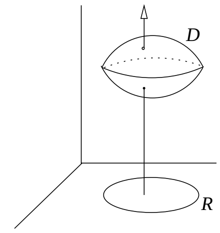
Limits in Iterated Integrals



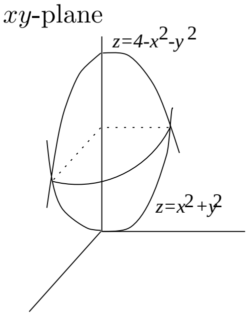
3. Triple integrals in rectangular and cylindrical coordinates.

You do these the same way, basically. To supply limits for $\iiint_D dz\,dy\,dx$ over the region D, we integrate first with respect to z. Therefore we

1. Hold x and y fixed, and let z increase. This gives us a vertical line.

2. Integrate from the z-value where the vertical line enters the region D to the z-value where it leaves D.

3. Supply the remaining limits (in either xy-coordinates or polar coordinates) so that you include all vertical lines which intersect D. This means that you will be integrating the remaining double integral over the region R in the xy-plane which D projects onto.



For example, if D is the region lying between the two paraboloids

$$z = x^2 + y^2 \qquad\qquad z = 4 - x^2 - y^2,$$

we get by following steps 1 and 2,

$$\iiint_D dz\,dy\,dx \;=\; \iint_R \int_{x^2+y^2}^{4-x^2-y^2} dz\,dA$$

where R is the projection of D onto the xy-plane. To finish the job, we have to determine what this projection is. From the picture, what we should determine is the xy-curve over which the two surfaces intersect. We find this curve by eliminating z from the two equations, getting

$$\begin{aligned} x^2 + y^2 &= 4 - x^2 - y^2, \qquad \text{which implies} \\ x^2 + y^2 &= 2. \end{aligned}$$

Thus the xy-curve bounding R is the circle in the xy-plane with center at the origin and radius $\sqrt{2}$.

This makes it natural to finish the integral in polar coordinates. We get

$$\iiint_D dz\,dy\,dx \;=\; \int_0^{2\pi} \int_0^{\sqrt{2}} \int_{x^2+y^2}^{4-x^2-y^2} dz\,r\,dr\,d\theta \;;$$

the limits on z will be replaced by r^2 and $4 - r^2$ when the integration is carried out.

Problems: Triple Integrals

1. Set up, but do not evaluate, an integral to find the volume of the region below the plane $z = y$ and above the paraboloid $z = x^2 + y^2$.

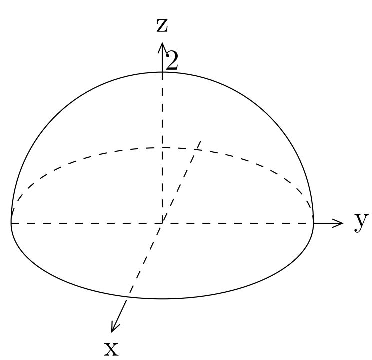
Answer: Draw a picture. The plane $z = y$ slices off an thin oblong from the side of the paraboloid. We'll compute the volume of this oblong by integrating vertical strips in the z direction over a region in the xy-plane.

To describe the planar region below the volume, we study the curve of intersection of the plane and the paraboloid: $y = x^2 + y^2$. Completing the square gives us $\frac{1}{4} = x^2 + \left(y - \frac{1}{2}\right)^2$. This is the equation of a circle with radius $1/2$ about the center $(0, 1/2)$. (We might also discover this by solving to get $x = \pm\sqrt{y - y^2}$ and using a computer graphing utility.)

The most natural set of limits seems to be:
Inner: z from $x^2 + y^2$ to y.
Middle: x from $-\sqrt{y - y^2}$ to $\sqrt{y - y^2}$.
Outer: y from 0 to 1.

Thus, Volume $= \displaystyle\int_0^1 \int_{-\sqrt{y-y^2}}^{\sqrt{y-y^2}} \int_{x^2+y^2}^{y} 1\, dz\, dx\, dy.$

2. Use cylindrical coordinates to find the center of mass of the hemisphere shown. (Assume $\delta = 1$.)



$x^2 + y^2 + z^2 = 4$

Answer: By symmetry it's clear $x_{cm} = 0$ and $y_{cm} = 0$.

$z_{cm} = \dfrac{1}{M}\displaystyle\iiint_D z\, dm = \dfrac{1}{M}\iiint_D z\,\delta\, dV.$

Clearly R is a disc of radius 2 and $M = \dfrac{16}{3}\pi$.

Limits: inner z: from 0 to $\sqrt{4 - x^2 - y^2} = \sqrt{4 - r^2}$.
middle r: from 0 to 2.
outer θ: from 0 to 2π.

$\Rightarrow \; z_{cm} = \dfrac{3}{16\pi}\displaystyle\int_0^{2\pi}\int_0^2\int_0^{\sqrt{4-r^2}} zr\, dz\, dr\, d\theta.$

Inner: $\displaystyle \frac{3}{16\pi}\frac{z^2 r}{2}\bigg|_0^{\sqrt{4-r^2}} = \frac{3}{16\pi}\,\frac{4-r^2}{2}\cdot r = \frac{3}{16\pi}\,\frac{4r-r^3}{2}.$

Middle: $\displaystyle \frac{3}{16\pi}\left[r^2 - \frac{r^4}{8}\right|_0^2 = \frac{3}{8\pi}.$

Outer: $\displaystyle \frac{3}{8\pi}\,2\pi = \frac{3}{4} \;\Rightarrow\; z_{cm} = \frac{3}{4}.$

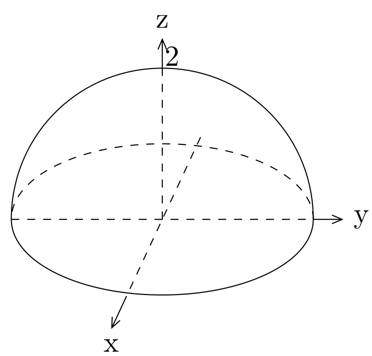
It makes sense that the center of mass would lie between $(0,0,0)$ and $(0,0,1)$.

18.02SC Multivariable Calculus
Fall 2010

Problems: Triple Integrals

1. Set up, but do not evaluate, an integral to find the volume of the region below the plane $z = y$ and above the paraboloid $z = x^2 + y^2$.

2. Use cylindrical coordinates to find the center of mass of the hemisphere shown. (Assume $\delta = 1$.)



$x^2 + y^2 + z^2 = 4$

Triple Integrals

1. Find the moment of inertia of the tetrahedron shown about the z-axis. Assume the tetrahedron has density 1.

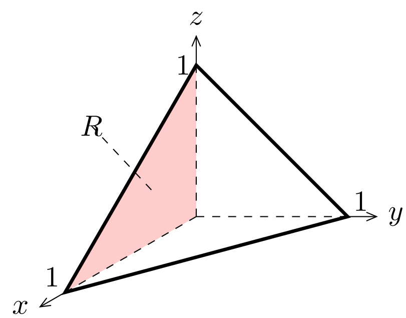


Figure 1: The tetrahedron bounded by $x+y+z=1$ and the coordinate planes.

Answer: To compute the moment of inertia, we integrate distance squared from the z-axis times mass:

$$\iiint_D (x^2+y^2)\cdot 1\, dV.$$

Using cylindrical coordinates about the axis of rotation would give us an "easy" integrand (r) with complicated limits. The integrand x^2+y^2 is not particularly intimidating, so we instead use rectangular coordinates. Integrating first with respect to y or x is preferable; $(x^2+y^2)(1-x-y)$ is a somewhat more intimidating integrand.

To find our limits of integration, we let y go from 0 to the slanted plane $x+y+z=1$. The x and z coordinates are in R, the *projection* of D to the xz-plane which is bounded by the x and z axes and the line $x+z=1$.

$$\text{Moment of Inertia } = \int_0^1 \int_0^{1-z} \int_0^{1-x-z} (x^2+y^2)\, dy\, dx\, dz.$$

Inner: $(x^2y+\frac{1}{3}y^3)\Big|_0^{1-x-z} = x^2-x^3-x^2z+\frac{1}{3}(1-x-z)^3.$

Middle:

$$\begin{aligned}\int_0^{1-z} x^2(1-z)-x^3+\frac{1}{3}(1-x-z)^3\, dx &= \frac{1}{3}x^3(1-z)-\frac{1}{4}x^4-\frac{1}{12}(1-x-z)^4\Big|_0^{1-z} \\ &= \frac{1}{3}(1-z)^4-\frac{1}{4}(1-z)^4+\frac{1}{12}(1-z)^4 \\ &= \frac{1}{6}(1-z)^4.\end{aligned}$$

Outer: $\frac{1}{30}(1-z)^5\Big|_0^1 = \frac{1}{30}.$

2. Find the mass of a cylinder centered on the z-axis which has height h, radius a and density $\delta = x^2+y^2$.

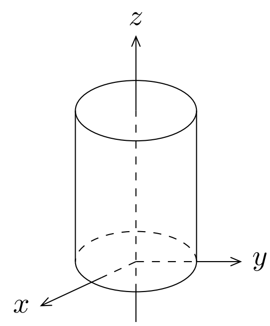


Figure 2: Cylinder.

Answer: To find the mass we integrate the product of density and volume:

$$\text{Mass} = \iiint_D \delta \, dV = \iiint_D r^2 \, dV.$$

Naturally, we'll use cylindrical coordinates in this problem. The limits on z run from 0 to h. The x and y coordinates lie in a disk of radius a, so $0 \le r \le a$ and $0 < \theta \le 2\pi$.

$$\text{Mass} = \iiint_D r^2 \, dV = \int_0^{2\pi} \int_0^a \int_0^h r^2 \, dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^a \int_0^h r^3 \, dz \, dr \, d\theta.$$

Inner integral: $r^3 z\big|_0^h = hr^3$.

Middle integral: $\displaystyle\int_0^a hr^3 \, dr = \frac{ha^4}{4}$.

Outer integral: $2\pi \dfrac{ha^4}{4} = \dfrac{\pi h a^4}{2}$.

Problems: Practice with Triple Integrals

Find the moment of inertia about the z-axis of a solid bounded by the paraboloid $z = x^2+y^2$ and the plane $z = 1$. Assume the solid has uniform density 1.

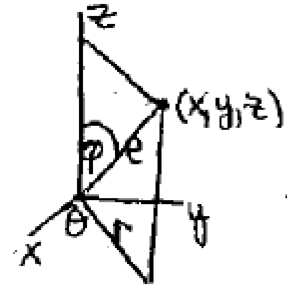
Answer: We use the formula $I = \iiint \rho r^2\, dV$ with density $\rho = 1$. Converting to polar coordinates, the equation of the paraboloid becomes $z = r^2$ and we get the limits of integration $0 \le r \le \sqrt{z}$.

$$\begin{aligned}
I &= \iiint_{\text{solid}} \rho r^2\, dV \\
&= \int_0^1 \int_0^{2\pi} \int_0^{\sqrt{z}} r^2 \cdot r\, dr\, d\theta\, dz \\
&= \int_0^1 \int_0^{2\pi} \frac{z^2}{4}\, d\theta\, dz \\
&= \int_0^1 \pi \frac{z^2}{2}\, dz \\
&= \frac{\pi}{6}.
\end{aligned}$$

Problems: Practice with Triple Integrals

Find the moment of inertia about the z-axis of a solid bounded by the paraboloid $z = x^2+y^2$ and the plane $z = 1$. Assume the solid has uniform density 1.

Limits in Spherical Coordinates



Definition of spherical coordinates

$\rho =$ distance to origin, $\rho \geq 0$

$\phi =$ angle to z-axis, $0 \leq \phi \leq \pi$

$\theta =$ usual $\theta =$ angle of projection to xy-plane with x-axis, $0 \leq \theta \leq 2\pi$

Easy trigonometry gives:

$$\begin{aligned} z &= \rho \cos \phi \\ x &= \rho \sin \phi \cos \theta \\ y &= \rho \sin \phi \sin \theta. \end{aligned}$$

The equations for x and y are most easily deduced by noticing that for r from polar coordinates we have

$$r = \rho \sin \phi.$$

This implies

$$x = r \cos \theta = \rho \sin \phi \cos \theta, \text{ and } y = r \sin \theta = \rho \sin \phi \sin \theta.$$

Going the other way:

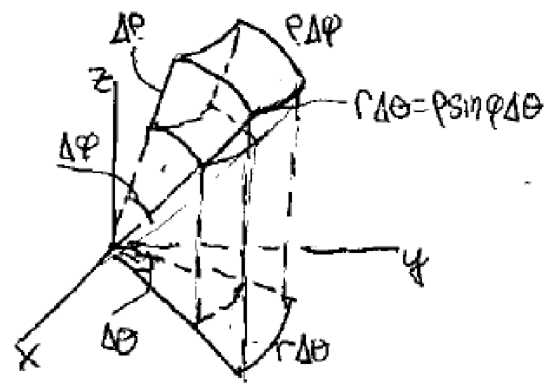
$\rho = \sqrt{z^2 + y^2 + z^2} \quad \phi = \cos^{-1}(z/\rho) \quad \theta = \tan^{-1}(y/x).$

Example: $(x, y, z) = (1, 0, 0) \Rightarrow \rho = 1,\ \phi = \pi/2,\ \theta = 0$

$(x, y, z) = (0, 1, 0) \Rightarrow \rho = 1,\ \phi = \pi/2,\ \theta = \pi/2$

$(x, y, z) = (0, 0, 1) \Rightarrow \rho = 1,\ \phi = 0,\ \theta$ –can be anything

The volume element in spherical coordinates



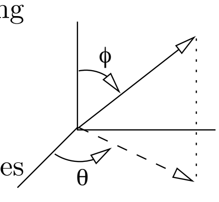
$$dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta.$$

The figure at right shows how we get this. The volume of the curved box is

$$\Delta V \approx \Delta\rho \cdot \rho\Delta\phi \cdot \rho \sin \phi \Delta\theta = \rho^2 \sin \phi \, \Delta\rho \, \Delta\phi \, \Delta\theta.$$

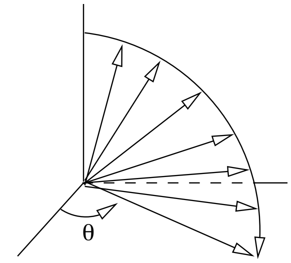
Finding limits in spherical coordinates

We use the same procedure as for rectangular and cylindrical coordinates. To calculate the limits for an iterated integral $\int\int\int_D d\rho \, d\phi \, d\theta$ over a region D in 3-space, we are integrating first with respect to ρ. Therefore we



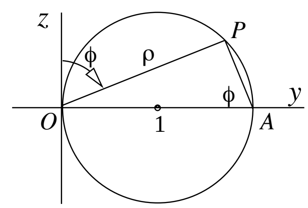
1. Hold ϕ and θ fixed, and let ρ increase. This gives us a ray going out from the origin.

2. Integrate from the ρ-value where the ray enters D to the ρ-value where the ray leaves D. This gives the limits on ρ.



3. Hold θ fixed and let ϕ increase. This gives a family of rays, that form a sort of fan. Integrate over those ϕ-values for which the rays intersect the region D.

4. Finally, supply limits on θ so as to include all of the fans which intersect the region D.



For example, suppose we start with the circle in the yz-plane of radius 1 and center at $(1, 0)$, rotate it about the z-axis, and take D to be that part of the resulting solid lying in the first octant.

First of all, we have to determine the equation of the surface formed by the rotated circle. In the yz-plane, the two coordinates ρ and ϕ are indicated. To see the relation between them when P is on the circle, we see that also angle $OAP = \phi$, since both the angle ϕ and OAP are complements of the same angle, AOP. From the right triangle, this shows the relation is $\rho = 2\sin\phi$.
As the circle is rotated around the z-axis, the relationship stays the same, so $\rho = 2\sin\phi$ is the equation of the whole surface.

To determine the limits of integration, when ϕ and θ are fixed, the corresponding ray enters the region where $\rho = 0$ and leaves where $\rho = 2\sin\phi$.

As ϕ increases, with θ fixed, it is the rays between $\phi = 0$ and $\phi = \pi/2$ that intersect D, since we are only considering the portion of the surface lying in the first octant (and thus above the xy-plane).

Again, since we only want the part in the first octant, we only use θ values from 0 to $\pi/2$. So the iterated integral is

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{2\sin\phi} d\rho\, d\phi\, d\theta.$$

Problems: Limits in Spherical Coordinates

1. Find the limits needed to use spherical coordinates to compute the volume of a sphere of radius a.

Answer: Limits: inner ρ: 0 to a –radial segments
middle ϕ: 0 to π –fan of rays.
outer θ: 0 to 2π –volume.

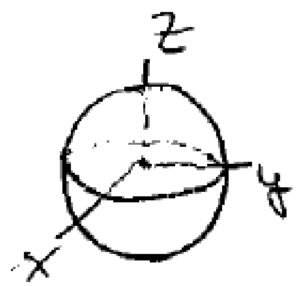
To set up and evaluate the integral (optional):

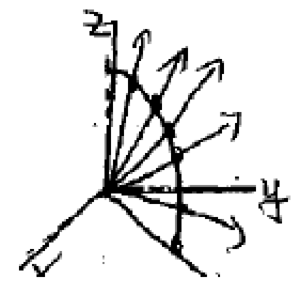
$$V = \iiint_D dV = \int_0^{2\pi} \int_0^{\pi} \int_0^{a} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

Inner: $\dfrac{\rho^3}{3} \sin\phi \Big|_0^a = \dfrac{a^3}{3} \sin\phi$

Middle: $-\dfrac{a^3}{3} \cos\phi \Big|_0^{\pi} = \dfrac{2}{3} a^3$

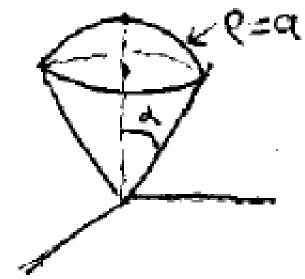
Outer: $\frac{4}{3}\pi a^3$ –as it should be.





2. Find limits in spherical coordinates which describe the region bounded by the sphere $\rho = a$ and the cone $\phi = \alpha$.

Answer: Limits: ρ: 0 to a, ϕ: 0 to α, θ: 0 to 2π.



3. Find limits for a solid spherical cap obtained by slicing a solid sphere of radius $a\sqrt{2}$ by a plane at a distance a from the center.

Answer: See the picture.

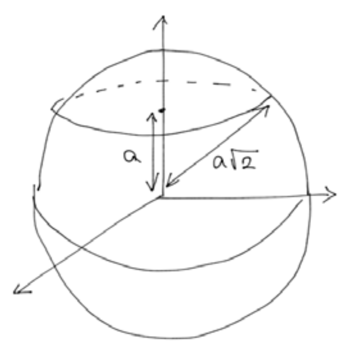


Figure 1: Sphere of radius $a\sqrt{2}$ sliced by the plane $z = a$.

Inner ρ: $a/\cos\phi$ to $a\sqrt{2}$, middle ϕ: 0 to $\pi/4$, outer θ: 0 to 2π.

Problems: Limits in Spherical Coordinates

1. Find the limits needed to use spherical coordinates to compute the volume of a sphere of radius a.

2. Find limits in spherical coordinates which describe the region bounded by the sphere $\rho = a$ and the cone $\phi = \alpha$.

3. Find limits for a solid spherical cap obtained by slicing a solid sphere of radius $a\sqrt{2}$ by a plane at a distance a from the center.

Changing Variables in Multiple Integrals

4. Changing coordinates in triple integrals

Here the coordinate change will involve three functions

$$u = u(x, y, z), \quad v = v(x, y, z) \quad w = w(x, y, z)$$

but the general principles remain the same. The new coordinates u, v, and w give a three-dimensional grid, made up of the three families of contour surfaces of u, v, and w. Limits are put in by the kind of reasoning we used for double integrals. What we still need is the formula for the new volume element dV.

To get the volume of the little six-sided region ΔV of space bounded by three pairs of these contour surfaces, we note that nearby contour surfaces are approximately parallel, so that ΔV is approximately a parallelepiped, whose volume is (up to sign) the 3×3 determinant whose rows are the vectors forming the three edges of ΔV meeting at a corner. These vectors are calculated as in section 2; after passing to the limit we get

$$dV = \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\, dv\, dw\ , \tag{24}$$

where the key factor is the **Jacobian**

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix} . \tag{25}$$

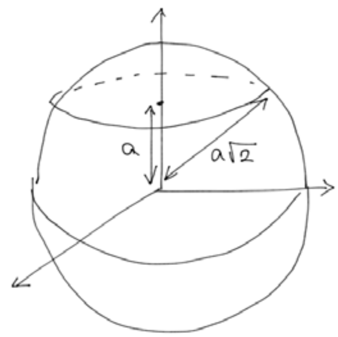
As an example, you can verify that this gives the correct volume element for the change from rectangular to spherical coordinates:

$$x = \rho \sin\phi \cos\theta, \qquad y = \rho \sin\phi \sin\theta, \qquad z = \rho \cos\phi; \tag{26}$$

while this is a good exercise, it will make you realize why most people prefer to derive the volume element in spherical coordinates by geometric reasoning.

Problems: Spherical Coordinates

1. Find the volume of a solid spherical cap obtained by slicing a solid sphere of radius $a\sqrt{2}$ by a plane at a distance a from the center. (See picture.)



Answer: In session 76 we found the limits:

inner ρ: $a/\cos\phi$ to $a\sqrt{2}$,

middle ϕ: 0 to $\pi/4$,

outer θ: 0 to 2π.

$$\text{Volume} = \int_0^{2\pi}\int_0^{\pi/4}\int_{a/\cos\phi}^{a\sqrt{2}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

Inner: $\displaystyle \frac{1}{3}\rho^3 \sin\phi\Big|_{a/\cos\phi}^{a\sqrt{2}} = \frac{2a^3\sqrt{2}}{3}\sin\phi - \frac{a^3\sin\phi}{3\cos^3\phi}.$

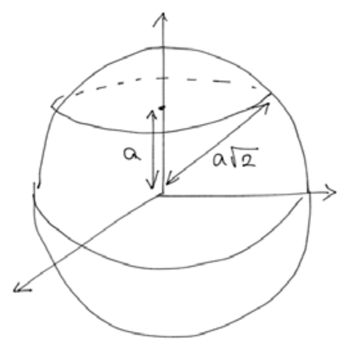
Middle: $\displaystyle \left[-\frac{2a^3\sqrt{2}}{3}\cos\phi - \frac{a^3}{6\cos^2\phi}\right]_{\phi=0}^{\pi/4} = -\frac{2a^3}{3} - \frac{a^3}{3} - \left(-\frac{2\sqrt{2}a^3}{3} - \frac{a^3}{6}\right) = \frac{2\sqrt{2}a^3}{3} - \frac{5a^3}{6}.$

Outer: $\displaystyle 2\pi\left(\frac{2\sqrt{2}a^3}{3} - \frac{5a^3}{6}\right) = \frac{a^3\pi}{3}(4\sqrt{2}-5) \approx 0.7a^3.$

The volume of the entire sphere is about $12a^3$ and we're looking at approximately the top sixth of its height. The sphere has more volume near its midpoint than at its top and bottom, so this answer seems reasonable.

Problems: Spherical Coordinates

1. Find the volume of a solid spherical cap obtained by slicing a solid sphere of radius $a\sqrt{2}$ by a plane at a distance a from the center. (See picture.)



Problems: Jacobian for Spherical Coordinates

Use the Jacobian to show that the volume element in spherical coordinates is the one we've been using.

Answer: $z = \rho\cos\phi, \quad x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta$

$$\begin{aligned}
\Rightarrow \quad \frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} &= \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} \\
&= \cos\phi \begin{vmatrix} \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix} + \rho\sin\phi \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix} \\
&= \cos\phi(\rho^2\sin\phi\cos\phi) + \rho\sin\phi\,\rho\sin^2\phi \\
&= \rho^2\sin\phi(\cos^2\phi + \sin^2\phi) \\
&= \rho^2\sin\phi \qquad \text{(as promised.)}
\end{aligned}$$

Problems: Jacobian for Spherical Coordinates

Use the Jacobian to show that the volume element in spherical coordinates is the one we've been using.

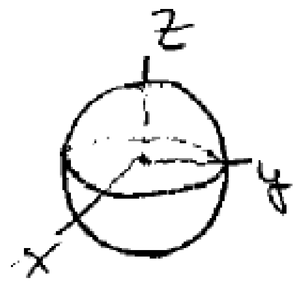
Integrals in Spherical Coordinates

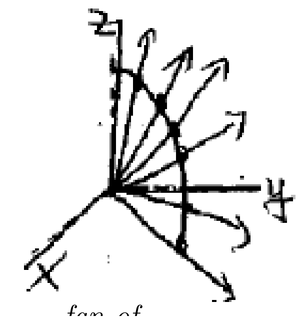
1. Find the volume of a sphere of radius a.

Answer: From the problems on limits in spherical coordinates (Session 76), we have limits: inner ρ: 0 to a –radial segments

middle ϕ: 0 to π –fan of rays.

outer θ: 0 to 2π –volume.





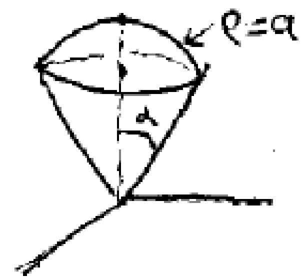
$$V = \iiint_D dV = \int_0^{2\pi} \int_0^{\pi} \int_0^{a} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

Inner: $\left. \dfrac{\rho^3}{3} \sin\phi \right|_0^a = \dfrac{a^3}{3} \sin\phi$

Middle: $\left. -\dfrac{a^3}{3} \cos\phi \right|_0^{\pi} = \dfrac{2}{3} a^3$

Outer: $\frac{4}{3}\pi a^3$ –as it should be.

2. Find the centroid of the region bounded by the sphere $\rho = a$ and the cone $\phi = \alpha$.



Answer: In Session 76 we computed the limits:

ρ: 0 to a, ϕ: 0 to α, θ: 0 to 2π.

By symmetry, $x_{cm} = y_{cm} = 0$.

$$z_{cm} = \frac{1}{V} \iiint_D z \, dV = \frac{1}{V} \int_0^{2\pi} \int_0^{\pi} \int_0^{a} \rho \cos\phi \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{1}{V} \int_0^{2\pi} \int_0^{\pi} \int_0^{a} \rho^3 \cos\phi \sin\phi \, d\rho \, d\phi \, d\theta.$$

Inner, middle and outer integrals are easy to compute: $z_{cm} = \dfrac{1}{V} \cdot \dfrac{\pi a^4 \sin^2\alpha}{4}$.

$$V = \iiint_D dV = \int_0^{2\pi} \int_0^{\pi} \int_0^{a} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{2}{3} \pi a^3 (1 - \cos\alpha).$$

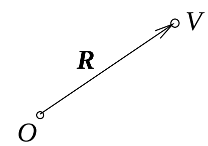
$$\Rightarrow \; z_{cm} = \frac{a^4 \sin^2\alpha \, \pi}{4} \cdot \frac{3}{2\pi a^3 (1 - \cos\alpha)} = \frac{3a}{8} \cdot \frac{\sin^2\alpha}{1 - \cos\alpha} = \frac{3}{8} a (1 + \cos\alpha).$$

Gravitational Attraction

We use triple integration to calculate the gravitational attraction that a solid body V of mass M exerts on a unit point mass placed at the origin.

If the solid V is also a point mass, then according to Newton's law of gravitation, the force it exerts is given by

$$\mathbf{F} \;=\; \frac{GM}{|\mathbf{R}|^2}\,\mathbf{r}, \tag{1}$$



where $\mathbf{R}$ is the position vector from the origin $\mathbf{0}$ to the point V, and the unit vector $\mathbf{r} = \mathbf{R}/|\mathbf{R}|$ is its direction.

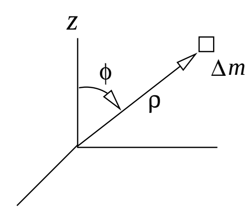
If however the solid body V is not a point mass, we have to use integration. We concentrate on finding just the $\mathbf{k}$ component of the gravitational attraction — all our examples will have the solid body V placed symmetrically so that its pull is all in the $\mathbf{k}$ direction anyway.

To calculate this force, we divide up the solid V into small pieces having volume ΔV and mass Δm. If the density function is $\delta(x,y,z)$, we have for the piece containing the point (x,y,z)

$$\Delta m \;\approx\; \delta(x,y,z)\Delta V, \tag{2}$$

Thinking of this small piece as being essentially a point mass at (x,y,z), the force $\mathbf{\Delta F}$ it exerts on the unit mass at the origin is given by (1), and its $\mathbf{k}$ component ΔF_z is therefore

$$\Delta F_z = \; G\,\frac{\Delta m}{|\mathbf{R}|^2}\,\mathbf{r}\cdot\mathbf{k}\;\;,$$



which in spherical coordinates becomes, using (2), and the picture,

$$\Delta F_z = \; G\,\frac{\cos\phi}{\rho^2}\,\delta\,\Delta V \;\;=\;\; G\,\frac{\delta\,\Delta V}{\rho^2}\;\cos\phi\;.$$

If we sum all the contributions to the force from each of the mass elements Δm and pass to the limit, we get for the $\mathbf{k}$-component of the gravitational force

$$F_z = G\iiint_V \frac{\cos\phi}{\rho^2}\,\delta\,dV\;. \tag{3}$$

If the integral is in spherical coordinates, then $dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$, and the integral becomes

$$F_z = G\iiint_V \delta\cos\phi\sin\phi\,d\rho\,d\phi\,d\theta\;. \tag{4}$$

Example 1. Find the gravitational attraction of the upper half of a solid sphere of radius a centered at the origin, if its density is given by $\delta = \sqrt{x^2+y^2}$.

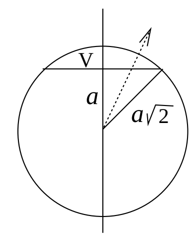
Solution. Since the solid and its density are symmetric about the z-axis, the force will be in the $\mathbf{k}$-direction, and we can use (3) or (4). Since

$$\sqrt{x^2+y^2} = r = \rho \sin\phi \ ,$$

the integral is

$$F_z = G \int_0^{2\pi} \int_0^{\pi/2} \int_0^a \rho \sin^2\phi \cos\phi \, d\rho \, d\phi \, d\theta$$

which evaluates easily to $\pi G a^2/3$.



Example 2. Let V be the solid spherical cap obtained by slicing a solid sphere of radius $a\sqrt{2}$ by a plane at a distance a from the center of the sphere. Find the gravitational attraction of V on a unit point mass at the center of the sphere. (Take the density to be 1.)

Solution. To take advantage of the symmetry, place the origin at the center of the sphere, and align the axis of the cap along the z-axis (so the flat side of the cap is parallel to the xy-plane).

We use spherical coordinates; the main problem is determining the limits of integration. If we fix ϕ and θ and let ρ vary, we get a ray which enters V at its flat side

$$z = a, \quad \text{or} \quad \rho \cos\phi = a,$$

and leaves V on its spherical side, $\rho = a\sqrt{2}$. The rays which intersect V in this way are those for which $0 \le \phi \le \pi/4$, as one sees from the picture. Thus by (4),

$$F_z \;=\; G \int_0^{2\pi} \int_0^{\pi/4} \int_{a/\cos\phi}^{a\sqrt{2}} \sin\phi\cos\phi \, d\rho \, d\phi \, d\theta,$$

which after integrating with respect to ρ (and θ) becomes

$$\begin{aligned} &= \; 2\pi G \int_0^{\pi/4} a\Big(\sqrt{2} - \frac{1}{\cos\phi}\Big) \sin\phi\cos\phi \, d\phi \\ &= \; 2\pi G a \Big(\frac{3\sqrt{2}}{4} - 1\Big) \ . \end{aligned}$$

Remark. Newton proved that a solid sphere of uniform density and mass M exerts the same force on an external point mass as would a point mass M placed at the center of the sphere. (See Problem 6a in problem section 5C).

This does *not* however generalize to other uniform solids of mass M — it is not true that the gravitational force they exert is the same as that of a point mass M at their center of mass. For if this were so, a unit test mass placed on the axis between two equal point masses M and M' ought to be pulled toward the midposition, whereas actually it will be pulled toward the closer of the two masses.

Problems: Applications of Spherical Coordinates

Find the average distance of a point in a solid sphere of radius a from:

(a) the center,

(b) a fixed diameter, and

(c) a fixed plane through the center.

Answer: Recall that the average value of a function $f(x, y, z)$ over a volume D is given by $\frac{1}{V}\iiint_D f(x, y, z)\, dV$. We know $V = \frac{4}{3}\pi a^3$. For each of these problems, we'll assume D is a sphere centered at the origin.

(a) In this case, $f(x, y, z) = \rho$ and so:

$$\begin{aligned}
\text{A.V.} &= \frac{1}{4\pi a^3/3}\int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^3 \sin\phi\, d\rho\, d\phi\, d\theta \\
&= \frac{1}{4\pi a^3/3}\int_0^{2\pi}\int_0^{\pi} \frac{1}{4}a^4 \sin\phi\, d\phi\, d\theta \\
&= \frac{3a}{16\pi}\int_0^{2\pi} 2\, d\theta \\
&= \frac{3a}{8\pi}(2\pi) = 3a/4.
\end{aligned}$$

(b) Here we'll use the z-axis as the diameter in question, in which case $f = r = \rho\sin\phi$.

$$\begin{aligned}
\text{A.V.} &= \frac{1}{4\pi a^3/3}\int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^3 \sin^2\phi\, d\rho\, d\phi\, d\theta \\
&= \frac{3}{4\pi a^3}\int_0^{2\pi}\int_0^{\pi} \frac{a^4}{4} \sin^2\phi\, d\phi\, d\theta \\
&= \frac{3a}{16\pi}\int_0^{2\pi} \frac{\pi}{2}\, d\theta \\
&= \frac{3a}{32}\cdot(2\pi) = 3\pi a/16.
\end{aligned}$$

(c) If we choose the xy-plane, $f = |z|$. Because spheres are symmetric the average value of the upper half will equal the average value over the whole sphere, so we compute just that ($V = \frac{2}{3}\pi a^3$).

$$\begin{aligned}
\text{A.V.} &= \frac{3}{2\pi a^3}\int_0^{2\pi}\int_0^{\pi/2}\int_0^{a} \rho^3 \cos\phi\sin\phi\, d\rho\, d\phi\, d\theta \\
&= \frac{3}{2\pi a^3}\int_0^{2\pi}\int_0^{\pi/2} \frac{a^4}{4} \cos\phi\sin\phi\, d\phi\, d\theta \\
&= \frac{3a}{8\pi}\int_0^{2\pi} \frac{1}{2}\, d\theta \\
&= \frac{3a}{16\pi}(2\pi) = 3a/8.
\end{aligned}$$

Problems: Applications of Spherical Coordinates

Find the average distance of a point in a solid sphere of radius a from:

(a) the center,

(b) a fixed diameter, and

(c) a fixed plane through the center.