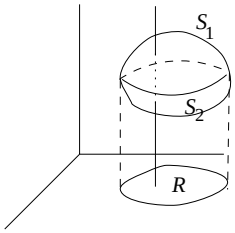
V10.2 The Divergence Theorem

2. Proof of the divergence theorem.

We give an argument assuming first that the vector field $\mathbf{F}$ has only a $\mathbf{k}$-component: $\mathbf{F} = P(x,y,z)\,\mathbf{k}$. The theorem then says

$$\iint_S P\,\mathbf{k}\cdot\mathbf{n}\,dS \;=\; \iiint_D \frac{\partial P}{\partial z}\,dV\ . \tag{4}$$

The closed surface S projects into a region R in the xy-plane. We assume S is vertically simple, i.e., that each vertical line over the interior of R intersects S just twice. (S can have vertical sides, however — a cylinder would be an example.) S is then described by two equations:



$$z = g(x,y) \quad \textit{(lower surface);} \qquad z = h(x,y) \quad \textit{(upper surface)} \tag{5}$$

The strategy of the proof of (4) will be to reduce each side of (4) to a double integral over R; the two double integrals will then turn out to be the same.

We do this first for the triple integral on the right of (4). Evaluating it by iteration, we get as the first step in the iteration,

$$\begin{aligned} \iiint_D \frac{\partial P}{\partial z}\,dV \;&=\; \iint_R \int_{g(x,y)}^{h(x,y)} \frac{\partial P}{\partial z}\,dz\,dx\,dy \\ &=\; \iint_R \big(P(x,y,h) - P(x,y,g)\big)\,dx\,dy \end{aligned} \tag{6}$$

To calculate the surface integral on the left of (4), we use the formula for the surface area element $d\mathbf{S}$ given in V9, (13):

$$d\mathbf{S} \;=\; \pm(-z_x\,\mathbf{i} - z_y\,\mathbf{j} + k)\,dx\,dy,$$

where we use the + sign if the normal vector to S has a positive k-component, i.e., points generally upwards (as on the upper surface here), and the − sign if it points generally downwards (as it does for the lower surface here).

This gives for the flux of the field $P\,\mathbf{k}$ across the upper surface S_2, on which $z = h(x,y)$,

$$\iint_{S_2} P\,\mathbf{k}\cdot d\mathbf{S} \;=\; \iint_R P(x,y,z)\,dx\,dy \;=\; \iint_R P(x,y,h(x,y))\,dx\,dy\ ,$$

while for the flux across the lower surface S_1, where $z = g(x,y)$ and we use the − sign as described above, we get

$$\iint_{S_1} P\,\mathbf{k}\cdot d\mathbf{S} \;=\; \iint_R -P(x,y,z)\,dx\,dy \;=\; \iint_R -P(x,y,g(x,y))\,dx\,dy\ ;$$

adding up the two fluxes to get the total flux across S, we have

$$\iint_S P\,\mathbf{k}\cdot d\mathbf{S} \;=\; \iint_R P(x,y,h)\,dx\,dy \;-\; \iint_R P(x,y,g)\,dx\,dy$$

which is the same as the double integral in (6). This proves (4). □

In the same way, if $\mathbf{F} = M(x, y, z)\ \mathbf{i}$ and the surface is simple in the $\mathbf{i}$ direction, we can prove

$$(4') \qquad \iint_S M\,\mathbf{i}\cdot\mathbf{n}\,dS \;=\; \iiint_D \frac{\partial M}{\partial x}\,dV$$

while if $\mathbf{F} = N(x, y, z)\ \mathbf{j}$ and the surface is simple in the $\mathbf{j}$ direction,

$$(4'') \qquad \iint_S N\,\mathbf{j}\cdot\mathbf{n}\,dS \;=\; \iiint_D \frac{\partial N}{\partial y}\,dV\ .$$

Finally, for a general field $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ and a closed surface S which is simple in all three directions, we have only to add up (4), (4′), and (4″). and we get the divergence theorem.

If the domain D is not bounded by a closed surface which is simple in all three directions, it can usually be divided up into smaller domains D_i which are bounded by such surfaces S_i. Adding these up gives the divergence theorem for D and S, since the surface integrals over the new faces introduced by cutting up D each occur twice, with the opposite normal vectors $\mathbf{n}$, so that they cancel out; after addition, one ends up just with the surface integral over the original S.

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Partial Differential Equations

An important application of the higher partial derivatives is that they are used in partial differential equations to express some laws of physics which are basic to most science and engineering subjects. In this section, we will give examples of a few such equations. The reason is partly cultural, so you meet these equations early and learn to recognize them, and partly technical: to give you a little more practice with the chain rule and computing higher derivatives.

A **partial differential equation**, PDE for short, is an equation involving some unknown function of several variables and one or more of its partial derivatives. For example,

$$x\,\frac{\partial w}{\partial x} \;-\; y\,\frac{\partial w}{\partial y} \;=\; 0$$

is such an equation. Evidently here the unknown function is a function of two variables

$$w \;=\; f(x,y)\;;$$

we infer this from the equation, since only x and y occur in it as independent variables. In general a **solution** of a partial differential equation is a differentiable function that satisfies it. In the above example, the functions

$$w \;=\; x^n y^n \qquad \text{any } n$$

all are solutions to the equation. In general, PDE's have many solutions, far too many to find all of them. The problem is always to find the one solution satisfying some extra conditions, usually called either *boundary conditions* or *initial conditions* depending on their nature.

Our first important PDE is the **Laplace equation** in three dimensions:

$$\frac{\partial^2 w}{\partial x^2} \;+\; \frac{\partial^2 w}{\partial y^2} \;+\; \frac{\partial^2 w}{\partial z^2} \;=\; 0\;. \tag{1}$$

Any steady-state temperature distribution in three-space

$$w \;=\; T(x,y,z), \qquad\qquad T = \text{temperature at the point } \;(x,y,z) \tag{2}$$

satisfies Laplace's equation. (Here *steady-state* means that it is unchanging over time, here reflected in the fact that T is not a function of time. For example, imagine a solid object made of some uniform heat-conducting material (say a solid metal ball), and imagine a steady temperature distribution on its surface is maintained somehow (say with some arrangement of wires and thermostats). Then after a while the temperature at each point inside the ball will come to equilibrium — reach a steady state — and the resulting temperature function (2) inside the ball will then satisfy Laplace's equation.

As another example, the *gravitational potential*

$$w \;=\; \phi(x,y,z)$$

resulting from some arrangement of masses in space satisfies Laplace's equation in any region R of space not containing masses. The same is true of the *electrostatic potential* resulting from some collection of electric charges in space: (1) is satisfied in any region which is free of charge. This potential function measures the work done (against the field) carrying a unit test mass (or charge) from a fixed reference point to the point (x, y, z) in the gravitational (or electrostatic) field. Knowing ϕ, the field itself can be recovered as its negative gradient:

$$\mathbf{F} \;=\; -\nabla\phi \;.$$

All of this is just to stress the fundamental character of Laplace's equation — we live our lives surrounded by its solutions.

The *two-dimensional* Laplace equation is similar — you just drop the term involving z. The steady-state temperature distribution in a flat metal plate would satisfy the two-dimensional Laplace equation, if the faces of the plate were kept insulated and a steady-state temperature distribution maintained around the edges of the plate.

If in the temperature model we include also heat sources and sinks in the region, unchanging over time, the temperature function satisfies the closely related **Poisson equation**

$$\frac{\partial^2 w}{\partial x^2} \;+\; \frac{\partial^2 w}{\partial y^2} \;+\; \frac{\partial^2 w}{\partial z^2} \;=\; f(x, y, z), \tag{3}$$

where f is some given function related to the sources and sinks.

Another important PDE is the **wave equation**; given below are the one-dimensional and two-dimensional versions; the three dimensional version would add a similar term in z to the left:

$$\frac{\partial^2 w}{\partial x^2} \;=\; \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2}; \qquad\qquad \frac{\partial^2 w}{\partial x^2} \;+\; \frac{\partial^2 w}{\partial y^2} \;=\; \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2}. \tag{4}$$

Here $x, y, \ldots$ are the space variables, t is the time, and c is the velocity with which the wave travels — this depends on the medium and the type of wave (light, sound, etc.). A solution, respectively

$$w \;=\; w(x, t), \qquad\qquad w \;=\; w(x, y, t),$$

gives for each moment t_0 of time the shape $w(x, t_0),\ w(x, y, t_0)$ of the wave.

The third PDE goes by two names, depending on the context: **heat equation** or **diffusion equation**. The one- and two-dimensional versions are respectively

$$\frac{\partial^2 w}{\partial x^2} \;=\; \frac{1}{a^2}\frac{\partial w}{\partial t}; \qquad\qquad \frac{\partial^2 w}{\partial x^2} \;+\; \frac{\partial^2 w}{\partial y^2} \;=\; \frac{1}{a^2}\frac{\partial w}{\partial t}. \tag{5}$$

It looks a lot like the wave equation (4), but the right-hand side this time involves only the first derivative, which gives it mathematically and physically an entirely different character.

When it is called the (one-dimensional) heat equation, a solution $w(x, t)$ represents a time-varying temperature distribution in say a uniform conducting metal rod, with insulated sides. In the same way, $w(x, y, t)$ would be the time-varying temperature distribution in a flat metal plate with insulated faces. For each moment t_0 in time, $w(x, y, t_0)$ gives the temperature distribution at that moment.

For example, if we assume the distribution is steady-state, i.e., not changing with time, then

$$\frac{\partial w}{\partial t} = 0 \qquad \text{(steady-state condition)}$$

and the two-dimensional heat equation would turn into the two-dimensional Laplace equation (1).

When (5) is referred to as the *diffusion equation*, say in one dimension, then $w(x,t)$ represents the concentration of a dissolved substance diffusing along a uniform tube filled with liquid, or of a gas diffusing down a uniform pipe.

Notice that all of these PDE's are second-order, that is, involve derivatives no higher than the second. There is an important fourth-order PDE in elasticity theory (the bilaplacian equation), but by and large the general rule seems to be either that Nature is content with laws that only require second partial derivatives, or that these are the only laws that humans are intelligent enough to formulate.

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Problems: Harmonic Functions and Averages

A function u is called *harmonic* if $\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = 0$. In this problem we will see that the average value of a harmonic function over any sphere is exactly its value at the center of the sphere.

For simplicity, we'll take the center to be the origin and show the average is $u(0,0,0)$.

Let u be a harmonic function and S_R the sphere of radius R centered at the origin. The average value of u over S is given by $A = \frac{1}{4\pi R^2} \iint_S u(x,y,z)\, dS$.

1. Write this integral explicitly using spherical coordinates.

Answer:

$$\begin{aligned} A &= \frac{1}{4\pi R^2} \int_0^{2\pi} \int_0^{\pi} u(R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi)\, R^2 \sin\phi\, d\phi\, d\theta \\ &= \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} u(R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi)\, \sin\phi\, d\phi\, d\theta. \end{aligned}$$

2. Differentiate A with respect to R

Answer: $\dfrac{dA}{dR} = \dfrac{1}{4\pi} \displaystyle\int_0^{2\pi} \int_0^{\pi} (u_x \sin\phi\cos\theta + u_y \sin\phi\sin\theta + u_z \cos\phi)\, \sin\phi\, d\phi\, d\theta.$

3. Rewrite the formula in part (2) in terms of $\nabla u \cdot \mathbf{n}$.

Answer: On S we have $\mathbf{n} = \dfrac{\langle x,y,z\rangle}{R} = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle$ and $dS = R^2 \sin\phi\, d\phi\, d\theta$

$$\Rightarrow \frac{dA}{dR} = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \langle u_x, u_y, u_z\rangle \cdot \mathbf{n}\, \frac{dS}{R^2} = \frac{1}{4\pi R^2} \iint_{S_R} \nabla u \cdot \mathbf{n}\, dS.$$

4. Use the divergence theorem to show $\frac{dA}{dR} = 0$ and conclude the average $A = u(0,0,0)$.

Answer: Let D be the solid ball of radius R. Applying the divergence theorem to part (3) we get

$$\frac{dA}{dR} = \frac{1}{4\pi R^2} \iiint_D \nabla \cdot \nabla u\, dV = \frac{1}{4\pi R^2} \iiint_D \nabla^2 u\, dV = 0.$$

For R near 0 the average is approximately $u(0,0,0)$.

Since the derivative is 0 the average is the same for any radius and we can let R go to 0 to conclude $A = u(0,0,0)$.

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