2. APPLICATIONS OF DERIVATIVES

Let us Study

- Applications of Drivatives to Tangents and Normals
- Derivative as a rate measure
- Approximations
- Rolle's Theorem and Lagrange's Mean Value Theorem.
- Increasing and Decreasing Functions
- Maxima and Minima

Let us Recall

- Continuous functions.
- Derivatives of Composite, Inverse Trigonometric, Logarithmic, Parametric functions.
- Relation between derivative and slope.
- Higher Order Derivatives.

2.1.1 Introduction :

In the previous chapter we have studied the derivatives of various functions such as composite functions, Inverse Trigonometric functions, Logarithmic functions etc. and also the relation between Derivative and slope of the tangent. In this chapter we are going to study various applications of differentiation such as application to (i) Geometry, (ii) Rate measure (iii) Approximations (iv) Rolle's Theorem and Lagrange's Mean Value Therorem (v) Increasing and Decreasing functions and (vi) Maxima and Minima.

Let us Learn

2.1.2 Application of Derivative in Geometry :

In the previous chapter we have studied the relation between derivative and slope of a line or slope of a tangent to the curve at a given point on it.

Let $y = f(x)$ be a continuous function of x representing a curve in XY- plane and $P(x_1, y_1)$ be any point on the curve.

Then $\left[\dfrac{dy}{dx}\right]_{(x_1,\, y_1)} = [f'(x)]_{(x_1,\, y_1)}$ represents slope, also called gradient, of the tangent to the curve at $P(x_1, y_1)$. The normal is perpendicular to the tangent. Hence, the slope of the normal at P will be the negative of reciprocal of the slope of tangent at P. Let m and m' be the slopes of tangent and normal respectively, then $m = \left[\frac{dy}{dx}\right]_{(x_1, y_1)}$ and $m' = -\frac{1}{\left[\frac{dy}{dx}\right]_{(x_1, y_1)}}$ if $\left[\frac{dy}{dx}\right]_{(x_1, y_1)} \neq 0$.

Equation of tangent at $P(x_1, y_1)$ is given by $y - y_1 = m(x - x_1)$ i.e. $y - y_1 = \left[\frac{dy}{dx}\right]_{(x_1, y_1)} (x - x_1)$

and equation of normal at $P(x_1, y_1)$ is given by

$y - y_1 = m'(x - x_1)$ where $m' = -\frac{1}{\left[\frac{dy}{dx}\right]_{(x_1, y_1)}}$

SOLVED EXAMPLES

Ex. 1 : Find the equations of tangent and normal to the curve at the given point on it.

(i) $y = 2x^3 - x^2 + 2$ at $\left(\frac{1}{2}, 2\right)$

(ii) $x^3 + 2x^2y - 9xy = -2$ at $(2, 1)$

(iii) $x = 2\sin^3\theta$, $y = 3\cos^3\theta$ at $\theta = \frac{\pi}{4}$

Solution :

(i) Given that : $y = 2x^3 - x^2 + 2$

Differentiate *w. r. t. x*

$$\frac{dy}{dx} = \frac{d}{dx}(2x^3 - x^2 + 2) = 6x^2 - 2x$$

Slope of tangent at $\left(\frac{1}{2}, 2\right) = m = 6\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right)$

$$\therefore \quad m = \frac{1}{2}$$

Slope of normal at $\left(\frac{1}{2}, 2\right) = m' = -2$

Equation of tangent is given by

$$y - 2 = \frac{1}{2}\left(x - \frac{1}{2}\right) \Rightarrow 2y - 4 = \frac{2x - 1}{2}$$

$$4y - 8 = 2x - 1 \Rightarrow 2x - 4y + 7 = 0$$

Equation of normal is given by

$$y - 2 = -2\left(x - \frac{1}{2}\right) \Rightarrow y - 2 = -2x + 1$$

$$2x + y - 3 = 0$$

(ii) Given that : $x^3 + 2x^2y - 9xy = -2$

Differentiate *w. r. t. x*

$$3x^2 + 2\left(x^2\frac{dy}{dx} + y\frac{d}{dx}(x^2)\right) - 9\left(x\frac{dy}{dx} + y\frac{d}{dx}(x)\right) = 0$$

$$3x^2 + 2x^2\frac{dy}{dx} + 4xy - 9x\frac{dy}{dx} - 9y = 0$$

$$(2x^2 - 9x)\frac{dy}{dx} = 9y - 4xy - 3x^2 \therefore \frac{dy}{dx} = \frac{9y - 4xy - 3x^2}{2x^2 - 9x}$$

Slope of tangent at $(2, 1)$

$$\left(\frac{dy}{dx}\right)_{(2,\,1)} = m = \frac{9(1) - 4(2)(1) - 3(4)}{2(4) - 9(1)} = \frac{9 - 8 - 12}{8 - 9}$$

$$m = \frac{-11}{-1} \quad \therefore m = 11$$

Slope of normal at $(2, 1) = m' = -\frac{1}{11}$

Equation of tangent is given by

$$y - 1 = 11(x - 2) \Rightarrow 11x - y - 21 = 0$$

Equation of normal is given by

$$y - 1 = -\frac{1}{11}(x - 2) \Rightarrow 11y - 11 = -x + 2$$

$$x + 11y - 13 = 0$$

(iii) Given that : $y = 3\cos^3\theta$

Differentiate *w. r. t.* θ

$$\frac{dy}{d\theta} = 3\frac{d}{d\theta}(\cos\theta)^3 = 9\cos^2\theta\frac{d}{d\theta}(\cos\theta)$$

$$\therefore \quad \frac{dy}{d\theta} = -9\cos^2\theta\sin\theta$$

Now, $x = 2\sin^3\theta$

Differentiate *w. r. t.* θ

$$\frac{dx}{d\theta} = 2\frac{d}{d\theta}(\sin\theta)^3 = 6\sin^2\theta\frac{d}{d\theta}(\sin\theta)$$

$$\therefore \quad \frac{dx}{d\theta} = 6\sin^2\theta\cos\theta$$

We know that

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = -\frac{9\cos^2\theta\sin\theta}{6\sin^2\theta\cos\theta} = -\frac{3}{2}\cot\theta$$

Slope of tangent at $\theta = \frac{\pi}{4}$ is

$$\left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{4}} = m = -\frac{3}{2}\cot\left(\frac{\pi}{4}\right) = -\frac{3}{2}$$

Slope of normal at $\left(\theta = \frac{\pi}{4}\right) = m' = \frac{2}{3}$

When, $\theta = \frac{\pi}{4}$

$$x = 2\sin^3\left(\frac{\pi}{4}\right) = 2\left(\frac{1}{\sqrt{2}}\right)^3 = \frac{1}{\sqrt{2}}$$

$$y = 3\cos^3\left(\frac{\pi}{4}\right) = 3\left(\frac{1}{\sqrt{2}}\right)^3 = \frac{3}{2\sqrt{2}}$$

$\therefore$ The point is $P = \left(\frac{1}{\sqrt{2}}, \frac{3}{2\sqrt{2}}\right)$

Equation of tangent at P is given by

$$y - \frac{3}{2\sqrt{2}} = -\frac{3}{2}\left(x - \frac{1}{\sqrt{2}}\right) \Rightarrow y - \frac{3}{2\sqrt{2}} = -\frac{3x}{2} + \frac{3}{2\sqrt{2}}$$

$$\frac{3x}{2} + y - \frac{3}{\sqrt{2}} = 0 \qquad \text{i.e. } 3x + 2y - 3\sqrt{2} = 0$$

Equation of normal is given by

$$y - \frac{3}{2\sqrt{2}} = \frac{2}{3}\left(x - \frac{1}{\sqrt{2}}\right) \Rightarrow y - \frac{3}{2\sqrt{2}} = \frac{2x}{2} - \frac{2}{3\sqrt{2}}$$

$$\frac{2x}{3} - y - \frac{2}{3\sqrt{2}} + \frac{3}{2\sqrt{2}} = 0$$

i.e. $4\sqrt{2}x - 6\sqrt{2}y + 5 = 0$ . . . [Multiply by $6\sqrt{2}$]

Ex. 2 : Find points on the curve given by $y = x^3 - 6x^2 + x + 3$ where the tangents are parallel to the line $y = x + 5$.

Solution : Equation of curve is $y = x^3 - 6x^2 + x + 3$

Differentiate *w. r. t.* x

$$\frac{dy}{dx} = \frac{d}{dx}(x^3 - 6x^2 + x + 3) = 3x^2 - 12x + 1$$

Given that the tangent is parallel to $y = x + 5$ whose slope is 1.

$\therefore$ Slope of tangent $= \dfrac{dy}{dx} = 1 \Rightarrow 3x^2 - 12x + 1 = 1$

$3x(x - 4) = 0$ so, $x = 0$ or $x = 4$

When $x = 0$, $y = (0)^3 - 6(0)^2 + (0) + 3 = 3$

When $x = 4$, $y = (4)^3 - 6(4)^2 + (4) + 3 = -25$

So the required points on the curve are $(0, 3)$ and $(4, -25)$.

2.1.3 Derivative as a Rate measure :

If $y = f(x)$ is the given function then a change in x from x_1 to x_2 is generally denoted by $\delta x = x_2 - x_1$ and the corresponding change in y is denoted by $\delta y = f(x_2) - f(x_1)$. The difference quotient $\dfrac{\delta y}{\delta x} = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$ is called the **average rate of change** with respect to x. This can also be interpreted geometrically as the slope of the secant line joining the points $P(x_1, f(x_1))$ and $Q(x_2, f(x_2))$ on the graph of function $y = f(x)$.

Consider the average rate of change over smaller and smaller intervals by letting x_2 to approach x_1 and therefore letting δx to approach 0. The limit of these average rates of change is called the **instantaneous rate of change** of y with respect to x at $x = x_1$, which is interpreted as the slope of the tangent to the curve $y = f(x)$ at $P(x_1, f(x_1))$. Therefore instantaneous rate of change is given by

$$\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \lim_{x_2 \to x_1}\left(\frac{f(x_2) - f(x_1)}{x_2 - x_1}\right)$$

We recognize this limit as being the derivative of $f(x)$ at $x = x_1$, i.e. $f'(x_1)$. We know that one interpretation of the derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect x when $x = a$. The other interpretation is $f(x)$ at $f'(a)$ is the slope of the tangent to $y = f(x)$ at $(a, f(a))$.

SOLVED EXAMPLES

Ex. 1 : A stone is dropped in to a quiet lake and waves in the form of circles are generated, radius of the circular wave increases at the rate of 5 cm/ sec. At the instant when the radius of the circular wave is 8 cm, how fast the area enclosed is increasing ?

Solution : Let R be the radius and A be the area of the circular wave.

$\therefore \quad A = \pi \cdot R^2$

Differentiate *w. r. t. t*

$$\frac{dA}{dt} = \pi \frac{d}{dt}(R^2)$$

$$\frac{dA}{dt} = 2\pi R \frac{dR}{dt} \qquad \ldots \text{(I)}$$

Given that $\frac{dR}{dt} = 5$ cm/sec.

Thus when $R = 8$ cm, from (I) we get,

$$\frac{dA}{dt} = 2\pi(8)(5) = 80\pi$$

Hence when the radius of the circular wave is 8 cm, the area of the circular wave is increasing at the rate of 80π cm^2/ sec.

Ex. 2 : The volume of the spherical ball is increasing at the rate of 4π cc/sec. Find the rate at which the radius and the surface area are changing when the volume is 288π cc.

Solution : Let R be the radius, S be the surface area and V be the volume of the spherical ball.

$$V = \frac{4}{3}\pi R^3 \qquad \ldots \text{(I)}$$

Differentiate *w. r. t. t*

$$\frac{dV}{dt} = \frac{4\pi}{3} \cdot \frac{d}{dt}(R^3)$$

$$4\pi = \frac{4\pi}{3} \cdot 3R^2 \frac{dR}{dt} \qquad \ldots \left[\text{Given } \frac{dV}{dt} = 4\pi \text{ cc/sec}\right]$$

$$\frac{dR}{dt} = \frac{1}{R^2} \qquad \ldots \text{(II)}$$

When volume is 288π cc.

i.e. $\frac{4}{3}\pi \cdot R^3 = 288\pi$ we get, $R^3 = 216 \Rightarrow R = 6$ ... [From (I)]

From (II) we get, $\frac{dR}{dt} = \frac{1}{36}$

So, the radius of the spherical ball is increasing at the rate of $\frac{1}{36}$ cc/sec.

Now, $S = 4\pi R^2$

Differentiate *w. r. t. t.*

$$\frac{dS}{dt} = 4\pi \frac{d}{dt}(R^2) = 8\pi R \frac{dR}{dt}$$

So, when $R = 6$ cm

$$\left[\frac{dS}{dt}\right]_{R=6} = 8\pi(6)\frac{1}{36} = \frac{4\pi}{3}$$

$\therefore$ Surface area is increasing at the rate of $\frac{4\pi}{3}$ cm^2/ sec.

Ex. 3 : Water is being poured at the rate of 36 m^3/sec in to a cylindrical vessel of base radius 3 meters. Find the rate at which water level is rising.

Solution : Let R be the radius of the base, H be the height and V be the volume of the cylindrical vessel at any time t. R, V and H are functions of t.

$$V = \pi R^2 H$$

$$V = \pi(3)^2 H = 9\pi H \quad \ldots [\text{ Given : } R = 3]$$

Differentiate $w.\ r.\ t.\ t$

$$\frac{dV}{dt} = 9\pi \frac{dH}{dt}$$

$$\frac{dH}{dt} = \frac{1}{9\pi} \cdot \frac{dV}{dt} \quad \ldots (I)$$

Given that,

$$\frac{dV}{dt} = 36 \text{ m}^3/\text{sec} \quad \ldots (II)$$

From (I) we get, $\quad \dfrac{dH}{dt} = \dfrac{1}{9\pi} \cdot (36) = \dfrac{4}{\pi}$

$\therefore$ Water level is rising at the rate of $\dfrac{4}{\pi}$ meter/sec.

Ex. 4 : A man of height 180 cm is moving away from a lamp post at the rate of 1.2 meters per second. If the height of the lamp post is 4.5 meters, find the rate at which (i) his shadow is lengthening. (ii) the tip of the shadow is moving.

Solution : Let OA be the lamp post, MN be the man, $MB = x$ be the length of shadow and $OM = y$ be the distance of the man from the lamp post at time t. Given that man is moving away from the lamp post at the rate of 1.2 meter/sec. x and y are functions of t.

Hence $\dfrac{dy}{dt} = 1.2$. The rate at which shadow is lengthening $= \dfrac{dx}{dt}$.

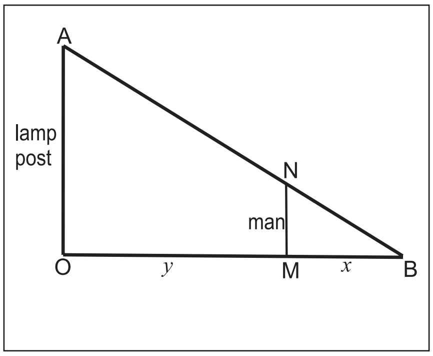
B is the tip of the shadow and it is at a distance of $(x + y)$ from the post.

$\dfrac{x}{1.8} = \dfrac{x+y}{4.5}$ i.e. $45x = 18x + 18y$ i.e. $27x = 18y$

$\therefore \quad x = \dfrac{2y}{3}$

Differentiate $w.\ r.\ t.\ t$

$$\frac{dx}{dt} = \frac{2}{3} \times \frac{dy}{dt} = \frac{2}{3} \times 1.2 = 0.8 \text{ meter/sec.}$$



rate at which the tip of the shadow is moving is given by

$$\frac{d}{dt}(x+y) = \frac{dx}{dt} + \frac{dy}{dt}$$

$$\therefore \quad \frac{d}{dt}(x+y) = 0.8 + 1.2 = 2 \text{ meter/sec.}$$

Shadow is lengthening at the rate of 0.8 meter/ sec. and its tip is moving at the rate of 2 meters/sec.

2.1.4 Velocity, Acceleration and Jerk :

If $s = f(t)$ is the desplacement function of a particle that moves along a straight line, then $f'(t)$ is the rate of change of the displacement s with respect to the time t. In other words, $f'(t)$ is the **velocity** of the particle. The **speed** of the particle is the absolute value of the velocity, that is, $|f'(t)|$.

The rate of change of velocity with respect to time is valled the **acceleration** of the particle denoted by $a(t)$. Thus the acceleration function is the derivative of the velocity function and is therefore the second derivative of the position function $s = f(t)$.

Thus, $a = \dfrac{dy}{dt} = \dfrac{d^2s}{dt^2}$ i.e. $a(t) = v'(t) = s''(t)$.

Let us consider the third derivative of the position function $s = f(t)$ of an object that moves along a straight line. $s'''(t) = v''(t) = a'(t)$ is derivative of the acceleration function and is called the **Jerk** (j).

Thus, $j = \dfrac{da}{dt} = \dfrac{d^3s}{dt^3}$. Hence the jerk j is the rate of change of acceleration. It is aptly named because a jerk means a sudden change in acceleration, which causes an abrupt movement in a vehicle.

SOLVED EXAMPLES

Ex. 1 : A car is moving in such a way that the distance it covers, is given by the equation $s = 4t^2 + 3t$ where s is in meters and t is in seconds. What would be the velocity and the acceleration of the car at time $t = 20$ second ?

Solution : Let v be the velocity and a be the acceleration of the car.

Distance traveled by the car is given by

$s = 4t^2 + 3t$

Differentiate *w. r. t. t.*

$\therefore$ Velocity of the car is given by

$$v = \frac{ds}{dt} = \frac{d}{dt}(4t^2 + 3t) = 8t + 3 \quad \ldots \text{(I)}$$

and Acceleration of the car is given by

$$a = \frac{d}{dt}\left(\frac{dv}{dt}\right) = \frac{d}{dt}(8t + 3) = 8 \quad \ldots \text{(II)}$$

Put $t = 20$ in (I),

$\therefore$ Velocity of the car, $v_{t=20} = 8(20) + 3 = 163$ m/sec.

Put $t = 20$ in (II),

$\therefore$ Acceleration of the car, $a_{t=20} = 8$ m/sec^2.

Note : In this problem, the acceleration is independent of time. Such a motion is said to be uniformly accelerated motion.

Ex. 2 : The displacement of a particle at time t is given by $s = 2t^3 - 5t^2 + 4t - 3$. Find the time when the acceleration is 14 ft/ sec^2, the velocity and the displacement at that time.

Solution : Displacement of a particle is given by

$$s = 2t^3 - 5t^2 + 4t - 3 \quad \ldots \text{(I)}$$

Differentiate *w. r. t. t.*

$$\text{Velocity, } v = \frac{ds}{dt} = \frac{d}{dt}(2t^3 - 5t^2 + 4t - 3)$$

$$\therefore \quad v = 6t^2 - 10t + 4 \quad \ldots \text{(II)}$$

$$\text{Acceleration, } a = \frac{dv}{dt} = \frac{d}{dt}(6t^2 - 10t + 4)$$

$$\therefore \quad a = 12t - 10 \quad \ldots \text{(III)}$$

Given : Acceleration = 14 ft/ sec^2.

$\therefore$ $12t - 10 = 14 \Rightarrow 12t = 24 \Rightarrow t = 2$

So, the particle reaches an acceleration of 14 ft/ sec^2 in 2 seconds.

Velocity, when $t = 2$ is

$\therefore$ $v_{t=2} = 6(2)^2 - 10(2) + 4 = 8$ ft/ sec.

Displacement when $t = 2$ is

$\therefore$ $s_{t=2} = 2(2)^3 - 5(2)^2 + 4(2) - 3 = 1$ foot.

Hence the velocity is 8 ft/ sec and the displacement is 1 foot after 2 seconds.

EXERCISE 2.1

(1) Find the equations of tangents and normals to the curve at the point on it.

(i) $y = x^2 + 2e^x + 2$ at $(0, 4)$

(ii) $x^3 + y^3 - 9xy = 0$ at $(2, 4)$

(iii) $x^2 - \sqrt{3}xy + 2y^2 = 5$ at $(\sqrt{3}, 2)$

(iv) $2xy + \pi \sin y = 2\pi$ at $\left(1, \frac{\pi}{2}\right)$

(v) $x \sin 2y = y \cos 2x$ at $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

(vi) $x = \sin \theta$ and $y = \cos 2\theta$ at $\theta = \frac{\pi}{6}$

(vii) $x = \sqrt{t}$, $y = t - \frac{1}{\sqrt{t}}$ at $t = 4$.

(2) Find the point on the curve $y = \sqrt{x-3}$ where the tangent is perpendicular to the line $6x + 3y - 5 = 0$.

(3) Find the points on the curve $y = x^3 - 2x^2 - x$ where the tangents are parallel to $3x - y + 1 = 0$.

(4) Find the equations of the tangents to the curve $x^2 + y^2 - 2x - 4y + 1 = 0$ which are parallel to the X-axis.

(5) Find the equations of the normals to the curve $3x^2 - y^2 = 8$, which are parallel to the line $x + 3y = 4$.

(6) If the line $y = 4x - 5$ touches the curve $y^2 = ax^3 + \text{b}$ at the point (2, 3) find a and b.

(7) A particle moves along the curve $6y = x^3 + 2$ Find the points on the curve at which y-coordinate is changing 8 times as fast as the X-coordinate.

(8) A spherical soap bubble is expanding so that its radius is increasing at the rate of 0.02 cm/sec. At what rate is the surface area is increasing, when its radius is 5 cm?

(9) The surface area of a spherical balloon is increasing at the rate of 2 cm^2/ sec. At what rate the volume of the balloon is increasing when radius of the balloon is 6 cm?

(10) If each side of an equilateral triangle increases at the rate of $\sqrt{2}$ cm/ sec, find the rate of increase of its area when its side of length 3 cm .

(11) The volume of a sphere increase at the rate of 20 cm^3/ sec. Find the rate of change of its surface area when its radius is 5 cm.

(12) The edge of a cube is decreasing at the rate of 0.6 cm/sec. Find the rate at which its volume is decreasing when the edge of the cube is 2 cm.

(13) A man of height 2 meters walks at a uniform speed of 6 km/hr away from a lamp post of 6 meters high. Find the rate at which the length of the shadow is increasing.

(14) A man of height 1.5 meters walks toward a lamp post of height 4.5 meters, at the rate of $\left(\frac{3}{4}\right)$ meter/sec. Find the rate at which

(i) his shadow is shortening. (ii) the tip of the shadow is moving.

(15) A ladder 10 meter long is leaning against a vertical wall. If the bottom of the ladder is pulled horizontally away from the wall at the rate of 1.2 meters per second, find how fast the top of the ladder is sliding down the wall when the bottom is 6 meters away from the wall.

(16) If water is poured into an inverted hollow cone whose semi-vertical angel is 30°, so that its depth (measured along the axis) increases at the rate of 1 cm/ sec. Find the rate at which the volume of water increasing when the depth is 2 cm.

2.2.1 Approximations

If $f(x)$ is a differentiable function of x, then its derivative at $x = a$ is given by

$$f'(a) = \lim_{h \to 0} \left[\frac{f(a+h) - f(a)}{h}\right]$$

Here we use $\doteqdot$ sign for approximation.

For a sufficiently small h we have,

$$f'(a) \doteqdot \left[\frac{f(a+h) - f(a)}{h}\right]$$

i.e. $\quad h f'(a) \doteqdot f(a+h) - f(a)$

$\therefore \quad f(a+h) \doteqdot f(a) + h f'(a)$

This is the formula to find the approximate value of the function at $x = a + h$, when $f'(a)$ exists. Let us solve some problems by using this formula.

SOLVED EXAMPLES

Ex. 1 : Find the approximate value of $\sqrt{64.1}$.

Solution :

Let $f(x) = \sqrt{x}$ $\quad \ldots$ (I)

Differentiate *w. r. t. x.*

$f'(x) = \dfrac{1}{2\sqrt{x}}$ $\quad \ldots$ (II)

Let $a = 64$, $h = 0.1$

For $x = a = 64$, from (I) we get

$f(a) = f(64) = \sqrt{64} = 8$ $\quad \ldots$ (III)

For $x = a = 64$, from (II) we get

$f'(a) = f'(64) = \dfrac{1}{2\sqrt{64}} = \dfrac{1}{16}$

$\therefore \quad f'(a) = 0.0625$ $\quad \ldots$ (IV)

We have, $f(a+h) \doteqdot f(a) + h f'(a)$

$f(64 + 0.1) \doteqdot f(64) + (0.1) \cdot f'(64)$

$f(64.1) \doteqdot 8 + (0.1) \cdot (0.0625) \ldots$ [From (III) and (IV)]

$\doteqdot 8 + 0.00625$

$\therefore \quad f(64.1) = \sqrt{64.1} \doteqdot 8.00625$

Ex. 2 : Find the approximate value of $(3.98)^3$.

Solution :

Let $f(x) = x^3$ $\quad \ldots$ (I)

Differentiate *w. r. t. x.*

$f'(x) = 3x^2$ $\quad \ldots$ (II)

Let $a = 4$, $h = -0.02$

For $x = a = 4$, from (I) we get

$f(a) = f(4) = (4)^3 = 64$ $\quad \ldots$ (III)

For $x = a = 4$, from (II) we get

$f'(a) = f'(4) = 3(4)^2 = 48$ $\quad \ldots$ (IV)

We have, $f(a+h) \doteqdot f(a) + h f'(a)$

$f[4 + (-0.02)] \doteqdot f(4) + (-0.02) \cdot f'(4)$

$f(3.98) \doteqdot 64 + (-0.02).(48)$ $\quad \ldots$ [From (III) and (IV)]

$f(3.98) \doteqdot 64 - 0.96$

$\therefore \quad f(3.98) = (3.98)^3 \doteqdot 63.04$

Ex. 3 : Find the approximate value of $\sin(30° \, 30')$. Given that $1° = 0.0175^c$ and $\cos 30° = 0.866$.

Solution : Let $f(x) = \sin x$ . . . (I)

Differentiate $w.\ r.\ t.\ x.$

$f'(x) = \cos x$

Now, $30° \, 30' = 30° + 30' = 30° + \left(\frac{1}{2}\right)^°$

$= \frac{\pi}{6} + \frac{0.1750^c}{2}$

$30° \, 30' = \frac{\pi}{6} + 0.00875$ . . . (II)

Let $a = \frac{\pi}{6}$, $h = 0.00875$

For $x = a = \frac{\pi}{6}$, from (I) we get

$f(a) = f\left(\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2} = 0.5$ . . . (III)

For $x = a = \frac{\pi}{6}$, from (II) we get

$f'(a) = f'\left(\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right) = 0.866$ . . . (IV)

We have, $f(a+h) \doteqdot f(a) + h f'(a)$

$f\left(\frac{\pi}{6} + 0.00875^c\right) \doteqdot f\left(\frac{\pi}{6}\right) + (0.00875)\cdot f'\left(\frac{\pi}{6}\right)$

$f(30° \, 30') \doteqdot 0.5 + (0.00875)\cdot(0.866)$ ... . . . [From (III) and (IV)]

$\doteqdot 0.5 + 0.0075775$

$\therefore \ f(30° \, 30') = \sin(30° \, 30') \doteqdot 0.5075775$

Ex. 4 : Find the approximate value of $\tan^{-1}(0.99)$, Given that $\pi \doteqdot 3.1416$.

Solution : Let $f(x) = \tan^{-1} x$ . . . (I)

Differentiate $w.\ r.\ t.\ x.$

$f'(x) = \frac{1}{1+x^2}$ . . . (II)

Let $a = 1$, $h = -0.01$

For $x = a = 1$, from (I) we get

$f(a) = f(1) = \tan^{-1}(1) = \frac{\pi}{4}$ . . . (III)

For $x = a = 1$, from (II) we get

$f'(a) = f'(1) = \frac{1}{1+1^2} = 0.5$ . . . (IV)

We have, $f(a+h) \doteqdot f(a) + h f'(a)$

$f[(1) + (-0.01)] \doteqdot f(1) + (-0.01)\cdot f'(1)$

$f(0.99) \doteqdot \frac{\pi}{4} - (0.01)\cdot(0.5)$ . . . [From (III) and (IV)]

$\doteqdot \frac{\pi}{4} - 0.005$

$\doteqdot \frac{3.1416}{4} - 0.005$

$\doteqdot 0.7854 - 0.005 = 0.7804$

$\therefore \ f(0.99) = \tan^{-1}(0.99) \doteqdot 0.7804$

Ex. 5 : Find the approximate value of $e^{1.005}$. Given that $e = 2.7183$.

Solution : Let $f(x) = e^x$ . . . (I)

Differentiate $w.\ r.\ t.\ x.$

$f'(x) = e^x$ . . . (II)

Let $a = 1$, $h = 0.005$

For $x = a = 1$, from (I) we get

$f(a) = f(1) = e^1 = 2.7183$ . . . (III)

For $x = a = 1$, from (II) we get

$f'(a) = f'(1) = e^1 = 2.7183$ . . . (IV)

We have, $f(a+h) \doteqdot f(a) + h f'(a)$

$f(1 + 0.005) \doteqdot f(1) + (0.005)\cdot f'(1)$

$f(1.005) \doteqdot 2.7183 + (0.005)(2.7183)$ ... . . . [From (III) and (IV)]

$f(1.005) \doteqdot 2.7183 + 0.0135915$

$\doteqdot 2.7318915$

$f(1.005) = e^{1.005} \doteqdot 2.73189$

Ex. 6 : Find the approximate value of $\log_{10}(998)$. Given that $\log_{10} e = 0.4343$.

Solution : Let $f(x) = \log_{10} x = \dfrac{\log x}{\log 10}$

$\therefore \quad f(x) = (\log_{10} e) \cdot \log x \quad \ldots \text{(I)}$

Differentiate w. r. t. x.

$f'(x) = \dfrac{\log_{10} e}{x} = \dfrac{0.4343}{x} \quad \ldots \text{(II)}$

Let $a = 1000$, $h = -2$

For $x = a = 1000$, from (I) we get

$f(a) = f(1000) = \log_{10} 1000$

$\therefore \quad f(a) = 3\log_{10} 10 = 3 \quad \ldots \text{(III)}$

For $x = a = 1000$, from (II) we get

$f'(a) = f'(1000) = \dfrac{0.4343}{1000}$

$\therefore \quad f'(a) = 0.0004343 \quad \ldots \text{(IV)}$

We have, $f(a+h) \doteqdot f(a) + h f'(a)$

$f[1000 + (-2)] \doteqdot f(1000) + (-2) f'(1000)$

$f(998) \doteqdot 3 - (2)(0.0004343) \ldots$ [From (III) and (IV)]

$\doteqdot 3 - 0.0008686$

$f(998) = \log(998) \doteqdot 2.9991314$

Ex. 7 : Find the approximate value of $f(x) = x^3 + 5x^2 - 2x + 3$ at $x = 1.98$.

Solution : Let $f(x) = x^3 + 5x^2 - 2x + 3 \quad \ldots \text{(I)}$

Differentiate w. r. t. x.

$f'(x) = 3x^2 + 10x - 2 \quad \ldots \text{(II)}$

Let $a = 2$, $h = -0.02$

For $x = a = 2$, from (I) we get

$f(a) = f(2) = (2)^3 + 5(2)^2 - 2(2) + 3$

$\therefore \quad f(a) = 27 \quad \ldots \text{(III)}$

For $x = a = 2$, from (II) we get

$f'(a) = f'(2) = 3(2)^2 + 10(2) - 2$

$\therefore \quad f'(a) = 30 \quad \ldots \text{(IV)}$

We have, $f(a+h) \doteqdot f(a) + h f'(a)$

$f[(2) + (-0.02)] \doteqdot f(2) + (-0.02) \cdot f'(2)$

$f(1.98) \doteqdot 27 - (0.02) \cdot (30) \ldots$ [From (III) and (IV)]

$\doteqdot 27 - 0.6$

$f(1.98) \doteqdot 26.4$

EXERCISE 2.2

(1) Find the approximate value of given functions, at required points.

(i) $\sqrt{8.95}$ (ii) $\sqrt[3]{28}$ (iii) $\sqrt[5]{31.98}$

(iv) $(3.97)^4$ (v) $(4.01)^3$

(2) Find the approximate value of

(i) $\sin(61°)$ given that $1° = 0.0175^c$, $\sqrt{3} = 1.732$

(ii) $\sin(29° 30')$ given that $1° = 0.0175^c$, $\sqrt{3} = 1.732$

(iii) $\cos(60° 30')$ given that $1° = 0.0175^c$, $\sqrt{3} = 1.732$

(iv) $\tan(45° 40')$ given that $1° = 0.0175^c$.

(3) Find the approximate value of

(i) $\tan^{-1}(0.999)$ (ii) $\cot^{-1}(0.999)$

(iii) $\tan^{-1}(1.001)$

(4) Find the approximate value of

(i) $e^{0.995}$ (ii) $e^{2.1}$ given that $e^2 = 7.389$

(iii) $3^{2.01}$ given that $\log 3 = 1.0986$

(5) Find the approximate value of

(i) $\log_e(101)$ given that $\log_e 10 = 2.3026$

(ii) $\log_e(9.01)$ given that $\log 3 = 1.0986$

(iii) $\log_{10}(1016)$ given that $\log_{10} e = 0.4343$

(6) Find the approximate value of

(i) $f(x) = x^3 - 3x + 5$ at $x = 1.99$

(ii) $f(x) = x^3 + 5x^2 - 7x + 10$ at $x = 1.12$

2.3.1 Rolle's Theorem or Rolle's Lemma :

If a real-valued function f is continous on $[a, b]$, differentiable on the open interval (a, b) and $f(a) = f(b)$, then there exists at least one c in the open interval (a, b) such that $f'(c) = 0$.

Rolle's Theorem essentially states that any real-valued differentiable function that attains equal values at two distinct points on it, must have at least one stationary point somewhere in between them, that is, a point where the first derivative (the slope of the tangent line to the graph of the function) is zero.

Geometrical Significance :

Let $f(x)$ be a real valued function defined on $[a, b]$ and it is continuous on $[a, b]$. This means that we can draw the graph $f(x)$ between the values $x = a$ and $x = b$. Also $f(x)$ is differentiable on (a, b) which means the graph of $f(x)$ has a tangent at each point of (a, b). Now the existence of real number $c \in (a, b)$ such that $f'(c) = 0$ shows that the tangent to the curve at $x = c$ has slope zero, that is, tangent is parallel to X-axis since $f(a) = f(b)$.

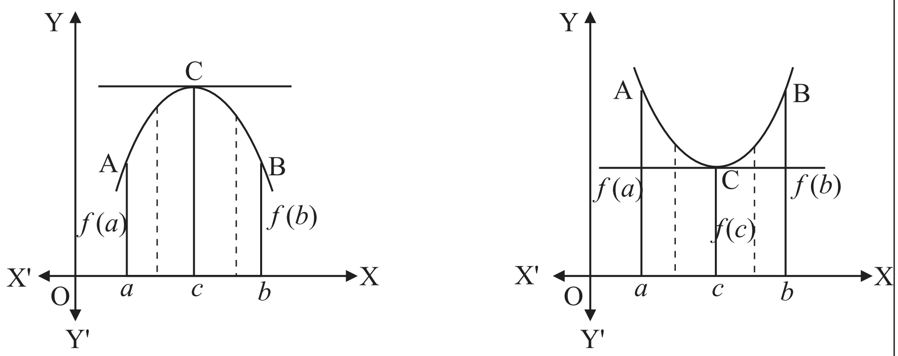


Fig. 2.3.1

SOLVED EXAMPLES

Ex. 1 : Check whether conditions of Rolle's theorem are satisfied by the following functions.

(i) $f(x) = 2x^3 - 5x^2 + 3x + 2,\ x \in \left[0, \frac{3}{2}\right]$ (ii) $f(x) = x^2 - 2x + 3,\ x \in [1, 4]$

Solution :

(i) Given that $\quad f(x) = 2x^3 - 5x^2 + 3x + 2 \quad \ldots$ (I)

$f(x)$ is a polynomial which is continuous on $\left[0, \frac{3}{2}\right]$ and it is differentiable on $\left(0, \frac{3}{2}\right)$.

Let $a = 0$, and $b = \frac{3}{2}$,

For $x = a = 0$ from (I) we get,

$f(a) = f(0) = 2(0)^3 - 5(0)^2 + 3(0) + 2 = 2$

For $x = b = \left(\frac{3}{2}\right)$ from (I) we get,

$$f(b) = f\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^3 - 5\left(\frac{3}{2}\right)^2 + 3\left(\frac{3}{2}\right) + 2 = \frac{54}{8} - \frac{45}{4} + \frac{9}{2} + 2$$

$$f(b) = f\left(\frac{3}{2}\right) = \frac{54 - 90 + 36}{8} + 2 = 2$$

So, here $f(a) = f(b)$ i.e. $f(0) = f\left(\frac{3}{2}\right) = 2$

Hence conditions of Rolle's Theorem are satified.

(ii) Given that $f(x) = x^2 - 2x + 3$ . . . (I)

$f(x)$ is a polynomial which is continuous on [1, 4] and it is differentiable on (1, 4).

Let $a = 1$, and $b = 4$

For $x = a = 1$ from (I) we get,

$f(a) = f(1) = (1)^2 - 2(1) + 3 = 2$

For $x = b = 4$ from (I) we get,

$f(b) = f(4) = (4)^2 - 2(4) + 3 = 11$

So, here $f(a) \neq f(b)$ i.e. $f(1) \neq f(4)$

Hence conditions of Rolle's theorem are not satisfied.

Ex. 2 : Verify Rolle's theorem for the function $f(x) = x^2 - 4x + 10$ on [0, 4].

Solution :

Given that $f(x) = x^2 - 4x + 10$ . . . (I)

$f(x)$ is a polynomial which is continuous on [0, 4] and it is differentiable on (0, 4).

Let $a = 0$, and $b = 4$

For $x = a = 0$ from (I) we get,

$f(a) = f(0) = (0)^2 - 4(0) + 10 = 10$

For $x = b = 4$ from (I) we get,

$f(b) = f(4) = (4)^2 - 4(4) + 10 = 10$

So, here $f(a) = f(b)$ i.e. $f(0) = f(4) = 10$

All the conditions of Rolle's theorem are satisfied.

To get the value of c, we should have

$f'(c) = 0$ for some $c \in (0, 4)$

Differentiate (I) *w. r. t. x.*

$f'(x) = 2x - 4 = 2(x - 4)$

Now, for $x = c$,

$f'(c) = 0 \Rightarrow 2(c - 2) = 0 \Rightarrow c = 2$

Also $c = 2 \in (0, 4)$

Thus Rolle's theorem is verified.

Ex. 3 : Given an interval $[a, b]$ that satisfies hypothesis of Rolle's theorem for the function $f(x) = x^3 - 2x^2 + 3$. It is known that $a = 0$. Find the value of b.

Solution :

Given that $f(x) = x^3 - 2x^2 + 3$ . . . (I)

Let $g(x) = x^3 - 2x^2 = x^2(x - 2)$

From (I), $f(x) = g(x) + 3$

We see that $g(x)$ becomes zero for $x = 0$ and $x = 2$.

We observe that for $x = 0$,

$f(0) = g(0) + 3 = 3$

and for $x = 2$,

$f(2) = g(2) + 3 = 3$

$\therefore$ We can write that $f(0) = f(2) = 3$

It is obvious that the function $f(x)$ is everywhere continuous and differentiable as a cubic polynomial. Consequently, it satisfies all the conditions of Rolle's theorem on the interval [0, 2].

So $b = 2$.

Ex. 4 : Verify Rolle's theorem for the function $f(x) = e^x(\sin x - \cos x)$ on $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$.

Solution : Given that, $f(x) = e^x(\sin x - \cos x)$ . . . (I)

We know that e^x, $\sin x$ and $\cos x$ are continuous and differentiable on their domains. Therefore $f(x)$ is continuous and differentiable on $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$ and $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ respectively.

Let $a = \frac{\pi}{4}$, and $b = \frac{5\pi}{4}$

For $x = a = \frac{\pi}{4}$ from (I) we get,

$$f(a) = f\left(\frac{\pi}{4}\right) = e^{\frac{\pi}{4}}\left[\sin\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{4}\right)\right] = e^{\frac{\pi}{4}}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = 0$$

For $x = b = \left(\frac{5\pi}{4}\right)$ from (I) we get,

$$f(a) = f\left(\frac{5\pi}{4}\right) = e^{\frac{5\pi}{4}}\left[\sin\left(\frac{5\pi}{4}\right) - \cos\left(\frac{5\pi}{4}\right)\right] = e^{\frac{5\pi}{4}}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = 0$$

$\therefore$ $f(a) = f(b)$ i.e. $f\left(\frac{\pi}{4}\right) = f\left(\frac{5\pi}{4}\right)$.

All the conditions of Rolle's theorem are satisfied.

To get the value of c, we should have $f'(c) = 0$ for some $c \in \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$.

Differentiate (I) *w. r. t. x.*

$f'(x) = e^x(\cos x + \sin x) + (\sin x - \cos x)\, e^x = 2e^x \sin x$

Now, for $x = c$, $f'(c) = 0 \Rightarrow 2e^c \sin c = 0$. As $e^c \neq 0$ for any $c \in R$

$\sin c = 0 \Rightarrow c = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

It is clearly seen that $\pi \in \left[\frac{\pi}{4}, \frac{5\pi}{4}\right] \therefore c = \pi$

Thus Rolle's theorem is verified.

2.3.2 Lagrange's Mean Value Theorem (LMVT) :

If a real-valued function f is continous on a closed $[a, b]$ and differentiable on the open interval (a, b) then there exists at least one c in the open interval (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Lagrange's mean value theorem states, that for any real-valued diffenentiable function which is continuous at the two end points, there is at least one point at which the tangent is parallel to the the secant through its end points.

Geometrical Significance :

Draw the curve $y = f(x)$ (see Figure 2.3.2) and take the end points $A\,(a, f(a))$ and $B\,(b, f(b))$ on the curve, then

Slope of the chord $AB = \dfrac{f(b) - f(a)}{b - a}$

Since by statement of Lagrange's Mean Value Theorem

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$f'(c)$ = Slope of the chord AB.

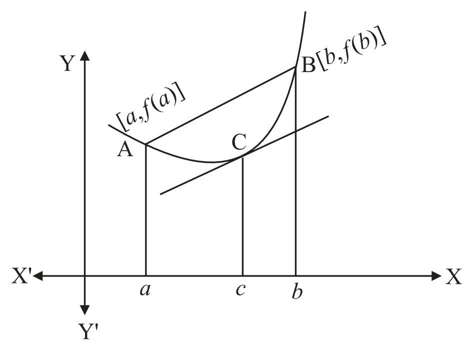


Fig. 2.3.2

This shows that the tangent to the curve $y = f(x)$ at the point $x = c$ is parallel to the chord AB.

SOLVED EXAMPLES

Ex. 1 : Verify Lagrange's mean value theorem for the function $f(x) = \sqrt{x + 4}$ on the interval $[0, 5]$.

Solution : Given that $f(x) = \sqrt{x+4}$ . . . (I)

The function $f(x)$ is continuous on the closed interval $[0, 5]$ and differentiable on the open interval $(0, 5)$, so the LMVT is applicable to the function.

Differentiate (I) *w. r. t. x*.

$$f'(x) = \frac{1}{2\sqrt{x+4}} \quad \ldots \text{(II)}$$

Let $a = 0$ and $b = 5$

From (I), $f(a) = f(0) = \sqrt{0+4} = 2$

$f(b) = f(5) = \sqrt{5+4} = 3$

Let $c \in (0, 5)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\frac{1}{2\sqrt{c+4}} = \frac{3-2}{5-0} = \frac{1}{5}$$

$$\therefore \sqrt{c+4} = \frac{5}{2} \Rightarrow c + 4 = \frac{25}{4} \;\therefore\; c = \frac{9}{4} \in (0, 5)$$

Thus Lagrange's Mean Value Theorem is verified.

Ex. 2 : Verify Lagrange's mean value theorem for the function $f(x) = x + \dfrac{1}{x}$ on the interval $[1, 3]$.

Solution : Given that $f(x) = x + \dfrac{1}{x}$ . . . (I)

The function $f(x)$ is continuous on the closed interval $[1, 3]$ and differentiable on the open interval $(1, 3)$, so the LMVT is applicable to the function.

Differentiate (I) *w. r. t. x*.

$$f'(x) = 1 - \frac{1}{x^2} \quad \ldots \text{(II)}$$

Let $a = 1$ and $b = 3$

From (I), $f(a) = f(1) = 1 + \dfrac{1}{1} = 2$

$f(b) = f(3) = 3 + \dfrac{1}{3} = \dfrac{10}{3}$

Let $c \in (1, 3)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$1 - \frac{1}{c^2} = \frac{\frac{10}{3} - 2}{3 - 1}$$

$$1 - \frac{1}{c^2} = \frac{\frac{4}{3}}{2} = \frac{2}{3}$$

$\therefore\; c^2 = 3 \Rightarrow c = \pm\sqrt{3}$

$\therefore\; c = \sqrt{3} \in (1, 3)$ and $c = -\sqrt{3} \notin (1, 3)$

EXERCISE 2.3

(1) Check the validity of the Rolle's theorem for the following functions.

(i) $f(x) = x^2 - 4x + 3, x \in [1, 3]$

(ii) $f(x) = e^{-x} \sin x, x \in [0, \pi]$

(iii) $f(x) = 2x^2 - 5x + 3, x \in [1, 3]$

(iv) $f(x) = \sin x - \cos x + 3, x \in [0, 2\pi]$

(v) $f(x) = x^2$ if $0 \le x \le 2$
$\quad = 6 - x$ if $2 \le x \le 6$

(vi) $f(x) = x^{\frac{2}{3}}, x \in [-1, 1]$

(2) Given an interval $[a, b]$ that satisfies hypothesis of Rolle's thorem for the function $f(x) = x^4 + x^2 - 2$. It is known that $a = -1$. Find the value of b.

(3) Verify Rolle's theorem for the following functions.

(i) $f(x) = \sin x + \cos x + 7, x \in [0, 2\pi]$

(ii) $f(x) = \sin\left(\frac{x}{2}\right), x \in [0, 2\pi]$

(iii) $f(x) = x^2 - 5x + 9, x \in [1, 4]$

(4) If Rolle's theorem holds for the function $f(x) = x^3 + px^2 + qx + 5, x \in [1, 3]$ with $c = 2 + \frac{1}{\sqrt{3}}$, find the values of p and q.

(5) Rolle's theorem holds for the function $f(x) = (x - 2) \log x, x \in [1, 2]$, show that the equation $x \log x = 2 - x$ is satisfied by at least one value of x in $(1, 2)$.

(6) The function $f(x) = x(x + 3) e^{-\frac{x}{2}}$ satisfies all the conditions of Rolle's theorem on $[-3, 0]$. Find the value of c such that $f'(c) = 0$.

(7) Verify Lagrange's mean value theorem for the following functions.

(i) $f(x) = \log x$, on $[1, e]$

(ii) $f(x) = (x - 1)(x - 2)(x - 3)$ on $[0, 4]$

(iii) $f(x) = x^2 - 3x - 1, x \in \left[-\frac{11}{7}, \frac{13}{7}\right]$

(iv) $f(x) = 2x - x^2, x \in [0, 1]$

(v) $f(x) = \frac{x-1}{x-3}$ on $[4, 5]$

2.4.1 Increasing and decreasing functions :

Increasing functions :

Definition : A function f is said to be a monotonically (or strictly) increasing function on an interval (a, b) if for any $x_1, x_2 \in (a, b)$ with if $x_1 < x_2$, we have $f(x_1) < f(x_2)$.

Consider an increasing function $y = f(x)$ in (a, b). Let $h > 0$ be a small increment in x then,

$$x < x + h \qquad [x = x_1, x + h = x_2]$$

$$f(x) < f(x + h) \qquad [f(x_1) < f(x_2)]$$

$$\therefore \quad f(x + h) > f(x)$$

$$\therefore \quad f(x + h) - f(x) > 0$$

$$\therefore \quad \frac{f(x + h) - f(x)}{h} > 0$$

$$\therefore \quad \lim_{h \to 0}\left[\frac{f(x + h) - f(x)}{h}\right] \ge 0$$

$$\therefore \quad f'(x) \ge 0$$

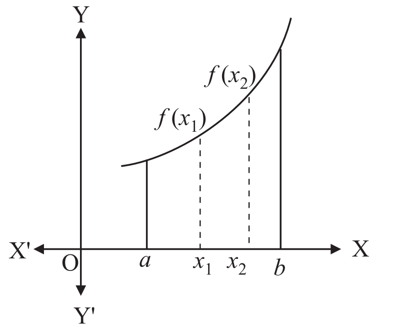


Fig. 2.4.1

If $f'(a) > 0$, then in a small δ-neighborhood of a i.e. $(a - \delta, a + \delta)$, we have f strictly increasing if

$$\frac{f(a+h)-f(a)}{h} > 0 \qquad \text{for } |h| < \delta$$

Hence if $0 < h < \delta$, $f(a+h) - f(a) > 0$ and $f(a-h) - f(a) < 0$

Thus for $0 < h < \delta$, $f(a-h) < f(a) < f(a+h)$

Decreasing functions :

Definition : A function f is said to be a monotonically (strictly) decreasing function on an interval (a, b) if for any $x_1, x_2 \in (a, b)$ with $x_1 < x_2$, we have $f(x_1) > f(x_2)$.

Consider a decreasing function $y = f(x)$ in (a, b). Let $h > 0$ be a small increment in x then,

$$x + h > x \qquad [x = x_1, x + h = x_2]$$

$$f(x) > f(x+h) \qquad [f(x_1) > f(x_2)]$$

$$\therefore \quad f(x+h) < f(x)$$

$$\therefore \quad f(x+h) - f(x) < 0$$

$$\therefore \quad \frac{f(x+h)-f(x)}{h} < 0$$

$$\therefore \quad \lim_{h \to 0}\left[\frac{f(x+h)-f(x)}{h}\right] \le 0$$

$$\therefore \quad f'(x) \le 0$$

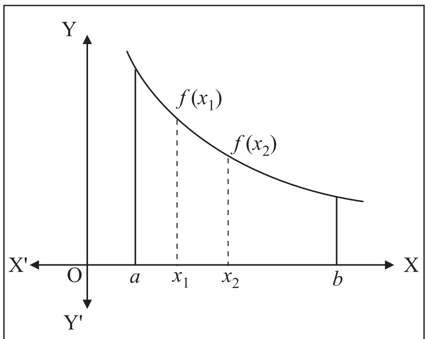


Fig. 2.4.2

If $f'(a) < 0$, then in a small δ-neighborhood of a i.e. $(a - \delta, a + \delta)$, we have f strictly decreasing

because $$\frac{f(a+h)-f(a)}{h} < 0 \qquad \text{for } |h| < \delta$$

Hence for $0 < h < \delta$, $f(a-h) > f(a) > f(a+h)$

Note : Whenever $f'(x) = 0$, at that point the tangent is parallel to X-axis, we cannot deduce that whether $f(x)$ is increasing or decreasing at that point.

SOLVED EXAMPLES

Ex. 1: Show that the function $f(x) = x^3 + 10x + 7$ for $x \in R$ is strictly increasing.

Solution : Given that $f(x) = x^3 + 10x + 7$

Differentiate *w. r. t. x*.

$f'(x) = 3x^2 + 10$

Here, $3x^2 \ge 0$ for all $x \in R$ and $10 > 0$.

$\therefore \quad 3x^2 + 10 > 0 \Rightarrow f'(x) > 0$

Thus $f(x)$ is a strictly increasing function.

Ex. 2: Test whether the function $f(x) = x^3 + 6x^2 + 12x - 5$ is increasing or decreasing for all $x \in R$.

Solution : Given that $f(x) = x^3 + 6x^2 + 12x - 5$

Differentiate *w. r. t. x*.

$f'(x) = 3x^2 + 12x + 12 = 3(x^2 + 4x + 4)$

$f'(x) = 3(x + 2)^2$

$3(x + 2)^2$ is always positive for $x \neq -2$

$\therefore f'(x) \ge 0$ for all $x \in R$

Hence $f(x)$ is an increasing function for all $x \in R$.

Ex. 3: Find the values of x, for which the funciton $f(x) = x^3 + 12x^2 + 36x + 6$ is (i) monotonically increasing. (ii) monotonically decreasing.

Solution : Given that $f(x) = x^3 + 12x^2 + 36x + 6$

Differentiate *w. r. t. x*.

$$f'(x) = 3x^2 + 24x + 36$$

$$= 3(x^2 + 8x + 12)$$

$$f'(x) = 3(x+2)(x+6)$$

(i) $f(x)$ is monotonically increasing if $f'(x) > 0$

i.e. $3(x+2)(x+6) > 0$, $(x+2)(x+6) > 0$

then either $(x+2) < 0$ and $(x+6) < 0$ OR $(x+2) > 0$ and $(x+6) > 0$

Case (I) : $x + 2 < 0$ and $x + 6 < 0$

$x < -2$ and $x < -6$

Thus for every $x < -6$, $(x+2)(x+6) > 0$, hence f is monotonically increasing.

Case (II) : $x + 2 > 0$ and $x + 6 > 0$

$x > -2$ and $x > -6$

Thus for every $x > -2$, $(x+2)(x+6) > 0$ and f is monotonically increasing.

$\therefore$ From Case (I) and Case (II), $f(x)$ is monotonically increasing if and only if $x < -6$ or $x > -2$.

Hence, $x \in (\infty, -6)$ or $x \in (-2, \infty) \Rightarrow f$ is monotonically increasing.

(ii) $f(x)$ is said to be monotonically decreasing if $f'(x) < 0$

i.e. $3(x+2)(x+6) < 0$, $(x+2)(x+6) < 0$

then either $(x+2) < 0$ and $(x+6) > 0$ OR $(x+2) > 0$ and $(x+6) < 0$

Case (I) : $x + 2 < 0$ and $x + 6 > 0$

$x < -2$ and $x > -6$

Thus for $x \in (-6, -2)$, f is monotonically decreasing.

Case (II) : $x + 2 > 0$ and $x + 6 < 0$

$x > -2$ and $x < -6$

$\therefore$ This case does not arise. . . . [check. why ?]

2.4.2 Maxima and Minima :

Maxima of a function $f(x)$: A function $f(x)$ is said to have a maxima at $x = c$ if the value of the function at $x = c$ is greater than any other value of $f(x)$ in a δ-neighborhood of c. That is for a small $\delta > 0$ and for $x \in (c-\delta, c+\delta)$ we have $f(c) > f(x)$. The value $f(c)$ is called a Maxima of $f(x)$. Thus the function $f(x)$ will have maxima at $x = c$ if $f(x)$ is increasing in $c - \delta < x < c$ and decreasing in $c < x < c + \delta$.

Minima of a function $f(x)$: A function $f(x)$ is said to have a minima at $x = c$ if the value of the function at $x = c$ is less than any other value of $f(x)$ in a δ-neighborhood of c. That is for a small $\delta > 0$ and for $x \in (c-\delta, c+\delta)$ we have $f(c) < f(x)$. The value $f(c)$ is called a Minima of $f(x)$. Thus the function $f(x)$ will have minima at $x = c$ if $f(x)$ is decreasing in $c - \delta < x < c$ and increasing in $c < x < c + \delta$.

If $f'(c) = 0$ then at $x = c$ the function is neither increasing nor decrasing, such a point on the curve is called **turning point** or **stationary point** of the function. Any point at which the tangent to the graph is horizontal is a turning point. We can locate the turn points by looking for points at which $\frac{dy}{dx} = 0$. At these points if the function has Maxima or Minima then these are called extreme values of the function.

Note : The maxima and the minima of a function are not necessarily the greatest and the least values of the function in the whole domain. Actually these are the greatest and the least values of the function in a small interval. Hence the maxima or the minima defined above are known as **local (or relative) maximum** and **the local (or relative) minimum** of the function $f(x)$.

To find the extreme values of the function let us use following tests.

2.4.3 First derivative test :

A function $f(x)$ has a maxima at $x = c$ if

(i) $f'(c) = 0$

(ii) $f'(c-h) > 0$ [$f(x)$ is increasing for values of $x < c$]

(iii) $f'(c+h) < 0$ [$f(x)$ is decreasing for values of $x > c$]

where h is a small positive number.

A function $f(x)$ has a minima at $x = c$ if

(i) $f'(c) = 0$

(ii) $f'(c-h) < 0$ [$f(x)$ is decreasing for values of $x < c$]

(iii) $f'(c+h) > 0$ [$f(x)$ is increasing for values of $x > c$]

where h is a small positive number.

Note : If $f'(c) = 0$ and $f'(c-h) > 0, f'(c+h) > 0$ or $f'(c-h) < 0, f'(c+h) < 0$ then $f(x)$ in neither maxima nor minima. In such a case $x = c$ is called a **point of inflexion**. e.g. $f(x) = x^3$, $f(x) = x^5$ in $[-2, 2]$.

SOLVED EXAMPLES

Ex. 1: Find the local maxima or local minima of $f(x) = x^3 - 3x$.

Solution : Given that $f(x) = x^3 - 3x$ . . . (I)

Differentiate (I) *w. r. t. x*.

$f'(x) = 3x^2 - 3 = 3(x^2 - 1)$ . . . (II)

For extreme values, $f'(x) = 0$

$3x^2 - 3 = 0$ i.e. $3(x^2 - 1) = 0$

i.e. $x^2 - 1 = 0 \Rightarrow 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

The turning points are $x = 1$ and $x = -1$

Let's consider the turning point, $x = 1$

Let $x = 1 - h$ for a small, $h > 0$, from (II) we get,

$f'(1 - h) = 3[(1 - h)^2 - 1] = 3(1 - 2h + h^2 - 1) = 3h(h - 2)$

$\therefore$ $f'(1 - h) < 0$ . . . [since, $h > 0, h - 2 < 0$]

$\therefore$ $f'(x) > 0$ for $x = 1 - h \Rightarrow f(x)$ is decreasing for, $x > 1$.

Now for $x = 1 + h$ for a small, $h > 0$, from (II) we get,

$f'(1 + h) = 3[(1 + h)^2 - 1] = 3(1 + 2h + h^2 - 1) = 3(h^2 + 2h)$

$\therefore$ $f'(1 + h) > 0$ . . . [since, $h > 0, h^2 + 2h > 0$]

$\therefore$ $f'(x) < 0$ for $x = 1 + h \Rightarrow f(x)$ is increasing for, $x < 1$.

$\therefore$ $f'(x) < 0$ for $1 - h < x < 1$

$\therefore$ $f'(x) > 0$ for $1 < x < 1 + h$.

$\therefore$ $x = 1$ is the point of local minima.

Minima of $f(x)$, is $f(1) = 1^3 - 3(1) = -2$

Now, let's consider the turning point, $x = -1$

Let $x = -1 - h$ for a small, $h > 0$, from (II) we get,

$\therefore$ $f'(-1 - h) = 3[(-1 - h)^2 - 1] = 3(1 + 2h + h^2 - 1) = 3(h^2 + 2h)$

$\therefore$ $f'(-1 - h) > 0$ . . . [since, $h > 0, h^2 + 2h > 0$]

$\therefore$ $f'(x) > 0$ for $x = -1 - h \Rightarrow f(x)$ is increasing for, $x < -1$.

Now for $x = -1 + h$ for a small, $h > 0$, from (II) we get,

$\therefore$ $f'(-1 + h) = 3[(-1 + h)^2 - 1] = 3(1 - 2h + h^2 - 1) = -3h(2 - h)$

$\therefore$ $f'(-1 + h) < 0$ . . . [since, $h > 0, 2 - h > 0$]

$\therefore$ $f'(x) < 0$ for $x = -1 + h \Rightarrow f(x)$ is decreasing for, $x > -1$.

$\therefore$ $f'(x) > 0$ for $-1 - h < x < -1$

$\therefore$ $f'(x) > 0$ for $-1 < x < -1 + h$.

$\therefore$ $x = -1$ is the point of local maxima.

Maxima of $f(x)$, is $f(-1) = (-1)^3 - 3(-1) = -1 + 3 = 2$

Hence, Maxima of $f(x)$ is 2 and Minima of $f(x)$ is -2

2.4.4 Second derivative test :

A function $f(x)$ has a maxima at $x = c$ if $f'(c) = 0$ and $f''(c) < 0$

A function $f(x)$ has a minima at $x = c$ if $f'(c) = 0$ and $f''(c) > 0$

Note : If $f''(c) = 0$ then second derivative test fails so, you may try using first derivative test.

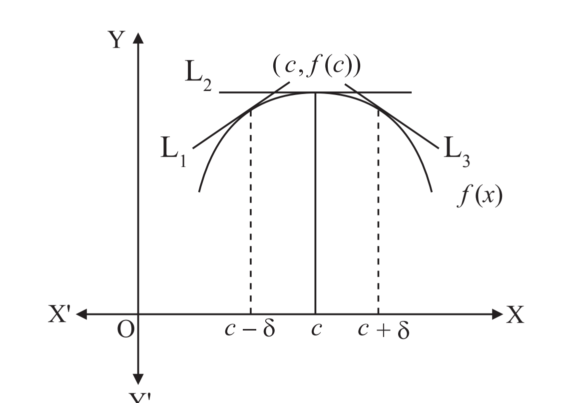


Fig. 2.4.4 (a)

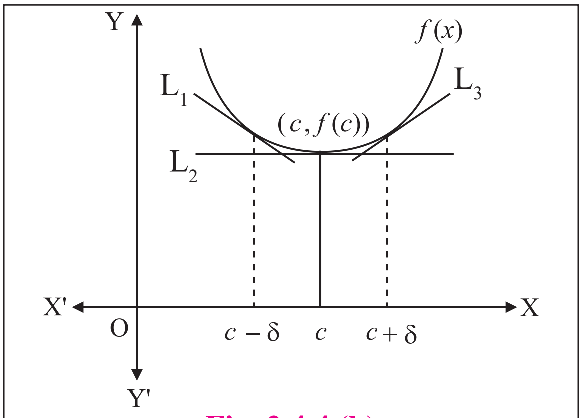


Fig. 2.4.4 (b)

Maxima at A : Consider the slopes of the tangents (See Fig 2.4.4a) Slope of L_1 is +ve, slope of $L_2 = 0$ and slope of L_3 is −ve. Thus the slope is seen to be decreasing if there is a maximum at A.

Minima at A : Consider the slopes of the tangents (See Fig 2.4.4b) slope of L_1 is −ve, slope of $L_2 = 0$ and slope of L_3 is +ve. Thus the slope is seen to be increasing if there is a minima at A.

SOLVED EXAMPLES

Ex. 1: Find the local maximum and local minimum value of $f(x) = x^3 - 3x^2 - 24x + 5$.

Solution : Given that $f(x) = x^3 - 3x^2 - 24x + 5$ . . . (I)

Differentiate (I) *w. r. t. x.*

$f'(x) = 3x^2 - 6x - 24$ . . . (II)

For extreme values, $f'(x) = 0$

$3x^2 - 6x - 24$ i.e. $3(x^2 - 2x - 8) = 0$

i.e. $x^2 - 2x - 8 = 0$ i.e. $(x + 2)(x - 4) = 0$

$\Rightarrow x + 2 = 0$ or $x - 4 = 0 \Rightarrow x = -2$ or $x = 4$

The stationary points are $x = -2$ and $x = 4$.

Differentiate (II) *w. r. t. x.*

$f''(x) = 6x - 6$ . . . (III)

For $x = -2$, from (III) we get,

$f''(-2) = 6(-2) - 6 = -18 < 0$

$\therefore$ At $x = -2$, $f(x)$ has a maximum value.

For maximum of $f(x)$, put $x = -2$ in (I)

$f(-2) = (-2)^3 - 3(-2)^2 - 24(-2) + 5 = 33.$

For $x = 4$, from (III) we get

$f''(4) = 6(4) - 6 = 18 > 0$

$\therefore$ At $x = 4$, $f(x)$ has a minimum value.

For minima of $f(x)$, put $x = 4$ in (I)

$f(4) = (4)^3 - 3(4)^2 - 24(4) + 5 = -75$

$\therefore$ Local maximum of $f(x)$ is 33 when $x = -2$

and

Local minimum of $f(x)$ is −75 when $x = 4$.

Ex. 2: A wire of length 120 cm is bent in the form of a rectangle. Find its dimensions if the area of the rectangle is maximum.

Solution : Let x cm and y cm be the length and the breadth of the rectangle. Perimeter of rectangle = 120 cm.

$\therefore \quad 2(x + y) = 120$ so, $x + y = 60$

$\therefore \quad y = 60 - x \quad \ldots (I)$

Let A be the area of the rectangle

$\therefore \quad A = xy = x(60 - x) = 60x - x^2 \ldots$ [From (I)]

Differentiate *w. r. t.* x.

$$\frac{dA}{dx} = 60 - 2x \quad \ldots (II)$$

For maximum area $\frac{dA}{dx} = 0$

i.e. $60 - 2x = 0 \Rightarrow x = 30$

Differentiate (II) *w. r. t.* x.

$$\frac{d^2A}{dx^2} = -2 \quad \ldots (III)$$

For, $x = 30$ from (III) we get,

$$\left(\frac{d^2A}{dx^2}\right)_{x=30} = -2 < 0$$

When, $x = 30$, Area of the rectangle is maximum.

Put $x = 30$ in (I) we get $y = 60 - 30 = 30$

$\therefore$ Area of the rectangle is maximum if length = breadth = 30 cm.

Ex. 3: A Rectangular sheet of paper has it area 24 sq. meters. The margin at the top and the bottom are 75 cm each and at the sides 50 cm each. What are the dimensions of the paper, if the area of the printed space is maximum ?

Solution : Let x m and y m be the width and the length of the rectangular sheet of paper respectively. Area of the paper = 24 sq. m.

$$\therefore \quad xy = 24 \Rightarrow y = \frac{24}{x} \quad \ldots (I)$$

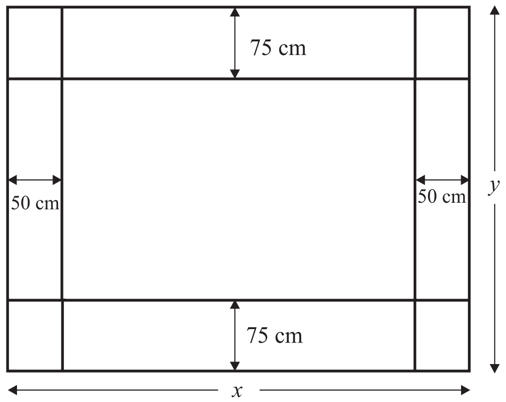


Fig. 2.4.5

After leaving the margins, length of the printing space is $(x - 1)$ m and breadth of the printing space is $(y - 1.5)$ m.

Let A be the area of the printing space

$$A = (x-1)(y-1.5) = (x-1)\left(\frac{24}{x} - 1.5\right)$$

$$= 24 - 1.5x - \frac{24}{x} + 1.5 \quad \ldots \text{[From (I)]}$$

$$A = 25.5 - 1.5x - \frac{24}{x} \quad \ldots (II)$$

Differentiate *w. r. t.* x.

$$\frac{dA}{dx} = -1.5 + \frac{24}{x^2} \quad \ldots (III)$$

For maximum printing space $\frac{dA}{dx} = 0$

i.e. $\quad -1.5x + \frac{24}{x^2} = 0 \Rightarrow 1.5x^2 = 24 \Rightarrow x = \pm 4, x \neq -4$

$\therefore \quad x = 4$

Differentiate (III) *w. r. t.* x.

$$\frac{d^2A}{dx^2} = -\frac{48}{x^3} \quad \ldots (IV)$$

For, $x = 4$, from (IV) we get,

$$\left(\frac{d^2A}{dx^2}\right)_{x=4} = -\frac{48}{(4)^3} < 0$$

When, $x = 4$ Area of the rectangular printing space is maximum.

Put $x = 4$ in (I) we get $y = \frac{24}{4} = 6$

$\therefore$ Area of the printing space is maximum when width and length of sheet are 4 meter and 6 meter respectively.

Ex. 4: An open box is to be cut out of piece of square card of side 18 cm by cutting of equal squares from the corners and turning up the sides. Find the maximum volume of the box.

Solution : Let the side of each of the small squares cut be x cm, so that each side of the box to be made is $(18 - 2x)$ cm. and height x cm.

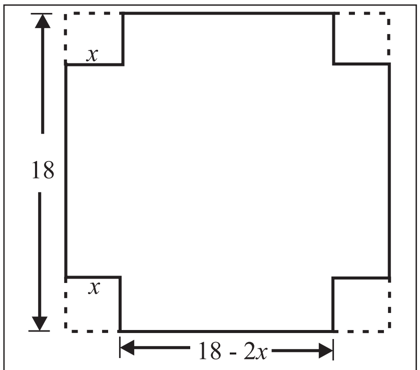


Fig. 2.4.6

Let V be the volume of the box.

V = Area of the base × Height

$= (18 - 2x)^2 x = (324 - 72x + 4x^2) x$

$V = 4x^3 - 72x^2 + 324x \qquad \ldots \text{(I)}$

Differentiate *w. r. t.* x

$$\frac{dV}{dx} = 12x^2 - 144x + 324 \qquad \ldots \text{(II)}$$

For maximum volume $\frac{dV}{dx} = 0$

i.e. $12x^2 - 144x + 324 = 0 \Rightarrow x^2 - 12x + 27 = 0$

$(x - 3)(x - 9) = 0 \Rightarrow x - 3 = 0$ or $x - 9 = 0$

$\therefore \quad x = 3$ or $x = 9$, but $x \neq 9 \quad \therefore \quad x = 3$

Differentiate (II) *w. r. t.* x

$$\frac{d^2V}{dx^2} = 24x - 144 \qquad \ldots \text{(III)}$$

For, $x = 3$ from (III) we get,

$$\left(\frac{d^2V}{dx^2}\right)_{x=3} = 24(3) - 144 = -72 < 0$$

Volume of the box is maximum when $x = 3$.

Maximum volume of the box

$= (18 - 6)^2 (3) = 432$ c.c.

Ex. 5: Two sides of a triangle are given, find the angle between them such that the area of the triangle is maximum.

Solution : Let ABC be a triangle. Let the given sides be $AB = c$ and $AC = b$.

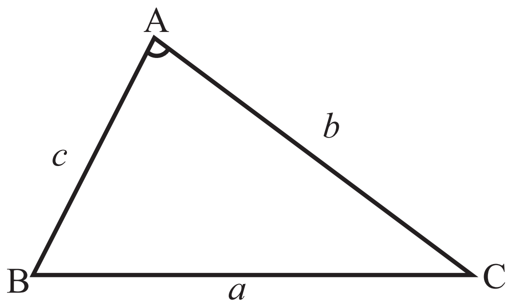


Fig. 2.4.7

Let Δ be the area of the triangle.

$$\Delta = \frac{1}{2} bc \sin A \qquad \ldots \text{(I)}$$

Differentiate *w. r. t.* A.

$$\frac{d\Delta}{dA} = \frac{bc}{2} \cos A \qquad \ldots \text{(II)}$$

For maximum area $\frac{d\Delta}{dA} = 0$

i.e. $\frac{bc}{2} \cos A = 0 \Rightarrow \cos A = 0 \Rightarrow A = \frac{\pi}{2}$

Differentiate (II) *w. r. t.* A.

$$\frac{d^2\Delta}{dA^2} = -\frac{bc}{2} \sin A \qquad \ldots \text{(III)}$$

For, $A = \frac{\pi}{2}$ from (III) we get,

$$\left(\frac{d^2\Delta}{dA^2}\right)_{A=\frac{\pi}{2}} = -\frac{bc}{2} \sin\left(\frac{\pi}{2}\right) = \frac{-bc}{a} < 0$$

When, $A = \frac{\pi}{2}$ Area of the triangle is maximum.

Hence, the area of the triangle is maximum when the angle between the given sides $\frac{\pi}{2}$.

Note : $\sin A$ is maximum (=1), when $A = \frac{\pi}{2}$

Ex. 6: The slant side of a right circular cone is l. Show that the semi-vertical angle of the cone of maximum volume is $\tan^{-1}(\sqrt{2})$.

Solution : Let x be the height of the cone and r be the radius of the base.

So, $r^2 = l^2 - x^2$ . . . (I)

Let V be the volume of the cone.

$$V = \frac{1}{3}\pi r^2 x = \frac{\pi}{3}(l^2 - x^2)\,x$$

$$\therefore \quad V = \frac{\pi}{3}(l^2 x - x^3)$$

Differentiate *w. r. t.* x

$$\frac{dV}{dx} = \frac{\pi}{3}(l^2 - 3x^2) \quad \text{. . . (II)}$$

For maximum volume $\frac{dV}{dx} = 0$

i.e. $\frac{\pi}{3}(l^2 - 3x^2) = 0 \Rightarrow x^2 = \frac{l^2}{3}$

$x = \pm\frac{l}{\sqrt{3}} \Rightarrow x = \frac{l}{\sqrt{3}}$ or $x = -\frac{l}{\sqrt{3}}$ is the stataionary point but, $x \neq -\frac{l}{\sqrt{3}}$ $\therefore$ $x = \frac{l}{\sqrt{3}}$

Differentiate (II) *w. r. t.* x

$$\left(\frac{d^2V}{dx^2}\right) = -2\pi x \quad \text{. . . (III)}$$

For, $x = \frac{l}{\sqrt{3}}$ from (III) we get,

$$\left(\frac{d^2V}{dx^2}\right)_{x=\frac{l}{\sqrt{3}}} = -\frac{2\pi l}{\sqrt{3}} < 0$$

Volume of the cone is maximum when height of the cone is $x = \frac{l}{\sqrt{3}}$.

Put $x = \frac{l}{\sqrt{3}}$ in (I) we get, $r = \sqrt{l^2 - \left(\frac{l}{\sqrt{3}}\right)^2} = \frac{l\sqrt{2}}{\sqrt{3}}$

Let α be the semi-vertical angle.

Then $\tan\alpha = \frac{r}{x} = \frac{\frac{l\sqrt{2}}{\sqrt{3}}}{\frac{l}{\sqrt{3}}} = \sqrt{2}$

$$\therefore \quad \alpha = \tan^{-1}(\sqrt{2})$$

Ex. 7: Find the height of a covered box of fixed volume so that the total surface area of the box is minimum whose base is a rectangle with one side three times as long as the other.

Solution : Given that, box has a rectangular base with one side three times as long as other.

Let x and $3x$ be the sides of the rectangular base.

Let h be the height of the box and V be its volume.

$V = (x)(3x)(h) = 3x^2h$ . . . [Observe that V is constant]

Differentiate *w. r. t. x.*

$$\frac{dV}{dx} = 3x^2 \frac{dh}{dx} + h \frac{d}{dx}(3x^2)$$

$$\therefore \quad 3x^2 \frac{dh}{dx} + 6xh = 0 \Rightarrow \frac{dh}{dx} = -\frac{2h}{x} \qquad \text{. . . (I)}$$

Let S be the surface area of the box.

$$\therefore \quad S = (2 \times 3x^2) + (2 \times 3xh) + (2 \times xh) = 6x^2 + 8xh$$

Differentiate *w. r. t. x.*

$$\frac{dS}{dx} = 12x + 8\left(x\frac{dh}{dx} + h\frac{d}{dx}(x)\right)$$

$$\frac{dS}{dx} = 12x + 8\left[x\left(-\frac{2h}{x}\right) + h\right] \qquad \text{. . . [from (I)]}$$

$$= 12x + 8(-2h + h)$$

$$\therefore \quad \frac{dS}{dx} = 12x - 8h \qquad \text{. . . (II)}$$

For minimum surface area

$$\frac{dS}{dx} = 0 \Rightarrow 12x - 8h = 0 \Rightarrow h = \frac{3x}{2}$$

Differentiate (II) *w. r. t. x.*

$$\left(\frac{d^2S}{dx^2}\right) = 12 - 8\frac{dh}{dx} = 12 - 8\left(-\frac{2h}{x}\right) = 12 + \frac{16h}{x} \qquad \text{. . . (III) . . . [from (I)]}$$

Both x and h are positive, from (III) we get,

$$\left(\frac{d^2S}{dx^2}\right) = 12 + \frac{16h}{x} > 0$$

Surface area of the box is minimum if height $= \frac{3}{2} \times$ shorter side of base.

EXERCISE 2.4

(1) Test whether the following functions are increasing or decreasing.

(i) $f(x) = x^3 - 6x^2 + 12x - 16, x \in R$

(ii) $f(x) = 2 - 3x + 3x^2 - x^3, x \in R$

(iii) $f(x) = x - \frac{1}{x}, x \in R$ and $x \neq 0$

(2) Find the values of x for which the following functions are strictly increasing -

(i) $f(x) = 2x^3 - 3x^2 - 12x + 6$

(ii) $f(x) = 3 + 3x - 3x^2 + x^3$

(iii) $f(x) = x^3 - 6x^2 - 36x + 7$

(3) Find the values of x for which the following functions are strictly decreasing -

(i) $f(x) = 2x^3 - 3x^2 - 12x + 6$

(ii) $f(x) = x + \frac{25}{x}$

(iii) $f(x) = x^3 - 9x^2 + 24x + 12$

(4) Find the values of x for which the function $f(x) = x^3 - 12x^2 - 144x + 13$

(a) Increasing (b) Decreasing

(5) Find the values of x for which

$f(x) = 2x^3 - 15x^2 - 144x - 7$ is

(a) strictly increasing

(b) strictly decreasing

(6) Find the values of x for which $f(x) = \dfrac{x}{x^2+1}$ is

(a) strictly increasing

(b) strictly decreasing

(7) Show that $f(x) = 3x + \dfrac{1}{3x}$ increasing in $\left(\dfrac{1}{3}, 1\right)$ and decreasing in $\left(\dfrac{1}{9}, \dfrac{1}{3}\right)$.

(8) Show that $f(x) = x - \cos x$ is increasing for all x.

(9) Find the maximum and minimum of the following functions -

(i) $y = 5x^3 + 2x^2 - 3x$

(ii) $f(x) = 2x^3 - 21x^2 + 36x - 20$

(iii) $f(x) = x^3 - 9x^2 + 24x$

(iv) $f(x) = x^2 + \dfrac{16}{x^2}$

(v) $f(x) = x \log x$ (vi) $f(x) = \dfrac{\log x}{x}$

(10) Divide the number 30 in to two parts such that their product is maximum.

(11) Divide the number 20 in to two parts such that sum of their squares is minimum.

(12) A wire of length 36 meter is bent in the form of a rectangle. Find its dimensions if the area of the rectangle is maximum.

(13) A ball is thrown in the air. Its height at any time t is given by $h = 3 + 14t - 5t^2$. Find the maximum height it can reach.

(14) Find the largest size of a rectangle that can be inscribed in a semi circle of radius 1 unit, So that two vertices lie on the diameter.

(15) An open cylindrical tank whose base is a circle is to be constructed of metal sheet so as to contain a volume of πa^3 cu. cm of water. Find the dimensions so that sheet required is minimum.

(16) The perimeter of a triangle is 10 cm. If one of the side is 4 cm. What are the other two sides of the triangle for its maximum area ?

(17) A box with a square base is to have an open top. The surface area of the box is 192 sq.cm. What should be its dimensions in order that the volume is largest ?

(18) The profit function $P(x)$ of a firm, selling x items per day is given by

$P(x) = (150 - x)x - 1625$. Find the number of items the firm should manufacture to get maximum profit. Find the maximum profit.

(19) Find two numbers whose sum is 15 and when the square of one number multiplied by the cube of the other is maximum.

(20) Show that among rectangles of given area, the square has the least perimeter.

(21) Show that the height of a closed right circular cylinder, of a given volume and least surface area, is equal to its diameter.

(22) Find the volume of the largest cylinder that can be inscribed in a sphere of radius 'r' cm.

(23) Show that $y = \log(1 + x) - \dfrac{2x}{2+x}$, $x > -1$ is an increasing function on its domain.

(24) Prove that $y = \dfrac{4\sin\theta}{2+\cos\theta} - \theta$ is an increasing function of $\theta \in \left[0, \dfrac{\pi}{2}\right]$.

Let us Remember

- Equations of tangent and Normal at $P(x_1, y_1)$ respectively are given by

  $y - y_1 = m(x - x_1)$ where $m = \left[\frac{dy}{dx}\right]_{(x_1, y_1)}$

  $y - y_1 = m'(x - x_1)$ where $m' = -\frac{1}{\left[\frac{dy}{dx}\right]_{(x_1, y_1)}}$, if $\left[\frac{dy}{dx}\right]_{(x_1, y_1)} \neq 0$

- Approximate value of the function $f(x)$ at $x = a + h$ is given by $f(a + h) \doteqdot f(a) + h f'(a)$

- **Rolle's theorem :** If real-valued function f is continous on a closed $[a, b]$, differentiable on the open interval (a, b) and $f(a) = f(b)$, then there exists at least one c in the open interval (a, b) such that $f'(c) = 0$.

- **Lagrange's Mean Value Theorem (LMVT) :** If a real-valued function f is continous on a closed $[a, b]$ and differentiable on the open interval (a, b) then there exists at least one c in the open interval (a, b) such that $\quad f'(c) = \frac{f(b) - f(a)}{b - a}$

- **Increasing and decreasing functions :**

  (i) A function f is monotonically increasing if $f'(x) > 0$.

  (ii) A function f is monotonically decreasing if $f'(x) < 0$.

  (iii) A function f is increasing if $f'(x) \geq 0$.

  (iv) A function f is decreasing if $f'(x) \leq 0$.

- **(i) First Derivative test :**

  A function $f(x)$ has a maxima at $x = c$ if

  (i) $f'(c) = 0$

  (ii) $f'(c - h) > 0$ $\quad$ [$f(x)$ is increasing for values of $x < c$]

  (iii) $f'(c + h) < 0$ $\quad$ [$f(x)$ is decreasing for values of $x > c$]

  where h is a small positive number.

  A function $f(x)$ has a minima at $x = c$ if

  (i) $f'(c) = 0$

  (ii) $f'(c - h) < 0$ $\quad$ [$f(x)$ is decreasing for values of $x < c$]

  (iii) $f'(c + h) > 0$ $\quad$ [$f(x)$ is increasing for values of $x > c$]

  where h is a small positive number.

  (ii) Second Derivative test :

  A function $f(x)$ has a maxima at $x = c$ if $f'(c) = 0$ and $f''(c) < 0$.

  A function $f(x)$ has a minimum at $x = c$ if $f'(c) = 0$ and $f''(c) > 0$.

MISCELLANEOUS EXERCISE 2

(I) Choose the correct option from the given alternatives :

(1) If the function $f(x) = ax^3 + bx^2 + 11x - 6$ satisfies conditions of Rolle's theorem in [1, 3] and $f'\left(2 + \frac{1}{\sqrt{3}}\right) = 0$, then values of a and b are respectively.

(A) 1, −6 (B) −2, 1 (C) −1, −6 (D) −1, 6

(2) If $f(x) = \frac{x^2 - 1}{x^2 + 1}$, for every real x, then the minimum value of f is -

(A) 1 (B) 0 (C) −1 (D) 2

(3) A ladder 5 m in length is resting against vertical wall. The bottom of the ladder is pulled along the ground away from the wall at the rate of 1.5 m/ sec. The length of the higher point of ladder when the foot of the ladder is 4.0 m away from the wall decreases at the rate of

(A) 1 (B) 2 (C) 2.5 (D) 3

(4) Let $f(x)$ and $g(x)$ be differentiable for $0 < x < 1$ such $f(0) = 0$, $g(0) = 0$, $f(1) = 6$. Let there exist a real number c in (0, 1) such that $f'(c) = 2g'(c)$, then the value of $g(1)$ must be

(A) 1 (B) 3 (C) 2.5 (D) −1

(5) If $f(x) = x^3 - 6x^2 + 9x + 18$, then $f(x)$ is strictly decreasing in -

(A) $(-\infty, 1)$ (B) $[3, \infty)$ (C) $(-\infty, 1]$ U $[3, \infty)$ (D) $(1, 3)$

(6) If $x = -1$ and $x = 2$ are the extreme points of $y = \alpha \log x + \beta x^2 + x$ then

(A) $\alpha = -6, \beta = \frac{1}{2}$ (B) $\alpha = -6, \beta = -\frac{1}{2}$ (C) $\alpha = 2, \beta = -\frac{1}{2}$ (D) $\alpha = 2, \beta = \frac{1}{2}$

(7) The normal to the curve $x^2 + 2xy - 3y^2 = 0$ at (1, 1)

(A) Meets the curve again in second quadrant. (B) Does not meet the curve again.

(C) Meets the curve again in third quadrant. (D) Meets the curve again in fourth quadrant.

(8) The equation of the tangent to the curve $y = 1 - e^{\frac{x}{2}}$ at the point of intersection with Y-axis is

(A) $x + 2y = 0$ (B) $2x + y = 0$ (C) $x - y = 2$ (D) $x + y = 2$

(9) If the tangent at (1, 1) on $y^2 = x(2 - x)^2$ meets the curve again at P then P is

(A) (4, 4) (B) (−1, 2) (C) (3, 6) (D) $\left(\frac{9}{4}, \frac{3}{8}\right)$

(10) The appoximate value of tan (44° 30′) given that 1° = 0.0175.

(A) 0.8952 (B) 0.9528 (C) 0.9285 (D) 0.9825

(II) (1) If the curves $ax^2 + by^2 = 1$ and $a' x^2 + b' y^2 = 1$ intersect orthogonally, then prove that $\frac{1}{a} - \frac{1}{b} = \frac{1}{a'} - \frac{1}{b'}$.

(2) Determine the area of the triangle formed by the tangent to the graph of the function $y = 3 - x^2$ drawn at the point (1, 2) and the cordinate axes.

(3) Find the equation of the tangent and normal drawn to the curve $y^4 - 4x^4 - 6xy = 0$ at the point M (1, 2).

(4) A water tank in the form of an inverted cone is being emptied at the rate of 2 cubic feet per second. The height of the cone is 8 feet and the radius is 4 feet. Find the rate of change of the water level when the depth is 6 feet.

(5) Find all points on the ellipse $9x^2 + 16y^2 = 400$, at which the y-coordinate is decreasing and the x-coordinate is increasing at the same rate.

(6) Verify Rolle's theorem for the function $f(x) = \frac{2}{e^x + e^{-x}}$ on $[-1, 1]$.

(7) The position of a particle is given by the function $s(t) = 2t^2 + 3t - 4$. Find the time $t = c$ in the interval $0 \leq t \leq 4$ when the instantaneous velocity of the particle equals to its average velocity in this interval.

(8) Find the approximate value of the function $f(x) = \sqrt{x^2 + 3x}$ at $x = 1.02$.

(9) Find the approximate value of $\cos^{-1}(0.51)$ given $\pi = 3.1416$, $\frac{2}{\sqrt{3}} = 1.1547$.

(10) Find the intervals on which the function $y = x^x$, $(x > 0)$ is increasing and decreasing.

(11) Find the intervals on the which the function $f(x) = \frac{x}{\log x}$, is increasing and decreasing.

(12) An open box with a square base is to be made out of a given quantity of sheet of area a^2, Show the maximum volume of the box is $\frac{a^3}{6\sqrt{3}}$.

(13) Show that of all rectangles inscribed in a given circle, the square has the maximum area.

(14) Show that a closed right circular cyclinder of given surface area has maximum volume if its height equals the diameter of its base.

(15) A window is in the form of a rectangle surmounted by a semi-circle. If the perimeter be 30 m, find the dimensions so that the greatest possible amount of light may be admitted.

(16) Show that the height of a right circular cylinder of greatest volume that can be inscribed in a right circular cone is one-third of that of the cone.

(17) A wire of length l is cut in to two parts. One part is bent into a circle and the other into a square. Show that the sum of the areas of the circle and the square is least, if the radius of the circle is half the side of the square.

(18) A rectangular sheet of paper of fixed perimeter with the sides having their length in the ratio 8 : 15 converted in to an open rectangular box by folding after removing the squares of equal area from all corners. If the total area of the removed squares is 100, the resulting box has maximum valume. Find the lengths of the sides of rectangular sheet of paper.

(19) Show that the altitude of the right circular cone of maximum volume that can be inscribed in a shpere of radius r is $\frac{4r}{3}$.

(20) Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume.

(21) Find the maximum and minimum values of the function $f(x) = \cos^2 x + \sin x$.

❖ ❖ ❖

MISCELLANEOUS EXERCISE 1

(I)

1	2	3	4	5	6	7	8	9	10	11	12
D	C	C	B	A	C	D	C	B	C	A	B

(II) (1) $\frac{3}{4}$ (ii) Does not exist (iii) -2

(2) (A) 3, (B) 5, (C) 4, (D) 1.

(3) (i) $-\frac{1}{9}$ (ii) $-\frac{40}{3}$ (iii) $-\frac{29}{96}$

(iv) $-\frac{4}{9}$

(4) (i) $-\frac{x}{\sqrt{1-x^2}}$ [Hint : $x = \cos 2\theta$]

(ii) $-\frac{1}{2}$ [Hint : $x = \cos 2\theta$]

(iii) $\frac{3}{2\sqrt{x}\,(1+x)}$ [Hint : $\sqrt{x} = \tan\theta$]

(iv) $-\frac{1}{2\cdot\sqrt{1-x^2}}$ [Hint : $x = \cos 2\theta$]

(v) $\frac{3}{1+9x^2} + \frac{5}{1+25x^2}$

(vi) $\frac{1}{2\,(1+x^2)}$ [Hint : $x = \tan\theta$]

(6) (i) $\frac{\sqrt{1-x^2}}{4\,(1+x^2)}$

(ii) $-\frac{2x}{\sqrt{1+x^2}.\,\sin(\log x)}$ (iii) 1

2. APPLICATIONS OF DERIVATIVES

EXERCISE 2.1

(1) (i) $2x - y + 4 = 0,\ x + 2y - 8 = 0$

(ii) $4x - 5y + 12 = 0,\ 5x + 4y - 26 = 0,$

(iii) $y = 2,\ x = \sqrt{3}$

(iv) $\pi x + 2y - 2\pi = 0,$

$4x - 2\pi y + \pi^2 - 4 = 0$

(v) $2x - y = 0,\ 4x + 8y - 5\pi = 0$

(vi) $4x + 2y - 3 = 0,\ 2x - 4y + 1 = 0$

(vii) $17x - 4y - 20 = 0,\ 8x + 34y - 135 = 0$

(2) (4, 1)

(3) $(2, -2)\left(-\frac{2}{3}, -\frac{14}{27}\right)$ (4) $y = 0$ and $y = 4$

(5) $x + 3y - 8 = 0,\ x + 3y + 8 = 0$

(6) $a = 2,\ b = -7$ (7) (4, 11) and $\left(-4, -\frac{31}{3}\right)$

(8) 0.8 π cm²/sec. (9) 6 cm³/ sec.

(10) $\frac{3\sqrt{6}}{2}$ cm²/ sec. (11) 8 cm²/ sec

(12) 7.2 cm³/ sec (13) 3 km/hr

(14) (i) $\left(\frac{3}{8}\right)$ meter/sec. (ii) $\frac{9}{8}$ meter/sec.

(15) 0.9 meter/sec. (16) $\left(\frac{4\pi}{3}\right)$ cm³/ sec

EXERCISE 2.2

(1) (i) 2.9168 (ii) 3.03704 (iii) 1.9997

(iv) 248.32 (v) 64.48

(2) (i) 0.953 (ii) 0.42423 (iii) 0.4924

(iv) 1.02334

(3) (i) 0.7845 (ii) 0.7859 (iii) 0.7859

(4) (i) 2.70471 (ii) 8.1279 (iii) 9.09887

(5) (i) 4.6152 (ii) 2.1983 (iii) 3.006049

(6) (i) 6.91 (ii) 9.72

EXERCISE 2.3

(1) (i) Valid (ii) Valid

(iii) Invalid (iv) Valid

(v) Invalid (vi) Invalid

(2) $b = 1$

(3) (i) $\frac{\pi}{4}$ or $\frac{5\pi}{4}$ (ii) $c = \pi$ (iii) $c = \frac{5}{2}$

(4) $p = -6, q = 11$ (6) $c = -2$

(7) (i) $e - 1$ (ii) $2 \pm \frac{2}{\sqrt{3}}$ (iii) $\frac{1}{7}$

(iv) $\frac{1}{2}$ (v) $3 + \sqrt{2}$

EXERCISE 2.4

(1) (i) Increasing $\forall\ x \in R$

(ii) Decreasing $\forall\ x \in R$

(iii) Increasing $\forall\ x \in R$

(2) (i) $x < -1$ and $x > 2$ (ii) $R - \{1\}$

(iii) $x < -2$ and $x > 6$

(3) (i) $-1 < x < 2$ (ii) $(-5, 5) - \{0\}$

(iii) $x \in (2, 4)$

(4) (a) $(-\infty, -4] \cup [12, \infty)$

(b) $-4 \le x \le 12$ i.e. $[-4, 12]$

(5) (a) $x < -3$ and $x > 8$ (b) $-3 < x < 8$

(6) (a) $-1 < x < 1$ (b) $(-\infty, -1) \cup (1, \infty)$

(9) (i) Max $= \frac{36}{25}$, Min $= -\frac{16}{27}$

(ii) Max $= -3$, Min $= -128$

(iii) Max $= 20$, Min $= 16$ (iv) Min $= 8$

(v) Min $= -\frac{1}{e}$ (vi) Max $= \frac{1}{e}$

(10) 15, 15 (11) 10, 10 (12) 9 (13) 12.8

(14) $l = \sqrt{2}$ and $b = \frac{1}{\sqrt{2}}$

(15) Radius = Height = a (16) 3, 3

(17) Side of square base = 8 cm, Height = 4 cm

(18) $x = 75$, $P = 4000$ (19) 6, 9

(22) $\frac{4\pi r^3}{3\sqrt{3}}$ cm^3

MISCELLANEOUS EXERCISE 2

(I)

1	2	3	4	5	6	7	8	9	10
A	C	B	B	D	C	D	A	D	D

(II) (2) 4

(3) $14x - 13y + 12 = 0,\ 13x + 14y - 41 = 0$

(4) $\frac{2}{9\pi}$ ft/sec (5) $\left(\frac{16}{3}, 3\right), \left(-\frac{16}{3}, -3\right)$

(6) $c = 0$ (7) $c = 2$ (8) 2.025

(9) 1.03565

(10) Decreasing in $\left(0, \frac{1}{e}\right]$ and

Increasing in $\left[\frac{1}{e}, \infty\right)$

(11) Increasing in $[e, \infty)$, Decreasing in $(1, e]$

(15) $l = \frac{60}{\pi + 4}, b = \frac{30}{\pi + 4}, r = \frac{30}{\pi + 4}$

(17) Side $= \frac{l}{\pi + 4}$, Radius $= \frac{l}{2(\pi + 4)} = \frac{x}{2}$

(18) 24, 45 (21) Max $= \frac{5}{4}$, Min $= 1$

3. INDEFINITE INTEGRATION

EXERCISE 3.1

(1) (i) $\frac{x^4}{4} + \frac{x^3}{3} - \frac{x^2}{2} + x + c$ (ii) $\frac{x^3}{3} - 2x^2 + 4x + c$

(iii) $3 \tan x - 4 \log x - \frac{2}{\sqrt{x}} - 7x + c$

(iv) $\frac{x^2}{4} - \frac{5x^2}{2} + 3 \log x - \frac{1}{x^4} + c$

(v) $\frac{6}{5} x^2 \sqrt{x} - 4\sqrt{x} - \frac{10}{\sqrt{x}} + c$

(2) (i) $\tan x - x + c$ (ii) $-2 \cos x + c$

(iii) $\sec x + c$ (iv) $-\cot x - 2x + c$

(v) $-\cot x - \tan x + x + c$

(vi) $\sec x - \tan x + x + c$

(vii) $\sec x - \tan x + x + c$

(viii) $\sin x - \cos x + c$ (ix) $-\sqrt{2} \cos x + c$

(x) $-\frac{1}{14} \cos 7x - \frac{1}{2} \cos x + c$

(3) (i) $x - 2 \log (x + 2) + c$

(ii) $2x + \frac{1}{2} \log (2x + 1) + c$

(iii) $\frac{5}{3} x - \frac{26}{9} \log (3x - 4) + c$

(iv) $\frac{2 (x + 5)^{\frac{3}{2}}}{3} - 14 \sqrt{x + 5} + c$

(v) $\frac{1}{12} (4x - 1)^{\frac{3}{2}} - \frac{13}{4} \sqrt{4x - 1} + c$

(vi) $-\cos 2x + c$

(vii) $\frac{2}{5} \left(\sin \frac{5x}{2} - \cos \frac{5x}{2} \right) + c$

(viii) $\frac{1}{4} (2x + \sin 2x) + c$

(ix) $-\frac{4}{9} \left[x^{\frac{3}{2}} + (x + 3)^{\frac{3}{2}} \right] + c$

(x) $\frac{2}{21} \left[(7x - 2)^{\frac{3}{2}} + (7x - 5)^{\frac{3}{2}} \right] + c$

(4) $f(x) = \frac{x^2}{2} + \frac{3}{2x^2} + \frac{7}{2}$

EXERCISE 3.2 (A)

I. 1. $\frac{(\log x)^{n+1}}{n + 1} + c$ 2. $\frac{2}{5} (\sin^{-1} x)^{\frac{5}{2}} + c$

3. $\log \left(\operatorname{cosec} (x + \log x) - \cot (x + \log x) \right) + c$

4. $\frac{-1}{\sqrt{\tan (x^2)}} + c$ 5. $\frac{1}{3} (e^{3x} + 1) + c$