V8. Vector Fields in Space

Just as in Section V1 we considered vector fields in the plane, so now we consider vector fields in three-space. These are fields given by a vector function of the type

$$\mathbf{F}(x,y,z) \;=\; M(x,y,z)\,\mathbf{i} + N(x,y,z)\,\mathbf{j} + P(x,y,z)\,\mathbf{k}\;. \tag{1}$$

Such a function assigns the vector $\mathbf{F}(x_0,y_0,z_0)$ to a point (x_0,y_0,z_0) where M, N, and P are all defined. We place the vector so its tail is at (x_0,y_0,z_0), and in this way get the vector field. Such a field in space looks a little like the interior of a haystack.

As before, we say $\mathbf{F}$ is *continuous* in some domain D of 3-space (we will usually use "domain" rather than "region" , when referring to a portion of 3-space) if M, N, and P are continuous in that domain. We say $\mathbf{F}$ is *continuously differentiable* in the domain D if all nine first partial derivatives

$$M_x,\ M_y,\ M_z;\quad N_x,\ N_y,\ N_z;\quad P_x,\ P_y,\ P_z$$

exist and are continuous in D.

Again as before, we give two physical interpretations for such a vector field.

The three-dimensional **force fields** of different sorts — gravitational, electrostatic, electromagnetic — all give rise to such a vector field: at the point (x_0,y_0,z_0) we place the vector having the direction and magnitude of the force which the field would exert on a unit test particle placed at the point.

The three-dimensional **flow fields** and **velocity fields** arising from the motion of a fluid in space are the other standard example. We assume the motion is steady-state (i.e., the direction and magnitude of the flow at any point does not change over time). We will call this a *three-dimensional flow.*

As before, we allow sources and sinks — places where fluid is being added to or removed from the flow. Obviously, we can no longer appeal to people standing overhead pouring fluid in at various points (they would have to be aliens in four-space), but we could think of thin pipes inserted into the domain at various points adding or removing fluid.

The velocity field of such a flow is defined just as it was previously: $\mathbf{v}(x,y,z)$ gives the direction and magnitude (speed) of the flow at (x,y,z).

The flow field $\mathbf{F} = \delta\mathbf{v}$, where $\delta(x,y,z)$ is the density) may be similarly interpreted:

$$\begin{aligned} \text{dir}\ \mathbf{F} \;&=\; \text{the direction of flow} \\ |\mathbf{F}| \;&=\; \text{mass transport rate (per unit area) at } (x,y,z) \text{ in the flow direction;} \end{aligned} \tag{2}$$

that is, $|\mathbf{F}|$ is the rate per unit area at which mass is transported across a small piece of plane perpendicular to the flow at the point (x,y,z)..

The derivation of this interpretation is exactly as in Sections V1 and V3, replacing the small line segment Δl by a small plane area ΔA perpendicular to the flow.

Example 1. Find the three-dimensional electrostatic force field $\mathbf{F}$ arising from a unit positive charge placed at the origin, given that in suitable units $\mathbf{F}$ is directed radially outward from the origin and has magnitude $1/\rho^2$, where ρ is the distance from the origin.

Solution. The vector $x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ with tail at (x, y, z) is directed radially outward and has magnitude ρ. Therefore

$$\mathbf{F} \;=\; \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{\rho^3}\;, \qquad \rho = \sqrt{x^2 + y^2 + z^2}$$

Example 2. a) Find the velocity field of a fluid rotating with constant angular velocity ω around the z-axis, in the direction given by the right-hand rule (right-hand fingers curl in direction of flow when thumb points in the $\mathbf{k}$-direction).

b) Find the analogous field if the flow is rotating about the y-axis.

Solution. a) The flow doesn't depend on z — it is really just a two-dimensional problem, whose solution is the same as before (section V1, Example 4):

$$\mathbf{F}(x, y, z) \;=\; \omega(-y\,\mathbf{i} + x\,\mathbf{j})\;.$$

b) If the axis of flow is the y-axis, the flow will have no $\mathbf{j}$-component and will not depend on y. However, by the right-hand rule, the flow in the xz-plane is *clockwise*, when the positive x and z axes are drawn so as to give a right-handed system. Thus

$$\mathbf{F}(x, y, z) \;=\; \omega(z\,\mathbf{i} - x\,\mathbf{k})\;.$$

Example 3. Find the three-dimensional flow field of a gas streaming radially outward with constant velocity from a source at the origin of constant strength.

Solution. This is like the corresponding two-dimensional problem (section V1, Example 3), except that the area of a sphere increases like the *square* of its radius. Therefore, to maintain constant velocity, the density of flow must decrease like $1/\rho^2$ as you go out from the origin; letting δ be the density and c_i be constants, we get

$$\mathbf{F}(x, y, z) \;=\; \delta\mathbf{v} \;=\; \frac{c_1}{\rho^2}\,\frac{c_2(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})}{\rho} \;=\; \frac{c(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})}{\rho^3}\;.$$

Notice that in the three-dimensional case, this field is the same as the one in Example 1 above, with the magnitude falling off like $1/\rho^2$. For the two-dimensional case, the analogue of a point fluid source at the origin is not a point charge at the origin, but a uniform charge along a vertical wire; both give the field whose magnitude falls off like $1/r$.

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Problems: Vector Fields in Space

Find the gravitational attraction of an upper solid half-sphere of radius a and center $(0,0,0)$ on a mass m_0 at $(0,0,0)$. Assume this half-sphere has density $\delta = z$.

Answer: Draw a picture.

We follow the steps outlined in recitation, changing only the density.

The force is $\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$. (This notation is unfortunately standard. The subscript indicates component, not partial derivative.) By symmetry we know $F_x = F_y = 0$.

At (x, y, z) a small volume dV has mass $dm = \delta(x, y, z)\, dV = z\, dV$. This mass dm exerts a force $\dfrac{Gm_0\, dm}{\rho^2}\dfrac{\langle x, y, z\rangle}{\rho}$ on the test mass. The z-component of this force is $\dfrac{zGm_0\, dm}{\rho^3}$, so $F_z = \displaystyle\iiint_D \frac{zGm_0\, z\, dV}{\rho^3}$.

The limits in spherical coordinates are: ρ from 0 to a, ϕ from 0 to $\pi/2$, θ from 0 to 2π. Recall that $z = \rho\cos\phi$. Then:

$$F_z = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a Gm_0\,\frac{z^2}{\rho^3}\, dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a Gm_0\,\frac{\rho^2\cos^2\phi}{\rho^3}\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^a Gm_0\,\rho\cos^2\phi\,\sin\phi\, d\rho\, d\phi\, d\theta.$$

Inner integral: $\dfrac{Gm_0a^2}{2}\cos^2\phi\sin\phi$

Middle integral: $\dfrac{Gm_0a^2}{2}\left(\dfrac{-\cos^3\phi}{3}\right)\Bigg|_0^{\pi/2} = \dfrac{Gm_0a^2}{6}$

Outer integral: $\dfrac{Gm_0\pi a^2}{3} \quad\Rightarrow\quad \boxed{\mathbf{F} = \langle 0, 0, Gm_0\pi a^2/3\rangle.}$

Surprisingly, this is the same as the force exerted by a half-sphere with density $\sqrt{x^2+y^2}$.

Problems: Vector Fields in Space

Find the gravitational attraction of an upper solid half-sphere of radius a and center $(0, 0, 0)$ on a mass m_0 at $(0, 0, 0)$. Assume this half-sphere has density $\delta = z$.

V9.1 Surface Integrals

Surface integrals are a natural generalization of line integrals: instead of integrating over a curve, we integrate over a surface in 3-space. Such integrals are important in any of the subjects that deal with continuous media (solids, fluids, gases), as well as subjects that deal with force fields, like electromagnetic or gravitational fields.

Though most of our work will be spent seeing how surface integrals can be calculated and what they are used for, we first want to indicate briefly how they are defined. The surface integral of the (continuous) function $f(x, y, z)$ over the surface S is denoted by

$$\iint_S f(x, y, z)\, dS \ . \tag{1}$$

You can think of dS as the area of an infinitesimal piece of the surface S. To define the integral (1), we subdivide the surface S into small pieces having area ΔS_i, pick a point (x_i, y_i, z_i) in the i-th piece, and form the Riemann sum

$$\sum f(x_i, y_i, z_i)\Delta S_i \ . \tag{2}$$

w As the subdivision of S gets finer and finer, the corresponding sums (2) approach a limit which does not depend on the choice of the points or how the surface was subdivided. The surface integral (1) is defined to be this limit. (The surface has to be smooth and not infinite in extent, and the subdivisions have to be made reasonably, otherwise the limit may not exist, or it may not be unique.)

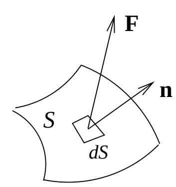
1. The surface integral for flux.

The most important type of surface integral is the one which calculates the flux of a vector field across S. Earlier, we calculated the flux of a plane vector field $\mathbf{F}(x, y)$ across a directed curve in the xy-plane. What we are doing now is the analog of this in space.

We assume that S is *oriented*: this means that S has two sides and one of them has been designated to be the *positive side*. At each point of S there are two unit normal vectors, pointing in opposite directions; the *positively directed* unit normal vector, denoted by $\mathbf{n}$, is the one standing with its base (i.e., tail) on the positive side. If S is a closed surface, like a sphere or cube — that is, a surface with no boundaries, so that it completely encloses a portion of 3-space — then by convention it is oriented so that the outer side is the positive one, i.e., so that $\mathbf{n}$ always points towards the outside of S.

Let $\mathbf{F}(x, y, z)$ be a continuous vector field in space, and S an oriented surface. We define

$$\textsf{\textbf{flux of }} F \textsf{\textbf{ through }} S \ = \ \iint_S (\mathbf{F}\cdot\mathbf{n})\, dS \ = \ \iint_S \mathbf{F}\cdot d\mathbf{S} \ ; \tag{3}$$



the two integrals are the same, but the second is written using the common and suggestive abbreviation $d\mathbf{S} = \mathbf{n}\, dS$.

If $\mathbf{F}$ represents the velocity field for the flow of an incompressible fluid of density 1, then $\mathbf{F}\cdot\mathbf{n}$ represents the component of the velocity in the positive perpendicular direction to the surface, and $\mathbf{F}\cdot\mathbf{n}\, dS$ represents the flow rate across the little infinitesimal piece of surface having area dS. The integral in (3) adds up these flows across the pieces of surface, so that we may interpret (3) as saying

$$\textit{flux of } F \textit{ through } S \;=\; \textit{net flow rate across } S, \tag{4}$$

where we count flow in the direction of $\mathbf{n}$ as positive, flow in the opposite direction as negative. More generally, if the fluid has varying density, then the right side of (4) is the net mass transport rate of fluid across S (per unit area, per time unit).

If $\mathbf{F}$ is a force field, then nothing is physically flowing, and one just uses the term "flux" to denote the surface integral, as in (3).

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Problems: Calculating Flux

1. Find the flux of $\mathbf{F} = \langle x, y, z \rangle$ through the surface $x^2 + y^2 + z^2 = 1$, where $z \geq 0$.

Answer: The surface in question is the upper unit half-sphere and $\mathbf{F}$ is identical to the outward unit normal. Therefore, $\mathbf{F} \cdot \mathbf{n} = 1$ and $\displaystyle\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \text{Area} = 2\pi r^2$.

2. Find the flux of $\mathbf{F} = \langle 0, x, 0 \rangle$ through the portion of the plane $x + z = 1$ for which $x > 0$, $0 < y < 1$ and $z > 0$.

Answer: The surface in question is a rectangle in the first octant. It has constant normal $\langle 1, 0, 1 \rangle$ which is everywhere orthogonal to $\mathbf{F}$, so $\mathbf{F} \cdot \mathbf{n} = 0$ over the surface and the flux is 0.

Problems: Calculating Flux

1. Find the flux of $\mathbf{F} = \langle x, y, z \rangle$ through the surface $x^2 + y^2 + z^2 = 1$, where $z \geq 0$.

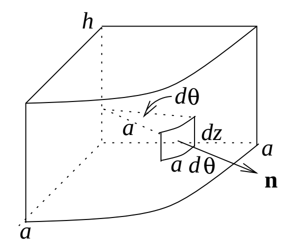
2. Find the flux of $\mathbf{F} = \langle 0, x, 0 \rangle$ through the portion of the plane $x + z = 1$ for which $x > 0$, $0 < y < 1$ and $z > 0$.

V9.2 Surface Integrals

2. Flux through a cylinder and sphere.

We now show how to calculate the flux integral, beginning with two surfaces where $\mathbf{n}$ and dS are easy to calculate — the cylinder and the sphere.

Example 1. Find the flux of $\mathbf{F} = z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$ outward through the portion of the cylinder $x^2 + y^2 = a^2$ in the first octant and below the plane $z = h$.



Solution. The piece of cylinder is pictured. The word "outward" suggests that we orient the cylinder so that $\mathbf{n}$ points outward, i.e., away from the z-axis. Since by inspection $\mathbf{n}$ is radially outward and horizontal,

$$\mathbf{n} = \frac{x\,\mathbf{i} + y\,\mathbf{j}}{a}\ . \tag{5}$$

(This is the outward normal to the circle $x^2 + y^2 = a^2$ in the xy-plane; $\mathbf{n}$ has no z-component since it is horizontal. We divide by a to make its length 1.)

To get dS, the infinitesimal element of surface area, we use cylindrical coordinates to parametrize the cylinder:

$$x = a\cos\theta, \qquad y = a\sin\theta \qquad z = z\ . \tag{6}$$

As the parameters θ and z vary, the whole cylinder is traced out ; the piece we want satisfies $0 \le \theta \le \pi/2,\ 0 \le z \le h$. The natural way to subdivide the cylinder is to use little pieces of curved rectangle like the one shown, bounded by two horizontal circles and two vertical lines on the surface. Its area dS is the product of its height and width:

$$dS = dz \cdot a\,d\theta\ . \tag{7}$$

Having obtained $\mathbf{n}$ and dS, the rest of the work is routine. We express the integrand of our surface integral (3) in terms of z and θ:

$$\begin{aligned}\mathbf{F}\cdot\mathbf{n}\,dS &= \frac{zx + xy}{a}\cdot a\,dz\,d\theta\ , && \text{by (5) and (7);}\\ &= (az\cos\theta + a^2\sin\theta\cos\theta)\,dz\,d\theta, && \text{using (6).}\end{aligned}$$

This last step is essential, since the dz and $d\theta$ tell us the surface integral will be calculated in terms of z and θ, and therefore the integrand must use these variables also. We can now calculate the flux through S:

$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{n}\,dS &= \int_0^{\pi/2}\int_0^h (az\cos\theta + a^2\sin\theta\cos\theta)\,dz\,d\theta\\ \text{inner integral} &= \frac{ah^2}{2}\cos\theta + a^2h\sin\theta\cos\theta\\ \text{outer integral} &= \left[\frac{ah^2}{2}\sin\theta + a^2h\frac{\sin^2\theta}{2}\right]_0^{\pi/2} \;=\; \frac{ah}{2}(a+h)\ .\end{aligned}$$

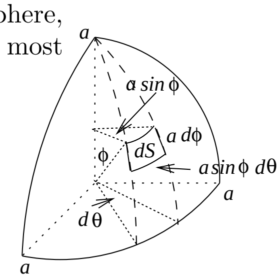
Example 2. Find the flux of $\mathbf{F} = xz\,\mathbf{i} + yz\,\mathbf{j} + z^2\,\mathbf{k}$ outward through that part of the sphere $x^2 + y^2 + z^2 = a^2$ lying in the first octant $(x, y, z, \ge 0)$.

Solution. Once again, we begin by finding $\mathbf{n}$ and dS for the sphere. We take the outside of the sphere as the positive side, so $\mathbf{n}$ points radially outward from the origin; we see by inspection therefore that

$$\mathbf{n} = \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{a}, \tag{8}$$

where we have divided by a to make $\mathbf{n}$ a unit vector.

To do the integration, we use spherical coordinates ρ, ϕ, θ. On the surface of the sphere, $\rho = a$, so the coordinates are just the two angles ϕ and θ. The area element dS is most easily found using the volume element:



$$dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = dS \cdot d\rho \quad = \quad \text{area} \cdot \text{thickness}$$

so that dividing by the thickness $d\rho$ and setting $\rho = a$, we get

$$dS = a^2 \sin\phi\, d\phi\, d\theta. \tag{9}$$

Finally since the area element dS is expressed in terms of ϕ and θ, the integration will be done using these variables, which means we need to express x, y, z in terms of ϕ and θ. We use the formulas expressing Cartesian in terms of spherical coordinates (setting $\rho = a$ since (x, y, z) is on the sphere):

$$x = a\sin\phi\cos\theta, \qquad y = a\sin\phi\sin\theta, \qquad z = a\cos\phi\ . \tag{10}$$

We can now calculate the flux integral (3). By (8) and (9), the integrand is

$$\mathbf{F}\cdot\mathbf{n}\, dS \;=\; \frac{1}{a}\,(x^2z + y^2z + z^2z)\cdot a^2 \sin\phi\, d\phi\, d\theta\ .$$

Using (10), and noting that $x^2 + y^2 + z^2 = a^2$, the integral becomes

$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{n}\; dS \;&=\; a^4 \int_0^{\pi/2}\int_0^{\pi/2} \cos\phi\sin\phi\, d\phi\, d\theta \\ &=\; a^4\,\frac{\pi}{2}\,\frac{1}{2}\,\sin^2\phi\Big]_0^{\pi/2} \;=\; \frac{\pi a^4}{4}\ .\end{aligned}$$

Problems: Flux Through Surfaces

Let $\mathbf{F} = \langle x, y, z \rangle$.

1. Find the flux of $\mathbf{F}$ through the square with vertices $(0,0,0)$, $(1,0,0)$, $(1,1,0)$, $(0,1,0)$.

Answer: The square in question lies in the plane $z = 0$, so $\mathbf{n} = \langle 0, 0, 1 \rangle$. $\mathbf{F} \cdot \mathbf{n} = z = 0$ on the whole square, so the flux is zero.

2. Find the flux of $\mathbf{F}$ through the square with vertices $(0,0,1)$, $(1,0,1)$, $(1,1,1)$, $(0,1,1)$.

Answer: Again $\mathbf{n} = \langle 0, 0, 1 \rangle$ and $\mathbf{F} \cdot \mathbf{n} = z$.

$$\text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \int_0^1 \int_0^1 1\, dx\, dy = 1.$$

3. Find the flux of $\mathbf{F}$ through the surface $x^2 + y^2 = 1$ with $0 \leq z \leq 1$.

Answer: Here $\mathbf{n} = \langle x, y, 0 \rangle$, so $\mathbf{F} \cdot \mathbf{n} = x^2 + y^2 = 1$. We can parametrize the surface by $x = \cos\theta$, $y = \sin\theta$ with $dS = d\theta\, dz$ and integrate, or we can observe that the result of that calculation will just be the surface area of the cylinder. Flux $= 2\pi$.

Problems: Flux Through Surfaces

Let $\mathbf{F} = \langle x, y, z \rangle$.

1. Find the flux of $\mathbf{F}$ through the square with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(1, 1, 0)$, $(0, 1, 0)$.

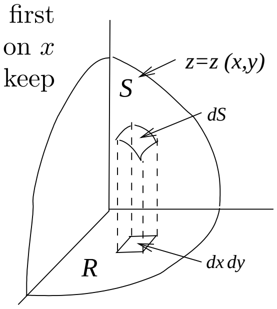
2. Find the flux of $\mathbf{F}$ through the square with vertices $(0, 0, 1)$, $(1, 0, 1)$, $(1, 1, 1)$, $(0, 1, 1)$.

3. Find the flux of $\mathbf{F}$ through the surface $x^2 + y^2 = 1$ with $0 \leq z \leq 1$.

V9.3-4 Surface Integrals

3. Flux through general surfaces.

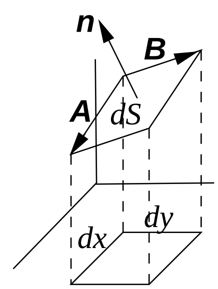
For a general surface, we will use xyz-coordinates. It turns out that here it is simpler to calculate the infinitesimal vector $d\mathbf{S} = \mathbf{n}\,dS$ directly, rather than calculate $\mathbf{n}$ and dS separately and multiply them, as we did in the previous section. Below are the two standard forms for the equation of a surface, and the corresponding expressions for $d\mathbf{S}$. In the first we use z both for the dependent variable and the function which gives its dependence on x and y; you can use $f(x,y)$ for the function if you prefer, but that's one more letter to keep track of.



(11a) $\qquad z = z(x,y), \qquad d\mathbf{S} = (-z_x\,\mathbf{i} - z_y\,\mathbf{j} + \mathbf{k})\,dx\,dy \qquad$ ($\mathbf{n}$ points "up")

(11b) $\qquad F(x,y,z) = c, \qquad d\mathbf{S} = \pm\dfrac{\nabla F}{F_z}\,dx\,dy \qquad$ (choose the right sign);

Derivation of formulas for $d\mathbf{S}$.



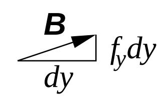
Refer to the pictures at the right. The surface S lies over its projection R, a region in the xy-plane. We divide up R into infinitesimal rectangles having area $dx\,dy$ and sides parallel to the xy-axes — one of these is shown. Over it lies a piece dS of the surface, which is approximately a parallelogram, since its sides are approximately parallel.

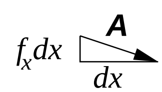
The infinitesimal vector $d\mathbf{S} = \mathbf{n}\,dS$ we are looking for has

direction: perpendicular to the surface, in the "up" direction;
magnitude: the area dS of the infinitesimal parallelogram.

This shows our infinitesimal vector is the cross-product

$$d\mathbf{S} = \mathbf{A} \times \mathbf{B}$$





where $\mathbf{A}$ and $\mathbf{B}$ are the two infinitesimal vectors forming adjacent sides of the parallelogram. To calculate these vectors, from the definition of the partial derivative, we have

$\mathbf{A}$ lies over the vector $dx\,\mathbf{i}$ and has slope f_x in the $\mathbf{i}$ direction, so $\mathbf{A} = dx\,\mathbf{i} + f_x\,dx\,\mathbf{k}$;
$\mathbf{B}$ lies over the vector $dy\,\mathbf{j}$ and has slope f_y in the $\mathbf{j}$ direction, so $\mathbf{B} = dy\,\mathbf{j} + f_y\,dy\,\mathbf{k}$.

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ dx & 0 & f_x dx \\ 0 & dy & f_y dy \end{vmatrix} = (-f_x\,\mathbf{i} - f_y\,\mathbf{j} + \mathbf{k})\,dx\,dy\ ,$$

which is (11a).

To get (11b) from (11a), , our surface is given by

(12) $\qquad F(x,y,z) = c, \qquad z = z(x,y)$

where the right-hand equation is the result of solving $F(x,y,z) = c$ for z in terms of the independent variables x and y. We differentiate the left-hand equation in (12) with respect to the independent variables x and y, using the chain rule and remembering that $z = z(x,y)$:

$$F(x,y,z) = c \quad \Rightarrow \quad F_x\frac{\partial x}{\partial x} + F_y\frac{\partial y}{\partial x} + F_z\frac{\partial z}{\partial x} = 0 \quad \Rightarrow \quad F_x + F_z\frac{\partial z}{\partial x} = 0$$

from which we get

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \qquad \text{and similarly,} \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} \ .$$

Therefore by (11a),

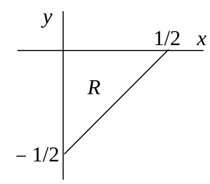
$$d\mathbf{S} \ = \ \left(-\frac{\partial z}{\partial x}\,\mathbf{i} - \frac{\partial z}{\partial y}\,\mathbf{j} + 1\right) dx\, dy \ = \ \left(\frac{F_x}{F_z}\,\mathbf{i} + \frac{F_y}{F_z}\,\mathbf{j} + 1\right) dx\, dy \ = \ \frac{\nabla F}{F_z}\, dx\, dy \ ,$$

which is (11b).

Example 3. The portion of the plane $2x - 2y + z = 1$ lying in the first octant forms a triangle S. Find the flux of $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ through S; take the positive side of S as the one where the normal points "up".

Solution. Writing the plane in the form $z = 1 - 2x + 2y$, we get using (11a),

$$\begin{aligned} d\mathbf{S} \ &= \ (2\,\mathbf{i} - 2\,\mathbf{j} + \mathbf{k})\, dx\, dy \ , \qquad \text{so} \\ \iint_S \mathbf{F}\cdot d\mathbf{S} \ &= \ \iint_S (2x - 2y + z)\, dy\, dx \\ &= \ \iint_R \big(2x - 2y + (1 - 2x + 2y)\big)\, dy\, dx \ , \end{aligned}$$



where R is the region in the xy-plane over which S lies. (Note that since the integration is to be in terms of x and y, we had to express z in terms of x and y for this last step.) To see what R is explicitly, the plane intersects the three coordinate axes respectively at $x = 1/2,\ y = -1/2,\ z = 1$. So R is the region pictured; our integral has integrand 1, so its value is the area of R, which is $1/8$.

Remark. When we write $z = f(x, y)$ or $z = z(x, y)$, we are agreeing to parametrize our surface using x and y as parameters. Thus the flux integral will be reduced to a double integral over a region R in the xy-plane, involving only x and y. Therefore you must *get rid of z by using the relation $z = z(x, y)$* after you have calculated the flux integral using (11a). Then determine the region R (the projection of S onto the xy-plane), and supply the limits for the iterated integral over R.

Example 4. Set up a double integral in the xy-plane which gives the flux of the field $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ through that portion of the ellipsoid $4x^2 + y^2 + 4z^2 = 4$ lying in the first octant; take $\mathbf{n}$ in the "up" direction.

Solution. Using (11b), we have $d\,\mathbf{S} = \dfrac{\langle 8x, 2y, 8z\rangle}{8z}\, dx\, dy$. Therefore

$$\iint_S \mathbf{F}\cdot d\,\mathbf{S} = \iint_S \frac{8x^2 + 2y^2 + 8z^2}{8z}\, dx\, dy = \iint_S \frac{1}{z}\, dx\, dy = \iint_R \frac{dx\, dy}{\sqrt{1 - x^2 - (y/2)^2}} \ ,$$

where R is the portion of the ellipse $4x^2 + y^2 = 4$ lying the the first quadrant.

The double integral would be most simply evaluated by making the change of variable $u = y/2$, which would convert it to a double integral over a quarter circle in the xu-plane easily evaluated by a change to polar coordinates.

4. General surface integrals.* The surface integral $\iint_S f(x,y,z)\,dS$ that we introduced at the beginning can be used to calculate things other than flux.

a) **Surface area.** We let the function $f(x,y,z)=1$. Then the area of $S \;=\; \iint_S dS$.

b) **Mass, moments, charge.** If S is a thin shell of material, of uniform thickness, and with density (in gms/unit area) given by $\delta(x,y,z)$, then

$$\text{mass of } S \;=\; \iint_S \delta(x,y,z)\,dS, \tag{13}$$

$$x\text{-component of center of mass} \;=\; \overline{x} \;=\; \frac{1}{\text{mass } S}\iint_S x\cdot\delta\,dS \tag{14}$$

with the y- and z-components of the center of mass defined similarly. If $\delta(x,y,z)$ represents an electric charge density, then the surface integral (13) will give the total charge on S.

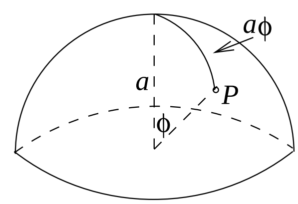
c) **Average value.** The average value of a function $f(x,y,z)$ over the surface S can be calculated by a surface integral:

$$\text{average value of } f \text{ on } S \;=\; \frac{1}{\text{area } S}\iint_S f(x,y,z)\,dS\;. \tag{15}$$

Calculating general surface integrals; finding dS.

To evaluate general surface integrals we need to know dS for the surface. For a sphere or cylinder, we can use the methods in section 2 of this chapter.

Example 5. Find the average distance along the earth of the points in the northern hemisphere from the North Pole. (Assume the earth is a sphere of radius a.)



Solution. — We use (15) and spherical coordinates, choosing the coordinates so the North Pole is at $z=a$ on the z-axis. The distance of the point (a,ϕ,θ) from $(a,0,0)$ is $a\phi$, measured along the great circle, i.e., the longitude line — see the picture). We want to find the average of this function over the upper hemisphere S. Integrating, and using (9), we get

$$\iint_S a\phi\,dS \;=\; \int_0^{2\pi}\int_0^{\pi/2} a\phi a^2\sin\phi\,d\phi\,d\theta \;=\; 2\pi a^3\int_0^{\pi/2}\phi\,\sin\phi\,d\phi \;=\; 2\pi a^3\;.$$

(The last integral used integration by parts.) Since the area of $S = 2\pi a^2$, we get using (15) the striking answer: average distance $= a$.

For more general surfaces given in xyz-coordinates, since $d\mathbf{S} = \mathbf{n}\,dS$, the area element dS is the magnitude of $d\mathbf{S}$. Using (11a) and (11b), this tells us

$$z=z(x,y), \qquad dS=\sqrt{z_x^2+z_y^2+1}\;dx\,dy \tag{16a}$$

$$F(x,y,z)=c, \qquad dS=\frac{|\nabla F|}{|F_z|}\;dx\,dy \tag{16b}$$

Example 6. The area of the piece S of $z=xy$ lying over the unit circle R in the xy-plane is calculated by (a) above and (16a) to be:

$$\iint_S dS = \iint_R \sqrt{y^2+x^2+1}\,dx\,dy = \int_0^{2\pi}\int_0^1 \sqrt{r^2+1}\,r\,dr\,d\theta = 2\pi\cdot\frac{1}{3}(r^2+1)^{3/2}\Big]_0^1 = \frac{2\pi}{3}(2\sqrt{2}-1).$$

Problems: Flux Through General Surfaces

1. Let $\mathbf{F} = -y\mathbf{i} + x\mathbf{k}$ and let S be the graph of $z = x^2 + y^2$ above the unit square in the xy-plane. Find the *upward flux* of $\mathbf{F}$ through S.

Answer: We can save time by noting that $\mathbf{F}$ is a tangential vector field and the vectors in $\mathbf{F}$ are parallel to S.

Otherwise, for a surface $z = f(x, y)$ we know that (for the upward normal)

$$\mathbf{n}\, dS = \langle -f_x, -f_y, 1\rangle\, dx\, dy.$$

In this case, $\mathbf{n}\, dS = \langle -2x, -2y, 1\rangle\, dx\, dy$.

Then $\mathbf{F} \cdot \mathbf{n}\, dS = (2xy - 2xy)\, dx\, dy = 0\, dx\, dy$.

Hence, Flux $= \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = 0$.

2. Let $\mathbf{F} = -y\mathbf{i} + x\mathbf{k}$ and let S be the graph of $z = x^2 + y$ above the square with vertices at $(0, 0, 0)$, $(2, 0, 0)$, $(2, 2, 0)$ and $(0, 2, 0)$. Find the upward flux of $\mathbf{F}$ through S.

Answer:

Figure 1: The surface $z = x^2 + y$.

Step 1. Find $\mathbf{n}\, dS$: Here $\mathbf{n}\, dS = \langle -f_x, -f_y, 1\rangle\, dx\, dy = \langle -2x, -1, 1\rangle\, dx\, dy$.

Step 2. $\mathbf{F} \cdot \mathbf{n}\, dS = \langle -y, x, 0\rangle \cdot \langle -2x, -1, 1\rangle\, dx\, dy = (2xy - x)\, dx\, dy$.

Step 3. Flux $= \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_R (2xy - x)\, dx\, dy$, where R is the region in the xy-plane below S, i.e. the region 'holding' the parameters x and y.

Step 4. Compute the integral:

Limits: inner x: from 0 to 2, outer y: from 0 to 2.

$\Rightarrow$ flux $= \displaystyle\int_0^2 \int_0^2 2xy - x\, dx\, dy.$

Inner: $2(2y - 1)$.

Outer: $2\left(y^2 - y\right)\Big|_0^2 = 4 =$ upward flux.

Note that this implies that the *downward flux* is -4; upward and downward flux are about the choice of $\mathbf{n}$, not $\mathbf{F}$.

Problems: Flux Through General Surfaces

1. Let $\mathbf{F} = -y\mathbf{i} + x\mathbf{k}$ and let S be the graph of $z = x^2 + y^2$ above the unit square in the xy-plane. Find the *upward flux* of $\mathbf{F}$ through S.

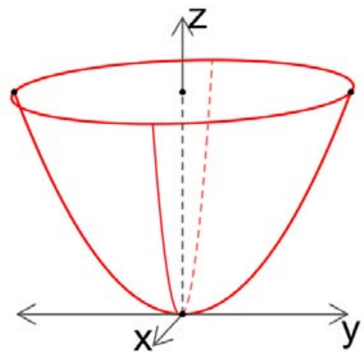
2. Let $\mathbf{F} = -y\mathbf{i} + x\mathbf{k}$ and let S be the graph of $z = x^2 + y$ above the square with vertices at $(0, 0, 0)$, $(2, 0, 0)$, $(2, 2, 0)$ and $(0, 2, 0)$. Find the upward flux of $\mathbf{F}$ through S.

Problems: Flux Through a Paraboloid

Consider the paraboloid $z = x^2 + y^2$. Let S be the portion of this surface that lies below the plane $z = 1$. Let $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + (1 - 2z)\mathbf{k}$.

Calculate the flux of $\mathbf{F}$ across S using the outward normal (the normal pointing away from the z-axis).

<u>Answer:</u> First, draw a picture:



The surface S is a bowl centered on the z-axis. The outward normal $\mathbf{n}$ points away from the outside of the bowl and downward. The region R is the shadow of the bowl – the unit circle in the xy-plane.

We know the z component of $\mathbf{n}$ is negative, so $\mathbf{n}\,dS = \langle z_x, z_y, -1\rangle\,dx\,dy = \langle 2x, 2y, -1\rangle\,dx\,dy$. Thus, $\mathbf{F}\cdot\mathbf{n}\,dS = (2x^2 + 2y^2 + 2z - 1)\,dx\,dy = (4z - 1)\,dx\,dy = (4r^2 - 1)\,dx\,dy$.

$$
\begin{aligned}
\iint_S \mathbf{F}\cdot\mathbf{n}\,dS &= \iint_R (4r^2 - 1)\,dx\,dy \\
&= \int_0^{2\pi}\int_0^1 (4r^2 - 1)r\,dr\,d\theta \\
&= \int_0^{2\pi} \frac{1}{2}\,d\theta \\
&= \pi.
\end{aligned}
$$

Problems: Flux Through a Paraboloid

Consider the paraboloid $z = x^2 + y^2$. Let S be the portion of this surface that lies below the plane $z = 1$. Let $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + (1 - 2z)\mathbf{k}$.

Calculate the flux of $\mathbf{F}$ across S using the outward normal (the normal pointing away from the z-axis).

General Formula for $\mathbf{n}\,dS$

Suppose S is a surface parametrized by x and y and $\mathbf{N}$ is any vector normal to S (not necessarily unit length). Then $\mathbf{n}\,dS = \dfrac{\mathbf{N}}{\mathbf{N}\cdot\mathbf{k}}\,dx\,dy$. Here $\mathbf{n}$ is the upward unit normal

Example: for the sphere $x^2+y^2+z^2=a^2$ with $\mathbf{N}=\langle x,y,z\rangle$, find $\mathbf{n}\,dS$.

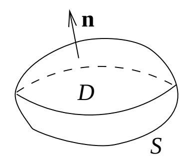
<u>Answer:</u> $\mathbf{n}\,dS = \dfrac{\mathbf{N}}{\mathbf{N}\cdot\mathbf{k}}\,dx\,dy = \langle \dfrac{x}{z}, \dfrac{y}{z}, 1\rangle\,dx\,dy.$

(Just like if we wrote $z=\sqrt{a^2-x^2-y^2}$, $\mathbf{n}\,dS = \langle -z_x, -z_y, 1\rangle\,dx\,dy$.)

V10.1 The Divergence Theorem

1. Introduction; statement of the theorem.

The divergence theorem is about closed surfaces, so let's start there. By a **closed** surface S we will mean a surface consisting of one connected piece which doesn't intersect itself, and which completely encloses a single finite region D of space called its *interior*. The closed surface S is then said to be the *boundary* of D; we include S in D. A sphere, cube, and torus (an inflated bicycle inner tube) are all examples of closed surfaces. On the other hand, these are not closed surfaces: a plane, a sphere with one point removed, a tin can whose cross-section looks like a figure-8 (it intersects itself), an infinite cylinder.



A closed surface always has two sides, and it has a natural positive direction — the one for which $\mathbf{n}$ points away from the interior, i.e., points toward the outside. We shall always understand that the closed surface has been oriented this way, unless otherwise specified.

We now generalize to 3-space the normal form of Green's theorem (Section V4).

Definition. Let $\mathbf{F}(x, y, z) = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ be a vector field differentiable in some region D. By the **divergence** of $\mathbf{F}$ we mean the scalar function $\text{div}\,\mathbf{F}$ of three variables defined in D by

$$(1) \qquad \text{div}\,\mathbf{F} \;=\; \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\ .$$

The divergence theorem. *Let S be a positively-oriented closed surface with interior D, and let $\mathbf{F}$ be a vector field continuously differentiable in a domain contatining D. Then*

$$(2) \qquad \iint_S \mathbf{F}\cdot d\mathbf{S} \;=\; \iiint_D \text{div}\,\mathbf{F}\,dV$$

We write dV on the right side, rather than $dx\,dy\,dz$ since the triple integral is often calculated in other coordinate systems, particularly spherical coordinates. The theorem is sometimes called **Gauss' theorem**.

Physically, the divergence theorem is interpreted just like the normal form for Green's theorem. Think of $\mathbf{F}$ as a three-dimensional flow field. Look first at the left side of (2). The surface integral represents the mass transport rate across the closed surface S, with flow out of S considered as positive, flow into S as negative.

Look now at the right side of (2). In what follows, we will show that the value of div $\mathbf{F}$ at (x, y, z) can be interpreted as the **source rate** at (x, y, z): the rate at which fluid is being added to the flow at this point. (Negative rate means fluid is being removed from the flow.) The integral on the right of (2) thus represents the *source rate for D*. So what the divergence theorem says is:

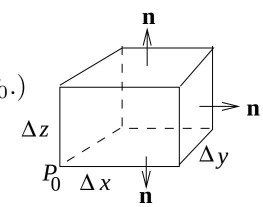
$$(3) \qquad \textit{flux across } S = \textit{source rate for } D\ ;$$

i.e., the net flow outward across S is the same as the rate at which fluid is being produced (or added to the flow) inside S.

To complete the argument for (3) we still have to show that

$$\text{div}\,\mathbf{F} \;=\; \textit{source rate at } (x, y, z)\ . \tag{3}$$

To see this, let $P_0 : (x_0, y_0, z_0)$ be a point inside the region D where $\mathbf{F}$ is defined. (To simplify, we denote by $(\text{div}\,\mathbf{F})_0, (\partial M/\partial x)_0$, etc., the value of these functions at P_0.)



Consider a little rectangular box, with edges $\Delta x, \Delta y, \Delta z$ parallel to the coordinate axes, and one corner at P_0. We take $\mathbf{n}$ to be always pointing outwards, as usual; thus on top of the box $\mathbf{n} = \mathbf{k}$, but on the bottom face, $\mathbf{n} = -\mathbf{k}$.

The flux across the top face in the $\mathbf{n}$ direction is approximately

$$\mathbf{F}(x_0, y_0, z_0 + \Delta z) \cdot \mathbf{k}\ \Delta x \Delta y \;=\; P(x_0, y_0, z_0 + \Delta z)\ \Delta x \Delta y,$$

while the flux across the bottom face in the $\mathbf{n}$ direction is approximately

$$\mathbf{F}(x_0, y_0, z_0) \cdot -\mathbf{k}\ \Delta x \Delta y \;=\; -P(x_0, y_0, z_0)\ \Delta x \Delta y\ .$$

So the net flux across the two faces combined is approximately

$$[P(x_0, y_0, z_0 + \Delta z) - P(x_0, y_0, z_0)]\ \Delta x \Delta y \;=\; \left(\frac{\Delta P}{\Delta z}\right) \Delta x \Delta y \Delta z\ .$$

Since the difference quotient is approximately equal to the partial derivative, we get the first line below; the reasoning for the following two lines is analogous:

$$\begin{aligned}
\textit{net flux across top and bottom} \;&\approx\; \left(\frac{\partial P}{\partial z}\right)_0 \Delta x \Delta y \Delta z;\\
\textit{net flux across two side faces} \;&\approx\; \left(\frac{\partial N}{\partial y}\right)_0 \Delta x \Delta y \Delta z;\\
\textit{net flux across front and back} \;&\approx\; \left(\frac{\partial M}{\partial x}\right)_0 \Delta x \Delta y \Delta z;
\end{aligned}$$

Adding up these three net fluxes, and using (3), we see that

$$\begin{aligned}
\textit{source rate for box} \;&=\; \textit{net flux across faces of box}\\
&\approx \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\right)_0 \Delta x \Delta y \Delta z\ .
\end{aligned}$$

Using this, we get the interpretation for div $\mathbf{F}$ we are seeking:

$$\textit{source rate at } P_0 \;=\; \lim_{\text{box}\to 0} \frac{\textit{source rate for box}}{\textit{volume of box}} \;=\; (\text{div}\,\mathbf{F})_0\ .$$

Example 1. Verify the theorem when $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and S is the sphere $\rho = a$.

Solution. For the sphere, $\mathbf{n} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{a}$; thus $\mathbf{F} \cdot \mathbf{n} = a$, and $\displaystyle\iint_S \mathbf{F} \cdot d\mathbf{S} = 4\pi a^3$.

On the other side, div $\mathbf{F} = 3$, $\iiint_D 3\,dV = 3 \cdot \frac{4}{3}\pi a^3$; thus the two integrals are equal. □

Example 2. Use the divergence theorem to evaluate the flux of $\mathbf{F} = x^3\,\mathbf{i} + y^3\,\mathbf{j} + z^3\,\mathbf{k}$ across the sphere $\rho = a$.

Solution. Here div $\mathbf{F} = 3(x^2 + y^2 + z^2) = 3\rho^2$. Therefore by (2),

$$\iint_S \mathbf{F} \cdot d\mathbf{S} \;=\; 3\iiint_D \rho^2\,dV \;=\; 3\int_0^a \rho^2 \cdot 4\pi\rho^2\,d\rho \;=\; \frac{12\pi a^5}{5}\;;$$

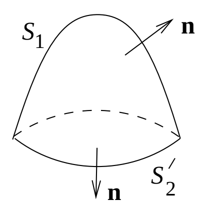
we did the triple integration by dividing up the sphere into thin concentric spheres, having volume $dV = 4\pi\rho^2\,d\rho$.

Example 3. Let S_1 be that portion of the surface of the paraboloid $z = 1 - x^2 - y^2$ lying above the xy-plane, and let S_2 be the part of the xy-plane lying inside the unit circle, directed so the normal $\mathbf{n}$ points upwards. Take $\mathbf{F} = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$; evaluate the flux of $\mathbf{F}$ across S_1 by using the divergence theorem to relate it to the flux across S_2.

Solution. We see immediately that div $\mathbf{F} = 0$. Therefore, if we let S_2' be the same surface as S_2, but oppositely oriented (so $\mathbf{n}$ points downwards), the surface $S_1 + S_2'$ is a closed surface, with $\mathbf{n}$ pointing outwards everywhere. Hence by the divergence theorem,

$$\iint_{S_1+S_2'} \mathbf{F} \cdot d\mathbf{S} \;=\; 0 \;=\; \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} - \iint_{S_2} \mathbf{F} \cdot d\mathbf{S}$$

Therefore, since we have $\mathbf{n} = \mathbf{k}$ on S_2,



$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} \;=\; \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} \;=\; \iint_{S_2} \mathbf{F} \cdot \mathbf{k}\;dS \;=\; \iint_{S_2} xy\,dx\,dy$$
$$=\; 0,$$

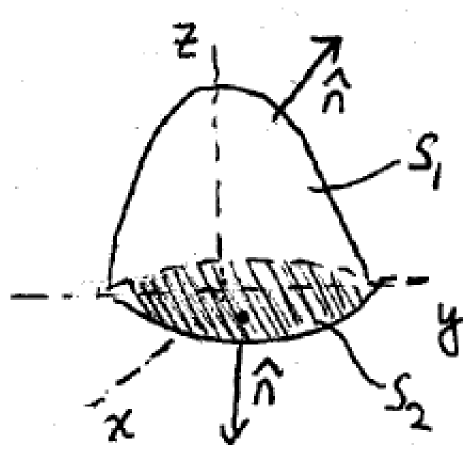
by integrating in polar coordinates (or by symmetry).

Problems: Divergence Theorem

Let S_1 be the part of the paraboloid $z = 1 - x^2 - y^2$ which is above the xy-plane and S_2 be the unit disk in the xy-plane. Use the divergence theorem to find the flux of $\mathbf{F}$ upward through S_1, where $\mathbf{F} = \langle yz, xz, xy \rangle$.

Answer: Write $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$, where $M = yz$, $N = xz$, and $P = xy$. Then

$$\mathrm{div}\mathbf{F} = M_x + N_y + P_z = 0.$$



The divergence theorem says: flux $= \iint_{S_1+S_2} \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_D \mathrm{div}\mathbf{F}\,dV = \iiint_D 0\,dV = 0$

$\Rightarrow \iint_{S_1} \mathbf{F}\cdot\mathbf{n}\,dS + \iint_{S_2} \mathbf{F}\cdot\mathbf{n}\,dS = 0 \Rightarrow \iint_{S_1} \mathbf{F}\cdot\mathbf{n}\,dS = -\iint_{S_2} \mathbf{F}\cdot\mathbf{n}\,dS.$

Therefore to find what we want we only need to compute the flux through S_2.

But S_2 is in the xy-plane, so $dS = dx\,dy$, $\mathbf{n} = -\mathbf{k} \Rightarrow \mathbf{F}\cdot\mathbf{n}\,dS = -xy\,dx\,dy$ on S_2.

Since S_2 is the unit disk, symmetry gives

$$\iint_{S_2} -xy\,dx\,dy = 0 \Rightarrow \iint_{S_1} \mathbf{F}\cdot\mathbf{n}\,dS = -\iint_{S_2} \mathbf{F}\cdot\mathbf{n}\,dS = 0.$$

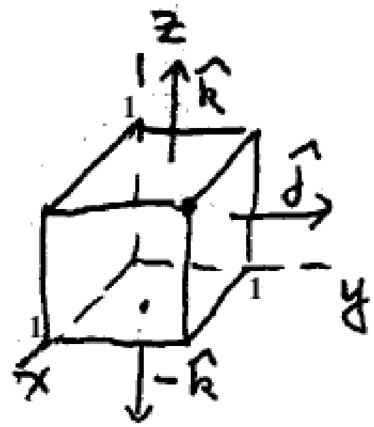
Problems: Divergence Theorem

Let S_1 be the part of the paraboloid $z = 1 - x^2 - y^2$ which is above the xy-plane and S_2 be the unit disk in the xy-plane. Use the divergence theorem to find the flux of $\mathbf{F}$ upward through S_1, where $\mathbf{F} = \langle yz, xz, xy \rangle$.

Problems: Del Notation; Flux

1. Verify the divergence theorem if $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and S is the surface of the unit cube with opposite vertices $(0,0,0)$ and $(1,1,1)$.

Answer: To confirm that $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_D \operatorname{div}\mathbf{F}\,dV$ we calculate each integral separately. The surface integral is calculated in six parts – one for each face of the cube.



$$\begin{aligned}
\text{Flux through top:}\quad & \mathbf{n} = \mathbf{k} \;\Rightarrow\; \mathbf{F}\cdot\mathbf{n}\,dS = z\,dx\,dy = dx\,dy \\
& \Rightarrow \iint_{\text{top}} \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^1\int_0^1 dx\,dy = 1. \\
\text{bottom:}\quad & \mathbf{n} = -\mathbf{k} \;\Rightarrow\; \mathbf{F}\cdot\mathbf{n}\,dS = -z\,dx\,dy = 0\,dx\,dy \;\Rightarrow\; \iint_{\text{bottom}} \mathbf{F}\cdot\mathbf{n}\,dS = 0. \\
\text{right:}\quad & \mathbf{n} = \mathbf{j} \;\Rightarrow\; \mathbf{F}\cdot\mathbf{n}\,dS = y\,dx\,dz = dx\,dz \\
& \Rightarrow \iint_{\text{right}} \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^1\int_0^1 dx\,dz = 1. \\
\text{left:}\quad & \mathbf{n} = -\mathbf{j} \;\Rightarrow\; \mathbf{F}\cdot\mathbf{n}\,dS = -y\,dx\,dz = 0\,dx\,dz \;\Rightarrow\; \iint_{\text{left}} \mathbf{F}\cdot\mathbf{n}\,dS = 0. \\
\text{front:}\quad & \mathbf{n} = \mathbf{i} \;\Rightarrow\; \mathbf{F}\cdot\mathbf{n}\,dS = x\,dy\,dz = dy\,dz \\
& \Rightarrow \iint_{\text{front}} \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^1\int_0^1 dy\,dz = 1. \\
\text{back:}\quad & \mathbf{n} = -\mathbf{i} \;\Rightarrow\; \mathbf{F}\cdot\mathbf{n}\,dS = -x\,dy\,dz = 0\,dy\,dz \;\Rightarrow\; \iint_{\text{back}} \mathbf{F}\cdot\mathbf{n}\,dS = 0.
\end{aligned}$$

The total flux through the surface of the cube is 3. (We could have used geometric reasoning to see that the flux through the back, left and bottom sides is 0; the vectors of F are parallel to the surface along those sides.)

To calculate the divergence we start by noting $\operatorname{div}\mathbf{F} = 1+1+1 = 3$. Then

$$\iiint_D \operatorname{div}\mathbf{F}\,dV = \iiint_D 3\,dV = 3\cdot(\text{Volume}) = 3.$$

We have verified that $\displaystyle\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_D \operatorname{div}\mathbf{F}\,dV$ in this example.

2. Prove that $\frac{1}{2}\boldsymbol{\nabla}(\mathbf{F}\cdot\mathbf{F}) = \mathbf{F}\times(\boldsymbol{\nabla}\times\mathbf{F}) + (\mathbf{F}\cdot\boldsymbol{\nabla})\mathbf{F}$, where $\langle P, Q, R\rangle\cdot\boldsymbol{\nabla}$ is the differential operator $P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}$.

Answer: We expand the left hand side, then the right hand side, then note that the expansions are equal. As usual, we assume $\mathbf{F} = \langle P, Q, R\rangle$.

LHS:

$$\begin{aligned}
\frac{1}{2}\boldsymbol{\nabla}\left(\mathbf{F}\cdot\mathbf{F}\right) &= \frac{1}{2}\boldsymbol{\nabla}(P^2+Q^2+R^2)\\
&= \left(P\frac{\partial P}{\partial x}+Q\frac{\partial Q}{\partial x}+R\frac{\partial R}{\partial x}\right)\mathbf{i}+\left(P\frac{\partial P}{\partial y}+Q\frac{\partial Q}{\partial y}+R\frac{\partial R}{\partial y}\right)\mathbf{j}\\
&\quad+\left(P\frac{\partial P}{\partial z}+Q\frac{\partial Q}{\partial z}+R\frac{\partial R}{\partial z}\right)\mathbf{k}.
\end{aligned}$$

RHS (in two parts):

$$\begin{aligned}
\mathbf{F}\times(\boldsymbol{\nabla}\times\mathbf{F}) &= \mathbf{F}\times\left[\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\mathbf{i}+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\mathbf{j}+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathbf{k}\right]\\
&= \left(Q\frac{\partial Q}{\partial x}-Q\frac{\partial P}{\partial y}-R\frac{\partial P}{\partial z}+R\frac{\partial R}{\partial x}\right)\mathbf{i}\\
&\quad+\left(R\frac{\partial R}{\partial y}-R\frac{\partial Q}{\partial z}-P\frac{\partial Q}{\partial x}+P\frac{\partial P}{\partial y}\right)\mathbf{j}\\
&\quad+\left(P\frac{\partial P}{\partial z}-P\frac{\partial R}{\partial x}-Q\frac{\partial R}{\partial y}+Q\frac{\partial Q}{\partial z}\right)\mathbf{k}\\
&= \left(Q\frac{\partial Q}{\partial x}+R\frac{\partial R}{\partial x}\right)\mathbf{i}+\left(R\frac{\partial R}{\partial y}+P\frac{\partial P}{\partial y}\right)\mathbf{j}+\left(P\frac{\partial P}{\partial z}+Q\frac{\partial Q}{\partial z}\right)\mathbf{k}\\
&\quad-\left(Q\frac{\partial P}{\partial y}+R\frac{\partial P}{\partial z}\right)\mathbf{i}-\left(R\frac{\partial Q}{\partial z}-P\frac{\partial Q}{\partial x}\right)\mathbf{j}-\left(P\frac{\partial R}{\partial x}+Q\frac{\partial R}{\partial y}\right)\mathbf{k}.
\end{aligned}$$

$$\begin{aligned}
(\mathbf{F}\cdot\boldsymbol{\nabla})\mathbf{F} &= \left(P\frac{\partial}{\partial x}+Q\frac{\partial}{\partial y}+R\frac{\partial}{\partial z}\right)\mathbf{F}\\
&= \left(P\frac{\partial P}{\partial x}+Q\frac{\partial P}{\partial y}+R\frac{\partial P}{\partial z}\right)\mathbf{i}+\left(P\frac{\partial Q}{\partial x}+Q\frac{\partial R}{\partial y}+R\frac{\partial Q}{\partial z}\right)\mathbf{j}\\
&\quad+\left(P\frac{\partial R}{\partial x}+Q\frac{\partial Q}{\partial y}+R\frac{\partial R}{\partial z}\right)\mathbf{k}.
\end{aligned}$$

Note that the negative terms in $\mathbf{F}\times(\boldsymbol{\nabla}\times\mathbf{F})$ are cancelled by positive terms in $(\mathbf{F}\cdot\boldsymbol{\nabla})\mathbf{F}$, leading to the desired result.

Problems: Del Notation; Flux

1. Verify the divergence theorem if $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and S is the surface of the unit cube with opposite vertices $(0, 0, 0)$ and $(1, 1, 1)$.

2. Prove that $\dfrac{1}{2}\boldsymbol{\nabla}(\mathbf{F}\cdot\mathbf{F}) = \mathbf{F}\times(\boldsymbol{\nabla}\times\mathbf{F}) + (\mathbf{F}\cdot\boldsymbol{\nabla})\mathbf{F}$, where $\langle P, Q, R\rangle\cdot\boldsymbol{\nabla}$ is the differential operator $P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}$.

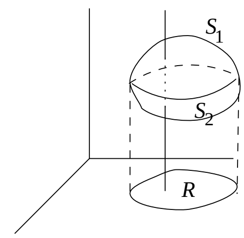
V10.2 The Divergence Theorem

2. Proof of the divergence theorem.

We give an argument assuming first that the vector field $\mathbf{F}$ has only a $\mathbf{k}$-component: $\mathbf{F} = P(x,y,z)\,\mathbf{k}$. The theorem then says

$$(4) \qquad \iint_S P\,\mathbf{k}\cdot\mathbf{n}\,dS \;=\; \iiint_D \frac{\partial P}{\partial z}\,dV\ .$$

The closed surface S projects into a region R in the xy-plane. We assume S is vertically simple, i.e., that each vertical line over the interior of R intersects S just twice. (S can have vertical sides, however — a cylinder would be an example.) S is then described by two equations:



$$(5) \qquad z = g(x,y) \quad \textit{(lower surface)}; \qquad z = h(x,y) \quad \textit{(upper surface)}$$

The strategy of the proof of (4) will be to reduce each side of (4) to a double integral over R; the two double integrals will then turn out to be the same.

We do this first for the triple integral on the right of (4). Evaluating it by iteration, we get as the first step in the iteration,

$$\begin{aligned} \iiint_D \frac{\partial P}{\partial z}\,dV \;&=\; \iint_R \int_{g(x,y)}^{h(x,y)} \frac{\partial P}{\partial z}\,dz\,dx\,dy \\ (6) \qquad &=\; \iint_R \big(P(x,y,h) - P(x,y,g)\big)\,dx\,dy \end{aligned}$$

To calculate the surface integral on the left of (4), we use the formula for the surface area element $d\mathbf{S}$ given in V9, (13):

$$d\mathbf{S} \;=\; \pm(-z_x\,\mathbf{i} - z_y\,\mathbf{j} + k)\,dx\,dy,$$

where we use the $+$ sign if the normal vector to S has a positive k-component, i.e., points generally upwards (as on the upper surface here), and the $-$ sign if it points generally downwards (as it does for the lower surface here).

This gives for the flux of the field $P\,\mathbf{k}$ across the upper surface S_2, on which $z = h(x,y)$,

$$\iint_{S_2} P\,\mathbf{k}\cdot d\mathbf{S} \;=\; \iint_R P(x,y,z)\,dx\,dy \;=\; \iint_R P(x,y,h(x,y))\,dx\,dy\ ,$$

while for the flux across the lower surface S_1, where $z = g(x,y)$ and we use the $-$ sign as described above, we get

$$\iint_{S_1} P\,\mathbf{k}\cdot d\mathbf{S} \;=\; \iint_R -P(x,y,z)\,dx\,dy \;=\; \iint_R -P(x,y,g(x,y))\,dx\,dy\ ;$$

adding up the two fluxes to get the total flux across S, we have

$$\iint_S P\,\mathbf{k}\cdot d\mathbf{S} \;=\; \iint_R P(x,y,h)\,dx\,dy \;-\; \iint_R P(x,y,g)\,dx\,dy$$

which is the same as the double integral in (6). This proves (4). □

In the same way, if $\mathbf{F} = M(x,y,z)\,\mathbf{i}$ and the surface is simple in the $\mathbf{i}$ direction, we can prove

$$\iint_S M\,\mathbf{i}\cdot\mathbf{n}\,dS \;=\; \iiint_D \frac{\partial M}{\partial x}\,dV \tag{4'}$$

while if $\mathbf{F} = N(x,y,z)\,\mathbf{j}$ and the surface is simple in the $\mathbf{j}$ direction,

$$\iint_S N\,\mathbf{j}\cdot\mathbf{n}\,dS \;=\; \iiint_D \frac{\partial N}{\partial y}\,dV\;. \tag{4''}$$

Finally, for a general field $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ and a closed surface S which is simple in all three directions, we have only to add up (4), (4′), and (4″). and we get the divergence theorem.

If the domain D is not bounded by a closed surface which is simple in all three directions, it can usually be divided up into smaller domains D_i which are bounded by such surfaces S_i. Adding these up gives the divergence theorem for D and S, since the surface integrals over the new faces introduced by cutting up D each occur twice, with the opposite normal vectors $\mathbf{n}$, so that they cancel out; after addition, one ends up just with the surface integral over the original S.

Partial Differential Equations

An important application of the higher partial derivatives is that they are used in partial differential equations to express some laws of physics which are basic to most science and engineering subjects. In this section, we will give examples of a few such equations. The reason is partly cultural, so you meet these equations early and learn to recognize them, and partly technical: to give you a little more practice with the chain rule and computing higher derivatives.

A **partial differential equation**, PDE for short, is an equation involving some unknown function of several variables and one or more of its partial derivatives. For example,

$$x\frac{\partial w}{\partial x} - y\frac{\partial w}{\partial y} = 0$$

is such an equation. Evidently here the unknown function is a function of two variables

$$w = f(x,y) \ ;$$

we infer this from the equation, since only x and y occur in it as independent variables. In general a **solution** of a partial differential equation is a differentiable function that satisfies it. In the above example, the functions

$$w = x^n y^n \qquad \text{any } n$$

all are solutions to the equation. In general, PDE's have many solutions, far too many to find all of them. The problem is always to find the one solution satisfying some extra conditions, usually called either *boundary conditions* or *initial conditions* depending on their nature.

Our first important PDE is the **Laplace equation** in three dimensions:

$$(1) \qquad \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = 0\ .$$

Any steady-state temperature distribution in three-space

$$(2) \qquad w = T(x,y,z), \qquad T = \text{temperature at the point } (x,y,z)$$

satisfies Laplace's equation. (Here *steady-state* means that it is unchanging over time, here reflected in the fact that T is not a function of time. For example, imagine a solid object made of some uniform heat-conducting material (say a solid metal ball), and imagine a steady temperature distribution on its surface is maintained somehow (say with some arrangement of wires and thermostats). Then after a while the temperature at each point inside the ball will come to equilibrium — reach a steady state — and the resulting temperature function (2) inside the ball will then satisfy Laplace's equation.

As another example, the *gravitational potential*

$$w = \phi(x,y,z)$$

resulting from some arrangement of masses in space satisfies Laplace's equation in any region R of space not containing masses. The same is true of the *electrostatic potential* resulting from some collection of electric charges in space: (1) is satisfied in any region which is free of charge. This potential function measures the work done (against the field) carrying a unit test mass (or charge) from a fixed reference point to the point (x, y, z) in the gravitational (or electrostatic) field. Knowing ϕ, the field itself can be recovered as its negative gradient:

$$\mathbf{F} = -\nabla\phi .$$

All of this is just to stress the fundamental character of Laplace's equation — we live our lives surrounded by its solutions.

The *two-dimensional* Laplace equation is similar — you just drop the term involving z. The steady-state temperature distribution in a flat metal plate would satisfy the two-dimensional Laplace equation, if the faces of the plate were kept insulated and a steady-state temperature distribution maintained around the edges of the plate.

If in the temperature model we include also heat sources and sinks in the region, unchanging over time, the temperature function satisfies the closely related **Poisson equation**

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = f(x, y, z), \tag{3}$$

where f is some given function related to the sources and sinks.

Another important PDE is the **wave equation**; given below are the one-dimensional and two-dimensional versions; the three dimensional version would add a similar term in z to the left:

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2}; \qquad \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2}. \tag{4}$$

Here $x, y, \ldots$ are the space variables, t is the time, and c is the velocity with which the wave travels — this depends on the medium and the type of wave (light, sound, etc.). A solution, respectively

$$w = w(x, t), \qquad w = w(x, y, t),$$

gives for each moment t_0 of time the shape $w(x, t_0)$, $w(x, y, t_0)$ of the wave.

The third PDE goes by two names, depending on the context: **heat equation** or **diffusion equation**. The one- and two-dimensional versions are respectively

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{a^2}\frac{\partial w}{\partial t}; \qquad \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{1}{a^2}\frac{\partial w}{\partial t}. \tag{5}$$

It looks a lot like the wave equation (4), but the right-hand side this time involves only the first derivative, which gives it mathematically and physically an entirely different character.

When it is called the (one-dimensional) heat equation, a solution $w(x, t)$ represents a time-varying temperature distribution in say a uniform conducting metal rod, with insulated sides. In the same way, $w(x, y, t)$ would be the time-varying temperature distribution in a flat metal plate with insulated faces. For each moment t_0 in time, $w(x, y, t_0)$ gives the temperature distribution at that moment.

For example, if we assume the distribution is steady-state, i.e., not changing with time, then

$$\frac{\partial w}{\partial t} = 0 \qquad \text{(steady-state condition)}$$

and the two-dimensional heat equation would turn into the two-dimensional Laplace equation (1).

When (5) is referred to as the *diffusion equation*, say in one dimension, then $w(x, t)$ represents the concentration of a dissolved substance diffusing along a uniform tube filled with liquid, or of a gas diffusing down a uniform pipe.

Notice that all of these PDE's are second-order, that is, involve derivatives no higher than the second. There is an important fourth-order PDE in elasticity theory (the bilaplacian equation), but by and large the general rule seems to be either that Nature is content with laws that only require second partial derivatives, or that these are the only laws that humans are intelligent enough to formulate.

Problems: Harmonic Functions and Averages

A function u is called *harmonic* if $\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = 0$. In this problem we will see that the average value of a harmonic function over any sphere is exactly its value at the center of the sphere.

For simplicity, we'll take the center to be the origin and show the average is $u(0,0,0)$.

Let u be a harmonic function and S_R the sphere of radius R centered at the origin. The average value of u over S is given by $A = \frac{1}{4\pi R^2}\iint_S u(x,y,z)\,dS$.

1. Write this integral explicitly using spherical coordinates.

Answer:

$$\begin{aligned} A &= \frac{1}{4\pi R^2}\int_0^{2\pi}\int_0^{\pi} u(R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi)\,R^2\sin\phi\,d\phi\,d\theta \\ &= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} u(R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi)\,\sin\phi\,d\phi\,d\theta. \end{aligned}$$

2. Differentiate A with respect to R

Answer: $\displaystyle \frac{dA}{dR} = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}(u_x\sin\phi\cos\theta + u_y\sin\phi\sin\theta + u_z\cos\phi)\,\sin\phi\,d\phi\,d\theta.$

3. Rewrite the formula in part (2) in terms of $\nabla u\cdot\mathbf{n}$.

Answer: On S we have $\mathbf{n} = \dfrac{\langle x,y,z\rangle}{R} = \langle\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle$ and $dS = R^2\sin\phi\,d\phi\,d\theta$

$$\Rightarrow \frac{dA}{dR} = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\langle u_x,u_y,u_z\rangle\cdot\mathbf{n}\,\frac{dS}{R^2} = \frac{1}{4\pi R^2}\iint_{S_R}\nabla u\cdot\mathbf{n}\,dS.$$

4. Use the divergence theorem to show $\frac{dA}{dR} = 0$ and conclude the average $A = u(0,0,0)$.

Answer: Let D be the solid ball of radius R. Applying the divergence theorem to part (3) we get

$$\frac{dA}{dR} = \frac{1}{4\pi R^2}\iiint_D \nabla\cdot\nabla u\,dV = \frac{1}{4\pi R^2}\iiint_D \nabla^2 u\,dV = 0.$$

For R near 0 the average is approximately $u(0,0,0)$.

Since the derivative is 0 the average is the same for any radius and we can let R go to 0 to conclude $A = u(0,0,0)$.

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