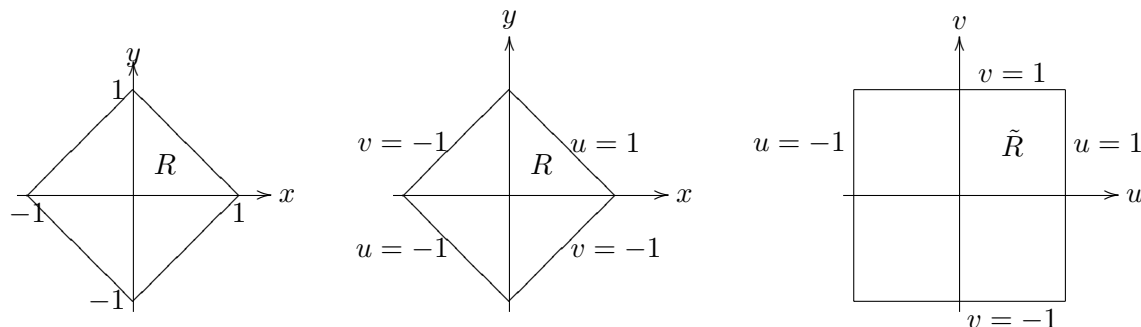# Problems: Change of Variables

Compute $\iint_R \left(\dfrac{x+y}{2-x+y}\right)^4 dx\,dy$, where R is the square with vertices at $(1,0)$, $(0,1)$, $(-1,0)$ and $(0,-1)$.

Answer: Since the region is bounded by the lines $x+y = \pm 1$ and $x - y = \pm 1$, we make a change of variables:

$$u = x + y \quad v = x - y.$$

Computing the Jacobian: $\dfrac{\partial(u,v)}{\partial(x,y)} = -2 \Rightarrow \dfrac{\partial(x,y)}{\partial(u,v)} = -1/2.$

Thus, $dx\,dy = \dfrac{1}{2}\,du\,dv.$

Using either method 1 or method 2 we see the boundaries are given by $u = \pm 1$, $v = \pm 1 \Rightarrow$

the integral is $\iint_R \left(\dfrac{x+y}{w-x+y}\right)^4 dx\,dy = \int_{-1}^1 \int_{-1}^1 \left(\dfrac{u}{2-v}\right)^4 \dfrac{1}{2}\,du\,dv.$

Inner integral $= \dfrac{u^5}{10(2-v)^4}\bigg|_{u=-1}^{u=1} = \dfrac{1}{5(2-v)^4}.$

Outer integral $= \dfrac{1}{15(2-v)^3}\bigg|_{-1}^{1} = \dfrac{26}{405} \approx .06.$

We're integrating the fourth power of a fraction whose numerator ranges between -1 and 1 and whose denominator ranges between 1 and 3. The value of this integrand will always be positive and will often be small, so this answer seems reasonable.

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Problems: Change of Variables

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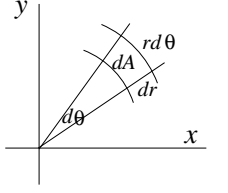
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Changing Variables in Multiple Integrals

2. The area element.

In polar coordinates, we found the formula $dA = r\,dr\,d\theta$ for the area element by drawing the grid curves $r = r_0$ and $\theta = \theta_0$ for the r, θ-system, and determining (see the picture) the infinitesimal area of one of the little elements of the grid.

For general u, v-coordinates, we do the same thing. The grid curves (4) divide up the plane into small regions ΔA bounded by these contour curves. If the contour curves are close together, they will be approximately parallel, so that the grid element will be approximately a small parallelogram, and

(13) $$\Delta A \approx \text{ area of parallelogram PQRS } = |PQ \times PR|$$

In the uv-system, the points P, Q, R have the coordinates

$$P : (u_0, v_0), \qquad Q : (u_0 + \Delta u, v_0), \qquad R : (u_0, v_0 + \Delta v) \ ;$$

to use the cross-product however in (13), we need PQ and PR in $\mathbf{i}\,\mathbf{j}$- coordinates. Consider PQ first; we have

(14) $$PQ = \Delta x\,\mathbf{i} + \Delta y\,\mathbf{j} \ ,$$

where Δx and Δy are the changes in x and y as you hold $v = v_0$ and change u_0 to $u_0 + \Delta u$. According to the definition of partial derivative,

$$\Delta x \approx \left(\frac{\partial x}{\partial u}\right)_0 \Delta u, \qquad \Delta y \approx \left(\frac{\partial y}{\partial u}\right)_0 \Delta u;$$

so that by (14),

(15) $$PQ \approx \left(\frac{\partial x}{\partial u}\right)_0 \Delta u\ \mathbf{i}\ + \ \left(\frac{\partial y}{\partial u}\right)_0 \Delta u\ \mathbf{j}\ .$$

In the same way, since in moving from P to R we hold u fixed and increase v_0 by Δv,

(16) $$PR \approx \left(\frac{\partial x}{\partial v}\right)_0 \Delta v\ \mathbf{i}\ + \ \left(\frac{\partial y}{\partial v}\right)_0 \Delta v\ \mathbf{j}\ .$$

We now use (13); since the vectors are in the xy-plane, $PQ \times PR$ has only a $\mathbf{k}$-component, and we calculate from (15) and (16) that

$$\mathbf{k}\text{-component of } PQ \times PR \approx \begin{vmatrix} x_u \Delta u & y_u \Delta u \\ x_v \Delta v & y_v \Delta v \end{vmatrix}_0$$

(17) $$= \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}_0 \Delta u \Delta v \ ,$$

where we have first taken the transpose of the determinant (which doesn't change its value), and then factored the Δu and Δv out of the two columns. Finally, taking the absolute value, we get from (13) and (17), and the definition (5) of Jacobian,

$$\Delta A \approx \left|\frac{\partial(x, y)}{\partial(u, v)}\right|_0 \Delta u \Delta v \ ;$$

passing to the limit as $\Delta u, \Delta v \to 0$ and dropping the subscript 0 (so that P becomes any point in the plane), we get the desired formula for the area element,

$$dA = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\,dv \ .$$
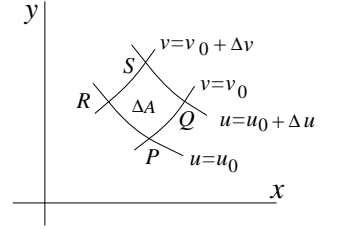
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Problems: Polar Coordinates and the Jacobian

1. Let $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\dfrac{y}{x}$. Directly calculate the Jacobian $\dfrac{\partial(r, \theta)}{\partial(x, y)} = \dfrac{1}{r}$.

Answer: Because we are familiar with the change of variables from rectangular to polar coordinates and we know that $\dfrac{\partial(r, \theta)}{\partial(x, y)} \cdot \dfrac{\partial(x, y)}{\partial(r, \theta)} = 1$, this result should not come as a surprise.

$$
\begin{aligned}
\frac{\partial(r, \theta)}{\partial(x, y)} &= \begin{vmatrix} r_x & r_y \\ \theta_x & \theta_y \end{vmatrix} \\[2mm]
&= \begin{vmatrix} \dfrac{2x}{2\sqrt{x^2+y^2}} & \dfrac{2y}{2\sqrt{x^2+y^2}} \\[2mm] \dfrac{-y/x^2}{1+(y/x)^2} & \dfrac{1/x}{1+(y/x)^2} \end{vmatrix} \\[2mm]
&= \begin{vmatrix} \dfrac{x}{\sqrt{x^2+y^2}} & \dfrac{y}{\sqrt{x^2+y^2}} \\[2mm] -\dfrac{y}{x^2+y^2} & \dfrac{x}{x^2+y^2} \end{vmatrix} \\[2mm]
&= \begin{vmatrix} \dfrac{x}{r} & \dfrac{y}{r} \\[2mm] -\dfrac{y}{r^2} & \dfrac{x}{r^2} \end{vmatrix} \\[2mm]
&= \frac{x^2 + y^2}{r^3} = \frac{1}{r}.
\end{aligned}
$$

2. For the change of variables $x = u$, $y = \sqrt{r^2 - u^2}$, write $dx\,dy$ in terms of u and r.

Answer: We know $dx\,dy = \left| \dfrac{\partial(x, y)}{\partial(u, r)} \right| du\,dr$.

$$
\begin{aligned}
\frac{\partial(x, y)}{\partial(u, r)} &= \begin{vmatrix} x_u & x_r \\ y_u & y_r \end{vmatrix} \\[2mm]
&= \begin{vmatrix} 1 & 0 \\[2mm] \dfrac{-u}{\sqrt{r^2-u^2}} & \dfrac{r}{\sqrt{r^2-u^2}} \end{vmatrix} \\[2mm]
&= \frac{r}{\sqrt{r^2 - u^2}}
\end{aligned}
$$

Hence $dx\,dy = \dfrac{r}{\sqrt{r^2 - u^2}}\, du\,dr$.

Problems: Polar Coordinates and the Jacobian

1. Let $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\frac{y}{x}$. Directly calculate the Jacobian $\dfrac{\partial(r, \theta)}{\partial(x, y)} = \dfrac{1}{r}$.

2. For the change of variables $x = u$, $y = \sqrt{r^2 - u^2}$, write $dx\, dy$ in terms of u and r.

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Changing Variables in Multiple Integrals

3. Examples and comments; putting in limits.

If we write the change of variable formula as

$$(18) \qquad \iint_R f(x,y)\,dx\,dy \;=\; \iint_R g(u,v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,du\,dv \;,$$

where

$$(19) \qquad \frac{\partial(x,y)}{\partial(u,v)} \;=\; \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}, \qquad g(u,v) = f(x(u,v), y(u,v)),$$

it looks as if the essential equations we need are the inverse equations:

$$(20) \qquad\qquad x = x(u,v), \qquad y = y(u,v)$$

rather than the direct equations we are usually given:

$$(21) \qquad\qquad u = u(x,y), \qquad v = v(x,y) \;.$$

If it is awkward to get (20) by solving (21) simultaneously for x and y in terms of u and v, sometimes one can avoid having to do this by using the following relation (whose proof is an application of the chain rule, and left for the Exercises):

$$(22) \qquad\qquad \frac{\partial(x,y)}{\partial(u,v)}\,\frac{\partial(u,v)}{\partial(x,y)} \;=\; 1$$

The right-hand Jacobian is easy to calculate if you know $u(x,y)$ and $v(x,y)$; then the left-hand one — the one needed in (19) — will be its reciprocal. Unfortunately, it will be in terms of x and y instead of u and v, so (20) still ought to be needed, but sometimes one gets lucky. The next example illustrates.

Example 3. Evaluate $\displaystyle\iint_R \frac{y}{x}\,dx\,dy$, where R is the region pictured, having as boundaries the curves $x^2 - y^2 = 1, \quad x^2 - y^2 = 4, \quad y = 0, \quad y = x/2 \;.$
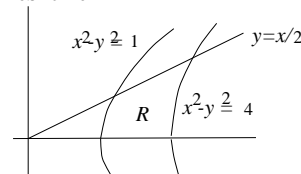
Solution. Since the boundaries of the region are contour curves of $x^2 - y^2$ and y/x, and the integrand is y/x, this suggests making the change of variable

$$(23) \qquad\qquad u \;=\; x^2 - y^2, \qquad v \;=\; \frac{y}{x} \;.$$

We will try to get through without solving these backwards for x, y in terms of u, v. Since changing the integrand to the u, v variables will give no trouble, the question is whether we can get the Jacobian in terms of u and v easily. It all works out, using (22):

$$\frac{\partial(u,v)}{\partial(x,y)} \;=\; \begin{vmatrix} 2x & -2y \\ -y/x^2 & 1/x \end{vmatrix} \;=\; 2 - 2y^2/x^2 \;=\; 2 - 2v^2; \qquad \text{so} \quad \frac{\partial(x,y)}{\partial(u,v)} \;=\; \frac{1}{2(1-v^2)} \;,$$

according to (22). We use now (18), put in the limits, and evaluate; note that the answer is positive, as it should be, since the integrand is positive.

$$
\begin{aligned}
\iint_R \frac{y}{x}\, dx\, dy &= \iint_R \frac{v}{2(1-v^2)}\, du\, dv \\
&= \int_0^{1/2} \int_1^4 \frac{v}{2(1-v^2)}\, du\, dv \\
&= -\frac{3}{4}\, \ln(1-v^2)\Big]_0^{1/2} \;=\; -\frac{3}{4}\, \ln\, \frac{3}{4}\ .
\end{aligned}
$$

Putting in the limits

In the examples worked out so far, we had no trouble finding the limits of integration, since the region R was bounded by contour curves of u and v, which meant that the limits were constants.

If the region is not bounded by contour curves, maybe you should use a different change of variables, but if this isn't possible, you'll have to figure out the uv-equations of the boundary curves. The two examples below illustrate.

Example 4. Let $u = x + y$, $v = x - y$; change $\displaystyle\int_0^1 \int_0^x dy\, dx$ to an iterated integral $du\, dv$.

Solution. Using (19) and (22), we calculate $\dfrac{\partial(x,y)}{\partial(u.v)} = -1/2$, so the Jacobian factor in the area element will be $1/2$.

To put in the new limits, we sketch the region of integration, as shown at the right. The diagonal boundary is the contour curve $v = 0$; the horizontal and vertical boundaries are not contour curves — what are their uv-equations? There are two ways to answer this; the first is more widely applicable, but requires a separate calculation for each boundary curve.
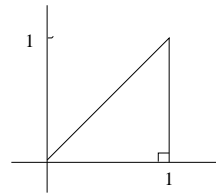
Method 1 *Eliminate x and y from the three simultaneous equations $u = u(x,y)$, $v = v(x,y)$, and the xy-equation of the boundary curve.* For the x-axis and $x = 1$, this gives

$$
\begin{cases} u = x + y \\ v = x - y \\ y = 0 \end{cases} \Rightarrow \quad u = v; \qquad\qquad
\begin{cases} u = x + y \\ v = x - y \\ x = 1 \end{cases} \Rightarrow \quad
\begin{cases} u = 1 + y \\ v = 1 - y \end{cases} \Rightarrow \quad u + v = 2.
$$

Method 2 *Solve for x and y in terms of u, v; then substitute $x = x(u,v)$, $y = y(u,v)$ into the xy-equation of the curve.*

Using this method, we get $x = \frac{1}{2}(u+v)$, $y = \frac{1}{2}(u-v)$; substituting into the xy-equations:

$$
y = 0 \;\Rightarrow\; \frac{1}{2}(u - v) = 0 \;\Rightarrow\; u = v; \qquad\qquad x = 1 \;\Rightarrow\; \frac{1}{2}(u + v) = 1 \;\Rightarrow\; u + v = 2.
$$

To supply the limits for the integration order $\iint du\, dv$, we

1. first hold v fixed, let u increase; this gives us the dashed lines shown;

2. integrate with respect to u from the u-value where a dashed line enters R (namely, $u = v$), to the u-value where it leaves (namely, $u = 2 - v$).

3. integrate with respect to v from the lowest v-values for which the dashed lines intersect the region R (namely, $v = 0$), to the highest such v-value (namely, $v = 1$).

Therefore the integral is $\displaystyle\int_0^1 \int_v^{2-v} \frac{1}{2} \, du \, dv$.
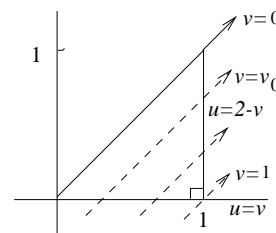
(As a check, evaluate it, and confirm that its value is the area of R. Then try setting up the iterated integral in the order $dv \, du$; you'll have to break it into two parts.)

Example 5. Using the change of coordinates $u = x^2 - y^2$, $v = y/x$ of Example 3, supply limits and integrand for $\displaystyle\iint_R \frac{dxdy}{x^2}$, where R is the infinite region in the first quadrant under $y = 1/x$ and to the right of $x^2 - y^2 = 1$.

Solution. We have to change the integrand, supply the Jacobian factor, and put in the right limits.

To change the integrand, we want to express x^2 in terms of u and v; this suggests eliminating y from the u, v equations; we get

$$u = x^2 - y^2, \quad y = vx \quad \Rightarrow \quad u = x^2 - v^2 x^2 \quad \Rightarrow \quad x^2 = \frac{u}{1 - v^2} \ .$$

From Example 3, we know that the Jacobian factor is $\dfrac{1}{2(1 - v^2)}$; since in the region R we have by inspection $0 \le v < 1$, the Jacobian factor is always positive and we don't need the absolute value sign. So by (18) our integral becomes

$$\iint_R \frac{dx \, dxy}{x^2} \ = \ \iint_R \frac{1 - v^2}{2u(1 - v^2)} \, du \, dv \ = \ \iint_R \frac{du \, dv}{2u}$$

Finally, we have to put in the limits. The x-axis and the left-hand boundary curve $x^2 - y^2 = 1$ are respectively the contour curves $v = 0$ and $u = 1$; our problem is the upper boundary curve $xy = 1$. To change this to $u - v$ coordinates, we follow Method 1:

$$\begin{cases} u = x^2 - y^2 \\ y = vx \\ xy = 1 \end{cases} \quad \Rightarrow \quad \begin{cases} u = x^2 - 1/x^2 \\ v = 1/x^2 \end{cases} \quad \Rightarrow \quad u = \frac{1}{v} - v \ .$$

The form of this upper limit suggests that we should integrate first with respect to u. Therefore we hold v fixed, and let u increase; this gives the dashed ray shown in the picture; we integrate from where it enters R at $u = 1$ to where it leaves, at $u = \dfrac{1}{v} - v$.
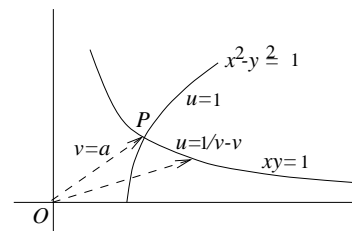
The rays we use are those intersecting R: they start from the lowest ray, corresponding to $v = 0$, and go to the ray $v = a$, where a is the slope of OP. Thus our integral is

$$\int_0^a \int_1^{1/v - v} \frac{du \, dv}{2u} \ .$$

To complete the work, we should determine a explicitly. This can be done by solving $xy = 1$ and $x^2 - y^2 = 1$ simultaneously to find the coordinates of P. A more elegant approach is to add $y = ax$ (representing the line OP) to the list of equations, and solve all three simultaneously for the slope a. We substitute $y = ax$ into the other two equations, and get

$$\begin{cases} ax^2 = 1 \\ x^2(1 - a^2) = 1 \end{cases} \quad \Rightarrow \quad a = 1 - a^2 \quad \Rightarrow \quad a = \frac{-1 + \sqrt{5}}{2} \, ,$$
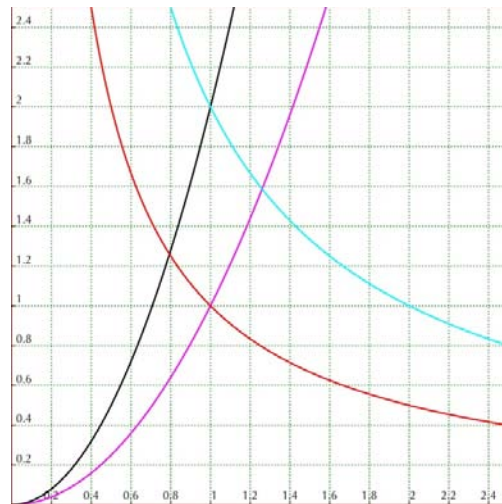
by the quadratic formula.

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Problems: Change of Variables Example

Use a change of variables to find the area of the region bounded by the curves $y = x^2$, $y = 2x^2$, $y = 1/x$, and $y = 2/x$.

Answer: Draw a picture:

The region is roughly diamond shaped. The top and bottom of the diamond have x coordinate 1, so we could split the area into left and right halves and compute the area using techniques from single variable calculus.

Instead, we rewrite the equations describing the boundary of the region as follows:

$$xy = 1, \quad xy = 2, \quad y/x^2 = 1, \quad y/x^2 = 2.$$

If we let $u = xy$ and $v = y/x^2$, the boundary curves become $u = 1$, $u = 2$, $v = 1$ and $v = 2$. The Jacobian is then:

$$\frac{\partial(u, v)}{\partial(x, y)} = \left| \begin{array}{cc} y & x \\ -\frac{2y}{x^3} & \frac{1}{x^2} \end{array} \right| = \frac{3y}{x^2} = 3v.$$

Noting that v is positive throughout the region, we get:

$$dx \, dy = \frac{1}{\left| \frac{\partial(u,v)}{\partial(x,y)} \right|} du \, dv = \frac{1}{3v} du \, dv.$$

Finally, we compute:

$$\text{Area} = \iint_R dx \, dy = \int_1^2 \int_1^2 \frac{1}{3v} du \, dv = \frac{1}{3} \ln 2 \approx 0.23.$$

We refer to our original sketch to confirm that this is a plausible result. We could also check our work by computing the area using single variable calculus.

Problems: Change of Variables Example

Use a change of variables to find the area of the region bounded by the curves $y = x^2$, $y = 2x^2$, $y = 1/x$, and $y = 2/x$.

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