Parametric equations of lines

General parametric equations

In this part of the unit we are going to look at parametric curves. This is simply the idea that a point moving in space traces out a path over time. Thus there are four variables to consider, the position of the point (x, y, z) and an independent variable t, which we can think of as time. (If the point is moving in plane there are only three variables, the position of the point (x, y) and the time t.)

Since the position of the point depends on t we write

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

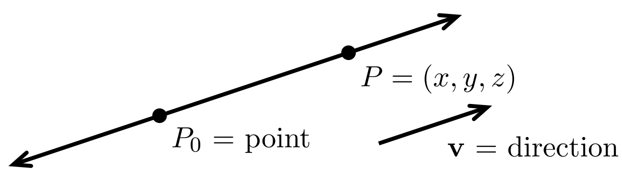
to indicate that x, y and z are functions of t. We call t the parameter and the equations for x, y and z are called *parametric equations*.

In physical examples the parameter often represents time. We will see other cases where the parameter has a different interpretation, or even no interpretation.

Parametric equations of lines

Later we will look at general curves. Right now, let's suppose our point moves on a line.

The basic data we need in order to specify a line are a point on the line and a vector parallel to the line. That is, we need a point and a direction.



Example 1: Write parametric equations for a line through the point $P_0 = (1, 2, 3)$ and parallel to the vector $\mathbf{v} = \langle 1, 3, 5 \rangle$.

__**Answer:**__ If $P = (x, y, z)$ is on the line then the vector

$$\overrightarrow{\mathbf{P_0P}} = \langle x-1, y-2, z-3 \rangle$$

is parallel to $\langle 1, 3, 5 \rangle$. That is, $\overrightarrow{\mathbf{P_0P}}$ is a scalar multiple of $\langle 1, 3, 5 \rangle$. We call the scale t and write:

$$\begin{aligned} & \langle x, y, z \rangle = \langle x-1, y-2, z-3 \rangle = t\langle 1, 3, 5 \rangle \\ \Leftrightarrow \quad & x - 1 = t, \ \ y - 2 = 3t, \quad z - 3 = 5t \\ \Leftrightarrow \quad & x = 1 + t, \ \ y = 2 + 3t, \quad z = 3 + 5t. \end{aligned}$$

Example 2: In example 1, if our direction vector was $\langle 2, 6, 10 \rangle = 2\mathbf{v}$ we would get the same line with a different parametrization. That is, the moving point's trajectory would follow the same path as the trajectory in example 1 , but would arrive at each point on the line at a different time.

Example 3: In general, the line through $P_0 = (x_0, y_0, z_0)$ in the direction of (i.e., parallel to) $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ has parametrization

$$\begin{aligned} & \langle x, y, z \rangle = \langle x_0 + tv_1, y_0 + tv_2, z_0 + tv_3 \rangle \\ \Leftrightarrow \quad & x = x_0 + tv_1, \quad y = y_0 + tv_2 \quad z = z_0 + tv_3. \end{aligned}$$

Example 4: Find the line through the point $P_0 = (1,2,3)$ and $P_1 = (2,5,8)$.

Answer: We use the data given to find the basic data (a point and direction vector) for the line.

We're given a point, $P_0 = (1,2,3)$. The direction vector $\mathbf{v} = \overrightarrow{\mathbf{P_0P_1}} = \langle 1,3,5\rangle$. So, we get

$$\begin{aligned} & \langle x,y,z\rangle = \overrightarrow{\mathbf{OP_0}} + t\mathbf{v} = \langle 1+t, 2+3t, 3+5t\rangle \\ \Leftrightarrow \quad & x = 1+t, \quad y = 2+3t, \quad z = 3+5t. \end{aligned}$$

Parametric Equations of Lines

1. Give parametric equations for x, y, z on the line through $(1, 1, 2)$ in a direction parallel to $\langle 2, -3, -1\rangle$.

Answer: We're given the basic data for a line of a point and a direction:

$$\langle x, y, z\rangle = \langle 1 + 2t, 1 - 3t, 2 - t\rangle \Leftrightarrow x = 1 + 2t, \quad y = 1 - 3t, \quad z = 2 - t$$

2. Give parametric equations for the intersection of the planes $x + y + z = 1$ and $x + 2y + 3z = 2$.

Answer: We need to find the basic data.

A point on the intersection: we take $z = 0$ and solve for x and y $\Rightarrow$ $P_0 = (0, 1, 0)$.

The line of intersection is perpendicular to both normals (to the planes), thus

$$\mathbf{v} = \langle 1, 1, 1\rangle \times \langle 1, 2, 3\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = \langle 1, -2, 1\rangle.$$

We get parametric equations

$$x = t, \quad y = 1 - 2t, \quad z = t.$$

Remark: The parametrization is not unique. You might have described the same line using a different point P_0 or a scaled version of $\mathbf{v}$.

Parametric Equations of Lines

1. Give parametric equations for x, y, z on the line through $(1, 1, 2)$ in a direction parallel to $\langle 2, -3, -1 \rangle$.

2. Give parametric equations for the intersection of the planes $x + y + z = 1$ and $x + 2y + 3z = 2$.

Intersection of a line and a plane

1. Find the intersection of the line through the points $(1, 3, 0)$ and $(1, 2, 4)$ with the plane through the points $(0, 0, 0)$, $(1, 1, 0)$ and $(0, 1, 1)$.

<u>**Answer:**</u> This brings together a number of things we've learned. We must find the equations of the line and the plane and then find the intersection.

The basic data specifying a line are a point and a direction. We have

$$P_0 = (1, 3, 0) \quad \text{and} \quad \mathbf{v} = \langle 1, 2, 4 \rangle - \langle 1, 3, 0 \rangle = \langle 0, -1, 4 \rangle.$$

Therefore, the equations for the line are

$$x = 1, \quad y = 3 - t, \quad z = 4t.$$

The basic data specifying a plane are a point and a normal vector. We have

$$Q_0 = (0, 0, 0) \quad \text{and} \quad \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \langle 1, -1, 1 \rangle.$$

Therefore the equation of the plane is

$$x - y + z = 0.$$

Substituting the line equations into the plane equation gives

$$1 - (3 - t) + 4t = 0 \;\Leftrightarrow\; t = 2/5.$$

We use this to find the point of intersection

$$(x, y, z) = (1, 13/5, 8/5).$$

Intersection of a line and a plane

1. Find the intersection of the line through the points $(1, 3, 0)$ and $(1, 2, 4)$ with the plane through the points $(0, 0, 0)$, $(1, 1, 0)$ and $(0, 1, 1)$.

Intersection of a line and a plane

1. Consider the plane $\mathcal{P} = 2x + y - 4z = 4$.

a) Find all points of intersection of $\mathcal{P}$ with the line

$$x = t, \quad y = 2 + 3t, \quad z = t.$$

b) Find all points of intersection of $\mathcal{P}$ with the line

$$x = 1 + t, \quad y = 4 + 2t, \quad z = t.$$

c) Find all points of intersection of $\mathcal{P}$ with the line

$$x = t, \quad y = 4 + 2t, \quad z = t.$$

<u>**Answer:**</u> a) To find the intersection we substitute the formulas for x, y and z into the equation for $\mathcal{P}$ and solve for t.

$$2(t) + (2 + 3t) - 4(t) = 4 \;\Leftrightarrow\; t = 2.$$

Now use $t = 2$ to find the point of intersection: $(x, y, z) = (2, 8, 2)$.

b) Substituting gives

$$2(1 + t) + (4 + 2t) - 4(t) = 4 \;\Leftrightarrow\; 6 = 4. \;\Leftrightarrow\; \text{no values of } t \text{ satisfy this equation.}$$

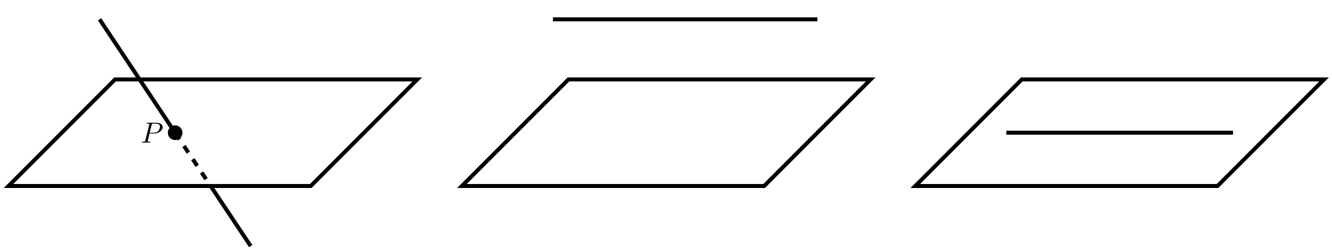
There are no points of intersection.

c) Substituting gives

$$2(t) + (4 + 2t) - 4(t) = 4 \;\Leftrightarrow\; 4 = 4. \;\Leftrightarrow\; \text{all values of } t \text{ satisfy this equation.}$$

The line is contained in the plane, i.e., all points of the line are in its intersection with the plane.

Here are cartoon sketches of each part of this problem.



(a) line intersects the plane in a point

(b) line is parallel to the plane

(c) line is in the plane

Parametric Curves

General parametric equations

We have seen parametric equations for lines. Now we will look at parametric equations of more general trajectories. Repeating what was said earlier, a parametric curve is simply the idea that a point moving in the space traces out a path.

We can use a parameter to describe this motion. Quite often we will use t as the parameter and think of it as time. Since the position of the point depends on t we write

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

to indicate that x, y and z are functions of t. We call t the parameter and the equations for x, y and z are called *parametric equations*.

It is not always necessary to think of the parameter as representing time. We will see cases where it is more convenient to express the position as a function of some other variable.

The position vector

In order to use vector techniques we define the *position vector*

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} = \langle x(t), y(t), z(t)\rangle.$$

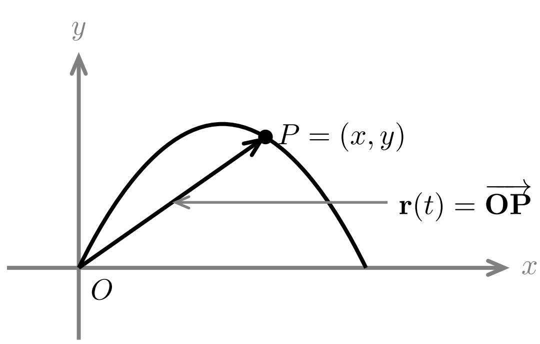
This is just the vector from the origin to the moving point. As the point moves so does the position vector –see the figure with example 1.

Example 1: Thomas Pynchon fires a rocket from the origin. Its initial x-velocity is $v_{0,x}$ and its initial y-velocity is $v_{0,y}$.

You've probably seen this, but in any case, physics tells us that the parametric equations for its parabolic trajectory are

$$x(t) = v_{0,x}t, \quad y(t) = -\frac{1}{2}gt^2 + v_{0,y}t.$$

At time t the rocket is at point $P = (x(t), y(t))$. The position vector can be written in many different ways: $\mathbf{r}(t) = \overrightarrow{\mathbf{OP}} = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} = \langle x, y\rangle$.

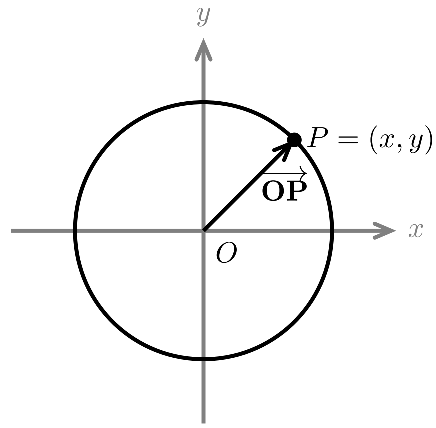


Next we will give a series of examples of parametrized curves. The most important are circles and lines. The last one is the *cycloid*. It is an important example which combines lines and circles.

Circles and ellipses
Consider the parametric curve in the plane

$$x(t) = a\cos t, \quad y(t) = a\sin t.$$

Easily we get the relation $x^2 + y^2 = a^2\cos^2 t + a^2\sin^2 t = a^2$. Therefore the trajectory is on a circle of radius a centered at O.



We will call $x(t) = a\cos t$, $y(t) = a\sin t$ the *parametric form* of the curve and $x^2 + y^2 = a^2$ the *symmetric form*.

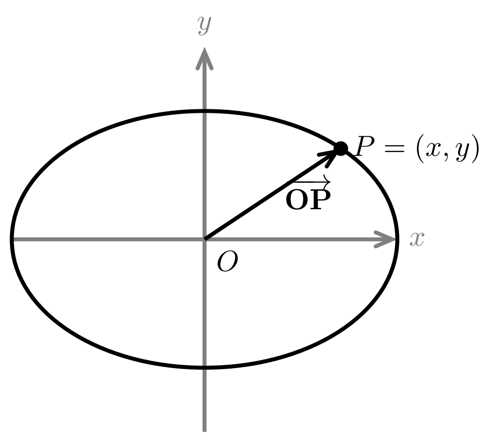
Note, a different parametrization, say

$$x(t) = a\cos(3t), \quad y(t) = a\sin(3t)$$

results in the same path, i.e. the circle $x^2 + y^2 = a^2$, but the two trajectories differ by how fast they travel around the circle.

The circle is easily changed to an ellipse by

$$\begin{aligned} &\text{parametric form:} \quad x(t) = a\cos t, \quad y(t) = b\cos t \\ &\text{symmetric form:} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \end{aligned}$$



Lines

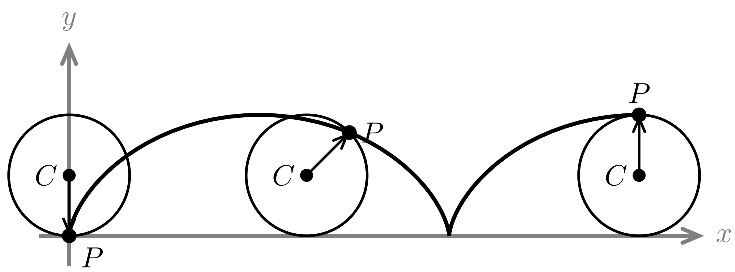
We review parametric equations of lines by writing the the equation of a general line in the plane. We know we can parametrize the line through (x_0, y_0) parallel to $\langle b_1, b_2\rangle$ by

$$x(t) = x_0 + tb_1, \quad y(t) = y_0 + tb_2 \;\Leftrightarrow\; \mathbf{r}(t) = \langle x, y\rangle = \langle x_0 + tb_1, y_0 + tb_2\rangle = \langle x_0, y_0\rangle + t\langle b_1, b_2\rangle.$$

The cycloid

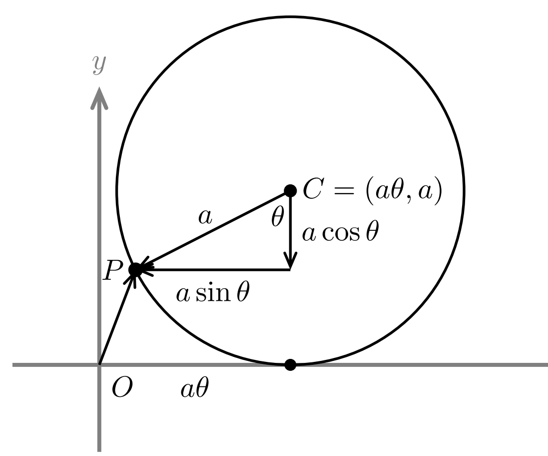
The cycloid has a long and storied history and comes up surprisingly often in physical problems. For us it is a curve that has no simple symmetric form, so we will only work with it in its parametric form.

The cycloid is the trajectory of a point on a circle that is rolling without slipping along the x-axis. To be specific, we'll follow the point P that starts at the origin.



The natural parameter to use is the angle θ that the wheel has turned. We'll use vector methods to find the position vector for P as a function of θ.

Our strategy is to break the motion up into translation of the center and rotation about the center. The figure shows the wheel after it has turned through a small θ. We see the position vector



$$\overrightarrow{\mathbf{OP}} = \overrightarrow{\mathbf{OC}} + \overrightarrow{\mathbf{CP}}.$$

We'll compute each piece separately.

After turning θ radians the wheel has rolled a distance $a\theta$, so the center of the circle is at $(a\theta, a)$, i.e.,

$$\overrightarrow{\mathbf{OC}} = \langle a\theta, a\rangle.$$

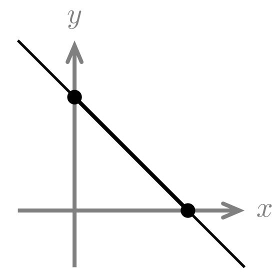
The figure also shows that

$$\overrightarrow{\mathbf{CP}} = \langle -a\sin\theta, -a\cos\theta\rangle.$$

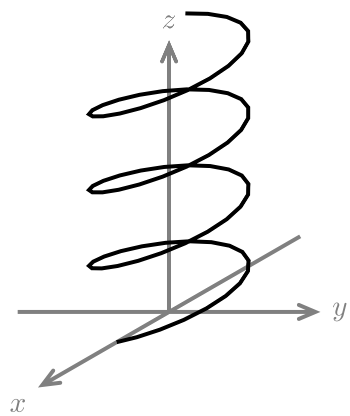
Putting the pieces together we get parametric equations for the cycloid

$$\overrightarrow{\mathbf{OP}} = \langle a\theta - a\sin\theta, a - a\cos\theta\rangle$$
$$\Leftrightarrow \quad x(\theta) = a\theta - a\sin\theta, \quad y(\theta) = a - a\cos\theta.$$

Example 2: (Where the symmetric form loses information.)
Find the symmetric form for $x = 3\cos^2 t,\ y = 3\sin^2 t$.
Easily we get: $x + y = 3$, with x, y non-negative.
The symmetric form shows a line, but the parametric trajectory only traces out a part of the line. In fact, it goes back an forth over the part of the line in the first quadrant.



Example 3: The curve $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + at\,\mathbf{k}$ is a helix winding around the z-axis.



Parametric curves

1. A disk of radius 2 cm slides at a speed $12\sqrt{2}$ cm/sec in the direction of $\langle 1, 1\rangle$. As it slides it spins counterclockwise at 3 revolutions per second. Measuring time in seconds, at time $t = 0$ the disk's center is at the origin (0,0).

Find parametric equations for the trajectory of the point P on the edge of the disk, which is initially at $(2, 0)$.

<u>**Answer:**</u> We will parametrize the curve by time t in seconds. To do this we split the motion into translation of the center and rotation about the center and use vectors to do the analysis.

See the figure below. At time t the center has moved to C and the edge point P has rotated $6\pi t$ radians. (3 rev./sec = 6π radians/sec.) Thus

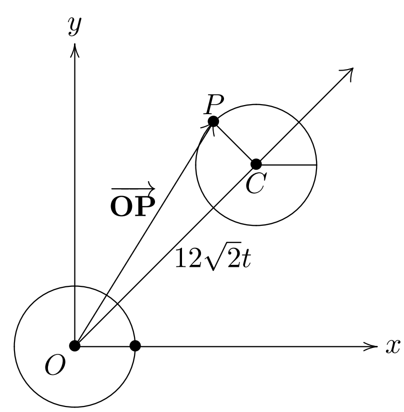
$$\overrightarrow{\mathbf{OC}} = 12\sqrt{2}t\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \langle 12t, 12t\rangle$$

and

$$\overrightarrow{\mathbf{CP}} = \langle 2\cos(6\pi t),\ 2\sin(6\pi t)\rangle\,.$$

Putting these together we get

$$\begin{aligned} & \overrightarrow{\mathbf{OP}} = \overrightarrow{\mathbf{OC}} + \overrightarrow{\mathbf{CP}} = \langle 12t + 2\cos(6\pi t),\ 12t + 2\sin(6\pi t)\rangle \\ \Leftrightarrow \quad & x = 12t + 2\cos(6\pi t), \quad y = 12t + 2\sin(6\pi t). \end{aligned}$$



Parametric curves

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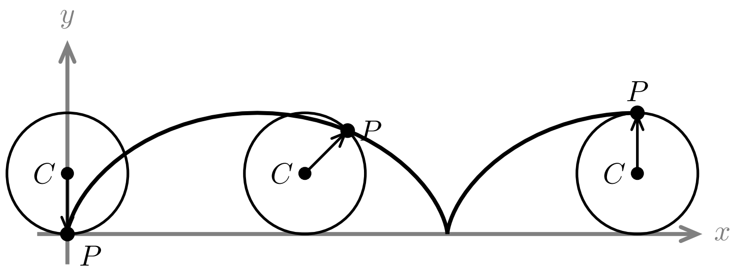
Find parametric equations for the trajectory of the point P on the edge of the disk, which is initially at $(2, 0)$.

Cusp on the cycloid

The graph of the cycloid has point where the graph touches the x-axis. These points are usually called *cusps*.

What you saw in the previous video was an analysis of the behavior of the trajectory near the cusps. We will go through that analysis again and discuss what's happening physically on a rolling wheel.



In order to simplify the way our equations look, let's take the radius of the wheel to be $a = 1$. Then the parametric equations for the cycloid are

$$x(\theta) = \theta - \sin\theta, \quad y(\theta) = 1 - \cos\theta.$$

Taking derivatives we get $\dfrac{dx}{d\theta} = 1 - \cos\theta$ and $\dfrac{dy}{d\theta} = \sin\theta$.
Thus the slope of the curve is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin\theta}{1-\cos\theta}.$$

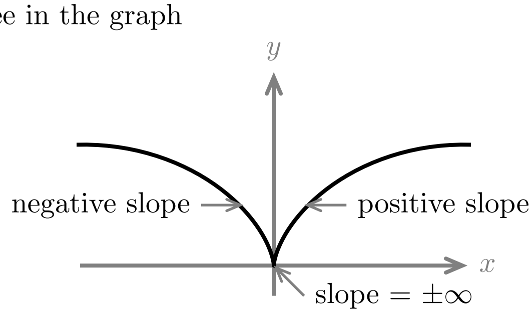
As $\theta \to 0$ this is of indeterminant form $0/0$. Using L'Hospital's rule we get

$$\lim_{\theta\to 0} \frac{\sin\theta}{1-\cos\theta} = \lim_{\theta\to 0} \frac{\cos\theta}{\sin\theta}$$

Since $\cos\theta$ goes to 1 and $\sin\theta$ goes to 0 this limit does not exist.

But looking at it more carefully we see that as $\theta \to 0^-$ the limit goes to $-\infty$ and as $\theta \to 0^+$ it goes to $+\infty$. That is, right at the cusp the slope of the curve is $-\infty$ to the left and $+\infty$ to the right.

This mirrors what we see in the graph



Later, when we learn about velocity we'll see that, at $\theta = 0$, $\dfrac{dx}{d\theta} = 0$ and $\dfrac{dy}{d\theta} = 0$ means the velocity is 0. At the cusp, the point changes abruptly from moving down to moving up. Physically this can only happen if the velocity is 0 at the changeover point.

Velocity and acceleration

Now we will see one of the benefits of using the position vector. Let's assume we have a moving point with position vector

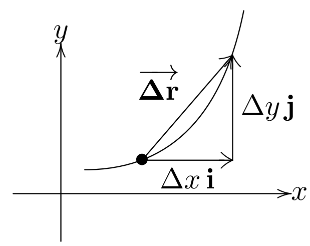
$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}.$$

(We assume the point moves in the plane. The extension to a point moving in space is trivial.)

Velocity

Over a short time Δt the position changes by $\Delta\mathbf{r}$. The average velocity over this time is simply

$$\frac{\Delta\mathbf{r}}{\Delta t},\ \text{ i.e., displacement/time.}$$



The figure shows $\Delta\mathbf{r} = \Delta x\,\mathbf{i} + \Delta y\,\mathbf{j}$. Dividing by Δt we get

$$\text{average velocity } = \frac{\Delta\mathbf{r}}{\Delta t} = \frac{\Delta x}{\Delta t}\,\mathbf{i} + \frac{\Delta y}{\Delta t}\,\mathbf{j}$$

Now, as we usually do in calculus, we let $\Delta t \to 0$. The average velocity becomes the (instantaneous) velocity and the ratios in the formula above become derivatives. For completeness we write the velocity vector in a number of different forms

$$\text{velocity } = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j} = x'\mathbf{i} + y'\mathbf{j} = \langle x', y'\rangle.$$

Tangent vector: (same thing as velocity)

In the picture above, we see that as Δt shrinks to 0 the vector $\frac{\Delta\mathbf{r}}{\Delta t}$ becomes tangent to the curve. When the parameter is time we can rightfully refer to $\mathbf{r}'(t)$ as the *velocity.* In general, we will abuse the language and refer to the derivative of position with respect to any parameter as velocity. If we are thinking geometrically or want to be precise, we will call the derivative by its geometric name: the *tangent vector.*

As always, we encourage you to remember the geometric view of the velocity vector. Knowing it is tangent to the curve will be important as we develop the subject and solve problems.

Acceleration

There is no reason to stop taking derivatives after one. Since acceleration is change in velocity per unit time, we get

$$\text{acceleration } = \mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = x''(t)\,\mathbf{i} + y''(t)\,\mathbf{j} = \langle x'', y''\rangle.$$

Example: A rocket follows a trajectory

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} = v_{0,x}\,t\,\mathbf{i} + (-\frac{g}{2}t^2 + v_{0,y}t)\mathbf{j}.$$

Find its velocity and acceleration vectors.

Answer:

$$\text{velocity } = \mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = v_{0,x}\mathbf{i} + (-gt + v_{0,y})\mathbf{j},$$

$$\text{acceleration } = \mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = -g\mathbf{j}.$$

Example: Find the velocity and acceleration vectors for the cycloid

$$x = \theta - \sin\theta, \quad y = 1 - \cos\theta.$$

Answer: As noted, this a slight abuse of language,

$$\text{velocity } = \text{tangent vector } = \mathbf{v}(\theta) = \frac{d\mathbf{r}}{d\theta} = \langle x'(\theta), y'(\theta)\rangle = \langle 1 - \cos\theta, \sin\theta\rangle.$$

$$\text{acceleration } = \mathbf{a}(\theta) = \frac{d\mathbf{v}}{d\theta} = \langle \sin\theta, \cos\theta\rangle.$$

Example: In the cycloid above, suppose the wheel rolls at 3 revolutions per second. Write the parametric equations in terms of time, and compute the velocity.

Answer: Since 3 rev/second = 6π radians/sec, we have $\theta = 6\pi t$. Therefore,

$$x(t) = 6\pi t - \sin(6\pi t), \quad y(t) = 1 - \cos(6\pi t).$$

$$\mathbf{v}(t) = \langle 6\pi - 6\pi\cos(6\pi t), 6\pi\sin(6\pi t)\rangle.$$

Product rule for vector derivatives

1. If $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ are two parametric curves show the product rule for derivatives holds for the cross product.

<u>**Answer:**</u> As with the dot product, this will follow from the usual product rule in single variable calculus. We want to show

$$\frac{d(\mathbf{r}_1 \times \mathbf{r}_2)}{dt} = \mathbf{r}_1' \times \mathbf{r}_2 \,+\, r_1 \times \mathbf{r}_2'.$$

Let $\mathbf{r}_1 = \langle x_1, y_1, z_1 \rangle$ and $\mathbf{r}_2 = \langle x_2, y_2, z_2 \rangle$. We have,

$$\mathbf{r}_1 \times \mathbf{r}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \langle y_1 z_2 - z_1 y_2,\; z_1 x_2 - x_1 z_2,\; x_1 y_2 - y_1 x_2 \rangle.$$

Taking derivatives using the product rule from single variable calculus, we get a lot of terms, which we can group to prove the vector formula.

$$\begin{aligned}
\frac{d(\mathbf{r}_1 \times \mathbf{r}_2)}{dt} &= \langle y_1' z_2 + y_1 z_2' - z_1' y_2 - z_1 y_2',\; z_1' x_2 + z_1 x_2' - x_1' z_2 - x_1 z_2',\; x_1' y_2 + x_1 y_2' - y_1' x_2 - y_1 x_2' \rangle \\
&= \langle (y_1' z_2 - z_1' y_2) + (y_1 z_2' - z_1 y_2'),\; (z_1' x_2 - x_1' z_2) + (z_1 x_2' - x_1 z_2'),\; (x_1' y_2 - y_1' x_2) + (x_1 y_2' - y_1 x_2') \rangle \\
&= \langle x_1', y_1', z_1' \rangle \times \langle x_2, y_2, z_2 \rangle + \langle x_1, y_1, z_1 \rangle \times \langle x_2', y_2', z_2' \rangle \\
&= \mathbf{r}_1' \times \mathbf{r}_2 \,+\, \mathbf{r}_1 \times \mathbf{r}_2'. \quad \blacksquare
\end{aligned}$$

Product rule for vector derivatives

1. If $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ are two parametric curves show the product rule for derivatives holds for the cross product.

Product rule for vector derivatives

1. If $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ are two parametric curves show the product rule for derivatives holds for the dot product.

Answer: This will follow from the usual product rule in single variable calculus. Lets assume the curves are in the plane. The proof would be exactly the same for curves in space. We want to prove that

$$\frac{d(\mathbf{r}_1 \cdot \mathbf{r}_2)}{dt} = \mathbf{r}_1' \cdot \mathbf{r}_2 \; + \; r_1 \cdot \mathbf{r}_2'.$$

Let $\mathbf{r}_1 = \langle x_1, y_1 \rangle$ and $\mathbf{r}_2 = \langle x_2, y_2 \rangle$. We have,

$$\mathbf{r}_1 \cdot \mathbf{r}_2 = x_1 x_2 + y_1 y_2.$$

Taking derivatives using the product rule from single variable calculus, we get

$$\begin{aligned}
\frac{d(\mathbf{r}_1 \cdot \mathbf{r}_2)}{dt} &= \frac{d(x_1 x_2 + y_1 y_2)}{dt} \\
&= x_1' x_2 + x_1 x_2' + y_1' y_2 + y_1 y_2' \\
&= (x_1' x_2 + y_1' y_2) + (x_1 x_2' + y_1 y_2') \\
&= \langle x_1', y_1' \rangle \cdot \langle x_2, y_2 \rangle + \langle x_1, y_1 \rangle \cdot \langle x_2', y_2' \rangle \\
&= \mathbf{r}_1' \cdot \mathbf{r}_2 \; + \; \mathbf{r}_1 \cdot \mathbf{r}_2'. \quad \blacksquare
\end{aligned}$$

Velocity and acceleration

Now we will see one of the benefits of using the position vector. Let's assume we have a moving point with position vector

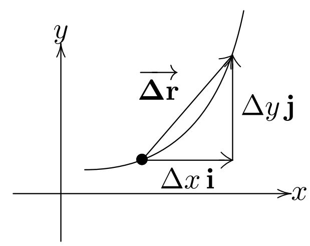
$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}.$$

(We assume the point moves in the plane. The extension to a point moving in space is trivial.)

Velocity

Over a short time Δt the position changes by $\Delta \mathbf{r}$. The average velocity over this time is simply

$$\frac{\Delta \mathbf{r}}{\Delta t}, \text{ i.e., displacement/time.}$$



The figure shows $\Delta \mathbf{r} = \Delta x\,\mathbf{i} + \Delta y\,\mathbf{j}$. Dividing by Δt we get

$$\text{average velocity } = \frac{\Delta \mathbf{r}}{\Delta t} = \frac{\Delta x}{\Delta t}\,\mathbf{i} + \frac{\Delta y}{\Delta t}\,\mathbf{j}$$

Now, as we usually do in calculus, we let $\Delta t \to 0$. The average velocity becomes the (instantaneous) velocity and the ratios in the formula above become derivatives. For completeness we write the velocity vector in a number of different forms

$$\text{velocity } = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j} = x'\mathbf{i} + y'\mathbf{j} = \langle x', y' \rangle.$$

Tangent vector: (same thing as velocity)

In the picture above, we see that as Δt shrinks to 0 the vector $\frac{\Delta \mathbf{r}}{\Delta t}$ becomes tangent to the curve. When the parameter is time we can rightfully refer to $\mathbf{r}'(t)$ as the *velocity*. In general, we will abuse the language and refer to the derivative of position with respect to any parameter as velocity. If we are thinking geometrically or want to be precise, we will call the derivative by its geometric name: the *tangent vector*.

As always, we encourage you to remember the geometric view of the velocity vector. Knowing it is tangent to the curve will be important as we develop the subject and solve problems.

Acceleration

There is no reason to stop taking derivatives after one. Since acceleration is change in velocity per unit time, we get

$$\text{acceleration } = \mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = x''(t)\,\mathbf{i} + y''(t)\,\mathbf{j} = \langle x'', y'' \rangle.$$

Example: A rocket follows a trajectory

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} = v_{0,x}\,t\,\mathbf{i} + (-\frac{g}{2}t^2 + v_{0,y}t)\mathbf{j}.$$

Find its velocity and acceleration vectors.

Answer:

$$\text{velocity } = \mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = v_{0,x}\mathbf{i} + (-gt + v_{0,y})\mathbf{j},$$

$$\text{acceleration } = \mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = -g\mathbf{j}.$$

Example: Find the velocity and acceleration vectors for the cycloid

$$x = \theta - \sin\theta, \quad y = 1 - \cos\theta.$$

Answer: As noted, this a slight abuse of language,

$$\text{velocity } = \text{tangent vector } = \mathbf{v}(\theta) = \frac{d\mathbf{r}}{d\theta} = \langle x'(\theta), y'(\theta)\rangle = \langle 1 - \cos\theta, \sin\theta\rangle.$$

$$\text{acceleration } = \mathbf{a}(\theta) = \frac{d\mathbf{v}}{d\theta} = \langle \sin\theta, \cos\theta\rangle.$$

Example: In the cycloid above, suppose the wheel rolls at 3 revolutions per second. Write the parametric equations in terms of time, and compute the velocity.

Answer: Since 3 rev/second = 6π radians/sec, we have $\theta = 6\pi t$. Therefore,

$$x(t) = 6\pi t - \sin(6\pi t), \quad y(t) = 1 - \cos(6\pi t).$$

$$\mathbf{v}(t) = \langle 6\pi - 6\pi\cos(6\pi t), 6\pi\sin(6\pi t)\rangle.$$

Product rule for vector derivatives

1. If $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ are two parametric curves show the product rule for derivatives holds for the cross product.

Answer: As with the dot product, this will follow from the usual product rule in single variable calculus. We want to show

$$\frac{d(\mathbf{r}_1 \times \mathbf{r}_2)}{dt} = \mathbf{r}_1' \times \mathbf{r}_2 \,+\, r_1 \times \mathbf{r}_2'.$$

Let $\mathbf{r}_1 = \langle x_1, y_1, z_1 \rangle$ and $\mathbf{r}_2 = \langle x_2, y_2, z_2 \rangle$. We have,

$$\mathbf{r}_1 \times \mathbf{r}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \langle y_1 z_2 - z_1 y_2,\ z_1 x_2 - x_1 z_2,\ x_1 y_2 - y_1 x_2 \rangle.$$

Taking derivatives using the product rule from single variable calculus, we get a lot of terms, which we can group to prove the vector formula.

$$\begin{aligned}
\frac{d(\mathbf{r}_1 \times \mathbf{r}_2)}{dt} &= \langle y_1' z_2 + y_1 z_2' - z_1' y_2 - z_1 y_2',\ z_1' x_2 + z_1 x_2' - x_1' z_2 - x_1 z_2',\ x_1' y_2 + x_1 y_2' - y_1' x_2 - y_1 x_2' \rangle \\
&= \langle (y_1' z_2 - z_1' y_2) + (y_1 z_2' - z_1 y_2'),\ (z_1' x_2 - x_1' z_2) + (z_1 x_2' - x_1 z_2'),\ (x_1' y_2 - y_1' x_2) + (x_1 y_2' - y_1 x_2') \rangle \\
&= \langle x_1', y_1', z_1' \rangle \times \langle x_2, y_2, z_2 \rangle + \langle x_1, y_1, z_1 \rangle \times \langle x_2', y_2', z_2' \rangle \\
&= \mathbf{r}_1' \times \mathbf{r}_2 \,+\, \mathbf{r}_1 \times \mathbf{r}_2'. \quad \blacksquare
\end{aligned}$$

Product rule for vector derivatives

1. If $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ are two parametric curves show the product rule for derivatives holds for the cross product.

Product rule for vector derivatives

1. If $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ are two parametric curves show the product rule for derivatives holds for the dot product.

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$$\frac{d(\mathbf{r}_1 \cdot \mathbf{r}_2)}{dt} = \mathbf{r}_1' \cdot \mathbf{r}_2 \, + \, r_1 \cdot \mathbf{r}_2'.$$

Let $\mathbf{r}_1 = \langle x_1, y_1 \rangle$ and $\mathbf{r}_2 = \langle x_2, y_2 \rangle$. We have,

$$\mathbf{r}_1 \cdot \mathbf{r}_2 = x_1 x_2 + y_1 y_2.$$

Taking derivatives using the product rule from single variable calculus, we get

$$\begin{aligned}
\frac{d(\mathbf{r}_1 \cdot \mathbf{r}_2)}{dt} &= \frac{d(x_1 x_2 + y_1 y_2)}{dt} \\
&= x_1' x_2 + x_1 x_2' + y_1' y_2 + y_1 y_2' \\
&= (x_1' x_2 + y_1' y_2) + (x_1 x_2' + y_1 y_2') \\
&= \langle x_1', y_1' \rangle \cdot \langle x_2, y_2 \rangle + \langle x_1, y_1 \rangle \cdot \langle x_2', y_2' \rangle \\
&= \mathbf{r}_1' \cdot \mathbf{r}_2 \, + \, \mathbf{r}_1 \cdot \mathbf{r}_2'. \quad \blacksquare
\end{aligned}$$

Velocity, speed and arc length

Speed

Velocity, being a vector, has a magnitude and a direction. The direction is tangent to the curve traced out by $\mathbf{r}(t)$. The magnitude of its velocity is the speed.

$$\text{speed } = |\mathbf{v}| = \left|\frac{d\mathbf{r}}{dt}\right|.$$

Speed is in units of distance per unit time. It reflects how fast our moving point is moving.

Example: A point goes one time around a circle of radius 1 unit in 3 seconds. What is its average velocity and average speed.

Answer: The distance the point traveled equals the circumference of the circle, 2π. Its net displacement is $\mathbf{0}$, since it ends where it started. Thus, its average speed = distance/time $= 2\pi/3$ and its average velocity = displacement/time = $\mathbf{0}$.

If you look carefully, we've used a boldface $\mathbf{0}$ because velocity is a vector.

Our usual symbol for distance traveled is s. For a point moving along a curve the distance traveled is the length of the curve. Because of this we also refer to s as *arc length.*

Notation and nomenclature summary:

Since we will use a variety of notations, we'll collect them here. The unit tangent vector will be explained below. As you should expect, we will also be able to view everything from a geometric perspective.

$\mathbf{r}(t) =$ position.

In the plane $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} = \langle x, y\rangle$

In space $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$.

$\dfrac{d\mathbf{r}}{dt} = \mathbf{v}(t) =$ velocity = tangent vector.

In the plane $\mathbf{v} = x'(t)\mathbf{i} + y'(t)\mathbf{j} = \langle x', y'\rangle$

In space $\mathbf{v} = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k} = \langle x', y', z'\rangle$.

$\mathbf{T} = \dfrac{\mathbf{v}}{|\mathbf{v}|} =$ unit tangent vector.

$s =$ arclength, speed $= \dfrac{ds}{dt} = |\mathbf{v}|$.

In the plane $\dfrac{ds}{dt} = \sqrt{(x')^2 + (y')^2}$.

In space $\dfrac{ds}{dt} = \sqrt{(x')^2 + (y')^2 + (z')^2}$.

$\mathbf{v} = \dfrac{ds}{dt}\mathbf{T}, \quad \mathbf{T} = \dfrac{\mathbf{v}}{ds/dt}$

$\mathbf{a}(t) = \dfrac{d\mathbf{v}}{dt} = \dfrac{d^2\mathbf{r}}{dt^2} =$ acceleration.

In the plane $\mathbf{a}(t) = x''(t)\mathbf{i} + y''(t)\mathbf{j} = \langle x'', y''\rangle$

In space $\mathbf{a} = x''(t)\mathbf{i} + y''(t)\mathbf{j} + z''(t)\mathbf{k} = \langle x'', y'', z''\rangle$.

Unit tangent vector
As its name implies, the *unit tangent vector* is a unit vector in the same direction as the tangent vector. We usually denote it $\mathbf{T}$. We compute it by dividing the tangent vector by its length. Here are several ways of writing this.

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{d\mathbf{r}/dt}{ds/dt} = \frac{\mathbf{v}}{ds/dt}.$$

Multiply $\mathbf{T}$ by ds/dt gives the formula

$$\mathbf{v} = \frac{ds}{dt}\mathbf{T},$$

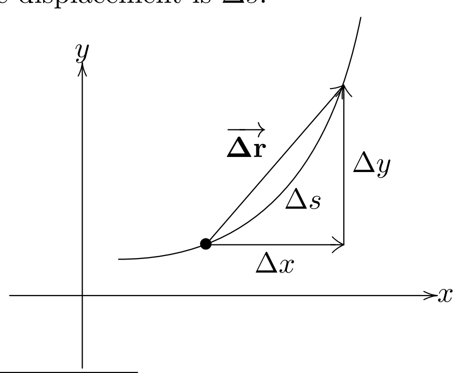
which expresses velocity as a magnitute, ds/dt and a direction $\mathbf{T}$.

Geometric considerations
Here we'll offer a mathematical justification for our statement that

$$\text{speed } = \frac{ds}{dt} = |\mathbf{v}|.$$

We'll work in two dimensions. The extension to 3D is straightforward.

The figure below shows a curve, and a small displacement $\Delta\mathbf{r}$. The length along the curve from the start to end of the displacement is Δs.



We see $\Delta s \approx |\Delta\mathbf{r}| = \sqrt{(\Delta x)^2 + (\Delta y)^2}$. Dividing by Δt gives

$$\frac{\Delta s}{\Delta t} \approx \left|\frac{\Delta\mathbf{r}}{\Delta t}\right| = \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}$$

Taking the limit as $\Delta t \to 0$ gives

$$\frac{ds}{dt} = \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

Vector derivatives and arc length

1. Let $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$.

a) Compute, velocity, speed, unit tangent vector and acceleration.

b) Write down the integral for arc length from $t = 1$ to $t = 4$. *(Do not compute the integral.)*

<u>**Answer:**</u> a) Velocity $= \mathbf{v} = \dfrac{d\mathbf{r}}{dt} = \langle 2t,\, 3t^2\rangle$.

Speed $= |\mathbf{v}| = \sqrt{4t^2 + 9t^4}$.

Unit tangent vector $= \mathbf{T} = \dfrac{\mathbf{v}}{ds/dt} = \left\langle \dfrac{2t}{\sqrt{4t^2+9t^4}},\, \dfrac{3t^2}{\sqrt{4t^2+9t^4}} \right\rangle$.

b) Arc length $= \displaystyle\int_1^4 \frac{ds}{dt}\, dt = \int_1^4 \sqrt{4t^2 + 9t^4}\, dt$.

2. Consider the parametric curve

$$x(t) = 3t + 1, \quad y(t) = 4t + 3.$$

a. Compute, velocity, speed, unit tangent vector and acceleration.

b. Compute the arc length of the trajectory from $t = 0$ to $t = 2$.

<u>**Answer:**</u> a) Velocity $= \mathbf{v} = \dfrac{d\mathbf{r}}{dt} = \langle 3,\, 4\rangle$.

Speed $= |\mathbf{v}| = \sqrt{9 + 16} = 5$.

Unit tangent vector $= \mathbf{T} = \dfrac{\mathbf{v}}{ds/dt} = \left\langle \dfrac{3}{5}, \dfrac{4}{5} \right\rangle$.

b) Arc length $= \displaystyle\int_0^2 \frac{ds}{dt}\, dt = \int_0^2 5\, dt = 10$.

Vector derivatives and arc length

1. Let $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$.

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b) Write down the integral for arc length from $t = 1$ to $t = 4$. *(Do not compute the integral.)*

2. Consider the parametric curve

$$x(t) = 3t + 1, \quad y(t) = 4t + 3.$$

a. Compute, velocity, speed, unit tangent vector and acceleration.

b. Compute the arc length of the trajectory from $t = 0$ to $t = 2$.

Speed and arc length

1. A rocket follows a trajectory

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} = 10t\,\mathbf{i} + (-5t^2 + 10t)\mathbf{j}.$$

Find its speed and the arc length from $t = 0$ to $t = 1$.

<u>**Answer:**</u>

velocity $= \mathbf{v}(t) = \dfrac{d\mathbf{r}}{dt} = 10\mathbf{i} + (-10t+10)\mathbf{j} \Rightarrow \dfrac{ds}{dt} = \sqrt{10^2 + (-10t+10)^2} = 10\sqrt{1 + (1-t)^2}.$

Arc length $L = \displaystyle\int_0^1 \frac{ds}{dt}\,dt = 10\int_0^1 \sqrt{1 + (-t+1)^2}\,dt$

Make the change of variables $u = -t + 1$

$\Rightarrow\ du = -dt,\ t = 0 \to u = 1,\ t = 1 \to u = 0.\ \Rightarrow\ L = 10\displaystyle\int_0^1 \sqrt{1+u^2}\,du.$

We can compute this integral with the trig. substitution $u = \tan\theta$ or by use of tables

$\Rightarrow\ L = 10\displaystyle\int_0^{\pi/4} \sec^3\theta\,d\theta = 5\left[\sec\theta\,\tan\theta + \ln(\sec\theta + \tan\theta)\right|_0^{\pi/4} = 5(\sqrt{2} + \ln(\sqrt{2}+1)).$

2. For the cycloid $x = a\theta - a\sin\theta$, $y = a - a\cos\theta$ find the velocity, speed, unit tangent vector and arc length of one arch.

We will use the trigonometric formulas

$$\sin^2(\theta/2) = \frac{1-\cos\theta}{2} \qquad \text{and} \qquad \sin\theta = 2\sin(\theta/2)\cos(\theta/2).$$

Computing,

$$\frac{d\mathbf{r}}{d\theta} = a\langle 1-\cos\theta,\, \sin\theta\rangle = 2a\langle \sin^2(\theta/2),\, \sin(\theta/2)\cos(\theta/2)\rangle,$$

which implies

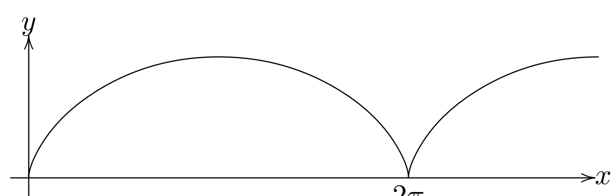
$$\left|\frac{d\mathbf{r}}{d\theta}\right| = \frac{ds}{d\theta} = 2a\sqrt{\sin^2(\theta/2)} = 2a|\sin(\theta/2)|.$$

So, $\mathbf{T} = \dfrac{2a\langle \sin^2(\theta/2),\, \sin(\theta/2)\cos(\theta/2)\rangle}{2a|\sin(\theta/2)|} = \pm\langle \sin(\theta/2),\, \cos(\theta/2)\rangle$ (a unit vector!)

Note, at the cusp ($\theta = 2\pi$) $ds/d\theta = 0$, i.e., you must stop to make a sudden 180 degree turn.

For one arch, $0 < \theta < 2\pi$, $\frac{ds}{d\theta} = 2a\sin(\theta/2)$

$$\begin{aligned}\Rightarrow\ \text{arc length} &= \int_0^{2\pi} \frac{ds}{d\theta}\,d\theta \\ &= \int_0^{2\pi} 2a\sin(\theta/2)\,d\theta \\ &= -4a\cos(\theta/2)\big|_0^{2\pi} \\ &= 8a \text{ (this is sometimes called Wren's theorem).}\end{aligned}$$



Kepler's Second Law

By studying the Danish astronomer Tycho Brahe's data about the motion of the planets, Kepler formulated three empirical laws; two of them can be stated as follows:

Second Law A planet moves in a plane, and the radius vector (from the sun to the planet) sweeps out equal areas in equal times.

First Law The planet's orbit in that plane is an ellipse, with the sun at one focus.

From these laws, Newton deduced that the force keeping the planets in their orbits had magnitude $1/d^2$, where d was the distance to the sun; moreover, it was directed toward the sun, or as was said, *central*, since the sun was placed at the origin.

Using a little vector analysis (without coordinates), this section is devoted to showing that *the Second Law is equivalent to the force being central.*

> It is harder to show that an elliptical orbit implies the magnitude of the force is of the form K/d^2, and vice-versa; this uses vector analysis in polar coordinates and requires the solution of non-linear differential equations.

1. Differentiation of products of vectors

Let $\mathbf{r}(t)$ and $\mathbf{s}(t)$ be two differentiable vector functions in 2- or 3-space. Then

$$(1) \qquad \frac{d}{dt}(\mathbf{r}\cdot\mathbf{s}) = \frac{d\mathbf{r}}{dt}\cdot\mathbf{s} + \mathbf{r}\cdot\frac{d\mathbf{s}}{dt}; \qquad \frac{d}{dt}(\mathbf{r}\times\mathbf{s}) = \frac{d\mathbf{r}}{dt}\times\mathbf{s} + \mathbf{r}\times\frac{d\mathbf{s}}{dt}.$$

These rules are just like the product rule for differentiation. Be careful in the second rule to get the multiplication order correct on the right, since $\mathbf{a}\times\mathbf{b} \neq \mathbf{b}\times\mathbf{a}$ in general. The two rules can be proved by writing everything out in terms of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ components and differentiating. They can also be proved directly from the definition of derivative, without resorting to components, as follows:

Let t increase by Δt. Then $\mathbf{r}$ increases by $\Delta\mathbf{r}$, and $\mathbf{s}$ by $\Delta\mathbf{s}$, and the corresponding change in $\mathbf{r}\cdot\mathbf{s}$ is given by

$$\Delta(\mathbf{r}\cdot\mathbf{s}) = (\mathbf{r}+\Delta\mathbf{r})\cdot(\mathbf{s}+\Delta\mathbf{s}) - \mathbf{r}\cdot\mathbf{s},$$

so if we expand the right side out and divide all terms by Δt, we get

$$\frac{\Delta(\mathbf{r}\cdot\mathbf{s})}{\Delta t} = \frac{\Delta\mathbf{r}}{\Delta t}\cdot\mathbf{s} + \mathbf{r}\cdot\frac{\Delta\mathbf{s}}{\Delta t} + \frac{\Delta\mathbf{r}}{\Delta t}\cdot\Delta\mathbf{s}.$$

Now let $\Delta t \to 0$; then $\Delta\mathbf{s} \to 0$ since $\mathbf{s}(t)$ is continuous, and we get the first equation in (1). The second equation in (1) is proved the same way, replacing $\cdot$ by $\times$ everywhere.

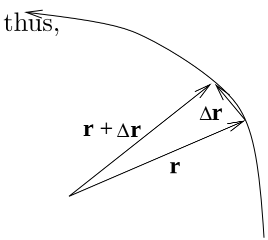
2. Kepler's second law and the central force. To show that the force being central (i.e., directed toward the sun) is equivalent to Kepler's second law, we need to translate that law into calculus. "Sweeps out equal areas in equal times" means:

the radius vector sweeps out area at a constant rate .

The first thing therefore is to obtain a mathematical expression for this rate. Referring to the picture, we see that as the time increases from t to $t+\Delta t$, the corresponding change in the area A is given approximately by

$$\Delta A \;\approx\; \text{area of the triangle} \;=\; \frac{1}{2}\,|\mathbf{r}\times\Delta\mathbf{r}|\ ,$$

since the triangle has half the area of the parallelogram formed by $\mathbf{r}$ and $\Delta\mathbf{r}$; thus,



$$2\,\frac{\Delta A}{\Delta t} \;\approx\; \left|\mathbf{r}\times\frac{\Delta\mathbf{r}}{\Delta t}\right|,$$

and as $\Delta t\to 0$, this becomes

$$(2)\qquad 2\,\frac{dA}{dt} \;=\; \left|\mathbf{r}\times\frac{d\mathbf{r}}{dt}\right| \;=\; |\mathbf{r}\times\mathbf{v}|. \qquad \text{where } \mathbf{v}=\frac{d\mathbf{r}}{dt}\ .$$

Using (2), we can interpret Kepler's second law mathematically. Since the area is swept out at a constant rate, dA/dt is constant, so according to (2),

$$(3)\qquad |\mathbf{r}\times\mathbf{v}| \quad \textit{is a constant.}$$

Moreover, since Kepler's law says $\mathbf{r}$ lies in a plane, the velocity vector $\mathbf{v}$ also lies in the same plane, and therefore

$$(4)\qquad \mathbf{r}\times\mathbf{v} \quad \textit{has constant direction (perpendicular to the plane of motion).}$$

Since the direction and magnitude of $\mathbf{r}\times\mathbf{v}$ are both constant,

$$(5)\qquad \mathbf{r}\times\mathbf{v} \;=\; \mathbf{K}, \text{ a constant vector,}$$

and from this we see that

$$(6)\qquad \frac{d}{dt}\,(\mathbf{r}\times\mathbf{v}) \;=\; \mathbf{0}\ .$$

But according to the rule (1) for differentiating a vector product,

$$(7)\qquad \begin{aligned} \frac{d}{dt}\,(\mathbf{r}\times\mathbf{v}) \;&=\; \mathbf{v}\times\mathbf{v}+\mathbf{r}\times\mathbf{a}, && \text{where } \mathbf{a}=\frac{d\mathbf{v}}{dt},\\ &=\; \mathbf{r}\times\mathbf{a}, && \text{since } \mathbf{s}\times\mathbf{s}=\mathbf{0} \text{ for any vector } \mathbf{s}. \end{aligned}$$

Now (6) and (7) together imply

$$(8)\qquad \mathbf{r}\times\mathbf{a} \;=\; \mathbf{0},$$

which shows that *the acceleration vector* $\mathbf{a}$ *is parallel to* $\mathbf{r}$, but in the opposite direction, since the planets do go around the sun, not shoot off to infinity.

Thus $\mathbf{a}$ is directed toward the center (i.e., the sun), and since $\mathbf{F}=m\mathbf{a}$, the force $\mathbf{F}$ is also directed toward the sun. (Note that "center" does not mean the center of the elliptical orbits, but the mathematical origin, i.e., the tail of the radius vector $\mathbf{r}$, which we are taking to be the sun's position.)

The reasoning is reversible, so for motion under any type of central force, the path of motion will lie in a plane and area will be swept out by the radius vector at a constant rate.

Vector derivatives

1. Let $\mathbf{r}(t)$ be a vector function. Prove by using components that

$$\frac{d\mathbf{r}}{dt} = \mathbf{0} \;\Rightarrow\; \mathbf{r}(t) = \mathbf{K}, \text{ where } \mathbf{K} \text{ is a constant vector.}$$

<u>Answer:</u> In two dimensions $\mathbf{r}(t) = \langle x(t), y(t)\rangle, \mathbf{r}'(t) = \langle x'(t), y'(t)\rangle$.
Therefore,

$$\begin{aligned} \mathbf{r}'(t) = 0 \quad &\Rightarrow \quad x'(t) = 0 \text{ and } y'(t) = 0 \\ &\Rightarrow \quad x(t) = k_1 \text{ and } y(t) = k_2 \\ &\Rightarrow \quad \mathbf{r}(t) = \langle k_1, k_2\rangle, \text{ where } k_1 \text{ and } k_2 \text{ are contants.} \end{aligned}$$

Vector derivatives

1. Let $\mathbf{r}(t)$ be a vector function. Prove by using components that

$$\frac{d\mathbf{r}}{dt} = \mathbf{0} \;\Rightarrow\; \mathbf{r}(t) = \mathbf{K}, \text{ where } \mathbf{K} \text{ is a constant vector.}$$