3 PERMUTATIONS AND COMBINATIONS

Let's Study

- Fundamental principles of counting
- Factorial notation
- Permutations
- Permutations of distinct objects
- Permutations when some objects are identical
- Circular permutations
- Combinations

Let's Recall

- The number system.
- The four basic mathematical operations: addition, subtraction, multiplication, division.

3.1 INTRODUCTION

Counting is a fundamental activity in Mathematics. Learning to count was our first step in learning Mathematics. After learning to count objects one by one, we used operations of addition and multiplication to make counting easy. We shall now learn two more methods of counting to make complicated counting easier. These two methods are known as permutations and combinations.

Permutations refer to the number of different arrangements of given objects, when the order of objects is important. Combinations are related to the number of different selections from a given set of objects, when the order of objects in the selections is immaterial.

The theory of permutations and combinations is central in problems of counting a large number of objects that are impossible to count manually. The theory of permutations and combinations enables us to count objects without listing or enumerating them.

Let us begin with the following simple example. Every smartphone requires a passcode to unlock it. A passcode is formed by four of the ten digits on the screen. The order of these four digits cannot be changed for passcode to work. How many distinct passcodes are possible? Note that a passcode consists of four digits. The first digit of a passcode can be any of the ten digits, the second digit can be any of the ten digits, and similarly for the third and fourth digits. This gives a total of $10 \times 10 \times 10 \times 10 = 10{,}000$ as the number of distinct possible passcodes.

Consider one more example which is not as easy as the last example. The school cricket team has eleven players. The school wants a photograph of these players, along with the principal and the two vice principals of the school, for school magazine. Seven chairs are arranged in a row for the photograph. Three chairs in the middle are reserved for the principal and the two vice principals. Four players will occupy the remaining four chairs and seven players will stand behind the chairs. The question then is: “In how many different ways can the eleven players take positions for the photograph?” This example will be considered later in the chapter. Till then, we can try, on our own, to find the number of different ways in which the eleven players can sit or stand for the photo.

Let us understand three principles of counting that are fundamental to all methods of counting, including permutations and combinations.

3.2 Fundamental principles of counting

Tree Diagram

We have learnt in set theory that subsets of a set can be represented in the form of a Venn diagram. An alternative method is to draw a tree diagram if the subsets are disjoint. For example, The games students play at school are of two types 1) Indoor games, 2) Outdoor games. Available indoor games in school are chess, carrom and table tennis. While outdoor, games in school are cricket, volleyball, basketball and badminton. Such information is presented with the help of tree-diagram as follows: (see fig 3.1)

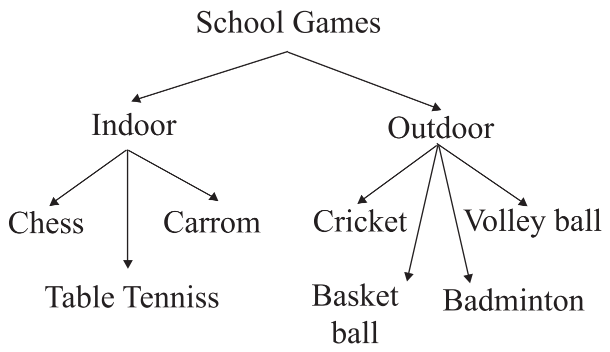

Fig. 3.1

Diagrams of this nature are called tree diagrams. A tree diagram shows the division of a set into disjoint subsets.

The fundamental principles of counting provide an efficient way of finding the number of different ways to carry out two or more activities, either simultaneously or successively (that is, one after another).

3.2.1 Addition Principle:

Consider the situation where a boy wants to go for movie. He has three T–shirts of three different colours: white, green, and blue. He also has four shirts of four different colours: red, green, yellow, and orange. How many choices does he have to wear? This situation can be represented using a tree diagram as follows. (see fig 3.2)

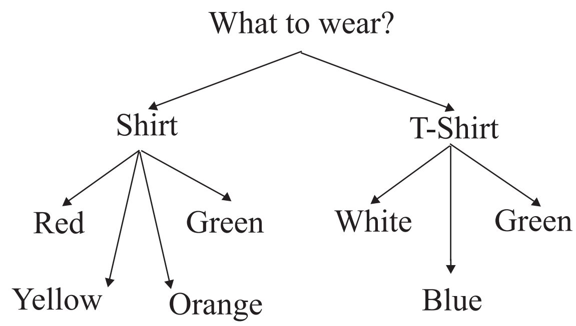

Fig. 3.2

The tree diagram shows that the boy has 4 choices in shirt and 3 choices in T-shirt, i.e. seven choices in all.

The boy can choose from 4 Shirts OR from 3 T-shirts. Hence, the number of ways of choosing are $3 + 4 = 7$ in all.

This example shows that the total number of outcomes is obtained by adding the number of outcomes of each characteristic when only one characteristic is to be chosen.

Statement of The Addition Principle.

Addition Principle : Suppose one operation can be done in m ways and another operation can be done in n ways, with no common way among them. If one of these operations is to be performed then there are $m + n$ ways to do it.

Ex. 1) A restaurant offers five types of fruit juices and three types of milk shakes. If a customer wants to order a drink, how many choices does the customer have?

Solution : Since the restaurant offers five fruit juices, the customer has five ways of selecting a fruit juices. Similarly, since the restaurant offers three types of milk shakes, the customer has three ways of selecting a milk shake. Finally, since the customer wants to select only one drink, there are $5 + 3 = 8$ choices for the customer.

Ex. 2) Consider an experiment of drawing a card from a pack of 52 playing cards. What is the number of ways in which the drawn card is a spade or a club?

Solution : Since there are 13 spade cards and 13 club cards hence the number of ways in which the drawn card is spade or club is $13 + 13 = 26$ ways.

Note : The word 'OR' in the statement suggests addition, i.e. '+'.

3.2.2 Multiplication principle

Now, consider the following situation. An Ice cream is served either in a cup or in a cone. Also, Ice cream is available in three flavours: vanilla, chocolate, and strawberry. This information can be represented in the form of a tree diagram as follows. (see Fig 3.3)

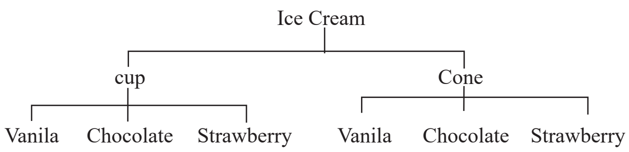

Fig. 3.3

There are 2 ways of choosing mode of serving and 3 ways of choosing flavours.

So, the tree diagram shows that there are six possible outcomes, i.e. $2 \times 3 = 6$ ways to serve Ice cream.

In the example of Ice cream, the final count of six is obtained by multiplying 3 (number of flavours) by 2 (number of serving options), in either order.

Definition of The Multiplication Principle

If one operation can be carried out in m ways, followed by the second operation that can be carried out in n ways, and these two operations are independent then the two operations can be carried out in $m \times n$ ways.

Ex. 1) : Samadhan Bhojanalay offers a thali that has four items: roti, rice, vegetable and dal. Following choices are available and one option is to be selected for each item.

Roti: chapati, tandoor roti

Rice: plain rice, jeera rice, dal khichadi

Vegetable: dum aloo, paneer masala, mixed veg

Dal: dal fry, dal tadka

How many different menus are possible?

Solution : Since one option is to be selected for each item, the number of different possible thali choices are identified as follows.

Roti 2, Rice 3, Vegetable 3, Dal 2 and hence the total number of different possible thali menus is $2 \times 3 \times 3 \times 2 = 36$.

Ex. 2) A company decides to label each of its different products with a code that consists of two letters followed by three digits. How many different products can be labeled in this way?

Solution : Since the first two characters in a label are letters, they can be formed in $26 \times 26 = 676$ ways. The next three characters are digits and can be formed in $10 \times 10 \times 10 = 1000$ ways. The total number of distinct labels is, by the multiplication principle, given by $676 \times 1000 = 6{,}76{,}000$.

Activity 1) Suresh has 4 pencils and 2 erasers. He wants to take one pencil and one eraser for the examination. Can we find the number of ways in which he can select a pencil and an eraser?

Activity 2) Sunil has 4 ballpens of one company and 3 ballpens of another company. In how many ways can he select a ball pen?

In the above activities, can we decide when to use the addition principle and when to use the multiplication principle? Can we give reasons? What are answers in the above examples?

Remark: The addition and multiplication principles can be extended from two to any finite number of activities, experiments, events, or operations.

Extended Addition Principle : Suppose there are three possible choices with no common outcome for any two, the first choice can be made in m ways, second in n ways and third in r ways. If only one of the choices is to be made, it has $m + n + r$ possible ways.

Extended Multiplication Principle: Suppose an experiment consists of three independent activities, where first activity has m possible outcomes, second has n possible outcomes, third has r possible outcomes. Then the total number of different possible outcomes of the experiment is $m \times n \times r$.

3.3 Invariance Principle

The result of counting objects in a set does not depend on the order in which these objects are counted or on the method used for counting these objects.

For example, 1) In earlier example of addition principle of choosing first from 4 Shirts, then from 3 T-Shirts is same as choosing first from 3 T-Shirts and then from 4 Shirts.

2) In earlier example of multiplication principle choosing first from 2 modes of servings and then from 3 flavours is same as choosing first from 3 flavours and then from the 2 modes of servings.

SOLVED EXAMPLES

Ex. 1) : From the figure below, find the total number of routes from A to B. e.g. one upper route is $A \to D \to N \to E \to B$. (See fig 3.4)

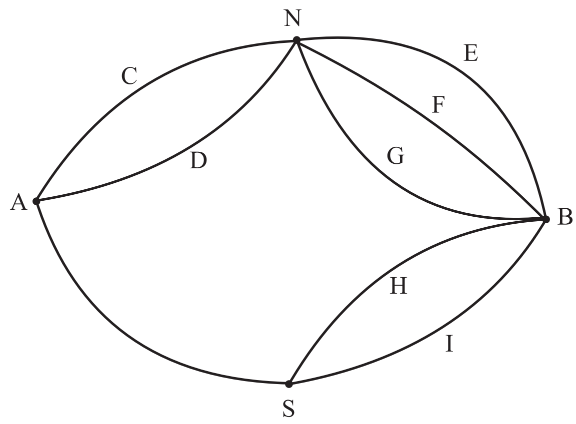

Fig 3.4

Solution: $A \xrightarrow{D} N \xrightarrow{E} B$ is one route through N. $A \xrightarrow{C} N \xrightarrow{F} B$ is another route through N. Thus there are in all $2 \times 3 = 6$ routes through N. There are 2 routes through S. Hence the total number of routes is $6 + 2 = 8$.

Ex. 2) : Suppose 5 chocoloates of different type are to be distributed among 4 children and there is no condition on how many chocolates a child can get (including zero.) How many different ways are possible for doing so?

Solution: The first chocolate can be given to any of the four children. Therefore, there are four different ways of giving the first chocolate. Similarly, the second chocolate can be given in four different ways, and similarly for each of the remaining chocolates. The multiplication principle then gives the total number of different ways as $4 \times 4 \times 4 \times 4 \times 4 = 4^5 = 1024$.

Ex. 3) : How many even numbers can be formed using the digits 2, 3, 7, 8 so that the number formed is less than 1000?

Solution: The condition that the number formed from the given digits is less that 1000 means that this number can have upto 3 digits. Let us therefore consider the three cases separately,

case i) One digit numbers

Since the number must be even, the one–digit number can be only 2 or 8. Hence, there are 2 ways of forming a one digit even number.

case ii) Two digit numbers

Since the required number is even, the units place of a two–digit number must be either 2 or 8 i.e. 2 ways. The ten's place can be filled with any of the four given digits. Therefore, there are $4 \times 2 = 8$ ways of forming a two–digit even number.

case iii) Three digit numbers

Finally,since the requried number is even, the units place can be filled in two ways, ten's place can be filled in four ways, and hundred's place can also be filled in four ways. The number of ways of forming a three digit even number is $4 \times 4 \times 2 = 32$

The addition principle finally gives the total number of ways of forming an even number less than 1000 using digits 2,3,7,8 is $2+8+32=42$.

EXERCISE 3.1

1. A teacher wants to select the class monitor in a class of 30 boys and 20 girls. In how many ways can the monitor be selected if the monitor must be a girl?
2. A Signal is generated from 2 flags by putting one flag above the other. If 4 flags of different colours are available, how many different signals can be generated?
3. How many two letter words can be formed using letters from the word SPACE, when repetition of letters (i) is allowed, (ii) is not allowed?
4. How many three–digit numbers can be formed from the digits 0, 1, 3, 5, 6 if repetitions of digits (i) are allowed, (ii) are not allowed?
5. How many three–digit numbers can be formed using the digits 2, 3,4,5,6 if digits can be repeated?
6. A letter lock has 3 rings and each ring has 5 letters. Determine the maximum number of trials that may be required to open the lock.
7. In a test, 5 questions are of the form 'state, true or false'. No student has got all answers correct. Also, the answer of every student is different. Find the number of students appeared for the test.
8. How many numbers between 100 and 1000 have 4 in the units place?
9. How many numbers between 100 and 1000 have the digit 7 exactly once?
10. How many four digit numbers will not exceed 7432 if they are formed using the digits 2,3,4,7 without repetition?

11. If numbers are formed using digits 2, 3, 4, 5, 6 without repetition, how many of them will exceed 400?

12. How many numbers formed with the digits 0, 1, 2, 5, 7, 8 will fall between 13 and 1000 if digits can be repeated?

13. A school has three gates and four staircases from the first floor to the second floor. How many ways does a student have to go from outside the school to his classroom on the second floor?

14. How many five–digit numbers formed using the digit 0, 1, 2, 3, 4, 5 are divisible by 5 if digits are not repeated?

3.4 Factorial Notation.

The theory of permutations and combinations uses a mathematical notation known as the factorial notation. Let us first understand the factorial notation, which is defined for natural number.

Definition: For a natural number n, the factorial of n, written as n! or $\lfloor n$ and read as "n factorial", is the product of n natural numbers from 1 to n.

That is, n! or $\lfloor n$ is expressed as

$1 \times 2 \times 3 \times .... \times (n-2) \times (n-1) \times n$.

Note: The factorial nonation can also be defined as the product of the natural numbers from n to 1.

That is, $n! = n \times (n-1) \times .... \times 3 \times 2 \times 1$

For example, $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

It is read as "5 factorial is equal to 120."

Illustrations:

Factorial 1 = $1! = 1$

Factorial 2 = $2! = 2 \times 1 = 2$

Factorial 3 = $3! = 3 \times 2 \times 1 = 6$

Factorial 4 = $4! = 4 \times 3 \times 2 \times 1 = 24$, and so on.

Note: Though 0 is not a natural number, we define $0! = 1$.

Properties of the factorial notaction.

For any positive integers m,n.,

1) $n! = n \times (n-1)!$
2) $n > 1,\ n! = n \times (n-1) \times (n-2)!$
3) $n > 2,\ n! = n \times (n-1) \times (n-2) \times (n-3)!$
4) $(m+n)!$ is always divisible by $m!$ as well as by $n!$ e.g. $(3+4)!$ is divisible by $3!$ as well as $4!$
5) $(m \times n)! \neq m! \times n!$
6) $(m+n)! \neq m! + n!$
7) $m > n$, $(m-n)! \neq m! - n!$ but $m!$ is divisible by $n!$
8) $(m \div n)! \neq m! \div n!$

Ex. 1) : Find the value of 6!

Solution : $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

Ex. 2) : Show that $(7-3)! \neq 7! - 3!$

Solution: $(7-3)! = 4! = 4 \times 3 \times 2 \times 1 = 24.$

$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$

Also, $3! = 3 \times 2 \times 1 = 6.$

$7! - 3! = 5040 - 6 = 5034$

Therefore, $(7-3)! \neq 7! - 3!$

Ex. 3): Find n if (n+6)! = 56 (n+4)!

Solution: (n+6)! = 56 (n+4)!

$\therefore$ (n+6) (n+5) (n+4)! = 56 (n+4)!

$\therefore$ (n+6) (n+5) = 56

Here instead of solving quadratic in n, we write 56 as product of 2 consecutive numbers as 56 = 8×7

$\therefore$ (n+6) (n+5) = 8×7

Equating bigger factors from either side

$\therefore$ (n+6) = 8

n = 8 – 6 = 2

Ex. 4) : Show that $\frac{12!}{5!7!}+\frac{12!}{6!6!}=\frac{13!}{6!7!}$

Solution: Consider,

$$\text{L. H. S.} = \frac{12!}{5!7!}+\frac{12!}{6!6!}$$

$$= 12!\left[\frac{1}{5!\times 7\times 6!}+\frac{1}{5!\times 6\times 6!}\right]$$

$$(\because\ 7! = 7\times 6! \quad 6! = 6\times 5!)$$

$$= \frac{12!}{5!6!}\left[\frac{1}{7}+\frac{1}{6}\right]$$

$$= \frac{12!}{5!6!}\left[\frac{13}{6\times 7}\right]$$

$$= \frac{12!\times 13}{(5!\times 6)\times(6!\times 7)}$$

$$= \frac{13!}{6!\times 7!}$$

$$= \text{R. H. S.}$$

EXERCISE 3.2

1. Evaluate:

(i) 8! (ii) 10!

(iii) 10! – 6! (iv) (10 – 6)!

2. Compute:

(i) $\frac{12!}{6!}$ (ii) $\left(\frac{12}{6}\right)!$

(iii) $(3\times 2)!$ (iv) $3!\times 2!$

(v) $\frac{9!}{3!\,6!}$ (vi) $\frac{6!-4!}{4!}$

(vii) $\frac{8!}{6!-4!}$ (viii) $\frac{8!}{(6-4)!}$

3. Write in terms of factorials

(i) $5\times 6\times 7\times 8\times 9\times 10$

(ii) $3\times 6\times 9\times 12\times 15$

(iii) $6\times 7\times 8\times 9$

(iv) $5\times 10\times 15\times 20$

4. Evaluate : $\frac{n!}{r!(n-r)!}$ for

(i) n = 8, r = 6 (ii) n = 12, r = 12,

(iii) n = 15, r = 10 (iv) n = 15, r = 8

5. Find n, if

(i) $\frac{n}{8!}=\frac{3}{6!}+\frac{1!}{4!}$ (ii) $\frac{n}{6!}=\frac{4}{8!}+\frac{3}{6!}$

(iii) $\frac{1!}{n!}=\frac{1!}{4!}-\frac{4}{5!}$

(iv) $(n+1)!=42\times(n-1)!$

(v) $(n+3)!=110\times(n+1)!$

6. Find n, if:

(i) $\frac{(17-n)!}{(14-n)!}=5!$ (ii) $\frac{(15-n)!}{(13-n)!}=12$

(iii) $\frac{n!}{3!(n-3)!}:\frac{n!}{5!(n-5)!}=5:3$

(iv) $\frac{n!}{3!(n-3)!}:\frac{n!}{5!(n-7)!}=1:6$

(v) $\frac{(2n)!}{7!(2n-7)!}:\frac{n!}{4!(n-4)!}=24:1$

7. Show that

$$\frac{n!}{r!(n-r)!}+\frac{n!}{(r-1)!(n-r+1)!}=\frac{(n+1)!}{r!(n-r+1)}$$

8. Show that

$$\frac{9!}{3!6!}+\frac{9!}{4!5!}=\frac{10!}{4!6!}$$

9. Show that

$$\frac{(2n)!}{n!}=2^n(2n-1)(2n-3)...5.3.1$$

10. Simplify

(i) $\frac{(2n+2)!}{(2n)!}$ (ii) $\frac{(n+3)!}{(n^2-4)(n+1)!}$

(iii) $\frac{1}{n!}-\frac{1}{(n-1)!}-\frac{1}{(n-2)!}$

(iv) $n[n!+(n-1)!]+n^2(n-1)!+(n+1)!$

(v) $\frac{n+2}{n!}-\frac{3n+1}{(n+1)!}$

(vi) $\frac{1}{(n-1)!}+\frac{1-n}{(n+1)!}$

(vii) $\frac{1}{n!}-\frac{3}{(n+1)!}-\frac{n^2-4}{(n+2)!}$

(viii) $\frac{n^2-9}{(n+3)!}+\frac{6}{(n+2)!}-\frac{1}{(n+1)!}$

3.5 Permutations: (When all objects are distinct)

If we want to place 3 persons on 3 chairs in a row, in how many ways can we obtain the seating arrangement?

Suppose the 3 seats are numbered 1, 2 and 3 and the 3 persons are named A, B and C.

We can fill the 1st chair in 3 ways. Having done that, we can fill the 2nd chair with any of the remaining two people, hence in 2 ways. The 3rd chair is then filled in a unique way as there is only one person left. Thus the total number of the seating arrangements are

$3 \times 2 \times 1 = 3! = 6$

This can also be checked by listing them as follows

ABC	BAC	CAB
ACB	BCA	CBA.

Extending this result for n persons to be placed in n chairs in a row, we get $n!$ ways of arrangement.

Note that all persons are distinct and chairs have their own ordinal numbers (1st, 2nd, 3rd etc.)

Now we see a different problem. There are 4 persons and 2 chairs in a row to be filled. The 1st chair can be filled in 4 ways. Having done this, the 2nd chair can be filled in 3 ways. So the number of different arrangements is $4 \times 3 = 12$.

See the enumeration

AB	BA	CA	DA
AC	BC	CB	DB
AD	BD	CD	DC

Extending this argument if 7 different objects are available and 3 boxes are in a row. We want to place one object in each box. There are $7 \times (7-1) \times (7-2) = 7 \times 6 \times 5$ different ways to do it.

A permutation is formally defined as follows.

Permutation : A permutation is an arrangement, in a definite order of a number of objects, taken some or all at a time.

The number of distinct permutations of r distinct objects chosen from a given collection of n distinct objects is denoted by nPr, ^nP_r, or P(n,r).

3.5.1 Permutations when all objects are distinct [$r \leq n$]:

Theorem 1. The number of permutations of n distinct objects taken r ($r \leq n$) at a time, without repetitions, is

$$n \times (n-1) \times (n-2) \times \ldots \times (n-r+1).$$

Proof. The number of ways of arranging n distinct objects taken r at a time without repetitions is same as the number of ways r places can be filled using n objets ($r \leq n$).

For this, consider the following table.

Place	1st	2nd	3rd	...	(r–2)th	(r–1)th	rth
Number of ways	n	n–1	n–2	...	[n–(r–3)]	[n–(r–2)]	[n–(r–1)]

The table shows that the first place can be filled with any of the n objects. As the result, there are n ways of filling the the first place. After putting one object in the first place, only n–1 objects are available because repetitions are not allowed. Therefore, the second place can be filled in n–1 ways. After putting two distinct objects in the first two places, only n–2 objects are available for the third place, so that the third place can filled in n–2 ways.

Continuing in this way, after putting r–1 objects in the first r–1 places, the number of available objects is [n – (r–1)] for the r^{th} place. Hence, the r^{th} place can be filled in [n–(r–1)] ways.

Now, using the multiplication principle of counting, the total number of ways of filling r places using n distinct objects is denoted by ^nP_r and given by the product

$n \times (n-1) \times (n-2) \times \ldots \times ([n-(r-2)] \times [n-(r-1)]$

$= n \times (n-1) \times (n-2) \times \ldots \times (n-r+2) \times (n-r+1)$

Hence $^nP_r = n \times (n-1) \times (n-2) \times \ldots \times (n-r+2) \times (n-r+1)$

If we multiply and divide this product by

$(n-r) \times (n-r-1) \times \ldots \times 3 \times 2 \times 1$,

we find that

$$^nP_r = n \times (n-1) \times \ldots \times (n-r+1) \times \frac{(n-r) \times (n-r-1) \times \ldots \times 3 \times 2 \times 1}{(n-r) \times (n-r-1) \times \ldots \times 3 \times 2 \times 1}$$

$$= \frac{n \times (n-1) \times (n-2) \times \ldots \times (n-r+1) \times (n-r) \times (n-r-1) \times \ldots \times 3 \times 2 \times 1}{(n-r) \times (n-r-1) \times \ldots \times 3 \times 2 \times 1}$$

$$= \frac{n!}{(n-r)!}$$

We have thus found that

$$^nP_r = \frac{n!}{(n-r)!} \quad (\text{for } r \leq n)$$

Note:

When $r = n$ i.e. all n objects are placed in a row.

$$^nP_n = n \times (n-1) \times (n-2) \times \ldots \times [n-(n-1)]$$

$$= n \times (n-1) \times (n-2) \times \ldots \times 1 = n!$$

Alternatively, from the above formula, we obtain

$$^nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

(since 0!=1, by definition)

SOLVED EXAMPLES

Ex. 1) Find the value of $^5P_2$.

Solution: $5P_2 = \frac{5!}{(5-2)!} = \frac{5!}{3!} = \frac{5\times4\times3!}{3!}$

$= 5 \times 4 = 20.$

Ex. 2) How many different ways are there to arrange letters of the word 'WORLD'? How many of these arrangements begin with the letter R? How many arrangements can be made taking three letters at a time?

Solution: The word WORLD has 5 letters W, O, R, L, D. These can be arranged among themselves in $^5P_5$=5!=120! different ways.

If an arrangement begins with R, the remaining four letters can be arranged among themselves in $^4P_4$=4!=24 ways.

The Number of arrangements of 5 letters, taken 3 at a time, is $^5P_3 = \frac{5!}{2!} = 60$

Ex. 3) How many three digit numbers can be formed from the digits 2,4,5,6,7 if no digit is repeated?

Solution: Every arrangement of digits gives a different number. Therefore, the problem is to find the number of arrangements of five digits taken two at a time This is given by $^5P_3 = \frac{5!}{2!} = 60$

Ex 4) How many numbers can be formed with the digits 3, 4, 6, 7, 8 taken all at a time? Find the sum of all such numbers.

Solution : The five digits can be arranged in $^5P_5$ = 5! = 120 ways.

Now, consider any one of the five given digits, say 3. Suppose the digit 3 is in the unit's place. The other four digits can be arranged in $^4P_4$ = 4! = 24 ways to form numbers that have 3 in the unit's place. This shows that 24 of the 120 numbers have 3 in the unit's place.

Similarly, each of the other four digits is in the unit's place in 24 of the 120 numbers.

The sum of the digits in the units place among all 120 numbers is 24 (3+4+6+7+8) = 24×28 = 672.

Similary, the sum of the digits in the ten's place among all 120 numbers is 672. The same is also the sum of the digits in each place among all 120 numbers.

The required sum is 672 units + 672 tens + 672 hundreds + 672 thousands + 672 ten thousands.

$= 672 \times (1+10+100+1000+10000)$

$= 672\times11111 = 74{,}66{,}592$

Ex 5) A teacher has 2 different books on English, 3 different books on Physics, and 4 different books on Mathematics. These books are to be placed in a shelf so that all books on any one subjects are together. How many different ways are there to do this?

Solution : First, let us consider all books on each subjects to be one set, so that there are three sets, say E,P,M. These three can be arranged in $^3P_3$ = 3! = 6 different ways. Now, in each of these ways, the 2 books on English can be arranged in $^2P_2$ = 2! = 2 different ways the 3 books on physics can be arragned in $^3P_3$ =3! = 6 different ways, and the four books on Mathematics can be arranged in $^4P_4$ = 4! = 24 different ways.

The required number of arrangments is then given by 6×2×6×24 = 1728

Ex 6) Find n if ${}^{n}P_5 = 42 \times {}^{n}P_3$

Solution: We are given that ${}^{n}P_5 = 42 \times {}^{n}P_3$

That is, $\dfrac{n!}{(n-5)!} = 42\dfrac{n!}{(n-3)!}$

$$\therefore \frac{n!}{(n-5)!} = 42\frac{n!}{(n-3)(n-4)(n-5)!}$$

$\therefore (n-3)(n-4) = 42$

$\therefore (n-3)(n-4) = 7\times 6$

$\therefore n-3 = 7$

$\therefore n = 3+7 = 10$

3.5.2 Permutations when repetitions are allowed:

We now consider problems of arranging n objects taken r at a time when repetitions are allowed.

Theorem 2. The number of arrangements of n distinct objects taken r at a time, when repetitions are allowed, is same as the number of ways of filling r places using n distinct objects when repetitions are allowed. Consider the following table.

Place	1st	2nd	3rd	…	(r–2)th	(r–1)th	rth
Number of ways	n	n	n	…	n	n	n

Because repetitions are allowed, each place can be filled in n different ways.

Using multiplication principle, it can be concluded that the number of permutations of n distinct objects taken r at a time, when repetitions are allowed, is given by

$n\times n\times \ldots \times n(\text{r times}) = n^r$

SOLVED EXAMPLES

Ex 1: It is required to arrange 8 books on a shelf. Find the number of ways to do this if two specified books are

(i) always together (ii) never together.

Solution:

(i) Consider 2 books as one single set. These 2 books can be arrannged among themselves in 2! ways. The total 7 books can be arranged among themselves in 7! ways.

The total number of ways = 2! 7! = 2 × 5040 = 10080.

(ii) We take one of the two books and remaining 6 books and arrange these in 7! ways. Then we can place the remaining book in such a way to avoid 2 places adjacent to the 1st book, so there are 6 places available hence 6 × 7! = 6 × 5040 = 30240.

Alternative Method :

The number is also obtained by subtracting from the total arrangements those when these books are together i.e.

8! – 2 × 7! = (8 – 2) × 7! = 6 × 7! = 6 × 5040 = 30240.

Ex 2: In how many ways can 7 examination papers be arranged so that papers 6 and 7 are never together?

Solution : By the same argument as in example above, the number of ways in which any two papers are never together is.

(7–2) (7–1)! = 5×6! = 3600

Ex 3: A family of 3 brothers and 5 sisters is to be arranged for a photograph in such a ways that, (i) all brothers sit together. (ii) no two brothers sit together.

Solution :

(i) Since all 3 brothers are together, treat them as one person, so that there are 5+1=6 persons. the number of arranging them is

$^6P_6 = 6! = 720.$

Once this is done, the three brothers can be arranged among themselves in $^3P_3 = 3! = 6$ ways.

The total number of arrangements is then given by $6!\times3! = 720 \times 6 = 4320$.

(ii) 5 sisters can be arranged among themselves in $^5P_5 = 5! = 120$ ways.

Consider the following arrangement.

$$* \; S_1 * S_2 * S_3 * S_4 * S_5 *$$

Where * indicates a position where one brother can be placed so that no two brothers are together. Since there are 6 such positions and 3 brothers, the number of arrangements is

$$^6P_3 = \frac{6!}{3!} = 120$$

The required number of arrangements is then given by

$^5P_5 \times {}^6P_3 = 120 \times 120$

$= 14400.$

If some of the n objects are always kept together in a permutation problem, then the following theorem is useful for such cases.

Theorem. The number of permutations taken all at a time, when m specified objects among n always come together, is $m!(n-m+1)!$

Proof. Since the specified m objects always come together, let us consider them as a single object. This makes the number of distinct objects $n-m+1$ for the purpose of permutations. The number of permutations of $n-m+1$ objects taken all at a time is $(n-m+1)!$. The m specified objects are together, but can be rearranged among themselves. The number of permutations of these m objects taken all at a time is m!. Since this argument holds for each of the $(n-m+1)!$ permutations, the required number of permutations of n objects taken all together, when m specified objects are always together is $m!\,(n-m+1)!$

Remarks:

1. The number of permutations of n distinct objects taken all at a time, when 2 specified objects are always together, is $2\times(n-1)!$.

2. The number of permutations of n distinct objects taken r at a time, when a specified object is always to be included, is.

   $r \times {}^{(n-1)}P_{(r-1)}$.

   First keep the specified object aside. Arrange (r–1) from the remaining (n–1) objects in $^{(n-1)}P_{(r-1)}$ ways. Then place the specified object in r possible ways. Hence the total number of arrangements is $r \times {}^{(n-1)}P_{(r-1)}$.

3. The number of permutations of n distinct objects taken r at a time, when a specified object is not be included in any permutation, is $^{(n-1)}P_r$.

EXERCISE 3.3

1. Find n, if $^nP_6 : {}^nP_3 = 120:1$

2. Find m and n, if $^{(m+n)}P_2 = 56$ and

   $^{(m-n)}P_2 = 12$

3. Find r, if $^{12}P_{r-2} : {}^{11}P_{r-1} = 3:14$

4. Show that $(n+1)\,({}^nP_r) = (n-r+1)\,[{}^{(n+1)}P_r]$

5. How many 4 letter words can be formed using letters in the word MADHURI if (a) letters can be repeated (b) letters cannot be repeated.

6. Determine the number of arrangements of letters of the word ALGORITHM if.
   (a) vowels are always together.
   (b) no two vowels are together.
   (c) consonants are at even positions.
   (d) O is the first and T is the last letter.

7. In a group photograph, 6 teachers are in the first row and 18 students are in the second row. There are 12 boys and 6 girls among the students. If the middle position is reserved for the principal and if no two girls are together, find the number of arrangements.

8. Find the number of ways so that letters of the word HISTORY can be arranged as,
   (a) Y and T are together
   (b) Y is next to T.
   (c) there is no restriction
   (d) begin and end with vowel
   (e) end in ST
   (f) begin with S and end with T

9. Find the number of arrangements of the letters in the word SOLAPUR so that consonents and vowels are placed alternately.

10. Find the number of 4–digit numbers that can be formed using the digits 1, 2, 4, 5, 6, 8 if
    (a) digits can be repeated
    (b) digits cannot be repeated

11. How many numbers can be formed using the digits 0, 1, 2, 3, 4, 5 without repetition so that resulting numbers are between 100 and 1000?

12. Find the number of 6–digit numbers using the digits 3,4,5,6,7,8 without repetition. How many of these numbers are
    (a) divisible by 5, (b) not divisible by 5.

13. A code word is formed by two different English letters followed by two non–zero distinct digits. Find the number of such code words. Also, find the number of such code words that end with an even digit.

14. Find the number of ways in which 5 letters can be posted in 3 post boxes if any number of letters can be posted in a post box.

15. Find the number of arranging 11 distinct objects taken 4 at a time so that a specified object.
    (a) always occurs (b) never occurs.

16. In how many ways can 5 different books be arranged on a shelf if
    (i) there are no restrictions
    (ii) 2 books are always together
    (iii) 2 books are never together

17. 3 boys and 3 girls are to sit in a row. How many ways can this be done if
    (i) there are no restrictions
    (ii) there is a girl at each end
    (iii) boys and girls are at alternate places
    (iv) all boys sit together

3.5.3 Permutations when some objects are identical

Consider the problem of arranging letters in the words like GOOD, INDIA, GEOLOGY, MATHEMATICS, or PHILOSOPHY, where some letters occur more than once and hence all letters are not distinct.

Consider the word ODD. First we consider the 2 D's as distinct objects D_1 and D_2.

The total number of words is OD_1D_2, OD_2D_1, D_1OD_2, D_2OD_1, D_1D_2O, $D_2D_1O = 6 = 3!$

But D_1 and D_2 are identical so, $OD_1D_2 = OD_2D_1$, $D_1D_2O = D_2D_1O$, $D_2OD_1 = D_1OD_2$ and as D_1 and D_2 can be arranged among themselves in 2! ways, (D_1D_2, D_2D_1)

Thus there are $\frac{6}{2} = \frac{3!}{2!} = 3$ different words formed (ODD,DOD,DDO)

Theorem : Consider a set of n objects where n_1 objects are identical and the remaining $n - n_1$ are distinct. The number of permutations of these n objects is $\frac{n!}{n_1!}$. Note that $n_1 < n$.

Proof : Let m be the total number of arrangements, where n_1 out of n objects are identical. The number of permutations of n different objects is $n!$. Now n_1 objects can be rearranged in $n_1!$ ways among themselves if they were all different. Thus each arrangement where these n_1 objects are identical corresponds to $n_1!$ different arrangements if they were all different.

Hence $m \times n_1! = n!$. Therefore $m = \frac{n!}{n_1!}$

Remarks.

1. The number of permutations of n objects, not all distinct, where n_1 objects are of one type and n_2 objects are of a second type, taken all at a time is $\frac{n!}{n_1!n_2!}$ and note that $n_1 + n_2 \leq n$.

   The proof is similar to the proof of the above theorem.

2. The number of permutations of n objects, not all distinct, where n_i objects are of type i, i=1, 2, …, k, taken all at a time is $\frac{n!}{n_1!n_2!...n_k!}$, $n_1 + n_2 + ... n_k \leq n$.

SOLVED EXAMPLES

Ex 1 : Find he number of permutations of the letters of the word UBUNTU.

Solution : The word UBUNTU consists of 6 letters, in which letter ‘U’ is repeated 3 times.

Therefore, number of permutations of the letters of the word UBUNTU = $\frac{6!}{3!} = 120$.

Ex 2 : How many arrangements can be made, with the letters of the word CALCULATOR? In how many of these arrangements, vowels occur together?

Solution : The word CALCULATOR consists of 10 letters, in which ‘C’ is repeated two times, ‘A’ is repeated two times, ‘L’ is repreated two times and rest all are different.

Therefore, number of permutations of the letters of the word CALCULATOR = $\frac{10!}{2!\,2!\,2!}$.

The word CALCULATOR consists of 4 vowels A, U, A, O. Let us consider them as a single letter say P.

Therefore, now we have 7 letters P, C, L, C, L, T, R in which ‘C’ is repeated two times, ‘L’ is repeated two times. The number of arrangement of these 7 letters is given $\frac{7!}{2!\,2!} = 1260$. After this is done, 4 vowels (in which ‘A’ is repeated 2 times) can be arranged in $\frac{4!}{2!} = 12$ ways

Therefore, number of arrangements of the letters of the word CALCULATOR in which vowels are together = $1260 \times 12 = 15120$.

EXERCISE 3.4

1. Find the number of permutations of letters in each of the following words.
   (i) DIVYA
   (ii) SHANTARAM
   (iii) REPRESENT
   (iv) COMBINE
   (v) BALBHARATI

2. You have 2 identical books on English, 3 identical books on Hindi, and 4 identical books on Mathematics. Find the number of distinct ways of arranging them on a shelf .

3. A coin is tossed 8 times. In how many ways can we obtain. (a) 4 heads and 4 tails? (b) at least 6 heads?

4. A bag has 5 red, 4 blue, and 4 green marbles. If all are drawn one by one and their colours are recorded, how many different arrangements can be found?

5. Find the number of ways of arranging letters of the word MATHEMATICAL How many of these arrangements have all vowels together?

6. Find the number of different arrangements of letters in the word MAHARASHTRA. How many of these arrangements have (a) letters R and H never together? (b) all vowels together?

7. How many different words are formed if the letters R is used thrice and letters S and T are used twice each?

8. Find the number of arrangements of letters in the word MUMBAI so that the letter B is always next to A.

9. Find the number of arrangements of letters in the word CONSTITUTION that begin and end with N.

10. Find the number of different ways of arranging letters in the word ARRANGE. How many of these arrangement do not have the two R's nor A's together?

11. How many distinct 5 digit numbers can be formed using the digits 3, 2, 3, 2, 4, 5.

12. Find the number of distinct numbers formed using the digits 3, 4, 5, 6, 7, 8, 9, so that odd positions are occupied by odd digits.

13. How many different 6–digit numbers can be formed using digits in the number 659942? How many of them are divisible by 4?

14. Find the number of distinct words formed from letters in the word INDIAN. How many of them have the two N's togethers?

15. Find the number of different ways of arranging letters in the word PLATOON if. (a) the two O's are never together. (b) consonants and vowels occupy alternate positions.

3.5.4 Circular permutation:

We can imagine a circular arrangement of n different objects to be transformed into an arrangement in a line by cutting the circle in a place. This cut can be made at n different places.

For example, consider $n = 4$

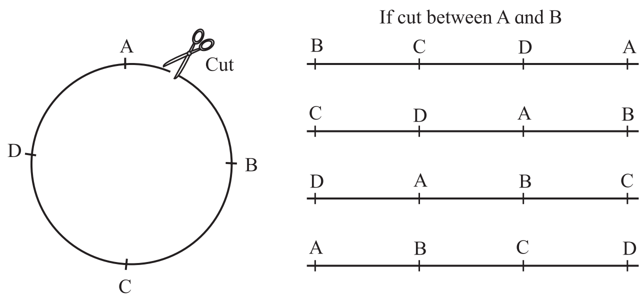

Fig. 3.5

Thus given a single circular arrangement of 4 objects, it corresponds to 4 different linear arrangements (arrangements in a row).

Let the circular arrangements be m.

We know that the total number of linear arrangement of 4 different objects is 4!

Each arrangement in row corresponds to some circular arrangement.

Therefore number of circular arrangements is m = $\frac{4!}{4}$ = 3!

Similarly, each circular arrangement of n objects corresponds to n different arrangements in a row.

Therefore number of circular arrangements is $\frac{n!}{n}$ = (n − 1)!

Note:

1) A circular arrangement does not have a fixed starting point and any rotation of it is considered to be the same circular arrangement. This arrangement will be the same with respect to each other if a rotation is made but clockwise and anticlockwise arrangements are different.

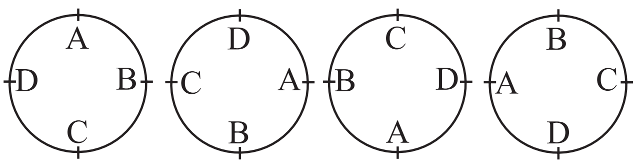

Fig. 3.5 (a)

In circular arrangement, (with respect to each other) all these count as one arrangement.

2) If clockwise and anticlockwise circular arrangements are considered to be the same then each circular arrangement corresponds to 2n different linear arrangements. Thus the numbers of such circular arrangement is $\frac{n!}{2n} = \frac{(n-1)!}{2}$.

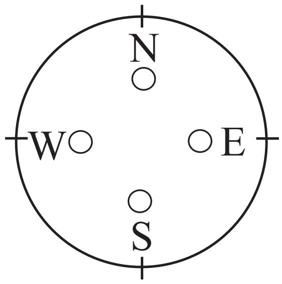

Fig. 3.5 (b)

Consider the 4 directions, E, N, W, S on Mariner's compass. If 4 stickers of different colours are to be placed at the different letters.

The number of arrangements is 4!. Here rotation of W, S, E, N is not allowed. Hence arrangement is like a linear arrangement. Such arrangement is called as an arrangement with respect to round table.

Now consider the situation where m out of the n objects are alike and cannot be distinguished from one another. The following result gives the formula for the number of circular arrangements of these n objects.

Theorem. The number of circular arrangements of n objects, of which m objects are alike (identical), is given by $\frac{(n-1)!}{m!}$

Proof. As argued earlier, m like (indistinguishable) objects can be rearranged among themselves in m! ways without affecting the arrangements of the n objects. The number of circular arrangements of n objects is (n−1)! and the number of arrangements of m like (indistinguishable) objects among themselves is m!, the required number of arrangements is given by $\frac{(n-1)!}{m!}$

Does this argument remind us of a similar argument that we came across earlier?

Remark: The number of circular permutations of r objects taken from n distinct objects can be found under two different conditions as follows.

(a) When clockwise and anticlockwise arrangements are considered to be different, then the required number of circular arrangements is given by $\frac{{}^{n}p_{r}}{r}$

(b) When clockwise and anticlockwise arrangements are not to be considered different, then the required number of circular arrangements is given by $\frac{{}^{n}p_{r}}{2r}$

Verify these two statements for n = 6, r = 3.

SOLVED EXAMPLES

Ex 1 : In how many ways can 8 students be arranged at a round table so that 2 particular students are together, if

(i) students are arranged with respect to each other? (That is the seats are not numbered.)

(ii) students are arranged with respect to the table? (That is the seats are numbered serially.)

Solution : Considering those 2 particular students as one student, we have 7 students.

(i) In circular arrangement 7 students can be arranged at a round table, in 6! ways and 2 students can be arranged among themselves in ${}^{2}p_{2} = 2!$ ways. Hence, the required number of ways in which two particular students come together = $6! \times 2! = 1440$.

(ii) Here the arrangement is like an arrangement in a row.

So 7 students can be arranged in 7! ways and 2 students can be arranged amongst themselves in ${}^{2}p_{2} = 2!$ ways. Hence, the required number of ways in which two particular students come together $= 7! \times 2! = 10080$.

Ex 2 : In how many ways 6 men and 3 women can be seated at a round table so that every man has woman by his side.

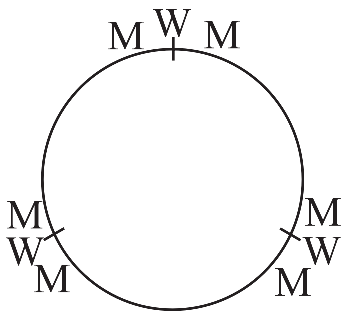

Fig. 3.6

Solution : 3 women have $(3 - 1)! = 2$ ways of circular seatings.

In each seating a women has one place on each side for a man. Thus there are 6 different places for 6 men which can be filled in 6! ways.

Hence, the total number of required arrangements $= 6! \times 2 = 720 \times 2 = 1440$.

Ex 3 : Find the number of ways in which 12 different flowers can be arranged in a garland so that 4 particular flowers are always together.

Solution : Considering 4 particular flowers as a single flower, we have 9 flowers which can be arranged to form a garland in 8! ways. But 4 particular flowers can be arranged in 4! ways. Hence, the required number of ways $= \frac{1}{2}(8! \times 4!) = 483840$.

Ex 4 : How many necklaces of 12 bead each, can be made from 18 beads of different colours?

Solution : Here, clockwise and anticlockwise arrangements are same. Hence, total number of circular permutations = $\frac{^{18}P_{12}}{2\times12} = \frac{^{18}P_{12}}{24}$.

Ex 5 : Three boys and three girls are to be seated around a table in a circle. Among them, the boy X does not want any girl as neighbour and girl Y does not want any boy as neighbour. How many such distinct arrangements are possible?

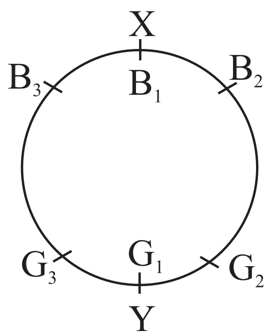

Fig. 3.7

Solution : The arrangement is as shown in the figure (fig. 3.7). The boy X will have B_2 and B_3 as neighbours. The girl Y will have G_2, G_3 as neighbours. The two boys B_2, B_3 can be arranged in two ways. The two girls G_2, G_3 can be arranged in two ways.

Hence, the total number of arrangements $= 2 \times 2 = 4$.

Ex 6 : In how many different arrangements can 6 gentlemen and 6 ladies sit around a table if (i) there is no restriction. (ii) no two ladies sit side by side?

Solution : (i) There is no restriction.

Here, the total number = 6 + 6 = 12.

12 persons can be arranged in circular permutation in (12–1)! = 11! ways.

(ii) No two ladies sit side by side.

When 6 gentlemen are arranged around a table, there are 6 positions, each being between two gentlemen, for 6 ladies when no two ladies sit side by side. Now, the number of ways in which 6 gentlemen can be seated around a table $= (6 - 1)! = 5!$

Now, corresponding to each seating arrangement for the gentlemen, the 6 ladies can be seated in 6! ways.

Thus, the required number of arrangements $= (5!)\ (6!) = 86400$.

Ex 7 : In how many different ways can 4 married couples occupy seats around a circular table if (i) Spouses sit opposite to each other? (ii) Men and women alternate?

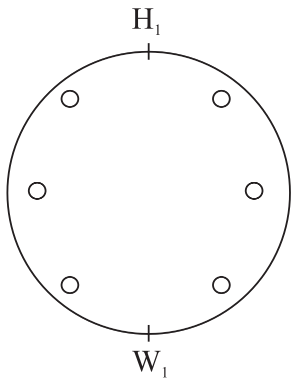

Fig. 3.8

Solution : (i) Spouses sit opposite to each other. Since there is no fixed starting point let one woman occupy one seat. Let, W_1H_1, W_2H_2, W_3H_3 and W_4H_4 be the four couples. Since there is no fixed starting point of the circle, Let W_1 occupy one seat then H_1 has to sit exactly opposite. There are 3 seats on either side of W_1. Then W_2 can occupy any of the 6 seats which confirms the seat of her husband H_2. Now there are 4 seats left. W_3 can occupy anyone of them so that her husband's seat is fixed. Now W_4 has 2 choices, fixing the seat of her husband. In all there are $6 \times 4 \times 2 = 48$ ways.

ii) Let 4 women sit on alternate seats around a table starting at any place. This is done in 3! ways. It leaves 4 alternate seats empty. They are fixd by 4 men in 4! ways. Hence, the total number of ways is $4! \times 3! = 144$.

EXERCISE 3.5

1. In how many different ways can 8 friends sit around a table?
2. A party has 20 participants. Find the number of distinct ways for the host to sit with them around a circular table. How many of these ways have two specified persons on either side of the host?
3. Delegates from 24 countries participate in a round table discussion. Find the number of seating arrangments where two specified delegates are. (a) always together, (b) never together.
4. Find the number of ways for 15 people to sit around the table so that no two arrangements have the same neighbours.
5. A committee of 10 members sits around a table. Find the number of arrangements that have the president and the vice president together.
6. Five men, two women, and a child sit around a table. Find the number of arrangements where the child is seated (a) between the two women. (b) between two men.
7. Eight men and six women sit around a table. How many of sitting arrangements will have no two women together?
8. Find the number of seating arrangements for 3 men and 3 women to sit around a table so that exactly two women are together.
9. Four objects in a set of ten objects are alike. Find the number of ways of arranging them in a circular order.
10. Fifteen persons sit around a table. Find the number of arrangements that have two specified persons not sitting side by side.

Properties of Permutations

(i) ${}^{n}P_{n} = n!$

(ii) ${}^{n}P_{0} = 1$

(iii) ${}^{n}P_{1} = n$

(iv) ${}^{n}P_{r} = n \times {}^{(n-1)}P_{(r-1)}$

$= n(n-1) \times {}^{(n-2)}P_{(r-2)}$

$= n(n-1)(n-2) \times {}^{(n-3)}P_{(r-3)}$ and so on.

(v) $\dfrac{{}^{n}p_{r}}{{}^{n}p_{(r-1)}} = n-r+1$

3.6 Combinations

Permutations involve ordered arrangements of objects. We shall now consider situations where the order of objects in an arrangement is immaterial, but only selection of a set of objects in the arrangement is considered. A selection of objects without any consideration of the order is called a combination.

Consider the earlier example from permutations where 2 chairs were filled from a group of 4 persons. We make a little change in the problem. We want to select a group of 2 people and not consider the order. So the arrangements AB and BA correspond to the same group. Similarly BC and CB are given in the same group. The list is given as follows.

$$\begin{pmatrix} AB \\ BA \end{pmatrix} \begin{pmatrix} BC \\ CB \end{pmatrix} \begin{pmatrix} CA \\ AC \end{pmatrix} \begin{pmatrix} AD \\ DA \end{pmatrix} \begin{pmatrix} BD \\ DB \end{pmatrix} \begin{pmatrix} CD \\ DC \end{pmatrix}$$

Thus there are altogether $\dfrac{{}^{4}P_{2}}{2} = \dfrac{4\times 3}{2\times 1} = 6$ different groups selected. This is called the combination number of selecting a group of 2 from 4 persons denoted by ${}^{4}C_{2}$.

Definition Combination. A combination of a set of n distinct objects taken r at a time without repetition is an r–element subset of the n objects.

Note: The order of arrangement of the elements is immaterial in a combination.

If we want to choose a team of 3 players from a set of 8 different players, we first get the number $^8P_3$, i.e. different ordered sets of 3 players and since any set of 3 gives 3! ordered sets, we divided $^8P_3$ by 3!. Thus the number of combinations of 3 players from 8 players is $\frac{^8P_3}{3!} = \frac{8!}{(8-3)!3!}$

Combination (^nC_r) : From n different objects, the number of ways of selecting a group or a set of r objects (without considering order) is denoted by ^nC_r or C(n,r) or nCr. It is the number of combinations of r ^nC_r objects from n distinct objects.

Theorem : $^nC_r = \frac{n!}{(n-r)!r!}$

Proof : First we find the number of 'ordered' sets of r objects from n distinct objects.

$\therefore\ ^nP_r = \frac{n!}{(n-r)!}$. Now, each set of r objects corresponds to r! different ordered arrangements. So, we count the number of different sets or r objects, without considering the order among them, is $= \frac{^nP_r}{r!} = \frac{n!}{(n-r)!r!}$

3.6.1 Properties of combinations.

1. Consider $^nC_{n-r} = \frac{n!}{(n-r)![n-(n-r)]!}$

$$= \frac{n!}{(n-r)!r!}$$

$$= {}^nC_r.$$

Thus, $^nC_{n-r} = {}^nC_r$ for $0 \le r \le n$.

2. $^nC_0 = \frac{n!}{0!(n-0)!} = \frac{n!}{n!} = 1$, because $0! = 1$ as has been stated earlier.

3. If $^nC_r = {}^nC_s$, then either s = r or s = n–r.

4. $^nC_r = \frac{^nP_r}{r!}$

5. $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

6. $^nC_0 + {}^nC_1 + \dots {}^nC_n = 2^n$

7. $^nC_0 + {}^nC_2 + {}^nC_4 + \dots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \dots = 2^{(n-1)}$

8. $^nC_r = \left(\frac{n}{r}\right)\,^{(n-1)}C_{(r-1)} = \left(\frac{n}{r}\right)\left(\frac{n-1}{r-1}\right)\,^{(n-2)}C_{(r-2)} = \dots$

9. ^nC_r has maximum value if (a) r = $\frac{n}{2}$ when n is even (b) r = $\frac{n-1}{2}$ or $\frac{n+1}{2}$ when n is odd.

SOLVED EXAMPLES

Ex 1 : Find the value of (i) $^7C_3$ (ii) $^{10}C_7$ (iii) $^{52}C_3$

Solution : We know that

$$^nC_r = \frac{n!}{r!(n-r)!}$$

(i) $$^7C_3 = \frac{7!}{3!(7-3)!} = \frac{7!}{3!(4)!} = \frac{7\times6\times5\times4!}{3\times2\times1\times4!} = \frac{7\times6\times5}{3\times2\times1} = 35$$

(ii) $$^{10}C_7 = \frac{10!}{7!(10-7)!} = \frac{10!}{7!(3)!} = \frac{10\times9\times8}{3\times2} = 120$$

(iii) $^{52}C_3 = \frac{52!}{3!(52-3)!} = \frac{52!}{3!(49)!}$

$$= \frac{52\times 51\times 50}{3\times 2\times 1} = 22100$$

Ex 2 : Find n and r

if $^nC_{r-1} : {^nC_r} : {^nC_{r+1}} = 14:8:3$

Solution : $\frac{^nC_{r-1}}{^nC_r} = \frac{14}{8} = \frac{7}{4}$ and $\frac{^nC_r}{^nC_{r+1}} = \frac{8}{3}$

$\therefore \quad \frac{r}{n-r+1} = \frac{7}{4}$ and $\frac{r+1}{n-r} = \frac{8}{3}$

$\therefore \quad (n-r) = \left(\frac{4}{7}r\right) - 1$ and $(n-r) = \frac{3}{8}(r+1)$

$\therefore \quad \left(\frac{4}{7}r\right) - 1 = \frac{3}{8}(r+1)$

Solving this we get r = 7 and n = 10

Ex 3 : There are n points in a plane. Find the number of straight lines and triangles that can be obtained by joining points on a plane if

(i) no three points are collinear

(ii) p-points are collinear ($p\geq 3$)

Solution : (i) Straight line can be drawn by joining any two points and traingle can be drawn by joining any three non-collinear points.

From n-points, any two points can be selected in $^nC_2$ ways.

$\therefore$ Number of straight lines = $^nC_2$

Since no three points are collinear, any three non-collinear points can be selected in $^nC_3$ ways.

$\therefore$ Number of triangles = $^nC_3$

(ii) If P points are non-collinear then, we can obtain $^PC_2$ straight lines and $^PC_3$ triangles from those p points. But we are given that p points are collinear, therefore they form only one line and no triangle i.e. we have counted ($^pC_2 - 1$) extra lines and $^pC_3$ extra triangles.

$\therefore$ If p-points ($p\geq 3$) are collinear, then number of straight lines = $^nC_2 - (^pC_2 - 1)$ and number of triangles = $^nC_3 - {^pC_3}$

Ex 4 : Four cards are drawn from a pack of 52 playing cards. In how many different ways can this be done? How many selections will contain

(i) exactly one card of each suit?

(ii) all cards of the same suit?

(iii) all club cards?

(iv) at least one club card?

(v) three kings and one queen?

(vi) three black and one red cards?

Solution : A pack of 52 cards contains 4 different suits, viz. Club, Spade, Diamond and Heart. Each suit contains 13 cards. Club and Spade are black coloured cards. Diamond and Heart are red colour cards. i.e. pack of 52 cards contains 26 black and 26 red colour cards.

From a pack of 52 cards, any 4 cards can be drawn in $^{52}C_4 = \frac{52!}{4!\times 48!}$ ways.

(i) Exactly one card of each suit.

One club card can be selected in $^{13}C_1 = 13$ ways. One heart card can be selected in $^{13}C_1 = 13$ ways. One spade card can be selected in $^{13}C_1 = 13$ ways. One diamond card can be selected in $^{13}C_1 =$ 13 ways.

$\therefore$ Using fundamental principle, exactly one card of each suit can be selected in $13 \times 13 \times 13 \times 13 = (13)^4$ ways.

(ii) All cards are of the same suit.

From 4 suits, any one suit can be selected in $^4C_1 = 4$ ways. After this is done, any four cards from selected suit can be drawn in $^{13}C_4 = 715$ ways.

$\therefore$ Using fundamental principle, all 4 cards of the same suit can be selected in $4 \times 715 = 2860$ ways.

(iii) All club cards

From 13 club cards, any 4 club cards can be drawn in $^{13}C_4 = 715$ ways.

(iv) At least one club card

From a pack of 52 cards, any 4 cards can be drawn in $^{52}C_4$ ways if there is no condition.

From $^{52}C_4$ selections, remove those selections that do not contain any club card, so that in the remaining selection we have at least one club card.

If the selection does not contain any club card i.e. 4 cards are drawn from remaining 39 non-club cards, then this can be done in $^{39}C_4$ ways.

$\therefore$ Number of selections which contain at least one club card = $^{52}C_4 - {}^{39}C_4$.

(v) Three king cards and one queen card

From 4 kings, any 3 king cards can be selected in $^4C_3 = 4$ ways and from 4 queen cards 1 queen card can be selected in $^4C_1 = 4$ ways.

$\therefore$ Using fundamental principle, three king cards and one queen card can be selected in $4 \times 4 = 16$ ways.

(vi) Three black cards and one red card

From 26 black cards, any 3 black cards can be selected in $^{26}C_3$ ways and from 26 red cards, any one red card can be selected in $^{26}C_1$ ways.

Using fundamental principle, three black and one red card can be selected in $^{26}C_3 \times {}^{26}C_1$ ways.

Ex 5 : Find n, if $^nC_8 = {}^nC_6$

Solution : If $^nC_x = {}^nC_y$, then either $x = y$ or $x = n-y$

$\therefore \quad 8 = n-6 \quad (\because x \neq y)$

$\therefore \quad n = 14$

Ex 6 : Find r, if $^{16}C_4 + {}^{16}C_5 + {}^{17}C_6 + {}^{18}C_7 = {}^{19}C_r$

Solution : $({}^{16}C_4 + {}^{16}C_5) + {}^{17}C_6 + {}^{18}C_7 = {}^{19}C_r$

$\therefore \quad ({}^{17}C_5 + {}^{17}C_6) + {}^{18}C_7 = {}^{19}C_r$

$\therefore \quad ({}^{18}C_6 + {}^{18}C_7) = {}^{19}C_r$

$\therefore \quad {}^{19}C_7 = {}^{19}C_r$

$\therefore \quad r = 7$ or $r = 19-7 = 12$

Ex 7 : Find the difference between the maximum values of ^8C_r and $^{11}C_r$

Solution :

Maximum value of ^8C_r occurs at $r = \frac{8}{2} = 4$

$\therefore \quad$ maximum value of $^8C_r = {}^8C_4 = 70$

Maximum value of $^{11}C_r$ occurs at $r = \frac{10}{2} = 5$

or at $\frac{12}{2} = 6$

$\therefore \quad$ Maximum value of $^{11}C_r = {}^{11}C_5 = {}^{11}C_6 = 462$

$\therefore \quad$ difference between the maximum values of ^8C_r and $^{11}C_r = {}^{11}C_5 - {}^8C_4 = 462 - 70 = 392$

EXERCISE 3.6

1. Find the value of (a) $^{15}C_4$ (b) $^{80}C_2$

   (c) $^{15}C_4 + {}^{15}C_5$ (d) $^{20}C_{16} - {}^{19}C_{16}$

2. Find n if

   (a) $^6P_2 = n\ {}^6C_2$

   (b) $^{2n}C_3 : {}^nC_2 = 52 : 3$

   (c) $^nC_{n-3} = 84$

3. Find r if ${}^{14}C_{3r} : {}^{10}C_{2r-4} = 143:10$

4. Find n and r if.

   (a) ${}^{n}P_{r} = 720$ and ${}^{n}C_{n-r} = 120$

   (b) ${}^{n}C_{r-1} : {}^{n}C_{r} : {}^{n}C_{r+1} = 20: 35:42$

5. If ${}^{n}P_{r} = 1814400$ and ${}^{n}C_{r} = 45$, find ${}^{n+4}C_{r+3}$

6. If ${}^{n}C_{r-1} = 6435$, ${}^{n}C_{r} = 5005$, ${}^{n}C_{r+1} = 3003$, find ${}^{r}C_{5}$.

7. Find the number of ways of drawing 9 balls from a bag that has 6 red balls, 8 green balls, and 7 blue balls so that 3 balls of every colour are drawn.

8. Find the number of ways of selecting a team of 3 boys and 2 girls from 6 boys and 4 girls.

9. After a meeting, every participant shakes hands with every other participants. If the number of handshakes is 66, find the number of participants in the meeting.

10. If 20 points are marked on a circle, how many chords can be drawn?

11. Find the number of diagonals of an n–sided polygon. In particular, find the number of diagonals when.

    (a) n = 10 (b) n = 15

    (c) n = 12 (d) n = 8

12. There are 20 straight lines in a plane so that no two lines are parallel and no three lines are concurrent. Determine the number of points of intersection.

13. Ten points are plotted on a plane. Find the number of straight lines obtained by joining these points if

    (a) no three points are collinear.

    (b) four points are collinear.

14. Find the number of triangles formed by joining 12 points if

    (a) no three points are collinear

    (b) four points are collineas.

15. A word has 8 consonants and 3 vowels. How many distinct words can be formed if 4 consonants and 2 vowels are chosen?

16. Find n if,

    (i) ${}^{n}C_{8} = {}^{n}C_{12}$

    (ii) ${}^{23}C_{3n} = {}^{23}C_{2n+3}$

    (iii) ${}^{21}C_{6n} = {}^{21}C_{(n^2+5)}$

    (iv) ${}^{2n}C_{r-1} = {}^{2n}C_{r+1}$

    (v) ${}^{n}C_{n-2}=15$

17. Find x if ${}^{n}P_{r} = x\,{}^{n}C_{r}$

18. Find r if ${}^{11}C_{4} + {}^{11}C_{5} + {}^{12}C_{6} + {}^{13}C_{7} = {}^{14}C_{r}$

19. Find the value of $\sum_{r=1}^{4} {}^{(21-r)}C_{4}$

20. Find the differences between the greatest values in the following:

    (a) ${}^{14}C_{r}$ and ${}^{12}C_{r}$, (b) ${}^{13}C_{r}$ and ${}^{8}C_{r}$,

    (c) ${}^{15}C_{r}$ and ${}^{11}C_{r}$,

21. In how many ways can a boy invite his 5 friends to a party so that at least three join the party?

22. A group consists of 9 men and 6 women. A team of 6 is to be selected. How many of possible selections will have at least 3 women?

23. A committee of 10 persons is to be formed from a group of 10 women and 8 men. How many possible committees will have at least 5 women? How many possible committees will have men in majority?

24. A question paper has two sections. section I has 5 questions and section II has 6 questions. A student must answer at least two question from each section among 6 questions he answers. How many different choices does the student have in choosing questions?

25. There are 3 wicketkeepers and 5 bowlers among 22 cricket players. A team of 11 players is to be selected so that there is exactly one wicketkeeper and at least 4 bowlers in the team. How many different teams can be formed?

26. Five students are selected from 11. How many ways can these students be selected if.

(a) two specified students are selected?

(b) two specified students are not selected?

Let's Remember

- **Factorial notation :**

n! or $\lfloor n$ = 1,2,3.... (n–2) (n–1) n,

(0! = 1)

- **Principle of Addition :**

If an event can occur either in m or n mutually exclusive alternate ways, then the total number of ways in which the event can occur is m + n.

- **Principle of Multiplication :**

If an event has m possible outcomes, and another independent event has n possible outcomes, then there are m.n possible outcomes for the two events together.

- **Permutation :**

A permutation is an arrangement, in a definite order, of a number of objects, taken some or all at a time.

(1) Linear Permutation :

(a) The number of permutation of n different objects taken r at a time when repetition of r objects in the permutation is not allowed is given by

$${}^{n}P_{r} = \frac{n!}{(n-r)!} \text{ where } r \leq n$$

(b) The number of permutations of n different objects, taken r objects at a time, when repetition of r objects in the permutation is allowed, is given by n^r.

(c) The number of permutations of n objects, when p objects are of one kind, q objects are of second kind, r objects are of third kind and the rest, (if any), are of different kind is $\frac{n!}{p!q!r!}$

(ii) Circular Permutation:

The arrangements in a circle are called circular permutations.

(a) The number of circular permutations of n different objects = (n–1)!

(b) The number of ways in which n things of which p are alike, can be arranged in a circular order is $\frac{(n-1)!}{p!}$

- **Combination :**

A combination is a selection. Total number of selections of 'n' different objects, taken 'r' at a time is denoted by ${}^{n}C_{r}$ or nCr or C(n,r), or or $\binom{n}{r}$ and is given by

$${}^{n}C_{r} = \frac{n!}{r!(n-r)!} = \frac{{}^{n}P_{r}}{r!}$$

- **Properties of ${}^nC_r$:**

(a) ${}^nC_r = {}^nC_{n-r}$

(b) If ${}^nC_x = {}^nC_y$, then either $x = y$ or $x + y = n$

(c) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r \quad 1 \leq r \leq n$ (Pascal's rule)

(d) The number of ways of selecting one or more things from n different things is given by 2^{n-1}.

(e) ${}^nC_r$ has maximum value, if

(i) $r = \dfrac{n}{2}$ when n is even

(ii) $r = \dfrac{n-1}{2}$ or $\dfrac{n+1}{2}$ when n is odd.

MISCELLANEOUS EXERCISE - 3

(I) Select the correct answer from the given alternatives.

1) A college offers 5 courses in the morning and 3 in the evening. The number of ways a student can select exactly one course, either in the morning or in the evening.

A) 5 B) 3 C) 8 D) 15

2) A college has 7 courses in the morning and 3 in the evening. The possible number of choices with the student if he wants to study one course in the morning and one in the evening is-.

A) 21 B) 4 C) 42 D) 10

3) In how many ways can 8 Indians and, 4 American and 4 Englishmen can be seated in a row so that all person of the same nationality sit together?

A) 3! 8! B) 3! 4! 8! 4!
C) 4! 4! D) 8! 4! 4!

4) In how many ways can 10 examination papers be arranged so that the best and the worst papers never come together?

A) 9×8! B) 8×8! C) 8×9! D) 8×9!

5) In how many ways 4 boys and 3 girls can be seated in a row so that they are alternate.

A) 12 B) 288 C) 144 D) 256

6) Find the number of triangles which can be formed by joining the angular points of a polygon of 8 sides as vertices.

A) 16 B) 56 C) 24 D) 8

7) A question paper has two parts, A and B, each containing 10 questions. If a student has to choose 8 from part A and 5 from part B, In how many ways can he choose the questions?

A) 320 B) 750 C) 40 D) 11340

8) There are 10 persons among whom two are brothers. The total number of ways in which these persons can be seated around a round table so that exactly one person sits between the brothers, is equal to:

A) 2!×7! B) 2!×8! C) 3!×7! D) 3!×8!

9) The number of arrangements of the letters of the word BANANA in which two N's do not appear adjacently.

A) 80 B) 60 C) 40 D) 100

10) The number of ways in which 5 male and 2 female members of a committee can be seated around a round table so that the two females are not seated together is

A) 840 B) 600 C) 720 D) 480

(II) Answer the following.

1) Find the value of r if ${}^{56}C_{r+2} : {}^{54}P_{r-1} = 30800:1$

2) How many words can be formed by writing letters in the word CROWN in different order?

3) Find the number of words that can be formed by using all the letters in the word REMAIN If these words are written in dictionary order, what will be the 40^{th} word?

4) Capital English alphabet has 11 symmetric letters that appear same when looked at in a mirror. These letters are A, H, I, M, O, T, U, V, W, X, and Y. How many symmentric three letter passwords can be formed using these letters?

5) How many numbers formed using the digits 3,2,0,4,3,2,3 exceed one million?

6) Ten students are to be selected for a project from a class of 30 stdudents. There are 4 students who want to be together either in the project or not in the project. Find the number of possible selections.

7) A student finds 7 books of his interest, but can borrow only three books. He wants to borrow Chemistry part II book only if Chemistry Part I can also be borrowed. Find the number of ways he can choose three books that he wants to borrow.

8) 30 objects are to be divided in three groups containing 7,10,13 objects. Find the number of distinct ways for doing so.

9) A student passes an examination if he secures a minimum in each of the 7 subjects. Find the number of ways a student can fail.

10) Nine friends decide to go for a picnic in two groups. One group decides to go by car and the other group decides to go by train. Find the number of different ways of doing so if there must be at least 3 friends in each group.

11) A hall has 12 lamps and every lamp can be switched on independently. Find the number of ways of illuminating the hall.

12) How many quadratic equations can be formed using numbers from 0,2,4,5 as coefficients if a coefficient can be repeated in an equation.

13) How many six−digit telephone numbers can be formed if the first two digits are 45 and no digit can appear more than once?

14) A question paper has 6 questions. How many ways does a student have to answer if he wants to solve at least one question?

15) Find the number of ways of dividing 20 objects in three groups of sizes 8,7,and 5.

16) There are 4 doctors and 8 lawyers in a panel. Find the number of ways for selecting a team of 6 if at least one doctor must be in the team.

17) Four parallel lines intersect another set of five parallel lines. Find the number of distinct parallelograms formed.

18) There are 12 distinct points A,B,C,.....,L, in order, on a circle. Lines are drawn passing through each pair of points

    i) How many lines are there in total.

    ii) How many lines pass through D.

    iii) How many triangles are determined by lines.

    iv) How many triangles have on vertex C.

4 METHOD OF INDUCTION AND BINOMIAL THEOREM

Let's Study

- Mathematical Induction
- Binomial Theorem
- General term of expansion
- Expansion for negative and fractional index
- Binomial coefficients

Let's Learn

Introduction :

The earliest implicit proof by induction was given by Al Karaji around 100 AD. The first explicit formulation of the principle was given by Pascal in 1665. The Mathematical Induction is a powerful method, easy to use for proving many theorems.

4.1 Principle of Mathematical Induction :

Principle of Mathematical Induction consists of the following four 4 steps:

Step 1 : (Foundation) To prove $P(n)$ is true for $n = 1$

(It is advisable to check if $P(n)$ is true for $n = 2, 3$ also if P(1) is trivial).

Step 2 : (Assumption) To assume $P(n)$ is true for $n = k$.

Step 3 : (Succession) To prove that $P(n)$ is true for $n = k + 1$.

Step 4 : (Induction) To conclude that $P(n)$ is true for all $n \in N$

Row of dominos standing close to each other gives us the idea of how the Principle of Mathematical Induction works.

Step 1 : (Foundation) The 1st domino falls down.

(followed by it 2^{nd} also falls down. Then 3^{rd}, 4^{th} and so on.)

Step 2 : (Assumption) Assume if k^{th} domino falls down.

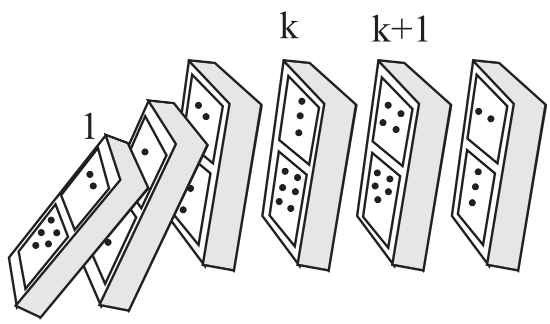

Fig. 4.1

Step 3 : (Succession) Followed by k^{th} domino, $(k + 1)^{th}$ domino will also fall down.

Step 4 : (Induction) It is true that all the dominos will fall down.

We will see how to use the principle of mathematical induction to prove statements.

Stepwise Explanation :

Step 1. (Foundation) Formulate the statement of the theorem as P(n) say, for any positive integer n and verify it for integer $n = 1$. In fact, it is often instructive, though not necessary, to verify the statement for $n = 2$ and 3. This gives better insight into the theorem.

Step 2. (Assumption) Assume that the statement P(n) is true for a positive integer k.

Step 3. (Succession) Prove the statement for $n = k + 1$.

Step 4. (Induction) Now invoke the principle of Mathematical induction. Conclude that the theorem is true for any positive integer n.

Illustration :

Let us prove a theorem with this method. The theorem gives the sum of the first n positive integers.

It is stated as P(n) : 1 + 2 + 3 . . . + n = n(n+1)/2.

Step 1 : (Foundation)

To prove *P(n)* is true for n = 1

L.H.S = 1 R.H.S = $\frac{1(1+1)}{2}$ = 1 which is trivially true.

Check that $1+2=\frac{2\times(2+1)}{2}$ and

$1+2+3=\frac{3\times(3+1)}{2}$, so P(2) and P(3) are also true.

Step 2 : (Assumption) Assume that P(*n*) is true for n = k and in particular,

$$1+2+3+\ldots+k=\frac{k(k+1)}{2}.$$

Step 3 : (Succession) To prove P(*n*) is true for n = k +1 that is

$$1+2+3+\ldots+k+(k+1)=\frac{(k+1)(k+2)}{2}$$

Here L.H.S. = 1 + 2 + 3 + . . . + k + (k+1)

$$=\frac{k(k+1)}{2}+(k+1) \quad \text{(by step 2)}$$

$$=(k+1)\left(\frac{k}{2}+1\right)$$

$$=\frac{(k+1)(k+2)}{2}=\text{R.H.S.}$$

Thus, P(k + 1) is proved.

Step 4 : (Induction) Now by the Principle of Mathematical induction, the statement P(n) is proved for all positive integers n.

SOLVED EXAMPLES

Ex.1 By method of induction, prove that.

$1.3+2.5+3.7+\ldots.+n(2n+1)=\frac{n}{6}(n+1)(4n+5)$ for all $n \in N$

Solution :

Let $P(n) \equiv 1.3+2.5+3.7+\ldots.+n(2n+1)$,

for all $n \in N$

$$=\frac{n}{6}(n+1)(4n+5)$$

Step (I) : (Foundation) To prove P(1) is true

Let n = 1

L. H. S. = 1.3 = 3

$$\text{R. H. S.} = \frac{1}{6}(1+1)(4.1+5)$$

$$=\frac{1}{6}(2)(9)=3$$

∴ L. H. S. = R. H. S.

∴ P(1) is true

Step (II) : (Assumption) Assume that let P(k) is true

$$\therefore 1.3+2.5+3.7+\ldots.+k(2k+1)$$

$$=\frac{k}{6}(k+1)(4k+5) \qquad \ldots(i)$$

Step (III) : (Succession) To prove that P(k+1) is true.

i.e. $1.3+2.5+3.7+\ldots.+(k+1)[2(k+1)+1]$

$$=\frac{(k+1)}{6}(k+1+1)[4(k+1)+5]$$

i.e. $1.3+2.5+3.7+\ldots.+(k+1)(2k+3)$

$$=\frac{(k+1)}{6}(k+2)(4k+9)$$

Now

$$\text{L.H.S.} = 1.3 + 2.5 + 3.7 + ... + (k+1)(2k+3)$$
$$= 1.3+2.5+...+k(2k+1)+(k+1)(2k+3)$$
$$= \frac{k}{6}(k+1)(4k+5)+(k+1)(2k+3)$$
... from (i)
$$= (k+1)\left[\frac{k(4k+5)}{6}+2k+3\right]$$
$$= (k+1)\frac{\left[4k^2+5k+12k+18\right]}{6}$$
$$= \frac{(k+1)(k+2)(4k+9)}{6}$$
$$= \text{R.H.S.}$$

$\therefore$ P (k+1) is true.

Step (IV) : (Induction) From all steps above by the principle of mathematical induction, P(n) is true for all $n \in N$.

$$\therefore\ 1.3 + 2.5 + 3.7 + ... + n(2n+1)$$
$$= \frac{n}{6}(n+1)(4n+5), \text{ for all } n \in N.$$

Ex.2 By method of induction, prove that.

$$\sum_{r=1}^{n} ax^{r-1} = a\left(\frac{1-x^n}{1-x}\right), \text{ for all } n \in N, x \neq 1.$$

Solution : Let $P(n) \equiv \sum_{r=1}^{n} ax^{r-1}$

$$= a + ax + ax^2 + ... + ax^{n-1} = a\left(\frac{1-x^n}{1-x}\right)$$

Step (I) : To prove that P(1) is true

Let n = 1

$\therefore$ L. H. S. = a

$$\text{R. H. S.} = a\left(\frac{1-x}{1-x}\right) = a$$

$\therefore$ L. H. S. = R. H. S.

$\therefore$ P(1) is true

Step (II) : Assume that P(k) is true.

$$\sum_{r=1}^{k} ax^{r-1} = a + ax + ax^2 + ... + ax^{k-1}$$
$$= a\left[\frac{1-x^k}{1-x}\right] \quad ....(i)$$

Step (III) : To prove that P(k+1) is true

i.e. $a + ax + ax^2 + ... + ax^k = a\left[\frac{1-x^{k+1}}{1-x}\right]$

Now, L.H.S. = $a + ax + ax^2 + ... + ax^{k-1} + ax^k$

$$= a\left[\frac{1-x^k}{1-x}\right] + ax^k \quad [\text{by (i)}]$$
$$= \frac{a(1-x^k)+ax^k(1-x)}{(1-x)}$$
$$= \frac{a[1-x^k+x^k-x^{k+1}]}{(1-x)}$$
$$= a\left[\frac{1-x^{k+1}}{1-x}\right]$$
$$= \text{R. H. S.}$$

$\therefore$ P(k+1) is true.

Step (IV) : From all steps above by the principle of mathematical induction, P(n) is true for all $n \in N$.

$$\therefore \sum_{r=1}^{n} ax^{r-1} = a\left(\frac{1-x^n}{1-x}\right), \text{ for all } n \in N, x \neq 1.$$

Ex.3 By method of induction, prove that.

$5^{2n} - 1$ is divisible by 6, for all $n \in N$.

Solution : $5^{2n} - 1$ is divisible by 6, if and only if

$5^{2n} - 1$ is a multiple of 6

Let P(n) be $5^{2n} - 1 = 6m$, $m \in N$.

Step (I) : To prove that P(1) is true, Let n = 1

$\therefore\ 5^{2n} - 1 = 25 - 1 = 24 = 6{\cdot}4$

$\therefore\ 5^{2n} - 1$ is a multiple of 6

$\therefore$ P(1) is true.

Step (II) : Assume that P(k) is true.

$\therefore\ 5^{2k} - 1 = 6a,$ where $a \in N$

$\therefore\ 5^{2k} = 6a + 1$ ...(i)

Step (III) : To prove that $P(k+1)$ is true

i.e. to prove that $5^{2(k+1)} - 1$ is a multiple of 6

i.e. $5^{2k+2} - 1 = 6b,$ $b \in N$

Now $5^{2k+2} - 1 = 5^{2k}.5^2 - 1$

$= (6a + 1)\ 25 - 1$ by (i)

$= 150a + 24 = 6(25a + 4)$

$= 6b$

Step (IV) : From all the steps above

$P(n) = 5^{2n} - 1$ is divisible by 6,

for all $n \in N$

Note :

1) 5<5 is not a true statement, whereas $5 \le 5$, $5 \ge 5$ are true statements.

2) $2 = 3$, $2 > 3$, $2 \ge 3$ are not true statements, whereas $2 < 3$, $2 \le 3$are true statements,

Ex. 4) By method of induction, prove that

$n\,! \ge 2^n$; $\forall\ n \in N, n \ge 4.$

Solution : Step I : (Foundation) Since $P(n)$ is stated for $n \ge 4$. Take $n = 4$

L.H.S. = 4! = 24, R.H.S. = 2^4 = 16.

Since $24 \ge 16$, $P(n)$ is true for $n = 4$

[$P(n)$ is not true for $n = 1, 2, 3$ (Verify!)]

Step (II) : (Assumption) Assume that let P(k) is true.

i.e. $k\,! \ge 2^k$; $k \in N, k \ge 4.$

Step (III) : (Succession) To prove that $P(k+1)$ is true.

i.e. to prove that $(k + 1)! \ge 2^{k+1}$, $k + 1 \ge 4.$

L.H.S. = $(k +1)! = (k + 1)k!$

Since $k \ge 4$, $k + 1 > 4+1$, i.e. $k + 1 \ge 5$, also $k +1 \ge 2$ (why?)

and from Step II, $k\,! \ge 2^k$; $k \ge 4.$

Therefore, L.H.S. = $(k+1)k! \ge 2.2^k = 2^{k+1}$ =R.H.S.

i.e. $(k+1)! \ge 2^{k+1}$, $k + 1 \ge 4$

Therefore $P(k+1)$ is true.

Step (IV) : (Induction) From all steps above, P(n) is true for $\forall\ n \in N, n \ge 4.$

Ex. 5) Given that (recurrence relation) $t_{n+1} = 3t_n + 4$, $t_1 = 1$, prove by induction that (general statement) $t_n = 3^n - 2$.

Solution : The statement $P(n)$ has L.H.S. a recurrence relation $t_{n+1} = 3t_n + 4$, $t_1 = 1$ and R.H.S. a general statement $t_n = 3^n - 2$.

Step I : (Foundation) To prove $P(1)$ is true.

L.H.S. = 1, R.H.S = $3^1 - 2 = 3 - 2 = 1$ So $P(1)$ is true.

For n = 2, L.H.S. = $t_2 = 3t_1 + 4 = 3(1) + 4 = 7$

Now R.H.S. = $t_2 = 3^2 - 2 = 9 - 2 = 7$. P(2) is also true.

Step II : (Assumption) Assume that $P(k)$ is true.

i.e. for $t_{k+1} = 3t_k + 4$, $t_1 = 1$, then $t_k = 3^k - 2$

Step III : (Sucession) To prove that $P(k + 1)$ is ture.

i.e. to prove $t_{k+1} = 3^{k+1} - 2$

Since $t_{k+1} = 3t_k + 4$, and $t_k = 3^k - 2$ (From Step II) $t_{k+1} = 3(3^k - 2) + 4 = 3^{k+1} - 6 + 4$

$= 3^{k+1} - 2.$

Therefore $P(k + 1)$ is true.

Step IV: (Induction) From all the steps above $P(n)$, $t_n = 3^n - 2$ is true for $\forall$ n $\in$ N, where $t_{n+1} = 3t_n + 4$, $t_1 = 1$.

Ex.6 By method of induction, prove that.

$2^n > n$, for all $n \in N$.

Solution : Let $P(n) = 2^n > n$

Step (I) : To prove that P(1) is true, Let n = 1

L.H.S. $= 2^1 = 2$

R.H.S. = 1

2 > 1 Which is true

$\therefore$ P(1) is true

Step (II) : Assume that P(k) is true, $k \in N$

$\therefore 2^k > k$ ...(i)

Step (III) : To prove that P(k+1) is true

i.e. $2^{k+1} > k + 1$

Now $2^{k+1} = 2^k.2^1 > k.2$ ...by (i)

$\therefore 2^{k+1} > 2k$

$\therefore 2^{k+1} > k + k$

$\therefore 2^{k+1} > k + 1$ ($\therefore k \geq 1$)

$\therefore$ P(k + 1) is true.

Step (IV) : From all steps above and by the principle of mathematical induction, P(n) is true for all $n \in N$.

$\therefore 2^n > n$, for all $n \in N$.

Remarks : (1) In the proof of P(n) by method of induction, both the conditions viz. (i) P(1) is true and (ii) P(k+1) is true when P(k) is true, must be satisfied. (2) In some problems, second step is satisfied but the first step is not satisfied. Hence the result is not valid for all $n \in N$.

for example,

let $P(n) \equiv 1.6+2.9+3.12+....+n(3n + 3) = n^3 + 3n^2 + 2n + 3$

Let us assume that P(k) is true.

$\therefore 1{\cdot}6+2{\cdot}9+3{\cdot}12+....+k(3k + 3) = k^3 + 3k^2 + 2k + 3$ ...(i)

We have to prove that P(k+1) is true,

i.e. to prove that

$1{\cdot}6 + 2{\cdot}9+ 3{\cdot}12 +....+ (k + 1) (3k + 6) = (k+1)^3+3(k+1)^2+2(k+1)+3$

L.H.S. $= 1{\cdot}6 + 2{\cdot}9+ 3{\cdot}12 +....+ (k + 1) (3k + 6)$

$= 1{\cdot}6+2{\cdot}9+3{\cdot}12 +....+ k(3k+3) + (k+1)(3k+6)$

$= k^3 + 3k^2 + 2k + 3 + (k + 1) (3k + 6)$ by (i)

$= k^3 + 3k^2 + 2k + 3 + 3k^2 + 6k + 3k + 6$

$= k^3 + 3k^2 + 3k + 1 + 3k^2+6k+3+2k+2+3$

$= (k + 1)^3 + 3(k^2 + 2k + 1) + 2(k + 1) + 3$

$= (k + 1)^3 + 3(k + 1)^2 + 2(k + 1) + 3$

= R. H. S.

$\therefore$ P(k + 1) is true.

If P(k) is true then P (k+1) is true.

Now we examine the result for n = 1

L.H.S. $= 1{\cdot}6 = 6$

R. H. S. $= 1^3 + 3(1)^2 + 2 (1) + 3$

$= 9$

$\therefore$ L. H. S. $\neq$ R. H. S.

$\therefore$ P(1) is not true

$\therefore$ P (n) is not true for all $n \in N$.

EXERCISE 4.1

Prove by method of induction, for all $n \in N$.

(1) $2 + 4 + 6 + ... + 2n = n (n+1)$

(2) $3 + 7 + 11 + ...$ + to n terms $= n(2n+1)$

(3) $1^2 + 2^2 + 3^2 + ... + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

(4) $1^2+3^2 + 5^2 + ... + (2n-1)^2 = \dfrac{n}{3}(2n-1)(2n+1)$

(5) $1^3+3^3 + 5^3 + ...$ to n terms $= n^2 (2n^2-1)$

(6) $1{\cdot}2+2{\cdot}3+3{\cdot}4+.....+n(n+1) = \dfrac{n}{3}(n+1)(n+2)$

(7) $1{\cdot}3+3{\cdot}5+5{\cdot}7+.....$ to n terms $= \dfrac{n}{3}(4n^2+6n-1)$

(8) $\dfrac{1}{1.3} + \dfrac{1}{3.5} + \dfrac{1}{5.7} +....+ \dfrac{1}{(2n-1)(2n+1)} = \dfrac{n}{2n+1}$

(9) $\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\ldots$ to n terms $=\frac{n}{3(2n+3)}$

(10) $(2^{3n}-1)$ is divisible by 7.

(11) $(2^{4n}-1)$ is divisible by 15.

(12) $3^n - 2n - 1$ is divisible by 4.

(13) $5 + 5^2 + 5^3 + \ldots + 5^n = \frac{5}{4}(5^n - 1)$

(14) $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$

(15) Given that $t_{n+1} = 5t_n + 4$, $t_1=4$, prove by method of induction that $t_n = 5^n - 1$

(16) Prove by method of induction

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix} \quad \forall\, n \in N$$

4.2 Binomial Theorem for positive integral index :

We know that

$(a+b)^0 =$ 1

$(a+b)^1 =$ 1a + 1b

$(a+b)^2 =$ $1a^2 + 2ab + 1b^2$

$(a+b)^3 =$ $1a^3 + 3a^2b + 3ab^2 + 1b^3$

$(a+b)^4 =$ $1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$

The coefficients of these expressions are arranged by Pascal's triangle as follows and are expressed in the form of ${}^nC_r$

Index	
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1

Index	
0	${}^0C_0 = 1$
1	${}^1C_0 = 1$ ${}^1C_1 = 1$
2	${}^2C_0 = 1$ ${}^2C_1 = 2$ ${}^2C_2 = 1$
3	${}^3C_0 = 1$ ${}^3C_1 = 3$ ${}^3C_2 = 3$ ${}^3C_3 = 1$
4	${}^4C_0=1$ ${}^4C_1=4$ ${}^4C_2= 6$ ${}^4C_3= 4$ ${}^4C_4=1$

Now, we will study how to expand binomials of higher powers.

Theorem : If a, b $\in$ R and n $\in$ N, then

$(a+b)^n = {}^nC_0\, a^nb^0 + {}^nC_1\, a^{n-1} b^1 + {}^nC_2\, a^{n-2}b^2 + \ldots + {}^nC_n\, a^0 b^n$

Proof : We prove this theorem by method of induction.

Let P(n) be $(a+b)^n =$

${}^nC_0\, a^nb^0 + {}^nC_1\, a^{n-1} b^1 + {}^nC_2\, a^{n-2}b^2 + \ldots + {}^nC_n\, a^0 b^n$

Step (I) : Let n = 1

$\therefore$ L. H. S. $= (a+b)^1 = a + b$

R. H. S. $= {}^1C_0\, a^1b^0 + {}^1C_1\, a^0b^1 = a + b$

$\therefore$ L. H. S. = R. H. S.

$\therefore$ P (1) is true.

Step (II) : Let P(k) be true.

$\therefore (a+b)^k = {}^kC_0\, a^kb^0 + {}^kC_1 a^{k-1} b^1 + {}^kC_2 a^{k-2}b^2 + .. + {}^kC_k a^0 b^k$ ....(i)

Step (III) : We have to prove that P(k+1) is true.

i.e. to prove that

$(a+b)^{k+1} =$

${}^{k+1}C_0\, a^{k+1}b^0 + {}^{k+1}C_1 a^kb^1 + {}^{k+1}C_2 a^{k-1}b^2 + \ldots$

$+ {}^{k+1}C_{k+1} a^0 b^{k+1}$

Now L. H. S. $= (a+b)^{k+1}$

$= (a+b)(a+b)^k$

$= (a+b)\,[{}^kC_0\, a^kb^0 + {}^kC_1 a^{k-1} b^1 + {}^kC_2 a^{k-2}b^2 + ..$

$+ {}^kC_k a^0 b^k]$ by (i)

$= a[{}^kC_0 a^kb^0 + {}^kC_1 a^{k-1}b + {}^kC_2 a^{k-2}b^2 + \ldots + {}^kC_k a^0b^k]$
$+ b[{}^kC_0 a^kb^0 + {}^kC_1 a^{k-1} b + {}^kC_2 a^{k-2}b^2 + \ldots + {}^kC_k a^0 b^k]$

$= [{}^kC_0a^{k+1}b^0+{}^kC_1a^kb^1+{}^kC_2a^{k-1}b^2+\ldots+{}^kC_kab^k]+$
$[{}^kC_0a^kb+{}^kC_1a^{k-1}\, b^2+{}^kC_2a^{k-2}b^3+\ldots+{}^kC_ka^0\, b^{k+1}]$
$= {}^kC_0a^{k+1}b^0+{}^kC_1a^kb^1+{}^kC_0a^kb^1+{}^kC_2a^{k-1}b^2+$
${}^kC_1a^{k-1}b^2+\ldots+{}^kC_ka^1\, b^k+{}^kC_{k-1}a^1b^k+{}^kC_ka^0\, b^{k+1}$
$= {}^kC_0a^{k+1}b^0+({}^kC_1+{}^kC_0)a^kb^1+({}^kC_2+{}^kC_1)a^{k-1}b^2+$
$\ldots+({}^kC_k+{}^kC_{k-1})\, a^1b^k+{}^kC_ka^0\, b^{k+1}$

But we know that

${}^kC_0 = 1 = {}^{k+1}C_{k+1},\ {}^kC_1+{}^kC_0 = {}^{k+1}C_1.$

${}^kC_2+{}^kC_1 = {}^{k+1}C_2,\ldots\ {}^kC_k+{}^kC_{k-1} = {}^{k+1}C_k,\ldots$

${}^kC_k = 1 = {}^{k+1}C_{k+1}$

$\therefore$ L.H.S. $= {}^{k+1}C_0a^{k+1}b^0+{}^{k+1}C_1a^kb^1+{}^{k+1}C_2a^{k-1}b^2 + \ldots\ldots + {}^{k+1}C_{k+1}a^0b^{k+1}$

= R.H.S.

$\therefore$ P (k+1) is true.

Step (IV) : From all steps above and by the principle of mathematical induction, P(n) is true for all $n \in N$.

$\therefore (a+b)^n={}^nC_0\, a^nb^0+{}^nC_1a^{n-1}\, b^1+{}^nC_2a^{n-2}b^2\ldots+{}^nC_na^0\, b^n$, for all $n \in N$.

Remarks :

(1) The expansion of $(a+b)^n$ contains n+1 terms.

(2) First term is a^n and last term is b^n.

(3) In each term, the sum of indices of a and b is always n.

(4) In successive terms, the index of a decreases by 1 and index of b increases by 1.

(5) Coefficients of the terms in binomial expansion equidistant from both the ends are equal. i.e. coefficients are symmetric.

(6) $(a-b)^n = {}^nC_0a^nb^0 - {}^nC_1a^{n-1}\, b^1+{}^nC_2a^{n-2}b^2 - \cdots + (-1)^nC_na^0\, b^n$.

In the RHS, the first term is positive and consequent terms are alternately negative and positive.

SOLVED EXAMPLES

Ex. 1 : Expand $(x^2 + 3y)^5$

Solution : Here $a = x^2$, $b = 3y$ and $n = 5$ using binomial theorem,

$$(x^2+3y)^5 = {}^5C_0(x^2)^5\,(3y)^0 + {}^5C_1(x^2)^4\,(3y)^1 + {}^5C_2(x^2)^3\,(3y)^2 + {}^5C_3(x^2)^2\,(3y)^3 + {}^5C_4(x^2)^1\,(3y)^4 + {}^5C_5(x^2)^0\,(3y)^5$$

Now ${}^5C_0 = {}^5C_5 = 1$, ${}^5C_1 = {}^5C_4 = 5$,

$${}^5C_2 = {}^5C_3 = \frac{5\cdot 4}{2\cdot 1} = 10$$

$$\therefore (x^2+3y)^5 = 1(x^{10})(1)+5(x^8)(3y)+10(x^6)(9y^2)+10(x^4)(27y^3)+5(x^2)(81y^4)+1(1)(243y^5)$$

$$\therefore (x^2+3y)^5 = x^{10}+15x^8y+90x^6y^2+270x^4y^3+405x^2y^4+243y^5$$

Ex. 2 : Expand $\left(2x - \frac{y}{2}\right)^5$

Solution : Here $a = 2x$, $b = \frac{y}{2}$ and $n = 5$

Using binomial theorem,

$$\left(2x - \frac{y}{2}\right)^5 = {}^5C_0(2x)^5\left(\frac{y}{2}\right)^0 - {}^5C_1(2x)^4\left(\frac{y}{2}\right)^1 + {}^5C_2(2x)^3\left(\frac{y}{2}\right)^2 - {}^5C_3(2x)^2\left(\frac{y}{2}\right)^3 + {}^5C_4(2x)^1\left(\frac{y}{2}\right)^4 - {}^5C_5(2x)^0\left(\frac{y}{2}\right)^5$$

Now ${}^5C_0 = {}^5C_5 = 1$, ${}^5C_1 = {}^5C_4 = 5$,

$${}^5C_2 = {}^5C_3 = \frac{5\cdot 4}{2\cdot 1} = 10$$

$$\therefore \left(2x - \frac{y}{2}\right)^5 = 1(32x^5)(1) - 5(16x^4)\left(\frac{y}{2}\right) + 10(8x^3)\left(\frac{y^2}{4}\right) - 10(4x^2)\left(\frac{y^3}{8}\right) + 5(2x)\left(\frac{y^4}{16}\right) - 1(1)\left(\frac{y^5}{32}\right)$$

$$\therefore \left(2x - \frac{y}{2}\right)^5 = 32x^5 - 40x^4y + 20x^3y^2 - 5x^2y^3 + \frac{5}{8}xy^4 - \frac{y^5}{32}$$

Ex.3 : Expand $\left(\sqrt{5}+\sqrt{3}\right)^4$

Solution : Here a = $\sqrt{5}$, b = $\sqrt{3}$ and n = 4

Using binomial theorem,

$$\left(\sqrt{5}+\sqrt{3}\right)^4 = {}^4C_0\left(\sqrt{5}\right)^4\left(\sqrt{3}\right)^0 + {}^4C_1\left(\sqrt{5}\right)^3\left(\sqrt{3}\right)^1 + {}^4C_2\left(\sqrt{5}\right)^2\left(\sqrt{3}\right)^2 + {}^4C_3\left(\sqrt{5}\right)^1\left(\sqrt{3}\right)^3 + {}^4C_4\left(\sqrt{5}\right)^0\left(\sqrt{3}\right)^4$$

Now ${}^4C_0 = {}^4C_4 = 1$, ${}^4C_1 = {}^4C_3 = 4$, ${}^4C_2 = \frac{4.3}{2.1} = 6$,

$$\therefore \left(\sqrt{5}+\sqrt{3}\right)^4 = 1(25)(1) + 4\left(5\sqrt{5}\right)\left(3\sqrt{3}\right) + 6(5)(3) + 4\left(\sqrt{5}\right)\left(3\sqrt{3}\right) + 1(1)(9)$$

$$\therefore \left(\sqrt{5}+\sqrt{3}\right)^4 = 25 + \left(20\sqrt{15}\right) + 90 + \left(12\sqrt{15}\right) + 9$$

$$\therefore \left(\sqrt{5}+\sqrt{3}\right)^4 = 124 + \left(32\sqrt{15}\right)$$

Ex.4 : Evaluate $\left(\sqrt{2}+1\right)^5 - \left(\sqrt{2}-1\right)^5$

Solution : $\left(\sqrt{2}+1\right)^5 = {}^5C_0\left(\sqrt{2}\right)^5 + {}^5C_1\left(\sqrt{2}\right)^4 + {}^5C_2\left(\sqrt{2}\right)^3 + {}^5C_3\left(\sqrt{2}\right)^2 + {}^5C_4\left(\sqrt{2}\right)^1 + {}^5C_5\left(\sqrt{2}\right)^0$

Now ${}^5C_0 = {}^5C_5 = 1$, ${}^5C_4 = {}^5C_1 = 5$, ${}^5C_2 = {}^5C_3 = \frac{5.4}{2.1} = 10$

$$\therefore \left(\sqrt{2}+1\right)^5 = 1\left(4\sqrt{2}\right) + 5(4) + 10\left(2\sqrt{2}\right) + 10(2) + 5\left(\sqrt{2}\right) + 1$$

$$\therefore \left(\sqrt{2}+1\right)^5 = \left(4\sqrt{2}\right) + 20 + \left(20\sqrt{2}\right) + 20 + \left(5\sqrt{2}\right) + 1 \quad \text{... (i)}$$

Similarly,

$$\left(\sqrt{2}-1\right)^5 = \left(4\sqrt{2}\right) - 20 + \left(20\sqrt{2}\right) - 20 + \left(5\sqrt{2}\right) - 1 \quad \text{... (ii)}$$

Subtracting (ii) from (i) we get,

$$\left(\sqrt{2}+1\right)^5 - \left(\sqrt{2}-1\right)^5$$

$$= \left(4\sqrt{2}+20+20\sqrt{2}+20+5\sqrt{2}+1\right) - \left(4\sqrt{2}-20+20\sqrt{2}-20+5\sqrt{2}-1\right)$$

$= 2(20 + 20 + 1)$

$= 82$

$$\therefore \left(\sqrt{2}+1\right)^5 - \left(\sqrt{2}-1\right)^5 = 82$$

Ex. 5 (Activity) : Using binomial theorem, find the value of $(99)^4$

Solution : We have $(99)^4 = (\square - 1)^4$

$$\therefore \quad (99)^4 = {}^4C_0(\square)^4 - {}^4C_1(\square)^3 + {}^4C_2(\square)^2 - {}^4C_3(\square)^1 + {}^4C_4(\square)^0$$

Now ${}^4C_0 = {}^4C_4 = 1$, ${}^4C_1 = {}^4C_3 = 4$, ${}^4C_2 = \frac{4.3}{2.1} = 6$

$$\therefore \quad (99)^4 = 1(10)^{\square} - 4(10)^{\square} + 6(10)^{\square} - 4(10)^{\square} + 1(1)$$

$$= \square - \square + \square - \square + \square = \square$$

Ex. 6 : Find the value of $(2.02)^5$ correct upto 4 decimal places.

Solution : $(2.02)^5 = [2+0.02]^5$

$$= {}^5C_0(2)^5(0.02)^0 + {}^5C_1(2)^4(0.02)^1 + {}^5C_2(2)^3(0.02)^2 + {}^5C_3(2)^2(0.02)^3 + {}^5C_4(2)^1(0.02)^4 + {}^5C_5(2)^0(0.02)^5$$

Now ${}^5C_0 = {}^5C_5 = 1$, ${}^5C_1 = {}^5C_4 = 5$, ${}^5C_2 = {}^5C_3 = 10$

$$\therefore \quad (2.02)^5 = 1(32)(1) + 5(16)(0.02) + 10(8)(0.0004) + 10(4)(0.000008) + 5(2)(0.00000016) + 1(0.0000000032)$$

Ignore last two terms for four decimal places

$\therefore \quad (2.02)^5 = 32 + 1.60 + 0.0320 + 0.0003$

$\therefore \quad (2.02)^5 = 33.6323.$

Ex. 7 : Without expanding, find the value of

$(2x-1)^5 + 5(2x-1)^4(1-x) + 10(2x-1)^3(1-x)^2 + 10(2x-1)^2(1-x)^3 + 5(2x-1)(1-x)^4 + (1-x)^5$

Solution : We notice that 1, 5, 10, 10, 5, 1 are the values of ${}^5C_0$, ${}^5C_1$, ${}^5C_2$, ${}^5C_3$, ${}^5C_4$ and ${}^5C_5$ respectively.

Hence, given expression can be written as

$${}^5C_0(2x-1)^5 + {}^5C_1(2x-1)^4(1-x)$$
$$+ {}^5C_2(2x-1)^3(1-x)^2 + {}^5C_3(2x-1)^2(1-x)^3$$
$$+ {}^5C_4(2x-1)(1-x)^4 + {}^5C_5(1-x)^5$$
$$=[(2x-1) + (1-x)]^5$$
$$= (2x - 1 + 1 - x)^5$$
$$= x^5$$

$\therefore \quad (2x-1)^5+5(2x-1)^4(1-x) +10(2x-1)^3(1-x)^2 + 10(2x-1)^2(1-x)^3 + 5(2x-1)(1-x)^4 + (1-x)^5 = x^5$

EXERCISE 4.2

(1) Expand (i) $\left(\sqrt{3}+\sqrt{2}\right)^4$ (ii) $\left(\sqrt{5}-\sqrt{2}\right)^5$

(2) Expand (i) $(2x^2 + 3)^4$ (ii) $\left(2x-\dfrac{1}{x}\right)^6$

(3) Find the value of

(i) $\left(\sqrt{3}+1\right)^4 - \left(\sqrt{3}-1\right)^4$

(ii) $\left(2+\sqrt{5}\right)^5 + \left(2-\sqrt{5}\right)^5$

(4) Prove that

(i) $\left(\sqrt{3}+\sqrt{2}\right)^6 + \left(\sqrt{3}-\sqrt{2}\right)^6 = 970$

(ii) $\left(\sqrt{5}+1\right)^5 - \left(\sqrt{5}-1\right)^5 = 352$

(5) Using binomial theorem, find the value of

(i) $(102)^4$ (ii) $(1.1)^5$

(6) Using binomial theorem, find the value of

(i) $(9.9)^3$ (ii) $(0.9)^4$

(7) Without expanding, find the value of

(i) $(x+1)^4 - 4(x+1)^3(x-1) + 6(x+1)^2(x-1)^2 - 4(x+1)(x-1)^3 + (x-1)^4$

(ii) $(2x-1)^4 + 4(2x-1)^3(3-2x) + 6(2x-1)^2(3-2x)^2 + 4(2x-1)^1(3-2x)^3 + (3-2x)^4$

(8) Find the value of $(1.02)^6$, correct upto four places of decimals.

(9) Find the value of $(1.01)^5$, correct upto three places of decimals.

(10) Find the value of $(0.9)^6$, correct upto four places of decimals.

4.3 General term in expansion of $(a+b)^n$

In the expansion of $(a+b)^n$, we denote the terms by $t_1, t_2, t_3, \ldots, t_r, t_{r+1}, \ldots t_n \ldots$ then

$$t_1 = {}^nC_0\, a^n b^0$$
$$t_2 = {}^nC_1\, a^{n-1} b^1$$
$$t_3 = {}^nC_2\, a^{n-2} b^2$$
$$\vdots$$
$$t_r = {}^nC_{r-1}\, a^{n-r+1} b^{r-1}$$
$$t_{r+1} = {}^nC_r\, a^{n-r} b^r$$

t_{r+1} is called a general term for all $r \in N$ and $0 \le r \le n$. Using this formula, we can find any term of the expansion.

4.3 Middle term (s) in the expansion of $(a+b)^n$:

(i) In $(a+b)^n$ if n is even then the number of terms in the expansion is odd. So the only middle term is $\left(\dfrac{n+2}{2}\right)^{th}$ term.

(ii) In $(a+b)^n$ if n is odd then the number of terms in the expansion is even. So the two middle terms are $\left(\frac{n+1}{2}\right)^{th}$ term and $\left(\frac{n+3}{2}\right)^{th}$ term.

SOLVED EXAMPLES

Ex. 1 : Find the fifth term in the expansion of $\left(2x^2+\frac{3}{2x}\right)^8$

Solution : Here $a = 2x^2$, $b = \frac{3}{2x}$, $n = 8$

For t_5, $r = 4$

Since, $t_{r+1} = {}^nC_r\, a^{n-r}\, b^r$,

$$t_5 = {}^8C_4 (2x^2)^{8-4}\left(\frac{3}{2x}\right)^4$$

$$= \frac{8.7.6.5}{4.3.2.1}(2x^2)^4\left(\frac{3}{2x}\right)^4$$

$$= 70(16x^8)\left(\frac{81}{16x^4}\right)$$

$$= 5670x^4$$

$\therefore$ The fifth term in the expansion of $\left(2x^2+\frac{3}{2x}\right)^8$ is $5670x^4$

Ex. 2 : Find the middle term(s) in the expansion of $\left(x^2+\frac{2}{x}\right)^8$

Solution : Here $a = x^2$, $b = \frac{2}{x}$, $n = 8$

Now n is even, hence $\left(\frac{n+2}{2}\right) = \left(\frac{8+2}{2}\right) = 5$

$\therefore$ Fifth term is the only middle term.

For t_5, $r = 4$

We have $t_{r+1} = {}^nC_r\, a^{n-r}\, b^r$,

$$t_5 = {}^8C_4 (x^2)^{8-4}\left(\frac{2}{x}\right)^4$$

$$= \frac{8.7.6.5}{4.3.2.1}(x^2)^4\left(\frac{2}{x}\right)^4$$

$$= 70\,(x^8)\,\frac{16}{x^4}$$

$$= 1120x^4$$

Ex. 3 : Find the middle terms in the expansion of $\left(2x-\frac{1}{4x}\right)^9$

Solution : Here $a = 2x$, $b = -\frac{1}{4x}$, $n = 9$

Now n is odd $\left(\frac{n+1}{2}\right) = 5$ $\left(\frac{n+3}{2}\right) = 6$

$\therefore$ Fifth and sixth terms are the middle terms.

We have $t_{r+1} = {}^nC_r\, a^{n-r}\, b^r$,

For t_5, $r = 4$

$$\therefore\quad t_5 = {}^9C_4\,(2x)^{9-4}\left(-\frac{1}{4x}\right)^4$$

$$= \frac{9\cdot8\cdot7\cdot6}{4\cdot3\cdot2\cdot1}(32x)^5\left(-\frac{1}{4x}\right)^4$$

$$= 126\,(2x^5)\left(\frac{1}{256x^4}\right)$$

$$= \left(\frac{63x}{4}\right)$$

For t_6, $r = 5$

$$\therefore\quad t_6 = {}^9C_5\,(2x)^{9-5}\left(\frac{-1}{4x}\right)^5$$

$$= \frac{9\cdot8\cdot7\cdot6}{4\cdot3\cdot2\cdot1}(2x)^4\left(\frac{-1}{4x}\right)^5$$

$$= 126\,(16x^4)\left(\frac{-1}{1024x^5}\right)$$

$$= -\frac{63}{32x}$$

$\therefore$ The middle terms are $\left(\frac{63x}{4}\right)$ and $-\frac{63}{32x}$

3. PERMUTIONS AND COMBINATIONS

EXERCISE 3.1

1) 50 ways
2) 12
3) i) 25 ii) 20
4) i) 100 ii) 48
5) 125
6) 124
7) 31
8) 90
9) 225
10) 23
11) 276
12) 207
13) 12
14) 216

EXERCISE 3.2

1) i) 40320 ii) 3628800
iii) 3628080 iv) 24

2) i) 665280 ii) 2
iii) 479001600 iv) 144
v) 84 vi) 29
vii) 57.93 viii) 20160

3) i) $\dfrac{10!}{4!}$ ii) $3^5 \times 5!$
iii) $\dfrac{9!}{5!}$ iv) $5^5 \times 5!$

4) i) 28 ii) 1
iii) 3003 iv) 6435

5) i) 1848 ii) 43/14
iii) 5 iv) 6
v) 8

6) i) 11 ii) 17
iii) 7 iv) 8
v) 5

10) i) $(2n + 1)(2n + 2)$ ii) $\dfrac{n+3}{n-2}$
iii) $\dfrac{-(n+1)}{n(n-2)!}$ iv) $(3n + 2)n!$
v) $\dfrac{(n-1)}{n!}$ vi) $\dfrac{(n^2+1)}{(n+1)!}$
vii) $\dfrac{6}{(n+2)!}$ viii) $\dfrac{1}{(n+2)!}$

EXERCISE 3.3

1) $n = 9$
2) $m = 6, \ n = 2$
3) $r = 6$
5) a) 2401 b) 840
6) a) 30240 b) 151200
c) 43200 d) 5040
7) $\dfrac{12! \times 13!}{7}$
8) a) 1440 b) 720
c) 7! d) 240
e) 120 f) 120

9) 144

10) a) 1296 b) 360

11) 100

12) 720 a) 120, b) 600

13) 46800, 20800

14) 243

15) i) 2880 ii) 5040

16) i) 120 ii) 148 iii) 72

17) i) 720 ii) 144 iii) 288 iv) 144

EXERCISE 3.4

1) i) 120 ii) 60480
iii) 30240 iv) 5040
v) 302400

2) 144

3) a) 70 b) 37

4) $\dfrac{13!}{5!4!4!}$

5) $\dfrac{12!}{2!3!2!}$

6) $\dfrac{11!}{4!2!2!}$ a) 414960 b) $\dfrac{8!}{2!2!}$

7) 210 8) 60

9) $\dfrac{10!}{2!3!2!}$ 10) 1260, 1230

11) 180 12) 144

13) 360, 96 14) 180, 60

15) a) 1800 b) 72

EXERCISE 3.5

1) $7! = 5040$ 2) $20!$, $2 \cdot 18!$

3) a) $2 \cdot 23!$ b) $2! \cdot 22!$

4) $\dfrac{14!}{2}$

5) $2 \cdot 8!$

6) a) $5! \times 2! = 240$ b) 9605

7) $7! \times 6P_5$ 8) 144

9) $\dfrac{9!}{4!}$ 10) $13 \cdot 14!$

EXERCISE 3.6

1) a) 1365 b) 3160 c) ${}^{16}C_5$ d) ${}^{19}C_{15}$

2) a) $n = 2$ b) $n = 7$
c) $n = 9$

3) $r = 4$

4) a) $n = 10,\ r = 3$ b) $n = 10, r = 4$

5) $r = 8$ 6) 126

7) 39200 8) 120

9) 12 10) 190

11) ${}^{n}C_2 - n$; a) 35 b) 90 c) 54 d) 20

12) 210

13) a) 45 b) 40

14) a) 220 b) 216

15) 151200

16) i) $n = 20$ ii) $n = 4, 3$
iii) $n = 1, 5$ iv) $n = 4$
v) $n = 6$

17) $x = r!$ 18) $r = 7$

19) 14161

20) a) 2508 b) 1646 c) 4125

21) 16 22) 2275

23) 36873 ; 6885 24) 425

25) 51051

26) a) 84 b) 126

MISCELLANEOUS EXERCISE - 3

(I)

1	2	3	4	5	6	7	8	9	10
C	A	B	D	A	C	A	A	C	D

(II)

1) 45
2) 120
3) 720 ; AENREM
4) 990
5) 360
6) 5541965
7) 15
8) $\frac{30!}{7!\ 10!\ 13!}$
9) 127
10) ${}^9C_3 + {}^9C_4 + {}^9C_5$
11) 4095
12) 48
13) 1680
14) 63
15) $\frac{20!}{8!\ 7!\ 5!}$
16) 896
17) 60
18) i) 66 ii) 11 iii) 220 iv) 55

4. METHOD OF INDUCTION AND BINOMIAL THEOREM

EXERCISE 4.1

Hints :

2) $p(n) = 3 + 7 + 11 + \ldots + (4n - 1) = n\,(2n + 1)$

5) $p(n) = 1^3 + 3^3 + 5^3 + \ldots + (2n - 1)^3 = n^2\,(2n^2 - 1)$

7) $p(n) = 1.3 + 3.5 + 5.7 + \ldots + (2n - 1)(2n + 1)$

$= \frac{n}{3}(4n^2 + 6n - 1)$

9) $p(n) = \frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \ldots + \frac{1}{(2n+1)(2n+3)}$

$= \frac{n}{3(2n+3)}$

EXERCISE 4.2

1) i) $49 + 20\sqrt{6}$ ii) $145\sqrt{5} - 229\sqrt{2}$

2) i) $16x^8 + 96x^6 + 216x^4 + 216x^2 + 81$

ii) $64x^6 - 192x^4 + 240x^2 - 160 + \frac{60}{x^2} - \frac{12}{x^4} + \frac{1}{x^6}$

3) i) $32\sqrt{3}$ ii) 1364

5) i) 108243216 ii) 1.61051

6) i) 970.299 ii) 0.6561

7) i) 16 ii) 16

8) 1.1262

9) 1.051

10) 0.5314

EXERCISE 4.3

1) i) $4032x^{10}$ ii) $84480x^2$

iii) $\frac{10500}{x^3}$ iv) $\frac{5a^{16}}{3}$

v) $\frac{{}^{13}C_9 \cdot 3^4 \cdot 4^9}{a^5}$

2) i) $122472\sqrt{2}$ ii) 700000

iii) 48620 iv) $\frac{5}{16}$

v) $\frac{-105}{8192}$

3) i) $\frac{1792}{9}$ ii) -96096

iii) 405 iv) 84

v) 10500000