# 6 FUNCTIONS

## Let's Study

- Function, Domain, Co-domain, Range
- Types of functions
- Representation of function
- Basic types of functions
- Piece-wise defined and special functions

## Let's : Learn

### 6.1 Function

**Definition :** A function (or mapping) $f$ from a set A to set B ($f$ : A→B) is a relation which associates for each element $x$ in A, a unique (exactly one) element $y$ in B.

Then the element $y$ is expressed as $y = f(x)$ .

$y$ is the image of $x$ under $f$.

$f$ is also called a map or transformation.

If such a function exists, then A is called the **domain** of $f$ and B is called the **co-domain** of $f$.

**Illustration:**

Examine the following relations which are given by arrows of line segments joining elements in A and elements in B.

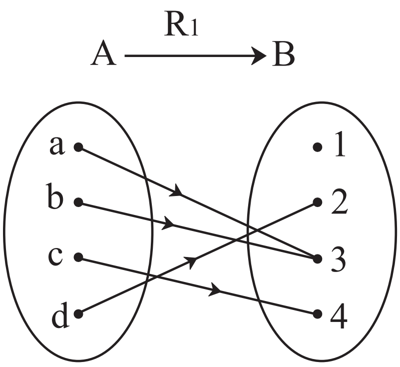



**Fig. 6.1**

Since, every element from A is associated to exactly one element in B, $R_1$ is a well defined function.

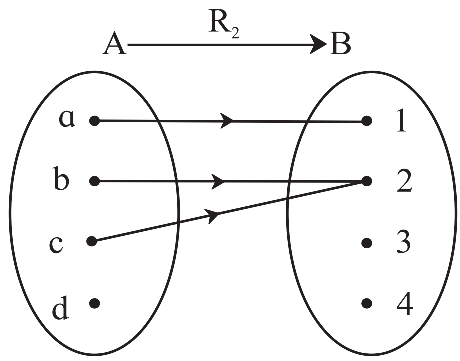



**Fig. 6.2**

$R_2$ is not a function because element 'd' in A is not associated to any element in B.

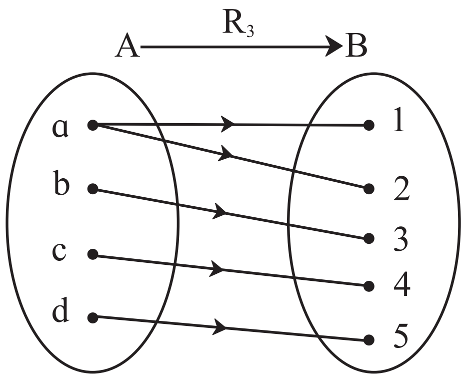



**Fig. 6.3**

$R_3$ is not a function because element $a$ in A is associated to two elements in B.

The relation which defines a function $f$ from domain A to co-domain B is often given by an algebraic rule.

For example, A = Z, the set of integers and B = Q the set of rational numbers and the function $f$ is given by $f(n) = \frac{n}{7}$ here $n \in$ Z, f(n) $\in$ Q.

### 6.1.1 Types of function

**One-one** or **One to one** or **Injective function**

**Definition :** A function $f : A \to B$ is said to be one-one if different elements in A have different images in B. The condition is also expressed as

$f(a) = f(b) \Rightarrow a = b$ [As $a \neq b \Rightarrow f(a) \neq f(b)$]

**Onto** or **Surjective function**

**Definition:** A function $f : A \to B$ is said to be onto if every element $y$ in B is an image of some $x$ in A (or $y$ in B has preimage $x$ in A)

The image of A can be denoted by $f(A)$.

$f(A) = \{y \in B \mid y = f(x) \text{ for some } x \in A\}$

$f(A)$ is also called the **range** of $f$.

Note that $f : A \to B$ is onto if $f(A) = f$.

Also range of $f = f(A) \subset$ co-domain of $f$.

**Illustration:**

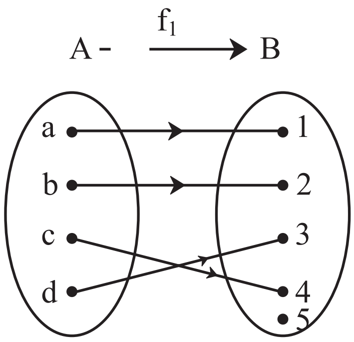



**Fig. 6.4**

$f_1$ is one-one, but not onto as element 5 is in B has no pre image in A

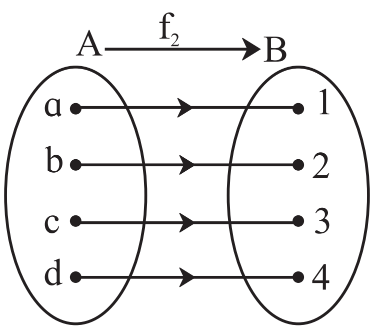



**Fig. 6.5**

$f_2$ is one-one, and onto

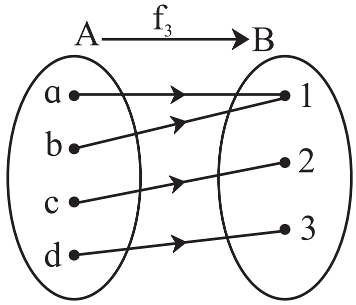



**Fig. 6.6**

$f_3$ is onto but not one-one as $f(a) = f(b) = 1$ but $a \neq b$.

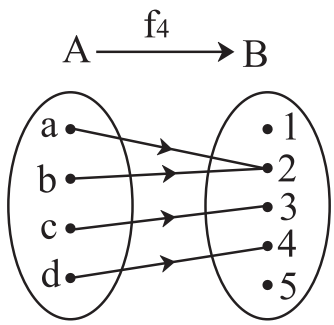
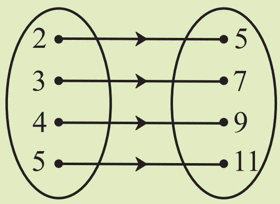



**Fig. 6.7**

$f_4$ is neither one-one, nor onto

### 6.1.2 Representation of Function

| | |
|---|---|
| Verbal form | Output exceeds twice the input by 1<br>Domain : Set of inputs<br>Range : Set of outputs |
| Arrow form on Venn Diagram | <br>**Fig. 6.8**<br>Domain : Set of pre-images<br>Range: Set of images |
| Ordered Pair $(x, y)$ | $f = \{(2,5), (3,7), (4,9), (5,11)\}$<br>Domain : Set of $1^{st}$ components from each ordered pair = $\{2, 3, 4, 5\}$<br>Range : Set of $2^{nd}$ components from each ordered pair = $\{5, 7, 9, 11\}$ |

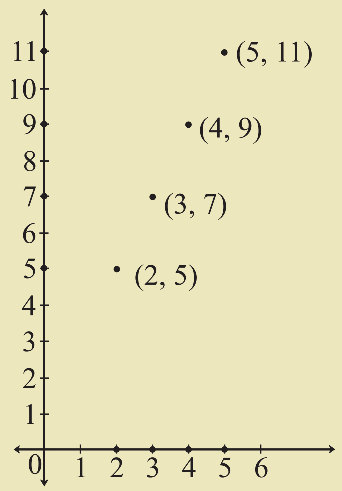
<table>
<tr><td>Rule / Formula</td><td>$y = f(x) = 2x + 1$<br>Where $x \in N$, $1 < x < 6$<br>$f(x)$ read as '$f$ of $x$' or 'function of $x$'<br>Domain : Set of values of $x$ for which $f(x)$ is defined<br>Range : Set of values of $y$ for which $f(x)$ is defined</td></tr>
<tr><td>Tabular Form</td><td><table><tr><th>$x$</th><th>$y$</th></tr><tr><td>2</td><td>5</td></tr><tr><td>3</td><td>7</td></tr><tr><td>4</td><td>9</td></tr><tr><td>5</td><td>11</td></tr></table>Domain : $x$ values<br>Range: $y$ values</td></tr>
<tr><td>Graphical form</td><td>

Fig. 6.9<br>Domain: Projection of graph on $x$-axis.<br>Range: Projection of graph on $y$-axis.</td></tr>
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### 6.1.3 Graph of a function:

If the domain of function is in R, we can show the function by a graph in xy plane. The graph consists of points $(x,y)$, where $y = f(x)$.

### Vertical Line Test

Given a graph, let us find if the graph represents a function of $x$ i.e. $f(x)$

A graph represents function of $x$, only if no vertical line intersects the curve in more than one point.

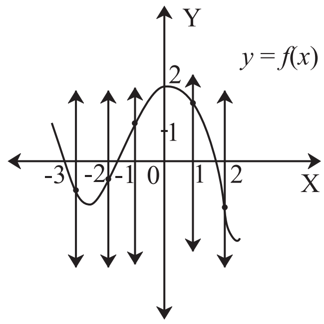



Fig. 6.10

Since every $x$ has a unique associated value of $y$.

It is a function.

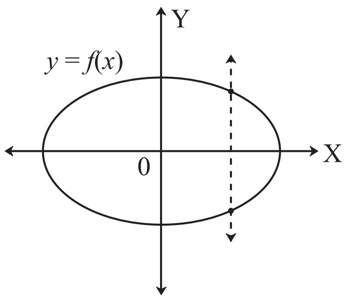



Fig. 6.11

This graph does not represent a function as vertical line intersects at more than one point some $x$ has more than one values of $y$.

### Horizontal Line Test:

If no horizontal line intersects the graph of a function in more than one point, then the function is one-one function.

### Illustration:

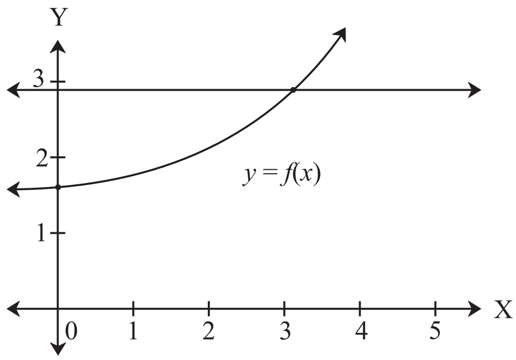



Fig. 6.12

The graph is a one-one function as a horizontal line intersects the graph at only one point.

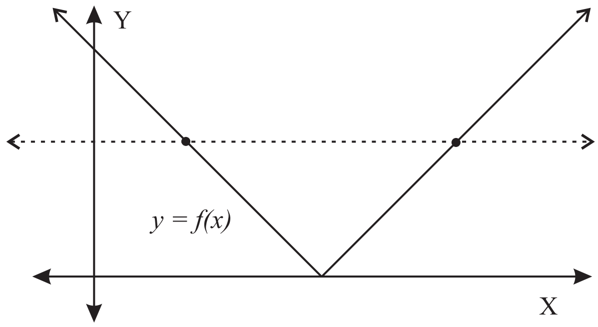



Fig. 6.13

The graph is a one-one function

**6.1.4 Value of funcation :** $f$(a) is called the value of funcation $f(x)$ at $x$ = a

**Evaluation of function:**

**Ex. 1)** Evaluate $f(x) = 2x^2 - 3x + 4$ at

$x = 7$ & $x = -2t$

**Solution :** $f(x)$ at $x = 7$ is $f(7)$

$f(7) = 2(7)^2 - 3(7) + 4$

$= 2(49) - 21 + 4$

$= 98 - 21 + 4$

$= 81$

$f(-2t) = 2(-2t)^2 - 3(-2t) + 4$

$= 2(4t^2) + 6t + 4$

$= 8t^2 + 6t + 4$

**Ex. 2)** Using the graph of $y = g(x)$ , find $g(-4)$ and $g(3)$

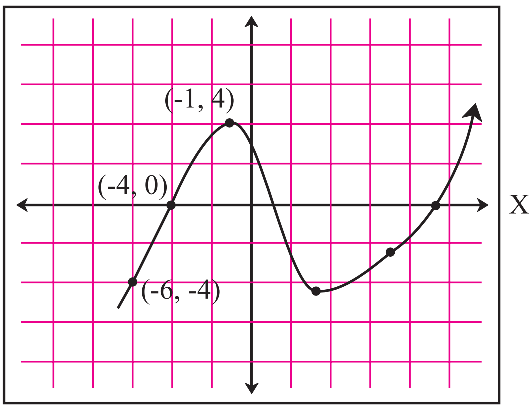



Fig. 6.14

**Solution :** From graph when $x = -4$, $y = 0$

so $g(-4) = 0$

From graph when $x = 3$, $y = -5$ so $g(3) = -5$

**Function Solution:**

**Ex. 3)** If $t(m) = 3m^2 - m$ and $t(m) = 4$, then find $m$

**Solution :** As

$t(m) = 4$

$3m^2 - m = 4$

$3m^2 - m - 4 = 0$

$3m^2 - 4m + 3m - 4 = 0$

$m(3m - 4) + 1(3m - 4) = 0$

$(3m - 4)(m + 1) = 0$

Therefore , $m = \frac{4}{3}$ or $m = -1$

**Ex. 4)** From the graph below find $x$ for which $f(x) = 4$

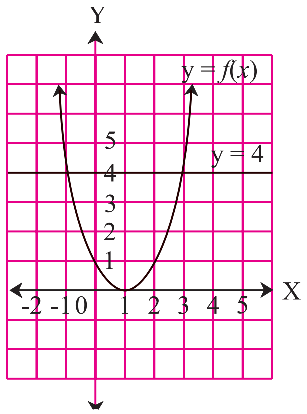



Fig. 6.15

**Solution :** To solve $f(x) = 4$ i.e. $y = 4$

Find the values of $x$ where graph intersects line $y = 4$

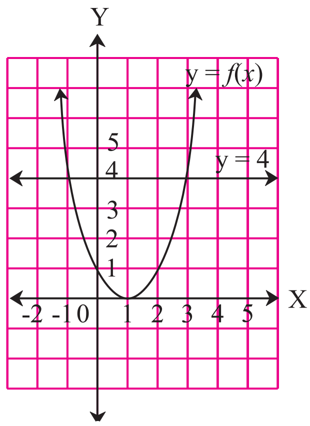



Fig. 6.16

Therefore , $x = -1$ and $x = 3$.

**Function from equation:**

**Ex. 5) (Activity)** From the equation $4x + 7y = 1$ express

i) $y$ as a function of $x$

ii) $x$ as a function of $y$

**Solution :** Given equation is $4x + 7y = 1$

i) From the given equation

$7y = \boxed{\phantom{x}}$

$y = \boxed{\phantom{x}}$ = function of $x$

So $y = f(x) = \boxed{\phantom{x}}$

ii) From the given equation

$4x = \boxed{\phantom{x}}$

$x = \boxed{\phantom{x}}$ = function of $y$

So $x = g(y) = \boxed{\phantom{x}}$

### 6.1.5 Some Basic Functions

(Here $f$ : R $\rightarrow$ R Unless stated otherwise)

**1) Constant Function**

**Form :** $f(x) = k$ , $k \in R$

**Example :** Graph of $f(x) = 2$

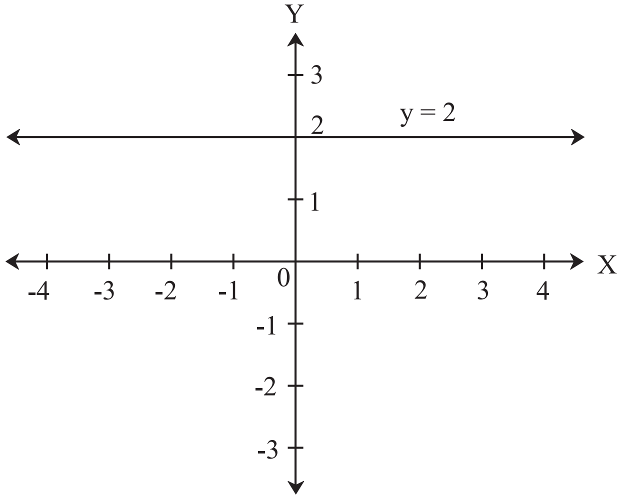



**Fig. 6.17**

**Domain :** R or $(-\infty, \infty)$ and **Range :** $\{2\}$

**2) Identity function**

If $f$ : R $\rightarrow$ R then identity function is defined by $f(x) = x$, for every $x \in R$.

Identity function is given in the graph below.

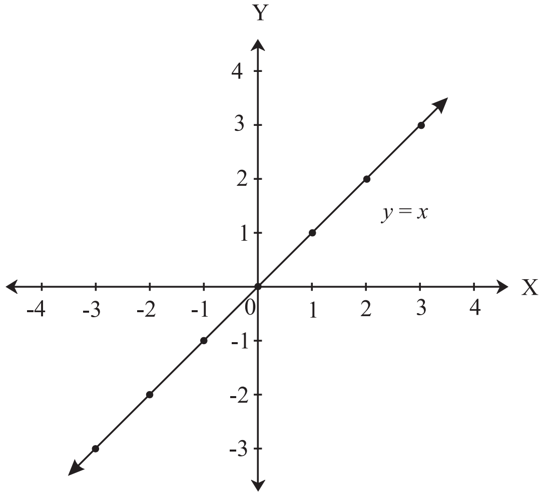



**Fig. 6.18**

**Domain :** R or $(-\infty, \infty)$ and **Range :** R or $(-\infty, \infty)$

[**Note :** Identity function is also given by I $(x) = x$].

**3) Power Functions :** $f(x) = ax^n$ , $n \in N$

(Note that this function is a multiple of $n^{th}$ power of $x$)

**i) Square Function**

**Example :** $f(x) = x^2$

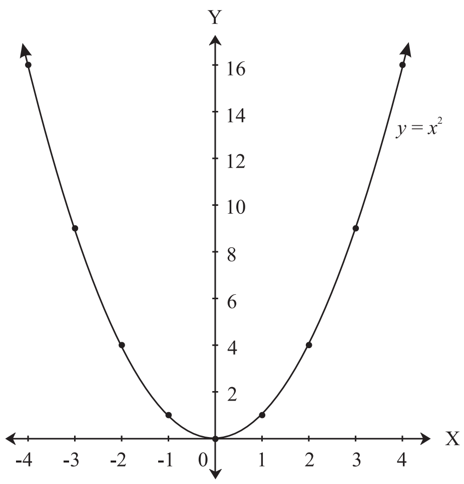



**Fig. 6.19**

**Domain :** R or $(-\infty, \infty)$ and **Range :** $[0, \infty)$

**Properties:**

1) Graph of $f(x) = x^2$ is a parabola opening upwards and with vertex at origin.
2) Graph is symmetric about $y$ - axis .
3) The graph of even powers of $x$ looks similar to square function. (verify !) e.g. $x^4, x^6$.
4) $(y - k) = (x - h)^2$ represents parabola with vertex at $(h, k)$
5) If $-2 \leq x \leq 2$ then $0 \leq x^2 \leq 4$ (see fig.) and if $-3 \leq x \leq 2$ then $0 \leq x^2 \leq 9$ (see fig).

### ii) Cube Function

**Example :** $f(x) = x^3$

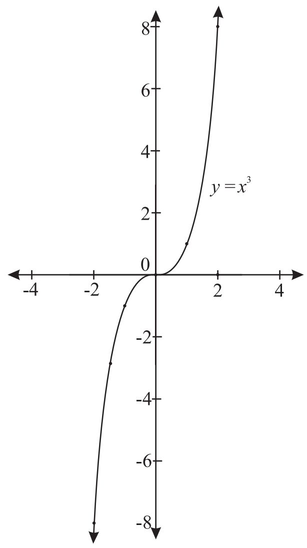



**Fig. 6.20**

**Domain :** R or $(-\infty, \infty)$ and **Range :** R or $(-\infty, \infty)$

**Properties:**

1) The graph of odd powers of $x$ (more than 1) looks similar to cube function. e.g. $x^5, x^7$.

### 4) Polynomial Function

$$f(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n$$

is polynomial function of degree $n$ , if $a_0 \neq 0$, and $a_i$ s are real.

### i) Linear Function

**Form :** $f(x) = ax + b \ (a \neq 0)$

**Example :** $f(x) = -2x + 3, \ x \in R$

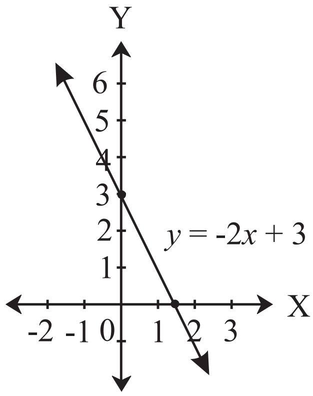



**Fig. 6.21**

**Domain :** R or $(-\infty, \infty)$ and **Range :** R or $(-\infty, \infty)$

**Properties :**

1) Graph of $f(x) = ax + b$ is a line with slope '$a$', $y$-intercept '$b$' and $x$-intercept $\left(-\dfrac{b}{a}\right)$.
2) Function : is increasing when slope is positive and deceasing when slope is negative.

### ii) Quadratic Function

**Form :** $f(x) = ax^2 + bx + c \ (a \neq 0)$

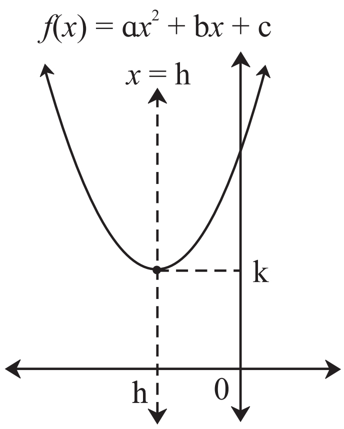



**Fig. 6.22**

**Domain :** R or $(-\infty, \infty)$ and **Range :** $[k, \infty)$

**Properties :**

1) Graph of $f(x) = ax^2 + bx + c$ and where $a > 0$ is a parabola.

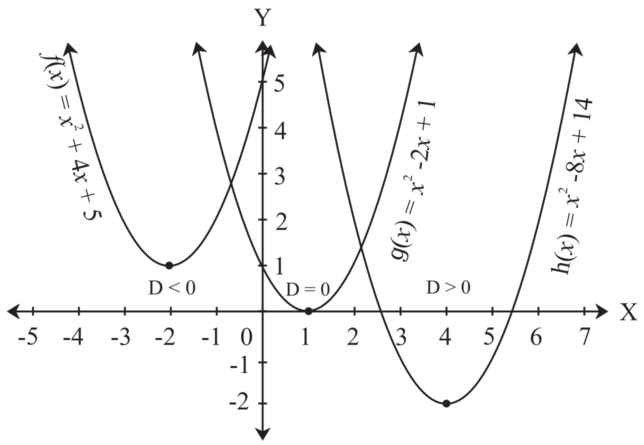



**Fig. 6.23**

Consider, $y = ax^2 + bx + c$

$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - \frac{b^2}{4a}$$

$$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}$$

$$\left(y + \frac{b^2 - 4ac}{4a}\right) = a\left(x + \frac{b}{2a}\right)^2$$

With change of variable

$$X = x + \frac{b}{2a},\ Y = y + \frac{b^2 - 4ac}{4a}$$

this is a parabola $Y = aX^2$

This is a parabola with vertex

$\left(-\frac{b}{2a}, \frac{b^2 - 4ac}{4a}\right)$ or $\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$ where

$D = b^2 - 4ac$ and the parabola is opening upwards.

There are three possibilities.

For a>0,

i) If $D = b^2 - 4ac = 0$, the parabola touches x-axis and $y \geq 0$ for all $x$. e.g. $g(x) = x^2 - 2x + 1$

ii) If $D = b^2 - 4ac > 0$, then parabola intersects x-axis at 2 distinct points. Here $y$ is negative for values of $x$ between the 2 roots and positive for large or small $x$.

iii) If $D = b^2 - 4ac < 0$, the parabola lies above x-axis and $y \neq 0$ for any $x$. Here $y$ is positive for all values of $x$. e.g. $f(x) = x^2 + 4x + 5$

### iii) Cubic Function

**Example :** $f(x) = ax^3 + bx^2 + cx + d\ (a \neq 0)$

**Domain :** R or $(-\infty, \infty)$ and

**Range :** R or $(-\infty, \infty)$

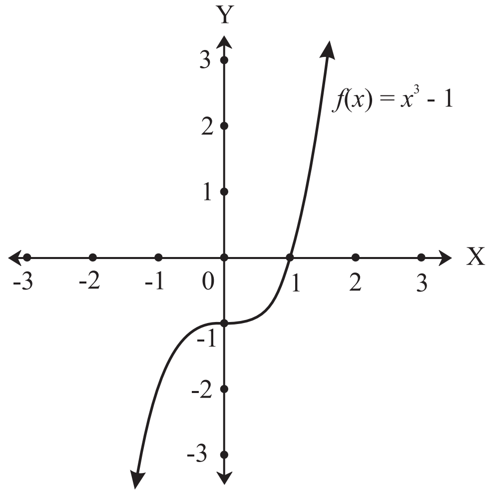



**Fig. 6.24**

**Property:**

1) Graph of $f(x) = x^3 - 1$

$f(x) = (x - 1)(x^2 + x + 1)$ cuts $x$-axis at only one point (1,0) , which means $f(x)$ has one real root & two complex roots.

Note that, any polynomial of odd degree must have at least one real root, because the complex roots appear in conjugate pairs.

### 5) Radical Function

**Ex:** $f(x) = \sqrt[n]{x}$ , $n \in N$

### 1. Square root function

$f(x) = \sqrt{x}$ , $x \geq 0$

(Since square root of negative number is not a real number, so the domain of $\sqrt{x}$ is restricted to positive values of $x$).

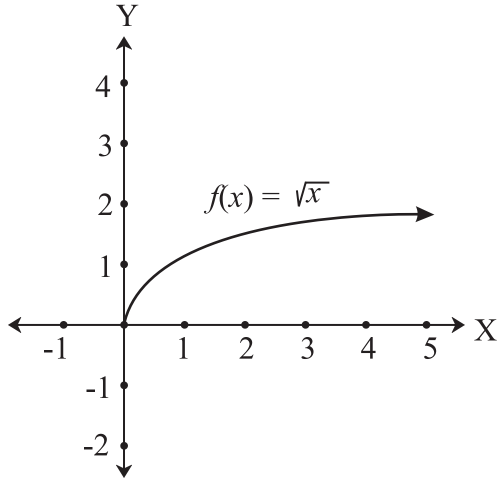



**Fig. 6.25**

**Domain :** $[0, \infty)$ and **Range :** $[0, \infty)$

**Note :**

1) If $x$ is positive, there are two square roots of $x$. By convention $\sqrt{x}$ is positive root and $-\sqrt{x}$ is the negattive root.

2) If $-4 < x < 9$, as $\sqrt{x}$ is only deifned for $x \geq 0$, so $0 \leq \sqrt{x} < 3$.

**Ex. 6 :** Find the domain and range of $f(x) = \sqrt{9-x^2}$.

**Soln. :** $f(x) = \sqrt{9-x^2}$ is defined for

$9 - x^2 \geq 0$, i.e. $x^2 - 9 \leq 0$ i.e. $(x-3)(x+3) \leq 0$

Therefore $[-3, 3]$ is domain of $f(x)$. (Verify !)

To find range, let $\sqrt{9-x^2} = y$

Since square root is always positive, so $y \geq 0$ ...(I)

Also, on squaring we get $9 - x^2 = y^2$

Since, $3 \leq x \leq 3$

i.e. $0 \leq x^2 \leq 9$

i.e. $0 \geq -x^2 \geq -9$

i.e. $9 \geq 9 - x^2 \geq 9 - 9$

i.e. $9 \geq 9 - x^2 \geq 0$

i.e. $3 \geq \sqrt{9-x^2} \geq 0$

$\therefore\ 3 \geq y \geq 0$ ...(II)

From (I) and (II), $y \in [0,3]$ is range of $f(x)$.

## 2. Cube root function

$f(x) = \sqrt[3]{x}$,

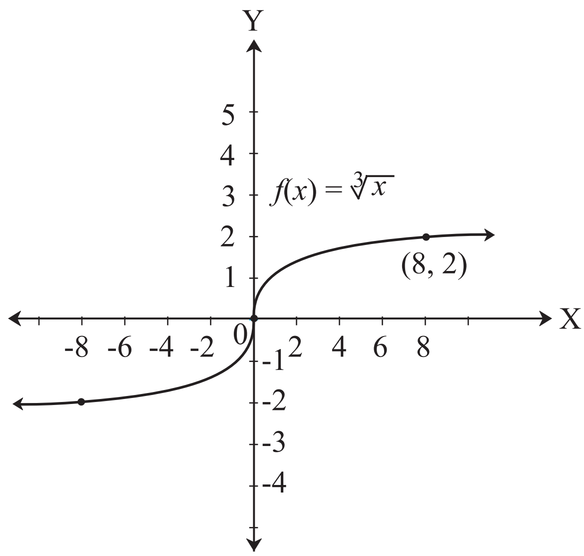



**Fig. 6.26**

**Domain :** R and **Range :** R

**Note :** If $-8 \leq x \leq 1$ then $-2 \leq \sqrt[3]{x} \leq 1$.

**Ex. 7 :** Find the domain $f(x) = \sqrt{x^3 - 8}$.

**Soln. :** $f(x)$ is defined for $x^3 - 8 \geq 0$

i.e. $x^3 - 2^3 \geq 0$, $(x-2)(x^2 + 2x + 4) \geq 0$

In $x^2 + 2x + 4$, $a = 1 > 0$ and $D = b^2 - 4ac = 2^2 - 4 \times 1 \times 4 = -12 < 0$

Therefore, $x^2 + 2x + 4$ is a positive quadratic.

i. e. $x^2 + 2x + 4 > 0$ for all $x$

Therefore $x - 2 \geq 0$, $x \geq 2$ is the domain.

i.e. Domain is $x \in [2, \infty)$

### 6) Rational Function

**Definition:** Given polynomials

$p(x)$, $q(x)$ $f(x) = \dfrac{p(x)}{q(x)}$ is defined for $x$ if $q(x) \neq 0$.

**Example :** $f(x) = \dfrac{1}{x}$ , $x \neq 0$

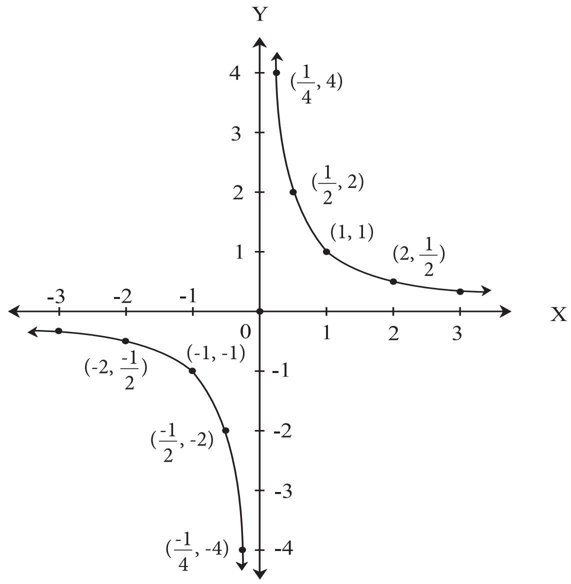



**Fig. 6.27**

**Domain :** R-{0} and **Range :** $R - \{0\}$

**Properties:**

1) As $x \to 0$ i.e. (As $x$ approaches 0) $f(x) \to \infty$ or $f(x) \to -\infty$ , so the line $x = 0$ i.e $y$-axis is called vertical asymptote.( A straight line which does not intersect the curve but as $x$ approaches to $\infty$ or $-\infty$ the distance between the line and the curve tends to 0, is called an asymptote of the curve.)

2) As As $x \to \infty$ or $x \to -\infty$, $f(x) \to 0$, $y = 0$ the line i.e $y$-axis is called horizontal asymptote.

3) The domain of rational function $f(x) = \dfrac{p(x)}{q(x)}$ is all the real values of except the zeroes of $q(x)$.

**Ex. 8 :** Find domain and range of the function $f(x) = \dfrac{6-4x^2}{4x+5}$

**Solution :** $f(x)$ is defined for all $x \in R$ exccept when denominators is 0.

Since, $4x + 5 = 0 \Rightarrow x = -\dfrac{5}{4}$.

So Domain of $f(x)$ is $R - \left\{-\dfrac{5}{4}\right\}$.

To find the range, let $y = \dfrac{6-4x^2}{4x+5}$

i.e. $y(4x + 5) = 6 - 4x^2$

i.e. $4x^2 + (4y)x + 5y - 6 = 0$.

This is a quadratic equation in $x$ with $y$ as constant.

Since $x \in R - \{-5/4\}$, i.e. $x$ is real, we get

Solution if, $D = b^2 - 4ac \geq 0$

i.e. $(4y)^2 - 4(4)(5y - 6) \geq 0$

$16y^2 - 16(5y - 6) \geq 0$

$y^2 - 5y + 6 \geq 0$

$(y - 2)(y - 3) \geq 0$

Therefore $y \leq 2$ or $y \geq 3$ (Verify!)

Range of $f(x)$ is $(-\infty, 2] \cup [3, \infty)$

### 7) Exponential Function

**Form :** $f(x) = a^x$ is an exponential function with base $a$ and exponent (or index) $x$, $a \neq 0$,

$a > 0$ and $x \in R$.

**Example :** $f(x) = 2^x$ and $f(x) = 2^{-x}$

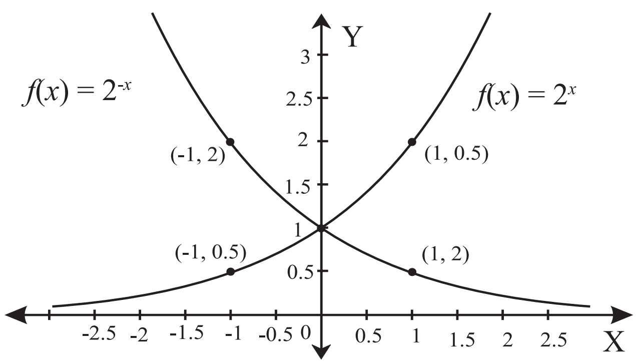



**Fig. 6.28**

**Domain:** R and **Range :** $(0, \infty)$

**Properties:**

1) As $x \to -\infty$ , then $f(x) = 2^x \to 0$, so the graph has horizontal asymptote ($y = 0$)

2) By taking the natural base $e$ ($\approx$ 2.718), graph of $f(x) = e^x$ is similar to that of $2^x$ in appearance

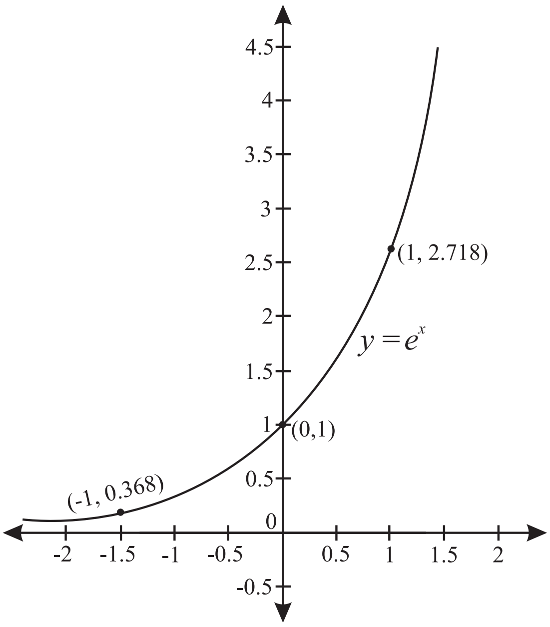



**Fig. 6.29**

3) For $a > 0$, $a \neq 1$, if $a^x = a^y$ then $x = y$. So $a^x$ is one-one function. (check graph for horizontal line test).

4) $r > 1, m > n \Rightarrow r^m > r^n$ and
$r < 1, m > n \Rightarrow r^m < r^n$

**Ex. 9 :** Solve $5^{2x+7} = 125$.

**Solution :** As $5^{2x+7} = 125$

i.e : $5^{2x+7} = 5^3$, $\therefore$ $2x + 7 = 3$

and $x = \dfrac{3-7}{2} = \dfrac{-4}{2} = -2$

**Ex. 10 :** Find the domain of $f(x) = \sqrt{6-2^x-2^{3-x}}$

**Solution :** Since $\sqrt{x}$ is defined for $x \geq 0$

$f(x)$ is defined for $6 - 2^x - 2^{3-x} \geq 0$

i.e. $6 - 2^x - \dfrac{2^3}{2^x} \geq 0$

i.e. $6.2^x - (2^x)^2 - 8 \geq 0$

i.e. $(2^x)^2 - 6.2^x + 8 \leq 0$

i.e. $(2^x - 4)(2^x - 2) \leq 0$

$2^x \geq 2$ and $2^x \leq 4$ (Verify !)

$2^x \geq 2^1$ and $2^x \leq 2^2$

$x \geq 1$ and $x \leq 2$ or $1 \leq x \leq 2$

Doamin is [1,2]

## 8) Logarithmic Function:

Let, $a > 0$, $a \neq 1$, Then logarithmic function $\log_a x$, $y = \log_a x$ if $x = a^y$.

for $x > 0$, is defined as

$$\underset{\text{log arithmic form}}{y = \log_a x} \Leftrightarrow \underset{\text{exponential form}}{a^y = x}$$

**Properties:**

1) As $a^0 = 1$, so $\log_a 1 = 0$ and as $a^1 = a$,

so $\log_a a = 1$

2) As $a^x = a^y \Leftrightarrow x = y$ so $\log_a x = \log_a y \Leftrightarrow$ $x = y$

3) Product rule of logarithms.

For $a, b, c > 0$ and $a \neq 1$,

$\log_a bc = \log_a b + \log_a c$ (Verify !)

4) Quotient rule of logarithms.

For $a, b, c > 0$ and $a \neq 1$,

$\log_a \dfrac{b}{c} = \log_a b - \log_a c$ (Verify !)

5) Power/Exponent rule of logarithms.

For $a, b, c > 0$ and $a \neq 1$,

$\log_a b^c = c \log_a b$ (Verify !)

6) For natural base $e$, $\log_e x = ln\, x$ as Natural Logarithm Function.

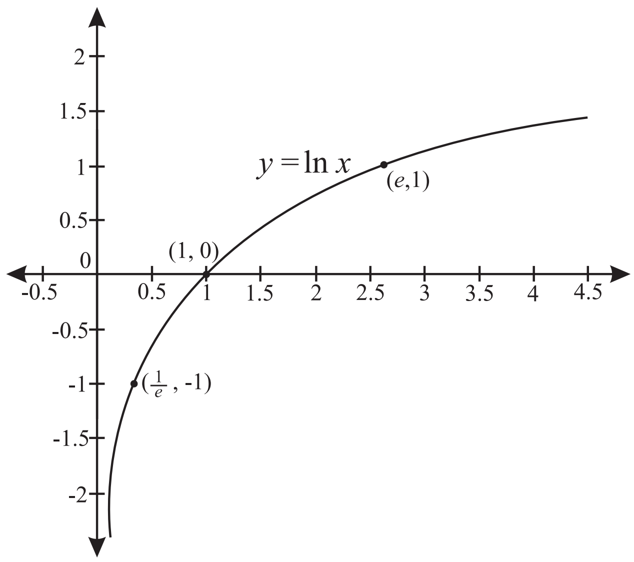



Fig. 6.30

Here domain of $ln\, x$ is $(0, \infty)$ and range is $(-\infty, \infty)$.

8) Logarithmic inequalities:

(i) If $a>1$, $0<m<n$ then $\log_a m < \log_a n$

e.g. $\log_{10} 20 < \log_{10} 30$

(ii) If $0>a<1$, $0<m<n$ then $\log_a m > \log_a n$

e.g. $\log_{0.1} 20 > \log_{0.1} 30$

(iii) For $a$, $m>0$ if $a$ and $m$ lies on the same side of unity (i.e. 1) then $\log_a m>0$.

e.g. $\log_2 3>0$, $\log_{0.3} 0.5>0$

(iv) For $a$, $m>0$ if $a$ and $m$ lies on the different sides of unity (i.e. 1) then $\log_a m>0$.

e.g. $\log_{0.2} 3<0$, $\log_3 0.5>0$

**Ex. 11 :** Write log72 in terms of log2 and log3.

**Solution :** $\log 72 = \log(2^3.3^2)$

$= \log 2^3 + \log 3^2$ ($\because$ Power rule)

$= 3 \log 2 + 2 \log 3$ ($\because$ Power rule)

**Ex. 12 :** Evaluate $\ln e^9 - \ln e^4$.

**Solution :** $\ln e^9 - \ln e^4 = \log_e e^9 - \log_e e^4$

$= 9 \log_e e - 4 \log_e e$

$= 9(1) - 4(1)$ ($\therefore \ln e = 1$)

$= 5$

**Ex. 13 :** Expand $\log\left[\dfrac{x^3(x+3)}{2(x-4)^2}\right]$

**Solution :** Using Quotient rule

$= \log [x^3(x+3)] - \log [2(x-4)^2]$

Using Product rule

$= [\log x^3 + \log (x+3)] - [\log 2 + \log (x-4)^2]$

Using Power rule

$= [3\log x + \log (x+3)] - [\log 2 - 2\log (x-4)]$

$= 3\log x + \log (x+3) - \log 2 + 2\log (x-4)$

**Ex. 14 :** Combine

$3\ln (p + 1) - \frac{1}{2} \ln r + 5\ln(2q + 3)$ into single logarithm.

**Solution :** Using Power rule,

$= \ln (p + 1)^3 - \ln r^{\frac{1}{2}} + \ln(2q + 3)^5$

Using Quotient rule

$= \ln \dfrac{(p+1)^3}{\sqrt{r}} + \ln(2q + 3)^5$

Using Product rule

$= \ln \left[\dfrac{(p+1)^3}{\sqrt{r}}(2q+3)^5\right]$

**Ex. 15 :** Find the domain of $ln(x - 5)$.

**Solution :** As $ln(x - 5)$ is defined for $(x - 5) > 0$

that is $x > 5$ so domain is $(5, \infty)$.

**Let's note:**

1) $\log(x + y) \neq \log x + \log y$

2) $\log x \log y \neq \log(xy)$

3) $\dfrac{\log x}{\log y} \neq \log\left(\dfrac{x}{y}\right)$

4) $(\log x)^n \neq n \log^n$

9) Change of base formula:

For $a, x, b > 0$ and $a, b \neq 1$, $\log_a x = \dfrac{\log_b x}{\log_b a}$

Note: $\log_a x = \dfrac{\log_x x}{\log_x a} = \dfrac{1}{\log_x a}$ (Verify !)

**Ex. 16 :** Evaluate $\dfrac{\log_4 81}{\log_4 9}$

**Solution :** By Change of base law, as the base is same (that is 4)

$$\frac{\log_4 81}{\log_4 9} = \log_9 81 = 2$$

**Ex. 17 :** Prove that, $2\log_b a^4 . \log_c b^3 . \log_a c^5 = 120$

**Solution :** L.H.S. $= 2\log_b a^4 . \log_c b^3 . \log_a c^5$

$= 4 \times 2\log_b a \times 3\log_c b \times 5\log_a c$

Using change of base law,

$= 4 \times 2 \dfrac{\log a}{\log b} \times 3 \dfrac{\log b}{\log c} \times 5 \dfrac{\log c}{\log a}$

$= 120$

**Ex. 18 :** Find the domain of $f(x) = \log_{x+5}(x^2 - 4)$

**Solution :** SInce $\log_a x$ is definedd for a, $x > 0$ and $a \neq 1$ $f(x)$ is defined for $(x^2 - 4) > 0$, $x + 5 > 0$, $x + 5 \neq 1$.

i.e. $(x - 2)(x + 2) > 0$, $x > -5$, $x \neq -4$

i.e. $x < -2$ or $x > 2$ and $x > -5$ and $x \neq -4$

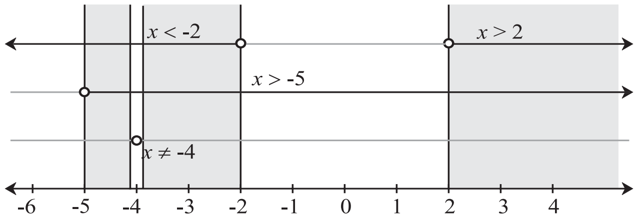



Fig. 6.31

## 9) Trigonometric function

The graphs of trigonometric functions are discusse in chapter 2 of Mathematics Book I.

| $f(x)$ | Domain | Range |
|---|---|---|
| $\sin x$ | R | $[-1,1]$ |
| $\cos x$ | R | $[-1,1]$ |
| $\tan x$ | $R - \left\{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2} \ldots\right\}$ | R |

**EXERCISE 6.1**

1) Check if the following relations are functions.

(a)

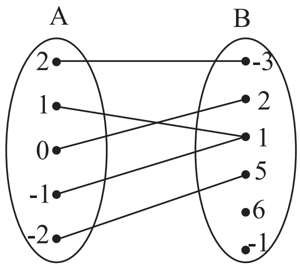



Fig. 6.32

(b)

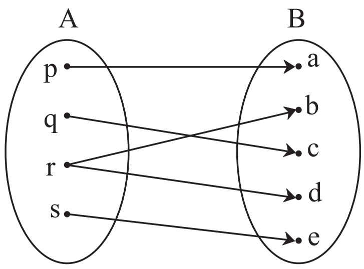



Fig. 6.33

(c)

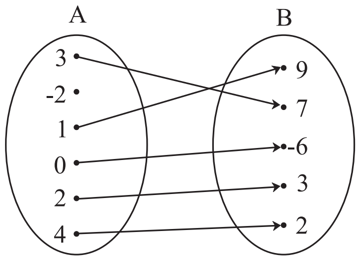



**Fig. 6.34**

2) Which sets of ordered pairs represent functions from A = {1, 2, 3, 4} to B = {−1, 0, 1, 2, 3}? Justify.

(a) $\{(1,0), (3,3), (2,-1), (4,1), (2,2)\}$

(b) $\{(1,2), (2,-1), (3,1), (4,3)\}$

(c) $\{(1,3), (4,1), (2,2)\}$

(d) $\{(1,1), (2,1), (3,1), (4,1)\}$

3) Check if the relation given by the equation represents $y$ as function of $x$.

(a) $2x + 3y = 12$ (b) $x + y^2 = 9$

(c) $x^2 - y = 25$ (d) $2y + 10 = 0$

(e) $3x - 6 = 21$

4) If $f(m) = m^2 - 3m + 1$, find

(a) $f(0)$ (b) $f(-3)$

(c) $f\left(\frac{1}{2}\right)$ (d) $f(x + 1)$

(e) $f(-x)$

(f) $\left(\frac{f(2+h)-f(2)}{h}\right)$, $h \neq 0$.

5) Find $x$, if $g(x) = 0$ where

(a) $g(x) = \frac{5x-6}{7}$ (b) $g(x) = \frac{18-2x^2}{7}$

(c) $g(x) = 6x^2 + x - 2$

(d) $g(x) = x^3 - 2x^2 - 5x + 6$

6) Find $x$, if $f(x) = g(x)$ where

(a) $f(x) = x^4 + 2x^2$, $g(x) = 11x^2$

(b) $f(x) = \sqrt{x} - 3$, $g(x) = 5 - x$

7) If $f(x) = \frac{a-x}{b-x}$, $f(2)$ is undefined, and $f(3) = 5$, find $a$ and $b$.

8) Find the domain and range of the follwoing functions.

(a) $f(x) = 7x^2 + 4x - 1$

(b) $g(x) = \frac{x+4}{x-2}$

(c) $h(x) = \frac{\sqrt{x+5}}{5+x}$

(d) $f(x) = \sqrt[3]{x+1}$

(e) $f(x) = \sqrt{(x-2)(5-x)}$

(f) $f(x) = \sqrt{\frac{x-3}{7-x}}$

(g) $f(x) = \sqrt{16-x^2}$

9) Express the area A of a square as a function of its (a) side $s$ (b) perimeter $P$.

10) Express the area A of circle as a function of its (a) radius $r$ (b) diameter $d$ (c) circumference C.

11) An open box is made from a square of cardboard of 30 cms side, by cutting squares of length $x$ centimeters from each corner and folding the sides up. Express the volume of the box as a function of $x$. Also find its domain.

Let $f$ be a subset of Z × Z defined by

12) $f = \{(ab, a+b) : a, b \in Z\}$. Is $f$ a function from Z to Z? Justify.

14) Check the injectivity and surjectivity of the following functions.

(a) $f : N \to N$ given by $f(x) = x^2$

(b) $f : Z \to Z$ given by $f(x) = x^2$

(c) $f : R \to R$ given by $f(x) = x^2$

(d) $f$ : N → N given by $f(x) = x^3$

(e) $f$ : R → R given by $f(x) = x^3$

14) Show that if $f : A \to B$ and $g : B \to C$ are one-one, then $g \circ f$ is also one-one.

15) Show that if $f : A \to B$ and $g : B \to C$ are onto, then $g \circ f$ is also onto.

16) If $f(x) = 3(4^{x+1})$ find $f(-3)$ .

17) Express the following exponential equations in logarithmic form

(a) $2^5 = 32$ (b) $54^0 = 1$

(c) $23^1 = 23$ (d) $9^{3/2} = 27$

(e) $3^{-4} = \frac{1}{81}$ (f) $10^{-2} = 0.01$

(g) $e^2 = 7.3890$ (h) $e^{1/2} = 1.6487$

(i) $e^{-x} = 6$

18) Express the following logarithmic equations in exponential form

(a) $\log_2 64 = 6$ (b) $\log_5 \frac{1}{25} = -2$

(c) $\log_{10} 0.001 = -3$ (d) $\log_{1/2} (-8) = 3$

(e) $\ln 1 = 0$ (f) $\ln e = 1$

(g) $\ln \frac{1}{2} = -0.693$

19) Find the domain of

(a) $f(x) = \ln (x-5)$

(b) $f(x) = \log_{10} (x^2-5x+6)$

20) Write the following expressions as sum or difference of logarithms

(a) $\log \left(\frac{pq}{rs}\right)$ (b) $\log \left(\sqrt{x}\sqrt[3]{y}\right)$

(c) $\ln \left(\frac{a^3(a-2)^2}{\sqrt{b^2+5}}\right)$

(d) $\ln \left[\frac{\sqrt[3]{x-2}\,(2x+1)^4}{(x+4)\sqrt{2x+4}}\right]^2$

21) Write the following expressions as a single logarithm.

(a) $5\log x + 7\log y - \log z$

(b) $\frac{1}{3}\log (x-1) + \frac{1}{2}\log (x)$

(c) $\ln (x+2) + \ln (x-2) - 3\ln (x+5)$

22) Given that $\log 2 = a$ and $\log 3 = b$,

write $\log \sqrt{96}$ in terms of $a$ and $b$.

23) Prove that

(a) $b^{\log_b a} = a$ (b) $\log_{b^m} a = \frac{1}{m}\log_b a$

(c) $a^{\log_c b} = b^{\log_c a}$

24) If $f(x) = ax^2 - bx + 6$ and $f(2) = 3$ and $f(4) = 30$, find $a$ and $b$

25) Solve for $x$.

(a) $\log 2 + \log(x+3) - \log(3x-5) = \log 3$

(b) $2\log_{10} x = 1 + \log_{10}\left(x + \frac{11}{10}\right)$

(c) $\log_2 x + \log_4 x + \log_{16} x = \frac{21}{4}$

(d) $x + \log_{10}(1+2^x) = x\log_{10} 5 + \log_{10} 6$

26) If $\log\left(\frac{x+y}{3}\right) = \frac{1}{2}\log x + \frac{1}{2}\log y$,

show that $\frac{x}{y} + \frac{y}{x} = 7$.

27) If $\log\left(\frac{x-y}{4}\right) = \log\sqrt{x} + \log\sqrt{y}$,

show that $(x+y)^2 = 20xy$

28) If $x = \log_a bc$, $y = \log_b ca$, $z = \log_c ab$

then prove that $\frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+z} = 1$

## 6.2 Algebra of functions:

Let $f$ and $g$ be functions with domains $A$ and $B$. Then the functions $f+g, f-g, fg, \frac{f}{g}$ are defined on $A \cap B$ as follows.

| Operations |
|---|
| $(f+g)(x) = f(x) + g(x)$ |
| $(f-g)(x) = f(x) - g(x)$ |
| $(f.g)(x) = f(x).g(x)$ |
| $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ where $g(x) \neq 0$ |

**Ex. 1 :** If $f(x) = x^2 + 2$ and $g(x) = 5x - 8$, then find

i) $(f+g)(1)$

ii) $(f-g)(-2)$

ii) $(f \circ g)(3m)$

iv) $\frac{f}{g}(0)$

**Solution :** i) As $(f+g)(x) = f(x) + g(x)$

$(f+g)(1) = f(1) + g(1)$

$= [(1)^2 + 2] + [5(1) - 8]$

$= 3 + (-3)$

$= 0$

ii) As $(f-g)(x) = f(x) - g(x)$

$(f-g)(-2) = f(-2) - g(-2)$

$= [(-2)^2 + 2] - [5(-2) - 8]$

$= [4 + 2] - [-10-8]$

$= 6 + 18$

$= 24$

iii) As $(fg)(x) = f(x)\, g(x)$

$(f \circ g)(3m) = f(3m)g(3m)$

$= [(3m)^2 + 2]\, [5(3m) - 8]$

$= [9m^2 + 2]\, [15m-8]$

$= 135m^3 - 72m^2 + 30m - 16$

iv) As $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, g(x) \neq 0$

$\left(\frac{f}{g}\right)(0) = \frac{f(0)}{g(0)} = \frac{0^2+2}{5(0)-8}$

$= \frac{2}{-8} = -\frac{1}{4}$

**Ex. 2 :** Given the function $f(x) = 5x^2$ and $g(x) = \sqrt{4-x}$ find the domain of

i) $(f+g)(x)$ ii) $(f \circ g)(x)$ iii) $\frac{f}{g}(x)$

**Solution :** i) Domain of $f(x) = 5x^2$ is $(-\infty, \infty)$ .

To find domain of $g(x) = \sqrt{4-x}$

$4 - x \geq 0$

$x - 4 \leq 0$

Let $x \leq 4$, So domain is $(-\infty, 4]$.

Therefore, domain of $(f+g)(x)$ is

$(-\infty, \infty) \cap (-\infty, 4]$, that is $(-\infty, 4]$

ii) Similarly, domain of $(f \circ g)(x) = 5x^2\sqrt{4-x}$ is $(-\infty, 4]$

iii) And domain of $\left(\frac{f}{g}\right)(x) = \frac{5x^2}{\sqrt{4-x}}$ is $(-\infty, 4)$

As , at $x = 4$ the denominator $g(x) = 0$ .

### 6.2.1 Composition of Functions:

A method of combining the function $f: A \to B$ with $g: B \to C$ is composition of functions, defined as $(f \circ g)(x) = f[g(x)]$ an read as '$f$ composed with $g$'

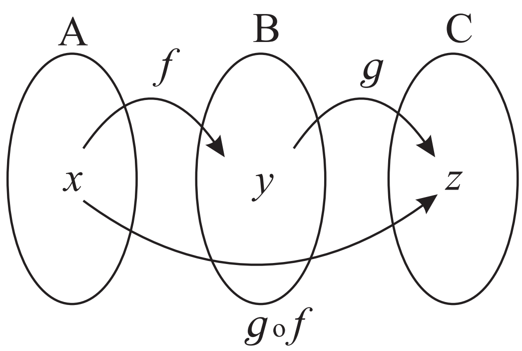



**Fig. 6.35**

**Note:**

1) The domain of $g \circ f$ is the set of all $x$ in A such that $f(x)$ is in the B. The range of $g \circ f$ is set of all g [$f(x)$] in C such that f(x) is in B.

2) Domain of $g \circ f \subseteq$ Domain of $f$ and Range of $g \circ f \subseteq$ Range of $g$.

**Illustration:**

A cow produces 4 liters of milk in a day. Then x number of cows produce $4x$ liters of milk in a day. This is given by function $f(x) = 4x =$ '$y$'. Price of one liter milk is Rs. 50. Then the price of $y$ liters of the milk is Rs. $50y$. This is given by another function $g(y) = 50y$. Now a function $h(x)$ gives the money earned from x number of cows in a day as a composite function of $f$ and $g$ as $h(x) = (g \circ f)(x) = g[f(x)] = g(4x) = 50(4x) = 200x$.

**Ex. 3 :** If $f(x) = \frac{2}{x+5}$ and $g(x) = x^2 - 1$, then find

i) $(f \circ g)(x)$ ii) $(g \circ f)(3)$

**Solution :**

i) As $(f \circ g)(x) = f[g(x)]$ and $f(x) = \frac{2}{x+5}$

Replace $x$ from $f(x)$ by $g(x)$, to get

$$(f \circ g)(x) = \frac{2}{g(x)+5}$$

$$= \frac{2}{x^2-1+5}$$

$$= \frac{2}{x^2+4}$$

ii) As $(g \circ f)(x) = g[f(x)]$ and $g(x) = x^2 - 1$

Replace $x$ by $f(x)$, to get

$$(g \circ f)(x) = [f(x)]^2 - 1$$

$$= \left(\frac{2}{x+5}\right)^2 - 1$$

Now let $x = 3$

$$(g \circ f)(3) = \left(\frac{2}{3+5}\right)^2 - 1$$

$$= \left(\frac{2}{8}\right)^2 - 1$$

$$= \left(\frac{1}{4}\right)^2 - 1$$

$$= \frac{1-16}{16}$$

$$= -\frac{15}{16}$$

**Ex 4 :** If $f(x) = x^2$, $g(x) = x + 5$, and $h(x) = \frac{1}{x}$, $x \neq 0$, find $(g \circ f \circ h)(x)$

**Solution :** $(g \circ f \circ h)(x)$

$$= g\{f[h(x)]\}$$

$$= g\left[f\left(\frac{1}{x}\right)\right]$$

$$= g\left[f\left(\frac{1}{x}\right)^2\right]$$

$$= \left(\frac{1}{x}\right)^2 + 5$$

$$= \frac{1}{x^2} + 5$$

**Ex. 5 :** If $h(x) = (x-5)^2$, find the functions $f$ and $g$, such that $h = f \circ g$.

→ In $h(x)$, 5 is subtracted from $x$ first and then squared. Let $g(x) = x - 5$ and $f(x) = x^2$, (verify)

**Ex. 6 :** Express $m(x) = \frac{1}{x^3+7}$ in the form of $f \circ g \circ h$

→ In $m(x)$, $x$ is cubed first then 7 is added and then its reciprocal taken. So,

$h(x) = x^3$, $g(x) = x + 7$ and $f(x) = \frac{1}{x}$, (verify)

### 6.2.2 Inverse functions:

Let $f : A \to B$ be one-one and onto function and $f(x) = y$ for $x \in A$. The inverse function $f^{-1} : B \to A$ is defined as $f^{-1}(y) = x$ if $f(x) = y$

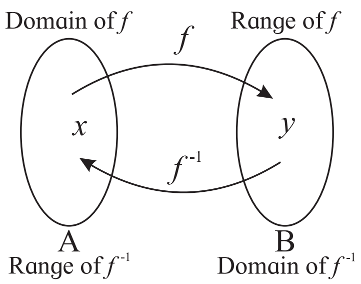



**Fig. 6.36**

**Note:**

1) As $f$ is one-one and onto every element $y \in B$ has a unique element $x \in A$ such that $y = f(x)$.

2) If $f$ and $g$ are one-one and onto functions such that $f[g(x)] = x$ for every $x \in$ Domain of $g$ and $g[f(x)] = x$ for every $x \in$ Domain of $f$, then $g$ is called inverse of function $f$. Function $g$ is denoted by $f^{-1}$ (read as f inverse).
i.e. $f[g(x)] = g[f(x)] = x$ then $g = f^{-1}$ which
Moreover this means $f[f^{-1}(x)] = f^{-1}[f(x)] = x$

3) $f^{-1}(x) \neq [f(x)]^{-1}$, because $[f(x)]^{-1} = \frac{1}{f(x)}$
$[f(x)]^{-1}$ is reciprocal of function $f(x)$ where as $f^{-1}(x)$ is the inverse function of $f(x)$.

e.g. If $f$ is one-one onto function with $f(3) = 7$ then $f^{-1}(7) = 3$.

**Ex. 7 :** If $f$ is one-one onto function with $f(x) = 9 - 5x$, find $f^{-1}(-1)$.

**Soln. :** $\to$ Let $f^{-1}(-1) = m$, then $-1 = f(m)$

Therefore,

$$-1 = 9 - 5m$$
$$5m = 9 + 1$$
$$5m = 10$$
$$m = 2$$

That is $f(2) = -1$, so $f^{-1}(-1) = 2$.

**Ex. 8 :** Verify that $f(x) = \frac{x-5}{8}$ and $g(x) = 8x + 5$ are inverse functions of each other.

**Solution :** As $f(x) = \frac{x-5}{8}$, replace $x$ in $f(x)$ with $g(x)$

$$f[g(x)] = \frac{g(x)-5}{8} = \frac{8x+5-5}{8} = \frac{8x}{8} = x$$

and $g(x) = 8x + 5$, replace $x$ in $g(x)$ with $f(x)$

$$g[f(x)] = 8f(x) + 5 = 8\left[\frac{x-5}{8}\right] + 5 = x - 5 + 5$$
$$= x$$

As $f[g(x)] = x$ and $g[f(x)] = x$, $f$ and $g$ are inverse functions of each other.

**Ex. 9 :** Determine whether the function

$f(x) = \frac{2x+1}{x-3}$ has inverse, if it exists find it.

**Solution :** $f^{-1}$ exists only if $f$ is one-one and onto.

Consider $f(x_1) = f(x_2)$,

Therefore,

$$\frac{2x_1+1}{x_1-3} = \frac{2x_2+1}{x_2-3}$$
$$(2x_1+1)(x_2-3) = (2x_2+1)(x_1-3)$$
$$2x_1x_2 - 6x_1 + x_2 - 3 = 2x_1x_2 - 6x_2 + x_1 - 3$$
$$-6x_1 + x_2 = -6x_2 + x_1$$
$$6x_1 + x_2 = 6x_2 + x_1$$
$$7x_2 = 7x_1$$
$$x_2 = x_1$$

Hence, $f$ is one-one function.

Let $f(x) = y$, so $y = \frac{2x+1}{x-3}$

Express $x$ as function of $y$, as follows

$$y = \frac{2x+1}{x-3}$$
$$y(x-3) = 2x + 1$$

$$xy - 3y = 2x + 1$$
$$xy - 2x = 3y + 1$$
$$x(y - 2) = 3y + 1$$
$$\therefore \quad x = \frac{3y+1}{y-2} \text{ for } y \neq 2.$$

Thus for any $y \neq 2$,

we have $x$ such that $f(x) = y$

$f^{-1}$ is well defined on R - {2}

If $f(x) = 2$ i.e. $2x + 1 = 2(x - 3)$

i.e. $2x + 1 = 2x - 6$ i.e. $1 = -6$

Which is contradiction.

$2 \notin$ Range of $f$.

Here the range of $f(x)$ is $R - \{2\}$.

$x$ is defined for all $y$ in the range.

Therefore $f(x)$ is onto function.

As $f$ is one-one and onto, so $f^{-1}$ exists.

As $f(x) = y$, so $f^{-1}(y) = x$

Therefore, $f^{-1}(y) = \dfrac{3y+1}{y-2}$

Replace $x$ by $y$, to get

$$f^{-1}(x) = \frac{3x+1}{x-2}.$$

### 6.2.3 Piecewise Defined Functions:

A function defined by two or more equations on different parts of the given domain is called piecewise defind function.

e.g.: If $f(x) = \begin{cases} x+1 & \text{if } x<1 \\ 4-x & \text{if } x\geq 1 \end{cases}$

Here $f(3) = 4 - 3 = 1$ as $3 > 1$,

whereas $f(-2) = -2 + 1 = -1$ as $-2 < 1$ and

$f(1) = 4 - 1 = 3$.

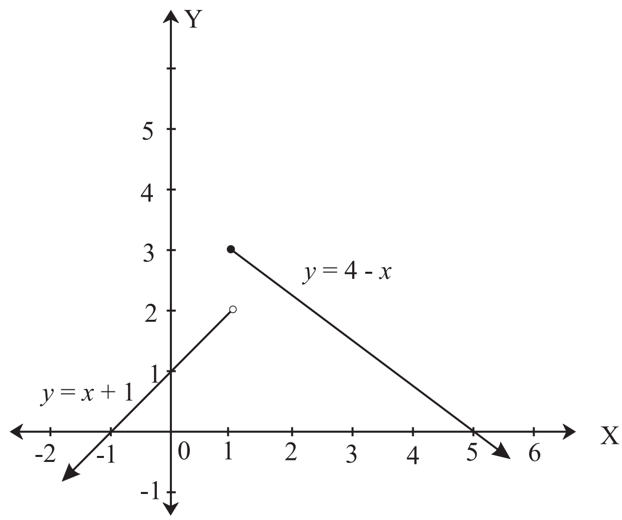



**Fig. 6.37**

As (1,3) lies on line $y = 4 - x$, so it is shown by small black disc on that line. (1,2) is shown by small white disc on the line $y = x + 1$, because it is not on the line.

**1) Signum function :**

**Definition:** $f(x) = \text{sgn}(x)$ is a piecewise function

$$f(x) = \begin{cases} -1 & \text{if } x<0 \\ 0 & \text{if } x=0 \\ 1 & \text{if } x>0 \end{cases}$$

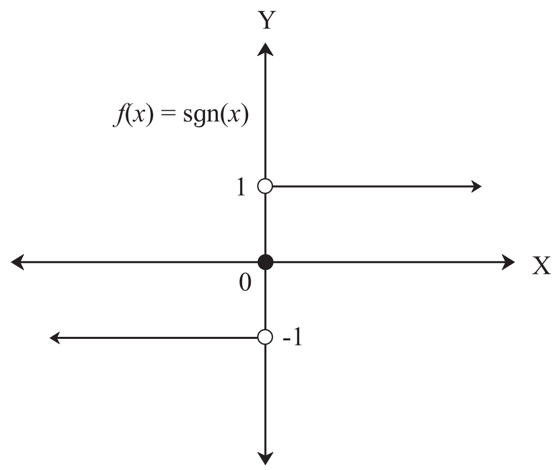



**Fig. 6.38**

**Domain:** R and **Range:** {−1, 0, 1}

**Properties:**

1) For $x > 0$, the graph is line $y = 1$ and for $x < 0$, the graph is line $y = -1$ .

2) For $f(0) = 0$ , so point (0,0) is shown by black disc, whereas points $(0,-1)$ and $(0,1)$ are shown by white discs.

## Absolute value function (Modulus function):

Definition: $f(x) = |x|$, is a piece wise function

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

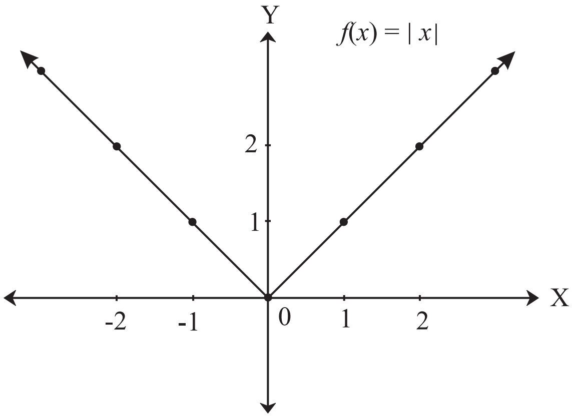



Fig. 6.39

**Domain :** R or $(-\infty,\infty)$ and **Range :** $[0,\infty)$

**Properties:**

1) Graph of $f(x) = |x|$ is union of line $y = x$ from quadrant I with the line $y = -x$ from quadrant II. As origin marks the change of directions of the two lines, we call it a critical point.

2) Graph is symmetric about $y$-axis .

3) Graph of $f(x) = |x-3|$ is the graph of $|x|$ shifted 3 units right and the critical point is (3,0).

4) $f(x) = |x|$ , represents the distance of $x$ from origin.

5) If $|x| = m$, then it represents every $x$ whose distance from origin is $m$, that is $x = + m$ or $x = -m$ .

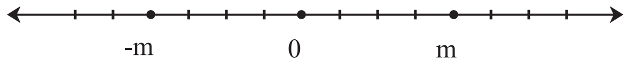



Fig. 6.40

6) If $|x| < m$ , then it represents every $x$ whose distance from origin is less than $m$, $0 \leq x < m$ and $0 \geq x > -m$ That is $-m < x < m$. In interval notation $x \in (-m, m)$


Fig. 6.41

7) If $|x| \geq m$ , then it represents every $x$ whose distance from origin is greater than or equal to $m$, so, $x \geq m$ and $x \leq -m$ . In interval notation $x \in (-\infty, m] \cup [m, \infty)$

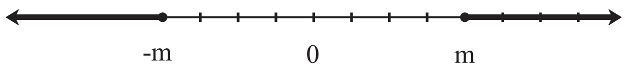



Fig. 6.42

8) If $m < |x| < n$, then it represents all $x$ whose distance from origin is greater than $m$ but less than $n$ . That is $x \in (-n,-m) \cup (m, n)$ .

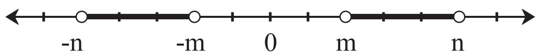



Fig. 6.43

9) Triangle inequality $|x + y| \leq |x| + |y|$ .

Verify by taking different values for $x$ and $y$ (positive or negative).

10) $|x|$ can also be defined as $|x| = \sqrt{x^2}$ $= \max\{x, -x\}$.

**Ex. 10 :** Solve $|4x - 5| \leq 3$.

**Solution :** If $|x| \leq m$ , then $-m \leq x \leq m$

Therefore

$$-3 \leq 4x - 5 \leq 3$$

$$-3 + 5 \leq 4x \leq 3 + 5$$

$$2 \leq 4x \leq 8$$

$$\frac{2}{4} \leq x \leq \frac{8}{4}$$

$$\frac{1}{2} \leq x \leq 2$$

**Ex. 11 :** Find the domain of $\dfrac{1}{\sqrt{||x|-1|-3}}$

**Solution :** As function is defined for $||x|-1|-3>0$

Therefore $||x|-1|>3$

So $|x|-1>3$ or $|x|-1<-3$

That is

$|x|>3+1$ or $|x|<-3+1$

$|x|>4$ or $|x|<-2$

But $|x|<-2$ is not possible as $|x|>0$ always

So $-4<x<4$ , $x \in (-4, 4)$ .

**Ex. 12 :** Solve $|x-1|+|x+2|=8$.

**Solution :** Let $f(x)=|x-1|+|x+2|$

Here the critical points are at $x=1$ and $x=-2$ .

They divide number line into 3 parts, as follows.

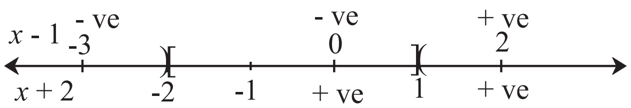



**Fig. 6.44**

| Region | Test Value | Sign | $f(x)$ |
|---|---|---|---|
| I<br>$x<-2$ | $-3$ | $(x-1)<0$,<br>$(x+2)<0$ | $-(x-1)-(x+2)$<br>$=-2x-1$ |
| II<br>$-2\le x\le 1$ | 0 | $(x-1)<0$,<br>$(x+2)>0$ | $-(x-1)+(x+2)$<br>$=3$ |
| III<br>$x>1$ | 2 | $(x-1)>0$<br>$(x+2)>0$ | $(x-1)+(x+2)$<br>$=2x+1$ |

As $f(x)=8$

From I, $-2x-1=8$ $\therefore$ $-2x=9$ $\therefore$ $x=-\dfrac{9}{2}$ .

From II, $3=8$ , which is impossible, hence there is no solution in this region.

From III, $2x+1=8$ $\therefore$ $2x=7$ $\therefore$ $x=\dfrac{7}{2}$.

Solutions are $x=-\dfrac{9}{2}$ and $x=\dfrac{7}{2}$.

**3) Greatest Integer Function (Step Function):**

**Definition:** For every real $x$ , $f(x)=[x]=$ The greatest integer less than or equal to $x$ . $[x]$ is also called as floor function and represented by $\lfloor x \rfloor$.

**Illustrations:**

1) $f(5.7)=[5.7]$ = greatest integer less than or equal to 5.7

Integers less than or equal to 5.7 are 5, 4, 3, 2 of which 5 is the greatest.

2) $f(-6.3)=[-6.3]$ = greatest integer less than or equal to $-6.3$.

Integers less than or equal to $-6.3$ are $-10, -9, -8, -7$ of which $-7$ is the greatest.

$\therefore$ $[-6.3]=-7$

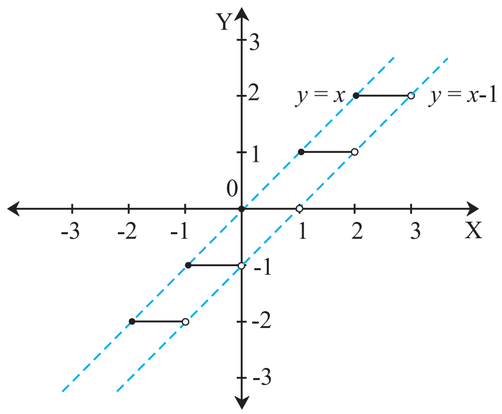
3) $f(2)=[2]$ = greatest integer less than or equal to $2=2$.

4) $[\pi]=3$  5) $[e]=2$

The function can be defined piece-wise as follows

$f(x)=n$, if $n\le x<n+1$ or $x \in [n, n+1)$, $n \in I$

$$f(x)=\begin{cases} -2 & \text{if } -2\le x<-1 \quad \text{or } x\in[-2,-1) \\ -1 & \text{if } -1\le x<0 \quad \text{or } x\in[-1,0) \\ 0 & \text{if } 0\le x<1 \quad \text{or } x\in[0,1) \\ 1 & \text{if } 1\le x<2 \quad \text{or } x\in[1,2) \\ 2 & \text{if } 2\le x<3 \quad \text{or } x\in[2,3) \end{cases}$$




Graph of $f(x)=[x]$

**Fig. 6.45**

Domain = R and Range = I (Set of integers)

**Properties:**

1) If $x \in [2,3)$, $f(x) = 2$ shown by horizontal line. At exactly $x = 2$ , $f(2) = 2$ , $2 \in [2,3)$ hence shown by black disc, whereas $3 \notin [2,3)$ hence shown by white disc.

2) Graph of $y = [x]$ lies in the region bounded by lines $y = x$ and $y = x-1$. So $x-1 \leq [x] < x$

3) $[x] + [-x] = \begin{cases} 0 & \text{if } x \in I \\ -1 & \text{if } x \notin I \end{cases}$

**Ex.** $[3.4] + [-3.4] = 3 + (-4) = -1$ where $3.4 \notin I$

$[5] + [-5] = 5 + (-5) = 0$ where $5 \in I$

4) $[x+n] = [x] + n$, where $n \in I$

**Ex.** $[4.5 + 7] = [11.5] = 11$ and

$[4.5] + 7 = 4 + 7 = 11$

### 4) Fractional part function:

**Definition:** For every real $x$, $f(x) = \{x\}$ is defined as $\{x\} = x - [x]$

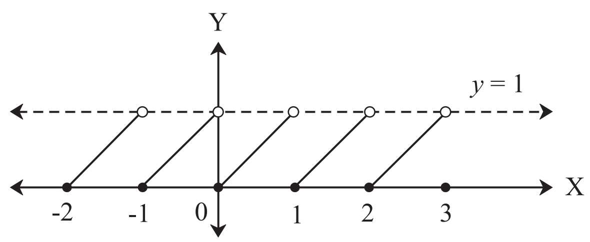
**Illustrations:**

$f(4.8) = \{4.8\} = 4.8 - [4.8] = 4.8 - 4 = 0.8$

$f(-7.1) = \{-7.1\} = -7.1 - [-7.1]$

$= -7.1 - (-8) = -7.1+8 = 0.9$

$f(8) = \{8\} = 8 - [8] = 8 - 8 = 0$




Graph of $f(x) = \{x\}$

**Fig. 6.46**

Domain = R and Range = [0,1)

**Properties:**

1) If $x \in [0,1]$, $f(x) = \{x\} \in [0,1)$ shown by slant line $y = x$ . At $x = 0$, $f(0) = 0$, $0 \in [0,1)$ hence shown by black disc, whereas at $x = 1$, $f(1) = 1$, $1 \notin [0,1)$ hence shown by white disc.

2) Graph of $y = \{x\}$ lies in the region bounded by $y = 0$ and $y = 1$. So $0 \leq \{x\} < 1$

3) $\{x\} + \{-x\} = \begin{cases} 0 & \text{if } x \in I \\ 1 & \text{if } x \notin I \end{cases}$

**Ex. 13:** $\{5.2\}+\{-5.2\} = 0.2+0.8 = 1$ where $5.2 \in 1$

$\{7\} + \{-7\} = 0 + (0) = 0$ where $7 \in I$

4) $\{x \pm n\} = \{x\}$, where $n \in I$

**Ex. 14 :** $\{2.8+5\} = \{7.8\} = 0.8$ and $\{2.8\} = 0.8$

$\{2.8 - 5\} = \{-2.2\} = -2.2 - (-2.2) = -2.2 - (-3)$

$= 0.8$ ($\because$ $\{x\} = x - [x]$)

**Ex. 15 :** If $\{x\}$ and $[x]$ are the fractional part function and greatest integer function of $x$ respectively. Solve for $x$, if $\{x + 1\} + 2x = 4[x + 1] - 6$.

**Solution :** $\{x + 1\} + 2x = 4[x + 1] - 6$

Since $\{x + n\} = \{x\}$ and $[x + n] = [x] + n$, for $n \in I$, also $x = [x] + \{x\}$

$\therefore$ $\{x\} + 2(\{x\} + [x]) = 4([x] + 1) - 6$

$\therefore$ $\{x\} + 2\{x\} + 2[x] = 4[x] + 4 - 6$

$\therefore$ $3\{x\} = 4[x] - 2[x] - 2$

$\therefore$ $3\{x\} = 2[x] - 2$ ... (I)

Since $0 \leq \{x\} < 1$

$\therefore$ $0 \leq 3\{x\} < 3$

$\therefore$ $0 \leq 2[x] - 2 < 3$ ($\because$ from I)

$\therefore$ $0 + 2 \leq 2[x] < 3 + 2$

$\therefore$ $2 \leq 2[x] < 5$

$\therefore$ $\frac{2}{2} \leq [x] < \frac{5}{2}$

$\therefore$ $1 \leq [x] < 2.5$

But as $[x]$ takes only integer values

$[x] = 1, 2$ since $[x] = 1 \Rightarrow 1 \le x < 2$ and $[x] = 2 \Rightarrow 2 \le x < 3$

Therefore $x \in [1,3)$

**Note:**

1)

| Property | $f(x)$ |
|---|---|
| $f(x+y) = f(x) + f(y)$ | $kx$ |
| $f(x+y) = f(x)\,f(y)$ | $a^{kx}$ |
| $f(xy) = f(x)\,f(y)$ | $x^n$ |
| $f(xy) = f(x) + f(y)$ | $\log x$ |

2) If $n(A) = m$ and $n(B) = n$ then

(a) number of functions from A and B is $n^m$ (b) for $m \le n$, number of one-one functions is $\dfrac{n!}{(n-m)!}$

(c) for $m > n$, number of one-one functions is 0

(d) for $m \ge n$, number of onto functions are $n^m - {}^nC_1(n-1)^m + {}^nC_2(n-2)^m - {}^nC_3(n-3)^m + \ldots + (-1)^{n-1}\,{}^nC_{n-1}$

(e) for $m < n$, number of onto functions are 0.

(f) number of constant fuctions is $m$.

3) Characteristic & Mantissa of Common Logarithm $\log_{10} x$ :

As $x = [x] + \{x\}$

$\log_{10} x = [\log_{10} x] + \{\log_{10} x\}$

Where, integral part $[\log_{10} x]$ is called Characteristic & fractional part $\{\log_{10} x\}$ is called Mantissa.

**Illustration :** For $\log_{10} 23$,

$\log_{10} 10 < \log_{10} 23 < \log_{10} 100$

$\log_{10} 10 < \log_{10} 23 < \log_{10} 10^2$

$\log_{10} 10 < \log_{10} 23 < 2\log_{10} 10$

$1 < \log_{10} 23 < 2 \qquad (\because \log_{10}^{10} = 1)$

Then $[\log_{10} 23] = 1$, hence Characteristic of $\log_{10} 23$ is 1.

The characteristic of the logarithm of a number N, with 'm' digits in its integral part is 'm−1'.

**Ex. 16 :** Given that $\log_{10} 2 = 0.3010$, find the number of digits in the number $20^{10}$.

**Solution :** Let $x = 20^{10}$, taking $\log_{10}$ on either sides, we get

$$\begin{aligned}
\log_{10} x &= \log_{10}(20^{10}) = 10\log_{10} 20 \\
&= 10\log_{10}(2\times 10) = 10\{\log_{10} 2 + \log_{10} 10\} \\
&= 10\{\log_{10} 2 + 1\} = 10\{0.3010 + 1\} \\
&= 10\,(1.3010) = 13.010
\end{aligned}$$

That is characteristic of $x$ is 13.

So number of digits in $x$ is $13 + 1 = 14$

## EXERCISE 6.2

1) If $f(x) = 3x + 5$, $g(x) = 6x - 1$, then find

(a) $(f+g)(x)$ (b) $(f-g)(2)$

(c) $(fg)(3)$ (d) $(f/g)(x)$ and its domain.

2) Let $f : \{2,4,5\} \to \{2,3,6\}$ and $g : \{2,3,6\} \to \{2,4\}$ be given by $f = \{(2,3), (4,6), (5,2)\}$ and $g = \{(2,4), (3,4), (6,2)\}$ . Write down $g \circ f$

3) If $f(x) = 2x^2 + 3$, $g(x) = 5x - 2$, then find

(a) $f \circ g$ (b) $g \circ f$

(c) $f \circ f$ (d) $g \circ g$

4) Verify that $f$ and $g$ are inverse functions of each other, where

(a) $f(x) = \dfrac{x-7}{4}$ , $g(x) = 4x + 7$

(b) $f(x) = x^3 + 4$, $g(x) = \sqrt[3]{x-4}$

(c) $f(x) = \dfrac{x+3}{x-2}$ , $g(x) = \dfrac{2x+3}{x-1}$

5) Check if the following functions have an inverse function. If yes, find the inverse function.

(a) $f(x) = 5x^2$ (b) $f(x) = 8$

(c) $f(x) = \dfrac{6x-7}{3}$ (d) $f(x) = \sqrt{4x+5}$

(e) $f(x) = 9x^3+8$

(f) $f(x) = \begin{cases} x+7 & x<0 \\ 8-x & x\geq 0 \end{cases}$

6) If $f(x) = \begin{cases} x^2+3, & x\leq 2 \\ 5x+7, & x>2 \end{cases}$, then find

(a) $f(3)$ (b) $f(2)$ (c) $f(0)$

7) If $f(x) = \begin{cases} 4x-2, & x\leq -3 \\ 5, & -3<x<3 \\ x^2, & x\geq 3 \end{cases}$, then find

(a) $f(-4)$ (b) $f(-3)$

(c) $f(1)$ (d) $f(5)$

8) If $f(x) = 2|x| + 3x$, then find

(a) $f(2)$ (b) $f(-5)$

9) If $f(x) = 4[x] - 3$, where $[x]$ is greatest integer function of $x$, then find

(a) $f(7.2)$ (b) $f(0.5)$

(c) $f\left(-\dfrac{5}{2}\right)$ (d) $f(2\pi)$, where $\pi = 3.14$

10) If $f(x) = 2\{x\} + 5x$, where $\{x\}$ is fractional part function of $x$, then find

(a) $f(-1)$ (b) $f\left(\dfrac{1}{4}\right)$

(c) $f(-1.2)$ (d) $f(-6)$

11) Solve the following for $x$, where $|x|$ is modulus function, $[x]$ is greatest integer function, $[x]$ is a fractional part function.

(a) $|x+4|\geq 5$ (b) $|x-4| + |x-2| = 3$

(b) $x^2 + 7|x| + 12 = 0$ (d) $|x| \leq 3$

(e) $2|x| = 5$

(f) $[x + [x + [x]]] = 9$ (g) $\{x\}>4$

(h) $\{x\} = 0$ (i) $\{x\} = 0.5$

(j) $2\{x\} = x + [x]$

## Let's Remember

- If $f$:A $\to$ B is a function and $f(x) = y$, where $x \in$ A and $y \in$ B, then

  **Domain** of $f$ is A = Set of Inputs = Set of Pre-images = Set of values of $x$ for which $y = f(x)$ is defined = Projection of graph of $f(x)$ on X-axis.

  **Range** of $f$ is $f$ (A) = Set of Outputs = Set of Images = Set of values of y for which $y = f(x)$ is defined = Projection of graph of $f(x)$ on Y-axis.

  **Co-domain** of $f$ is B.

- If $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ then $f$ is **one-one** and for every $y \in$ B, if there exists $x \in$ A such that $f(x) = y$ then $f$ is **onto**.

- If $f$:A $\to$ B. $g$:B $\to$ C then a function $g \circ f$:A $\to$ C is a **composite function**.

- If $f$:A $\to$ B, then $f^{-1}$:B $\to$ A is **inverse function** of $f$.

- If $f$:R $\to$ R is a real valued function of real variable, the following table is formed.

| Type of $f$ | Form of $f$ | Domain of $f$ | Range of $f$ |
|---|---|---|---|
| Constant function | $f(x) = k$ | R | $k$ |
| Identity function | $f(x) = x$ | R | R |
| Square function | $f(x) = x^2$ | R | $[0, \infty)$ or $R^+$ |
| Cube function | $f(x) = x^3$ | R | R |
| Linear function | $f(x) = ax + b$ | R | R |
| Quadratic function | $f(x) = ax^2 + bx + c$ | R | $\left(\dfrac{4ac-b^2}{4a}, \infty\right)$ |
| Cubic function | $f(x) = ax^3 + bx^2 + cx + d$ | R | R |
| Square root funtion | $f(x) = \sqrt{x}$ | $[0, \infty)$ | $[0, \infty)$ or $R^+$ |
| Cube root function | $f(x) = \sqrt[3]{x}$ | R | R |
| Rational function | $f(x) = \dfrac{p(x)}{q(x)}$ | $R - \{x \mid q(x) = 0\}$ | depends on $p(x)$ and $q(x)$ |
| Exponential function | $f(x) = a^x$, $a > 1$ | R | $(0, \infty)$ |
| Logarithmic function | $f(x) = \log_a x$, $a > 1$ | $(0, \infty)$ or $R^+$ | R |
| Absolute function | $f(x) = \lvert x \rvert$ | R | $[0, \infty)$ or $R^+$ |
| Signum function | $f(x) = \text{sgn}(x)$ | R | $\{-1, 0, 1\}$ |
| Greatest Integer function | $f(x) = [x]$ | R | I (set of integers) |
| Fractional Part function | $f(x) = \{x\}$ | R | $[0,1)$ |

## MISCELLANEOUS EXERCISE 6

**(I) Select the correct answer from given alternatives.**

1) If $\log(5x - 9) - \log(x + 3) = \log 2$ then $x =$ ...............

A) 3 B) 5 C) 2 D) 7

2) If $\log_{10}(\log_{10}(\log_{10}x)) = 0$ then $x =$

A) 1000 B) $10^{10}$

C) 10 D) 0

3) Find $x$, if $2\log_2 x = 4$

A) 4, −4 B) 4

C) −4 D) not defined

4) The equation $\log_{x^2} 16 + \log_{2x} 64 = 3$ has,

A) one irrational solution

B) no prime solution

C) two real solutions

D) one integral solution

5) If $f(x) = \dfrac{1}{1-x}$, then $f(f\{f(x)\}]$ is

A) $x - 1$ B) $1 - x$ C) $x$ D) $-x$

6) If $f : R \rightarrow R$ is defined by $f(x) = x^3$ then $f^{-1}$ (8) is euqal to :

A) {2}  B) {–2. 2}

C) {–2}  D) (–2. 2)

7) Let the function $f$ be defined by $f(x) = \frac{2x+1}{1-3x}$ then $f^{-1}(x)$ is:

A) $\frac{x-1}{3x+2}$  B) $\frac{x+1}{3x-2}$

C) $\frac{2x+1}{1-3x}$  C) $\frac{3x+2}{x-1}$

8) If $f(x) = 2x^2 + bx + c$ and $f(0) = 3$ and $f(2) = 1$, then $f(1)$ is equal to

A) –2  B) 0  C) 1  D) 2

9) The domain of $\frac{1}{[x]-x}$ where $[x]$ is greatest integer function is

A) R  B) Z  C) R–Z  D) Q - {o}

10) The domain and range of $f(x) = 2 - |x-5|$ is

A) $R^+, (-\infty, 1]$  B) $R, (-\infty, 2]$

C) $R, (-\infty, 2)$  D) $R^+, (-\infty, 2]$

## (II) Answer the following.

1) Which of the following relations are functions? If it is a function determine its domain and range.

i) {(2, 1), (4, 2), (6, 3), (8, 4), (10, 5), (12, 6), (14, 7)}

ii) {(0, 0), (1, 1), (1, –1), (4, 2), (4, –2), (9, 3), (9, –3), (16, 4), (16, –4)}

iii) {12, 1), (3, 1), (5, 2)}

2) Find whether following functions are one-one.

i) $f: R \rightarrow R$ defined by $f(x) = x^2+5$

ii) $f: R-\{3\} \rightarrow R$ defined by $f(x) = \frac{5x+7}{x-3}$ for $x \in R-\{3\}$

3) Find whether following functions are onto or not.

i) $f: Z \rightarrow Z$ defined by $f(x) = 6x-7$ for all $x \in Z$

ii) $f: R \rightarrow R$ defined by $f(x) = x^2+3$ for all $x \in R$

4) Let $f: R \rightarrow R$ be a function defined by $f(x) = 5x^3-8$ for all $x \in R$, show that f is one-one and onto. Hence find $f^{-1}$.

5) A function $f: R \rightarrow R$ defined by $f(x) = \frac{3x}{5} + 2$, $x \in R$. Show that $f$ is one-one and onto. Hence find $f^{-1}$.

6) A function $f$ is defined as $f(x) = 4x+5$, for $-4 \leq x < 0$. Find the values of $f(-1)$, $f(-2)$, $f(0)$, if they exist.

7) A function $f$ is defined as : $f(x) = 5-x$ for $0 \leq x \leq 4$. Find the value of $x$ such that (i) $f(x) = 3$ (ii) $f(x) = 5$

8) If $f(x) = 3x^4 - 5x^2 + 7$ find $f(x-1)$.

9) If $f(x) = 3x + a$ and $f(1) = 7$ find $a$ and $f(4)$.

10) If $f(x) = ax^2 + bx + 2$ and $f(1) = 3$, $f(4) = 42$, find $a$ and $b$.

11) Find composite of $f$ and g

i) $f$ = {(1, 3), (2, 4), (3, 5), (4, 6)}
$g$ = {(3, 6), (4, 8), (5, 10), (6, 12)}

ii) $f$ = {(1, 1), (2, 4), (3, 4), (4, 3)}
$g$ = {(1, 1), (3, 27), (4, 64)}

12) Find $fog$ and $gof$

i) $f(x) = x^2 + 5$, $g(x) = x-8$

ii) $f(x) = 3x - 2$, $g(x) = x^2$

iii) $f(x) = 256x^4$, $g(x) = \sqrt{x}$

13) If $f(x) = \frac{2x-1}{5x-2}$, $x \neq \frac{5}{2}$

Show that $(fof)(x) = x$.

14) If $f(x) = \frac{x+3}{4x-5}$, $g(x) = \frac{3+5x}{4x-1}$ then show that $(fog)(x) = x$.

15) Let $f: R - \{2\} \rightarrow R$ be defined by $f(x) = \frac{x^2-4}{x-2}$ and $g: R \rightarrow R$ be defined by $g(x) = x + 2$. Ex whether $f = g$ or not.

16) Let $f: R \rightarrow R$ be given by $f(x) = x + 5$ for all $x \in R$. Draw its graph.

17) Let $f: R \rightarrow R$ be given by $f(x) = x^3 + 1$ for all $x \in R$. Draw its graph.

18) For any base show that

$\log(1+2+3) = \log 1 + \log 2 + \log 3$.

19) Find $x$, if $x = 3^{3\log_3 2}$

20) Show that,

$$\log\left|\sqrt{x^2+1}+x\right| + \log\left|\sqrt{x^2+1}-x\right| = 0$$

21) Show that, $\log\frac{a^2}{bc} + \log\frac{b^2}{ca} + \log\frac{c^2}{ab} = 0$

22) Simplify, $\log(\log x^4) - \log(\log x)$.

23) Simplify

$$\log_{10}\frac{28}{45} - \log_{10}\frac{35}{324} + \log_{10}\frac{325}{432} - \log_{10}\frac{13}{15}$$

24) If $\log\left(\frac{a+b}{2}\right) = \frac{1}{2}(\log a + \log b)$, then show that $a=b$

25) If $b^2=ac$. prove that, $\log a + \log c = 2\log b$

26) Solve for $x$, $\log_x(8x-3) - \log_x 4 = 2$

27) If $a^2 + b^2 = 7ab$, show that,

$$\log\left(\frac{a+b}{2}\right) = \frac{1}{2}\log a + \frac{1}{2}\log b$$

28) If $\log\left(\frac{x-y}{5}\right) = \frac{1}{2}\log x + \frac{1}{2}\log y$, show that $x^2 + y^2 = 27xy$.

29) If $\log_3[\log_2(\log_3 x)] = 1$, show that $x = 6561$.

30) If $f(x) = \log(1-x)$, $0 \le x < 1$ show that

$$f\left(\frac{1}{1+x}\right) = f(1-x) - f(-x)$$

31) Without using log tables, prove that

$$\frac{2}{5} < \log_{10} 3 < \frac{1}{2}$$

32) Show that

$$7\log\left(\frac{15}{16}\right) + 6\log\left(\frac{8}{3}\right) + 5\log\left(\frac{2}{5}\right) + \log\left(\frac{32}{25}\right)$$

$= \log 3$

33) Solve : $\sqrt{\log_2 x^4} + 4\log_4\sqrt{\frac{2}{x}} = 2$

34) Find value of $\dfrac{3+\log_{10}343}{2+\frac{1}{2}\log_{10}\left(\frac{49}{4}\right)+\frac{1}{2}\log_{10}\left(\frac{1}{25}\right)}$

35) If $\frac{\log a}{x+y-2z} = \frac{\log b}{x+y-2x} = \frac{\log c}{x+y-2y}$, show that $abc = 1$.

36) Show that, $\log_y x^3 . \log_z y^4 . \log_x z^5 = 60$

37) If $\frac{\log_2 a}{4} = \frac{\log_2 b}{6} = \frac{\log_2 c}{3k}$ and $a^3b^2c = 1$ find the value of k.

38) If $a^2 = b^3 = c^4 d^5$, show that $\log_a bcd = \frac{47}{30}$.

39) Solve the following for $x$, where $|x|$ is modulus function, $[x]$ is greatest interger function, $\{x\}$ is a fractional part function.

a) $1 < |x-1| < 4$  c) $|x^2 - x - 6| = x + 2$

c) $|x^2 - 9| + |x^2 - 4| = 5$

d) $-2 < [x] \le 7$  e) $2[2x-5] - 1 = 7$

f) $[x^2] - 5[x] + 6 = 0$

g) $[x-2] + [x+2] + \{x\} = 0$

h) $\left[\frac{x}{2}\right] + \left[\frac{x}{3}\right] = \frac{5x}{6}$

40) Find the domain of the following functions.

a) $f(x) = \dfrac{x^2+4x+4}{x^2+x-6}$

b) $f(x) = \sqrt{x-3} + \dfrac{1}{\log(5-x)}$

c) $f(x) = \sqrt{1-\sqrt{1-\sqrt{1-x^2}}}$

d) $f(x) = x!$

e) $f(x) = {}^{5-x}P_{x-1}$

f) $f(x) = \sqrt{x-x^2} + \sqrt{5-x}$

g) $f(x) = \sqrt{\log(x^2-6x+6)}$

41) Find the range of the following functions.

a) $f(x) = |x-5|$ b) $f(x) = \dfrac{x}{9+x^2}$

c) $f(x) = \dfrac{1}{1+\sqrt{x}}$ d) $f(x) = [x] - x$

e) $f(x) = 1 + 2^x + 4^x$

42) Find $(f \circ g)\,(x)$ and $(g \circ f)\,(x)$

a) $f(x) = e^x$, $g\,(x) = \log x$

b) $f(x) = \dfrac{x}{x+1}$, $g(x) = \dfrac{x}{1-x}$

43) Find $f\,(x)$ if

a) $g(x) = x^2 + x - 2$ and $(g \circ f)\,(x) = 4x^2 - 10x + 4$

(b) $g(x) = 1 + \sqrt{x}$ and $f[g(x)] = 3 + 2\sqrt{x} + x$.

44) Find $(f \circ f)\,(x)$ if

(a) $f(x) = \dfrac{x}{\sqrt{1+x^2}}$

(b) $f(x) = \dfrac{2x+1}{3x-2}$

❖ ❖ ❖

# 6. FUNCTION

## EXERCISE 6.1

1) a) Yes  b) No
 c) No

2) a) No  b) Yes
 c) No  d) Yes

3) a) Yes  b) No
 c) Yes  d) Yes
 e) No

4) a) 1  b) 19  c) $-\frac{1}{4}$  d) $x^2 - x - 1$
 e) $x^2 + 3x + 1$  f) $h + 1$

5) a) $\frac{6}{5}$  b) $\pm 3$  c) $\frac{1}{2}, -\frac{2}{3}$  d) 1,–2,3

6) a) $0, \pm 3$  b) $\frac{17 \pm \sqrt{33}}{2}$

7) 1) a = –2, b = 2

8) a) $R; \left[-\frac{11}{7}, \infty\right)$  b) $R - \{2\};\ R - \{1\}$
 c) $(-5, \infty); R^+$  d) $R; R$  e) $[2,5]\ ;\ [0, \frac{3}{2}]$
 f) $[3,7)\ ;\ [0,\infty]$  g) $[-4,4]\ ;\ [0,4]$

9) a) $A = s^2$  b) $A = \frac{p^2}{16}$

10) a) $A = \pi r^2$  b) $A = \frac{\pi d^2}{4}$  c) $A = \frac{c^2}{4\pi}$

11) $x(30 - 2x)^2$ ; (0,15)

12) Not a function; f (0) has 2 values.

13) a) Injective but not surjective
 b) neither injective nor surjective
 c) Surjective but not injective
 d) injective but not surjective
 e) injective and surjective

16) $\frac{3}{16}$

17) a) $5 = \log_2 32$  b) $0 = \log_{54} 1$
 c) $1 = \log_{23} 23$  d) $\frac{3}{2} = \log_9 27$
 e) $-4 = \log_3 \left(\frac{1}{81}\right)$  f) $-2 = \log_{10} 0.01$
 g) $\ln 7.3890 = 2$  h) $\ln 1.6487 = \frac{1}{2}$
 i) $\ln 6 = -x$

18) a) $2^6 = 64$  b) $\frac{1}{25} = 5^{-2}$  c) $0.001 = 10^{-3}$
 d) $8 = \left(\frac{1}{2}\right)^{-3}$  e) $e^0 = 1$  f) $e^1 = e$  g) $\frac{1}{2} = e^{-0.693}$

19) a) $(5,\infty)$  b) $(-\infty,2) \cup (3,\infty)$

20) a) $\log p + \log q - \log r - \log s$
 b) $\frac{1}{2} \log x + \frac{1}{3} \log y$
 c) $3 \ln a + 2 \ln (a - 2) - \frac{1}{2} \ln (b^2 + 5)$
 d) $2\left[\frac{1}{3} \ln(x-2) + 4\ln(2x+1) - \ln(x+4) - \frac{1}{2}\ln(2x+4)\right]$

21) a) $\log\left(\frac{x^5 y^7}{z}\right)$  b) $\log\left(\sqrt[3]{x} - 2\sqrt{x}\right)$
 c) $\ln\left[\frac{x^2 - 4}{(x+5)^3}\right]$

22) $\frac{5a+b}{2}$

24) $a = \frac{15}{4}, b = 9$

25) a) 3  b) 11, –1  c) 8  d) 1

## EXERCISE 6.2

1) a) $9x+4$ b) 0 c) 238

d) $\dfrac{3x+5}{6x-1}$; $R-\left\{\dfrac{1}{6}\right\}$

2) {(2,4), (4,2), (5,4)}

3) a) $50x^2-40x+11$ b) $10x^2+13$

c) $8x^4+24x^2+21$

d) $25x-12$

5) a) $f^{-1}$ does not exist

b) $f^{-1}$ doesn not exist

c) $f^{-1}(x)=\dfrac{3x+7}{6}$

d) $f^{-1}$ does not exist

e) $f^{-1}=\sqrt[3]{\dfrac{x-8}{9}}$

f) $f^{-1}$ does not exist

6) a) 22 b) 7 c) 3

7) a) −18 b) −14 c) 5 d) 25

8) a) 10 b) −5

9) a) 25 b) −3 c) −15 d) 21

10) a) −5 b) 1.75 c) −4.4. d) 42

11) a) $(-\infty,-9]$, $[1,\infty)$ b) 1·5, 4·5

c) { } d) [−3,3]

f) $3+r$ ; $0\le r<1$ g) { }

h) N, Z i) $n+0.5$, $n\in Z$

j) $x=0$

## MISCELLANEOUS EXERCISE - 6

(I)

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| B | B | B | C | C | A | A | B | C | B |

(II)

1) i) Function ; {2,4,6,8,10,12,14};

{1,2,3,4,5,6,7}

ii) Not a function

iii) Function ; {2,3,5}; {1,2}

2) i) not one one ii) one one

3) i) not onto ii) not onto

4) $f^{-1}(x)=\left(\dfrac{x+8}{5}\right)^{\frac{1}{3}}$

5) $f^{-1}(x)=\dfrac{5(x-2)}{3}$

6) 1,−3, does not exist

7) i) 2 ii) 0

8) $3x^4-12x^3+13x^2-2x+5$

9) a = 4, $f(4)=16$

10) a = 3, b = −2

11) i) $g\circ f=\{(1,6),(2,8),(3,10),(4,12)\}$

ii) $g\circ f=\{(1,1),(2,64),(3,64),(4,27)\}$

12) i) $f\circ g=x^2-16x+69$, $gof=x^2-3$

ii) $f\circ g=3x^2-2$, $gog=9x^2-12x+4$

iii) $f\circ g=256x^2$, $gof=16x^2$

15) $f\neq g$

19) 8 22) log4

23) $\log_{10}5$

26) $\dfrac{3}{2}$, $\dfrac{1}{2}$ 33) 2

34) 3 37) −8

39) a) (−3,0) , (2,5) b) {−2,2,4}

c) [−3,2] , [2,3] d) [−7,7]

e) $[\dfrac{13}{5},7)$ f) [2,4)

g) $x = 0$ h) $x = 6k$ , $k \geq 0$

40) a) $(-\infty,-3) \cup (-3,2) \cup (2,\infty)$

b) $[3,4) \cup (4,5)$

c) $[-1,1]$ d) W

e) $\{1,2,3\}$ f) $[0,1]$

g) $(-\infty,3-\sqrt{3}) \cup (3+\sqrt{3},\infty)$

41) a) $[0,\infty)$ b) $[-\frac{1}{6}, \frac{1}{6}]$

c) $(0,\infty)$ d) $(-1,0]$

e) $(1,\infty)$

42) a) $\text{fog}(x) = x = \text{gof}(x)$

b) $\text{fog}(x) = x = \text{gof}(x)$

43) a) $f(x) = 2x - 3$

b) $f(x) = x^2 + 2$

44) a) $\frac{x}{\sqrt{1+2x^2}}$ b) $x$

# 7. LIMITS

## EXERCISE 7.1

I) 1) $-\frac{1}{\sqrt{3}}$ 2) 15 3) $-\frac{1}{25}$

II) 1) $\frac{2\sqrt{3}}{3}$ 2) $-\frac{3}{16}$ 3) $\frac{3}{125}$ 4) $\pm\frac{2}{\sqrt{3}}$

III) 1) $\frac{n(n+1)}{2}$ 2) $\frac{2}{3\sqrt[3]{7}}$ 3) 4 4) 4

5) $-\frac{1}{6}$ 6) 24 7) $\frac{3\sqrt{a+2}}{2}$

8) $294\sqrt{7}$ 9) $n^2$

## EXERCISE 7.2

I) 1) $-\frac{1}{4}$ 2) $-\frac{1}{2}$ 3) $-\frac{1}{2}$ 4) $-\frac{1}{2}$ 5) 8

II) 1) $\frac{4}{3}$ 2) 0 3) 0 4) $2x-2$ 5) $-3$

6) Does not exist

III) 1) 3 2) $-2$ 3) $\frac{1}{2}$ 4) 0 5) $-\frac{3}{a^2}$

## EXERCISE 7.3

I) 1) $\frac{1}{2\sqrt{6}}$ 2) $-\frac{1}{18}$ 3) $-1$ 4) $2\sqrt{2}$

II) 1) $\frac{2}{3\sqrt{3}}$ 2) $-8$ 3) $\frac{1}{8\sqrt{3}}$ 4) $-\frac{1}{2a}$ 5) $-\frac{2}{3}$

III) 1) $\frac{7}{2}$ 2) 1 3) 24 4) $-\frac{1}{3}$ 5) $\frac{1}{3}$

## EXERCISE 7.4

I) 1) $\frac{m}{n}$ 2) 0 3) 2 4) $\frac{1}{2}$

II) 1) $\frac{n}{m}$ 2) $-\frac{1}{4}$ 3) $\frac{1}{\sqrt{2}}$

III) 1) $\frac{a^2-b^2}{c^2}$ 2) $-\frac{1}{4\sqrt{2}}$ 3) $2\sqrt{2}$ 4) $-3$

## EXERCISE 7.5

I) 1) $\frac{1}{2}$ 2) $5a^{\frac{4}{5}}.\cos a$ 3) $\frac{1}{8}$

4) $\frac{1}{3}$ 5) $\frac{2}{\pi}$

II) 1) $-\frac{1}{2\sqrt{3}}$ 2) $\frac{1}{16\sqrt{2}}$ 3) $\frac{1}{36}$

4) $\frac{\cos\sqrt{a}}{2\sqrt{a}}$ 5) $-\frac{1}{2}$