# Identifying Potential Functions

**1**. Show $\mathbf{F} = \langle 3x^2 + 6xy, 3x^2 + 6y \rangle$ is conservative and find the potential function $f$ such that $\mathbf{F} = \boldsymbol{\nabla} f$.

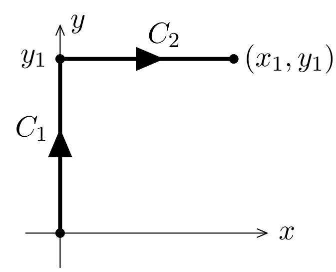
<u>**Answer:**</u> First, $M_y = 6x = N_x$. Since $\mathbf{F}$ is defined for all $(x, y)$, $\mathbf{F}$ is conservative.

Method 1 (for finding $f$): Use

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(P_1) - f(P_0) \;\Rightarrow\; f(P_1) = f(P_0) + \int_C \mathbf{F} \cdot d\mathbf{r}.$$

$P_1 = (x_1, y_1)$ must be arbitrary. We can fix $P_0$ and $C$ any way we want.

For this problem take $P_0 = (0, 0)$ and $C$ as the path shown.




Path for problem 1

$C_1 : x = 0, y = y, \;\Rightarrow\; dx = 0, dy = dy$

$C_2 : x = x, y = y_1, \;\Rightarrow\; dx = dx, dy = 0$

$$\Rightarrow \quad f(x_1, y_1, z_1) - f(0, 0, 0) = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M\,dx + N\,dy$$

$$= \int_0^{x_1} M(x, 0)\,dx + \int_0^{y_1} N(x_1, y)\,dy$$

$$= \int_0^{y_1} 6y\,dy + \int_0^{x_1} 3x^2 + 6y_1\,dx = 3y_1^2 + x_1^3 + 3x_1^2 y_1$$

$\Rightarrow\; f(x_1, y_1) - f(0, 0) = 3y_1^2 + x_1^3 + 3x_1^2 y_1 = 3y_1^2 + x_1^3 + 3x_1^2 y_1.$

$\Rightarrow\; \boxed{f(x, y) = 3y^2 + x^3 + 3x^2 y + C.}$

Method 2: $f_x = 3x^2 + 6xy \;\Rightarrow\; f = x^3 + 3x^2 y + g(y).$

$\Rightarrow\; f_y = 3x^2 + g'(y) = 3x^2 + 6y \;\Rightarrow\; g'(y) = 6y \;\Rightarrow\; g(y) = 3y^2 + C.$

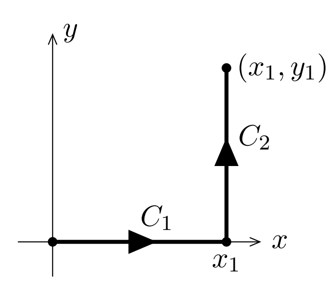
$\Rightarrow\; \boxed{f(x, y) = x^3 + 3x^2 y + 3y^2 + C.}$

In general method 1 is preferred because in 3 dimensions it will be easier.

**2**. Let $\mathbf{F} = (x + xy^2)\,\mathbf{i} + (x^2 y + 3y^2)\,\mathbf{j}$. Show $\mathbf{F}$ is a gradient field and find the potential function using both methods.

<u>**Answer:**</u> We have $M(x, y) = x + xy^2$ and $N(x, y) = x^2 y + 3y^2$, so $M_y = 2xy = N_x$ and $\mathbf{F}$ is defined on all $(x, y)$. Thusy, by Theorem 1, $\mathbf{F}$ is conservative.

Method 1: Use the path shown.




Path for problem 2

$f(P_1) - f(0, 0) = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M\,dx + N\,dy.$

$C_1 : x = x,\; y = 0, \;\Rightarrow\; dx = dx,\; dy = 0 \;\Rightarrow\; M(x, 0) = x.$

$C_2 : x = x_1,\; y = y, \;\Rightarrow\; dx = 0,\; dy = dy \;\Rightarrow\; N(x_1, y) = x_1^2 y + 3y^2.$

$$\Rightarrow \int_C \mathbf{F} \cdot d\mathbf{r} \;=\; \int_0^{x_1} x\,dx + \int_0^{y_1} x_1^2 y + 3y^2\,dy \;=\; x_1^2/2 + x_1^2 y_1^2/2 + y_1^3.$$

$\Rightarrow\; f(x_1, y_1) - f(0, 0) = x_1^2/2 + x_1^2 y_1^2/2 + y_1^3$

$\Leftrightarrow\; \boxed{f(x, y) = x^2/2 + x^2 y^2/2 + y^3 + C.}$

Method 2: $f_x = x + y^2 \;\Rightarrow\; f = x^2/2 + x^2 y^2/2 + g(y)$

$\Rightarrow\ f_y = x^2y + g'(y) = x^2y + 3y^2 \ \Rightarrow\ g'(y) = 3y^2 \ \Rightarrow\ g(y) = y^3 + C.$

$\Rightarrow\ \boxed{f(x,y) = x^2/2 + x^2y^2/2 + y^3 + C.}$

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# Two Dimensional Curl

We have learned about the curl for two dimensional vector fields.

By definition, if $\mathbf{F} = \langle M, N \rangle$ then the two dimensional curl of $\mathbf{F}$ is $\operatorname{curl}\mathbf{F} = N_x - M_y$

**Example:** If $\mathbf{F} = x^3y^2\,\mathbf{i} + x\,\mathbf{j}$ then $M = x^3y^2$ and $N = x$, so $\operatorname{curl}\mathbf{F} = 1 - 2x^3y$.

Notice that $\mathbf{F}(x, y)$ is a vector valued function and its curl is a scalar valued function. It is important that we label this as the two dimensional curl because it is only for vector fields in the plane. Later we will see that the two dimensional curl is really just the $\mathbf{k}$ component of the (vector valued) three dimensional curl.

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# Problems: Two Dimensional Curl

Imagine a flat arrangement of particles covering the plane. Suppose all the particles are moving in counterclockwise circles about the origin with constant angular speed $\omega$.

Let $\mathbf{F}(x, y)$ be the velocity field described by the velocity of the particles at point $(x, y)$. Find $\mathbf{F}$ and show $\text{curl}(\mathbf{F}) = 2\omega$.

**Answer:** Because the particles have a constant angular speed $\omega$ and no radial velocity, the motion of the particles can be parametrized by $r = r_0$, $\theta = \theta_0 + \omega t$. In polar coordinates we have $(x(t), y(t)) = (r_0 \cos(\theta_0 + \omega t), r_0 \sin(\theta_0 + \omega t))$.

Taking derivatives with respect to $t$ we find

$$\begin{aligned}
\mathbf{F} &= -\omega\, r_0 \sin(\theta_0 + \omega t)\mathbf{i} + \omega\, r_0 \cos(\theta_0 + \omega t)\mathbf{j} = \langle -\omega y, \omega x \rangle, \\
\text{curl}\mathbf{F} &= N_x - M_y \\
&= \omega - (-\omega) \\
&= 2\omega.
\end{aligned}$$

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