Parametric Curve

We're going to continue to work in three dimensional space, moving on to parametric equations; in particular we'll discuss parametric curves. This is another topic that will help us prepare for multivariable calculus; it's the beginning of the transition to multivariable thinking.

We're going to consider curves that are described by x being a function of t and y being a function of t.

$$\begin{aligned} x &= x(t) \\ y &= y(t). \end{aligned}$$

The variable t is called a *parameter*. The easiest way to think of parametric curves is as t equaling time and the position $(x(t), y(t))$ describing a *trajectory* in the plane.

The point $(x(0), y(0))$ describes a position at time $t = 0$. The point $(x(1), y(1))$ describes a later position at time $t = 1$. When we draw the trajectory it's a good idea to draw arrows on the curve that show what direction $(x(t), y(t))$ moves in as t increases.

Our first example will be to figure out what sort of curve is described by:

$$\begin{aligned} x &= a \cos t \\ y &= a \sin t. \end{aligned}$$

To do this we want to figure out what equation describes the curve in rectangular coordinates. Ideally, we quickly realize that:

$$\begin{aligned} x^2 + y^2 &= (x(t))^2 + (y(t))^2 \\ &= a^2 \cos^2 t + a^2 \sin^2 t \\ x^2 + y^2 &= a^2. \end{aligned}$$

The curve is a circle with radius a.

Another thing to keep track of is which direction we're going on this circle. There's more to this curve than just its shape; there's also where we are at what time, as with the trajectory of a planet. We'll figure this out by plotting a few points:

When $t = 0$, $(x, y) = (a \cos 0, a \sin 0) = (a, 0)$.

When $t = \frac{\pi}{2}$, $(x, y) = (a \cos \frac{\pi}{2}, a \sin \frac{\pi}{2}) = (0, a)$.

We deduce that the trajectory moves counterclockwise about the circle of radius a centered at the origin. (See Figure 1.)

Next time we'll learn to keep track of the arc length and to understand how fast $(x(t), y(t))$ is changing.

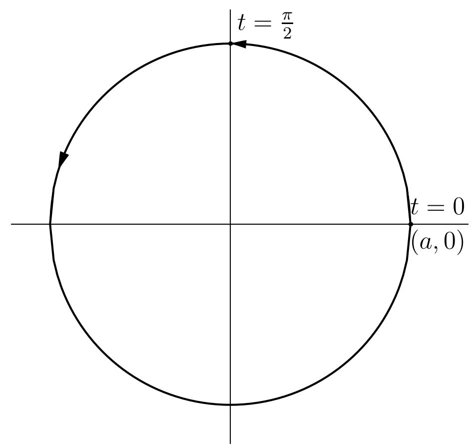

Figure 1: Parametrized circle.

Arc Length of Parametric Curves

We've talked about the following parametric representation for the circle:

$$\begin{aligned} x &= a\cos t \\ y &= a\sin t \end{aligned}$$

We noted that $x^2 + y^2 = a^2$ and that as t increases the point $(x(t), y(t))$ moves around the circle in the counterclockwise direction.

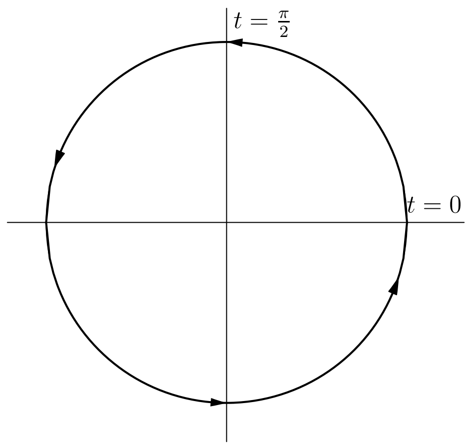

Figure 1: The parametrization $(a\cos t, a\sin t)$ has a counterclockwise trajectory.

We'll now learn how to compute the arc length of the path traced out by this trajectory; the result should match our previous result for the arc length of a circular curve.

Recall our basic relationship:

$$ds^2 = dx^2 + dy^2 \quad \text{or} \quad ds = \sqrt{dx^2 + dy^2}.$$

We incorporate parameter t into this formula as follows:

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

So, to compute the infinitesimal arc length ds we start by computing $\frac{dx}{dt}$ and $\frac{dy}{dt}$:

$$\frac{dx}{dt} = -a\sin t \quad \text{and} \quad \frac{dy}{dt} = a\cos t.$$

Hence,

$$ds = \sqrt{(-a\sin t)^2 + (a\cos t)^2}\, dt$$

$$
\begin{aligned}
&= \sqrt{a^2(\sin^2 t + \cos^2 t)}\, dt \\
&= \sqrt{a^2 \cdot 1}\, dt \\
ds &= a\, dt
\end{aligned}
$$

From this we conclude that the speed at which the point moves around the circle is: $\dfrac{ds}{dt} = a$. Because the speed is constant, we say that the point is moving with *uniform* speed.

Parametrizations such as:

$$
\begin{aligned}
x &= a \cos kt \\
y &= a \sin kt
\end{aligned}
$$

are common in math and physics classes. Again this is a parametrization of the circle, but this time the point is moving with uniform speed ak. (We'll assume that both a and k are positive.)

Remarks on Notation

We've been working with notation like ds^2 for a while now; what does this mean, what operations can we legitimately perform with these infinitesimals, and what isn't valid?

The basis for our arc length formula is that:

$$\Delta s^2 \approx \Delta x^2 + \Delta y^2.$$

We'll now see how our formula:

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

for parametric arc length can be more rigorously derived from the same basis.

Because Δt is not quite equal to 0, we can start by dividing both sides of the formula by Δt^2:

$$\begin{aligned} \Delta s^2 &\approx \Delta x^2 + \Delta y^2 \\ \left(\frac{\Delta s}{\Delta t}\right)^2 &\approx \left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2 \end{aligned}$$

Finally, we take the limit as t goes to zero of both sides to conclude that:

$$\left(\frac{ds}{dx}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2.$$

(This is what derivatives are all about.)

Warning: Never write $\left(\frac{dx}{dt}\right)^2 = (x'(t))^2$ as $\frac{dx^2}{dt^2}$. If you do, it could be incorrectly interpreted to mean $\frac{d^2x}{dt^2} = x''(t)$.

Another unfortunate thing is that we write $\sin^2 x$ to mean $(\sin x)^2$, perhaps because typographers are lazy. There is inconsistency in mathematical notation, and we have to work with the conventions that exist.

Non-Constant Speed Parametrization

Let's look at the following parametrization:

$$\begin{aligned} x &= 2\sin t \\ y &= \cos t. \end{aligned}$$

To solve this sort of problem we're going to need to convert this parametrization in terms of cosine, sine and t into a rectangular equation in x and y. (We'll see a few more examples of this process later today in a different context.)

To see the pattern here we'll use the relationship:

$$\sin^2 t + \cos^2 t = 1.$$

Since $\frac{x}{2} = \sin t$, we get that:

$$\frac{1}{4}x^2 + y^2 = \sin^2 t + \cos^2 t = 1.$$

So the trajectory described by this parametrization is an ellipse.

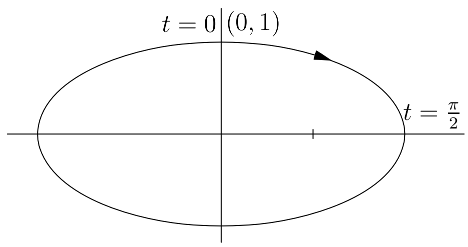

Figure 1: Ellipse described by $x = 2\sin t$, $y = \cos t$.

To sketch the ellipse we'll start by plotting a few points. When $t = 0$ we have:

$$(2\sin 0, \cos 0) = (0, 1),$$

so the ellipse "starts" at the point $(0, 1)$. When $t = \pi/2$ we get the point

$$\left(2\sin\frac{\pi}{2}, \cos\frac{\pi}{2}\right) = (2, 0).$$

We know that the trajectory follows an ellipse, and we can compute that the length of the minor axis is 1 and the major axis is 2, so we get the curve shown in Figure 1.

We also know that the motion is clockwise.

Next, let's examine the speed at which the point traces out arc length. We know:

$$\begin{aligned} \frac{ds}{dt} &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \\ &= \sqrt{(2\cos t)^2 + (-\sin t)^2} \end{aligned}$$

And so the total arc length covered by the point as it moves all the way around the ellipse (t varies from 0 to 2π) is:

$$\text{Arc length} = \int_0^{2\pi} \sqrt{4\cos^2 t + \sin^2 t}\, dt.$$

Unfortunately, this is not an elementary integral; we won't be able to find a formula for an antiderivative of $\sqrt{4\cos^2 t + \sin^2 t}$. That means we have to stop here; this is our final answer.

Note that it may be hard to tell whether or not it's possible to find a given antiderivative.

Question: When you draw the ellipse, don't you need to take into account what t is?

Answer: Good question. Our problem is to plot the curve parametrized by:

$$\begin{aligned} x &= 2\sin t \\ y &= \cos t. \end{aligned}$$

Obviously we can't simply graph y as a function of x; both y and x are functions of t. To draw this curve we did two things:

1. Plot points $(x(t), y(t))$ for "easy" values of t. (Here $t = 0$ and $t = \frac{\pi}{2}$.)

2. Find a relationship between x and y similar to one we can graph.

A calculator or computer would draw the curve by repeating the first step many, many times. We're not so patient, so we used the fact that $\left(\frac{1}{2}x\right)^2 + y^2 = 1$ to deduce that the curve must have an elliptical shape. Even if you don't recognize this as the equation of an ellipse, you should be able to guess that its graph will be a deformed circle.

By combining the information we have about locations of points on the curve (we might also want to find points for $t = \pi$ and $t = \frac{3\pi}{2}$) with the information that the curve looks like a deformed circle, we can get a fairly decent sketch of the parametrized curve.

You might be asked to give the rectangular equation for a parametric curve and then plot the curve. In this case, the answer would be $\frac{1}{4}x^2 + y^2 = 1$ followed by a picture of the ellipse.

Question: Do I need to know any specific formulas?

Answer: Any formulas that you know and remember may help you. You're not required to memorize the general equation of an ellipse, but you should recognize the equation of a circle.

Question: Arc length is the integral of $ds = \sqrt{dx^2 + dy^2}$. With ds, dy and dx all in the mix, how were you able to integrate just with respect to x in some examples?

Answer: When working with one dimensional objects like curves in space or in the plane, we're going to integrate with respect to a single variable. We get to choose which variable to use, but some choices are better than others.

For instance, if we're trying to find the arc length of a circle or the ellipse we just looked at, integrating with respect to x might be a problem because there are two different y values for most choices of x. However, if we can solve the problem by looking only at the top half of the circle or ellipse it could be ok to integrate with respect to x.

The uniform parameter (the one for which the point moves with uniform speed) may be the easiest one. In the previous example, the uniform parameter was t.

This returns us to the point that we're no longer tied to the view that y is a function of x. The variables x and y just represent the horizontal and vertical position of a point; they're no longer the input and output of a function. In fact, in this example both x and y are functions of t.

Surface Area of an Ellipsoid

Next we'll find the surface area of the surface formed by revolving our elliptical curve:

$$\begin{aligned} x &= 2\sin t \\ y &= \cos t \end{aligned}$$

about the y-axis.

Remember that our surface area element dA is the area of a thin circular ribbon with width ds. The radius of this circle is $x = 2\sin t$, which is the distance between the ribbon and the y-axis.

$$dA = 2\pi \underbrace{(2\sin t)}_{x} \underbrace{\sqrt{4\cos^2 t + \sin^2 t}dt}_{ds=\text{arc length}}.$$

To find the surface area we need to integrate dA between certain limits; what are they?

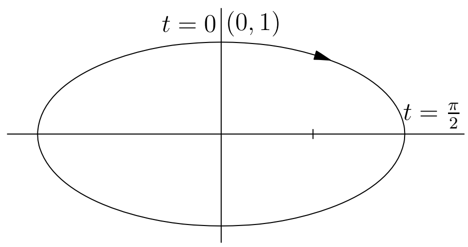

Figure 1: Elliptical path described by $x = 2\sin t$, $y = \cos t$.

By looking at Figure 1 we can see that we need to integrate from 0 to π. Remember that we only need to go from the top to the bottom of the ellipse to trace the right hand side; including the left hand side of the ellipse would double our result and give the wrong answer.

$$A = \int_0^\pi 2\pi(2\sin t)\sqrt{4\cos^2 t + \sin^2 t}dt.$$

Notice that we're integrating from the top of the ellipse to the bottom; if we think in terms of the y-variable we tend to think of going the opposite way.

This integral turns out to be do-able but long. Start by using the substitution $u = \cos t$, $du = -\sin t\, dt$.

$$\begin{aligned} A &= \int_0^\pi 2\pi(2\sin t)\sqrt{4\cos^2 t + \sin^2 t}dt \\ &= \int_{u=1}^{u=-1} -4\pi\sqrt{3u^2 + 1}du. \end{aligned}$$

Next would be another trigonometric substitution to deal with the square root, and so on.

Introduction to Polar Coordinates

Polar coordinates involve the geometry of circles. Just as Professor Jerison loves the number zero, the rest of MIT loves circles.

Polar coordinates are another way of describing points in the plane. Instead of giving x and y coordinates, we'll describe the location of a point by:

- r = distance to origin
- θ = angle between the ray from the origin to the point and the horizontal axis.

(This is the geometric idea, but is not a perfect match for how polar coordinates are actually used.)

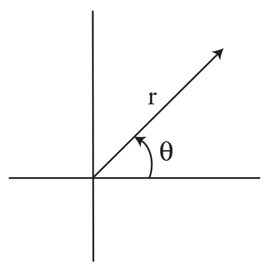

Figure 1: Polar coordinates describe a radius r and angle θ.

If we wish to relate polar coordinates back to rectangular coordinates (i.e. find the x and y coordinates of a point (r, θ)), we use the following formulas:

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

This is the official, unambiguous definition of polar coordinates, from which we get the geometric description above and also the following:

To convert rectangular coordinates to polar coordinates, use:

$$r = \sqrt{x^2 + y^2}, \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right).$$

This is close to being a good formula, and it's useful.

The ambiguity in these formulas comes from the fact that r could be negative $\sqrt{x^2+y^2}$, and θ could also be $\tan^{-1}\left(\frac{-y}{-x}\right)$. You must refer to your diagram when using these formulas to convert from rectangular to polar coordinates.

Two Coordinate Systems

The coordinate system that we're used to is the rectangular coordinate system. The notation (x, y) describes a location in that plane that is x units to the right of the origin and y units above the origin. As shown in Figure 2, the (green) lines $y = k$ are lines of constant height; the (red) lines $x = c$ are made up of all the points that are exactly c units to the right of the origin.

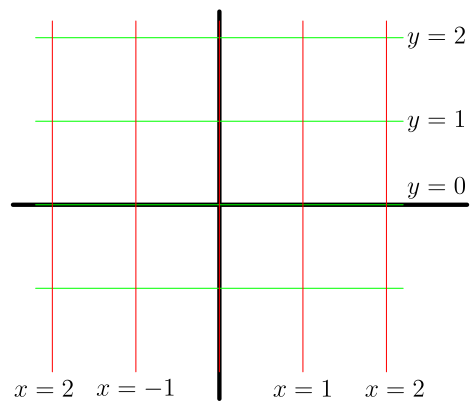

Figure 2: Lines $y = k$ and $x = c$ in a rectangular coordinate system.

In the polar coordinate system, the notation (r, θ) describes a point r units away from the origin at an angle of θ degrees. In Figure 3, each ray $\theta = c$ radiating from the origin is made up of points (r, θ) which all have the same angle θ. The circles $r = k$ about the origin are made up of points which are all the same distance k from the origin.

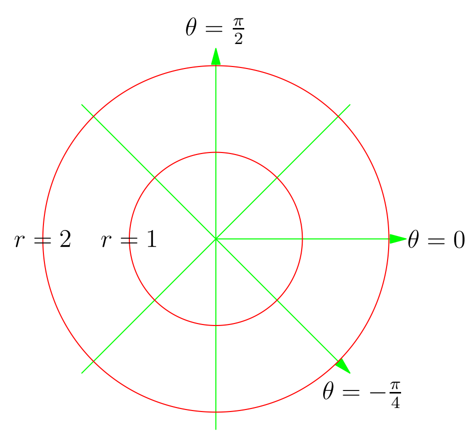

Figure 3: Lines $r = k$ and $\theta = c$ in a polar coordinate system.

The polar coordinate system is just a different way of describing the locations of points in the plane.

Simple Examples in Polar Coordinates

We've just learned about the polar coordinate system, which is very useful in multivariable calculus and in physics. Here are some examples to help you get used to it.

Example: $(x, y) = (1, -1)$

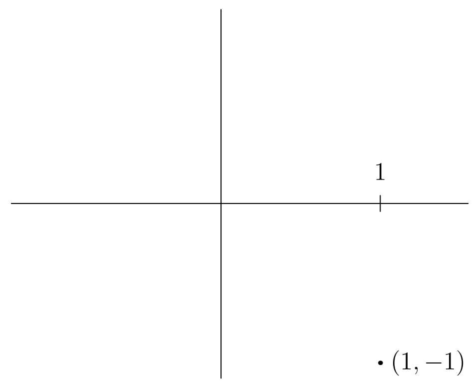

Figure 1: Point at $(1, 1)$ in rectangular coordinates.

How do you describe this point in polar coordinates? There's more than one right answer.

1. $r = \sqrt{2}, \qquad \theta = \frac{7\pi}{4}$
2. $r = \sqrt{2}, \qquad \theta = -\frac{\pi}{4}$
3. $r = -\sqrt{2}, \qquad \theta = \frac{3\pi}{4}$

Some of these answers are easier to make sense of than others, but they are all "legal" correct answers. This ambiguity is something you'll have to adapt to when working with polar coordinates.

Question: Don't the radii have to be positive because they represent a *distance* from the origin?

Answer: When we said r was the distance to the origin in polar coordinates, that was a lie. The only unambiguous description of polar coordinates is:

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

All the others are flawed in some way, but still useful.

It's useful to think of r as a distance, but it's not always accurate to think this way.

Example: $r = a$

In polar coordinates, $r = a$ describes a circle of radius a centered at the origin.

Example: $\theta = c$

The equation $\theta = c$ describes a ray in polar coordinates.

Warning: This implicitly assumes that $0 \le r < \infty$. If we instead assume $-\infty < r < \infty$ we'd get a line, not a ray.

Typical Conventions in Polar Coordinates

- $0 \le r < \infty$
- $-\pi < \theta \le \pi$ **or** $0 \le \theta < 2\pi$.

These are typical, but not universal; different assumptions and ranges are used in different contexts.

Translating $y = 1$ into Polar Coordinates

We'll take a simple description from rectangular coordinates, $y = 1$, and translate it into polar coordinates. To do this, we plug in the (definitive) formula $y = r\sin\theta$.

$$\begin{aligned} y &= r\sin\theta \\ 1 &= r\sin\theta \\ r &= \frac{1}{\sin\theta} \end{aligned}$$

In rectangular coordinates the line has equation $y = 1$. In polar coordinates its equation is $r = \dfrac{1}{\sin\theta}$.

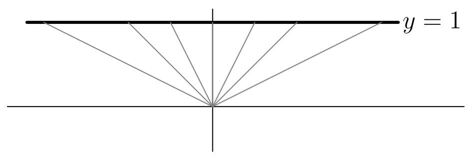

Figure 1: $r = \dfrac{1}{\sin\theta}$

As indicated in Figure 1, for different values of θ points on the horizontal line are different distances r from the origin. That distance r is $\frac{1}{\sin\theta}$.

We need one more piece of information to complete this problem; what is the range of θ? When $\theta = 0$ the denominator of the expression describing r is 0; this corresponds to one end of the line. As θ increases from 0 to π, r decreases to 1 at $\theta = \frac{\pi}{2}$ and then increases to infinity again.

Our final answer is:

$$r = \frac{1}{\sin\theta}, \qquad 0 < \theta < \pi.$$

Question: Is it typical to express r as a function of θ? Does it matter?

Answer: In this course our answers will almost always describe r as a function of θ, but it's not required. We do it this way because we like:

$$r = \frac{1}{\sin\theta}$$

better than:

$$\theta = \sin^{-1}\left(\frac{1}{r}\right).$$

Equation of an Off-Center Circle

This is a standard example that comes up a lot. Circles are easy to describe, unless the origin is on the rim of the circle. We'll calculate the equation in polar coordinates of a circle with center $(a, 0)$ and radius $(2a, 0)$. You should expect to repeat this calculation a few times in this class and then memorize it for multivariable calculus, where you'll need it often.

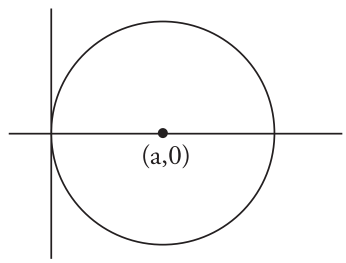

Figure 1: Off center circle through $(0, 0)$.

In rectangular coordinates, the equation of this circle is:

$$(x-a)^2 + y^2 = a^2.$$

We could plug in $x = r\cos\theta$, $y = \sin\theta$ to convert to polar coordinates, but there's a faster way. We start by expanding and simplifying:

$$\begin{aligned}
(x-a)^2 + y^2 &= a^2 \\
x^2 - 2ax + a^2 + y^2 &= a^2 \\
x^2 - 2ax + y^2 &= 0 \\
(x^2 + y^2) - 2ax &= 0 \\
r^2 - 2ar\cos\theta &= 0 \\
r^2 &= 2ar\cos\theta \\
\Longrightarrow r &= 2a\cos\theta \quad (\text{or } r = 0).
\end{aligned}$$

We used the facts that $x^2 + y^2 = r^2$ and $x = r\cos\theta$ to conclude that there were two values of r that satisfy this equation; $r = 2a\cos\theta$ and $r = 0$. These are the equations describing r in terms of θ that describe this circle in polar coordinates.

In order to use the equation $r = 2a\cos\theta$, we need to figure out the appropriate range of values for θ. By looking at the graph we see that $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Our final equation is:

$$r = 0 \qquad \text{or} \qquad r = 2a\cos\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

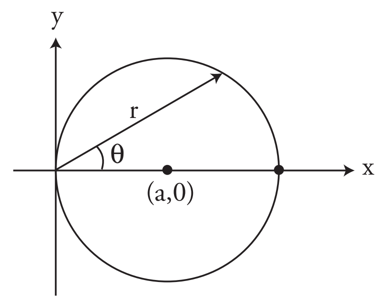

Figure 2: Off center circle in polar coordinates.

To check our work, let's find some points on this curve:

- At $\theta = 0$, $r = 2a$ and so $x = 2a$ and $y = 0$.
- At $\theta = \frac{\pi}{4}$, $r = 2a \cos \frac{\pi}{4} = a\sqrt{2}$. Hence $x = a$ and $y = a$.

Polar Coordinates and Area

How would we calculate an area using polar coordinates? Our basic increment of area will be shaped like a slice of pie. The slice of pie shown in Figure 1 has

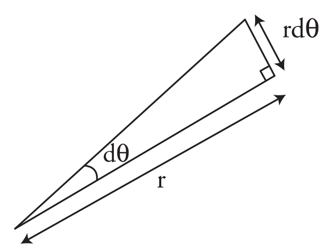

Figure 1: A slice of pie with radius r and angle $d\theta$.

a piece of a circular arc along its boundary with arc length $r\,d\theta$. We'll say that dA equals the area of the slice.

How do we express dA in terms of r and θ? The total area of the pie this was sliced from is πr^2. To find the area dA we note that the proportion of the total area covered equals the proportion of arc length covered. So:

$$\begin{aligned}\frac{dA}{\pi r^2} &= \frac{d\theta}{2\pi r} \\ dA &= \frac{r\,d\theta}{2\pi r}\cdot \pi r^2 \\ dA &= \frac{1}{2}r^2\,d\theta\end{aligned}$$

This is the basic formula for an increment of area in polar coordinates.

We want to use polar coordinates to compute areas of shapes other than circles. In this case r will be a function of θ. The distance between the curve and the origin changes depending on what angle our ray is at. Our center point of reference is the origin; we think of rays emerging from the origin at some angle θ; $r(\theta)$ is, roughly, the distance we must travel along that ray to get to the curve.

To find the area of a shape like this, we break it up into circular sectors with angle $\Delta\theta$. Since the curve is not a circle the circular sectors won't perfectly cover the region, so we just approximate the area of a wedge between the curve and the origin by:

$$\Delta A \approx \frac{1}{2}r^2\Delta\theta.$$

If we take the limit as $\Delta\theta$ approaches zero our sum of sector areas will approach

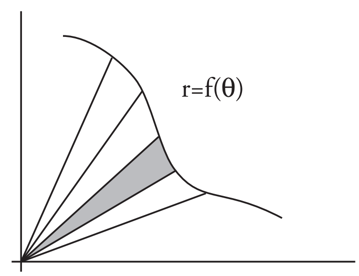

Figure 2: A slice from an oddly shaped pie.

the exact area and we get:

$$dA = \frac{1}{2} r^2 \, d\theta.$$

This is very similar to letting Δx go to zero in a Riemann sum of rectangle areas.

In the limit, we have:

$$A = \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 \, d\theta.$$

Remember that we're assuming r is a function of θ.

Area of an Off Center Circle

Let's find the area in polar coordinates of the region enclosed by the curve $r = 2a\cos\theta$. We've previously shown that this curve describes a circle with radius a centered at $(a, 0)$. In rectangular coordinates its equation is $(x - a)^2 + y^2 = a^2$.

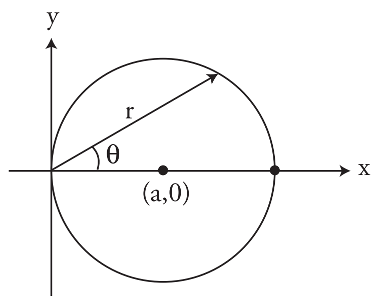

Figure 1: Off center circle $r = 2a\cos\theta$.

We're going to integrate an infintessimal amount of area dA. The integral will go from $\theta_1 = -\frac{\pi}{2}$ to $\theta_2 = \frac{\pi}{2}$. We could find these limits by looking at Figure 1; to draw the circle we might start by moving "down" at angle $-\frac{\pi}{2}$. As we move along the bottom of the circle toward $(2a, 0)$ the angle increases to 0, and as we trace out the top of the circle we're moving from angle 0 to angle $\frac{\pi}{2}$ ("up").

We might also find the limits of integration by looking at the formula and realizing that the cosine function is positive for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. When $\theta = \pm\frac{\pi}{2}$, $r = 2a\cos\theta$ is 0, so the two ends of the curve meet at the origin.

Our infinitessimal unit of area is $dA = \frac{1}{2}r^2\,d\theta$, so:

$$\begin{aligned}
A &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}r^2\,d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(2a\cos\theta)^2\,d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2a^2\cos^2\theta\,d\theta \\
&= 2a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\,d\theta \qquad \text{(half angle formula)} \\
&= 2a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}\,d\theta \\
&= a^2\left[\theta + \frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}
\end{aligned}$$

$$
\begin{aligned}
&= a^2\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) \\
&= \pi a^2.
\end{aligned}
$$

We know that the area of a circle of radius a is πa^2; our answer is correct.

Graph of $r = 2a\cos\theta$

Let's get some more practice in graphing and polar coordinates. We just found the area enclosed by the curve $r = 2a\cos\theta$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. What happens when θ doesn't lie in this range?

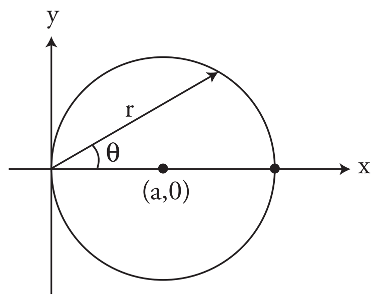

Figure 1: Off center circle $r = 2a\cos\theta$.

When $\frac{\pi}{2} < \theta < \pi$, r is negative. For example, when $\theta = \frac{3\pi}{4}$, $\cos\theta = -\frac{\sqrt{2}}{2}$ and $r = -a\sqrt{2}$. If we move a distance of *negative* $a\sqrt{2}$ in the direction of angle $\frac{3\pi}{4}$ we arrive at the point $(-a\sqrt{2}, \frac{3\pi}{4})$, which is $(a, -a)$ in rectangular coordinates.

In fact, because we know that the points on the curve must have the property:

$$(x-a)^2 + y^2 = a^2$$

in rectangular coordinates, we know that as θ increases, the point $(2a\cos\theta, \theta)$ must remain on that same curve. As θ ranges from 0 to 2π (or from $-\frac{\pi}{2}$ to $\frac{3\pi}{2}$), the point $(2a\cos\theta, \theta)$ travels around the circle twice.

A common mistake is to choose the wrong limits of integration and count the same area twice, or cancel a positive area with an overlapping negative one.

Question: Can you find the area using the limits of integration 0 and π?

Answer: Yes. The integral $\displaystyle\int_0^\pi \frac{1}{2}(2a\cos\theta)^2\, d\theta$ gives a correct answer.

However, $r = 2a\cos\theta$, $0 \le \theta \le \pi$ is an awkward way to describe a circle. As θ ranges from 0 to $\frac{\pi}{2}$, r is positive and (r, θ) moves along the top half of the circle. As θ sweeps through the second quadrant ($\frac{\pi}{2} < \theta < \pi$), r is negative and so the curve appears in the fourth quadrant.

When we work with negative values of r it's easy to get confused, so when possible it's a good idea to choose our limits of integration so that r is positive.

Graph of $r = \sin 2\theta$

This curve is a favorite; a similar curve appears in the homework. We'll plot a few points $(\sin 2\theta, \theta)$ to get an idea of what the graph of this curve looks like.

θ	$r = \sin 2\theta$
0	0
$\frac{pi}{4}$	1
$\frac{pi}{2}$	0

Note that $\sin 2\theta > 0$ for $0 < \theta < \frac{\pi}{2}$. So the curve starts at the origin, goes outward to the point $(1, \frac{\pi}{4})$ (which is $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ in rectangular coordinates), then returns to the origin as θ moves to point "up".

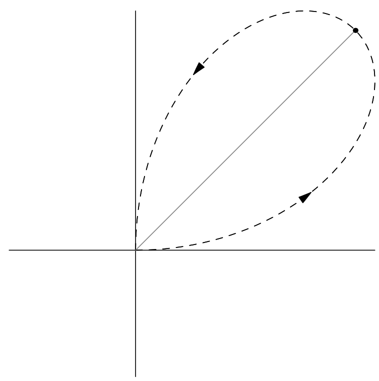

Figure 1: Graph of $r = \sin 2\theta$ for $0 < \theta < \frac{\pi}{2}$.

Because of the symmetries of the sine function, the curve will do something similar in each quadrant. However, it's useful to watch the curve being drawn in order to understand how its parts are connected.

The "loops" in the graph are caused by $r = \sin 2\theta$ changing sign each time the graph intersects the origin. When $\frac{\pi}{2} < \theta < \pi$, our angle is in the second quadrant; the portion of the graph corresponding to those values of θ appears opposite the angle, in the fourth quadrant.

If we want to compute the area of one petal of this rose, we have to be careful to use the right bounds for θ.

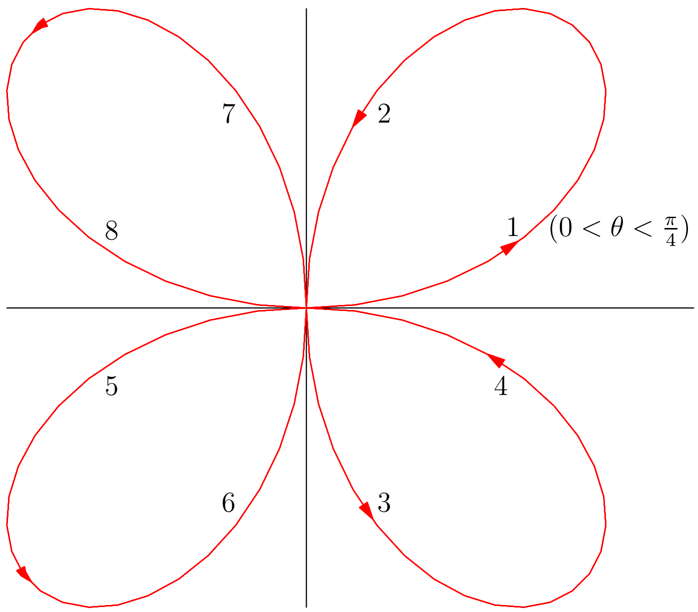

Figure 2: Graph of $r = \sin 2\theta$; a four leaf rose.

Polar Coordinates and Conic Sections

Suppose we want to graph the curve described by:

$$r = \frac{1}{1 + 2\cos\theta}.$$

Again we start by plotting some points on this curve:

θ	r
0	$\frac{1}{3}$
$\frac{\pi}{2}$	-1
$-\frac{\pi}{2}$	1

By using the equations:

$$x = r\cos\theta, \qquad y = r\sin\theta$$

we can convert these polar coordinates to rectangular coordinates, show in Figure 1. For example, when $\theta = \frac{\pi}{2}$ we know that $r = 1$ and so:

$$\begin{aligned} x = r\cos\theta &= 0 \\ y = r\sin\theta &= -1 \end{aligned}$$

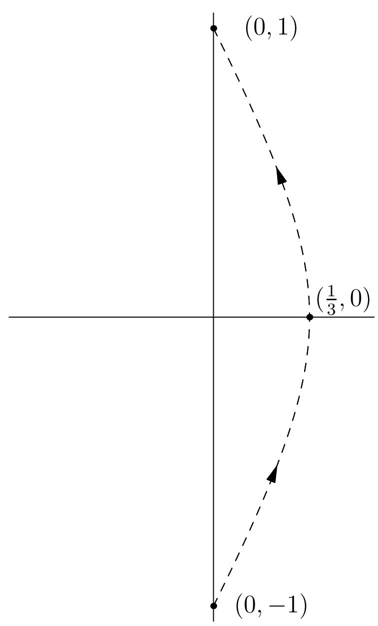

Figure 1: Rectangular coordinates of points on the curve $r = \dfrac{1}{1 + 2\cos\theta}$.

When $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, the denominator $1 + 2\cos\theta$ is positive and so r is positive; the curve traced over this interval must look something like the dotted line in Figure 1.

It's possible for the denominator to be 0:

$$\begin{aligned}1 + 2\cos\theta &= 0\\ 2\cos\theta &= -1\\ \cos\theta &= -\frac{1}{2}\\ \theta &= \arccos\left(-\frac{1}{2}\right)\\ \theta &= \pm\frac{2\pi}{3}\end{aligned}$$

The radius r goes to infinity as θ approaches $\frac{2\pi}{3}$ and $-\frac{2\pi}{3}$, so the curve will extend infinitely far in those directions.

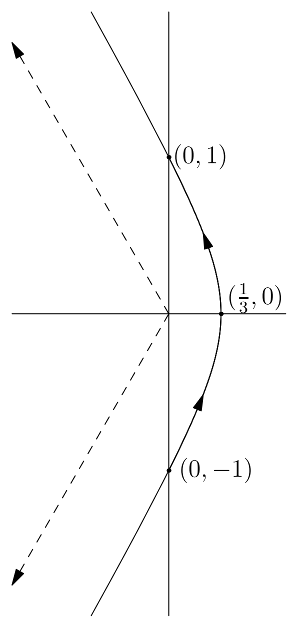

Figure 2: Graph of the curve $r = \dfrac{1}{1 + 2\cos\theta}$.

This is as much as we'll be able to figure out about the graph without converting its equation from polar to rectangular coordinates. Let's do that.

Rectangular Equation

What is the rectangular $(x,\, y)$ equation for $r = \dfrac{1}{1 + 2\cos\theta}$?

To answer this question we could use our formula $x = r\cos\theta$, $y = r\sin\theta$ and then try to simplify, but if we're clever we can manipulate our original formula until it appears in terms of x and y.

$$\begin{aligned} r &= \frac{1}{1+2\cos\theta} \\ r + 2r\cos\theta &= 1 \\ r &= 1 - 2r\cos\theta \\ r &= 1 - 2x \end{aligned}$$

Multiplying both sides by the denominator simplified the expression and allowed us to make the substitution $x = r\cos\theta$. The variable θ no longer appears.

If we square both sides of this new equation we can get rid of the variable r as well:

$$\begin{aligned} r &= 1 - 2x \\ r^2 &= (1-2x)^2 \\ x^2 + y^2 &= 1 - 4x + 4x^2 \\ -3x^2 + y^2 + 4x - 1 &= 0 \end{aligned}$$

This is a standard calculation with a standard result; whenever we start with have $\dfrac{1}{a + b\cos\theta}$ or $\dfrac{1}{a + b\sin\theta}$ we'll end up with an equation like this.

Because the signs of the coefficients of x^2 and y^2 are different, this is the equation of a hyperbola. (If the signs match, the equation describes an ellipse; if one of these coefficients is 0 the graph is a parabola.) We can now conclude that the dotted lines in Figure 1 are asymptotes of the graph.

To complete our understanding of the curve $r = \frac{1}{1+2\cos\theta}$ we ask what happens when the denominator $1 + 2\cos\theta$ is negative?

Since we know that the equation for the curve in rectangular coordinates is $-3x^2 + y^2 + 4x - 1 = 0$, we can guess that for $\frac{3\pi}{2} < \theta < \frac{5\pi}{2}$ the curve must trace out the other branch of the hyperbola.

Connection to Kepler's Second Law

There is a beautiful connection between the basic formula for area and these types of curve.

As you may know, the trajectories of comets are hyperbolas. Ellipses are the trajectories of planets or asteroids. When you represent hyperbolas and ellipses in polar coordinates like this, it turns out that:

$$r = 0 \qquad \text{is the focus of the hyperbola.}$$

Polar coordinates are the natural way to express the trajectory of a planet or comet if you want the center of gravity (the sun) to be at the origin.

The formula

$$dA = \frac{1}{2} r^2 \, d\theta$$

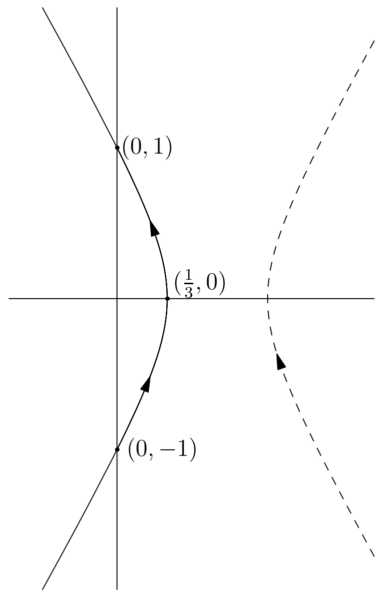

Figure 3: Both branches of the curve $r = \dfrac{1}{1+2\cos\theta}$.

is a central formula in astronomy. Kepler's second law says that a line joining a comet or planet to the sun sweeps out equal areas in equal time periods. In other words, the rate of change of area swept out is constant:

$$\frac{dA}{dt} = \text{constant}.$$

This tells us that as a comet travels around the sun, its speed varies, and varies predictably. Since we know $dA = \frac{1}{2}r^2\,d\theta$, we can conclude that:

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt}.$$

Combining this with Kepler's second law we get:

$$\frac{1}{2}r^2\frac{d\theta}{dt} = \text{constant}.$$

In modern-day terms, what this formula says is that angular momentum is conserved. The objects Kepler was observing weren't subject to friction or air resistance, but this equation is the same one used to describe why a top keeps spinning after you let it go, or why an ice skater spins faster when she pulls her arms and legs in.