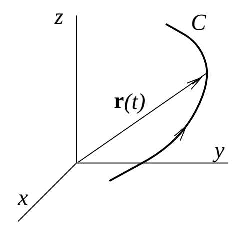
V11. Line Integrals in Space

1. Curves in space.

In order to generalize to three-space our earlier work with line integrals in the plane, we begin by recalling the relevant facts about parametrized space curves.

In 3-space, a vector function of one variable is given as

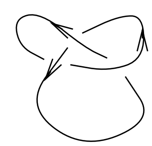


$$\mathbf{r}(t) \;=\; x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}\ . \tag{1}$$

It is called *continuous* or *differentiable* or *continuously differentiable* if respectively $x(t), y(t)$, and $z(t)$ all have the corresponding property. By placing the vector so that its tail is at the origin, its head moves along a curve C as t varies. This curve can be described therefore either by its position vector function (1), or by the three parametric equations

$$x = x(t), \qquad y = y(t), \qquad z = z(t)\ . \tag{2}$$

The curves we will deal with will be finite, connected, and piecewise smooth; this means that they have finite length, they consist of one piece, and they can be subdivided into a finite number of smaller pieces, each of which is given as the position vector of a continuously differentiable function (i.e., one whose derivative is continuous).



In addition, the curves will be *oriented*, or *directed*, meaning that an arrow has been placed on them to indicate which direction is considered to be the positive one. The curve is called *closed* if a point P moving on it always in the positive direction ultimately returns to its starting position, as in the accompanying picture.

The derivative of $\mathbf{r}(t)$ is defined in terms of components by

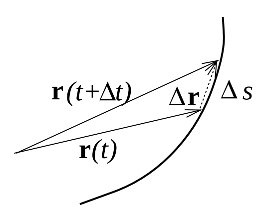
$$\frac{d\mathbf{r}}{dt} \;=\; \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j} + \frac{dz}{dt}\,\mathbf{k}\ . \tag{3}$$

If the parameter t represents time, we can think of $d\mathbf{r}/dt$ as the *velocity vector* $\mathbf{v}$. If we let s denote the arclength along C, measured from some fixed starting point in the positive direction, then in terms of s the magnitude and direction of $\mathbf{v}$ are given by

$$|\mathbf{v}| \;=\; \left|\frac{ds}{dt}\right|, \qquad \text{dir } \mathbf{v} \;=\; \begin{cases} \mathbf{t}, & \text{if } ds/dt > 0; \\ -\mathbf{t}, & \text{if } ds/dt < 0\ . \end{cases} \tag{4}$$

Here $\mathbf{t}$ is the unit tangent vector (pointing in the positive direction on C :

$$\mathbf{t} \;=\; \frac{d\mathbf{r}}{ds} \;=\; \frac{d\mathbf{r}/dt}{ds/dt}\ . \tag{5}$$



You can see from the picture that $\mathbf{t}$ is a unit vector, since

$$\left|\frac{d\mathbf{r}}{ds}\right| \;=\; \lim_{\Delta s \to 0} \left|\frac{\Delta\mathbf{r}}{\Delta s}\right| \;=\; 1\ .$$

2. Line integrals in space. Let $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ be a vector field in space, assumed continuous.

We define the *line integral of the tangential component of* $\mathbf{F}$ along an oriented curve C in space in the same way as for the plane. We approximate C by an inscribed sequence of directed line segments $\Delta\mathbf{r}_k$, form the approximating sum, then pass to the limit:

$$\int_C \mathbf{F}\cdot d\mathbf{r} \;=\; \lim_{k\to\infty} \sum_k \mathbf{r}_k \cdot \Delta\mathbf{r}_k \ .$$

The line integral is calculated just like the one in two dimensions:

$$(6) \qquad \int_C \mathbf{F}\cdot d\mathbf{r} \;=\; \int_{t_0}^{t_1} \mathbf{F}\cdot \frac{d\mathbf{r}}{dt}\, dt \ ,$$

if C is given by the position vector function $\mathbf{r}(t)$, $\quad t_0 \le t \le t_1$. Using x, y, z-components, one would write (6) as

$$(6') \qquad \int_C M\,dx + N\,dy + P\,dz \;=\; \int_{t_0}^{t_1} \left(M\frac{dx}{dt} + N\frac{dy}{dt} + P\frac{dz}{dt} \right) dt$$

In particular, if the parameter is the arclength s, then (6) becomes (since $\mathbf{t} = d\mathbf{r}/ds$)

$$(7) \qquad \int_C \mathbf{F}\cdot d\mathbf{r} \;=\; \int_{s_0}^{s_1} \mathbf{F}\cdot \mathbf{t}\, ds \ ,$$

which shows that the line integral is the integral along C of the tangential component of $\mathbf{F}$. As in two dimensions, this line integral represents the work done by the field $\mathbf{F}$ carrying a unit point mass (or charge) along the curve C.

Example 1. Find the work done by the electrostatic force field $\mathbf{F} = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$ in carrying a positive unit point charge from $(1,1,1)$ to $(2,4,8)$ along
a) a line segment b) the twisted cubic curve $\mathbf{r} = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$.

Solution. a) The line segment is given parametrically by

$$x - 1 = t, \quad y - 1 = 3t, \quad z - 1 = 7t, \qquad 0 \le t \le 1 \ .$$

$$\begin{aligned} \int_C y\,dx + z\,dy + x\,dz \;&=\; \int_0^1 (3t+1)\,dt + (7t+1)\cdot 3\,dt + (t+1)\cdot 7\,dt, \quad \text{using } (6') \\ &=\; \int_0^1 (31t + 11)\,dt \;=\; \frac{31}{2}t^2 + 11t\Big]_0^1 \;=\; \frac{31}{2} + 11 = 26.5 \ . \end{aligned}$$

b) Here the curve is given by $x = t,\ y = t^2,\ z = t^3, \quad 1 \le t \le 2$. For this curve, the line integral is

$$\begin{aligned} \int_1^2 t^2 dt + t^3 \cdot 2t\,dt + t\cdot 3t^2 dt \;&=\; \int_1^2 (t^2 + 3t^3 + 2t^4)dt \\ &=\; \frac{t^3}{3} + \frac{3t^4}{4} + \frac{2t^5}{5}\Big]_1^2 \;\approx\; 25.18 \ . \end{aligned}$$

The different results for the two paths shows that for this field, the line integral between two points depends on the path.

3. Gradient fields and path-independence.

The two-dimensional theory developed for line integrals in the plane generalizes easily to three-space. For the part where no new ideas are involved, we will be brief, just stating the results, and in places sketching the proofs.

Definition. Let $\mathbf{F}$ be a continuous vector field in a region D of space. The line integral $\int_P^Q \mathbf{F}\cdot d\mathbf{r}$ is called **path-independent** if, for any two points P and Q in the region D, the value of $\int_C \mathbf{F}\cdot d\mathbf{r}$ along a directed curve C lying in D and running from P to Q depends only on the two endpoints, and not on C.

An equivalent formulation is (the proof of equivalence is the same as before):

$$(8) \qquad \int_P^Q \mathbf{F}\cdot d\mathbf{r} \text{ is path independent } \Leftrightarrow \oint_C \mathbf{F}\cdot d\mathbf{r} = 0 \text{ for every closed curve } C \text{ in } D$$

Definition Let $f(x, y, z)$ be continuously differentiable in a region D. The vector field

$$(9) \qquad \nabla f \;=\; \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k}$$

is called the **gradient field** of f in D. Any field of the form ∇f is called a gradient field.

Theorem. First fundamental theorem of calculus for line integrals. *If $f(x, y, z)$ is continuously differentiable in a region D, then for any two points P_1, P_2 lying in D,*

$$(10) \qquad \int_{P_1}^{P_2} \nabla f\cdot d\mathbf{r} \;=\; f(P_2) - f(P_1),$$

where the integral is taken along any curve C lying in D and running from P_1 to P_2. In particular, the line integral is path-independent.

The proof is exactly the same as before — use the chain rule to reduce it to the first fundamental theorem of calculus for functions of one variable.

There is also an analogue of the second fundamental theorem of calculus, the one where we first integrate, then differentiate.

Theorem. Second fundamental theorem of calculus for line integrals.

Let $\mathbf{F}(x, y, z)$ be continuous and $\int_P^Q \mathbf{F}\cdot d\mathbf{r}$ path-independent in a region D; and define

$$(11) \qquad \begin{aligned} f(x,y,z) \;&=\; \int_{(x_0,y_0,z_0)}^{(x,y,z)} \mathbf{F}\cdot d\mathbf{r}; \qquad \textit{then} \\ \nabla f \;&=\; \mathbf{F} \qquad \textit{in } D. \end{aligned}$$

Note that since the integral is path-independent, no C need be specified in (11). The theorem is proved in your book for line integrals in the plane. The proof for line integrals in space is analogous.

Just as before, these two theorems produce the three equivalent statements: in D,

$$\text{(12)} \qquad \mathbf{F} = \nabla f \quad \Leftrightarrow \quad \int_P^Q \mathbf{F} \cdot d\mathbf{r} \ \text{ path-independent} \quad \Leftrightarrow \quad \oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \text{ for any closed } C$$

As in the two-dimensional case, if $\mathbf{F}$ is thought of as a force field, then the gradient force fields are called *conservative* fields, since the work done going around any closed path is zero (i.e., energy is conserved). If $\mathbf{F} = \nabla f$, then f is the called the (mathematical) *potential function* for $\mathbf{F}$; the physical potential function is defined to be $-f$.

Example 2. Let $f(x, y, z) = (x + y^2)z$. Calculate $\mathbf{F} = \nabla f$, and find $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the helix $x = \cos t, y = \sin t, z = t, 0 \le t \le \pi$.

Solution. By differentiating, $\mathbf{F} = z\,\mathbf{i} + 2yz\,\mathbf{j} + (x + y^2)\,\mathbf{k}$. The curve C runs from $(1, 0, 0)$ to $(-1, 0, \pi)$. Therefore by (10),

$$\int_C \mathbf{F} \cdot d\mathbf{r} \;=\; (x + y^2)z \Big]_{(1,0,0)}^{(-1,0,\pi)} \;=\; -\pi - 0 \;=\; -\pi.$$

No direct calculation of the line integral is needed, notice.

18.02SC Multivariable Calculus
Fall 2010

Problems: Work Along a Space Curve

1. Find the work done by the force $\mathbf{F} = -y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$ in moving a particle from $(0, 0, 0)$ to $(2, 4, 8)$
(a) along a line segment
(b) along the path $\mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$.

<u>Answer:</u>
(a) We use the parametrization $x = 2t$, $y = 4t$, $z = 8t$, where $0 \leq t \leq 1$. Other parametrizations should also work.

$$\begin{aligned} W &= \int_C M\,dx + N\,dy + P\,dz \\ &= \int_C -y\,dx + x\,dy + z\,dz \\ &= \int_0^1 -4t\,dt + 2t\,dt + 8t\,dt \\ &= \int_0^1 6t\,dt = 3. \end{aligned}$$

(b) We use the parametrization we were given:

$$\begin{aligned} W &= \int_C -y\,dx + x\,dy + z\,dz \\ &= \int_0^2 (-t^2)dt + t\,(2t\,dt) + t^3(3t^2\,dt) \\ &= \int_0^2 3t^5 + t^2\,dt = \frac{104}{3}. \end{aligned}$$

Note that for this force field, work done is not path independent.

2. Let $\mathbf{F} = \nabla f$, where $f = \dfrac{1}{(x+y+z)^2+1}$. Find the work done by $\mathbf{F}$ in moving a particle from the origin to infinity along a ray.

<u>Answer:</u> The fundamental theorem tells us that $\int_C \mathbf{F} \cdot d\mathbf{r} = f(P_1) - f(0)$ if C goes from 0 to P_1. In this example $f(0) = 1$, and as P_1 goes to infinity $f(P_1)$ approaches 0. Thus the work done is $0 - 1 = -1$.

Problems: Work Along a Space Curve

1. Find the work done by the force $\mathbf{F} = -y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$ in moving a particle from $(0, 0, 0)$ to $(2, 4, 8)$
(a) along a line segment
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2. Let $\mathbf{F} = \nabla f$, where $f = \dfrac{1}{(x+y+z)^2+1}$. Find the work done by $\mathbf{F}$ in moving a particle from the origin to infinity along a ray.

V12. Gradient Fields in Space

1. The criterion for gradient fields. The curl in space.

We seek now to generalize to space our earlier criterion (Section V2) for gradient fields in the plane.

Criterion for a Gradient Field. Let $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ be continuously differentiable. Then

$$(1)\quad \mathbf{F} = \nabla f \quad \text{for some } f(x,y,z) \quad \Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\,, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}\,, \quad \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}\,.$$

Proof. Since $\mathbf{F} = \nabla f$, when written out this says

$$(2)\qquad M = \frac{\partial f}{\partial x}, \qquad N = \frac{\partial f}{\partial y}. \qquad P = \frac{\partial f}{\partial z}\,; \qquad \text{therefore}$$

$$\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial N}{\partial x}\,.$$

The two mixed partial derivatives are equal since they are continuous, by the hypothesis that $\mathbf{F}$ is continuously differentiable.

The other two equalities in (1) are proved similarly. □

Though the criterion looks more complicated to remember and to check than the one in two dimensions, which involves just a single equation, it is not difficult to learn and apply. For theoretical purposes, it can be expressed more elegantly by using the three-dimensional vector curl $\mathbf{F}$.

Definition. Let $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ be differentiable. We define **curl F** by

$$(3)\qquad \operatorname{curl}\mathbf{F} = (P_y - N_z)\,\mathbf{i} + (M_z - P_x)\,\mathbf{j} + (N_x - M_y)\,\mathbf{k}$$

$$(3')\qquad = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix} \qquad \text{(symbolic notation; } \partial_x = \frac{\partial}{\partial x}\text{, etc.)}$$

$$(3'')\qquad = \nabla \times \mathbf{F}, \qquad \text{where } \nabla = \frac{\partial}{\partial x}\,\mathbf{i} + \frac{\partial}{\partial y}\,\mathbf{j} + \frac{\partial}{\partial z}\,\mathbf{k}\,.$$

The equation (3) is the definition. The other two lines give symbolic ways of writing and of remembering the right side of (3). Neither the first nor second row of the determinant contains the sort of thing you are allowed to put into a determinant; however, if you "evaluate" it using the Laplace expansion by the first row, what you get is the right side of (3). Similarly, to evaluate the symbolic cross-product in (3″), we use the determinant (3′). In doing these, by the "product" of $\dfrac{\partial}{\partial x}$ and M we mean $\dfrac{\partial M}{\partial x}$.

By using the vector field curl $\mathbf{F}$, our criterion (1) becomes

$$(1')\qquad \mathbf{F} = \nabla f \quad \Rightarrow \quad \operatorname{curl}\mathbf{F} = \mathbf{0}\,.$$

In dealing with a plane vector field $\mathbf{F} = M(x,y)\,\mathbf{i} + N(x,y)\,\mathbf{j}$, we gave the name curl $\mathbf{F}$ to the *scalar* function $N_x - M_y$, whereas for a vector field $\mathbf{F}$ in space, curl $\mathbf{F}$ is a *vector* function. However, if we think of the two-dimensional field $\mathbf{F}$ as a field in space (i.e., one with zero $\mathbf{k}$-component and not depending on z), then using definition (3) you can compute that

$$\text{curl}\ \mathbf{F} \;=\; (N_x - M_y)\,\mathbf{k}\ .$$

Thus curl $\mathbf{F}$ has only a $\mathbf{k}$-component, so if we are dealing just with two-dimensional fields, it is natural to give the name curl $\mathbf{F}$ just to this $\mathbf{k}$-component. This is not a universally accepted terminology, however; some call it the "scalar curl", others don't use any name at all for $N_x - M_y$.

Naturally, the question arises as to whether the converse of ($1'$) is true — if curl $\mathbf{F} = \mathbf{0}$, is $\mathbf{F}$ a gradient field? As in two dimensionas, this requires some sort of restriction on the domain, and we will return to this point after we have studied Stokes' theorem. For now we will assume the domain is the whole three-space, in which case it is true:

Theorem. *If* $\mathbf{F}$ *is continuously differentiable for all* x, y, z,

$$\text{(4)}\qquad \textit{curl}\ \mathbf{F} \;=\mathbf{0} \quad\Rightarrow\quad \mathbf{F} \;=\; \nabla f, \qquad \textit{for some differentiable } f(x,y,z).$$

We will prove this later. If $\mathbf{F}$ is a gradient field, we can calculate the corresponding (mathematical) potential function $f(x,y,z)$ by the three-dimensional analogue of either of the two methods described before (Section V2). We illustrate with an example.

Example 1. For what value(s), if any, of c will $\mathbf{F} \;=\; y\,\mathbf{i} + (x+cyz)\,\mathbf{j} + (y^2+z^2)\,\mathbf{k}$ be a conservative (i.e., gradient) field? For each such c, find a corresponding potential function $f(x,y,z)$.

Solution. Using (1) and (4), we calculate the relevant partial derivatives:

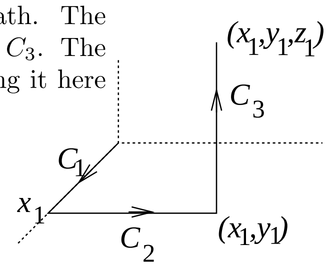
$$M_y = 1,\ N_x = 1; \quad N_z = cy,\ P_y = 2y; \quad M_z = 0,\ P_x = 0$$

Thus all three equations in (1) are satisfied $\Leftrightarrow$ $c = 2$. For this value of c, we now find $f(x,y,z)$ by two methods.

Method 1. We use the second fundamental theorem (Section V11, (11)), taking $(0,0,0)$ as a convenient lower limit for the integral, and using the subscript 1 on the upper limit to avoid confusion with the variables of integration. This gives

$$\text{(5)}\qquad f(x_1,y_1,z_1) \;=\; \int_{(0,0,0)}^{(x_1,y_1,z_1)} y\,dx + (x+2yz)\,dy + (y^2+z^2)\,dz$$

Since the integral is path-independent for the choice $c = 2$, we can use any path. The usual choice is the path illustrated, consisting of three line segments C_1, C_2 and C_3. The parametrizations for them are (don't write these out yourself — we are only doing it here this first time to make it clear how the line integral is being calculated):



$$\begin{array}{lll} C_1: & x = x,\ y = 0,\ z = 0; & \text{thus}\ \ dx = dx,\ dy = 0,\ dz = 0; \\ C_2: & x = x_1,\ y = y,\ z = 0; & \text{thus}\ \ dx = 0,\ dy = dy,\ dz = 0; \\ C_3: & x = x_1,\ y = y_1,\ z = z; & \text{thus}\ \ dx = 0,\ dy = 0,\ dz = dz\ . \end{array}$$

Using these, we calculate the line integral (5) over each of the C_i in turn:

$$\int_{C_1+C_2+C_3} y\,dx + (x+2yz)\,dy + (y^2+z^2)\,dz \;=\; \int_0^{x_1} 0\cdot dx \;+\; \int_0^{y_1} (x_1 + 2y\cdot 0)\,dy \;+\; \int_0^{z_1} (y_1^2+z^2)\,dz$$
$$= \; 0 + x_1 y_1 + (y_1^2 z_1 + \frac{1}{3} z_1^3)\;.$$

Dropping subscripts, we have therefore by (5),

$$f(x,y,z) \;=\; xy + y^2 z + \frac{1}{3}z^3 + c, \tag{6}$$

where we have added an arbitrary constant of integration to compensate for our arbitary choice of $(0,0,0)$ as the lower limit of integration — a different choice would have added a constant to the right side of (6).

The work should always be checked; from (6) one sees easily that $\nabla f \;=\; \mathbf{F}$, the field we started with.

Method 2. This requires no line integrals, but the work must be carried out systematically, otherwise you'll get lost in a mess of equations.

We are looking for an $f(x,y,z)$ such that $(f_x, f_y, f_z) \;=\; (y,\; x+2yz,\; y^2+z^2)$. This is equivalent to the three equations

$$f_x \;=\; y, \qquad f_y \;=\; x+2yz, \qquad f_z \;=\; y^2+z^2\;. \tag{7}$$

From the first equation, integrating with respect to x (holding y and z fixed), we get

$$\begin{array}{lrcll}
(8) & f(x,y,z) & = & xy + g(y,z), & g \text{ is an arbitrary function} \\
 & \dfrac{\partial f}{\partial y} & = & x + \dfrac{\partial g}{\partial y}, & \text{from (8)} \\
 & & = & x + 2yz & \text{from (7), second equation; comparing,} \\
 & \dfrac{\partial g}{\partial y} & = & 2yz. & \text{Integrating with respect to } y, \\
 & g(y,z) & = & y^2 z + h(z), & h \text{ is an arbitrary function; thus} \\
(9) & f(x,y,z) & = & xy + y^2 z + h(z), & \text{from the preceding and (8)} \\
 & \dfrac{\partial f}{\partial z} & = & y^2 + h'(z) & \\
 & & = & y^2 + z^2, & \text{from (7), third equation; comparing,} \\
 & h'(z) & = & z^2, & \\
 & h(z) & = & \dfrac{1}{3} z^3 + c; & \text{finally, by (9)} \\
 & f(x,y,z) & = & xy + y^2 z + \dfrac{1}{3} z^3 + c & \text{as in Method 1.}
\end{array}$$

2. Exact differentials

Just as we did in the two-dimensional case, we translate the previous ideas into the language of differentials.

The formal expression

$$M(x, y, z)\, dx + N(x, y, z)\, dy + P(x, y, z)\, dz \tag{10}$$

which appears as the integrand in our line integrals is called a **differential**. If $f(x, y, z)$ is a differentiable function, then its **total differential** (or just *differential*) is defined to be

$$df \;=\; \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy + \frac{\partial f}{\partial z}\, dz \tag{11}$$

The differential (10) is said to be **exact**, in some domain D where M, N and P are defined, if it is the total differential of some differentiable function $f(x, y, z)$ in this domain, that is, if there exists an $f(x, y, z)$ in D such that

$$M \;=\; \frac{\partial f}{\partial x}, \quad N \;=\; \frac{\partial f}{\partial y}, \quad P \;=\; \frac{\partial f}{\partial z}\ . \tag{12}$$

Criterion for exact differentials. *Let D be a domain in which M, N, P are continuously differentiable. Then in D,*

$$M\, dx + N\, dy + P\, dz \quad \text{is exact} \quad \Rightarrow \quad P_y = N_z, \quad M_z = P_x, \quad N_x = M_y\ ; \tag{13}$$

if D is all of 3-space, then the converse is true:

$$P_y = N_z, \quad M_z = P_x, \quad N_x = M_y \quad \Rightarrow \quad M\, dx + N\, dy + P\, dz \quad \text{is exact.} \tag{14}$$

If the test in this criterion shows that the differential (10) is exact, the function $f(x, y, z)$ may be found be either method 1 or method 2. The converse (14) is true under weaker hypotheses about D, which we will come back to after we have taken up Stokes' Theorem.

Problems: Gradient Fields and Potential Functions

Is the differential $y^2z\,dx + 2xyz\,dy + xy^2\,dz$ exact? If so, find a potential function. If not, explain why not.

Answer: The differential $f = M\,dx + N\,dy + P\,dz$ is exact if M, N, and P are continuously differentiable in all of 3-space and $P_y = N_z$, $M_z = P_x$ and $N_x = M_y$. Here $M = y^2z$, $N = 2xyz$ and $P = xy^2$ are continuously differentiable in all of 3-space. We compute:

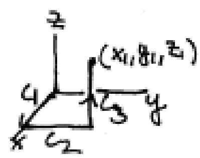
$$P_y = 2xy, \qquad N_z = 2xy,$$

$$M_z = y^2, \qquad P_x = y^2,$$

$$N_x = 2yz, \qquad M_y = 2yz.$$

This confirms that the differential is exact and so equal to df for some function $f(x, y, z)$. (Note that this method is equivalent to showing that $\text{curl}\langle M, N, P\rangle = \mathbf{0}$.)

To find a potential function we could "guess and check" or integrate along a path like the one shown below. (We carefully chose the path to make the integrals as simple as possible.)



$$\int_0^{x_1} 0\,dx + \int_0^{y_1} 0\,dy + \int_0^{z_1} x_1y_1^2\,dz = x_1y_1^2z_1.$$

We conclude that the potential functions for this differential are of the form $f = xy^2z + C$.

Problems: Gradient Fields and Potential Functions

Is the differential $y^2z\,dx + 2xyz\,dy + xy^2\,dz$ exact? If so, find a potential function. If not, explain why not.

V4.3 Physical meaning of curl

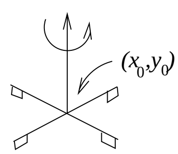
3. An interpretation for curl F.

We will start by looking at the two dimensional curl in the xy-plane. Our interpretation will be that the curl at a point represents twice the angular velocity of a small paddle wheel at that point. At the very end we will indicate how to extend this interpretation to 3 dimensions.

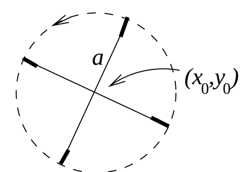
The function curl $\mathbf{F}$ can be thought of as measuring the tendency of $\mathbf{F}$ to produce rotation. Interpreting $\mathbf{F}$ either as a force field or a velocity field, $\mathbf{F}$ will make a suitable test object placed at a point P_0 spin about a vertical axis (i.e., one in the $\mathbf{k}$-direction), and the angular velocity of the spin will be proportional to (curl $\mathbf{F})_0$.

To see this for the velocity field $\mathbf{v}$ of a flowing liquid, place a paddle wheel of radius a so its center is at (x_0, y_0), and its axis is vertical. We ask how rapidly the flow spins the wheel.

If the wheel had only one blade, the velocity of the blade would be $\mathbf{F}\cdot\mathbf{t}$, the component of the flow velocity vector $\mathbf{F}$ perpendicular to the blade, i.e., tangent to the circle of radius a traced out by the blade.



paddlewheel



top view

Since $\mathbf{F}\cdot\mathbf{t}$ is not constant along this circle, if the wheel had only one blade it would spin around at an uneven rate. But if the wheel has many blades, this unevenness will be averaged out, and it will spin around at approximately the *average value* of the tangential velocity $\mathbf{F}\cdot\mathbf{t}$ over the circle. Like the average value of any function defined along a curve, this average tangential velocity can be found by integrating $\mathbf{F}\cdot\mathbf{t}$ over the circle, and dividing by the length of the circle. Thus,

$$\begin{aligned}
\text{speed of blade} &= \frac{1}{2\pi a}\oint_C \mathbf{F}\cdot\mathbf{t}\,ds \quad = \frac{1}{2\pi a}\oint_C \mathbf{F}\cdot d\mathbf{r} \\
&= \frac{1}{2\pi a}\iint_R (\text{curl }\mathbf{F})_0\,dx\,dy, \qquad \text{by Green's theorem,} \\
&\approx \frac{1}{2\pi a}(\text{curl }\mathbf{F})_0 \pi a^2\ , \qquad (8)
\end{aligned}$$

where (curl $\mathbf{F})_0$ is the value of the function curl $\mathbf{F}$ at (x_0, y_0). The justification for the last approximation is that if the circle formed by the paddlewheel is small, then curl $\mathbf{F}$ has approximately the value (curl $\mathbf{F})_0$ over the interior R of the circle, so that multiplying this constant value by the area πa^2 of R should give approximately the value of the double integral.

From (8) we get for the tangential speed of the paddlewheel:

$$\text{tangential speed} \quad \approx \quad \frac{a}{2}(\text{curl }\mathbf{F})_0\ . \qquad (9)$$

We can get rid of the a by using the angular velocity ω_0 of the paddlewheel; since the tangential speed is $a\omega_0$, (9) becomes

$$\omega_0 \quad \approx \quad \frac{1}{2}(\text{curl }\mathbf{F})_0\ . \qquad (10)$$

As the radius of the paddlewheel gets smaller, the approximation becomes more exact, and passing to the limit as $a \to 0$, we conclude that, for a two-dimensional velocity field $\mathbf{F}$,

$$(11) \qquad \boxed{\text{curl } \mathbf{F} = \textit{twice the angular velocity of an infinitesimal paddlewheel at } (x, y)\ .}$$

The curl thus measures the "vorticity" of the fluid flow — its tendency to produce rotation.

A consideration of curl $\mathbf{F}$ for a force field would be similar, interpreting $\mathbf{F}$ as exerting a torque on a spinnable object — a little dumbbell with two unit masses for a gravitational field, or with two unit positive charges for an electrostatic force field.

Example 1. Calculate and interpret curl $\mathbf{F}$ for (a) $x\,\mathbf{i} + y\,\mathbf{j}$ (b) $\omega(-y\,\mathbf{i} + x\,\mathbf{j})$

Solution. (a) curl $\mathbf{F} = 0$; this makes sense since the field is radially outward and radially symmetric, there is no favored angular direction in which the paddlewheel could spin.

(b) curl $\mathbf{F} = 2\omega$ at every point. Since this field represents a fluid rotating about the origin with constant angular velocity ω (see section V1), it is at least clear that curl $\mathbf{F}$ should be 2ω at the origin; it's not so clear that it should have this same value everywhere, but it is true.

Extension to Three Dimensions. To extend this interpretation to three dimensions note that any component of the flow of $\mathbf{F}$ in the $\mathbf{k}$ direction will not have any effect on a paddle wheel in the xy-plane. In fact, for any plane with normal $\mathbf{n}$ the component of $\mathbf{F}$ in the direction of $\mathbf{n}$ has no effect on a paddle wheel in the plane. This leads to the following interpretation of the three dimensional curl:

For any plane with unit normal $\mathbf{n}$, (curl $\mathbf{F}) \cdot \mathbf{n}$ is two times the angular velocity of a small paddle wheel in the plane.

We could force through a proof along the lines of the 2D proof above. Once we learn Stokes Theorem we can make a much simpler argument.

V15.1 Del Operator

1. Symbolic notation: the del operator

To have a compact notation, wide use is made of the symbolic operator "del" (some call it "nabla"):

(1) $$\nabla = \frac{\partial}{\partial x}\,\mathbf{i} + \frac{\partial}{\partial y}\,\mathbf{j} + \frac{\partial}{\partial z}\,\mathbf{k}$$

Recall that the "product" of $\dfrac{\partial}{\partial x}$ and the function $M(x,y,z)$ is understood to be $\dfrac{\partial M}{\partial x}$. Then we have

(2) $$\text{grad } f = \nabla f = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k}$$

The divergence is sort of a symbolic scalar product: if $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$,

(3) $$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

while the curl, as we have noted, as a symbolic cross-product:

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}.$$

Notice how this notation reminds you that $\nabla \cdot \mathbf{F}$ is a scalar function, while $\nabla \times \mathbf{F}$ is a vector function.

We may also speak of the Laplace operator (also called the "Laplacian"), defined by

(5) $$\text{lap } f = \nabla^2 f = (\nabla \cdot \nabla)\, f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\,.$$

Thus, Laplace's equation may be written: $\nabla^2 f = 0$. (This is for example the equation satisfied by the potential function for an electrostatic field, in any region of space where there are no charges; or for a gravitational field, in a region of space where there are no masses.)

In this notation, the divergence theorem and Stokes' theorem are respectively

(6) $$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_D \nabla \cdot \mathbf{F}\, dV \qquad\qquad \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$$

Two important relations involving the symbolic operator are:

(7) $$\text{curl (grad } f) = \mathbf{0} \qquad\qquad \text{div curl } \mathbf{F} = 0$$

(7′) $$\nabla \times \nabla f = \mathbf{0} \qquad\qquad \nabla \cdot \nabla \times \mathbf{F} = 0$$

The first we have proved (it was part of the criterion for gradient fields); the second is an easy exercise. Note however how the symbolic notation suggests the answer, since we know that for any vector $\mathbf{A}$, we have

$$\mathbf{A} \times \mathbf{A} = \mathbf{0}, \qquad\qquad \mathbf{A} \cdot \mathbf{A} \times \mathbf{F} = 0\,,$$

and (7′) says this is true for the symbolic vector ∇ as well.

Problems: Curl in 3D

1. Let $\mathbf{F} = \langle x, y, z \rangle$. Calculate and interpret curl$\mathbf{F}$.

Answer:

$$\begin{aligned}
\text{curl}\mathbf{F} &= \boldsymbol{\nabla} \times \mathbf{F} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} \\
&= \langle 0, 0, 0 \rangle.
\end{aligned}$$

We can interpret $\mathbf{F}$ as a velocity field in which particles race away from $(0, 0, 0)$ with speed equal to their distance from the origin. Because curl$\mathbf{F} = \mathbf{0}$, there is no rotational component in this field.

This makes sense. If we place a tiny paddle wheel at the point $(1, 0, 0)$, the torque about the axis from the force on a paddle just below the x-axis is exactly cancelled by the one symmetrically placed just above the x-axis. Since all the forces all the way around the wheel come in such pairs the net torque is 0 and the wheel will not start to spin. A similar argument holds for all other points in the plane except the origin. At the origin there is no force on any of the paddles, so the wheel doesn't spin.

2. Let $\mathbf{F} = \langle y, 0, 0 \rangle$. Calculate and interpret curl$\mathbf{F}$.

Answer:

$$\begin{aligned}
\text{curl}\mathbf{F} &= \boldsymbol{\nabla} \times \mathbf{F} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & 0 & 0 \end{vmatrix} \\
&= \langle 0, 0, -1 \rangle.
\end{aligned}$$

We can interpret $\mathbf{F}$ as a velocity field in which particles move away from the yz-plane with speed equal to their y coordinate. A tiny paddle wheel placed in this field with its axle pointing in the $\mathbf{k}$ direction will spin clockwise. Applying the right hand rule confirms the calculation that this field has a negative rotational component.

A paddle wheel placed with its axle pointing in the $\mathbf{i}$ direction will be completely unaffected by the velocity field, while the forces on the paddles of a wheel with its axle pointing in the $\mathbf{j}$ direction will cancel, as in problem 1.

Problems: Curl in 3D

1. Let $\mathbf{F} = \langle x, y, z \rangle$. Calculate and interpret curl$\mathbf{F}$.

2. Let $\mathbf{F} = \langle y, 0, 0 \rangle$. Calculate and interpret curl$\mathbf{F}$.

Testing for a Conservative Field

Let $\mathbf{F} = (3x^2y + az)\mathbf{i} + x^3\mathbf{j} + (3x + 3z^2)\mathbf{k}$.

1. For what value or values of a is $\mathbf{F}$ conservative?

Answer: We know $\mathbf{F}$ is conservative if it's continuously differentiable for all x, y, z and $\text{curl}F = 0$. We easily verify that $\mathbf{F}$ is continuously differentiable as required.

$$\text{curl}F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (3x^2y + az) & x^3 & (3x + 3z^2) \end{vmatrix} = 0\mathbf{i} - (3 - a)\mathbf{j} + (3x^2 - 3x^2)\mathbf{k} = (a - 3)\mathbf{j}.$$

If $a = 3$, $\text{curl}\mathbf{F} = 0$ and so $\mathbf{F}$ must be conservative.

Answer: $a = 3$.

2. Assuming a has the value(s) found in (1), find a potential function f for which $\mathbf{F} = \nabla f$.

Answer: As usual, there are two ways to find such a potential function. For variety, we'll use the second method.

Assume that $\mathbf{F} = \nabla f$. Then $f_x = 3x^2y + 3z$, so we have $f = x^3y + 3xz + g(y, z)$ for some function g.

Combine this with the fact that $f_y = x^3$ to get $x^3 + g_y = x^3$ so $g(y, z) = h(z)$ is constant with respect to y.

Finally, $f_z = 3x + h'(z) = 3x + 3z^2$ implies $h(z) = g(y, z) = z^3 + C$.

We conclude that $f(x, y, z) = x^3y + 3xz + z^3 + C$.

We can now calculate $f_x = 3x^2y + 3z$, $f_y = x^3$ and $f_z = 3x + 3z^2$ to check that $\mathbf{F} = \nabla f$.

V13.1-2 Stokes' Theorem

1. Introduction; statement of the theorem.

The normal form of Green's theorem generalizes in 3-space to the divergence theorem. What is the generalization to space of the tangential form of Green's theorem? It says

$$(1) \qquad \oint_C \mathbf{F} \cdot d\mathbf{r} \; = \; \iint_R \text{curl } \mathbf{F} \, dA$$

where C is a simple closed curve enclosing the plane region R.

Since the left side represents work done going around a closed curve in the plane, its natural generalization to space would be the integral $\oint \mathbf{F} \cdot d\mathbf{r}$ representing work done going around a closed curve in 3-space.

In trying to generalize the right-hand side of (1), the space curve C can only be the boundary of some piece of surface S — which of course will no longer be a piece of a plane. So it is natural to look for a generalization of the form

$$\oint_C \mathbf{F} \cdot d\mathbf{r} \; = \; \iint_S (\text{something derived from } \mathbf{F}) dS$$

The surface integral on the right should have these properties:

a) If curl $\mathbf{F} = 0$ in 3-space, then the surface integral should be 0; (for $\mathbf{F}$ is then a gradient field, by V12, (4), so the line integral is 0, by V11, (12)).

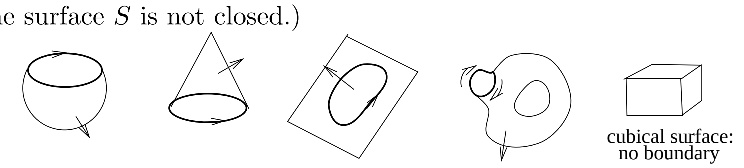
b) If C is in the xy-plane with S as its interior, and the field $\mathbf{F}$ does not depend on z and has only a $\mathbf{k}$-component, the right-hand side should be $\displaystyle\iint_S \text{curl } \mathbf{F} \, dS$.

These things suggest that the theorem we are looking for in space is

$$(2) \qquad \oint_C \mathbf{F} \cdot d\mathbf{r} \; = \; \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} \qquad\qquad \textbf{Stokes' theorem}$$

For the hypotheses, first of all C should be a closed curve, since it is the boundary of S, and it should be oriented, since we have to calculate a line integral over it.

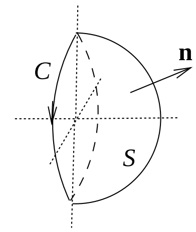
S is an oriented surface, since we have to calculate the flux of curl F through it. This means that S is two-sided, and one of the sides designated as positive; then the unit normal $\mathbf{n}$ is the one whose base is on the positive side. (There is no "standard" choice for positive side, since the surface S is not closed.)



It is important that C and S be *compatibly* oriented. By this we mean that the right-hand rule applies: when you walk in the positive direction on C, keeping S to your left, then your head should point in the direction of $\mathbf{n}$. The pictures give some examples.

The field $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ should have continuous first partial derivatives, so that we will be able to integrate curl F. For the same reason, the piece of surface S should be piecewise smooth and should be finite— i.e., not go off to infinity in any direction, and have finite area.

2. Examples and discussion.



Example 1. Verify the equality in Stokes' theorem when S is the half of the unit sphere centered at the origin on which $y \geq 0$, oriented so $\mathbf{n}$ makes an acute angle with the positive y-axis; take $\mathbf{F} = y\,\mathbf{i} + 2x\,\mathbf{j} + x\,\mathbf{k}$.

Solution. The picture illustrates C and S. Notice how C must be directed to make its orientation compatible with that of S.

We turn to the line integral first. C is a circle in the xz-plane, traced out *clockwise* in the plane. We select a parametrization and calculate:

$$x = \cos t, \qquad y = 0, \qquad z = -\sin t, \qquad\qquad 0 \leq t \leq 2\pi \ .$$

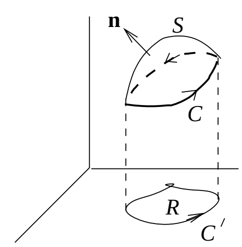
$$\oint_C y\,dx + 2x\,dy + x\,dz \ = \ \oint_C x\,dz \ = \ \int_0^{2\pi} -\cos^2 t\,dt \ = \ \left[-\frac{t}{2} - \frac{\sin 2t}{4}\right]_0^{2\pi} \ = \ -\pi \ .$$

For the surface S, we see by inspection that $\mathbf{n} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$; this is a unit vector since $x^2 + y^2 + z^2 = 1$ on S. We calculate

$$\text{curl } \mathbf{F} \ = \ \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ y & 2x & x \end{vmatrix} \ = \ -\mathbf{j} + \mathbf{k}\,; \qquad (\text{curl } \mathbf{F}) \cdot \mathbf{n} = -y + z$$

Integrating in spherical coordinates, we have $y = \sin\phi\sin\theta,\ z = \cos\phi,\ dS = \sin\phi\,d\phi\,d\theta,$ since $\rho = 1$ on S; therefore

$$\begin{aligned} \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} \ &= \ \iint_S (-y + z)\,dS \\ &= \ \int_0^{\pi}\int_0^{\pi} (-\sin\phi\sin\theta + \cos\phi)\sin\phi\,d\phi\,d\theta; \\ \text{inner integral} \ &= \ \sin\theta\left(\frac{\phi}{2} - \frac{\sin 2\phi}{4}\right) + \frac{1}{2}\,\sin^2\phi\,\Big]_0^{\pi} \ = \ \frac{\pi}{2}\,\sin\theta \\ \text{outer integral} \ &= -\frac{\pi}{2}\,\cos\theta\,\Big]_0^{\pi} \ = \ -\pi\ , \qquad \text{which checks.} \end{aligned}$$



Example 2. Suppose $\mathbf{F} = x^2\,\mathbf{i} + x\,\mathbf{j} + z^2\,\mathbf{k}$ and S is given as the graph of some function $z = g(x, y)$, oriented so $\mathbf{n}$ points upwards.

Show that $\oint_C \mathbf{F} \cdot d\mathbf{r} \ = \ $ *area of* R, where C is the boundary of S, compatibly oriented, and R is the projection of S onto the xy-plane.

Solution. We have $\text{curl } \mathbf{F} \ = \ \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ x^2 & x & z^2 \end{vmatrix} \ = \mathbf{k}$. By Stokes' theorem, (cf. V9, (12))

$$\oint_C \mathbf{F} \cdot d\mathbf{r} \ = \ \iint_S \mathbf{k} \cdot \mathbf{n}\,dS \ = \ \iint_R \mathbf{n} \cdot \mathbf{k}\,\frac{dA}{|\mathbf{n} \cdot \mathbf{k}\,|},$$

since $\mathbf{n}\cdot\mathbf{k} > 0$, $\quad |\mathbf{n}\cdot\mathbf{k}\,| = \mathbf{n}\cdot\mathbf{k}$; therefore

$$\oint_C \mathbf{F}\cdot d\mathbf{r} \;=\; \int_R dA \;=\; \text{area of } R\ .$$

The relation of Stokes' theorem to Green's theorem.

Suppose $\mathbf{F}$ is a vector field in space, having the form $\mathbf{F} = M(x,y)\,\mathbf{i} + N(x,y)\,\mathbf{j}$, and C is a simple closed curve in the xy-plane, oriented positively (so the interior is on your left as you walk upright in the positive direction). Let S be its interior, compatibly oriented — this means that the unit normal $\mathbf{n}$ to S is the vector $\mathbf{k}$, and $dS = dA$.

Then we get by the usual determinant method curl $\mathbf{F} = (N_x - M_y)\,\mathbf{k}$; since $\mathbf{n} = \mathbf{k}$, Stokes theorem becomes

$$\oint \mathbf{F}\cdot d\mathbf{r} \;=\; \iint_S \text{curl } \mathbf{F}\cdot\mathbf{n}\; dS \;=\; \iint_R (N_x - M_y)\, dA\ ,$$

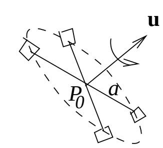
which is Green's theorem in the plane.

The same is true for other choices of the two variables; the most interesting one is $F = M(x,z)\,\mathbf{i} + P(x,z)\,\mathbf{k}$, where C is a simple closed curve in the xz-plane. If careful attention is paid to the choice of normal vector and the orientations, once again Stokes' theorem becomes just Green's theorem for the xz-plane. (See the Exercises.)

Interpretation of curl F.

Suppose now that $\mathbf{F}$ represents the velocity vector field for a three-dimensional fluid flow. Drawing on the interpretation we gave for the two-dimensional curl in Section V4, we can give the analog for 3-space.

The essential step is to interpret the $\mathbf{u}$-component of (curl $\mathbf{F})_0$ at a point P_0, , where $\mathbf{u}$ is a given unit vector, placed so its tail is at P_0.



Put a little paddlewheel of radius a in the flow so that its center is at P_0 and its axis points in the direction $\mathbf{u}$. Then by applying Stokes' theorem to a little circle C of radius a and center at P_0, lying in the plane through P_0 and having normal direction $\mathbf{u}$, we get just as in Section V4 (p. 4) that

$$\begin{aligned}\text{angular velocity of the paddlewheel} \;&=\; \frac{1}{2\pi a^2}\oint_C \mathbf{F}\cdot d\mathbf{r}\ ;\\ &=\; \frac{1}{2\pi a^2}\iint_S \text{curl } \mathbf{F}\cdot\mathbf{u}\; dS,\end{aligned}$$

by Stokes' theorem, S being the circular disc having C as boundary;

$$\approx\; \frac{1}{2\pi a^2}(\text{curl } \mathbf{F})_0\cdot\mathbf{u}\ (\pi a^2),$$

since curl $\mathbf{F}\cdot\mathbf{u}$ is approximately constant on S if a is small, and S has area πa^2; passing to the limit as $a \to 0$, the approximation becomes an equality:

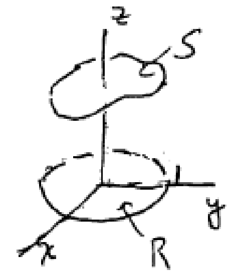
$$\text{angular velocity of the paddlewheel} \;=\; \frac{1}{2}\ (\text{curl } \mathbf{F})\cdot\mathbf{u}\ .$$

The preceding interprets (curl $\mathbf{F})_0\cdot\mathbf{u}$ for us. Since it has its maximum value when $\mathbf{u}$ has the direction of (curl $\mathbf{F})_0$, we conclude

$$\begin{aligned}\text{direction of } (\text{curl } \mathbf{F})_0 \;&=\; \text{axial direction in which wheel spins fastest}\\ \text{magnitude of } (\text{curl } \mathbf{F})_0 \;&=\; \text{twice this maximum angular velocity.}\end{aligned}$$

Problems: Stokes' Theorem

1. Let $\mathbf{F} = x^2\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}$ and let S be the graph of $z = x^3 + xy^2 + y^4$ over the unit disk. Use Stokes' Theorem to compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where C is the boundary of S.

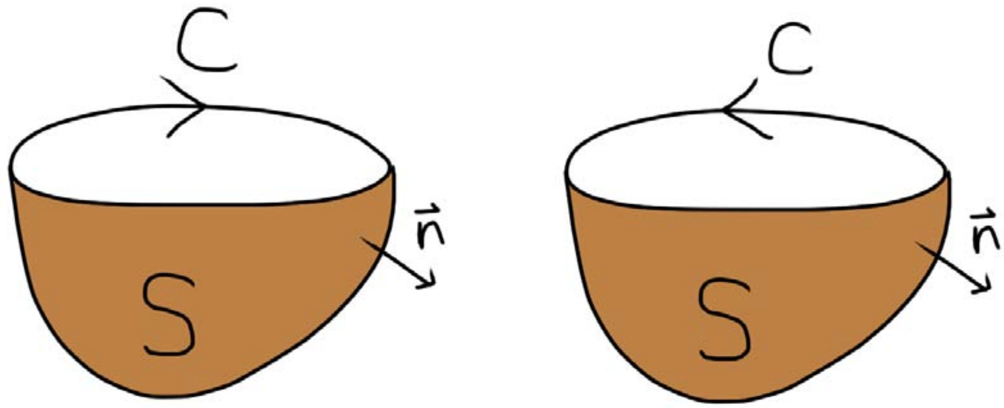


Answer: $\text{curl}\mathbf{F} = \langle 0, 0, 1 \rangle$, $\mathbf{n}\, dS = \langle -z_x, -z_y, 1 \rangle\, dx\, dy \;\Rightarrow\; \text{curl}\mathbf{F} \cdot \mathbf{n}\, dS = dx\, dy$.

$\Rightarrow \iint_S \text{curl}\mathbf{F} \cdot \mathbf{n}\, dS = \iint_R dx\, dy = \text{area } R = \pi$.

Therefore, by Stokes' Theorem $\boxed{\oint_C \mathbf{F} \cdot d\mathbf{r} = \pi.}$

2. Which of the figures below shows a compatibly oriented surface and curve?



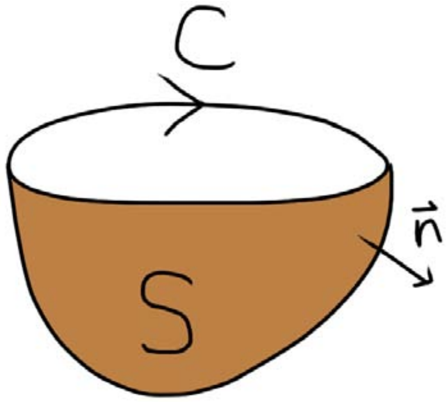
(a) (b)

Answer: On surface (a), the curve C is oriented compatibly with the surface S shown. To make this easier to see, add more arrows indicating the orientation of C.

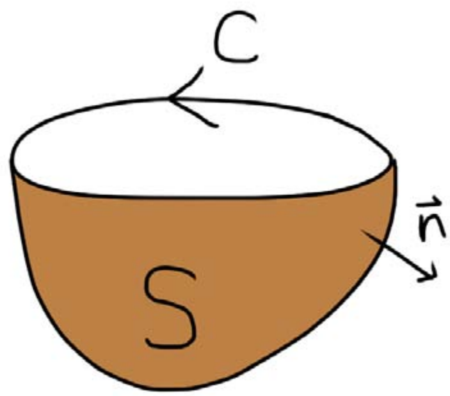
Problems: Stokes' Theorem

1. Let $\mathbf{F} = x^2\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}$ and let S be the graph of $z = x^3 + xy^2 + y^4$ over the unit disk. Use Stokes' Theorem to compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where C is the boundary of S.

2. Which of the figures below shows a compatibly oriented surface and curve?



(a)



(b)

V13.3 Stokes' Theorem

3. Proof of Stokes' Theorem.

We will prove Stokes' theorem for a vector field of the form $P(x,y,z)\,\mathbf{k}$. That is, we will show, with the usual notations,

$$\oint_C P(x,y,z)\,dz \;=\; \iint_S \text{curl}\;(P\,\mathbf{k}\,)\cdot\mathbf{n}\,dS\;. \tag{3}$$

We assume S is given as the graph of $z=f(x,y)$ over a region R of the xy-plane; we let C be the boundary of S, and C' the boundary of R. We take $\mathbf{n}$ on S to be pointing generally upwards, so that $|\mathbf{n}\cdot\mathbf{k}\,| = \mathbf{n}\cdot\mathbf{k}$.

To prove (3), we turn the left side into a line integral around C', and the right side into a double integral over R, both in the xy-plane. Then we show that these two integrals are equal by Green's theorem.

To calculate the line integrals around C and C', we parametrize these curves. Let

$$C':\;\; x=x(t),\;\; y=y(t), \qquad\qquad\qquad t_0\le t\le t_1$$

be a parametrization of the curve C' in the xy-plane; then

$$C:\;\; x=x(t),\;\; y=y(t),\;\; z=f(x(t),y(t)),\quad t_0\le t\le t_1$$

gives a corresponding parametrization of the space curve C lying over it, since C lies on the surface $z=f(x,y)$.

Attacking the line integral first, we claim that

$$\oint_C P(x,y,z)\,dz \;=\; \oint_{C'} P(x,y,f(x,y))(f_x dx + f_y dy)\;. \tag{4}$$

This looks reasonable purely formally, since we get the right side by substituting into the left side the expressions for z and dz in terms of x and y: $z=f(x,y), \quad dz = f_x dx + f_y dy$. To justify it more carefully, we use the parametrizations given above for C and C' to calculate the line integrals.

$$\begin{aligned}
\oint_C P(x,y,z)\,dz \;&=\; \int_{t_0}^{t_1} \big(P(x(t),y(t),z(t)\big)\,\frac{dz}{dt}\,dt \\
&=\; \int_{t_0}^{t_1} \big(P(x(t),y(t),z(t)\big)\,(f_x\frac{dx}{dt}+f_y\frac{dy}{dt})\,dt,\;\text{ by the chain rule} \\
&=\; \oint_{C'} P(x,y,f(x,y))(f_x dx + f_y dy), \quad \text{the right side of (4).}
\end{aligned}$$

We now calculate the surface integral on the right side of (3), using x and y as the variables. In the calculation, we must distinguish carefully between such expressions as $P_1(x,y,f)$ and $\frac{\partial}{\partial x}P(x,y,f)$. The first of these means: calculate the partial derivative with respect to the first variable x, treating x,y,z as independent; then substitute $f(x,y)$ for z. The second means: calculate the partial with respect to x, after making the substitution $z = f(x, y)$; the answer is

$$\frac{\partial}{\partial x} P(x, y, f) \;=\; P_1(x, y, f) + P_3(x, y, f)\, f_x \;.$$

(We use P_1 rather than P_x since the latter would be ambiguous — when you use numerical subscripts, everyone understands that the variables are being treated as independent.)

With this out of the way, the calculation of the surface integral is routine, using the standard procedure of an integral over a surface having the form $z = f(x, y)$ given in Section V9. We get

$$\begin{aligned}
d\mathbf{S} \;&=\; (-f_x\,\mathbf{i} - f_y\,\mathbf{j} + \mathbf{k})\,dx\,dy, \qquad\qquad \text{by V9, (13);}\\
\text{curl } (P(x,y,z)\,\mathbf{k}) \;&=\; \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & P \end{vmatrix} \;=\; P_2(x,y,z)\,\mathbf{i} - P_1(x,y,z)\,\mathbf{j}\\
\iint_S \text{curl } (P(x,y,z)\,\mathbf{k}) \cdot d\mathbf{S} \;&=\; \iint_S \big(-P_2(x,y,z) f_x + P_1(x,y,z) f_y\big)\,dx\,dy\\
&=\; \iint_R \big(-P_2(x,y,f) f_x + P_1(x,y,f) f_y\big)\,dx\,dy \qquad (5)
\end{aligned}$$

We have now turned the line integral into an integral around C' and the surface integral into a double integral over R. As the final step, we show that the right sides of (4) and (5) are equal by using Green's theorem:

$$\oint_{C'} U\,dx + V\,dy \;=\; \iint_R (V_x - U_y)\,dx\,dy \;.$$

(We have to state it using U and V rather than M and N, or P and Q, since in three-space we have been using these letters for the components of the general three-dimensional field $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$.) To substitute into the two sides of Green's theorem, we need four functions:

$$\begin{aligned}
V \;&=\; P(x,y,f(x,y)) f_y, \qquad \text{so} \qquad V_x \;=\; (P_1 + P_3 f_x) f_y + P(x,y,f) f_{yx}\\
U \;&=\; P(x,y,f(x,y)) f_x, \qquad \text{so} \qquad U_y \;=\; (P_2 + P_3 f_y) f_x + P(x,y,f) f_{xy}
\end{aligned}$$

Therefore, since $f_{xy} = f_{yx}$, four terms cancel, and the right side of Green's theorem becomes

$$V_x - U_y \;=\; P_1(x,y,f) f_y - P_2(x,y,f) f_x,$$

which is precisely the integrand on the right side of (5). This completes the proof of Stokes' theorem when $\mathbf{F} = P(x, y, z)\,\mathbf{k}$.

In the same way, if $\mathbf{F} = M(x, y, z)\,\mathbf{i}$ and the surface is $x = g(y, z)$, we can reduce Stokes' theorem to Green's theorem in the yz-plane.

If $\mathbf{F} = N(x, y, z)\,\mathbf{j}$ and $y = h(x, z)$ is the surface, we can reduce Stokes' theorem to Green's theorem in the xz-plane.

Since a general field $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ can be viewed as a sum of three fields, each of a special type for which Stokes' theorem is proved, we can add up the three Stokes' theorem equations of the form (3) to get Stokes' theorem for a general vector field.

A difficulty arises if the surface cannot be projected in a 1-1 way onto each of three coordinate planes in turn, so as to express it in the three forms needed above:

$$z = f(x, y), \qquad x = g(y, z), \qquad y = h(x, z) \ .$$

In this case, it can usually be divided up into smaller pieces which can be so expressed (if some of these are parallel to one of the coordinate planes, small modifications must be made in the argument). Stokes' theorem can then be applied to each piece of surface, then the separate equalities can be added up to get Stokes' theorem for the whole surface (in the addition, line integrals over the cut-lines cancel out, since they occur twice for each cut, in opposite directions). This completes the argument, *manus undulans*, for Stokes' theorem.

Problems: Extended Stokes' Theorem

Let $\mathbf{F} = \langle 2xz + y, 2yz + 3x, x^2 + y^2 + 5 \rangle$. Use Stokes' theorem to compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve shown on the surface of the circular cylinder of radius 1.

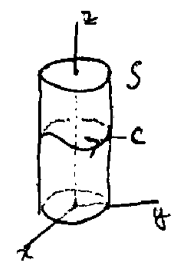


Figure 1: Positively oriented curve around a cylinder.

Answer: This is very similar to an earlier example; we can use Stokes' theorem to calculate this integral even though we don't have an exact description of C. We just make C into part of the boundary of a surface, as shown in the figure below.

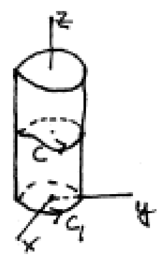


Figure 2: Curves C and C_1 bound part of a cylinder.

Let C_1 be the unit circle in the xy-plane oriented to match C and S the portion of the cylinder between C and C_1. Then Stokes' theorem says:

$$\oint_{C_1 - C} \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl}\mathbf{F} \cdot \mathbf{n}\, dS.$$

$$\text{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xz + y & 2yz + 3x & x^2 + y^2 + 5 \end{vmatrix} = 2\mathbf{k}.$$

Since the normal vector to S is always orthogonal to $\mathbf{k}$, $\iint_S \text{curl}\mathbf{F} \cdot \mathbf{n} = 0$.

Thus, $\oint_{C_1 - C} \mathbf{F} \cdot d\mathbf{r} = \oint_{C_1} \mathbf{F} \cdot d\mathbf{r} - \oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ and $\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_{C_1} \mathbf{F} \cdot d\mathbf{r}$.

To finish, parametrize C_1 by $x = \cos t$, $y = \sin t$, $z = 0$, $0 \leq t < 2\pi$ and calculate:

$$\begin{aligned}
\oint_{C_1} \mathbf{F} \cdot d\mathbf{r} &= \oint_C (2xz + y)dx + (2yz + 3x)dy + (x^2 + y^2)dz \\
&= \int_0^{2\pi} \sin t(-\sin t\, dt) + 3\cos t(\cos t\, dt) \\
&= \int_0^{2\pi} -1 + 4\cos^2 t\, dt \\
&= \left[-t + \frac{4}{2}(t + \sin t \cos t)\right]_0^{2\pi} = 2\pi.
\end{aligned}$$

Problems: Extended Stokes' Theorem

Let $\mathbf{F} = \langle 2xz + y, 2yz + 3x, x^2 + y^2 + 5\rangle$. Use Stokes' theorem to compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve shown on the surface of the circular cylinder of radius 1.

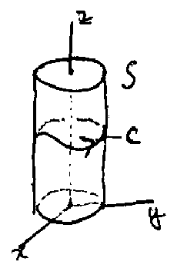
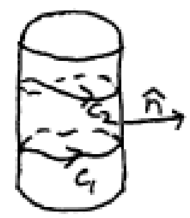


Figure 1: Positively oriented curve around a cylinder.

Extended Stokes' Theorem

Let $\mathbf{F} = \langle 2xz + 2y, 2yz + 2yx, x^2 + y^2 + z^2 \rangle$. Take C_1 and C_2 two curves going around the circular cylinder of radius a as shown. Show $\oint_{C_1} \mathbf{F} \cdot d\mathbf{r} = \oint_{C_2} \mathbf{F} \cdot d\mathbf{r}$.



Answer: We easily compute $\text{curl}\mathbf{F} = (2y - 2x)\mathbf{k} \;\Rightarrow\; \text{curl}\mathbf{F} \cdot \mathbf{n} = 0$, where $\mathbf{n}$ is the normal to the cylinder. Let S be the part of the cylinder between C_1 and C_2 then Stokes' theorem says

$$\oint_{C_1 - C_2} \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl}\mathbf{F} \cdot \mathbf{n}\, dS = 0. \;\Rightarrow\; \oint_{C_1} \mathbf{F} \cdot d\mathbf{r} = \oint_{C_2} \mathbf{F} \cdot d\mathbf{r}. \quad \text{QED}$$

V14. Some Topological Questions

We consider once again the criterion for a gradient field. We know that

$$\mathbf{F} = \nabla f \quad \Rightarrow \quad \text{curl } \mathbf{F} = \mathbf{0}\ , \tag{1}$$

and inquire about the converse. It is natural to try to prove that

$$\text{curl } \mathbf{F} = \mathbf{0} \quad \Rightarrow \quad \mathbf{F} = \nabla f \tag{2}$$

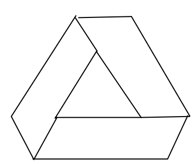
by using Stokes' theorem: if curl $\mathbf{F} = \mathbf{0}$, then for any closed curve C in space,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} \;=\; \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} \;=\; 0. \tag{3}$$

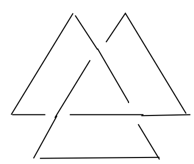
The difficulty is that we are given C, but not S. So we have to ask:

Question. Let D be a region of space in which $\mathbf{F}$ is continuously differentiable. Given a closed curve lying in D; is it the boundary of some two-sided surface S lying inside D?

We explain the "two-sided" condition. Some surfaces are only one-sided: if you start painting them, you can use only one color, if you don't allow abrupt color changes. An example is S below, formed giving three half-twists to a long strip of paper before joining the ends together.



Surface S



Boundary C (trefoil knot)

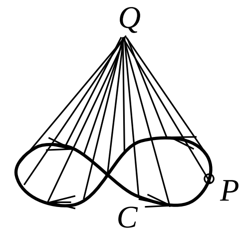
S has only one side. This means that it cannot be oriented: there is no continuous choice for the normal vector $\mathbf{n}$ over this surface. (If you start with a given $\mathbf{n}$ and make it vary continuously, when you return to the same spot after having gone all the way around, you will end up with $-\mathbf{n}$, the oppositely pointing vector.) For such surfaces, it makes no sense to speak of "the flux through S", because there is no consistent way of deciding on the positive direction for flow through the surface. Stokes' theorem does not apply to such surfaces.

To see what practical difficulties this causes, even when the domain is all of 3-space, consider the curve C in the above picture. It's called the trefoil knot. We know it is the boundary of the one-sided surface S, but this is no good for equation (3), which requires that we find a *two-sided* surface which has C for boundary.

> There are such surfaces; try to find one. It should be smooth and not cross itself. If successful, consider yourself a brown-belt topologist.

The preceding gives some ideas about the difficulties involved in finding a two-sided surface whose boundary is a closed curve C when the curve is knotted, i.e., cannot be continuously deformed into a circle without crossing itself at some point during the deformation. It is by no means clear that such a two-sided surface exists in general.

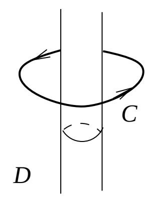
There are two ways out of the dilemma.



1. If we allow the surface to cross itself, and allow it to be not smooth along some lines, we can easily find such a two-sided surface whose boundary is a given closed curve C. The procedure is simple. Pick some fixed point Q not on the curve C, and join it to a point P on the curve (see the figure). Then as P moves around C, the line segment QP traces out a surface whose boundary is C. It will not be smooth at Q, and it will cross itself along a certain number of lines, but it's easy to see that this is a two-sided surface.

The point now is that Stokes' theorem can still be applied to such a surface: just use subdivision. Divide up the surface into skinny "triangles", each having one vertex at Q, and include among the edges of these triangles the lines where the surface crosses itself. Apply Stokes' theorem to each triangle, and add up the resulting equations.

2. Though the above is good enough for our purposes, it's an amazing fact that for any C there is always a smooth two-sided surface which doesn't cross itself, and whose boundary is C. (This was first proved around 1930 by van Kampen.)



The above at least answers our question affirmatively when D is all of 3-space. Suppose however that it isn't. If for instance D is the exterior of the cylinder $x^2 + y^2 = 1$, then it is intuitively clear that a circle C around the outside of this cylinder isn't the boundary of any finite surface lying entirely inside D.

A class of domains for which it is true however are the *simply-connected* ones.

Definition. A domain D in 3-space is **simply-connected** if each closed curve in it can be shrunk to a point without ever getting outside of D during the shrinking.

For example, 3-space itself is simply-connected, as is the interior or the exterior of a sphere. However the interior of a torus (a bagel, for instance) is not simply-connected, since any circle in it going around the hole cannot be shrunk to a point while staying inside the torus.

If D is simply-connected, then any closed curve C is the boundary of a two-sided surface (which may cross itself) lying entirely inside D. We can't prove this here, but it gives us the tool we need to establish the converse to the criterion for gradient fields in 3-space.

Theorem. *Let D be a simply-connected region in 3-space, and suppose that the vector field $\mathbf{F}$ is continuously differentiable in D. Then in D,*

$$\text{curl } \mathbf{F} = \mathbf{0} \quad \Rightarrow \quad \mathbf{F} = \nabla f\ . \tag{5}$$

Proof. According to the two fundamental theorems of calculus for line integrals (section V11.3), it is enough to prove that $\oint_C \mathbf{F} \cdot d\mathbf{r} \ = \ 0$ for every closed curve C lying in D.

Since D is simply-connected, given such a curve C, we can find a two-sided surface S lying entirely in D and having C as its boundary. Applying Stokes' theorem,

$$\oint_C \mathbf{F} \cdot d\mathbf{R} \ = \ \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} \ = \ 0,$$

which shows that $\mathbf{F}$ is conservative, and hence that $\mathbf{F}$ is a gradient field. $\square$

Summarizing, we can say that *if D is simply-connected, the following statements are equivalent—if one is true, so are the other two:*

$$(6) \qquad \mathbf{F} = \nabla f \quad \Leftrightarrow \quad \text{curl}\,\mathbf{F} = \mathbf{0} \quad \Leftrightarrow \quad \int_P^Q \mathbf{F}\cdot d\mathbf{r} \ \textit{ is path independent.}$$

Concluding remarks about Stokes' theorem.

Just as problems of sources and sinks lead one to consider Green's theorem in the plane for regions which are not simply-connected, it is important to consider such domains in connection with Stokes' theorem.

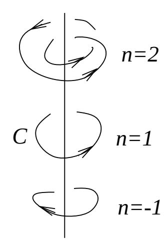
For example, if we put a closed loop of wire in space, the exterior of this loop — the region consisting of 3-space with the wire removed — is not simply-connected. If the wire carries current, the resulting electromagnetic force field $\mathbf{F}$ will satisfy curl $\mathbf{F} = \mathbf{0}$, but $\mathbf{F}$ will not be conservative. In particular, the value of $\oint \mathbf{F}\cdot d\mathbf{r}$ around a closed path which links with the loop will *not* in general be zero, (which explains why you can get power from a wire carrying current, even though the curl of its electromagnetic field is zero).

As an example, consider the vector field in 3-space

$$\mathbf{F} \;=\; \frac{-y\,\mathbf{i} + x\,\mathbf{j}}{r^2}, \qquad\qquad r = \sqrt{x^2+y^2}\ .$$

The domain of definition is xyz-space, with the z-axis removed (since the z-axis is where $r = 0$). Just as before, (Section V2,p.2), we can calculate that $\oint_C \mathbf{F}\cdot d\mathbf{r} = 2\pi$ if C is a suitably directed circle lying in a plane $z = z_0$ and centered on the z-axis.

Exercise. By using Stokes' theorem for a surface with more than one boundary curve, show informally that for the field above, $\oint_C \mathbf{F}\cdot d\mathbf{r} = 2\pi$ for any closed curve C going once around the z-axis, oriented so when that when the right thumb points in the direction $\mathbf{k}$, the fingers curl in the positive direction on C. Then show that if C goes around n times, the value of the integral will be $2n\pi$.



Suppose now the wire is a closed curve that is *knotted*, i.e., it cannot be continuously deformed to a circle, without crossing itself at some point in the deformation. Let D be the exterior of the wire loop, and consider the value of $\oint_C \mathbf{F}\cdot d\mathbf{r}$ for a vector field $\mathbf{F}$ defined in D and having curl $\mathbf{F} = \mathbf{0}$. If one closed curve C_1 can be deformed into another closed curve C_2 without leaving D (i.e., without crossing the wire), then by using Stokes' theorem for surfaces with two boundary curves, we conclude

$$(6) \qquad \oint_{C_1} \mathbf{F}\cdot d\mathbf{r} \;=\; \oint_{C_2} \mathbf{F}\cdot d\mathbf{r}\ .$$

More generally, two closed curves C_1 and C_2 are called *homologous*, written $C_1 \sim C_2$, if C_1 and C_2' (this means C_2 with its direction reversed) form the complete boundary of some surface lying entirely in D. Then by an extended form of Stokes' theorem, (6) will be true whenever $C_1 \sim C_2$. Thus the problem of determining the possible values for the line integral is reduced to the purely topological problem of finding a set of closed curves in D, no two of which are homologous, but such that every other closed curve is homologous to one of the curves in the set. For any particular knot in 3-space, such a set can be determined by an algorithm, but if one asks for general results relating the appearance of the knot to the number of such basic curves that will be needed, one gets into unsolved problems of topology.

In another vein, the theorems of Green, Stokes, and Gauss (as the divergence theorem is often called) all relate an integral over the interior of some closed curve or surface with an integral over its boundary. There is a much more general result — the generalized Stokes' theorem — which connects an integral over an n-dimensional hypersurface with an integral taken over its $n-1$-dimensional boundary. Green's and Stokes' theorems are the case $n = 2$ of this result, while the divergence theorem is closely related to the case $n = 3$ in 3-space. Just as the theorems we have studied are the key to an understanding of geometry and analysis in the plane and space, so this theorem is central to an understanding of n-dimensional space, and more general sorts of spaces called "n-dimensional manifolds".

Problems: Simply Connected Regions

1. Is the paraboloid described by $z = x^2 + y^2$ a simply connected surface? Why or why not?

Answer: Yes. Any closed curve C on the surface of the paraboloid can be shrunk to a point. To help visualize this, imagine taking the part of the paraboloid which contains C, smashing it flat, then shrinking C in the resulting planar surface.

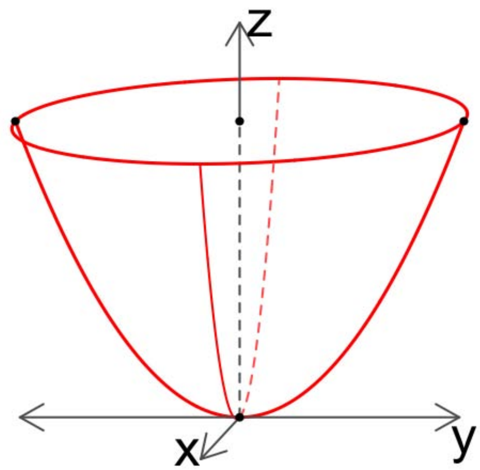


Figure 1: The graph of $z = x^2 + y^2$.

2. Is the Möbius strip described in lecture a simply connected surface? Why or why not?

Answer: No. Think of the Möbius strip as a long, thin rectangle whose ends have been joined to make a loop. A closed curve C around that loop cannot be shrunk to a point without "getting outside of" the Möbius strip.