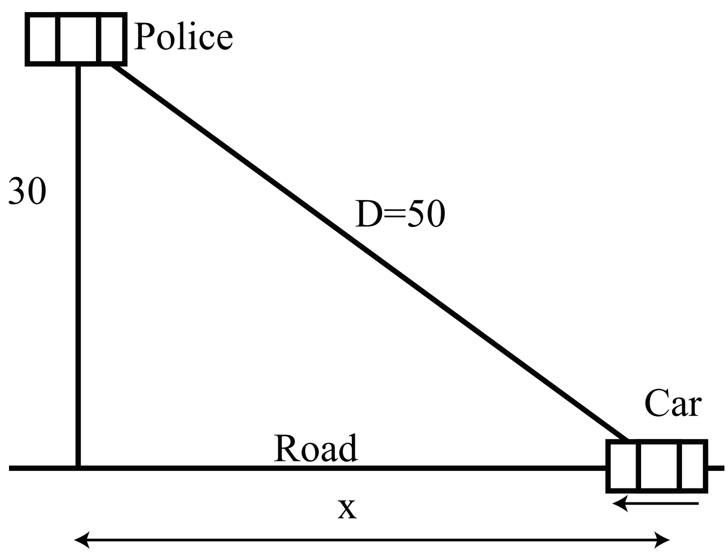

Figure 1: Illustration of example 1: triangle with the police, the car, the road, D and x labeled.

We chose t to stand for time in seconds, x to represent the distance along the road from your car to the police car, and D to represent the straight line distance between your car and the police car.

We know that $\frac{dD}{dt} = -80\text{ft/sec}$ and want to find out whether $\frac{dx}{dt} < -95\text{ft/sec}$ $\cong 65\text{mi/hr}$.

To answer this question we need to understand how x is related to D. First, we know from the Pythagorean theorem that:

$$30^2 + x^2 = D^2$$

We'll differentiate this equation with respect to time using implicit differentiation; we could solve for x, but this would take longer.

While we do this, we must be careful not to replace a variable like D by a constant like 50. The number 50 is a constant — its rate of change is 0. The rate of change of D is -80ft/sec. We must differentiate first before plugging in values.

$$\frac{d}{dt}\left(30^2 + x^2\right) = \frac{d}{dt}\left(D^2\right) \implies 2xx' = 2DD' \implies x' = \frac{2DD'}{2x}$$

Now we can plug in the instantaneous numerical values:

$$x' = \frac{2 \cdot 50 \cdot (-80)}{2 \cdot 40} = -100\frac{\text{feet}}{\text{s}} \cong -68\frac{\text{mi}}{\text{hr}}$$

This exceeds the speed limit of 95 feet per second. You are, in fact, speeding.

There is another, longer, way of solving this problem. Start with:

$$D = \sqrt{30^2 + x^2} = (30^2 + x^2)^{1/2}$$

$$\frac{d}{dt}D = \frac{1}{2}(30^2 + x^2)^{-1/2}(2x\frac{dx}{dt})$$

Plug in the values:

$$-80 = \frac{1}{2}(30^2 + 40^2)^{-1/2}(2)(40)\frac{dx}{dt}$$

and solve to find:

$$\frac{dx}{dt} = -100\frac{\text{feet}}{\text{s}}$$

A third strategy is to differentiate $x = \sqrt{D^2 - 30^2}$. It is easiest to differentiate the equation in its simplest algebraic form $30^2 + x^2 = D^2$, which was our first approach.

The general strategy for these types of problems is:

1. Draw a picture. Set up variables and equations.
2. Take derivatives.
3. Plug in the given values. Don't plug the values in until *after* taking the derivatives.

Related Rates, A Conical Tank

Example: Consider a conical tank whose radius at the top is 4 feet and whose depth is 10 feet. It's being filled with water at the rate of 2 cubic feet per minute. How fast is the water level rising when it is at depth 5 feet?

As always, our first step is to set up a diagram and variables.

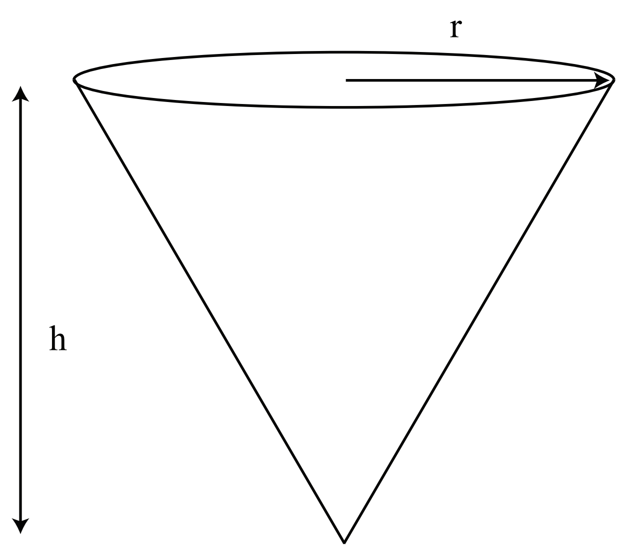

Figure 1: Illustration of example 2: inverted cone water tank.

This diagram just helps us to start thinking about the problem. For instance, we see that because the cone is narrower at the bottom the rate of change of the depth will vary; we need to depict the water level. We also realize that it's difficult to draw useful and accurate diagrams of three dimensional figures — a simple schematic may be more helpful.

The key here is to draw a two-dimensional cross-section. In the figure we're looking at one half of a vertical slice of the tank. The height of the slice equals 10 feet, which is the height of the tank. The widest part of the slice is 4 feet, which is the distance from center to edge of the top of the tank.

We'll use the variable r will represent the distance from center to edge of the top of the water, and h will represent the height of the top of the water (which is also the depth of the water). We can find the relationship between r and h from Fig. 2) using similar triangles:

$$\frac{r}{h} = \frac{4}{10}.$$

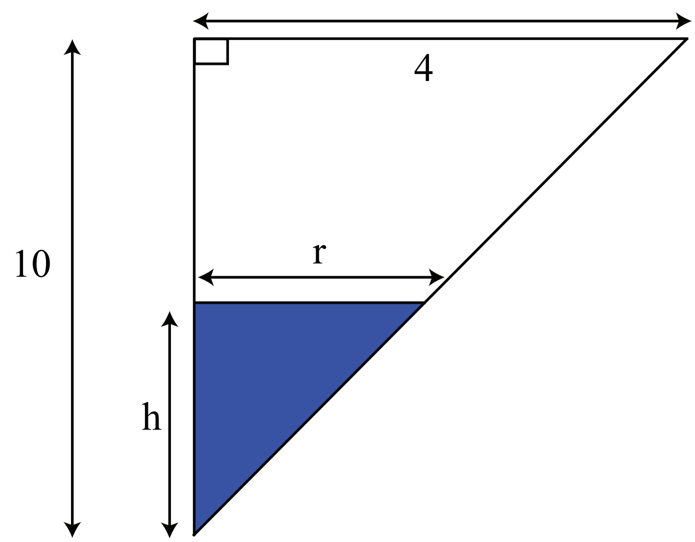

Figure 2: Relating r and h.

Our goal is to find out how fast the water is rising when the tank is half full. What we know is that the volume of water in the tank is changing at a rate of 2 cubic feet per minute. We need equations relating the volume of water in the tank to its depth, h.

The volume of a cone is $\frac{1}{3} \cdot \text{base} \cdot \text{height}$. From Fig. 1), the volume of this tank is given by:

$$V = \frac{1}{3} \cdot \underbrace{\pi r^2}_{\text{base}} \cdot \underbrace{h}_{\text{height}}$$

This relates the volume to the height and radius, and we know the relation between the hight and the radius. We have one more piece of information that we can use: $\frac{dV}{dt} = 2$.

The question is: "What is $\frac{dh}{dt}$ when $h = 5$?"

We've now translated all of the words in the original problem into formulas. Our word problem is now simply a calculus problem.

We could do this by implicit differentiation, but it's easy enough to solve for r in terms of h that there's no need to.

$$r = \frac{2}{5}h.$$

We plug this expression for r back into V to get:

$$V = \frac{1}{3}\pi\left(\frac{2}{5}h\right)^2 h = \frac{4}{3(25)}\pi h^3$$

At this point we could solve for h, but that turns out to be a bad idea. Implicit differentiation is much easier.

$$\begin{aligned}\frac{dV}{dt} &= \frac{dV}{dh}\frac{dh}{dt} \\ &= \frac{\pi}{3}\left(\frac{2}{5}\right)^2 3h^2\frac{dh}{dt} \\ &= \frac{4}{25}\pi h^2 h'\end{aligned}$$

Now that we've calculated the rates of change we can plug in the numbers $\frac{dV}{dt} = 2$ and $h = 5$:

$$2 = \left(\frac{4}{25}\right)\pi(5)^2 h'$$

$$2 = 4\pi h'$$

$$h' = \frac{1}{2\pi}\text{ft/min}$$

We were given the rate at which the volume of water in the tank was changing and we used that to compute the rate at which the water in the tank was rising. At the heart of this calculation was the chain rule:

$$\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt}.$$

Related rates problems are all about applying the chain rule to solve word problems.

Ring on a String

We're going to do one more max/min problem.

Consider a ring on a string held fixed at two ends at $(0, 0)$ and (a, b) (see Fig. 1). The ring is free to slide to any point. Find the position (x, y) that the ring slides to.

Note that if $b = 0$, i.e. if the two ends are at equal heights, the ring will settle midway between the two ends ($x = \frac{a}{2}$). We can perform this experiment physically and see the result; we now want to explain that result mathematically. One reason to be interested in this problem is that it's one of many problems that must be solved in order to build a suspension bridge.

Professor Jerison drew a diagram of the possible positions of the ring in lecture by tracing the position of an actual ring on a string held by two students. The next step after drawing this diagram is to name and label the variables, as shown in Figure 1.

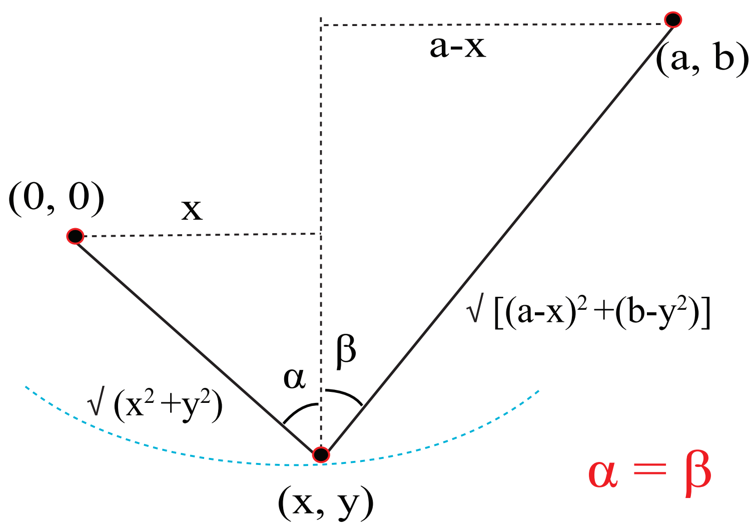

Figure 1: Illustration of the Ring on a String problem.

Physical Principle: The ring settles at the lowest height (lowest potential energy), so the problem is to minimize y subject to the constraint that (x, y) is on the string.

Constraint: The length L of the string is fixed.

$$\sqrt{x^2 + y^2} + \sqrt{(x - a)^2 + (y - b)^2} = L.$$

The function $y = y(x)$ is determined implicitly by the constraint equation above. We traced the constraint curve (possible positions of the ring) on the blackboard; the curve is also suggested in blue in Figure 1. This curve is an ellipse with foci at $(0,0)$ and (a,b), but knowing that the curve is an ellipse does not help us find the lowest point.

Experiments with the hanging ring show that the lowest point is somewhere between $x = 0$ and $x = a$. (This is one way we can confirm that the minimum solution isn't at one of the ends of the string; don't try to use the second derivative test.) Since the ends of the constraint curve are higher than the middle, the lowest point is a critical point (a point where $y'(x) = 0$). In class we also gave a physical demonstration of this by drawing the horizontal tangent at the lowest point.

To find the critical point, differentiate the constraint equation implicitly with respect to x:

$$\frac{x + yy'}{\sqrt{x^2 + y^2}} + \frac{x - a + (y - b)y'}{\sqrt{(x-a)^2 + (y-b)^2}} = 0.$$

Since $y' = 0$ a the critical point, the equation can be rewritten as:

$$\frac{x}{\sqrt{x^2 + y^2}} = \frac{a - x}{\sqrt{(x-a)^2 + (y-b)^2}}$$

From Fig. 1, we see that the last equation can be interpreted geometrically as saying that:

$$\sin\alpha = \sin\beta \quad \Longrightarrow \quad \alpha = \beta,$$

where α and β are the angles the left and right portions of the string make with the vertical.

Physical and geometric conclusions

The angles α and β are equal.

Using vectors to compute the force exerted by gravity on the two halves of the string, one finds that there is *equal tension* in the two halves of the string — a physical equilibrium. This is desirable in construction; if one end is under more stress than the other, it's more likely to break.

From another point of view, the equal angle property expresses a geometric property of ellipses: Suppose that the ellipse is a mirror. A ray of light from the focus $(0,0)$ reflects off the mirror according to the rule angle of incidence equals angle of reflection, and therefore the ray goes directly to the other focus at (a,b). This was used to good effect in the "Strokes of Genius: Mini Golf by Artists" exhibit at the DeCordova museum in the early 1990's; by placing the tee at one focus of an ellipse and the hole at the other, an artist created a golf course on which any stroke would end with a hole in one.

Formulae for x and y

We did not yet find the location of (x,y). We will now show that:

$$x = \frac{a}{2}\left(1 - \frac{b}{\sqrt{L^2 - a^2}}\right), \quad y = \frac{1}{2}\left(b - \sqrt{L^2 - a^2}\right).$$

Because $\alpha = \beta$,

$$x = \sqrt{x^2 + y^2}\ \sin\alpha; \quad a - x = \sqrt{(x-a)^2 + (y-b)^2}\ \sin\alpha$$

Adding these two equations,

$$a = \left(\sqrt{x^2+y^2} + \sqrt{(x-a)^2+(y-b)^2}\right)\sin\alpha = L\sin\alpha \implies \sin\alpha = \frac{a}{L}$$

The equations for the vertical legs of the right triangles are (note that $y < 0$):

$$-y = \sqrt{x^2 + y^2}\ \cos\alpha; \quad b - y = \sqrt{(x-a)^2 + (y-b)^2}\ \cos\beta.$$

Adding these two equations, and using $\alpha = \beta$, we get:

$$b - 2y = \left(\sqrt{x^2+y^2} + \sqrt{(x-a)^2+(y-b)^2}\right)\cos\alpha = L\cos\alpha$$

$$\implies$$

$$y = \frac{1}{2}(b - L\cos\alpha).$$

Use the relation $\sin\alpha = \dfrac{a}{L}$ to write:

$$\begin{aligned} L\cos\alpha &= L\sqrt{1-\sin^2\alpha} \\ &= \sqrt{L^2 - a^2}. \end{aligned}$$

Then the formula for y is:

$$y = \frac{1}{2}\left(b - \sqrt{L^2-a^2}\right).$$

Finally, to find the formula for x, use similar right triangles:

$$\tan\alpha = \frac{x}{-y} = \frac{a-x}{b-y} \implies x(b-y) = (-y)(a-x) \implies (b-2y)x = -ay$$

Therefore,

$$x = \frac{-ay}{b-2y} = \frac{a}{2}\left(1 - \frac{b}{\sqrt{L^2-a^2}}\right).$$

Thus we have formulae for x and y in terms of a, b and L.

This derivation of the formulae for x and y wasn't covered in lecture because it is long and because the most illuminating part of the problem is the balance condition $\alpha = \beta$ that is an immediate consequence of the critical point computation.

Final Remark. In 18.02, you will learn to treat constrained max/min problems in any number of variables using a method called Lagrange multipliers.

Newton's Method

Newton's method is a powerful tool for solving equations of the form $f(x) = 0$.

Example: Solve $x^2 = 5$.

We're going to use Newton's method to find a numerical approximation for $\sqrt{5}$. Any equation that you understand can be solved this way. In order to use Newton's method, we define $f(x) = x^2 - 5$. By finding the value of x for which $f(x) = 0$ we solve the equation $x^2 = 5$.

Our goal is to discover where the graph crosses the x-axis. We start with an initial guess — we'll guess $x_0 = 2$, since $\sqrt{5} \approx \sqrt{4} = 2$. This is not a very good guess; $f(2) = -1$, and we're looking for a number x for which $f(x) = 0$. We'll try to improve our guess.

We pretend that the function is linear, and look for the point where the tangent line to the function at x_0 crosses the x-axis: see Fig. 1. This point $(x_1, 0)$ gives us a new guess at our solution: x_1.

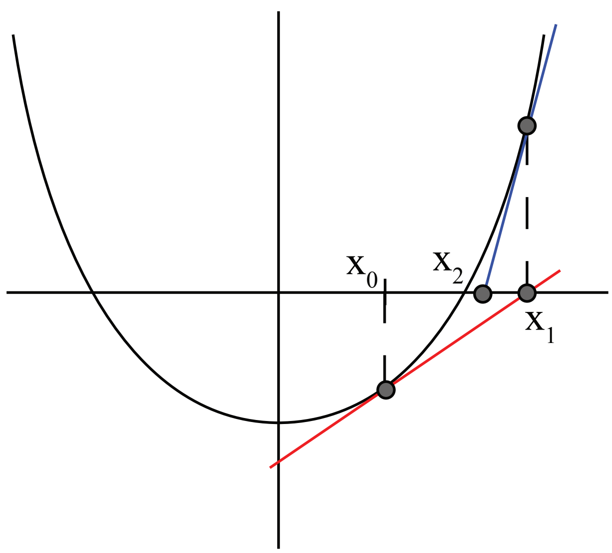

Figure 1: Illustration of Newton's Method

The equation for the tangent line is:

$$y - y_0 = m(x - x_0)$$

When the tangent line intercepts the x-axis $y = 0$, and the x coordinate of that point is our new guess x_1.

$$\begin{aligned} -y_0 &= m(x_1 - x_0) \\ -\frac{y_0}{m} &= x_1 - x_0 \\ x_1 &= x_0 - \frac{y_0}{m} \end{aligned}$$

In terms of f:

$$\begin{aligned} y_0 &= f(x_0) \\ m &= f'(x_0) \end{aligned}$$

because m is the slope of the tangent line to $y = f(x)$ at the point (x_0, y_0). Therefore,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

The point of Newton's method is that we can improve our new guess by repeating this process. To get our $(n+1)^{st}$ guess we apply this formula to our n^{th} guess:

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.}$$

In our example, $x_0 = 2$ and $f(x) = x^2 - 5$. We first calculate $f'(x) = 2x$. Thus,

$$\begin{aligned} x_1 &= x_0 - \frac{(x_0^2 - 5)}{2x_0} = x_0 - \frac{1}{2}x_0 + \frac{5}{2x_0} \\ x_1 &= \frac{1}{2}x_0 + \frac{5}{2x_0} \end{aligned}$$

The main idea is to repeat (iterate) this process:

$$\begin{aligned} x_2 &= \frac{1}{2}x_1 + \frac{5}{2x_1} \\ x_3 &= \frac{1}{2}x_2 + \frac{5}{2x_2} \end{aligned}$$

and so on. The procedure approximates $\sqrt{5}$ extremely well.

Let's see how well this works:

$$
\begin{aligned}
x_1 &= \frac{1}{2}2 + \frac{5}{2 \cdot 2} \\
&= 1 + \frac{5}{4} \\
&= \frac{9}{4} \\
x_2 &= \frac{1}{2}\frac{9}{4} + \frac{5}{2\frac{9}{4}} \\
&= \frac{9}{8} + \frac{5}{2}\frac{4}{9} \\
&= \frac{9}{8} + \frac{10}{9} \\
&= \frac{161}{72} \\
x_3 &= \frac{1}{2}\frac{161}{72} + \frac{5}{2}\frac{72}{161}
\end{aligned}
$$

n	x_n	$\sqrt{5} - x_n$
0	2	2×10^{-1}
1	$\frac{9}{4}$	10^{-2}
2	$\frac{161}{72}$	4×10^{-5}
3	$\frac{1}{2}\frac{161}{72} + \frac{5}{2}\frac{72}{161}$	10^{-10}

Notice that the number of digits of accuracy doubles with each iteration; x_2 is as good an approximation as you'll ever need, and x_3 is as good an approximation as the one displayed by your calculator.

Newton's Method

Today we'll discuss the accuracy of Newton's Method.

Recall how Newton's method works: to find the point at which a graph crosses the x-axis you make an initial guess x_0 at the x-coordinate of that crossing. You then find the tangent line to the graph at x_0 and use it to improve your guess: x_1 is the x-coordinate at which the tangent line crosses the x-axis. (See Fig. 1.) You can now draw the tangent line at x_1 to get a new guess x_2, and so on.

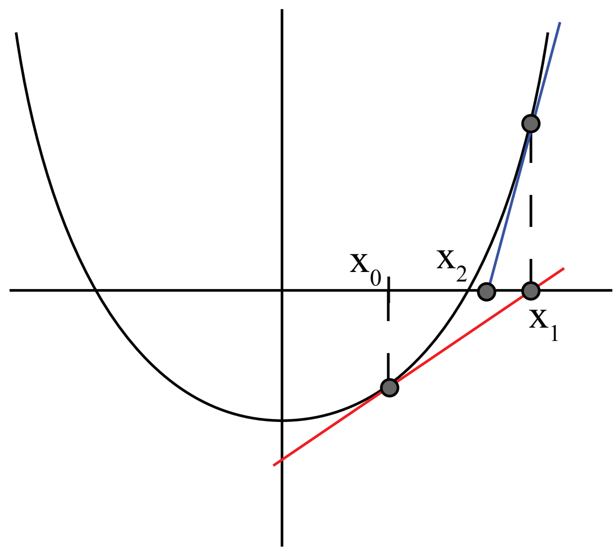

Figure 1: Illustration of Newton's Method

In algebraic terms,

$$\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}}$$

Figure 2 illustrates the k^{th} iteration of Newton's method.

If we're going to use this to get numerical approximations of solutions, we should know how accurate it is. If x is the exact value of the solution, then x_1 is $E_1 = |x - x_1|$ away from the exact answer. The error in our approximation at step n is $E_n = |x - x_n|$.

Last time we saw that error values of $E_n = |\sqrt{5} - x_n|$ quickly became very close to zero. It turns out that $E_2 \sim E_1^2$. So if $E_0 = 10^-1$, the size of the error can be expected to decrease as follows:

E_0	E_1	E_2	E_3	E_4
10^{-1}	10^{-2}	10^{-4}	10^{-8}	10^{-16}

The number of digits of accuracy doubles at each step!

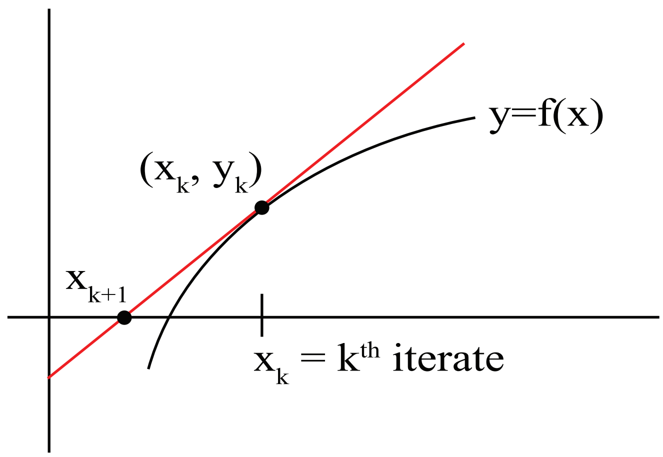

Figure 2: Illustration of Newton's Method.

Newton's method works (very) well if $|f'|$ is not too small, $|f''|$ is not too big, and x_0 starts near the solution x.

Newton's Method: What Could Go Wrong?

Newton's method works (very) well if $|f'|$ is not too small, $|f''|$ is not too big, and x_0 starts near the solution x.

We're not going to discuss these conditions in detail, but let's see why they're there. If f'' is too large the graph would be sharply curved, in which case the tangent line might not be a good approximation to the graph and x_1 might not be close to the solution. There are a couple of things that can go wrong if x_0 is too far from x_1, which we'll discuss now.

If the error $E_0 = |x - x_0|$ is greater than 1 and $E_1 \sim E_0^2$, the error of your estimate could actually *increase* as you apply Newton's method.

In the example $f(x) = x^2 - 5$, if we had chosen $x_0 = -2$ we would have found the solution $-\sqrt{5}$ and not $\sqrt{5}$. This convergence to an unexpected root is illustrated in Fig. 1

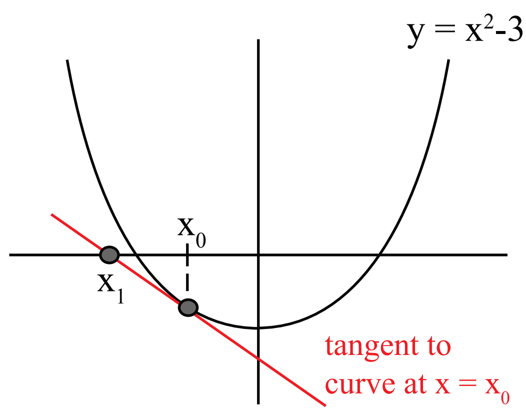

Figure 1: Newton's method converging to an unexpected root.

In the same example, if we chose $x_0 = 0$ then $f'(x_0) = 0$ and $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ is undefined.

Finally, there's a chance that Newton's method will cycle back and forth between two value and never converge at all. This failure is illustrated in Fig. 2; $x_2 = x_0$, $x_3 = x_1$, and so forth.

Newton's method is a good way of approximating solutions, but applying it requires some intelligence. You must beware of getting an unexpected result or no result at all. The better your initial guess at the solution, the more likely you are to get a correct result.

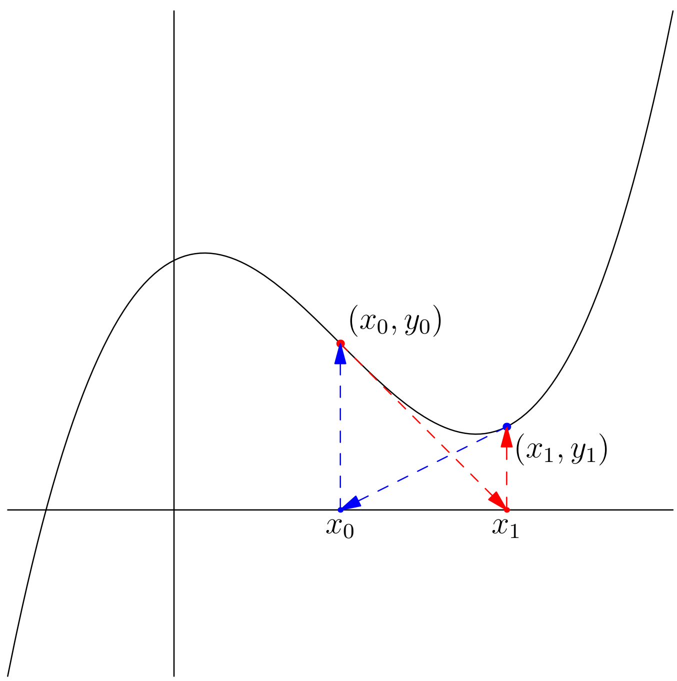

Figure 2: Newton's method cycling between x_0 and x_1.