Vectors

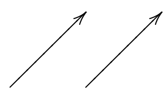
Our very first topic is unusual in that we will start with a brief written presentation. More typically we will begin each topic with a videotaped lecture by Professor Auroux and follow that with a brief written presentation.

As we pointed out in the introduction, vectors will be used throughout the course. The basic concepts are straightforward, but you will have to master some new terminology. Another important point we made earlier is that we can view vectors in two different ways: geometrically and algebraically. We will start with the geometric view and introduce terminology along the way.

Geometric view

A vector is defined as having a magnitude and a direction. We represent it by an arrow in the plane or in space. The length of the arrow is the vector's magnitude and the direction of the arrow is the vector's direction.

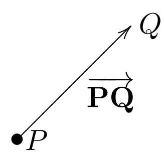
In this way, two arrows with the same magnitude and direction represent the same vector.



(same vector)

We will refer to the start of the arrow as the *tail* and the end as the *tip* or *head*.

The vector between two points will be denoted $\overrightarrow{\mathbf{PQ}}$.

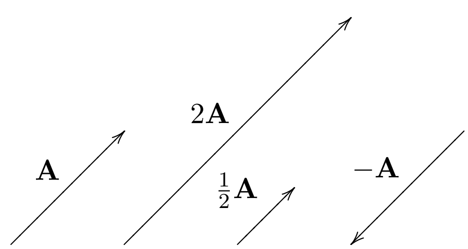


We call P the initial point and Q the terminal point of $\overrightarrow{\mathbf{PQ}}$.

The *magnitude* of the vector $\mathbf{A}$ will be denoted $|\mathbf{A}|$. Magnitude will also be called *length* or *norm*

Scaling, adding and subtracting vectors

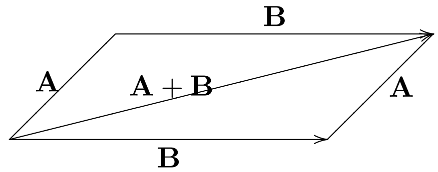
Scaling a vector means changing its length by a scale factor. For example,



(scaling a vector)

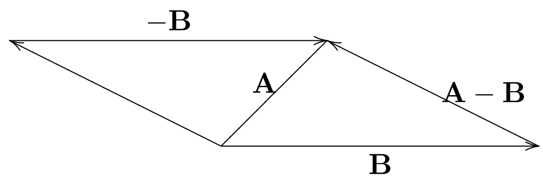
Because we use numbers to scale a vector we will often refer to real numbers as *scalars*

You add vectors by placing them head to tail. As the figure shows, this can be done in either order



It is often useful to think of vectors as *displacements*. In this way, $\mathbf{A}+\mathbf{B}$ can be thought of as the displacement $\mathbf{A}$ followed by the displacement $\mathbf{B}$.

You subtract vectors either by placing the tail to tail or by adding $\mathbf{A}+(-\mathbf{B})$.



Thought of as displacements $\mathbf{A}-\mathbf{B}$ is the displacement from the end of $\mathbf{B}$ to the end of $\mathbf{A}$.

Algebraic view

As is conventional, we label the origin O. In the plane $O=(0,0)$ and in space $O=(0,0,0)$.

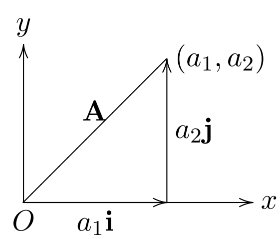
In the xy-plane if we place the tail of $\mathbf{A}$ at the origin, its head will be at the point with coordinates, say, (a_1,a_2). In this way, the coordinates of the head determine the vector $\mathbf{A}$. When we draw $\mathbf{A}$ from the origin we will refer to it as an *origin vector*.

Using the coordinates we write

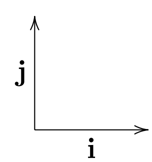
$$\mathbf{A}=\langle a_1,\, a_2\rangle.$$

Addition, subtraction and scaling using coordinates is discussed below.

Graphically:



The vectors $\mathbf{i}$ and $\mathbf{j}$ used in the figure above have coordinates $\mathbf{i}=\langle 1,\,0\rangle$, $\mathbf{j}=\langle 0,\,1\rangle$. We use them so often that they get their own symbols.



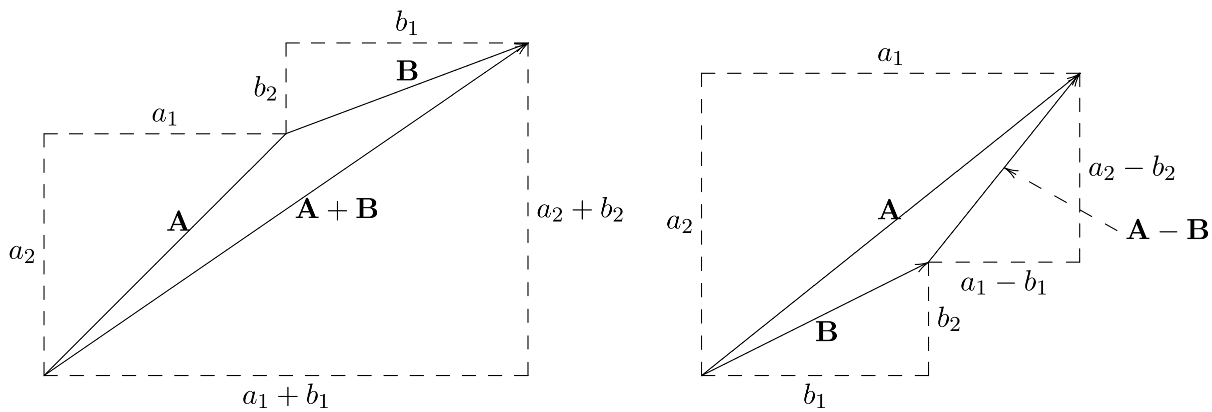
Notation and terminology

1. (a_1,a_2) indicates a point in the plane.
2. $\langle a_1,\,a_2\rangle = a_1\mathbf{i}+a_2\mathbf{j}$. This is equal to the vector drawn from the origin to the point (a_1,a_2).
3. For $\mathbf{A}=a_1\mathbf{i}+a_2\mathbf{j}$, a_1 and a_2 are called the $\mathbf{i}$ and $\mathbf{j}$ *components* of $\mathbf{A}$. (Note that they are scalars.)
5. $\overrightarrow{\mathbf{P}}=\overrightarrow{\mathbf{OP}}$ is the vector from the origin to P.

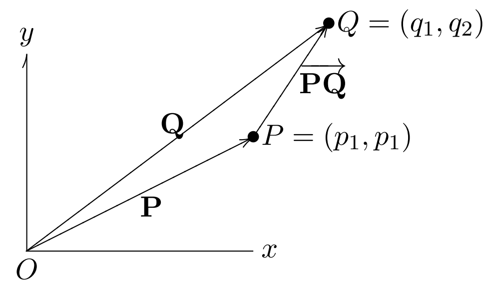
6. On the blackboard vectors will usually have an arrow above the letter. In print we will often drop the arrow and just use the bold face to indicate a vector, i.e. $\mathbf{P} \equiv \overrightarrow{\mathbf{P}}$.
7. A real number is a *scalar*, you can use it to scale a vector.

Vector algebra using coordinates
For the vectors $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j}$ we have the following algebraic rules. The figures below connect these rules to the geometric viewpoint.

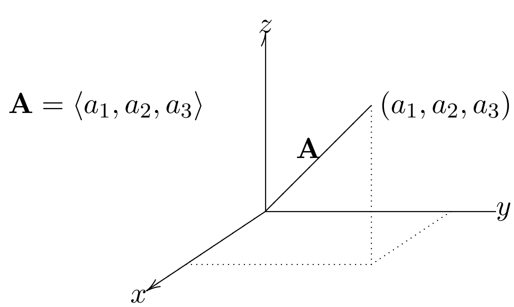
Magnitude: $|\mathbf{A}| = \sqrt{a_1^2 + a_2^2}$ (this is just the Pythagorean theorem)
Addition: $\mathbf{A} + \mathbf{B} = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j}$, that is, $\langle a_1, a_2\rangle + \langle b_1, b_2\rangle = \langle a_1 + b_1, a_2 + b_2\rangle$
Subtraction: $\mathbf{A} - \mathbf{B} = (a_1 - b_1)\mathbf{i} + (a_2 - b_2)\mathbf{j}$, that is, $\langle a_1, a_2\rangle - \langle b_1, b_2\rangle = \langle a_1 - b_1, a_2 - b_2\rangle$



For two points P and Q the vector $\overrightarrow{\mathbf{PQ}} = \overrightarrow{\mathbf{Q}} - \overrightarrow{\mathbf{P}}$ i.e., $\overrightarrow{\mathbf{PQ}}$ is the *displacement* from P to Q.



Vectors in three dimensions
We represent a three dimensional vector as an arrow in space. Using coordinates we need three numbers to represent a vector.



Geometrically nothing changes for vectors in three dimensions. They are scaled and added exactly as above.

Algebraically the origin vector $\mathbf{A} = \langle a_1, a_2, a_3\rangle$ starts at the origin and extends to the point (a_1, a_2, a_3). We have the special vectors $\mathbf{i} = \langle 1, 0, 0\rangle$, $\mathbf{j} = \langle 0, 1, 0\rangle$, $\mathbf{k} = \langle 0, 0, 1\rangle$. Using them

$$\langle a_1, a_2, a_3\rangle = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}.$$

Then, for $\mathbf{A} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{B} = \langle b_1, b_2, b_3\rangle$ we have

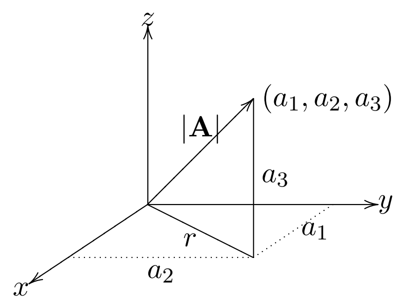
$$\langle a_1, a_2, a_3\rangle + \langle b_1, b_2, b_3\rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3\rangle.$$

exactly as in the two dimensional case.

Magnitude in three dimensions also follows from the Pythagorean theorem.

$$|a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}| = |\langle a_1, a_2, a_3\rangle| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

You can see this in the figure below, where $r = \sqrt{a_1^2 + a_2^2}$ and $|\mathbf{A}| = \sqrt{r^2 + a_3^2} = \sqrt{a_1^2 + a_2^2 + a_3^2}$.



Unit vectors

A unit vector is any vector with unit length. When we want to indicate that a vector is a unit vector we put a hat (circumflex) above it, e.g., $\widehat{\mathbf{u}}$.

The special vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors.

Since vectors can be scaled, any vector can be rescaled to be a unit vector.

Example: Find a unit vector that is parallel to $\langle 3, 4\rangle$.

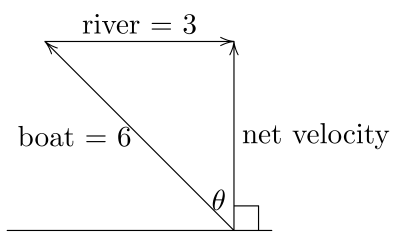
<u>Answer:</u> Since $|\langle 3, 4\rangle| = 5$ the vector $\frac{1}{5}\langle 3, 4\rangle = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$ has unit length and is parallel to $\langle 3, 4\rangle$.

Vector problems

1. a) A river flows at 3 mph and a rower rows at 6 mph. What heading should the rower take to go straight across a river?

b) Answer the same question if the river flows at 6 mph and the rower rows at 3 mph.

Answer:



a) Let θ be the angle the rower heads upstream from straight across the river.

The net velocity is the sum of the river and the rower's velocities. If the net velocity is straight across then the triangle shown is a right triangle which implies

$$\sin(\theta) = \frac{3}{6} \;\Rightarrow\; \theta = 30^\circ = \pi/6$$

Answer: Head at angle of 30° ($\pi/6$ radians) upstream from straight across.

b) In this case we need $\sin(\theta) = \frac{6}{3}$. Since this is impossible ($\sin(\theta) \le 1$) we conclude no heading will work.

This makes sense because the river flows faster than the rower rows, so the boat will be pushed downstream no matter what the heading.

2. Find a unit vector in the direction of $\langle 2, 3\rangle$.

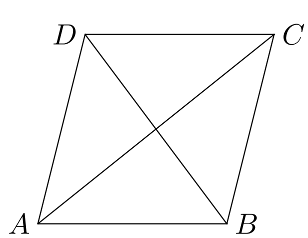
Answer: To get a unit vector we scale the original vector by one over its length.

$$\widehat{\mathbf{u}} = \frac{\langle 2,3\rangle}{\sqrt{13}} = \left\langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right\rangle$$

is a unit vector parallel to the original vector.

3. Use vectors to prove that the diagonals of a parallelogram bisect each other.

Answer:



We need to show that the two diagonals intersect at their mutual midpoints. Said differently we need to show that the midpoints of AC and BD are, in fact, the same point.

Let M_1 be the midpoint of AC and M_2 be the midpoint of BD. Rephrasing our goal yet again, we will show $M_1 = M_2$ by showing $\overrightarrow{\mathbf{AM_1}} = \overrightarrow{\mathbf{AM_2}}$.

Now, using vectors we have

$$\overrightarrow{\mathbf{AM_1}} = \frac{1}{2}\overrightarrow{\mathbf{AC}}$$

We need to use the fact that $ABCD$ is a parallelogram. From this we get

$$\overrightarrow{\mathbf{AB}} = \overrightarrow{\mathbf{DC}} \;\Rightarrow\; \overrightarrow{\mathbf{AB}} = \frac{1}{2}\overrightarrow{\mathbf{AB}} + \frac{1}{2}\overrightarrow{\mathbf{AB}} = \frac{1}{2}\overrightarrow{\mathbf{AB}} + \frac{1}{2}\overrightarrow{\mathbf{DC}}$$

We use this to write $\overrightarrow{\mathbf{AM_2}}$ in terms of A, B, C, D.

$$\overrightarrow{\mathbf{AM_2}} = \overrightarrow{\mathbf{AB}} + \frac{1}{2}\overrightarrow{\mathbf{BD}} = \frac{1}{2}\overrightarrow{\mathbf{AB}} + \frac{1}{2}\overrightarrow{\mathbf{DC}} + \frac{1}{2}\overrightarrow{\mathbf{BD}} = \frac{1}{2}(\overrightarrow{\mathbf{AB}} + \overrightarrow{\mathbf{BD}} + \overrightarrow{\mathbf{DC}}) = \frac{1}{2}\overrightarrow{\mathbf{AC}}$$

We have shown $\overrightarrow{\mathbf{AM_1}} = \overrightarrow{\mathbf{AM_2}}$, so we are done.

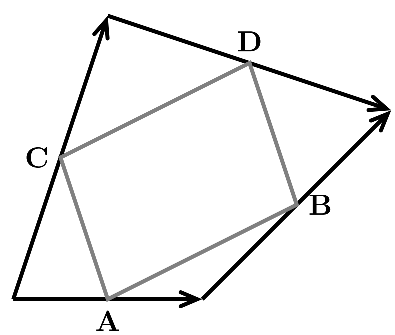
4. Prove using vector methods that the midpoints of the sides of a space quadrilateral form a parallelogram

<u>**Answer:**</u> Let **A**, **B**, **C**, **D** be the four sides; then if the vectors are oriented as shown in the figure below we have $\mathbf{A} + \mathbf{B} = \mathbf{C} + \mathbf{D}$.

The vector from the midpoint of **A** to the midpoint of C is $\frac{1}{2}\mathbf{C} - \frac{1}{2}\mathbf{A}$; similarly the vector joining the midpoints of the other two sides is $\frac{1}{2}\mathbf{B} - \frac{1}{2}\mathbf{D}$. Then

$$\mathbf{A} + \mathbf{B} = \mathbf{C} + \mathbf{D} \;\Rightarrow\; \mathbf{C} - \mathbf{A} = \mathbf{B} - \mathbf{D} \;\Rightarrow\; \frac{1}{2}(\mathbf{C} - \mathbf{A}) = \frac{1}{2}(\mathbf{B} - \mathbf{D}).$$

Thus two opposite sides are equal and parallel, which shows the figure is a parallelogram.



Some note about finding these proofs:
1. You need to use the hypotheses. For example, in this problem a key step made use of the fact that $ABCD$ is a parallelogram.

2. You should not expect to see the proofs immediately. Rather you will generally need to play around with the figure.

3. After arriving at a proof you should see if you can clean it up or simplify it.

Vector problems

1. a) A river flows at 3 mph and a rower rows at 6 mph. What heading should the rower take to go straight across a river?

b) Answer the same question if the river flows at 6 mph and the rower rows at 3 mph.

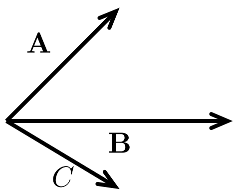
2. Find a unit vector in the direction of $\langle 2, 3\rangle$.

3. Use vectors to prove that the diagonals of a parallelogram bisect each other.

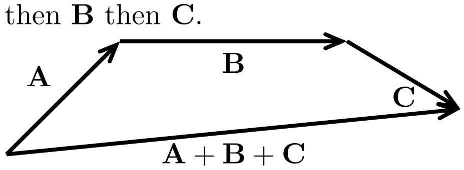
4. Prove using vector methods that the midpoints of the sides of a space quadrilateral form a parallelogram

Vector addition

1. Find $A + \mathbf{B} + \mathbf{C}$.



Answer: Vectors are displacements: $\mathbf{A} + \mathbf{B} + \mathbf{C}$ is the net displacement from following $\mathbf{A}$ then $\mathbf{B}$ then $\mathbf{C}$.



2. a) Find $\mathbf{A} + \mathbf{B} + \mathbf{C}$, where $\mathbf{A} = \langle 1, 2 \rangle$, $\mathbf{B} = \langle 1, 0 \rangle$, $\mathbf{C} = \langle 2, -1 \rangle$.

b) Find $\langle 1, 2, 5 \rangle + \langle -2, 1, 5 \rangle$.

Answer: a) Algebraically vectors add componentwise so

$$\mathbf{A} + \mathbf{B} + \mathbf{C} = \langle 1 + 1 + 2, 2 + 0 + -1 \rangle = \langle 4, 1 \rangle.$$

Remember $\langle 1, 2 \rangle$ is a shorthand for $\mathbf{i} + 2\mathbf{j}$. So another way to see this answer is that

$$\mathbf{A} + \mathbf{B} + \mathbf{C} = (\mathbf{i} + 2\mathbf{j}) + \mathbf{i} + (2\mathbf{i} - \mathbf{j}) = 4\mathbf{i} + \mathbf{j}.$$

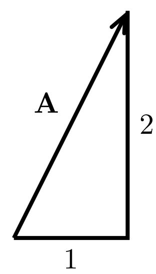
b) $\langle 1, 2, 5 \rangle + \langle -2, 1, 5 \rangle = \langle -1, 3, 10 \rangle$.

Hopefully you found these examples very simple. Things will get more complicated soon. For now you should just make sure you become very comfortable with vector computation done both geometrically and algebraically.

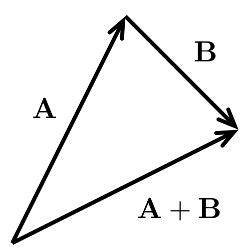
Vector lengths

1. Let $\mathbf{A} = \langle 1, 2 \rangle$, $\mathbf{B} = \langle 1, -1 \rangle$ and $\mathbf{C} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$. Find the lengths of $\mathbf{A}$, $\mathbf{A} + \mathbf{B}$ and $\mathbf{C}$.

Answer: Length is just an expression of the Pythagorean theorem. The picture below shows $|\mathbf{A}| = \sqrt{1 + 2^2} = \sqrt{5}$



Likewise $\mathbf{A} + \mathbf{B} = \langle 2, 1 \rangle \;\Rightarrow\; |\mathbf{A} + \mathbf{B}| = \sqrt{5}$. Perhaps, it's surprising that $\mathbf{A}$ and $\mathbf{A} + \mathbf{B}$ can have the same length.

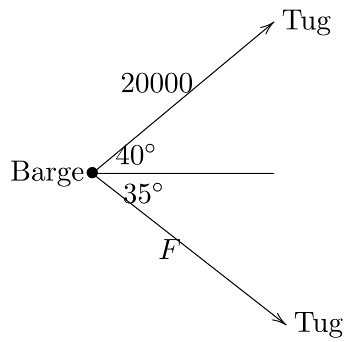


$\mathbf{C}$ is a vector in space, but the length is computed the same way, except there are 3 terms under the radical sign.

$$|\mathbf{C}| = \sqrt{1 + 2^2 + 3^2} = \sqrt{14}.$$

Force is a vector

1. The picture shows two tugs pulling a barge. If one pulls with 20000 Newtons of force what force should the other tug pull with to keep the barge going straight to the right.



Answer: We need the vertical components of the forces to cancel.

So, $20000 \sin(40°) = F \sin(35°) \;\Rightarrow\; F = \dfrac{20000 \sin(40°)}{\sin(35°)} \approx 22413.32.$

Proofs using vectors

1. The median of a triangle is a vector from a vertex to the midpoint of the opposite side. Show the sum of the medians of a triangle $= \mathbf{0}$.

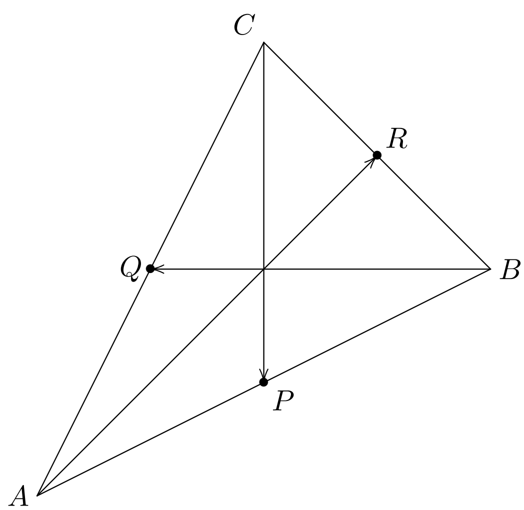
<u>Answer:</u> The median of side AB is the vector from vertex C to the midpoint of AB. Label this midpoint as P. As usual we write $\mathbf{P}$ for the origin vector $\overrightarrow{\mathbf{OP}}$.

The midpoint $\mathbf{P} = \frac{1}{2}(\mathbf{A} + \mathbf{B}) \;\Rightarrow\; \overrightarrow{\mathbf{CP}} = \frac{1}{2}(\mathbf{B} + \mathbf{A}) - \mathbf{C}$.

Likewise: $\overrightarrow{\mathbf{BQ}} = \frac{1}{2}(\mathbf{A} + \mathbf{C}) - \mathbf{B}$ and $\overrightarrow{\mathbf{AR}} = \frac{1}{2}(\mathbf{B} + \mathbf{C}) - \mathbf{A}$.

$\Rightarrow$ sum of medians is

$$\overrightarrow{\mathbf{CP}} + \overrightarrow{\mathbf{BQ}} + \overrightarrow{\mathbf{AR}} = \left(\frac{1}{2}(\mathbf{B} + \mathbf{A}) - \mathbf{C}\right) + \left(\frac{1}{2}(\mathbf{A} + \mathbf{C}) - \mathbf{B}\right) + \left(\frac{1}{2}(\mathbf{B} + \mathbf{C}) - \mathbf{A}\right) = 0.$$



Dot Product

The dot product is one way of combining ("multiplying") two vectors. The output is a scalar (a number). It is called the dot product because the symbol used is a dot. Because the dot product results in a scalar it, is also called the scalar product.

As with most things in 18.02, we have a geometric and algebraic view of dot product.

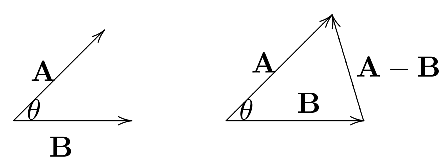
Algebraic definition (for 2D vectors):
If $\mathbf{A} = \langle a_1, a_2 \rangle$ and $\mathbf{B} = \langle b_1, b_2 \rangle$ then

$$\mathbf{A} \cdot \mathbf{B} = a_1 b_1 + a_2 b_2.$$

Example: $\langle 6, 5 \rangle \cdot \langle 1, 2 \rangle = 6 \cdot 1 + 5 \cdot 2 = 16.$

Geometric view:
The figure below shows $\mathbf{A}$, $\mathbf{B}$ with the angle θ between them. We get

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos\theta$$



Showing the two views (algebraic and geometric) are the same requires the law of cosines

$$|\mathbf{A} - \mathbf{B}|^2 = |\mathbf{A}|^2 + |\mathbf{B}|^2 - 2|\mathbf{A}||\mathbf{B}| \cos\theta$$

$\Rightarrow \ (a_1^2 + a_2^2) + (b_1^2 + b_2^2) - ((a_1 - b_1)^2 + (a_2 - b_2)^2) = 2|\mathbf{A}||\mathbf{B}| \cos\theta$
$\Rightarrow \ a_1 b_1 + a_2 b_2 = |\mathbf{A}||\mathbf{B}| \cos\theta.$
Since $\langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle = a_1 b_1 + a_2 b_2$, we have shown $\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos\theta$.

From the algebraic definition of dot product we easily get the the following algebraic law

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}.$$

Example: Find the dot product of $\mathbf{A}$ and $\mathbf{B}$.
i) $|\mathbf{A}| = 2$, $|\mathbf{B}| = 5$, $\theta = \pi/4$.
Answer: (draw the picture yourself) $\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos\theta = 10\sqrt{2}/2 = 5\sqrt{2}$.

ii) $\mathbf{A} = \mathbf{i} + 2\mathbf{j}$, $\mathbf{B} = 3\mathbf{i} + 4\mathbf{j}$.
Answer: $\mathbf{A} \cdot \mathbf{B} = 1 \cdot 3 + 2 \cdot 4 = 11.$

Three dimensional vectors
The dot product works the same in 3D as in 2D. If $\mathbf{A} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{B} = \langle b_1, b_2, b_3 \rangle$ then

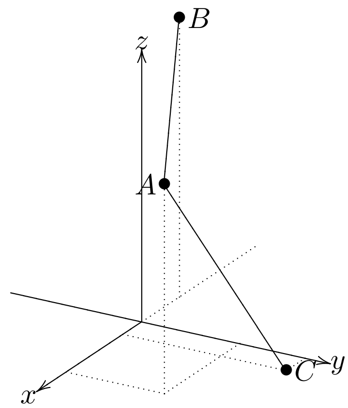
$$\mathbf{A} \cdot \mathbf{B} = a_1 \cdot b_1 + a_2 \cdot b_2 + a_3 \cdot b_3.$$

The geometric view is identical and the same proof shows

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos\theta$$

Example:
Show $A = (4, 3, 6)$, $B = (-2, 0, 8)$, $C = (1, 5, 0)$
are the vertices of a right triangle.

__**Answer:**__ Two legs of the triangle are $\overrightarrow{\mathbf{AC}} = \langle -3,\, 2,\, -6\rangle$ and $\overrightarrow{\mathbf{AB}} = \langle -6,\, -3,\, 2\rangle \;\Rightarrow$ $\overrightarrow{\mathbf{AC}} \cdot \overrightarrow{\mathbf{AB}} = 18 - 6 - 12 = 0$. The geometric view of dot product implies the angle between the legs is $\pi/2$ (i.e $\cos\theta = 0$).



Definition of the term orthogonal and the test for orthogonality

When two vectors are perpendicular to each other we say they are *orthogonal*.

As seen in the example, since $\cos(\pi/2) = 0$, the dot product gives a test for orthogonality between vectors:

$$\mathbf{A} \perp \mathbf{B} \;\Leftrightarrow\; \mathbf{A} \cdot \mathbf{B} = 0.$$

Dot product and length

Both the algebraic and geometric formulas for dot product show it is intimately connected to length. In fact, they show for a vector $\mathbf{A}$

$$\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}|^2.$$

Let's show this using both views.

Algebraically: suppose $\mathbf{A} = \langle a_1, a_2, a_3\rangle$ then

$$\mathbf{A} \cdot \mathbf{A} = \langle a_1, a_2, a_3\rangle \cdot \langle a_1, a_2, a_3\rangle = a_1^2 + a_2^2 + a_3^2 = |\mathbf{A}|^2.$$

Geometrically: the angle θ between $\mathbf{A}$ and itself is 0. Therefore,

$$\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}||\mathbf{A}|\cos\theta = |\mathbf{A}||\mathbf{A}| = |\mathbf{A}|^2.$$

As promised both views give the formula.

Dot product problems

1. a) Compute $\langle 1, 2, -4 \rangle \cdot \langle 2, 3, 5 \rangle$.

b) Is the angle between these two vectors acute, obtuse or right?

Answer: a) $\langle 1, 2, -4 \rangle \cdot \langle 2, 3, 5 \rangle = 1 \cdot 2 + 2 \cdot 3 - 4 \cdot 5 = -12$.

b) Let θ be the angle between the vectors. Since the dot product is negative we have $\cos\theta < 0$, which means $\theta > \pi/2$. The angle is obtuse.

2. Suppose $\mathbf{B} = \langle 2, 2, 1 \rangle$. Suppose also that $\mathbf{B}$ makes an angle of $30°$ with $\mathbf{A}$ and $\mathbf{A} \cdot \mathbf{B} = 6$. Find $|A|$.

Answer: Since $30^0 = \pi/6$ radians and $|\mathbf{B}| = 3$ we get

$$6 = \mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos(\pi/6) = |\mathbf{A}| \cdot 3 \cdot \frac{\sqrt{3}}{2} \;\Rightarrow\; |\mathbf{A}| = \frac{4}{\sqrt{3}}.$$

3. If $\mathbf{A} \cdot \mathbf{B} = 0$ what is the angle between $\mathbf{A}$ and $\mathbf{B}$?

Answer: $\pi/2$.

Dot product problems

1. a) Compute $\langle 1, 2, -4\rangle \cdot \langle 2, 3, 5\rangle$.

b) Is the angle between these two vectors acute, obtuse or right?

2. Suppose $\mathbf{B} = \langle 2, 2, 1\rangle$. Suppose also that $\mathbf{B}$ makes an angle of 30° with $\mathbf{A}$ and $\mathbf{A} \cdot \mathbf{B} = 6$. Find $|A|$.

3. If $\mathbf{A} \cdot \mathbf{B} = 0$ what is the angle between $\mathbf{A}$ and $\mathbf{B}$?

Uses of the Dot Product

1. Find the angle between the vectors $\mathbf{A} = \mathbf{i} + 8\mathbf{j}$ and $\mathbf{B} = \mathbf{i} + 2\mathbf{j}$.

Answer: As usual, call the angle in question θ. Since $\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos\theta$ we have

$$\cos\theta = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|} = \frac{\langle 1, 8\rangle \cdot \langle 1, 2\rangle}{\sqrt{65}\,\sqrt{5}} = \frac{17}{5\sqrt{13}}$$

Thus, $\theta = \cos^{-1}\left(\dfrac{17}{5\sqrt{13}}\right)$.

2. Take points $P = (a, 1, -1)$, $Q = (0, 1, 1)$, $R = (a, -1, 3)$. For what value(s) of a is PQR a right angle?

Answer: We need $\overrightarrow{\mathbf{QP}} \cdot \overrightarrow{\mathbf{QR}} = 0 \ \Rightarrow\ \langle a, 0, -2\rangle \cdot \langle a, -2, 2\rangle = a^2 - 4 = 0 \ \Rightarrow\ a = \pm 2$.

3. Show that the diagonals of a parallelogram are perpendicular if and only if it is a rhombus, i.e., its four sides have equal lengths.

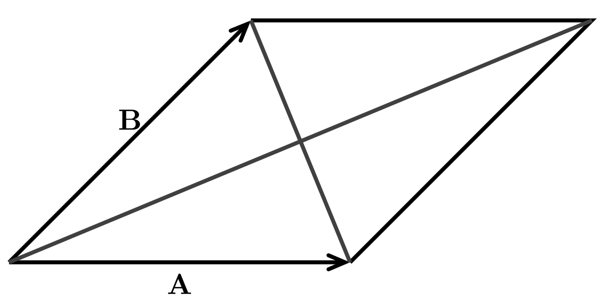
Answer: Let two adjacent sides of the parallelogram be the vectors $\mathbf{A}$ and $\mathbf{B}$ (as shown in the figure). Then we have the two diagonals are $\mathbf{A} + \mathbf{B}$ and $\mathbf{A} - \mathbf{B}$. We have

$$(\mathbf{A} + \mathbf{B}) \cdot (\mathbf{A} - \mathbf{B}) = \mathbf{A} \cdot \mathbf{A} - \mathbf{B} \cdot \mathbf{B}.$$

Therefore,

$$(\mathbf{A} + \mathbf{B}) \cdot (\mathbf{A} - \mathbf{B}) = 0 \ \Leftrightarrow\ \mathbf{A} \cdot \mathbf{A} = \mathbf{B} \cdot \mathbf{B}.$$

I.e., the diagonals are perpendicular if and only if two adjacent edges have equal lengths. In other words, if the parallelogram is a rhombus.



Uses of the Dot Product

1. Find the angle between the vectors $\mathbf{A} = \mathbf{i} + 8\mathbf{j}$ and $\mathbf{B} = \mathbf{i} + 2\mathbf{j}$.

2. Take points $P = (a, 1, -1)$, $Q = (0, 1, 1)$, $R = (a, -1, 3)$. For what value(s) of a is PQR a right angle?

3. Show that the diagonals of a parallelogram are perpendicular if and only if it is a rhombus, i.e., its four sides have equal lengths.

Uses of dot product

1. Find the angle between $\mathbf{i}+\mathbf{j}+2\mathbf{k}$ and $2\mathbf{i}-\mathbf{j}+\mathbf{k}$.

<u>Answer:</u> We call the angle θ and use both ways of computing the dot product. Algebraically we have

$$(\mathbf{i}+\mathbf{j}+2\mathbf{k})\cdot(2\mathbf{i}-\mathbf{j}+\mathbf{k}) = 2-1+2 = 3.$$

Geometrically

$$(\mathbf{i}+\mathbf{j}+2\mathbf{k})\cdot(2\mathbf{i}-\mathbf{j}+\mathbf{k}) = |\mathbf{i}+\mathbf{j}+2\mathbf{k}|\cdot|2\mathbf{i}-\mathbf{j}+\mathbf{k}|\cos\theta = \sqrt{6}\,\sqrt{6}\cos\theta.$$

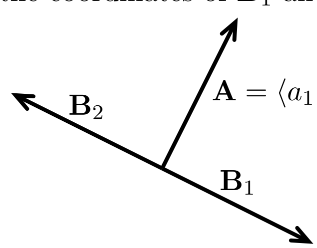
Combining these two we have

$$6\cos\theta = 3 \;\Rightarrow\; \cos\theta = \frac{3}{6} = \frac{1}{2} \;\Rightarrow\; \theta = \frac{\pi}{3}.$$

2. a) Are $\langle 1,3\rangle$ and $\langle -2,2\rangle$ orthogonal?

b) For what value of a are the vectors $\langle 1,a\rangle$ and $\langle 2,3\rangle$ at right angles?

c) In the figure the vectors $\mathbf{A}$ and $\mathbf{B}_1$ are orthogonal as are $\mathbf{A}$ and $\mathbf{B}_2$. If all the vectors are the same length what are the coordinates of $\mathbf{B}_1$ and $\mathbf{B}_2$?



<u>Answer:</u> a) Vectors are orthogonal if their dot product is 0. So, taking the dot product

$$\langle 1,3\rangle\cdot\langle -2,2\rangle = -2+6 = 4 \neq 0.$$

Thus the vectors are not orthogonal.

b) Setting the dot product to 0 and solving for a we get

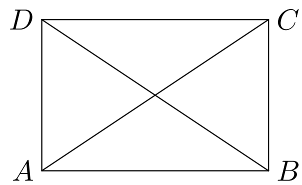
$$\langle 1,a\rangle\cdot\langle 2,3\rangle = 2+3a = 0 \;\Rightarrow\; a = -2/3.$$

c) $\mathbf{B}_1$ is $\mathbf{A}$ rotated 90° clockwise. We will show that $\mathbf{B}_1 = \langle a_2, -a_1\rangle$. It is easy to check that $|\langle a_2,-a_1\rangle| = |\mathbf{A}|$ and $\langle a_2,-a_1\rangle\cdot\mathbf{A} = 0$. The figure above shows that putting the negative sign on the a_1 means $\langle a_2,-a_1\rangle$ is turned clockwise from $\mathbf{A}$. Thus, $\langle a_2,-a_1\rangle = \mathbf{B}_1$.

$\mathbf{B}_2$ is $\mathbf{A}$ rotated 90° counterclockwise. Similarly to $\mathbf{B}_1$, we find $\mathbf{B}_2 = \langle -a_2, a_1\rangle$.

3. Using vectors and dot product show the diagonals of a parallelogram have equal lengths if and only if it's a rectangle

Answer:



We will make use of two properties of the dot product

1. $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$.

2. $\mathbf{v} \cdot \mathbf{w} = 0 \;\Leftrightarrow\; \mathbf{v} \perp \mathbf{w}$.

Referring to the figure, we will also need to use the fact that $ABCD$ is a parallelogram. That is, $\overrightarrow{\mathbf{AB}} = \overrightarrow{\mathbf{DC}}$.

We have $\overrightarrow{\mathbf{AC}} = \overrightarrow{\mathbf{AB}} + \overrightarrow{\mathbf{BC}}$ and $\overrightarrow{\mathbf{BD}} = \overrightarrow{\mathbf{BC}} + \overrightarrow{\mathbf{CD}} = \overrightarrow{\mathbf{BC}} - \overrightarrow{\mathbf{AB}}$.

Taking dot products:

$$|\overrightarrow{\mathbf{AC}}|^2 = \overrightarrow{\mathbf{AC}} \cdot \overrightarrow{\mathbf{AC}} = (\overrightarrow{\mathbf{AB}} + \overrightarrow{\mathbf{BC}}) \cdot (\overrightarrow{\mathbf{AB}} + \overrightarrow{\mathbf{BC}}) = |\overrightarrow{\mathbf{AB}}|^2 + 2\overrightarrow{\mathbf{AB}} \cdot \overrightarrow{\mathbf{BC}} + |\overrightarrow{\mathbf{BC}}|^2.$$

and

$$|\overrightarrow{\mathbf{BD}}|^2 = \overrightarrow{\mathbf{BD}} \cdot \overrightarrow{\mathbf{BD}} + (\overrightarrow{\mathbf{BC}} - \overrightarrow{\mathbf{AB}}) \cdot (\overrightarrow{\mathbf{BC}} - \overrightarrow{\mathbf{AB}}) = |\overrightarrow{\mathbf{BC}}|^2 - 2\overrightarrow{\mathbf{BC}} \cdot \overrightarrow{\mathbf{AB}} + |\overrightarrow{\mathbf{AB}}|^2$$

Comparing the two equations above we see

$$|\overrightarrow{\mathbf{AC}}|^2 = |\overrightarrow{\mathbf{BD}}|^2 \;\Leftrightarrow\; 4\overrightarrow{\mathbf{AB}} \cdot \overrightarrow{\mathbf{BC}} = 0.$$

This shows the diagonals have the same length if and only if $\overrightarrow{\mathbf{AB}} \perp \overrightarrow{\mathbf{BC}}$. That is, if and only if the sides of the parallelogram are orthogonal to each other. QED

Components and Projection

If $\mathbf{A}$ is any vector and $\widehat{\mathbf{u}}$ is a unit vector then the *component* of $\mathbf{A}$ in the direction of $\widehat{\mathbf{u}}$ is

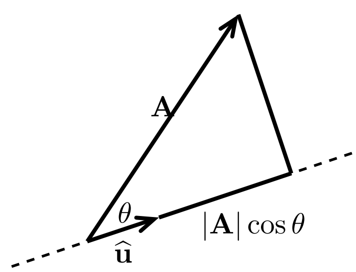
$$\mathbf{A} \cdot \widehat{\mathbf{u}}.$$

(Note: the component is a scalar.)

If θ is the angle between $\mathbf{A}$ and $\widehat{\mathbf{u}}$ then since $|\widehat{\mathbf{u}}| = 1$

$$A \cdot \widehat{\mathbf{u}} = |\mathbf{A}||\widehat{\mathbf{u}}| \cos\theta = |\mathbf{A}| \cos\theta.$$

The figure shows that geometrically this is the length of the leg of the right triangle with hypotenuse $\mathbf{A}$ and one leg parallel to $\widehat{\mathbf{u}}$.



We also call the leg parallel to $\widehat{\mathbf{u}}$ the *orthogonal projection* of $\mathbf{A}$ on $\widehat{\mathbf{u}}$.

For a non-unit vector: the component of $\mathbf{A}$ in the direction of $\mathbf{B}$ is simply the component of $\mathbf{A}$ in the direction of $\widehat{\mathbf{u}} = \frac{\mathbf{B}}{|\mathbf{B}|}$. ($\widehat{\mathbf{u}}$ is the unit vector in the same direction as $\mathbf{B}$.)

Example: Find the component of $\mathbf{A}$ in the direction of $\mathbf{B}$.

i) $|\mathbf{A}| = 2$, $|\mathbf{B}| = 5$, $\theta = \pi/4$.

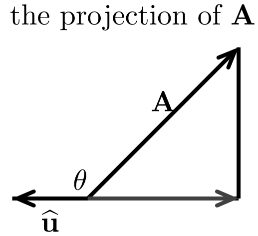
__**Answer:**__ Referring to the figure above: the component is $|\mathbf{A}|\cos\theta = 2\cos(\pi/4) = \sqrt{2}$. Note, the length of $\mathbf{B}$ given is irrelevant, since we only care about the unit vector parallel to $\mathbf{B}$.

ii) $\mathbf{A} = \mathbf{i} + 2\mathbf{j}$, $\mathbf{B} = 3\mathbf{i} + 4\mathbf{j}$.

__**Answer:**__ Unit vector in direction of $\mathbf{B}$ is $\frac{\mathbf{B}}{|\mathbf{B}|} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j} \;\Rightarrow\;$ component is $\mathbf{A} \cdot \mathbf{B}/|\mathbf{B}| = 3/5 + 8/5 = 11/5$.

iii) Find the component of $\mathbf{A} = \langle 2, 2\rangle$ in the direction of $\widehat{\mathbf{u}} = \langle -1, 0\rangle$

__**Answer:**__ The vector $\widehat{\mathbf{u}}$ is a unit vector, so the component is $\mathbf{A} \cdot \widehat{\mathbf{u}} = \langle 2, 2\rangle \cdot \langle -1, 0\rangle = -2$. The negative component is okay, it says the projection of $\mathbf{A}$ and $\widehat{\mathbf{u}}$ point in opposite directions.



We emphasize one more time that the component of a vector is a *scalar*.

Vector Components

1. a) Let $\mathbf{A} = \langle 1, 3 \rangle$ and $\mathbf{B} = \langle 3, 4 \rangle$.

(i) Find the component of **A** in the direction of **B**.

(ii) Find the component of **B** in the direction of **A**.

b) Let $\mathbf{A} = \langle 3, 5, 7 \rangle$ and $\mathbf{B} = \langle 3, 4, 0 \rangle$. Find the component **A** in the direction of **B**.

<u>Answer:</u> a) (i) $|\mathbf{B}| = 5 \Rightarrow$ the component is $\mathbf{A} \cdot \dfrac{\mathbf{B}}{|\mathbf{B}|} = \langle 1, 3 \rangle \cdot \dfrac{\langle 3, 4 \rangle}{5} = \dfrac{15}{5} = 3.$

(ii) $|\mathbf{A}| = \sqrt{10} \Rightarrow \mathbf{B} \cdot \dfrac{\mathbf{A}}{|\mathbf{A}|} = \langle 3, 4 \rangle \cdot \dfrac{\langle 1, 3 \rangle}{\sqrt{10}} = \dfrac{15}{\sqrt{10}}.$

b) In three dimensions the formula is the same. The component is $\mathbf{A} \cdot \dfrac{\mathbf{B}}{|\mathbf{B}|} = \langle 3, 5, 7 \rangle \cdot \dfrac{\langle 3, 4, 0 \rangle}{5} = \dfrac{29}{5}.$

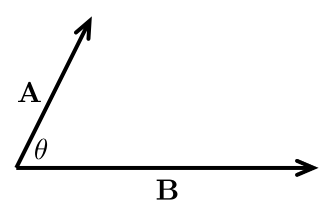
2. Let $\mathbf{A} = \langle a, 2 \rangle$ and $\mathbf{B} = \langle 1, 3 \rangle$. For what values of a is the component of **A** along **B** equal to 0? For what a is it negative?

<u>Answer:</u> The component is $\langle a, 2 \rangle \cdot \dfrac{\langle 1, 3 \rangle}{\sqrt{10}} = \dfrac{a + 6}{\sqrt{10}}.$

This is 0 if $a = -6$.

This is negative if $a < -6$.

3. For which angle θ is the component of **A** in the direction of **B** equal to 0.



<u>Answer:</u> $\theta = \pi/2$.

Vector Components

1. a) Let $\mathbf{A} = \langle 1, 3 \rangle$ and $\mathbf{B} = \langle 3, 4 \rangle$.

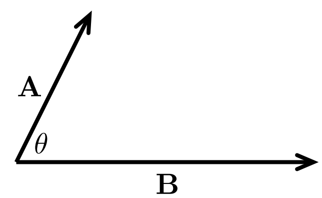
(i) Find the component of **A** in the direction of **B**.

(ii) Find the component of **B** in the direction of **A**.

b) Let $\mathbf{A} = \langle 3, 5, 7 \rangle$ and $\mathbf{B} = \langle 3, 4, 0 \rangle$. Find the component **A** in the direction of **B**.

2. Let $\mathbf{A} = \langle a, 2 \rangle$ and $\mathbf{B} = \langle 1, 3 \rangle$. For what values of a is the component of **A** along **B** equal to 0? For what a is it negative?

3. For which angle θ is the component of **A** in the direction of **B** equal to 0.



Determinants and areas

1. a) Compute $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$.

b) Compute $\begin{vmatrix} 1 & -2 \\ -3 & 4 \end{vmatrix}$.

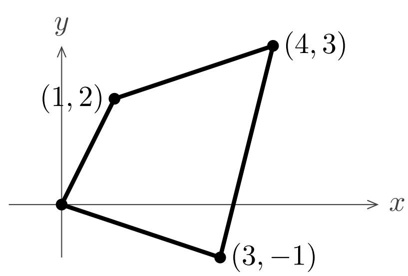
c) Compute $\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix}$.

<u>**Answer:**</u> a) $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1\cdot 4 - 2\cdot 3 = -2.$

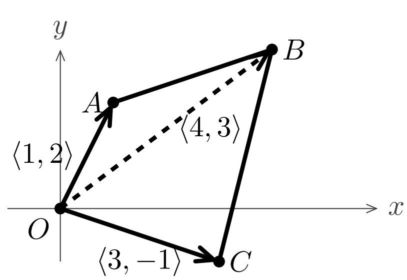
b) $\begin{vmatrix} 1 & -2 \\ -3 & 4 \end{vmatrix} = 1\cdot 4 - (-2)\cdot(-3) = -2.$

c) $\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 3\cdot 2 - 4\cdot 1 = 2.$

2. Find the area of the quadrilateral shown.



<u>**Answer:**</u>



We break the quadrilateral into two triangles. For convenience, on the figure, we have labeled the vertices $OABC$ and indicated the components of $\overrightarrow{\mathbf{OA}}$, $\overrightarrow{\mathbf{OB}}$ and $\overrightarrow{\mathbf{OC}}$.

Area $\triangle OAB = \dfrac{1}{2}\left|\det\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}\right| = \dfrac{1}{2}|-5| = \dfrac{5}{2}.$

Area $\triangle OBC = \dfrac{1}{2}\left|\det\begin{pmatrix} 4 & 3 \\ 3 & -1 \end{pmatrix}\right| = \dfrac{1}{2}|-13| = \dfrac{13}{2}.$

Thus, area of quadrilateral $OABC = \dfrac{18}{2} = 9.$

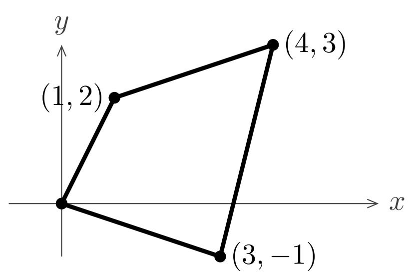
Determinants and areas

1. a) Compute $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$.

b) Compute $\begin{vmatrix} 1 & -2 \\ -3 & 4 \end{vmatrix}$.

c) Compute $\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix}$.

2. Find the area of the quadrilateral shown.

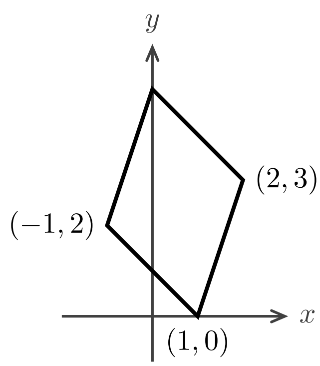


Areas and Determinants

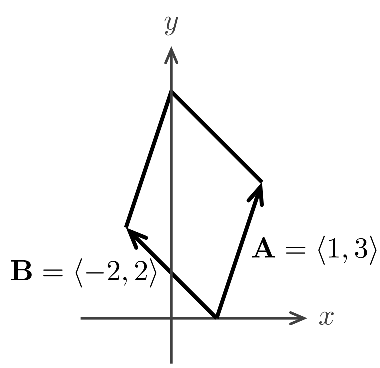
1. Compute $\begin{vmatrix} 6 & 5 \\ 1 & 2 \end{vmatrix}$.

Answer: $\begin{vmatrix} 6 & 5 \\ 1 & 2 \end{vmatrix} = 6 \cdot 2 - 5 \cdot 1 = 7.$

2. Compute the area of the parallelogram shown.



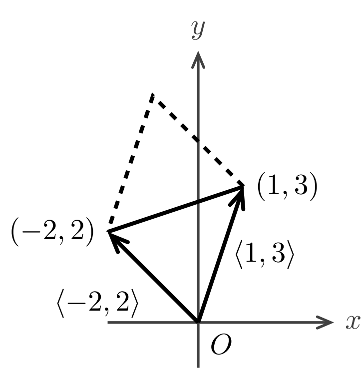
Answer: The area is given by the determinant of the vectors determining the parallelogram.



$$\text{Area} = |\det(\mathbf{A}, \mathbf{B})| = \left| \det \begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix} \right| = 2 + 6 = 8.$$

3. Find the area of the triangle with vertices $(0, 0)$, $(-2, 2)$ and $(1, 3)$.

Answer: The triangle is half a parallelogram. So the area is $\frac{1}{2} \left| \det \begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix} \right| = 2.$



Determinants 1.

Given a square array A of numbers, we associate with it a number called the **determinant** of A, and written either $\det(A)$, or $|A|$. For 2×2

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc. \tag{1}$$

Do not memorize this as a formula — learn instead the pattern which gives the terms. The 2×2 case is easy: the product of the elements on one diagonal (the "main diagonal"), minus the product of the elements on the other (the "antidiagonal").

Below we will see how to compute 3×3 determinants $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$. First, try the following 2×2 example on your own, then check your work against the solution.

Example 1.1 Evaluate $\begin{vmatrix} 1 & -2 \\ -1 & 3 \end{vmatrix}$ using (1).

Solution. Using the same order as in (1), we get $12 + (-8) + 1 - 6 - 8 - (-2) \;=\; -7$.

Important facts about $|A|$:

D-1. $|A|$ is multiplied by -1 if we interchange two rows or two columns.

D-2. $|A| = 0$ if one row or column is all zero, or if two rows or two columns are the same.

D-3. $|A|$ is multiplied by c, if every element of some row or column is multiplied by c.

D-4. The value of $|A|$ is unchanged if we add to one row (or column) a constant multiple of another row (resp. column).

All of these facts are easy to check for 2×2 determinants from the formula (1); from this, their truth also for 3×3 determinants will follow from the Laplace expansion.

Though the letters $a, b, c, \ldots$ can be used for very small determinants, they can't for larger ones; it's important early on to get used to the standard notation for the entries of determinants. This is what the common software packages and the literature use. The determinants of order two and three would be written respectively

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \qquad\qquad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

In general, the **ij-entry**, written a_{ij}, is the number in the i-th row and j-th column.

Its **ij-minor**, written $|A_{ij}|$, is the determinant that's left after deleting from $|A|$ the row and column containing a_{ij}.

Its **ij-cofactor**, written here A_{ij}, is given as a formula by $A_{ij} = (-1)^{i+j}|A_{ij}|$. For a 3×3 determinant, it is easier to think of it this way: we put $+$ or $-$ in front of the ij-minor, according to whether $+$ or $-$ occurs in the ij-position in the checkerboard pattern

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}. \tag{2}$$

Example 1.2 $\quad |A| = \begin{vmatrix} 1 & 0 & 3 \\ 1 & 2 & -1 \\ 2 & 1 & -1 \end{vmatrix}$. $\quad$ Find $|A_{12}|,\ A_{12},\ |A_{22}|,\ A_{22}$.

Solution. $|A_{12}| = \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = 1,\quad A_{12} = -1. \qquad |A_{22}| = \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} = -7,\quad A_{22} = -7.$

Laplace expansion by cofactors

This is another way to evaluate a determinant; we give the rule for a 3×3. It generalizes easily to an $n\times n$ determinant.

Select any row (or column) of the determinant. Multiply each entry a_{ij} in that row (or column) by its cofactor A_{ij}, and add the three resulting numbers; you get the value of the determinant.

As practice with notation, here is the formula for the Laplace expansion of a third order (i.e., a 3×3) determinant using the cofactors of the first row:

$$a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \ = \ |A| \tag{3}$$

and the formula using the cofactors of the j-th column:

$$a_{1j}A_{1j} + a_{2j}A_{2j} + a_{3j}A_{3j} \ = \ |A| \tag{4}$$

Example 1.3 Evaluate the determinant in Example 1.2 using the Laplace expansions by the first row and by the second column, and check by also using (1).

Solution. The Laplace expansion by the first row is

$$\begin{vmatrix} 1 & 0 & 3 \\ 1 & 2 & -1 \\ 2 & 1 & -1 \end{vmatrix} \ = \ 1\cdot\begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} - 0\cdot\begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} + 3\cdot\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 1\cdot(-1) - 0\cdot 1 + 3\cdot(-3) = -10.$$

The Laplace expansion by the second column would be

$$\begin{vmatrix} 1 & 0 & 3 \\ 1 & 2 & -1 \\ 2 & 1 & -1 \end{vmatrix} \ = \ -0\cdot\begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} + 2\cdot\begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} - 1\cdot\begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} = 0 + 2\cdot(-7) - 1\cdot(-4) = -10.$$

Checking by (1), we have $|A| \ = \ -2 + 0 + 3 - 12 - 0 - (-1) \ = \ -10.$

Example 1.4 Show the Laplace expansion by the first row gives the following formula (which you may have seen before).

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + chd - gec - hfa - ibd$$

Solution. We have

$$\begin{aligned} \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \cdot \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ &= a(ei - fh) - b(di - fg) + c(dh - eg), \end{aligned}$$

whose six terms agree with the six terms on the right of the formula above.

(A similar argument can be made for the Laplace expansion by any row or column.)

For $n \times n$ determinants, the **minor** $|A_{ij}|$ of the entry a_{ij} is defined to be the determinant obtained by deleting the i-th row and j-th column; the **cofactor** A_{ij} is the minor, prefixed by a + or − sign according to the natural generalization of the checkerboard pattern (2). Then the Laplace expansion by the i-th row would be

$$|A| \;=\; a_{i1}A_{i1} + a_{i2}A_{i2} + \ldots + a_{in}A_{in} \;.$$

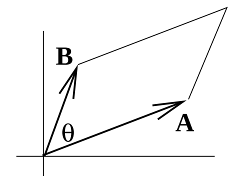
This is an inductive calculation — it expresses the determinant of order n in terms of determinants of order $n-1$. Thus, since we can calculate determinants of order 3, it allows us to calculate determinants of order 4; then determinants of order 5, and so on. If we take for definiteness $i = 1$, then the above Laplace expansion formula can be used as the basis of an inductive definition of the $n \times n$ determinant.

Example 1.5 Evaluate $\begin{vmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 1 & 4 \\ -1 & 4 & 1 & 0 \\ 0 & 4 & 2 & -1 \end{vmatrix}$ by its Laplace expansion by the first row.

Solution. $1 \cdot \begin{vmatrix} -1 & 1 & 4 \\ 4 & 1 & 0 \\ 4 & 2 & -1 \end{vmatrix} \;-\; 0 \cdot A_{12} \;+\; 2 \cdot \begin{vmatrix} 2 & -1 & 4 \\ -1 & 4 & 0 \\ 0 & 4 & -1 \end{vmatrix} \;-\; 3 \cdot \begin{vmatrix} 2 & -1 & 1 \\ -1 & 4 & 1 \\ 0 & 4 & 2 \end{vmatrix}$

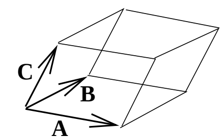
$$= \; 1 \cdot 21 \;+\; 2 \cdot (-23) \;-\; 3 \cdot 2 \;=\; -31.$$

Determinants 2. Area and Volume



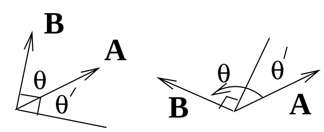
Area and volume interpretation of the determinant:

(1) $$\pm \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \text{area of parallelogram with edges } \mathbf{A} = (a_1, a_2),\ \mathbf{B} = (b_1, b_2).$$



(2) $$\pm \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \text{volume of parallelepiped with edges row-vectors } \mathbf{A}, \mathbf{B}, \mathbf{C}.$$

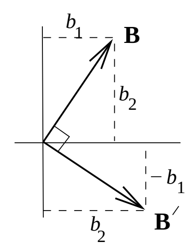
In each case, choose the sign which makes the left side non-negative.



Proof of (1). We begin with two preliminary observations.

Let θ be the positive angle from $\mathbf{A}$ to $\mathbf{B}$; we assume it is $< \pi$, so that $\mathbf{A}$ and $\mathbf{B}$ have the general positions illustrated.

Let $\theta' = \pi/2 - \theta$, as illustrated. Then $\cos\theta' = \sin\theta$.



Draw the vector $\mathbf{B}'$ obtained by rotating $\mathbf{B}$ to the right by $\pi/2$. The picture shows that $\mathbf{B}' = (b_2, -b_1)$, and $|\mathbf{B}'| = |\mathbf{B}|$.

To prove (1) now, we have a standard formula of Euclidean geometry,

$$\begin{aligned} \text{area of parallelogram} &= |\mathbf{A}||\mathbf{B}|\sin\theta \\ &= |\mathbf{A}||\mathbf{B}'|\cos\theta', && \text{by the above observations} \\ &= \mathbf{A}\cdot\mathbf{B}', && \text{by the geometric definition of dot product} \\ &= a_1b_2 - a_2b_1 && \text{by the formula for } \mathbf{B}' \end{aligned}$$

This proves the area interpretation (1) if $\mathbf{A}$ and $\mathbf{B}$ have the position shown. If their positions are reversed, then the area is the same, but the sign of the determinant is changed, so the formula has to read,

$$\text{area of parallelogram} = \pm \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}, \quad \text{whichever sign makes the right side } \geq 0.$$

The proof of the analogous volume formula (2) will be made when we study the scalar triple product $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}$.

Generalizing (1) and (2), $n \times n$ determinants can be interpreted as the hypervolume in n-space of a n-dimensional parallelotope.

Computing 3x3 determinants

1. a) Compute $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$. b) Compute $\begin{vmatrix} 2 & 1 & -5 \\ 0 & 0 & 4 \\ 3 & 1 & 2 \end{vmatrix}$.

Answer: a) Using Laplace expansion along the first row (and remembering to put minus signs at the appropriate place) we get

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1\cdot\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\cdot\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\cdot\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = (45-48)-2(36-42)+3(32-35) = -3+12-9 = 0.$$

To show everything we wrote out the arithmetic in detail. You would not need to show all this. It's usually easier to do the simple arithmetic in your head –but, never be afraid of writing out the details. The shorter answer would look something like

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = -3+12-9 = 0.$$

b) Because of the zeros we'll use Laplace expansion along the second row. (The signs are $-\ +\ -$, but the first two terms are 0.)

$$\begin{vmatrix} 2 & 1 & -5 \\ 0 & 0 & 4 \\ 3 & 1 & 2 \end{vmatrix} = -4\cdot\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = -4(2-3) = 4.$$

2. Compute $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 6 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 5 & 1 & 2 \end{vmatrix}$

Answer: In principle a 4x4 matrix requires us to compute four 3x3 determinants. Here we can expand along the second row so we'll only have one non-zero term. The signs for the second row are $-\ +\ -\ +$.

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 6 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 5 & 1 & 2 \end{vmatrix} = -6\begin{vmatrix} 1 & 2 & 4 \\ 1 & 0 & 0 \\ 1 & 5 & 2 \end{vmatrix}$$

This 3x3 determinant can be expanded along its second row.

$$\begin{vmatrix} 1 & 2 & 4 \\ 1 & 0 & 0 \\ 1 & 5 & 2 \end{vmatrix} = -(-16) = 16.$$

So the original determinant is $-6\cdot 16 = -96$.

Computing 3x3 determinants

1. a) Compute $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$. b) Compute $\begin{vmatrix} 2 & 1 & -5 \\ 0 & 0 & 4 \\ 3 & 1 & 2 \end{vmatrix}$.

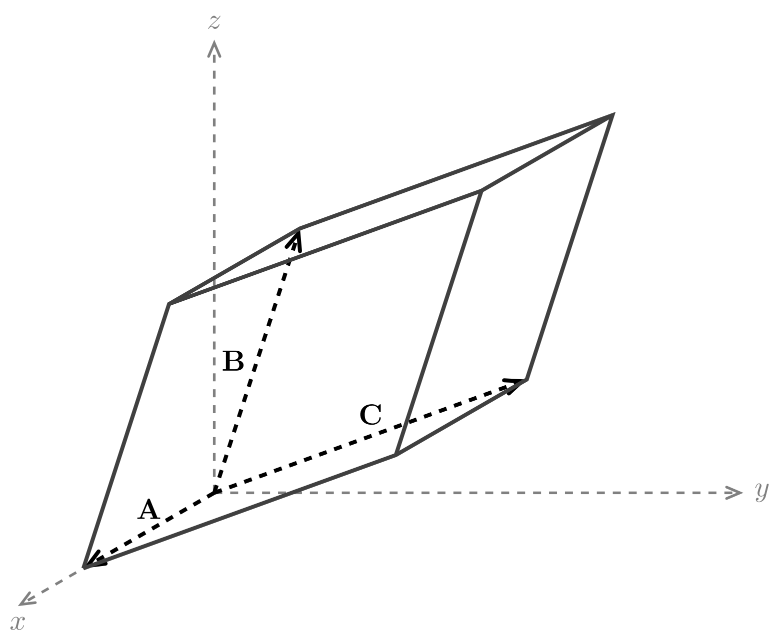
2. Compute $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 6 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 5 & 1 & 2 \end{vmatrix}$

Volumes and determinants

1. a) Find the volume of the parallelepiped with edges given by the origin vectors $\langle 1,2,4\rangle$, $\langle 2,0,0\rangle$, $\langle 1,5,2\rangle$

Answer: The figure below shows the box.

The volume is $|\det(\mathbf{A},\mathbf{B},\mathbf{C})| = \left|\det\begin{pmatrix} 1 & 2 & 4 \\ 2 & 0 & 0 \\ 1 & 5 & 2 \end{pmatrix}\right| = |-2\cdot(-16)| = 32.$



2. We know $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0.$

What does this say about the origin vectors $\langle 1,2,3\rangle$, $\langle 4,5,6\rangle$ and $\langle 7,8,9\rangle$?

Answer: Call the three vectors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$. Since $\det(\mathbf{A},\mathbf{B},\mathbf{C}) = 0$ the volume of the parallelepiped with these vectors as edges is 0. This means all three origin vectors lie in a plane.

To see this consider the figure in problem 1. It shows the opposite case, when the vectors are not in a plane the resulting parallelepiped is really three dimensional and has non-zero volume.

Volumes and determinants

1. a) Find the volume of the parallelepiped with edges given by the origin vectors $\langle 1, 2, 4\rangle$, $\langle 2, 0, 0\rangle$, $\langle 1, 5, 2\rangle$

2. We know $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0.$

What does this say about the origin vectors $\langle 1, 2, 3\rangle$, $\langle 4, 5, 6\rangle$ and $\langle 7, 8, 9\rangle$?

Cross Product

The cross product is another way of multiplying two vectors. (The name comes from the symbol used to indicate the product.) Because the result of this multiplication is *another vector* it is also called the *vector product*.

As usual, there is an algebraic and a geometric way to describe the cross product. We'll define it algebraically and then move to the geometric description.

Determinant definition for cross product

For the vectors $\mathbf{A} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{B} = \langle b_1, b_2, b_3 \rangle$ we define the cross product by the following formula

$$\begin{aligned}
\mathbf{A} \times \mathbf{B} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\
&= \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\
&= (a_2 b_3 - a_3 b_2)\mathbf{i} + (a_3 b_1 - a_1 b_3)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k} \\
&= \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle.
\end{aligned}$$

The bottom three equations above are easily seen to be equivalent and should be taken as the definition of the cross product. The top line is technically flawed because we are not really allowed to use vectors as entries in a determinant. Nonetheless is is an excellent way to remember how to compute the cross product.

Example: $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 0 \\ 3 & -2 & 0 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix} \mathbf{k} = -13\,\mathbf{k}$

Example: Compute $\mathbf{i} \times \mathbf{j}$.
<u>Answer:</u> $\mathbf{i} = \langle 1, 0, 0 \rangle$ and $\mathbf{j} = \langle 0, 1, 0 \rangle$ therefore

$$\mathbf{i} \times \mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \mathbf{i}(0) - \mathbf{j}(0) + \mathbf{k}(1) = \mathbf{k}.$$

Algebraic facts: (these follow easily from properties of determinant).

1. $\mathbf{A} \times \mathbf{A} = \mathbf{0}$
2. Anti-commutivity: $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$
3. Distributive law: $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$
4. Non-associativity: $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \neq \mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ (example in a moment).

For the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ we have

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}.$$

Example: (non-associativity) $(\mathbf{i} \times \mathbf{j}) \times \mathbf{j} = -\mathbf{i}$ but $\mathbf{i} \times (\mathbf{j} \times \mathbf{j}) = 0$.

Example: It is possible to compute a cross product using the algebraic facts and the known products of **i**, **j** and **k**. For example,

$$(2\mathbf{i}+3\mathbf{j})\times(3\mathbf{i}-2\mathbf{j}) = (6\mathbf{i}\times\mathbf{i}) - (4\mathbf{i}\times\mathbf{j}) + (9\mathbf{j}\times\mathbf{i}) - (6\mathbf{j}\times\mathbf{j}) = -13\mathbf{k}.$$

The first equation follows from the distributive law. In the second, we used $\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = 0$ (algebraic fact 1), $\mathbf{i}\times\mathbf{j} = \mathbf{k}$ (computed above) and $\mathbf{j}\times\mathbf{i} = -\mathbf{k}$ (anti-commutivity).

Geometric description

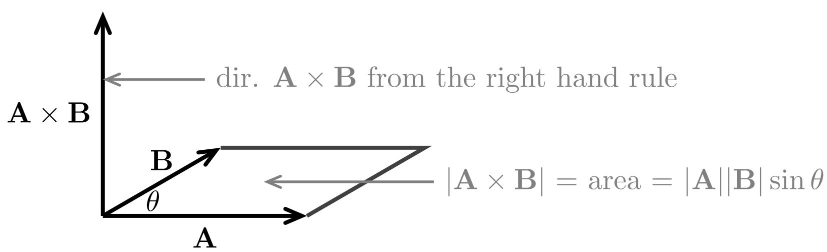
To describe the cross product geometrically we need to describe its magnitude and direction. This is done in the following theorem.

Theorem: The magnitude of $\mathbf{A}\times\mathbf{B}$ is

$$\begin{aligned}|\mathbf{A}\times\mathbf{B}| &= |\mathbf{A}||\mathbf{B}|\sin\theta, \text{ where } \theta \text{ is the angle between them} \\ &= \text{area of the parallelogram spanned by } \mathbf{A} \text{ and } \mathbf{B}.\end{aligned}$$

The direction of $\mathbf{A}\times\mathbf{B}$ is determined as follows.

$\mathbf{A}\times\mathbf{B}$ is perpendicular to the plane of **A** and **B**. In the figure below there are two directions perpendicular to the plane –up and down. The choice is made by the *right hand rule.* This rule says to take your right hand and point your fingers in the direction of **A** so that they curl towards **B**; then your thumb points in the direction of $\mathbf{A}\times\mathbf{B}$.

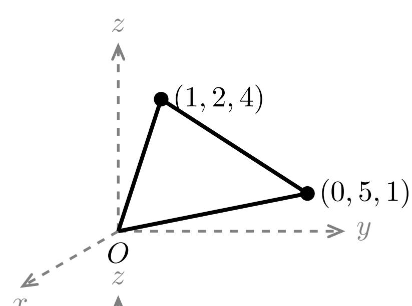


We will not go through the proof of this theorem. It makes use of the Lagrange identity

$$|\mathbf{A}\times\mathbf{B}|^2 = |\mathbf{A}|^2|\mathbf{B}|^2 - (\mathbf{A}\cdot\mathbf{B})^2.$$

This identity is easily show by expanding both sides using components.

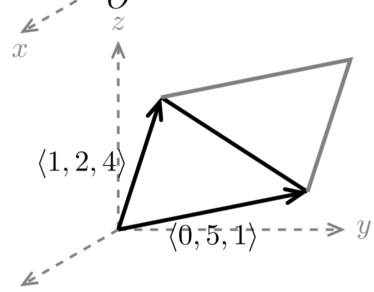
Example: Find the area of the triangle shown.



<u>**Answer:**</u>

The area of the triangle is half the area of the parallelogram (see figure).

So, area triangle $= \dfrac{1}{2}|\langle 1,2,4\rangle \times \langle 0,5,1\rangle|$.



$$\langle 1,2,4\rangle \times \langle 0,5,1\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 4 \\ 0 & 5 & 1 \end{vmatrix} = \mathbf{i}(-18) - \mathbf{j} + 5\mathbf{k}.$$

Area triangle $= \dfrac{1}{2}\sqrt{18^2+1^2+5^2} = \dfrac{1}{2}\sqrt{350}.$

DON'T FORGET THE GEOMETRY -it will be used to solve problems.

Cross product

1. a) Compute $\langle 1,3,1\rangle \times \langle 2,-1,5\rangle$.

b) Compute $(\mathbf{i}+2\mathbf{j}) \times (2\mathbf{i}-3\mathbf{j})$.

Answer: a) We use the determinant method:

$$\langle 1,3,1\rangle \times \langle 2,-1,5\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 1 \\ 2 & -1 & 5 \end{vmatrix} = \mathbf{i}(16) - \mathbf{j}(3) + k(-7) = \langle 16,-3,-7\rangle$$

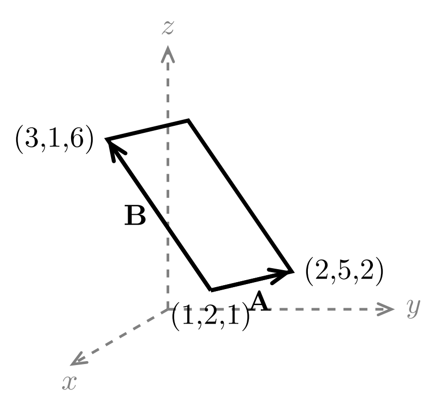
b) Using determinants we get

$$(\mathbf{i}+2\mathbf{j}) \times (2\mathbf{i}-3\mathbf{j}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 0 \\ 2 & -3 & 0 \end{vmatrix} = \mathbf{i}(0) - \mathbf{j}(0) + \mathbf{k}(-7) = -7\mathbf{k}.$$

Multiplying directly and using $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ etc we get

$$(\mathbf{i}+2\mathbf{j}) \times (2\mathbf{i}-3\mathbf{j}) = \mathbf{i}\times\mathbf{i} - 3\mathbf{i}\times\mathbf{j} + 4\mathbf{j}\times\mathbf{i} - 6\mathbf{j}\times\mathbf{j} = \mathbf{0} - 3\mathbf{k} - 4\mathbf{k} - 6\cdot\mathbf{0} = -7\mathbf{k}.$$

2. Find the area of the parallelogram shown.



Answer: The area is $|\mathbf{A}\times\mathbf{B}|$, where $\mathbf{A}$ and $\mathbf{B}$ are the vectors along two adjacent edges of the parallelogram. These vectors are

$$\mathbf{A} = \langle 2,5,2\rangle - \langle 1,2,1\rangle = \langle 1,3,1\rangle \quad \text{and } \mathbf{B} = \langle 3,1,6\rangle - \langle 1,2,1\rangle = \langle 2,-1,5\rangle.$$

We computed this cross product in problem (1a). So,

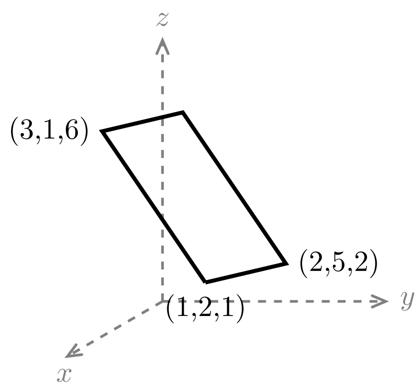
$$\text{area } = |\langle 16,-3,-7\rangle| = \sqrt{256+9+49} = \sqrt{314}.$$

Cross product

1. a) Compute $\langle 1, 3, 1\rangle \times \langle 2, -1, 5\rangle$.

b) Compute $(\mathbf{i} + 2\mathbf{j}) \times (2\mathbf{i} - 3\mathbf{j})$.

2. Find the area of the parallelogram shown.



Equation of a plane

1. Find the equation of the plane containing the three points $P_1 = (1, 0, 1)$, $P_2 = (0, 1, 1)$, $P_3 = (1, 1, 0)$.

<u>Answer:</u> This problem is identical (with changed numbers) to the worked example we just saw.

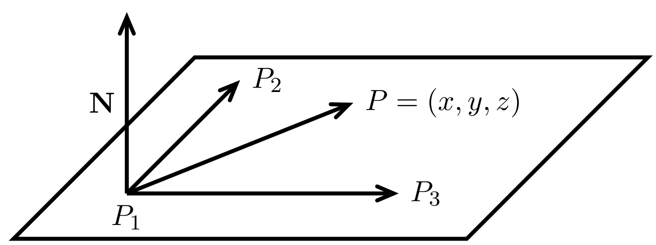
The vectors $\overrightarrow{\mathbf{P_1P_2}}$ and $\overrightarrow{\mathbf{P_1P_3}}$ are in the plane, so

$$\mathbf{N} = \overrightarrow{\mathbf{P_1P_2}} \times \overrightarrow{\mathbf{P_1P_3}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ 0 & 1 & -1 \end{vmatrix} = \mathbf{i}(-1) - \mathbf{j}(1) + \mathbf{k}(-1) = \langle -1, -1, -1 \rangle.$$

is orthogonal to the plane.

Now for any point $P = (x, y, z)$ in the plane, the vector $\overrightarrow{\mathbf{P_1P}}$ is also in the plane and is therefore orthogonal to $\mathbf{N}$. Expressing this with the dot product we get

$$\begin{aligned} & \mathbf{N} \cdot \overrightarrow{\mathbf{P_1P}} = 0 \\ \Leftrightarrow \quad & \langle -1, -1, -1 \rangle \cdot \langle x - 1, y, z - 1 \rangle = 0 \\ \Leftrightarrow \quad & -(x - 1) - y - (z - 1) = 0 \\ \Leftrightarrow \quad & x + y + z = 2. \end{aligned}$$



Equation of a plane

1. Find the equation of the plane containing the three points $P_1 = (1, 0, 1)$, $P_2 = (0, 1, 1)$, $P_3 = (1, 1, 0)$.

Equation of a Plane

1. Later we will return to the topic of planes in more detail. Here we will content ourself with one example.

Find the equation of the plane containing the three points $P_1 = (1, 3, 1)$, $P_2 = (1, 2, 2)$, $P_3 = (2, 3, 3)$.

<u>**Answer:**</u>

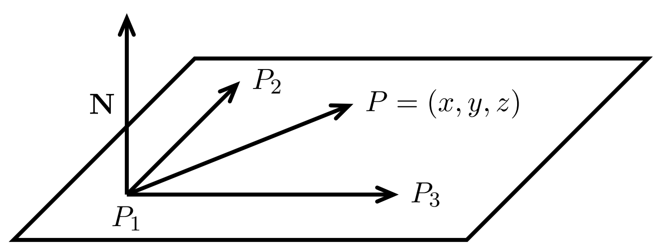
The vectors $\overrightarrow{\mathbf{P_1P_2}}$ and $\overrightarrow{\mathbf{P_1P_3}}$ are in the plane, so

$$\mathbf{N} = \overrightarrow{\mathbf{P_1P_2}} \times \overrightarrow{\mathbf{P_1P_3}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = \mathbf{i}(-2) - \mathbf{j}(-1) + \mathbf{k}(1) = \langle -2, 1, 1 \rangle.$$

is orthogonal to the plane.

Now for any point $P = (x, y, z)$ in the plane, the vector $\overrightarrow{\mathbf{P_1P}}$ is also in the plane and is therefore orthogonal to $\mathbf{N}$. Expressing this with the dot product we get

$$\begin{aligned} & \mathbf{N} \cdot \overrightarrow{\mathbf{P_1P}} = 0 \\ \Leftrightarrow \quad & \langle -2, 1, 1 \rangle \cdot \langle x - 1, y - 3, z - 1 \rangle = 0 \\ \Leftrightarrow \quad & -2(x - 1) + (y - 3) + (z - 1) = 0 \\ \Leftrightarrow \quad & -2x + y + z = 2. \end{aligned}$$



The equation of the plane is $-2x + y + z = 2$. You should check that the three points P_1, P_2, P_3 do, in fact, satisfy this equation.

The standard terminology for the vector $\mathbf{N}$ is to call it a *normal* to the plane.