7 Linear Programming

Let's Study

7.1 Linear Inequations in two variables.

7.1.1 Convex Sets.

7.1.2 Graphical representation of linear inequations in two variables.

7.1.3 Graphical solution of linear inequation.

7.2 Linear Programming Problem (L.P.P.).

7.2.1 Meaning of Linear Programming Problem.

7.2.2 Mathematical formulation of L.P. P.

7.2.3 Solution of L. P. P. by graphical methods.

Let's Recall

A linear equation in two variables $ax + by + c = 0$, where $a, b, c \in$ R, a and b are not zero simultaneously represents a straight line. A straight line makes disjoint parts of the plane. The points lying on the straight line and two half planes on either side, which are represented by $ax + by + c < 0$ and $ax + by + c > 0$. We will now study the two half planes made by a line.

The sets of points $\{(x, y) \mid ax + by + c < 0\}$ and $\{(x, y) \mid ax + by + c > 0\}$ are two open half planes. The sets of points $\{(x, y) \mid ax + by + c \leq 0\}$ and $\{(x, y) \mid ax + by + c \geq 0\}$ are two half, planes with common points. The sets $\{(x, y) \mid ax + by + c \leq 0\}$ and $\{(x, y) \mid ax + by + c \geq 0\}$ have the common boundary $\{(x, y) \mid ax + by + c = 0\}$.

Linear inequation in two variables.

Definition : A linear inequation in two variables x, y is a mathematical expression of the form $ax + by < c$ or $ax + by > c$ where $a \neq 0$, $b \neq 0$ simultaneously and $a, b \in$ R.

Activity : Check which of the following ordered pairs is a solution of $2x + 3y - 6 \leq 0$.

1. $(1, -1)$ 2. $(2, 1)$ 3. $(-2, 1)$ 4. $(-1, -2)$ 5. $(-3, 4)$

Solution :

Sr. No.	(x, y)	Inequation	Conclusion
1.	(1, -1)	$2(1) + 3(-1) - 6 = -1 < 0$	is a solution
2.	(2, 1)	$2(2) + 3(1) - 6 = 1 > 0$	
3.	(-2, 1)		
4.	(1, -2)		
5.	(-3, 4)		

Graphical representation of linear inequation in two variables $ax + by$ ($\leq$ or $\geq$) c is a region on any one side of the straight line ax + by = c in the coordinate system, depending on the sign of inequation.

7.1.1 : Convex set :

Definition : A set of points in a plane is said to be a convex set if line segment joining any two points of the set entirely lies within the set.

The following sets are convex sets :

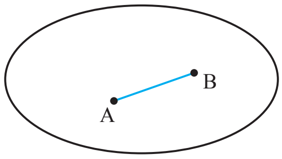

fig 7.1(a)

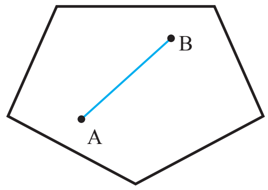

fig 7.1(b)

The following sets are not convex sets :

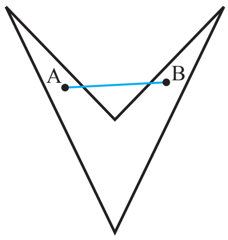

fig 7.1(c)

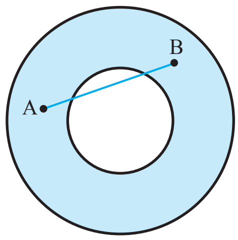

fig 7.1(d)

Note :

(i) The convex sets may be bounded.

Following are bounded convex sets.

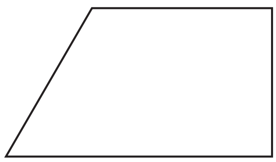

fig 7.2(a)

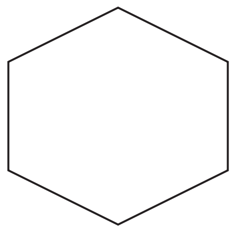

fig 7.2(b)

(ii) Convex sets may be unbounded.

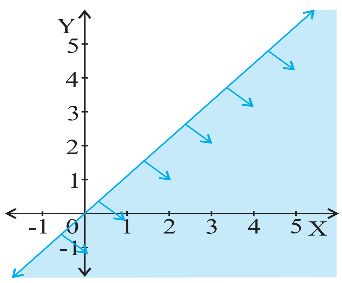

fig 7.3(a)

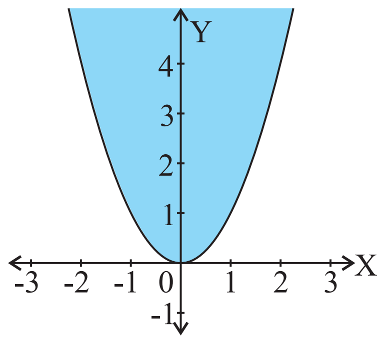

fig 7.3(b)

Note :

1) Graphical representations of $x \leq h$ and $x \geq h$ on the Cartesian coordinate system. Draw the line $x = h$ in XOY plane.

The solution set is the set of points lying on the Left side or Right side of the line $x = h$.

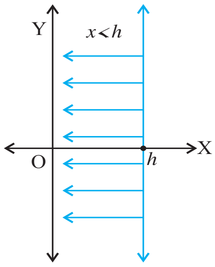

fig 7.4

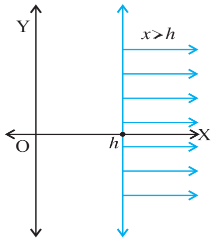

fig 7.5

2) Graphical representation of $y \leq k$ and $y \geq k$ on the Cartesian coordinate system.
Draw the line $y = k$ in XOY plane.
The solution set is the set of points lying below or above the line $y = k$.

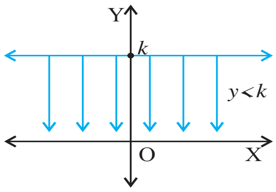

fig 7.6

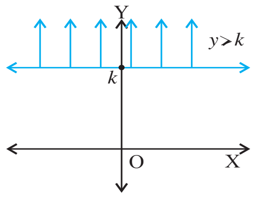

fig 7.7

3) Graphical representations of $ax + by \leq 0$ and ax+by ≥ 0 on the Cartesian coordinate system.
The line $ax + by = 0$ passes through the origin, see the following graphs.

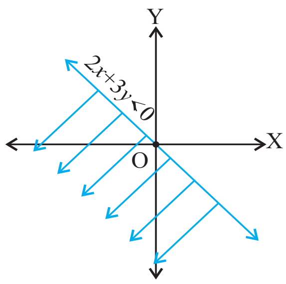

fig 7.8

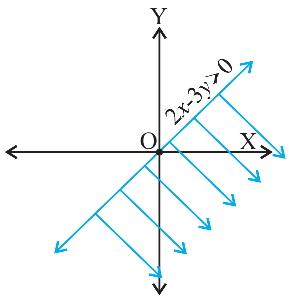

fig 7.9

4) To find the solutions of $ax + by \leq c$ and $ax + by \geq c$ graphically,
Draw the line $ax + by = c$ in XOY system. It divides the plane into two parts, each part is called half plane :

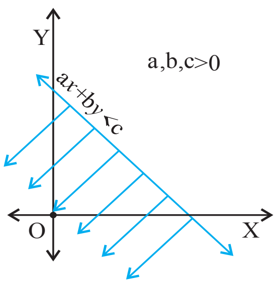

fig 7.10

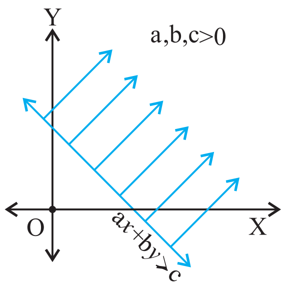

fig 7.11

The half plane H_1, containing origin is called origin side of the line and the other half plane H_2 is non-origin side of the line $ax + by = c$.

For convenience, consider O (0, 0) as test point. If O (0, 0) satisfies the given inequation

$ax + by \leq c$, then the required region is on origin side. Hence shade the region the H_1 otherwise shade the other half plane H_2.

The shaded portion represents the solution set of the given inequation.

Note that the points $\{(x, y)\} \mid ax + by = c\}$ form the common boundary of the two half planes.

Solved Examples

Ex.1 Show the solution sets for the following inequations graphically.

a) $x \leq 3$ b) $y \geq -2$ c) $x + 2y \leq 0$

d) $2x + 3y \geq 6$ e) $2x - 3y \geq -6$ f) $4x - 5y \leq 20$.

Solution :

a) To draw :$x \leq 3$

Draw line:$x = 3$

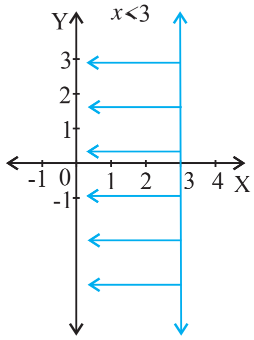

fig 7.12

b)To draw: $y \geq -2$

Draw line: $y = -2$

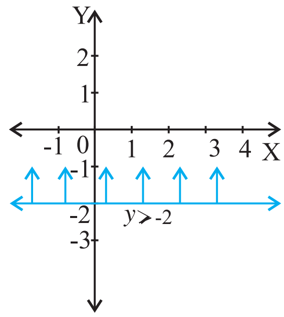

fig 7.13

c) To draw :$x + 2y \leq 0$

Draw line: $x = -2y$

x	0	–2	–4	2
$y = -\frac{x}{2}$	0	1	2	–1

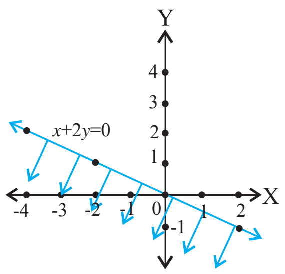

fig 7.14

As the origin lies on the line $x = -2y$, we have to choose another point as a test point. Let us choose $(-2,-2)$ =-2–4 = –6 < 0

$x + 2y = -2 - 4 = -6 < 0$

d) To draw : $2x + 3y \geq 6$

Draw line : $2x + 3y = 6$

x	y	(x, y)
3	0	(3, 0)
0	2	(0, 2)

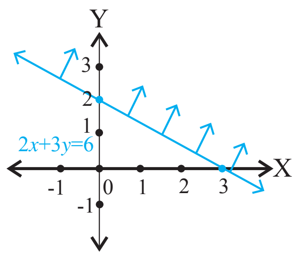

fig 7.15

$2x+3y\big|_{(0,0)} = 0<6$. Therefore, the required region is the non-origin side of the line.

e) To draw : $2x - 3y \geq -6$

Draw line : $2x - 3y = -6$

x	y	(x, y)
–3	0	(–3, 0)
0	2	(0, 2)

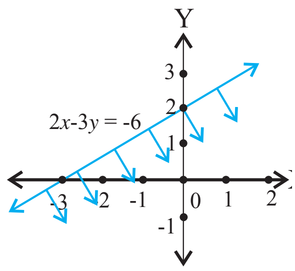

fig 7.16

$2x+3y\big|_{(0,0)} = 0 > -6$. Therefore, the required region is the origin side of the line.

f) To draw : $4x - 5y \leq 20$

Draw line : $4x - 5y = 20$

x	y	(x, y)
5	0	(5, 0)
0	–4	(0, –4)

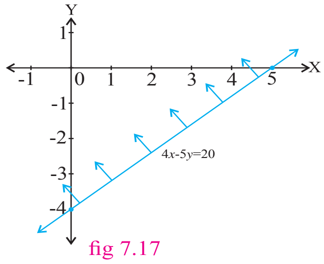

fig 7.17

$4x-5y\big|_{(0,0)} = 0 < 20$. Therefore, the required region is the origin side of the line.

Ex. 2 : Represent the solution set of inequation $3x + 2y \leq 6$ graphically.

Solution :

To draw : $3x + 2y \leq 6$

Draw the line : $3x + 2y = 6$

x	y	(x, y)
2	0	(2, 0)
0	3	(0, 3)

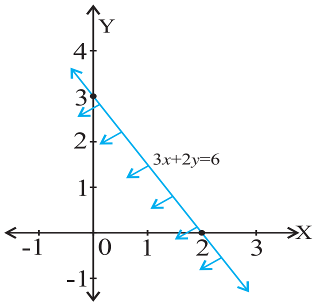

fig 7.18

$3x+2y\big|_{(0,0)} = 0<6.$ Therefore, the required region is the origin side of the line.

Ex. 3 : Find the common region of the solutions of the inequations $x + 2y \geq 4$, $2x - y \leq 6$.

Solution :

To find the common region of :$x + 2y \geq 4$ and $2x - y \leq 6$

Draw the lines : $x + 2y = 4$ and $2x - y = 6$

Equation of line	x	y	Line passes through (x, y)	Sign	Region
$x + 2y = 4$	4	0	(4, 0)	$\geq$	Non-origin side
	0	2	(0, 2)		
$2x - y = 6$	3	0	(3, 0)	$\leq$	Origin side
	0	–6	(0, –6)		

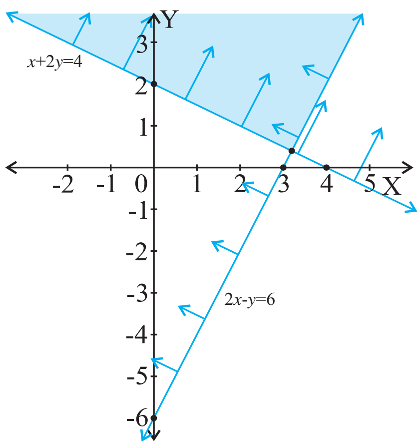

fig 7.19

Notes :

There are two methods of plotting the lines :

1) Find the intersection points of the line with X axis and Y axis.

2) Write the equation of the straight line in the double intercept form.

For example : consider the line $3x + 2y = 6$.

1) The intersection with X axis is given when y = 0, So A (2, 0) is the point of intersection with the x axis, The intersection with Y axis is given when x = 0.

So B (0, 3) is the point of intersection with the y axis. We draw the line through A and B.

2) The equation of line is $3x + 2y = 6$ Divide both sides by 6.

We get the double intercept form $\frac{x}{2} + \frac{y}{3} = 1$

$\therefore$ Intercepts on X axis and Y axis are 2 and 3 respectively.

The points (2, 0), (0, 3) lie on the line.

Ex. 4 : Find the graphical solution of $3x + 4y \le 12$, and $x - 4y \le 4$

Solution : To find the graphical solution of : $3x + 4y \le 12$ and $x - 4y \le 4$

Draw the lines : $L_1 : 3x + 4y = 12$ and $L_2 : x - 4y = 4$.

Equation of line	x	y	Line passes through (x, y)	Sign	Region
$3x + 4y = 12$	4	0	(4, 0)	$\le$	Origin side
	0	3	(0, 3)		
$x - 4y = 4$	4	0	(4, 0)	$\le$	Origin side
	0	–1	(0, –1)		

The common shaded region is graphical solution.

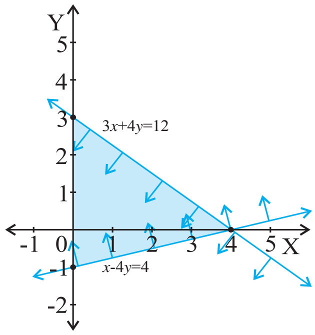

fig 7.20

Exercise 7.1

1) Solve graphically :

i) $x \ge 0$ ii) $x \le 0$ iii) $y \ge 0$ iv) $y \le 0$

2) Solve graphically :

i) $x \ge 0$ and $y \ge 0$ ii) $x \le 0$ and $y \ge 0$ iii) $x \le 0$ and $y \le 0$ iv) $x \ge 0$ and $y \le 0$

3) Solve graphically :

i) $2x - 3 \ge 0$ ii) $2y - 5 \ge 0$ iii) $3x + 4 \le 0$ iv) $5y + 3 \le 0$

4) Solve graphically :

i) $x + 2y \le 6$ ii) $2x - 5y \ge 10$ iii) $3x + 2y \ge 0$ iv) $5x - 3y \le 0$

5) Solve graphically :

i) $2x + y \geq 2$ and $x - y \leq 1$ ii) $x - y \leq 2$ and $x + 2y \leq 8$

iii) $x + y \geq 6$ and $x + 2y \leq 10$ iv) $2x + 3y \leq 6$ and $x + 4y \geq 4$

v) $2x + y \geq 5$ and $x - y \leq 1$

Solved Examples

Ex. 1 : Find the graphical solution of the system of inequations.

$$2x + y \leq 10,\ 2x - y \leq 2,\ x \geq 0,\ y \geq 0$$

Solution : To find the solution of the system of given inequations -

Draw lines	x	y	Line passes through (x, y)	Sign	Region
L_1 $2x + y = 10$	0	10	(0, 10)	$\leq$	Origin side of L_1
	5	0	(5, 0)		
L_2 $2x - y = 2$	0	–2	(0, –2)	$\leq$	Origin side of L_2
	1	0	(1, 0)		

The common shaded region OABCO is the graphical solution. This graphical solution is known as feasible solution.

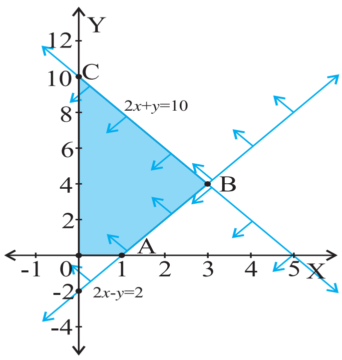

fig 7.21

Remark : The restrictions $x \geq 0$, $y \geq 0$, are called non-negativity constraints.

Definition : A solution which satisfies all the constraints is called a feasible solution.

Ex. 2 : Find the feasible solution of the system of inequations $3x + 4y \geq 12$, $2x + 5y \geq 10$, $x \geq 0$, $y \geq 0$.

To draw (Inequations)	Draw line	x	y	Line passes through (x, y)	Sign	Region
$3x + 4y \geq 12$	L_1 $3x + 4y = 12$	0	3	(0, 3)	$\geq$	Non-origin side of Line L_1
		4	0	(4, 0)		
$2x + 5y \geq 10$	L_2 $2x + 5y = 10$	0	2	(0, 2)	$\geq$	Non-origin side of Line L_2
		5	0	(5, 0)		

Solution :

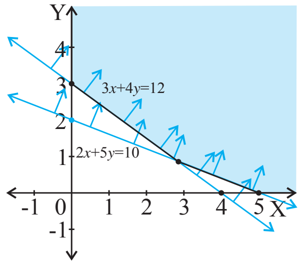

fig 7.22

Common shaded region is the feasible solution.

Ex. 3 :

A manufacturer produces two items A and B. Both are processed on two machines I and II. A needs 2 hours on machine I and 2 hours on machine II. B needs 3 hours on machine I and 1 hour on machine II. If machine I can run maximum 12 hours per day and II for 8 hours per day, construct a problem in the form of inequations and find its feasible solution graphically.

Solution :

Let x units of product A and y units of product B be produced.

$x \geq 0, y \geq 0$.

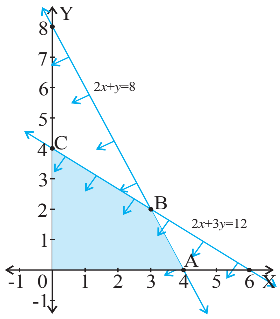

fig 7.23

Tabular form is:

Machine	Product A (x)	Product B (y)	Availability
I	2	3	12
II	2	1	08

Inequations are $2x + 3y \leq 12$, $2x + y \leq 8$, $x \geq 0$, $y \geq 0$.

To draw graphs of the above inequations :

To draw	Draw line	x	y	Line passes through (x, y)	Sign	Region lie on
$2x + 3y \leq 12$	L_1 $2x + 3y = 12$	0	4	(0, 4)	$\leq$	Origin side of Line L_1
		6	0	(6, 0)		
$2x + 4 \leq 8$	L_2 $2x + 4 = 8$	0	8	(0, 8)	$\leq$	Origin side of Line L_2
		4	0	(4, 0)		

The common shaded region OABCO the feasible region.

Exercise 7.2

I) Find the feasible solution of the following inequations graphically.

1) $3x + 2y \leq 18$, $2x + y \leq 10$, $x \geq 0$, $y \geq 0$
2) $2x + 3y \leq 6$, $x + y \geq 2$, $x \geq 0$, $y \geq 0$
3) $3x + 4y \geq 12$, $4x + 7y \leq 28$, $y \geq 1$, $x \geq 0$
4) $x + 4y \leq 24$, $3x + y \leq 21$, $x + y \leq 9$, $x \geq 0$, $y \geq 0$
5) $0 \leq x \leq 3$, $0 \leq y \leq 3$, $x + y \leq 5$, $2x + y \geq 4$
6) $x - 2y \leq 2$, $x + y \geq 3$, $-2x + y \leq 4$, $x \geq 0$, $y \geq 0$
7) A company produces two types of articles A and B which requires silver and gold. Each unit of A requires 3 gm of silver and 1 gm of gold, while each unit of B requires 2 gm of silver and 2 gm of gold. The company has 6 gm of silver and 4 gm of gold. Construct the inequations and find the feasible solution graphically.
8) A furniture dealer deals in tables and chairs. He has Rs.1,50,000 to invest and a space to store at most 60 pieces. A table costs him Rs.1500 and a chair Rs.750. Construct the inequations and find the feasible solution.

7.2 Linear Programming Problems (L.P.P.) :

L.P.P. is an optimization technique used in different fields such as management, planning, production, transportation etc. It is developed during the second world war to optimize the utilization of limited resources to get maximum returns. Linear Programming is used to minimize the cost of production and maximizing the profit. These problems are related to efficient use of limited resources like raw materials, man-power, availability of machine time and cost of the material and so on.

Linear Programming is mathematical technique designed to help managers in the planning and decision making. Programming problems are also known as optimization problems. The mathematical programming involves optimization of a certain function, called objective function, subject to given conditions or restrictions known as constraints.

7.2.1 Meaning of L.P.P. :

Linear implies all the mathematical functions contain variables of index at most one. A L.P.P. may be defined as the problem of maximizing or minimizing a linear function subject to linear constraints. These constraints may be equations or inequations.

Now, we formally define the terms related to L.P.P. as follows :

1) **Decision variables :** The variables involved in L.P.P. are called decision variables.
2) **Objective function :** A linear function of variables which is to be optimized, i.e. either maximized or minimized, is called an objective function.
3) **Constraints :** Conditions under which the objective function is to be optimized are called constraints. These constraints are in the form of equations or inequations.
4) **Non-negativity constraints :** In some situations, the values of the variables under considerations may be positive or zero due to the imposed constraints, Such constraints are referred as non-negativity constraints.

7.2.2 Mathematical formulations of L.P.P. :

Step 1) : Identify the decision variables (x, y) or (x_1, x_2).

Step 2) : Identify the objective function and write it as mathematical expression in terms of decision variables.

Step 3) : Identify the different constraints and express them as mathematical equations / inequations.

Note : i) We shall study L.P.P. with at most two variables.

ii) We shall restrict ourselves to L.P.P. involving non-negativity constraints.

Solved examples

Ex. 1 : A Toy manufacturer produces bicycles and tricycles, each of which must be processed through two machine A and B. Machine A has maximum of 120 hours available and machine B has a maximum of 180 hours available. Manufacturing a bicycle requires 4 hours on machine A and 10 hours on machine B. Manufacturing a tricycle required 6 hours on machine A and 3 hours on machine B. If profits are Rs.65 for a bicycle and Rs.45 for a tricycle, formulate L.P.P. to have maximum profit.

Solution :

Let z be the profit, which can be made by manufacturing and selling x tricycles and y bicycles.

$x \geq 0, y \geq 0$

$\therefore$ Total Profit $z = 45x + 65y$

Maximize $z = 45x + 65y$

It is given that

Machine	Tricycles (x)	Bicycles (y)	Availability
A	6	4	120
B	3	10	180

From the above table, remaining conditions are $6x + 4y \leq 120$, $3x + 10y \leq 180$.

$\therefore$ The required formulated L.P.P. is as follows :

Maximize $\quad z = 45x + 65y$

Subject to the constraints $\quad x \geq 0, y \geq 0$

$6x + 4y \leq 120$

$3x + 10y \leq 18\ 0$

Ex. 2 : A company manufactures two types of toys A and B. Each toy of type A requires 2 minutes for cutting and 1 minute for assembling. Each toy of type B requires 3 minutes for cutting and 4 minutes for assembling. There are 3 hours available for cutting and 2 hours are available for assembling. On selling a toy of type A the company gets a profit of Rs.10 and that on toy of type B is Rs. 20. Formulate the L.P.P. to maximize profit.

Solution :

Suppose, the company manufacters x toys of type A and y toys of type B.

$x \geq 0, y \geq 0$

Let P be the total profit

On selling a toy of type A, company gets Rs.10 and that on a toy of type B is Rs.20.

$\therefore$ total profit on selling x toy s of type A and y toys of type B is $p = 10x + 20y$.

$\therefore$ maximize $\quad p = 10x + 20y$.

The conditions are

$2x + 3y \leq 180, \qquad x + 4y \leq 120, \qquad x \geq 0, \qquad y \geq 0.$

Ex. 3 : A horticulturist wishes to mix two brands of fertilizers that will provide a minimum of 15 units of potash, 20 units of nitrate and 24 units of phosphate. One unit of brand I provides 3 units of potash, 1 unit of nitrate, 3 units of phosphate. One unit of brand II provides 1 unit of potash, 5 units of nitrate and 2 units of phosphates. One unit of brand I costs Rs. 120 and one unit of brand II costs Rs.60 per unit. Formulate this problems as L.P.P. to minimize the cost.

Solution :

Let z be the cost of mixture prepared by mixing x units of brand I and y units of brand II.

Then $x \geq 0$, $y \geq 0$.

Since, 1 unit of brand I costs Rs.120.

1 unit of brand II costs Rs.60.

$\therefore$ total cost $\quad z = 120x + 60y$.

$\therefore$ Minimize $\quad z = 120x + 60y$.

Brand / Content	I per unit	II per unit	Minimum requirement
Potash	3	1	15
Nitrate	1	5	20
Phosphate	3	2	24

The conditions are $\quad 3x + y \geq 15$,

$x + 5y \geq 20$,

$3x + 2y \geq 24$.

The L.P.P. is

Maximize $\quad z = 120x + 60y$ subject to the above constraints.

Exercise 7.3

1) A manufacturing firm produces two types of gadgets A and B, which are first processed in the foundry and then sent to machine shop for finishing. The number of man hours of labour required in each shop for production of A and B per unit and the number of man hours available for the firm are as follows :

Gadgets	Foundry	Machine Shop
A	10	5
B	6	4
Time available (hour)	60	35

Profit on the sale of A is Rs. 30 and B is Rs. 20 per units. Formulate the L.P.P. to have maximum profit.

2) In a cattle breading firm, it is prescribed that the food ration for one animal must contain 14, 22 and 1 units of nutrients A, B and C respectively. Two different kinds of fodder are available. Each unit of these two contains the following amounts of these three nutrients :

Nutrient \ Fodder	Fodder 1	Fodder 2
Nutrients A	2	1
Nutrients B	2	3
Nutrients C	1	1

The cost of fodder 1 is Rs.3 per unit and that of fodder Rs. 2, Formulate the L.P.P. to minimize the cost.

3) A company manufactures two types of chemicals A and B. Each chemical requires two types of raw material P and Q. The table below shows number of units of P and Q required to manufacture one unit of A and one unit of B and the total availability of P and Q.

Raw Material \ Chemical	A	B	Availability
P	3	2	120
Q	2	5	160

The company gets profits of Rs.350 and Rs.400 by selling one unit of A and one unit of B respectively. (Assume that the entire production of A and B can be sold). How many units of the chemicals A and B should be manufactured so that the company get maximum profit? Formulate the problem as L.P.P. to maximize the profit.

4) A printing company prints two types of magazines A and B. The company earns Rs. 10 and Rs. 15 on magazines A and B per copy. These are processed on three machines I, II, III. Magazine A requires 2 hours on Machine I, 5 hours on Machine II and 2 hours on Machine III. Magazine B requires 3 hours on Machine I, 2 hours on Machine II and 6 hours on Machine III. Machines I, II, III are available for 36, 50, 60 hours per week respectively. Formulate the L.P.P. to determine weekly production of A and B, so that the total profit is maximum.

5) A manufacture produces bulbs and tubes. Each of these must be processed through two machines M_1 and M_2. A package of bulbs require 1 hour of work on Machine M_1 and 3 hours of work on M_2. A package of tubes require 2 hours on Machine M_1 and 4 hours on Machine M_2. He earns a profit of Rs. 13.5 per package of bulbs and Rs. 55 per package of tubes. Formulate the LLP to maximize the profit, if he operates the machine M_1, for atmost 10 hours a day and machine M_2 for atmost 12 hours a day.

6) A company manufactures two types of fertilizers F_1 and F_2. Each type of fertilizer requires two raw materials A and B. The number of units of A and B required to manufacture one unit of fertilizer F_1 and F_2 and availability of the raw materials A and B per day are given in the table below :

Fertilizers / Raw Material	F1	F2	Availability
A	2	3	40
B	1	4	70

By selling one unit of F_1 and one unit of F_2, company gets a profit of Rs. 500 and Rs. 750 respectively. Formulate the problem as L.P.P. to maximize the profit.

7) A doctor has prescribed two different units of foods A and B to form a weekly diet for a sick person. The minimum requirements of fats, carbohydrates and proteins are 18, 28, 14 units respectively. One unit of food A has 4 units of fats. 14 units of carbohydrates and 8 units of protein. One unit of food B has 6 units of fat, 12 units of carbohydrates and 8 units of protein. The price of food A is 4.5 per unit and that of food B is 3.5 per unit. Form the L.P.P. so that the sick person's diet meets the requirements at a minimum cost.

8) If John drives a car at a speed of 60 kms/hour he has to spend Rs. 5 per km on petrol. If he drives at a faster speed of 90 kms/hour, the cost of petrol increases to 8 per km. He has Rs. 600 to spend on petrol and wishes to travel the maximum distance within an hour. Formulate the above problem as L.P.P.

9) The company makes concrete bricks made up of cement and sand. The weight of a concrete brick has to be least 5 kg. Cement costs Rs.20 per kg. and sand costs of Rs.6 per kg. strength consideration dictate that a concrete brick should contain minimum 4 kg. of cement and not more than 2 kg. of sand. Form the L.P.P. for the cost to be minimum.

7.2.3 Formal definitions related to L.P.P. :

1) **Solution of L.P.P. :** A set of values of the decision variables x_1, x_2, ..., x_n which satisfy the conditions of given linear programming problem is called a solution to that problem.

2) **Feasible solution :** A solution which satisfies the given constraints is called a feasible solution.

3) **Optimal feasible solution :** A feasible solution which maximizes or minimizes the objective function as per the requirements is called an optimal feasible solution.

4) **Feasible region :** The common region determined by all the constraints of the L.P.P. is called the feasible region.

Solution of L.P.P. :

There are two methods to find the solution of L.P.P. :

1) Graphical method, 2) Simplex method.

Note : We shall restrict ourselves to graphical method.

Some definitions :

Solution :

A set of values of the variables which satisfies all the constraints of the L.P.P. is called the solution of the L.P.P.

Optimum feasible solution :

A feasible solution which optimizes (either maximizes or minimizes) the objective function of L.P.P. is called optimum feasible solution.

Theorems (without proof) :

Theorem 1: The set of all feasible solutions of L.P.P. is a convex set.

Theorem 2: The objective function of L.P.P. attains its optimum value (either maximum or minimum) at least at one of the vertices of convex polygon. This is known as convex polygon theorem.

Corner - Point Method :

1) Convert all inequations of the constraints into equations..
2) Draw the lines in X - Y plane.
3) Locate common region indicated by the constraints. This common region is feasible region.
4) Find the vertices of feasible region.
5) Find the value of the objective function z at all vertices of feasible region.

Suppose, we are expected to maximize or minimize a given objective function $z = ax + by$ in the feasible region. The feasible region is a convex region bounded by straight lines. If any linear function $z = ax + by$, is maximized in the feasible region at some point, then the point is vertex of the polygon. This can be verified by drawing a line $ax + by = c$ which passes through the feasible region and moves with different values of c.

Solve graphically the following Linear Programming Problems :

Example 1 : Maximize : $z = 9x + 13y$ subject to $2x + 3y \le 18$, $2x + y \le 10$, $x \ge 0$, $y \ge 0$.

Solution : To draw $2x + 3y \le 18$ and $2x + y \le 10$

Draw line $2x + 3y = 18$ and $2x + y = 10$

To draw	x	y	Line passes through (x, y)	Sign	Region lies on
L_1 $2x + 3y = 18$	0	6	(0, 6)	$\le$	Origin side of Line L_1
	9	0	(9, 0)		
L_2 $2x + y = 10$	0	10	(0, 10)	$\le$	Origin side of Line L_2
	5	0	(5, 0)		

The common shaded region is O A B C O is a feasible region with vertices O (0, 0), A (5, 0), B (3, 4), C (0, 6).

(x, y) Vertex of S	Value of z = 9x + 13y at (x, y)
O (0, 0)	0
A (5, 0)	45
B (3, 4)	79
C (0, 6)	78

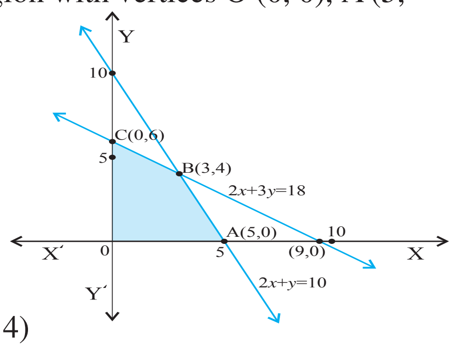

fig 7.24

From the table, maximum value of $z = 79$, occurs at B (3, 4) i.e. when $x = 3$, $y = 4$.

Solve graphically the following Linear Programming Problems :

Example 2 : Maximize : $z = 5x + 2y$ subject to $5x + y \geq 10$, $x + y \geq 6$, $x \geq 0$, $y \geq 0$.

Solution : To draw $5x + y \geq 10$ and $x + y \geq 6$

Draw line $5x + y = 10$ and $x + y = 6$.

To draw	x	y	Line passes through (x, y)	Sign	Region lies on
L_1 $5x + y = 10$	0	10	(0, 10)	$\geq$	Non-origin side of Line L_1
	2	0	(2, 0)		
L_2 $x + y = 6$	0	6	(0, 6)	$\geq$	Non-origin side of Line L_2
	6	0	(6, 0)		

The common shaded region is feasible region with vertices A (6, 0), B (1, 5), C (0, 10).

(x, y) Vertex of S	Value of z = 5x + 2y at (x, y)
A (6, 0)	30
B (1, 5)	15
C (0, 10).	20

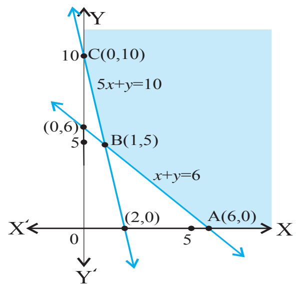

fig 7.25

From the table, maximum value of $z = 15$, occurs at B (1, 5) i.e. when $x = 1$, $y = 5$.

Example 3: Maximize: $z = 3x+4y$ subject to $x-y \geq 0$, $-x + 3y \leq 3$, $x \geq 0$, $y \geq 0$.

Solution : To draw $x - y \geq 0$ and $-x + 3y \leq 3$

Draw line $x - y \geq 0$ and $-x + 3y = 3$.

To draw	Draw line	x	y	Line passes through (x, y)	Sign	Region lies on
$x - y \geq 0$	L_1 $x = y$	0	0	(0, 0)	$\geq$	A side
		1	1	(1, 1)		
$-x + 3y \leq 3$	L_2 $-x+3y = 3$	0	1	(0, 1)	$\leq$	Origin side of Line L_2
		-3	0	$(-3, 0)$		

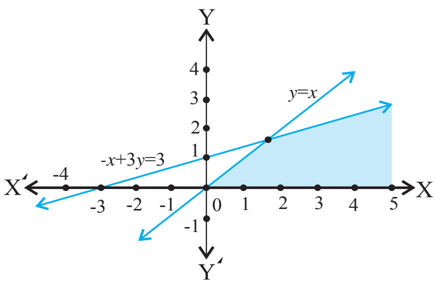

fig 7.26

From graph, we can see that the common shaded area is the feasible region which is unbounded (not a polygon). In such cases, the iso-profit lines can be moved away form the origin indefinitely. $\therefore$ There is no finite maximum value of z within the feasible region.

Example 4 : Maximize : $z = 5x + 2y$ subject to $3x + 5y \leq 15$, $5x + 2y \leq 10$, $x \geq 0$, $y \geq 0$.

Solution : To draw $3x + 5y \leq 15$ and $5x + 2y \leq 10$

Draw line $3x + 5y = 15$ and $5x + 2y = 10$.

To draw	x	y	Line passes through (x, y)	Sign	Region lies on
L_1 $3x + 5y = 15$	5	0	(5, 0)	$\leq$	Origin side of Line L_1
	0	3	(0, 3)		
L_2 $5x + 2y = 10$	2	0	(2, 0)	$\leq$	Origin side of Line L_2
	0	5	(0, 5)		

The shaded region O A B C is the feasible region with the vertices O (0, 0), A (2, 0), $B\left(\frac{20}{19}, \frac{45}{19}\right)$, C (0, 3)

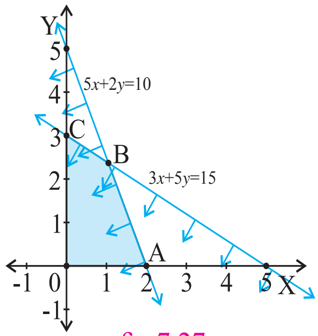

fig 7.27

$Z_0 = 0$, $Z_A = 10$, $Z_B = 10$, $Z_C = 6$.

Maximum value of z occurs at A and B and is $z = 10$.

Maximum value of z occurs at every point lying on the segment AB.

Hence there are infinite number of optimal solutions.

Note : If the two distinct points produce the same minimum value then the minimum value of objective function occurs at every point on the segment joining them.

Exercise 7.4

Solve the following L.P.P. by graphical method :

1) Maximize : $z = 11x + 8y$ subject to $x \leq 4$, $y \leq 6$, $x + y \leq 6$, $x \geq 0$, $y \geq 0$.

2) Maximize : $z = 4x + 6y$ subject to $3x + 2y \leq 12$, $x + y \geq 4$, $x, y \geq 0$.

3) Maximize : $z = 7x + 11y$ subject to $3x + 5y \leq 26$ $5x + 3y \leq 30$, $x \geq 0$, $y \geq 0$.

4) Maximize : $z = 10x + 25y$ subject to $0 \leq x \leq 3$, $0 \leq y \leq 3$, $x + y \leq 5$ also find maximum value of z.

5) Maximize : $z = 3x + 5y$ subject to $x + 4y \leq 24$, $3x + y \leq 21$, $x + y \leq 9$, $x \geq 0$, $y \geq 0$ also find maximum value of z.

6) Minimize : $z = 7x + y$ subject to $5x + y \geq 5$, $x + y \geq 3$, $x \geq 0$, $y \geq 0$.

7) Minimize : $z = 8x + 10y$ subject to $2x + y \geq 7$, $2x + 3y \geq 15$, $y \geq 2$, $x \geq 0$, $y \geq 0$.

8) Minimize : $z = 6x + 2y$ subject to $x + 2y \geq 3$, $x + 4y \geq 4$, $3x + y \geq 3$, $x \geq 0$, $y \geq 0$.

Let's remember!

* Working rule to formulate the L.P.P. :

Step 1 : Identify the decision variables and assign the symbols x, y or x_1, x_2 to them. Introduce non-negativity constraints.

Step 2 : Identify the set of constraints and express them as linear inequation in terms of the decision variables.

Step 3 : Identify the objective function to be optimized (i.e. maximized or minimized) and express it as a linear function of decision variables.

* Let R be the feasible region (convex polygon) for a L.P.P. and Let $z = ax + by$ be the objective functions then the optimal value (maximum or minimum) of z occurs at least one of the corner points (vertex) of the feasible region.

* Corner point method for solving L.P.P. graphically :

Step 1 : Find the feasible region of the L.P.P.

Step 2 : Determine the vertices of the feasible region either by inspection or by solving the two equations of the lines intersecting at that points.

Step 3 : Find the value of the objective function z, at all vertices of feasible region.

Step 4 : Determine the feasible solution which optimizes the value of the objective function.

Miscelleneous Exercise -7

I) Select the appropriate alternatives for each of the following :

1) The value of objective function is maximum under linear constraints _______.
   A) at the centre of feasible region
   B) at (0, 0)
   C) at a vertex of feasible region
   D) the vertex which is of maximum distance from (0, 0)

2) Which of the following is correct _______.
   A) every L.P.P. has an optimal solution
   B) a L.P.P. has unique optimal solution
   C) if L.P.P. has two optimal solutions then it has infinite number of optimal solutions
   D) the set of all feasible solution of L.P.P. may not be convex set

3) Objective function of L.P.P. is _______.
   A) a constraint
   B) a function to be maximized or minimized
   C) a relation between the decision variables
   D) equation of a straight line

4) The maximum value of $z = 5x + 3y$ subjected to the constraints $3x + 5y \leq 15$, $5x + 2y \leq 10$, $x, y \geq 0$ is _______.
   A) 235 B) $\frac{235}{9}$
   C) $\frac{235}{19}$ D) $\frac{235}{3}$

5) The maximum value of $z = 10x + 6y$ subjected to the constraints $3x + y \leq 12$, $2x + 5y \leq 34$, $x \geq 0$, $y \geq 0$. _______.
   A) 56 B) 65
   C) 55 D) 66

6) The point at which the maximum value of $x + y$ subject to the constraints $x + 2y \leq 70$, $2x + y \leq 95$, $x \geq 0$, $y \geq 0$ is obtained at _______.
   A) (30, 25) B) (20, 35)
   C) (35, 20) D) (40, 15)

7) Of all the points of the feasible region, the optimal value of z obtained at the point lies _______.
   A) inside the feasible region B) at the boundary of the feasible region
   C) at vertex of feasible region D) outside the feasible region

8) Feasible region is the set of points which satisfy ________.

A) the objective function B) all of the given constraints

C) some of the given constraints D) only one constraint

9) Solution of L.P.P. to minimize $z = 2x + 3y$ s.t. $x \geq 0$, $y \geq 0$, $1 \leq x + 2y \leq 10$ is ________.

A) $x = 0,\ y = \frac{1}{2}$ B) $x = \frac{1}{2},\ y = 0$

C)x = 1, y = 2 D) $x = \frac{1}{2},\ y = \frac{1}{2}$

10) The corner points of the feasible solution given by the inequation $x + y \leq 4$, $2x + y \leq 7$, $x \geq 0$, $y \geq 0$ are ________.

A) (0, 0), (4, 0), (7, 1), (0, 4) B) (0, 0), $(\frac{7}{2}, 0)$, (3, 1), (0, 4)

C) (0, 0), $(\frac{7}{2}, 0)$, (3, 1), (0, 7) D) (0, 0), (4, 0), (3, 1), (0, 7)

11) The corner points of the feasible solution are (0, 0), (2, 0), $\left(\frac{12}{7}, \frac{3}{7}\right)$, (0, 1). Then Z = $7x + y$ is maximum at ________.

A)(0, 0) B)(2, 0)

C)$\left(\frac{12}{7}, \frac{3}{7}\right)$ D)(0, 1)

12) If the corner points of the feasible solution are (0, 0), (3, 0), (2, 1) and $\left(0, \frac{7}{3}\right)$ the maximum value of $z = 4x + 5y$ is ________.

A) 12 B) 13

C) $\frac{35}{3}$ D) 0

13) If the corner points of the feasible solution are (0, 10), (2, 2) and (4, 0) then the point of minimum $z = 3x + 2y$ is ________.

A) (2, 2) B) (0, 10)

C) (4, 0) D) (3 ,4)

14) The half plane represented by $3x + 2y < 8$ contains the point ________.

A) $\left(1, \frac{5}{2}\right)$ B) (2, 1)

C) (0, 0) D) (5, 1)

15) The half plane represented by $4x + 3y > 14$ contains the point ________.

A) (0, 0) B) (2, 2)

C) (3, 4) D) (1, 1)

II) Solve the following :

1) Solve each of the following inequations graphically using X Y plane.

i) $4x - 18 \geq 0$ ii) $-11x - 55 \leq 0$

iii) $5y - 12 \geq 0$ iv) $y \leq -3.5$

2) Sketch the graph of each of following inequations in XOY co-ordinate system.

i) $x \geq 5y$ ii) $x + y \leq 0$

iii) $2y - 5x \geq 0$ iv) $1x + 51 \leq y$

3) Find graphical solution for each of the following system of linear inequation.

i) $2x + y \geq 2$, $x - y \leq 1$ ii) $x + 2y \geq 4$, $2x - y \leq 6$

iii) $3x + 4y \leq 12$, $x - 2y \geq 2$, $y \geq -1$

4) Find feasible solution for each of the following system of linear inequations graphically.

i) $2x + 3y \leq 12$, $2x + y \leq 8$, $x \geq 0$, $y \geq 0$

ii) $3x + 4y \geq 12$, $4x + 7y \leq 28$, $x \geq 0$, $y \geq 0$

5) Solve each of the following L.P.P.

i) Maximize $z = 5x_1 + 6x_2$ subject to $2x_1 + 3x_2 \leq 18$, $2x_1 + x_2 \leq 12$, $x_1 \geq 0$, $x_2 \geq 0$

ii) Maximize $z = 4x + 2y$ subject to $3x + y \geq 27$, $x + y \geq 21$

iii) Maximize $z = 6x + 10y$ subject to $3x + 5y \leq 10$, $5x + 3y \leq 15$, $x \geq 0$, $y \geq 0$

iv) Maximize $z = 2x + 3y$ subject to $x - y \geq 3$, $x \geq 0$, $y \geq 0$

6) Solve each of the following L.P.P.

i) Maximize $z = 4x_1 + 3x_2$ subject to $3x_1 + x_2 \leq 15$, $3x_1 + 4x_2 \leq 24$, $x_1 \geq 0$, $x_2 \geq 0$

ii) Maximize $z = 60x + 50y$ subject to $x + 2y \leq 40$, $3x + 2y \leq 60$, $x \geq 0$, $y \geq 0$

iii) Maximize $z = 4x + 2y$ subject to $3x + y \geq 27$, $x + y \geq 21$, $x + 2y \geq 30$; $x \geq 0$, $y \geq 0$

7) A carpenter makes chairs and tables. Profits are Rs.140/- per chair and Rs. 210/- per table. Both products are processed on three machines : Assembling, Finishing and Polishing. The time required for each product in hours and availability of each machine is given by following table:

Product Machine	Chair (x)	Table (y)	Available time (hours)
Assembling	3	3	36
Finishing	5	2	50
Polishing	2	6	60

Formulate the above problem as L.P.P. Solve it graphically to get maximum profit.

Formulate and solve the following Linear Programming Problems using graphical method :

8) A company manufactures bicycles and tricycles, each of which must be processed through two machines A and B. Maximum availability of Machine A and B is respectively 120 and 180 hours. Manufacturing a bicycle requires 6 hours on Machine A and 3 hours on Machine B. Manufacturing a tricycles requires 4 hours on Machine A and 10 hours on Machine B. If profits are Rs.180/- for a bicycle and Rs.220/- for a tricycle. Determine the number of bicycles and tricycles that should be manufactured in order to maximize the profit.

9) A factory produced two types of chemicals A and B. The following table gives the units of ingredients P and Q (per kg) of chemicals A and B as well as minimum requirements of P and Q and also cost per kg. chemicals A and B :

Chemicals in units Ingredients per kg.	A (x)	B (y)	Minimum requirements in units
P	1	2	80
Q	3	1	75
Cost (in Rs.)	4	6	--

Find the number of units of chemicals A and B should be produced so as to minimize the cost.

10) A company produces mixers and food processors. Profit on selling one mixer and one food processor is Rs. 2,000/- and Rs. 3,000/- respectively. Both the products are processed through three Machines A, B, C. The time required in hours by each product and total time available in hours per week on each machine are as follows :

Product Machine	Mixer (per unit)	Food Processor (per unit)	Available time
$A \to A$	3	3	36
$B \to B$	5	2	50
$C \to C$	2	6	60

How many mixers and food processors should be produced to maximize the profit?

11) A chemical company produces a chemical containing three basic elements A, B, C so that it has at least 16 liters of A, 24 liters of B and 18 liters of C. This chemical is made by mixing two compounds I and II. Each unit of compound I has 4 liters of A, 12 liters of B, 2 liters of C. Each unit of compound II has 2 liters of A, 2 liters of B and 6 liters of C. The cost per unit of compound I is Rs.800/- and that of compound II is Rs.640/-. Formulate the problem as L.P.P. and solve it to minimize the cost.

12) A person makes two types of gift items A and B requires the services of a cutter and a finisher. Gift item A requires 4 hours of cutter's time and 2 hours of finisher's time. B requires 2 hours of cutter's time and 4 hours of finisher's time. The cutter and finisher have 208 hours and 152 hours available times respectively every month. The profit of one gift item of type A is Rs.75/- and on gift item B is Rs.125/-. Assuming that the person can sell all the gift items produced, determine how many gift items of each type should he make every month to obtain the best returns?

13) A firm manufactures two products A and B on which profit earned per unit Rs.3/- and Rs.4/- respectively. Each product is processed on two machines M_1 and M_2. The product A requires one minute of processing time on M_1 and two minute of processing time on M_2, B requires one minute of processing time on M_1 and one minute of processing time on M_2. Machine M_1 is available for use for 450 minutes while M_2 is available for 600 minutes during any working day. Find the number of units of product A and B to be manufactured to get the maximum profit.

14) A firm manufacturing two types of electrical items A and B, can make a profit of Rs.20/- per unit of A and Rs.30/- per unit of B. Both A and B make use of two essential components a motor and a transformer. Each unit of A requires 3 motors and 2 transformers and each units of B requires 2 motors and 4 transformers. The total supply of components per month is restricted to 210 motors and 300 transformers. How many units of A and B should the manufacture per month to maximize profit? How much is the maximum profit?

7. Linear Programming

Exercise 7.3

1) maximize $z = 30x + 20y$ subject to $10x + 6y \leq 60$, $5x + 4y \leq 35$, $x \geq 0$, $y \geq 0$

2) maximize $z = 3x + 2y$ subject to $2x + y \geq 14$, $2x + 3y \geq 22$, $x + y \geq 1$, $x \geq 0$, $y \geq 0$

3) maximize $p = 350x + 400y$ subject to $3x + 2y \leq 120$, $2x + 5y \leq 160$, $x \geq 0$, $y \geq 0$

4) maximize $z = 10x + 15y$ subject to $2x + 3y \leq 36$, $5x + 2y \leq 50$, $2x + 6y \leq 60$, $x \geq 0$, $y \geq 0$

5) maximize $p = 13.5x + 55y$ subject to $x + 2y \leq 10$, $3x + 4y \leq 12$, $x \geq 0$, $y \geq 0$

6) maximize $z = 500x + 750y$ subject to $2x + 3y \leq 40$, $x + 4y \leq 70$, $x \geq 0$, $y \geq 0$

7) minimize $z = 4.5x + 3.5y$ subject to $4x + 6y \geq 18$, $14x + 12y \geq 28$, $7x + 8y \geq 14$, $x \geq 0$, $y \geq 0$

8) maximize $z = x_1 + x_2$ subject to $\frac{x_1}{60} + \frac{x_2}{90} \leq 1$, $5x_1 + 8x_2 \leq 600$, $x \geq 0$, $x_2 \geq 0$

9) minimize $C = 20\, x_1 + 6x_2$ s. t $x_1 > 4$, $x_2 < 2$, $x_1 + x_2 \geq 5$, $x \geq 0$, $x_2 \geq 0$.

Exercise 7.4

1) Maximum at (4, 2), 60

2) Maximum at (0, 6), maximum value = 36

3) Maximum at (4.5, 2.5), 59

4) Maximum at (2, 3), maximum value = 95

5) Maximum at (4, 5), maximum $z = 37$

6) minimum at (0, 5), 5

7) minimum at (1.5, 4), 52

8) minimum at (2, 0.5), 22.5

Miscellaneous exercise - 7

I.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
A	C	B	C	A	D	C	B	A	B	B	B	A	C	C

5) (i) $x_1 = 4.5$, $x_2 = 3$ max $z = 40.5$.

(ii) $x = 3$, $y = 18$ min $z = 48$.

(iii) infinite number of optimum solutions on the line $3x + 5y = 10$ between $A\left(\frac{45}{16}, \frac{5}{16}\right)$ and B(0, 2).

6) (i) $x = 4, y = 3$ maximize $z = 25$.

(ii) $x = 10, y = 15$maximize $z = 1350$.

(iii) $x = 3, y = 18$ maximize $z = 48$.

7) maximize $z = 140x + 210y$ s.t. $3x + 3y \le 36,\ 5x + 2y \le 50,\ 2x + 6y \le 60$

$x, y \ge 0$ where x = no. of tables = 3

y = no. of chairs = 9

maximize z = maximum profit = 2310

8) Maximize $z = 180x + 220y$ s.t. $6x + 4y \le 120,\ 3x + 10y \le 180,\ x \ge 0, y \ge 0$.

Ans. $x = 10, y = 15$.

9) Minimize $z = 4x + 6y$ s.t. $x + 2y \ge 80,\ 3x + y \ge 75,\ x \ge 0, y \ge 0$.

Ans. $x = 14, y = 33$.

10) Maximize $z = 2000x + 3000y$ s.t. $3x + 3y \le 36,\ 5x + 2y \le 50,\ 2x + 6y \le 60,\ x \ge 0, y \ge 0$.

Ans. $x = 3, y = 9$.

11) Minimize $z = 800x + 640y$ s.t. $4x + 2y \ge 16,\ 12x + 2y \ge 24,\ 2x + 6y \ge 18,\ x \ge 0, y \ge 0$.

Ans. Minimum cost ₹3680 when $x = 3, y = 2$.

12) Maximize $z = 75x + 125y$ s.t. $4x + 2y \le 208,\ 2x + 4y \le 152,\ x \ge 0, y \ge 0$.

Ans. $x = 44, y = 16$.

13) Maximize $z = -3x + 4y$ s.t. $x + y \le 450,\ 2x+y \le 600,\ x \ge 0,\ y \ge 0$

maximum profit = Rs. 1800 at (0, 450)

14) Maximize $z = 20x + 30y$ s.t. $2x + 2y \le 210,\ 3x + 4y \le 300,\ x \ge 0,\ y \ge 0$

maximum profit = Rs. 2400 at (30, 60)