Green's Theorem

Green's Theorem

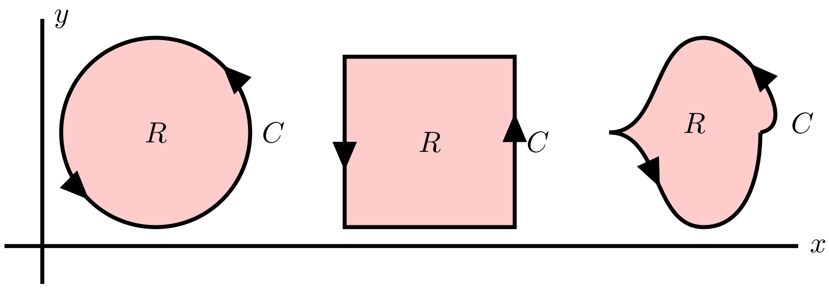
We start with the ingredients for Green's theorem.

(i) C a *simple* closed curve (simple means it never intersects itself)

(ii) R the interior of C.

We also require that C must be *positively oriented*, that is, it must be traversed so its interior is on the left as you move in around the curve. Finally we require that C be *piecewise smooth*. This means it is a smooth curve with, possibly a finite number of corners.

Here are some examples.



Green's Theorem

With the above ingredients for a vector field $\mathbf{F} = \langle M, N \rangle$ we have

$$\oint_C M\,dx + N\,dy = \iint_R N_x - M_y\,dA.$$

We call $N_x - M_y$ the two dimensional curl and denote it $\operatorname{curl}\mathbf{F}$.

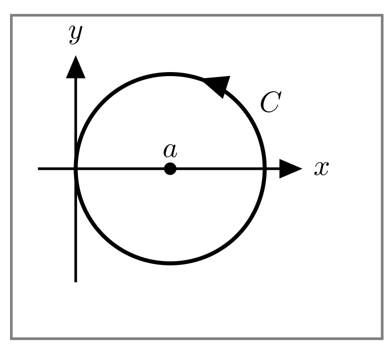
We can write also Green's theorem as

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \operatorname{curl}\mathbf{F}\,dA.$$

Example 1: (use the right hand side (RHS) to find the left hand side (LHS))

Use Green's Theorem to compute

$I = \oint_C 3x^2y^2\,dx + 2x^2(1+xy)\,dy$ where C is the circle shown.



By Green's Theorem $I = \iint_R 6x^2y + 4x - 6x^2y\,dA = 4\iint_R x\,dA.$

We could compute this directly, but we know $x_{cm} = \dfrac{1}{A}\iint_R x\,dA = a$

$\Rightarrow \iint_R x\,dA = \pi a^3 \Rightarrow \boxed{I = 4\pi a^3.}$

Example 2: (Use the LHS to find the RHS.)

Use Green's Theorem to find the area under one arch of the cycloid

$$x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta).$$

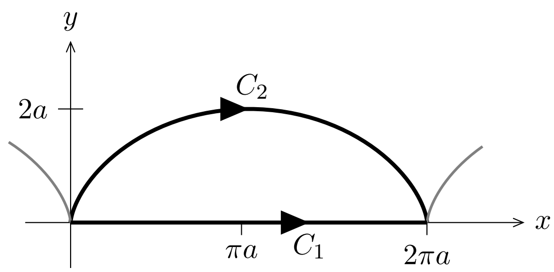
The picture shows the curve $C = C_1 - C_2$ surrounding the area we want to find.(Note the minus sign on C_2.)

By Green's Theorem,

$$\oint_C -y\,dx = \iint_R dA = \text{area}.$$

Thus,

$$\text{area} = \oint_{C_1 - C_2} -y\,dx = \int_{C_1} 0 \cdot dx - \int_{C_2} -y\,dx = \int_0^{2\pi} a^2(1-\cos\theta)^2\,d\theta = 3\pi a^2.$$



18.02SC Multivariable Calculus
Fall 2010

Using Green's Theorem

1. Show that $\oint_C -x^2y\,dx + xy^2\,dy > 0$ for all simple closed curves C.

Answer:

If R is the interior of C, then Green's Theorem tells us:

$$\oint_C M\,dx + N\,dy = \iint_R N_x - M_y\,dA.$$

Here, $M = -x^2y$ and $N = xy^2$, so $N_x - M_y = y^2 - (-x^2) = x^2 + y^2$. In other words, $N_x - M_y$ is the square of the distance from (x, y) to the origin. This distance is always positive, so the integral of this value over any non-empty region in the plane will be positive.

We conclude that $\oint_C -x^2y\,dx + xy^2\,dy = \iint_R (x^2 + y^2)\,dA > 0$.

2. Let $\mathbf{F} = 2y\mathbf{i} + x\mathbf{j}$ and let C be the positively oriented unit circle. Compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$ directly and by Green's theorem.

Answer: Using Green's theorem: $N_x - M_y = 1 - 2 = -1$, so $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (-1)\,dA = -\pi$.

Directly (using half-angle formulas): We parametrize C by $x = \cos t$, $y = \sin t$, $0 \leq t \leq 2\pi$. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} -2\sin^2 t + \cos^2 t\,dt = \int_0^{2\pi} -(1 - \cos 2t) + \frac{1 + \cos 2t}{2}\,dt = -\pi.$$

Using Green's Theorem

1. Show that $\oint_C -x^2y\,dx + xy^2\,dy > 0$ for all simple closed curves C.

2. Let $\mathbf{F} = 2y\mathbf{i} + x\mathbf{j}$ and let C be the positively oriented unit circle. Compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$ directly and by Green's theorem.

Problems: Green's Theorem

Calculate $\oint_C -x^2y\,dx + xy^2\,dy$, where C is the circle of radius 2 centered on the origin.

Answer: Green's theorem tells us that if $\mathbf{F} = \langle M, N\rangle$ and C is a positively oriented simple closed curve, then

$$\oint_C M\,dx + N\,dy = \iint_R N_x - M_y\,dA.$$

We let $M = -x^2y$ and $N = xy^2$ to get:

$$\begin{aligned}
\oint_C -x^2y\,dx + xy^2\,dy &= \iint_R y^2 - (-x^2)\,dA \\
&= \iint_R x^2 + y^2\,dA \\
&= \int_0^{2\pi}\int_0^2 r^2\,r\,dr\,d\theta \\
&= \int_0^{2\pi} \frac{8}{3} d\theta \\
&= \frac{16\pi}{3}.
\end{aligned}$$

This result is 4/3 times the area $\iint_R 1\,dA$ of the circle, and so is a plausible answer.

Green's Theorem and Conservative Fields

We can use Green's theorem to prove the following theorem.

Theorem

Suppose $\mathbf{F} = \langle M, N \rangle$ is a vector field which is defined and with continuous partial derivatives for all (x, y). Then

$$\mathbf{F} \text{ is conservative } \Leftrightarrow N_x = M_y \text{ or } N_x - M_y = \operatorname{curl} \mathbf{F} = 0.$$

Proof

This is a consequence of Green's theorem. First, suppose $\mathbf{F}$ is conservative, i.e., its work integral is 0 along all simple closed curves. Then Green's theorem says

$$0 = \oint_C \mathbf{F} \cdot d\mathbf{r} = \int\int_R \operatorname{curl} \mathbf{F} \, dA.$$

The only way for the integral of $\operatorname{curl} \mathbf{F}$ to be 0 over all regions R is if $\operatorname{curl} \mathbf{F}$ itself is 0. This implies $N_x = M_y$ as claimed.

For the converse, assume $N_x = M_y$. Then, for any closed curve C surrounding a region R, Green's theorem says,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int\int_R N_x - M_y \, dA = 0.$$

Therefore, the work integral of $\mathbf{F}$ is 0 over any closed curve, which means $\mathbf{F}$ is conservative.

Be careful, the requirement that $\mathbf{F}$ is defined and differentiable everywhere is important. The problem following this note will give an example of a nonconservative field with $\operatorname{curl} \mathbf{F} = 0$.

Later we will learn how to handle fields that aren't defined everywhere.

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Identifying Gradient Fields and Exact Differentials

1. Compute the curl of the tangential vector field $\mathbf{F} = \left\langle -\frac{y}{r^2}, \frac{x}{r^2} \right\rangle$.

Answer: We know that if $\mathbf{F} = \langle M, N \rangle$ then curl$\mathbf{F} = N_x - M_y$. In this case, $M = -\frac{y}{r^2}$ and $N = \frac{x}{r^2}$. Applying the chain rule and differentiating $r^2 = x^2 + y^2$ as needed, we get $N_x = \dfrac{y^2 - x^2}{r^4}$ and $M_y = \dfrac{y^2 - x^2}{r^4}$. Thus, curl$\mathbf{F} = 0$.

2. Show that $\mathbf{F}$ is not conservative by computing $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the unit circle.

Answer: Note: since $\mathbf{F}$ is not defined at $(0, 0)$, curl$\mathbf{F} = 0$ does not necessarily mean $\mathbf{F}$ is conservative.

We parametrize C by $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$. Then $dx = -\sin t\, dt$ and $dy = \cos t\, dt$.

$$\begin{aligned}
\oint_C \mathbf{F} \cdot d\mathbf{r} &= \oint_C M\, dx + N\, dy \\
&= \int_0^{2\pi} -\frac{\sin t}{1^2}(-\sin t)\, dt + \frac{\cos t}{1^2} \cos t\, dt \\
&= 2\pi
\end{aligned}$$

If $\mathbf{F}$ were conservative its line integral over a simple, closed curve (like the unit circle) would be zero. Since this is not the case, $\mathbf{F}$ must not be conservative.

3. Why do you think we refer to $\mathbf{F}$ as a "tangential" vector field?

Answer: Every vector in $\mathbf{F}$ is tangential to some circle centered at the origin. You can see this because $\mathbf{F}$ is clearly orthogonal to the "radial" vector field $\langle x, y \rangle$.

4 In polar coordinates, $\theta(x, y) = \tan^{-1} y/x$. Show that $\mathbf{F} = \nabla\theta$.

Answer: We wish to show that $M = \theta_x$ and $N = \theta_y$.

$$\theta_x = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \frac{-y}{x^2} = -\frac{y}{r^2} = M.$$

$$\theta_y = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \frac{1}{x} = \frac{x}{r^2} = N.$$

18.02SC Multivariable Calculus
Fall 2010

Identifying Gradient Fields and Exact Differentials

1. Compute the curl of the tangential vector field $\mathbf{F} = \left\langle -\frac{y}{r^2}, \frac{x}{r^2} \right\rangle$.

2. Show that $\mathbf{F}$ is not conservative by computing $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the unit circle.

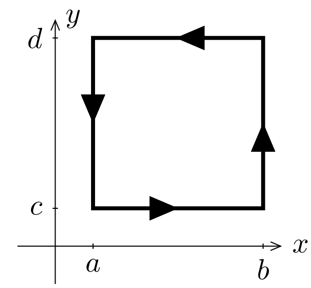
3. Why do you think we refer to $\mathbf{F}$ as a "tangential" vector field?

4 In polar coordinates, $\theta(x, y) = \tan^{-1} y/x$. Show that $\mathbf{F} = \nabla\theta$.

Green's Theorem: Sketch of Proof

Green's Theorem: $\displaystyle\oint_C M\,dx + N\,dy = \iint_R N_x - M_y\,dA.$

Proof:
i) First we'll work on a rectangle. Later we'll use a lot of rectangles to approximate an arbitrary region.



ii) We'll only do $\displaystyle\oint_C M\,dx$ ($\displaystyle\oint_C N\,dy$ is similar).

By direct calculation the right hand side of Green's Theorem

$$\iint_R -\frac{\partial M}{\partial y}\,dA = \int_a^b \int_c^d -\frac{\partial M}{\partial y}\,dy\,dx.$$

Inner integral: $\quad -M(x,y)|_c^d = -M(x,d) + M(x,c)$

Outer integral: $\quad \displaystyle\iint_R -\frac{\partial M}{\partial y}\,dA = \int_a^b M(x,c) - M(x,d)\,dx.$

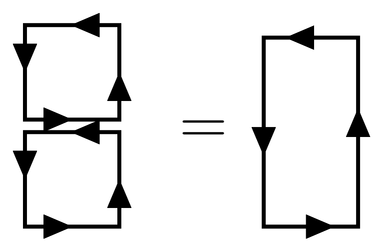
For the LHS we have

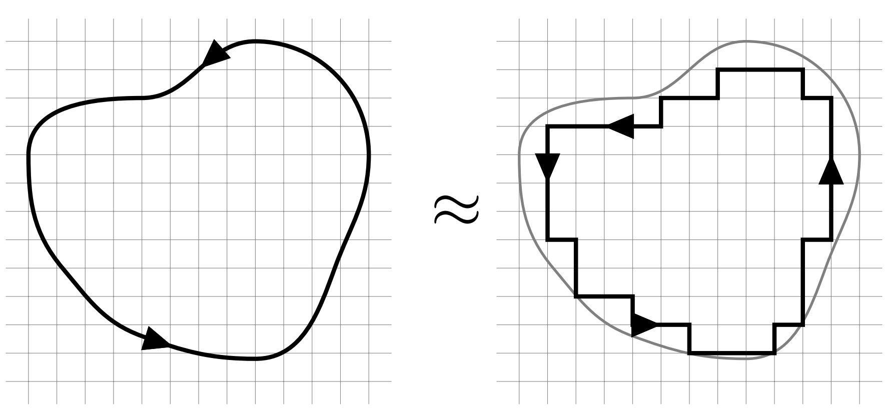
$$\begin{aligned}\oint_C M\,dx &= \int_{bottom} M\,dx + \int_{top} M\,dx \quad \text{(since } dx = 0 \text{ along the sides)}\\ &= \int_a^b M(x,c)\,dx + \int_b^a M(x,d)\,dx = \int_a^b M(x,c) - M(x,d)\,dx.\end{aligned}$$

So, for a rectangle, we have proved Green's Theorem by showing the two sides are the same.

In lecture, Professor Auroux divided R into "vertically simple regions". This proof instead approximates R by a collection of rectangles which are especially simple both vertically and horizontally.

For line integrals, when adding two rectangles with a common edge the common edges are traversed in opposite directions so the sum is just the line integral over the outside boundary. Similarly when adding a lot of rectangles: everything cancels except the outside boundary. This extends Green's Theorem on a rectangle to Green's Theorem on a sum of rectangles. Since any region can be approximated as closely as we want by a sum of rectangles, Green's Theorem must hold on arbitrary regions.





Problems: Green's Theorem and Area

1. Find M and N such that $\oint_C M\,dx + N\,dy$ equals the polar moment of inertia of a uniform density region in the plane with boundary C.

Answer: Let R be the region enclosed by C and ρ be the density of R. The polar moment of inertia is calculated by integrating the product mass times distance to the origin:

$$I = \iint_R dI = \iint_R r^2\,dm = \iint_R (x^2 + y^2) \cdot \rho\,dA.$$

Green's theorem now tells us that we're looking for functions M and N such that $N_x - M_y = \rho x^2 + \rho y^2$. The simplest choice is $N_x = \rho y^2$, $M_y = -\rho x^2$. This leads to $N = \rho x y^2$, $M = -\rho x^2 y$.

Use Green's theorem to check this answer:

$$\begin{aligned} \oint_C -\rho x^2 y\,dx + \rho x y^2\,dy &= \iint_R \rho y^2 - (-\rho x^2)\,dA \\ &= \iint_R r^2 \cdot \rho\,dA = I. \end{aligned}$$

Problems: Green's Theorem and Area

1. Find M and N such that $\oint_C M\,dx + N\,dy$ equals the polar moment of inertia of a uniform density region in the plane with boundary C.

Finding Area Using Line Integrals

Use a line integral (and Green's Theorem) to find the area of the unit circle.

Answer: Recall that Green's Theorem tells us $\displaystyle\oint_C M\,dx + N\,dy = \iint_R N_x - M_y\,dA$. To find the area of the unit circle we let $M = 0$ and $N = x$ to get $\iint_R 1\,dA = \oint_C x\,dy$.

We parametrize the circle by $x = \cos\theta$, $y = \sin\theta$, $0 < \theta \leq 2\pi$, so $x\,dy = \cos^2\theta\,d\theta$. Then

$$\begin{aligned}
\text{Area} &= \iint_R 1\,dA \\
&= \oint_C x\,dy \\
&= \int_0^{2\pi} \cos^2\theta\,d\theta \\
&= \int_0^{2\pi} \frac{1+\cos 2\theta}{2}\,d\theta \\
&= \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)\Bigg|_0^{2\pi} \\
&= \pi.
\end{aligned}$$

V3. Two-dimensional Flux

In this section and the next we give a different way of looking at Green's theorem which both shows its significance for flow fields and allows us to give an intuitive physical meaning for this rather mysterious equality between integrals.

We have seen that if $\mathbf{F}$ is a force field and C a directed curve, then

$$\text{(1)} \qquad \text{work done by } \mathbf{F} \text{ along } C \;=\; \int_C \mathbf{F}\cdot d\mathbf{r} \;=\; \int_C \mathbf{F}\cdot\mathbf{T}\,ds$$

In words, we are integrating $\mathbf{F}\cdot\mathbf{T}$, the *tangential component* of $\mathbf{F}$, along the curve C. In component notation, if $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$, then the above reads

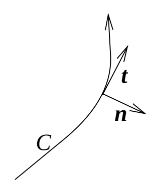
$$\text{(2)} \qquad \text{work} \;=\; \int_C M\,dx + N\,dy \;=\; \int_{t_0}^{t_1}\left(M\frac{dx}{dt} + N\frac{dy}{dt}\right)dt\ .$$

Analogously now, we may integrate $\mathbf{F}\cdot\mathbf{n}$, the *normal component* of $\mathbf{F}$ along C. To describe this, suppose the curve C is parametrized by the arclength s, increasing in the positive direction on C. The position vector for this parametrization and its corresponding tangent vector are given respectively by

$$\mathbf{r}(s) \;=\; x(s)\,\mathbf{i} + y(s)\,\mathbf{j}\,, \qquad\qquad \mathbf{t}(s) \;=\; \frac{dx}{ds}\,\mathbf{i} + \frac{dy}{ds}\,\mathbf{j}\,;$$

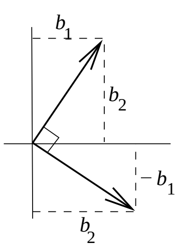
where we have used $\mathbf{t}$ instead of $\mathbf{T}$ since it is a unit vector— its length is 1, as one can see by dividing through by ds on both sides of

$$ds = \sqrt{(dx)^2 + (dy)^2}\ .$$



The unit normal vector $\mathbf{n}$ is the one shown in the picture, obtained by rotating $\mathbf{t}$ clockwise through a right angle.

> Unfortunately, this direction is opposite to the one customarily used in kinematics, where $\mathbf{t}$ and $\mathbf{n}$ form a right-handed coordinate system for motion along C. The choice of $\mathbf{n}$ depends therefore on the context of the problem; the choice we have given is the most natural for applying Green's theorem to flow problems.



The usual formula for rotating a vector clockwise by 90^o (see the figure) shows that

$$\text{(3)} \qquad \mathbf{n}(s) \;=\; \frac{dy}{ds}\,\mathbf{i} - \frac{dx}{ds}\,\mathbf{j}\,.$$

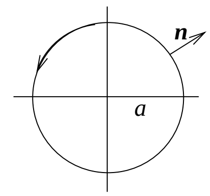
The line integral over C of the normal component $\mathbf{F}\cdot\mathbf{n}$ of the vector field $\mathbf{F}$ is called the **flux of F across** C. In symbols,

$$\text{(4)} \qquad \text{flux of } \mathbf{F} \text{ across } C \;=\; \int_C \mathbf{F}\cdot\mathbf{n}\,ds \;=\; \int_C \left(M\frac{dy}{ds} - N\frac{dx}{ds}\right)ds$$

In the notation of differentials, using (3) we write $\mathbf{n}\,ds = dy\,\mathbf{i} - dx\,\mathbf{j}\,,$ so that

$$\text{(5)} \qquad \text{flux of } F \text{ across } C \;=\; \int_C M\,dy - N\,dx \;=\; \int_C \left(M\frac{dy}{dt} - N\frac{dx}{dt}\right)dt\ ,$$

where $x(t),\ y(t)$ is any parametrization of C. We will need both (4) and (5).



Example 1. Calculate the flux of the field $\mathbf{F} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j}}{x^2+y^2}$ across a circle of radius a and center at the origin, by a) using (4); b) using (5).

Solutions. a) The field is directed radially outward, so that $\mathbf{F}$ and $\mathbf{n}$ have the same direction. (As usual, the circle is directed counterclockwise, which means that $\mathbf{n}$ points outward.) Therefore, at each point of the circle,

$$\mathbf{F}\cdot\mathbf{n} = |\mathbf{F}| = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{a}\ .$$

Therefore, by (4), we get

$$\text{flux} = \oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \oint_C \frac{1}{a}\,ds = 2\pi\ .$$

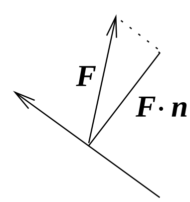
b) We can also get the same result by straightforward computation using a parametrization of the circle: $x = \cos t,\ y = \sin t$. Using this and (5) above,

$$\text{flux} = \oint_C \frac{x\,dy - y\,dx}{x^2+y^2} = \int_0^{2\pi} \frac{a^2\cos^2 t + a^2\sin^2 t}{a^2}\,dt = 2\pi\ .$$

The natural physical interpretation for flux calls for thinking of $\mathbf{F}$ as representing a two-dimensional flow field (see section V1). Then the line integral represents the *rate with respect to time at which mass is being transported across* C. (We think of the flow as taking place in a shallow tank of unit depth. The convention about $\mathbf{n}$ makes this mass-transport rate positive if the flow is from left to right as you face in the positive direction on C, and negative in the other case.)

To see this, we follow the same procedure that was used to interpret the tangential integral in a force field as work.

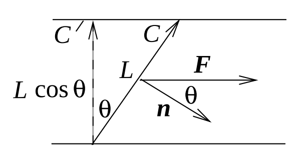
The essential step to see is that if $\mathbf{F}$ is a constant vector field representing a flow, and C is a directed line segment of length L, then



(6) $$\text{mass-transport rate across } C = (\mathbf{F}\cdot\mathbf{n})\,L$$

To see this, resolve the flow field into its components parallel to C and perpendicular to C. The component parallel to C contributes nothing to the flow rate *across* C, while the component perpendicular to C is $\mathbf{F}\cdot\mathbf{n}$.

Another way to se (6) is illustrated at the right. Letting C' be as shown, we see by conservation of mass that



$$\begin{aligned}\text{mass-transport rate across } C &= \text{mass-transport rate across } C'\\ &= |\mathbf{F}|(L\cos\theta)\\ &= (\mathbf{F}\cdot\mathbf{n})\,L\ .\end{aligned}$$

Once we have this, we follow the same procedure used to define work as a line integral. We divide up the curve and apply (6) to each of the approximating line segments, the k-th segment being of length approximately Δs_k. Thus

$$\text{mass-transport rate across } k\text{-th line segment } \approx (\mathbf{F}_k \cdot \mathbf{n}_k)\Delta s_k \ .$$

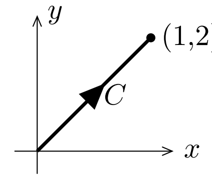
Adding these up and passing to the limit as the subdivision of the curve gets finer and finer then gives

$$\text{mass-transport rate across } C = \int_C \mathbf{F} \cdot \mathbf{n}\, ds \ .$$

This interpretation shows why we call the line integral the *flux* of $\mathbf{F}$ across C. This terminology however is used even when $\mathbf{F}$ no longer represents a two-dimensional flow field. We speak of the flux of an electromagnetic field, for example.

Referring back to Example 1, the field $\mathbf{F} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j}}{x^2 + y^2}$ discussed there represents a flow stemming from a single source of strength 2π at the origin; thus the flux across each circle centered at the origin should also be 2π, regardless of the radius of the circle. This is what we found by actual calculation.

Problems: Flux Across Curves



Compute the flux of $\mathbf{F} = x^2\,\mathbf{i} + y\,\mathbf{j}$ across the line segment from $(0, 0)$ to $(1, 2)$.

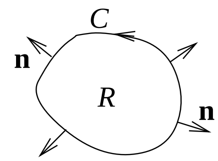
Answer: Parametrize the curve as $x = x$, $y = 2x$, $\quad 0 \le x \le 1$.

Flux $= \displaystyle\int_C M\,dy - N\,dx = \int_0^1 x^2\,2dx - 2x\,dx = \frac{2}{3} - 1 = -\frac{1}{3}.$

Problems: Flux Across Curves

Compute the flux of $\mathbf{F} = x^2\,\mathbf{i} + y\,\mathbf{j}$ across the line segment from $(0,0)$ to $(1,2)$.

V4.1-2 Green's Theorem in Normal Form



1. Green's theorem for flux.

Let $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$ represent a two-dimensional flow field, and C a simple closed curve, positively oriented, with interior R.

According to the previous section,

$$\text{(1)} \qquad \text{flux of } F \text{ across } C \;=\; \oint_C M\,dy - N\,dx\ .$$

Notice that since the normal vector points outwards, away from R, the flux is positive where the flow is out of R; flow into R counts as negative flux.

We now apply Green's theorem to the line integral in (1); first we write the integral in standard form (dx first, then dy):

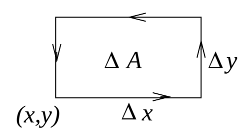
$$\oint_C M\,dy - N\,dx \;=\; \oint_C -N\,dx + M\,dy \;=\; \iint_R \big(M_x - (-N)_y\big)\,dA\ .$$

This gives us **Green's theorem in the normal form**

$$\text{(2)} \qquad \oint_C M\,dy - N\,dx \;=\; \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA\ .$$

Mathematically this is the same theorem as the tangential form of Green's theorem — all we have done is to juggle the symbols M and N around, changing the sign of one of them. What is different is the physical interpretation. The left side represents the flux of $\mathbf{F}$ across the closed curve C. What does the right side represent?

2. The two-dimensional divergence.



Once again, let $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$. We give a name to and a notation for the integrand of the double integral on the right of (2):

$$\text{(3)} \qquad \operatorname{div}\mathbf{F} \;=\; \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}, \qquad \text{the } \textbf{divergence} \text{ of } \mathbf{F}\ .$$

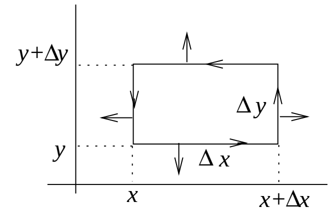
Evidently div $\mathbf{F}$ is a scalar function of two variables. To get at its physical meaning, look at the small rectangle pictured. If $\mathbf{F}$ is continuously differentiable, then div $\mathbf{F}$ is a continuous function, which is therefore approximately constant if the rectangle is small enough. We apply (2) to the rectangle; the double integral is approximated by a product, since the integrand is approximately constant:

$$\text{(4)} \quad \text{flux across sides of rectangle} \;\approx\; \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\Delta A\ , \qquad \Delta A \;=\; \text{area of rectangle.}$$

Because of the importance of this approximate relation, we give a more direct derivation of it which doesn't use Green's theorem. The reasoning which follows is widely used in mathematical modeling of physical problems.

Consider the small rectangle shown; we calculate approximately the flux over each side.

$$\begin{aligned}\text{flux across top} &\approx (\mathbf{F}(x,y+\Delta y)\cdot \mathbf{j}\,)\,\Delta x \;=\; N(x,y+\Delta y)\,\Delta x\\ \text{flux across bottom} &\approx (\mathbf{F}(x,y)\cdot -\mathbf{j}\,)\Delta x \;=\; -N(x,y)\Delta x\ ;\end{aligned}$$



adding these up,

$$\begin{matrix}\text{total flux across}\\ \text{top and bottom}\end{matrix} \;\approx\; \big(N(x,y+\Delta y)-N(x,y)\big)\Delta x \;\approx\; \left(\frac{\partial N}{\partial y}\,\Delta y\right)\Delta x.$$

By similar reasoning applied to the two sides,

$$\begin{matrix}\text{total flux across}\\ \text{left and right sides}\end{matrix} \;\approx\; \Big(M(x+\Delta x,y)-M(x,y)\Big)\,\Delta y \;\approx\; \left(\frac{\partial M}{\partial x}\,\Delta x\right)\Delta y\ .$$

Adding up the flux over the four sides, we get (4) again:

$$\begin{matrix}\text{total flux over four}\\ \text{sides of the rectangle}\end{matrix} \;\approx\; \left(\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}\right)\Delta x\,\Delta y\ .$$

Continuing our search for a physical meaning for the divergence, if the total flux over the sides of the small rectangle is positive, this means there is a net flow *out* of the rectangle. According to conservation of matter, the only way this can happen is if there is a *source* adding fluid directly to the rectangle. If the flow is taking place in a shallow tank of uniform depth, such a source can be visualized as someone standing over the tank, pouring fluid directly into the rectangle. Similarly, a net flow *into* the rectangle implies there is a *sink* withdrawing fluid from the rectangle. It is best to think of such a sink as a "negative source". The net rate (positive or negative) at which fluid is added directly to the rectangle from above may be called the "source rate" for the rectangle. Thus, since matter is conserved,

$$\text{flux over sides of rectangle} \;=\; \text{source rate for the rectangle;}$$

combining this with (4) shows that

$$\text{(5)}\qquad \text{source rate for the rectangle} \;\approx\; \left(\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}\right)\Delta A.$$

We now divide by ΔA and pass to the limit, getting by definition

$$\text{(6)}\qquad \text{the source rate at } (x,y) \;=\; \left(\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}\right) \;=\; \operatorname{div}\mathbf{F}\ .$$

The definition of the double integral as the limit of a sum shows in the usual way now that

$$\text{(7)}\qquad \text{source rate for } R \;=\; \iint_R \operatorname{div}\mathbf{F}\,dA\ .$$

These two relations (6) and (7) interpret the divergence physically, for a flow field, and they interpret also Green's theorem in the normal form:

$$\begin{aligned}\text{total flux across } C &= \text{source rate for } R\\ \oint_C M\,dy - N\,dx &= \iint_R \left(\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}\right)dA\end{aligned}$$

Since Green's theorem is a mathematical theorem, one might think we have "proved" the law of conservation of matter. This is not so, since this law was needed for our interpretation of div $\mathbf{F}$ as the source rate at (x, y).

We give side-by-side the two forms of Green's theorem, first in the vector form, then in the differential form used when calculations are to be done.

Tangential form

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R \text{curl}\, \mathbf{F}\, dA$$

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

work by $\mathbf{F}$ around C

Normal form

$$\oint_C \mathbf{F}\cdot \mathbf{n} ds = \iint_R \text{div}\, \mathbf{F}\, dA$$

$$\oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA$$

flux of $\mathbf{F}$ across C

source rate for R

Problems: Normal Form of Green's Theorem

Use geometric methods to compute the flux of $\mathbf{F}$ across the curves C indicated below, where the function $g(r)$ is a function of the radial distance r.

1. $\mathbf{F} = g(r)\langle x, y\rangle$ and C is the circle of radius a centered at the origin and traversed in a clockwise direction.

Answer: (Radial field) $\mathbf{F}$ is parallel to $\mathbf{n}$ with $\langle x, y\rangle = a\mathbf{n}$ on C, so we have $\mathbf{F} \cdot \mathbf{n} = g(a) \cdot a$ $\Rightarrow$ Flux $= g(a)2\pi a^2$.

2. $\mathbf{F} = g(r)\langle -y, x\rangle$; C as above.

Answer: (Tangential field) Since $\mathbf{F}$ is orthogonal to $\mathbf{n}$ the flux is 0.

3. $\mathbf{F} = 3\langle 1, 1\rangle$; C is the line segment from $(0, 0)$ to $(1, 1)$.

Answer: Since $\mathbf{F}$ is parallel to the line segment C we have $\mathbf{F} \cdot \mathbf{n} = 0$. $\Rightarrow$ flux $= 0$.

4. $\mathbf{F} = 3\langle -1, 1\rangle$; C is the line segment from $(0, 0)$ to $(1, 1)$.

Answer: $\mathbf{F}$ is orthogonal to C. $\mathbf{F}$ points in the opposite direction from $\mathbf{n}$ because $\mathbf{n}$ is clockwise from the direction vector for C.

$\Rightarrow$ flux $= \int \mathbf{F} \cdot \mathbf{n}\, dS = \int 3\sqrt{2}\, ds = 6$.

Problems: Normal Form of Green's Theorem

Use geometric methods to compute the flux of $\mathbf{F}$ across the curves C indicated below, where the function $g(r)$ is a function of the radial distance r.

1. $\mathbf{F} = g(r)\langle x, y\rangle$ and C is the circle of radius a centered at the origin and traversed in a clockwise direction.

2. $\mathbf{F} = g(r)\langle -y, x\rangle$; C as above.

3. $\mathbf{F} = 3\langle 1, 1\rangle$; C is the line segment from $(0, 0)$ to $(1, 1)$.

4. $\mathbf{F} = 3\langle -1, 1\rangle$; C is the line segment from $(0, 0)$ to $(1, 1)$.

18.02SC Multivariable Calculus
Fall 2010

Verify Green's Theorem in Normal Form

Verify that $\oint_C M\,dy - N\,dx = \iint_R \left(\dfrac{\partial M}{\partial x} + \dfrac{\partial N}{\partial y}\right) dA$ when $\mathbf{F} = x\hat{\mathbf{i}} + x\hat{\mathbf{j}}$ and C is the square with vertices $(0,0)$, (1,0), $(1,1)$ and $(0,1)$.

<u>Answer:</u>

Right hand side: Here $M = N = x$, so $\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA = \iint_R 1\,dA = 1$.

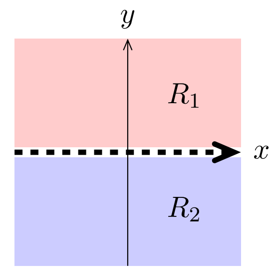
Left hand side: $\oint_C M\,dy - N\,dx = \oint_C x\,dy - x\,dx$. We evaluate this line integral in four parts.

- $(0,0)$ to $(1,0)$.
$$\int_{x=0}^{x=1} x \cdot 0 - x\,dx = \left.\frac{x^2}{2}\right|_0^1 = \frac{1}{2}.$$
- $(1,0)$ to $(1,1)$.
$$\int_{y=0}^{y=1} 1\,dy - 1 \cdot 0 = 1.$$
- $(1,1)$ to $(0,1)$.
$$\int_{x=1}^{x=0} x \cdot 0 - x\,dx = -\frac{1}{2}.$$
- $(0,1)$ to $(0,0)$.
$$\int_{y=1}^{y=0} 0\,dy - 0 \cdot 0 = 0.$$

Since the sum of the line integrals along the components of C is 1, $\oint_C x\,dy - x\,dx = 1$. This confirms that the normal form of Green's Theorem is true in this example.

Problems: Extended Green's Theorem

1. Is $\mathbf{F} = \dfrac{y\,dx - x\,dy}{y^2}$ exact? If so, find a potential function.



Answer: $M = \dfrac{1}{y}$ and $N = -\dfrac{x}{y^2}$ are continuously differentiable whenever $y \neq 0$, i.e. in the two half-planes R_1 and R_2 – both simply connected.
Since $M_y = -1/y^2 = N_x$ in each half-plane the field is exact where it is defined.
To find a potential function f for which $\mathbf{F} = df$ we use method 2.
$f_x = 1/y \;\Rightarrow\; f = x/y + g(y)$.
$f_y = -x/y^2 + g'(y) = -x/y^2 \;\Rightarrow\; g'(y) = 0 \;\Rightarrow\; g(y) = c$.
$\Rightarrow\; f(x, y) = x/y + c$.

(continued)

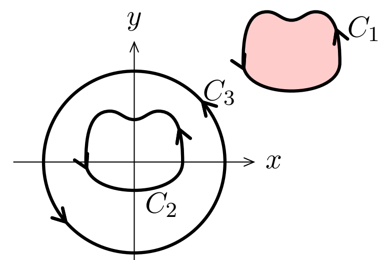
Example 3: Let $\mathbf{F} = r^n(x\mathbf{i} + y\mathbf{j})$. Use extended Green's Theorem to show that $\mathbf{F}$ is conservative for all integers n. Find a potential function.

First note, $M = r^n x$, $N = r^n y \;\Rightarrow\; M_y = nr^{n-2}xy = N_x \;\Leftrightarrow\;$ curl$\mathbf{F} = 0$.

We show $\mathbf{F}$ is conservative by showing $\int_C \mathbf{F}\cdot d\mathbf{r} = 0$ for all simple closed curves C.

If C_1 is a simple closed curve not around 0 then Green's Theorem implies $\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = 0$.

If C_3 is a circle centered on (0,0) then, since $\mathbf{F}$ is radial $\displaystyle\oint_{C_3} \mathbf{F}\cdot d\mathbf{r} = \oint_{C_3} \mathbf{F}\cdot\mathbf{T}\,ds = 0$.



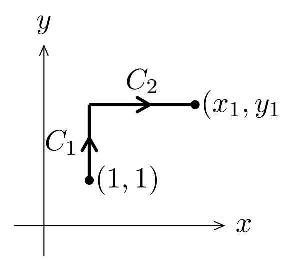
If C_3 completely surrounds C_2 then extended Green's Theorem implies $\displaystyle\oint_{C_2} \mathbf{F}\cdot d\mathbf{r} = \oint_{C_3} \mathbf{F}\cdot d\mathbf{r} = 0$.

Thus $\displaystyle\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ for all closed loops $\Rightarrow$ $\mathbf{F}$ is conservative.

To find the potential function we use method 1 over the curve C shown.

The calculation works for $n = 2$. For $n = 2$ everything is the same except we'd get natural logs instead of powers. (We also ignore the fact that if (x_1, y_1) is on the negative x-axis we shoud use a different path that doesn't go through the origin. This isn't really an issue since we already know a potential function exists, so continuity would handle these points without using an integral.)

$$\begin{aligned}
f(x_1, y_1) &= \int_C r^n x\,dx + r^n y\,dy \\
&= \int_1^{y_1} (1+y^2)^{n/2} y\,dy + \int_1^{x_1} (x^2+y_1^2)^{n/2} x\,dx \\
&= \left.\frac{(1+y^2)^{(n+2)/2}}{n+2}\right|_1^{y_1} + \left.\frac{(x^2+y_1^2)^{(n+2)/2}}{n+2}\right|_1^{x_1} \\
&= \frac{(1+y_1^2)^{(n+2)/2} - 2^{(n+2)/2}}{n+2} + \frac{(x_1^2+y_1^2)^{(n+2)/2} - (1+y_1^2)}{(n+2)/2} \\
&= \frac{(x_1^2+y_1^2)^{(n+2)/2} - 2^{(n+1)/2}}{n+2}
\end{aligned}$$



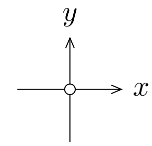
$$\Rightarrow \boxed{f(x,y) = \frac{r^{n+2}}{n+2} + C.}$$

If $n = -2$ we get $f(x, y) = \ln r + C$.

Problems: Extended Green's Theorem

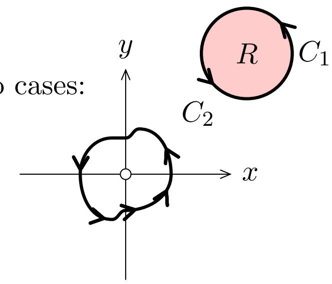
1. Is $\mathbf{F} = \dfrac{y\,dx - x\,dy}{y^2}$ exact? If so, find a potential function.

Extended Green's Theorem



Let $\mathbf{F}$ be the "tangential field" $\mathbf{F} = \dfrac{-y\mathbf{i} + x\mathbf{j}}{r^2}$, defined on the punctured plane $D = \mathbf{R}^2 - (0,0)$. It's easy to compute (we've done it before) that $\text{curl}\mathbf{F} = 0$ in D.

Question: For the tangential field $\mathbf{F}$, what do you think the possible values of $\oint_C \mathbf{F} \cdot d\mathbf{r}$ could be if C were allowed to be any closed curve?



<u>**Answer:**</u> As we saw in lecture, if C is simple and positively oriented we have two cases:
(i) C_1 not around 0 (ii) C_2 around 0

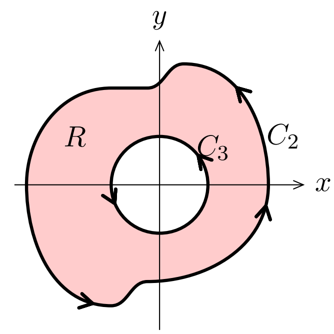
(i) Green's Theorem $\Rightarrow \oint_{C_1} \mathbf{F} \cdot d\mathbf{r} = \iint_R \text{curl}\mathbf{F} \cdot \mathbf{k}\, dA = 0.$

(ii) We show that $\oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = 2\pi.$

Let C_3 be a small circle of radius a, entirely inside C_2.
By extended Green's Theorem

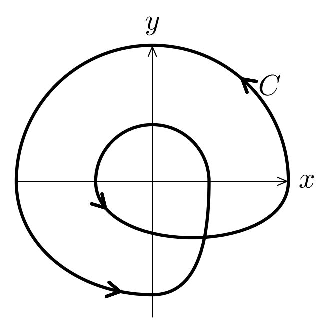
$$\oint_{C_2} \mathbf{F} \cdot d\mathbf{r} - \oint_{C_3} \mathbf{F} \cdot d\mathbf{r} = \iint_R \text{curl}\mathbf{F} \cdot \mathbf{k}\, dA = 0$$

$$\Rightarrow \oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = \oint_{C_3} \mathbf{F} \cdot d\mathbf{r}.$$



On the circle C_3 we can easily compute the line integral:

$$\mathbf{F} \cdot \mathbf{T} = 1/a \;\Rightarrow\; \oint_{C_3} \mathbf{F} \cdot \mathbf{T}\, ds = \int_{C_3} \frac{1}{a}\, ds = \frac{2\pi a}{a} = 2\pi. \quad \text{QED}$$



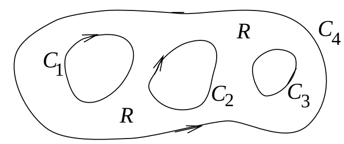
If C is positively oriented but not simple, the figure to the right suggests that we can break C into two curves around the origin at a point where it crosses itself. Repeating this as often as necessary, we find that $\oint_C \mathbf{F} \cdot d\mathbf{r} = 2\pi n$, where n is the number of times C goes counterclockwise around (0,0).
If C is negatively oriented $\oint_C \mathbf{F} \cdot d\mathbf{r} = -\oint_{C'} \mathbf{F} \cdot d\mathbf{r}$, where C' is an oppositely oriented copy of C. Hence, our final answer is that $\oint_C \mathbf{F} \cdot d\mathbf{r}$ may equal $2\pi n$ for any integer n.

An interesting aside: n is called the *winding number* of C around 0. n also equals the number of times C crosses the positive x-axis, counting $+1$ from below and -1 from above.

V5. Simply-Connected Regions

1. The Extended Green's Theorem.

In the work on Green's theorem so far, it has been assumed that the region R has as its boundary a single simple closed curve. But this isn't necessary. Suppose the region has a boundary composed of several simple closed curves, like the ones pictured. We suppose these boundary curves $C_1, \dots, C_m$ all lie within the domain where $\mathbf{F}$ is continuously differentiable. Most importantly, all the curves must be directed so that the normal $\mathbf{n}$ points *away* from R.

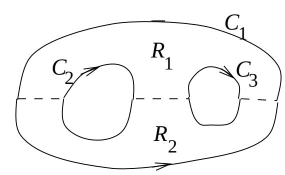


Extended Green's Theorem *With the curve orientations as shown,*

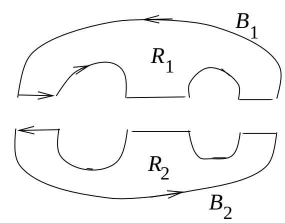
$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \ldots + \int_{C_m} \mathbf{F} \cdot d\mathbf{r} \;=\; \iint_R \text{curl}\, \mathbf{F}\, dA \tag{1}$$

In other words, Green's theorem also applies to regions with several boundary curves, provided that we take the line integral over the complete boundary, with each part of the boundary oriented so the normal $\mathbf{n}$ points outside R.

Proof. We use subdivision; the idea is adequately conveyed by an example. Consider a region with three boundary curves as shown. The three cuts illustrated divide up R into two regions R_1 and R_2, each bounded by a single simple closed curve, and Green's theorem in the usual form can be applied to each piece. Letting B_1 and B_2 be the boundary curves shown, we have therefore



$$\oint_{B_1} \mathbf{F} \cdot d\mathbf{r} \;=\; \iint_{R_1} \text{curl}\, \mathbf{F}\, dA \qquad \oint_{B_2} \mathbf{F} \cdot d\mathbf{r} \;=\; \iint_{R_2} \text{curl}\, \mathbf{F}\, dA \tag{2}$$

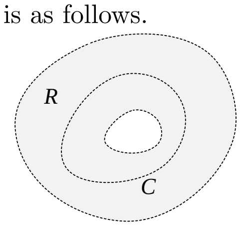


Add these two equations together. The right sides add up to the right side of (1). The left sides add up to the left side of (1) (for $m = 2$), since over each of the three cuts, there are two line integrals taken in opposite directions, which therefore cancel each other out. $\square$

2. Simply-connected and multiply-connected regions.

Though Green's theorem is still valid for a region with "holes" like the ones we just considered, the relation $\text{curl}\, \mathbf{F} = 0 \;\Rightarrow\; \mathbf{F} = \nabla f$ is not. The reason for this is as follows.

We are trying to show that $\quad \text{curl}\, \mathbf{F} = 0 \quad \Rightarrow \quad \oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve lying in R. We expect to be able to use Green's theorem. But if the region has a hole, like the one pictured, we cannot apply Green's theorem to the curve C because the interior of C is not entirely contained in R.



To see what a delicate affair this is, consider the earlier Example 2 in Section V2. The field $\mathbf{G}$ there satisfies curl $\mathbf{G} = 0$ everywhere but the origin. The region R is the xy-plane with $(0,0)$ removed. But $\mathbf{G}$ is not a gradient field, because $\oint_C \mathbf{G} \cdot d\mathbf{r} \neq 0$ around a circle C surrounding the origin.

This is clearer if we use Green's theorem in normal form (Section V4). If the flow field satisfies div $\mathbf{F} = 0$ everywhere except at one point, that doesn't guarantee that the flux through every closed curve will be 0. For the spot where div $\mathbf{F}$ is undefined might be a source, through which fluid is being added to the flow.

In order to be able to prove under reasonable hypotheses that curl $\mathbf{F} = 0 \;\Rightarrow\; \mathbf{F} = \nabla f$, we define our troubles away by assuming that R is the sort of region where the difficulties described above cannot occur—i.e., we assume that R has no holes; such regions are called *simply-connected.*

Definition. A two-dimensional region D of the plane consisting of one connected piece is called **simply-connected** if it has this property: whenever a simple closed curve C lies entirely in D, then its interior also lies entirely in D.

As examples: the xy-plane, the right-half plane where $x \geq 0$, and the unit circle with its interior are all simply-connected regions. But the xy-plane minus the origin is not simply-connected, since any circle surrounding the origin lies in D, yet its interior does not.

As indicated, one can think of a simply-connected region as one without "holes". Regions with holes are said to be *multiply-connected*, or *not simply-connected.*

Theorem. Let $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$ be continuously differentiable in a simply-connected region D of the xy-plane. Then in D,

(3) $\quad$ curl $F = 0 \quad \Rightarrow \quad \mathbf{F} = \nabla f, \qquad$ for some $f(x,y)$; $\qquad$ in terms of components,

(3′)
$$M_y = N_x \quad \Rightarrow \quad M\,\mathbf{i} + N\,\mathbf{j} = \nabla f, \qquad \text{for some } f(x,y).$$

Proof. Since a field is a gradient field if its line integral around any closed path is 0, it suffices to show

$$\text{curl}\,\mathbf{F} = 0 \quad \Rightarrow \quad \oint_C \mathbf{F}\cdot d\mathbf{r} = 0 \qquad \text{for every closed curve } C \text{ in } D. \tag{4}$$

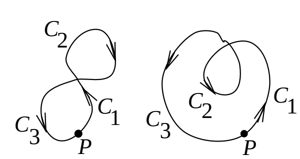
We prove (4) in two steps.

Assume first that C is a simple closed curve; let R be its interior. Then since D is simply-connected, R will lie entirely inside D. Therefore $\mathbf{F}$ will be continuously differentiable in R, and we can use Green's theorem:

$$\oint_C \mathbf{F}\cdot d\mathbf{r} \;=\; \iint_R \text{curl}\,\mathbf{F}\,dxdy \;=\; 0\ .$$

Next consider the general case, where C is closed but not simple—i.e., it intersects itself. Then C can be broken into smaller simple closed curves for which the above argument will be valid. A formal argument would be awkward to give, but the examples illustrate. In both cases, the path starts and ends at P, and

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} + \oint_{C_2} \mathbf{F}\cdot d\mathbf{r} + \int_{C_3} \mathbf{F}\cdot d\mathbf{r}\ .$$



In both cases, C_2 is a simple closed path, and also $C_1 + C_3$ is a simple closed path. Since D is simply-connected, the interiors automatically lie in D, so that by the first part of the argument,

$$\int_{C_1} \mathbf{F}\cdot d\mathbf{r} + \int_{C_3} \mathbf{F}\cdot d\mathbf{r} = 0 \qquad \text{and} \qquad \oint_{C_2} \mathbf{F}\cdot d\mathbf{r} = 0$$

Adding these up, we get $\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$. $\square$

The above argument works if C intersects itself a finite number of times. If C intersects itself infinitely often, we would have to resort to approximations to C; we skip this case.

We pause now to summarize compactly the central result, both in the language of vector fields and in the equivalent language of differentials.

Curl Theorem. Let $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$ be a continuously differentiable vector field in a simply-connected region D of the xy-plane. Then the following four statements are equivalent — if any one is true for $\mathbf{F}$ in D, so are the other three:

1. $\displaystyle\int_P^Q \mathbf{F}\cdot d\mathbf{r}$ is path-independent   1.′ $\displaystyle\int_P^Q M\,dx + N\,dy$ is path-independent

for any two points P, Q in D;

2. $\displaystyle\oint_C \mathbf{F}\cdot d\mathbf{r} = 0,$ 2.′ $\displaystyle\oint_C M\,dx + N\,dy = 0,$

for any simple closed curve C lying in D;

3. $\mathbf{F} = \nabla f$ for some f in D 3.′ $M\,dx + N\,dy = df$ for some f in D

4. curl $\mathbf{F} = 0$ in D 4.′ $M_y = N_x$ in D.

Remarks. We summarize below what still holds true even if one or more of the hypotheses doesn't hold: D is not simply-connected, or the field $\mathbf{F}$ is not differentiable everywhere in D.

1. Statements 1, 2, and 3 are equivalent even if $\mathbf{F}$ is only continuous; D need not be simply-connected..

2. Statements 1, 2, and 3 each implies 4, if $\mathbf{F}$ is continuously differentiable; D need not be simply-connected. (But 4 implies 1, 2, 3 only if D is simply-connected.)

Example 1. Is $\mathbf{F} = xy\,\mathbf{i} + x^2\,\mathbf{j}$ a gradient field?

Solution. We have curl $\mathbf{F} = x = 0$, so the theorem says it is not.

Example 2. Is $\dfrac{ydx - xdy}{y^2}$ an exact differential? If so, find all possible functions $f(x,y)$ for which it can be written df .

Solution. $M = 1/y$ and $N = -x/y^2$ are continuously differentiable wherever $y = 0$, i.e., in the two half-planes above and below the x-axis. These are both simply-connected. In each of them,

$$M_y = -1/y^2 = N_x$$

Thus in each half-plane the differential is exact, by the theorem, and we can calculate $f(x,y)$ by the standard methods in Sction V2. They give

$$f(x,y) \;=\; \frac{x}{y} + c$$

where c is an arbitrary constant. This constant need not be the same for the two regions, since they do not touch. Thus the most general function is

$$f(x,y) = \begin{cases} x/y \; + c, & y > 0 \\ x/y \; + c', & y < 0 \end{cases}; \qquad c, c' \text{ are arbitrary constants.}$$

Example 3. Let $\mathbf{F} = r^n(x\,\mathbf{i} + y\,\mathbf{j}),\ r = \sqrt{x^2+y^2}$. For which integers n is $\mathbf{F}$ conservative? For each such, find a corresponding $f(x,y)$ such that $\mathbf{F} = \nabla f$.

Solution. By the usual calculation, using the chain rule and the useful polar coordinate relations $r_x = x/r,\ r_y = y/r$, we find that curl $\mathbf{F} = 0$. There are two cases.

Case 1: $n \geq 0$. Then $\mathbf{F}$ is continuously differentiable in the whole xy-plane, which is simply-connected. Thus by the preceding theorem, $\mathbf{F}$ is conservative, and we can calculate $f(x,y)$ as in Section V2.

We use method 1 (line integration). The radial symmetry suggests using the ray C from $(0,0)$ to (x_1, y_1) as the path of integration, with the parametrization

$$x = x_1 t, \quad y = y_1 t, \quad 0 \leq t \leq 1;$$

also, let

$$r_1 = \sqrt{x_1^2 + y_1^2}; \qquad \text{then} \qquad r^n = r_1^n t^n, \quad x\,dx + y\,dy = r_1^2 t\,dt$$

and we get, by method 1 for finding $f(x,y)$,

$$\begin{aligned} f(x_1.y_1) &= \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C r^n(x\,dx + y\,dy) \\ &= \int_0^1 r_1^{n+2} t^{n+1} dt = r_1^{n+2} \frac{t^{n+2}}{n+2}\bigg|_0^1 = \frac{r_1^{n+2}}{n+2}. \end{aligned} \tag{6}$$

so that

$$f(x,y) = \frac{r^{n+2}}{n+2}, \qquad \mathbf{F} = \nabla f, \quad n \geq 0 \,. \tag{7}$$

Case 2: $n < 0$. The field $\mathbf{F}$ is not defined at $(0,0)$, so that its domain, the xy-plane with $(0,0)$ removed, is not simply-connected. So even though curl $\mathbf{F} = 0$ in this region, (3) is not immediately applicable.

Nonetheless, if $n = -2$, one can check by differentiation that (7) is still valid.

If $n = -2$, guessing, inspection, or method 2 give $f(x,y) = \ln r$.

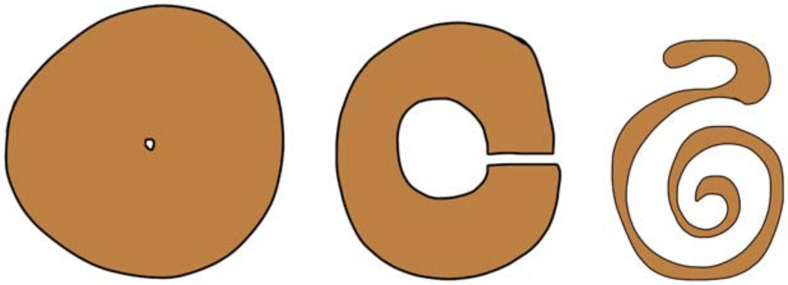
We conclude that the field in all cases is a gradient field. Note in particular that the two force fields given in section V1, representing respectively (apart from a constant factor) the fields arising from a positive charge at $(0,0)$ and a uniform positive charge along the z-axis, correspond to the respective cases $n = -3$ and $n = -2$, and are both gradient fields:

$$\frac{x\,\mathbf{i} + y\,\mathbf{j}}{r^3} = \nabla\left(-\frac{1}{r}\right) \qquad (n = -3: \text{ positive charge at } (0,0))$$

$$\frac{x\,\mathbf{i} + y\,\mathbf{j}}{r^2} = \nabla(\ln r) \qquad (n = -2: \text{ uniform } + \text{ charge on } z\text{-axis}).$$

Problems: Simply Connected Regions

1. Which of the regions shown below are simply connected?



(a) (b) (c)

Answer: Region (a) is not simply connected – the "puncture" at the center of the disk would prevent any simple closed curve around it from contracting to a point while remaining within the region.

Regions (b) and (c) are simply connected. We frequently "cut" a region like (a) to create a simply connected region similar to (b). Region (c) illustrates the fact that simply connected regions aren't always simple!

For each of the vector fields described below, find the domain on which it is defined and continuously differentiable. Is that domain simply connected?

2. $\sin(x^2 + y^2)\mathbf{i} + \cos(x^2 + y^2)\mathbf{j}$

Answer: The vector field is defined and differentiable at all points (x, y). This region is simply connected.

3. $|x|\mathbf{i} + 0\mathbf{j}$

Answer: This vector field is not differentiable when $x = 0$. The region is the union of the open left and right half-planes and is not simply connected.

4. $\dfrac{x\mathbf{i} + y\mathbf{j}}{x^2 + y^2}$

Answer: The region is the punctured plane $(x, y) \neq (0, 0)$. It is not simply connected.

5. $\dfrac{y\mathbf{i} - x\mathbf{j}}{y^2}$

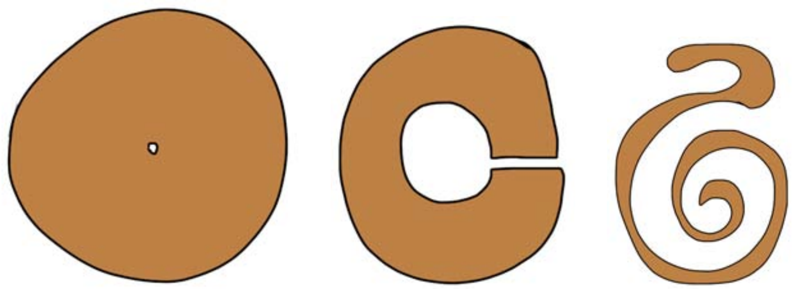
Answer: The region is the union of the upper and lower half planes; $y \neq 0$. It is not simply connected.

6. $\sqrt{x^2 - 1}\,\mathbf{i} + \sqrt{y^2 - 1}\mathbf{j}$

Answer: This vector field is defined and continuously differentiable when x^2 and y^2 are greater than 1. This region is the plane minus a square of side length two and is not simply connected.

Problems: Simply Connected Regions

1. Which of the regions shown below are simply connected?



(a) (b) (c)

For each of the vector fields described below, find the domain on which it is defined and continuously differentiable. Is that domain simply connected?

2. $\sin(x^2+y^2)\mathbf{i}+\cos(x^2+y^2)\mathbf{j}$

3. $|x|\mathbf{i}+0\mathbf{j}$

4. $\dfrac{x\mathbf{i}+y\mathbf{j}}{x^2+y^2}$

5. $\dfrac{y\mathbf{i}-x\mathbf{j}}{y^2}$

6. $\sqrt{x^2-1}\,\mathbf{i}+\sqrt{y^2-1}\mathbf{j}$