4 Pair of Straight Lines

Let's Study

4.1 Combined equation of a pair lines.

4.2 Homogeneous equation of degree two.

4.3 Angle between lines.

4.4 General second degree equation in x and y.

4.1 INTRODUCTION

We know that equation $ax + by + c = 0$, where $a, b, c, \in, R$, (a and b not zero simultaneously), represents a line in XY plane. We are familiar with different forms of equations of line. Now let's study two lines simultaneously. For this we need the concept of the combined equation of two lines.

Let's learn.

4.1 Combined equation of a pair of lines :

An equation which represents two lines is called the combined equation of those two lines. Let $u \equiv a_1x+b_1 y + c_1$ and $v \equiv a_2x+b_2 y + c_2$. Equation $u = 0$ and $v = 0$ represent lines. We know that equation $u + kv = 0$, $k \in R$ represents a family of lines. Let us interpret the equation $uv = 0$.

Theorem 4.1:

The equation $uv = 0$ represents,the combined equation of lines $u = 0$ and $v = 0$

Proof : Consider the lines represented by $u = 0$ and $v = 0$

$\therefore \quad a_1x + b_1 y + c_1 = 0$ and $a_2x + b_2 y + c_2 = 0$.

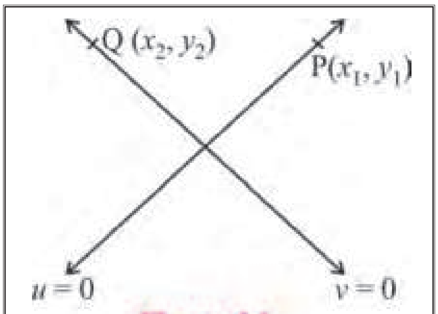


Figure 4.1

Let $P(x_1, y_1)$ be a point on the line $u = 0$.

$\therefore \quad (x_1, y_1)$ satisfy the equation $a_1x + b_1 y + c_1 = 0$

$\therefore \quad a_1x_1 + b_1 y_1 + c_1 = 0$

To show that (x_1, y_1) satisfy the equation $uv = 0$.

$(a_1x_1 + b_1 y_1 + c_1) (a_2x_1 + b_2 y_1 + c_2)$

$= 0 (a_2x_1 + b_2 y_1 + c_2)$

$= 0$

Therefore (x_1, y_1) satisfy the equation $uv = 0$.

This proves that every point on the line $u = 0$ satisfy the equation $uv = 0$.

Similarly we can prove that every point on the line $v = 0$ satisfies the equation $uv = 0$.

Now let $R(x',y')$ be any point which satisfy the equation $uv = 0$.

$\therefore \quad (a_1x' + b_1 y' + c_1) (a_2x' + b_2 y' + c_2) = 0$

$\therefore \quad (a_1 x' + b_1 y' + c_1) = 0$ or $(a_2 x' + b_2 y' + c_2) = 0$

Therefore R(x', y') lies on the line $u = 0$ or $v = 0$.

Every points which satisfy the equation $uv = 0$ lies on the line $u = 0$ or the line $v = 0$.

Therefore equation $uv = 0$ represents the combined equation of lines $u = 0$ and $v = 0$.

Remark :

1) The combined equation of a pair of lines is also called as the joint equation of a pair of lines.

2) Equations $u = 0$ *and* $v = 0$ are called separate equations of lines represented by $uv = 0$.

Solved Examples :

Ex. 1) Find the combined equation of lines $x + y - 2 = 0$ and $2x - y + 2 = 0$

Solution : The combined equation of lines $u = 0$ and $v = 0$ is $uv = 0$

$\therefore$ The combined equation of lines $x + y - 2 = 0$ and $2x - y + 2 = 0$ is

$(x + y - 2)(2x - y + 2) = 0$

$\therefore \quad x(2x - y + 2) + y(2x - y + 2) - 2(2x - y + 2) = 0$

$\therefore \quad 2x^2 - xy + 2x + 2xy - y^2 + 2y - 4x + 2y - 4 = 0$

$\therefore \quad 2x^2 + xy - y^2 - 2x + 4y - 4 = 0$

Ex. 2) Find the combined equation of lines $x - 2 = 0$ and $y + 2 = 0$.

Solution : The combined equation of lines $u = 0$ and $v = 0$ is $uv = 0$.

$\therefore$ The combined equation of lines $x - 2 = 0$ and $y + 2 = 0$ is

$(x - 2)(y + 2) = 0$

$\therefore xy + 2x - 2y - 4 = 0$

Ex. 3) Find the combined equation of lines $x - 2y = 0$ and $x + y = 0$.

Solution : The combined equation of lines $u = 0$ and $v = 0$ is $uv = 0$.

$\therefore$ The combined equation of lines $x - 2y = 0$ and $x + y = 0$ is

$(x - 2y)(x + y) = 0$

$\therefore x^2 - xy - 2y^2 = 0$

Ex. 4) Find separate equation of lines represented by $x^2 - y^2 + x + y = 0$.

Solution : We factorize equation $x^2 - y^2 + x + y = 0$ as

$(x + y)(x - y) + (x + y) = 0$

$\therefore (x + y)(x - y + 1) = 0$

Required separate equations are $x + y = 0$ and $x - y + 1 = 0$.

4.2 Homogeneous equation of degree two:

4.2.1 Degree of a term:

Definition: The sum of the indices of all variables in a term is called the degree of the term.

For example, in the expression $x^2 + 3xy - 2y^2 + 5x + 2$ the degree of the term x^2 is two, the degree of the term $3xy$ is two, the degree of the term $-2y^2$ is two, the degree of $5x$ is one. The degree of constant term 2 is zero. Degree of '0' is not defined.

4.2.2 Homogeneous Equation :

Definition: An equation in which the degree of every term is same, is called a homogeneous equation.

For example: $x^2 + 3xy = 0$, $7xy - 2y^2 = 0$, $5x^2 + 3xy - 2y^2 = 0$ are homogeneous equations.

But $3x^2 + 2xy + 2y^2 + 5x = 0$ is not a homogeneous equation.

Homogeneous equation of degree two in x and y has form $ax^2 + 2hxy + by^2 = 0$.

Theorem 4.2 :

The combined equation of a pair of lines passing through the origin is a homogeneous equation of degree two in x and y.

Proof : Let $a_1x + b_1y = 0$ and $a_2x + b_2y = 0$ be any two lines passing through the origin.

Their combined equation is $(a_1x + b_1y)(a_2x + b_2y) = 0$

$$a_1a_2x^2 + a_1b_2xy + a_2b_1xy + b_1b_2y^2 = 0$$

$$(a_1a_2)x^2 + (a_1b_2 + a_2b_1)xy + (b_1b_2)y^2 = 0$$

In this if we put a_1a_2 = $a, a_1 b_2+a_2b_1$ = $2h$, $b_1b_2 = b$, we get, $ax^2 + 2hxy + by^2 = 0$, which is a homogeneous equation of degree two in x and y.

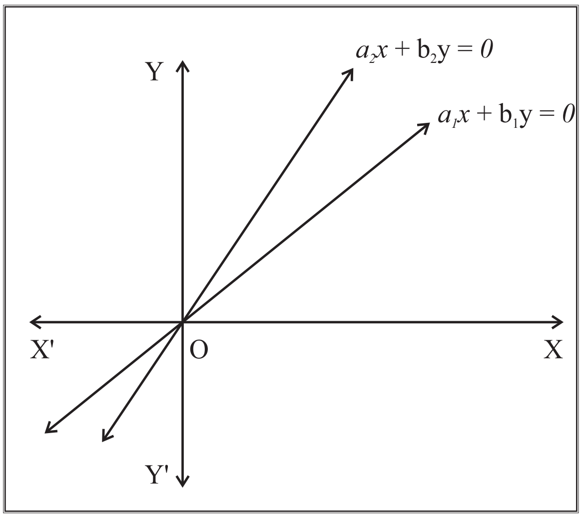


Figure 4.2

Ex.1) Verify that the combined equation of lines $2x + 3y = 0$ and $x - 2y = 0$ is a homogeneous equation of degree two.

Solution :

The combined equation of lines $u = 0$ and $v = 0$ is $uv = 0$.

$\therefore$ The combined equation of lines $2x + 3y = 0$ and $x - 2y = 0$ is

$(2x + 3y)(x - 2y) = 0$

$2x^2 - xy - 6y^2 = 0$, which is a homogeneous equation of degree two.

Remark :

The combined equation of a pair of lines passing through the origin is a homogeneous equation of degree two. But every homogeneous equation of degree two *need not* represent a pair of lines.

Equation $x^2 + y^2 = 0$ is a homogeneous equation of degree two but it does not represent a pair of lines.

How to test whether given homogeneous equation of degree two represents a pair of lines or not?

Let's have a theorem.

Theorem 3 : Homogenous equation of degree two in x and y, $ax^2 + 2hxy + by^2 = 0$ represents a pair of lines passing through the origin if $h^2 - ab \geq 0$.

Proof : Consider the homogeneous equation of degree two in x and y, $ax^2 + 2hxy + by^2 = 0 \cdots\cdots (1)$

Consider two cases $b = 0$ and $b \neq 0$. These two cases are exhaustive.

Case 1: If $b = 0$ then equation (1) becomes $ax^2 + 2hxy = 0$

$\therefore$ $x(ax + 2hy) = 0$, which is the combined equation of lines

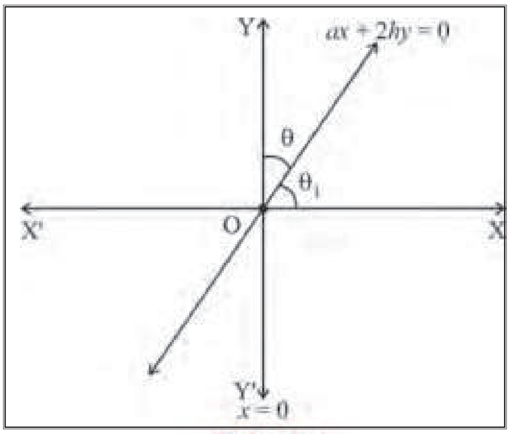


Figure 4.3

$x = 0$ and $ax + 2hy = 0$.

We observe that these lines pass through the origin.

Case 2: If $b \neq 0$ then we multiply equation (1) by b.

$abx^2 + 2hbxy + b^2y^2 = 0$

$b^2y^2 + 2hbxy = -abx^2$

To make L.H.S. complete square we add h^2x^2 to both sides.

$b^2y^2 + 2hbxy + h^2x^2 = h^2x^2 - abx^2$

$(by + hx)^2 = (h^2 - ab)x^2$

$(by + hx)^2 = \left(\sqrt{h^2 - ab}\right)^2 x^2$, as $h^2 - ab \geq 0$

$(by + hx)^2 - \left(\sqrt{h^2 - ab}\right)^2 x^2 = 0$

$(by + hx + \sqrt{h^2 - ab}\, x)(by + hx - \sqrt{h^2 - ab}\, x) = 0$

$[(h + \sqrt{h^2 - ab})\, x + by] \times [(h - \sqrt{h^2 - ab})\, x + by] = 0$

Which is the combined equation of lines $(h + \sqrt{h^2 - ab})\, x + by = 0$ and $(h - \sqrt{h^2 - ab})\, x + by = 0$.

As $b \neq 0$, we can write these equations in the form $y = m_1 x$ and $y = m_2 x$, where $m_1 = \dfrac{-h - \sqrt{h^2 - ab}}{b}$ and $m_2 = \dfrac{-h + \sqrt{h^2 - ab}}{b}$.

We observe that these lines pass through the origin.

Therefore equation $ax^2 + 2hbxy + by^2 = 0$ represents a pair of lines passing through the origin if $h^2 - ab \geq 0$.

Remarks:

1) If $h^2 - ab > 0$ then line represented by (1) are distinct.
2) If $h^2 - ab = 0$ then lines represented by (1) are coincident.
3) If $h^2 - ab < 0$ then equation (1) does not represent a pair of lines.
4) If $b = 0$ then one of the lines is the Y - axis, whose slope is not defined and the slope of the other line is $-\dfrac{a}{2h}$ (provided that $h \neq 0$).
5) If $h^2 - ab \geq 0$ and $b \neq 0$ then slopes of the lines are $m_1 = \dfrac{-h - \sqrt{h^2 - ab}}{b}$ and

   $m_2 = \dfrac{-h + \sqrt{h^2 - ab}}{b}$

   Their sum is $m_1 + m_2 = -\dfrac{2h}{b}$ and product is $m_1 m_2 = \dfrac{a}{b}$

   The quadratic equation in m whose roots are m_1 and m_2 is given by

   $m^2 - (m_1 + m_2)\, m + m_1 m_2 = 0$

   $\therefore\ m^2 - \left(-\dfrac{2h}{b}\right) m + \dfrac{a}{b} = 0$

$$bm^2 + 2hm + a = 0 \qquad .... (2)$$

Equation (2) is called the **auxiliary** equation of equation (1). Roots of equation (2) are slopes of lines represented by equation (1).

Solved Examples

Ex. 1) Show that lines represented by equation $x^2 - 2xy - 3y^2 = 0$ are distinct.

Solution : Comparing equation $x^2 - 2xy - 3y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get

$a = 1, h = -1$ and $b = -3$.

$$h^2 - ab = (-1)^2 - (1)(-3)$$
$$= 1 + 3$$
$$= 4 > 0$$

As $h^2 - ab > 0$, lines represented by equation $x^2 - 2xy - 3y^2 = 0$ are distinct.

Ex. 2) Show that lines represented by equation $x^2 - 6xy + 9y^2 = 0$ are coincident.

Solution : Comparing equation $x^2 - 6xy + 9y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get

$a = 1$, $h = -3$ and $b = 9$.

$$h^2 - ab = (-3)^2 - (1)(9)$$
$$= 9 - 9 = 0$$

As $h^2 - ab = 0$, lines represented by equation $x^2 - 6xy + 9y^2 = 0$ are coincident.

Ex. 3) Find the sum and the product of slopes of lines represented by $x^2 + 4xy - 7y^2 = 0$.

Solution : Comparing equation $x^2 + 4xy - 7y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get $a = 1$, $h = 2$ and $b = -7$.

If m_1 and m_2 are slopes of lines represented by this equation then

$$m_1 + m_2 = -\frac{2h}{b} \text{ and } m_1 m_2 = \frac{a}{b}.$$

$$\therefore \; m_1 + m_2 = \frac{-4}{-7} = \frac{4}{7} \text{ and } m_1 m_2 = \frac{1}{-7} = -\frac{1}{7}$$

Their sum is $\frac{4}{7}$ and products is $-\frac{1}{7}$

Ex. 4) Find the separate equations of lines represented by

i) $x^2 - 4y^2 = 0$

ii) $3x^2 - 7xy + 4y^2 = 0$

iii) $x^2 + 2xy - y^2 = 0$

iv) $5x^2 - 3y^2 = 0$

Solution : i) $x^2 - 4y^2 = 0$

$\therefore \; (x - 2y)(x + 2y) = 0$

Required separate equations are

$x - 2y = 0$ and $x + 2y = 0$

ii) $3x^2 - 7xy + 4y^2 = 0$

$\therefore\ 3x^2 - 3xy - 4xy + 4y^2 = 0$

$\therefore\ 3x(x - y) - 4y(x - y) = 0$

$\therefore\ (x - y)\,(3x - 4y) = 0$

Required separate equations are

$x - y = 0$ and $3x - 4y = 0$

iii) $x^2 + 2xy - y^2 = 0$

The corresponding auxiliary equation is $bm^2 + 2hm + a = 0$

$\therefore\ -m^2 + 2m + 1 = 0$

$m^2 - 2m - 1 = 0$

$\therefore\ m = \dfrac{2 \pm \sqrt{8}}{2} = \dfrac{2 \pm 2\sqrt{2}}{2}$

$= 1 \pm \sqrt{2}$

Slopes of these lines are

$m_1 = 1 + \sqrt{2}$ and $m_2 = 1 - \sqrt{2}$

$\therefore$ Required separate equations are

$y = m_1 x$ and $y = m_2 x$

$\therefore\ y = (1 + \sqrt{2}\,)\,x$ and $y = (1 - \sqrt{2}\,)x$

$\therefore\ (1 + \sqrt{2}\,)\,x - y = 0$ and $(1 - \sqrt{2}\,)x - y = 0$

iv) $5x^2 - 3y^2 = 0$

$\therefore\ (\sqrt{5}\,x)^2 - (\sqrt{3}\,y)^2 = 0$

$\therefore\ (\sqrt{5}\,x - \sqrt{3}\,y\,)\,(\sqrt{5}\,x + \sqrt{3}\,y\,) = 0$

$\therefore$ Required separate equations are

$\sqrt{5}\,x - \sqrt{3}\,y = 0$ and $\sqrt{5}\,x + \sqrt{3}\,y = 0$

Ex. 5) Find the value of k if $2x + y = 0$ is one of the lines represented by $3x^2 + kxy + 2y^2 = 0$.

Solution : Slope of the line $2x + y = 0$ is -2

As $2x + y = 0$ is one of the lines represented by $3x^2 + kxy + 2y^2 = 0$, -2 is a root of the auxiliary equation $2m^2 + km + 3 = 0$

$\therefore\ 2(-2)^2 + k(-2) + 3 = 0$

$\therefore\ 8 - 2k + 3 = 0$

$\therefore\ -2k + 11 = 0$

$\therefore\ 2k = 11 \qquad \therefore\ k = \dfrac{11}{2}.$

Alternative Method : As $2x + y = 0$ is one of the lines represented by $3x^2 + kxy + 2y^2 = 0$, co-ordinates of every point on the line $2x + y = 0$ satisfy the equation $3x^2 + kxy + 2y^2 = 0$.

As $(1, -2)$ is a point on the line $2x + y = 0$, it must satisfy the combined equation.

$\therefore\ 3(1)^2 + k(1)(-2) + 2(-2)^2 = 0$

$\therefore\ -2k + 11 = 0$

$\therefore\ 2k = 11 \qquad \therefore\ k = \dfrac{11}{2}.$

Ex. 6) Find the condition that the line $3x - 2y = 0$ coincides with one of the lines represented by $ax^2 + 2hxy + by^2 = 0$.

Solution : The corresponding auxiliary equation is $bm^2 + 2hm + a = 0$.

As line $3x - 2y = 0$ coincides with one of the lines given by $ax^2 + 2hxy + by^2 = 0$, its slope $\frac{3}{2}$ is a root of the auxiliary equation.

$\therefore\ \frac{3}{2}$ is a root of $bm^2 + 2hm + a = 0$

$$\therefore\ b\left(\frac{3}{2}\right)^2 + 2h\left(\frac{3}{2}\right) + a = 0$$

$$\therefore\ \frac{9}{4}b + 3h + a = 0$$

$\therefore\ 4a + 12h + 9b = 0$ is the required condition.

Ex.7) Find the combined equation of the pair of lines passing through the origin and perpendicular to the lines represented by $3x^2 + 2xy - y^2 = 0$.

Solution : Let m_1 and m_2 are slopes of lines represented by $3x^2 + 2xy - y^2 = 0$.

$$\therefore\ m_1 + m_2 = -\frac{2h}{b} = \frac{-2}{-1} = 2$$

$$\text{and } m_1 m_2 = \frac{a}{b} = \frac{3}{-1} = -3$$

Now required lines are perpendicular to given lines.

$\therefore$ Their slopes are $-\frac{1}{m_1}$ and $-\frac{1}{m_2}$

And required lines pass through the origin.

$\therefore$ Their equations are $y = -\frac{1}{m_1}x$ and $y = -\frac{1}{m_2}x$

$\therefore\ m_1 y = -x$ and $m_2 y = -x$

$\therefore\ x + m_1 y = 0$ and $x + m_2 y = 0$

Their combined equation is $(x + m_1 y)(x + m_2 y) = 0$

$\therefore\ x^2 + (m_1 + m_2)xy + m_1 m_2 y^2 = 0$

$\therefore\ x^2 + (2)xy + (-3)y^2 = 0$

$\therefore\ x^2 + 2xy - 3y^2 = 0$

Ex.8) Find the value of k, if slope of one of the lines represented by $4x^2 + kxy + y^2 = 0$ is four times the slope of the other line.

Solution : Let slopes of the lines represented by $4x^2 + kxy + y^2 = 0$ be m and $4m$,

their sum is $m + 4m = 5m$

But their sum is $\frac{-2h}{b} = \frac{-k}{1} = -k$

$\therefore\ 5m = -k$

$\therefore\ m = \frac{-k}{5}$ ... (1)

Now their product is $(m)(4m) = 4m^2$

But their product is $\frac{a}{b} = \frac{4}{1} = 4$

$\therefore\ 4m^2 = 4$

$\therefore\ m^2 = 1$..... (2)

From (1) and (2), we get

$$\left(\frac{-k}{5}\right)^2 = 1$$

$\therefore\ k^2 = 25$

$\therefore\ k = \pm 5$

Exercise 4.1

1) Find the combined equation of the following pairs of lines:

i) $2x + y = 0$ and $3x - y = 0$

ii) $x + 2y - 1 = 0$ and $x - 3y + 2 = 0$

iii) Passing through (2,3) and parallel to the co-ordinate axes.

iv) Passing through (2,3) and perpendicular to lines $3x + 2y - 1 = 0$ and $x - 3y + 2 = 0$

v) Passsing through $(-1,2)$,one is parallel to $x + 3y - 1 = 0$ and the other is perpendicular to $2x - 3y - 1 = 0$.

2) Find the separate equations of the lines represented by following equations:

i) $3y^2 + 7xy = 0$

ii) $5x^2 - 9y^2 = 0$

iii) $x^2 - 4xy = 0$

iv) $3x^2 - 10xy - 8y^2 = 0$

v) $3x^2 - 2\sqrt{3}\,xy - 3y^2 = 0$

vi) $x^2 + 2(\text{cosec}\ \alpha)xy + y^2 = 0$

vii) $x^2 + 2xy\tan\alpha - y^2 = 0$

3) Find the combined equation of a pair of lines passing through the origin and perpendicular to the lines represented by following equations :

i) $5x^2 - 8xy + 3y^2 = 0$

ii) $5x^2 + 2xy - 3y^2 = 0$

iii) $xy + y^2 = 0$

iv) $3x^2 - 4xy = 0$

4) Find k if,

i) the sum of the slopes of the lines represented by $x^2 + kxy - 3y^2 = 0$ is twice their product.

ii) slopes of lines represent by $3x^2 + kxy - y^2 = 0$ differ by 4.

iii) slope of one of the lines given by $kx^2 + 4xy - y^2 = 0$ exceeds the slope of the other by 8.

5) Find the condition that :

i) the line $4x + 5y = 0$ coincides with one of the lines given by $ax^2 + 2hxy + by^2 = 0$.

ii) the line $3x + y = 0$ may be perpendicular to one of the lines given by $ax^2 + 2hxy + by^2 = 0$.

6) If one of the lines given by $ax^2 + 2hxy + by^2 = 0$ is perpendicular to $px + qy = 0$ then show that $ap^2 + 2hpq + bq^2 = 0$.

7) Find the combined equation of the pair of lines passing through the origin and making an equilateral triangle with the line y = 3.

8) If slope of one of the lines given by $ax^2 + 2hxy + by^2 = 0$ is four times the other then show that $16h^2 = 25ab$.

9) If one of the lines given by $ax^2 + 2hxy + by^2 = 0$ bisects an angle between co-ordinate axes then show that $(a + b)^2 = 4h^2$.

4.3 Angle between lines represented by $ax^2 + 2hxy + by^2 = 0$:

If we know slope of a line then we can find the angles made by the line with the co-ordinate axes. In equation $ax^2 + 2hxy + by^2 = 0$ if $b = 0$ then one of the lines is the Y - axis. Using the slope of the other line we can find the angle between them. In the following discussion we assume that $b \neq 0$, so that slopes of both lines will be defined.

If m_1 and m_2 are slopes of these lines then $m_1 m_2 = \dfrac{a}{b}$

We know that lines having slopes m_1 and m_2 are perpendicular to each other if and only if $m_1 m_2 = -1$.

$$\therefore \ \frac{a}{b} = -1$$

$$\therefore \ a = -b$$

$$\therefore \ a + b = 0$$

Thus lines represented by $ax^2 + 2hxy + by^2 = 0$ are perpendicular to each other if and only if a + b = 0.

If lines are **not perpendicular** to each other then the acute angle between them can be obtained by using the following theorem.

Theorem 4.4 : The acute angle θ between the lines represented by $ax^2 + 2hxy + by^2 = 0$ is given by

$$\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right|$$

Proof : Let m_1 and m_2 be slopes of lines represented by the equation $ax^2 + 2hxy + by^2 = 0$.

$$\therefore \ m_1 + m_2 = -\frac{2h}{b} \text{ and } m_1 m_2 = \frac{a}{b}$$

$$\therefore \ (m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1 m_2$$

$$= \left(\frac{2h}{b}\right)^2 - 4\left(\frac{a}{b}\right)$$

$$= \frac{4h^2}{b^2} - \frac{4ab}{b^2}$$

$$= \frac{4h^2 - 4ab}{b^2}$$

$$= \frac{4\left(h^2 - ab\right)}{b^2}$$

$$\therefore m_1 - m_2 = \pm \frac{2\sqrt{h^2 - ab}}{b}$$

As θ is the acute angle between the lines,

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$= \left|\frac{\pm\frac{2\sqrt{h^2 - ab}}{b}}{1 + \frac{a}{b}}\right|$$

$$= \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$$

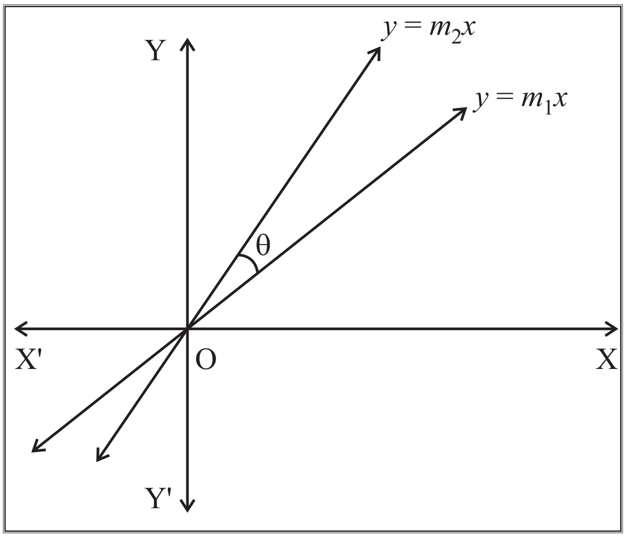


Figure 4.4

Remark : Lines represented by $ax^2 + 2hxy + by^2 = 0$ are coincident if and only if $m_1 = m_2$

$\therefore \quad m_1 - m_2 = 0$

$\therefore \quad \frac{2\sqrt{h^2 - ab}}{b} = 0$

$\therefore h^2 - ab = 0$

$\therefore h^2 = ab$

Lines represented by $ax^2 + 2hxy + by^2 = 0$ are coincident if and only if $h^2 = ab$.

Solved Examples

Ex.1) Show that lines represented by $3x^2 - 4xy - 3y^2 = 0$ are perpendicular to each other.

Solution : Comparing given equation with $ax^2 + 2hxy + by^2 = 0$ we get $a = 3$, $h = -2$ and $b = -3$.

As $a + b = 3 + (-3) = 0$, lines represented by $3x^2 - 4xy - 3y^2 = 0$ are perpendicular to each other.

Ex. 2) Show that lines represented by $x^2 + 4xy + 4y^2 = 0$ are coincident.

Solution : Comparing given equation with $ax^2 + 2hxy + by^2 = 0$, we get $a = 1$, $h = 2$ and $b = 4$.

As, $h^2 - ab = (2)^2 - (1)(4)$

$= 4 - 4 = 0$

$\therefore$ Lines represented by $x^2 + 4xy + 4y^2 = 0$ are coincident.

Ex.3) Find the acute angle between lines represented by:

i) $x^2 + xy = 0$

ii) $x^2 - 4xy + y^2 = 0$

iii) $3x^2 + 2xy - y^2 = 0$

iv) $2x^2 - 6xy + y^2 = 0$

v) $xy + y^2 = 0$

Solution :

i) Comparing equation $x^2 + xy = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get $a = 1$, $h = \frac{1}{2}$ and $b = 0$.

Let θ be the acute angle between them.

$$\therefore \quad \tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right| = \left|\frac{2\sqrt{\frac{1}{4} - 0}}{1}\right| = 1$$

$$\therefore \quad \theta = 45° = \frac{\pi}{4}$$

ii) Comparing equation $x^2 - 4xy + y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get $a = 1$, $h = -2$ and $b = 1$.

Let θ be the acute angle between them.

$$\therefore \tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right|$$

$$= \left|\frac{2\sqrt{4-1}}{2}\right| = \sqrt{3}$$

$$\therefore \theta = 60° = \frac{\pi}{3}$$

iii) Comparing equation $3x^2 + 2xy - y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get $a = 3$, $h = 1$ and $b = -1$.

Let θ be the acute angle between them.

$$\therefore \quad \tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right|$$

$$= \left|\frac{2\sqrt{1+3}}{2}\right| = 2.$$

$$\therefore \quad \theta = \tan^{-1}(2)$$

iv) Comparing equation $2x^2 - 6xy + y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get $a = 2$, $h = -3$ and $b = 1$.

Let θ be the acute angle between them.

$$\therefore \tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right|$$

$$= \left|\frac{2\sqrt{9-2}}{3}\right| = \frac{2\sqrt{7}}{3} \quad \therefore \theta = \tan^{-1}\left(\frac{2\sqrt{7}}{3}\right).$$

v) Comparing equation $xy + y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get $a = 0$, $h = \frac{1}{2}$ and $b = 1$.

Let θ be the acute angle between them.

$$\therefore \tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right|$$

$$= \left|\frac{2\sqrt{\frac{1}{4} - 0}}{1}\right| = 1$$

$$\therefore \theta = 45° = \frac{\pi}{4}.$$

Ex.4) Find the combined equation of lines passing through the origin and making angle $\frac{\pi}{6}$ with the line $3x + y - 6 = 0$.

Solution : Let m be the slope of one of the lines which make angle $\frac{\pi}{6}$ with the line $3x+y-6 = 0$. Slope of the given line is -3.

$$\therefore \tan\frac{\pi}{6} = \left|\frac{m-(-3)}{1+m(-3)}\right|$$

$$\therefore \frac{1}{\sqrt{3}} = \left|\frac{m+3}{1-3m}\right|$$

$\therefore (1 - 3m)^2 = 3(m + 3)^2$

$\therefore 9m^2 - 6m + 1 = 3(m^2 + 6m + 9)$

$\therefore 9m^2 - 6m + 1 = 3m^2 + 18m + 27$

$\therefore 6m^2 - 24m - 26 = 0$

$\therefore 3m^2 - 12m - 13 = 0$

This is the auxiliary equation of the required combined equation.

The required combined equation is $-13x^2 - 12xy + 3y^2 = 0$

$\therefore 13x^2 + 12xy - 3y^2 = 0$

Ex. 5) Find the combined equation of lines passing through the origin and each of which making angle 60° with the X - axis.

Solution :

Let m be the slope of one of the required lines.

The slope of the X - axis is 0. As required lines make angle 60° with the X - axis,

$$\tan 60° = \left|\frac{m-0}{1+(m)(0)}\right|$$

$\therefore \sqrt{3} = |m|$

$\therefore m^2 = 3$

$\therefore m^2 + 0m - 3 = 0$ is the auxiliary equation.

$\therefore$ The required combined equation is

$-3x^2 + 0xy + y^2 = 0$

$\therefore\ 3x^2 - y^2 = 0$

Alternative Method: As required lines make angle 60° with the X - axis, their inclination are 60° and 120°. Hence their slopes are $\sqrt{3}$ and $-\sqrt{3}$.

Lines pass thorugh the origin. Their equations are $y = \sqrt{3}x$ and $y = -\sqrt{3}x$

$\therefore\ \sqrt{3}x - y = 0$ and $\sqrt{3}x + y = 0$

Their combined equation is $(\sqrt{3}x - y)(\sqrt{3}x + y) = 0$

$\therefore\ 3x^2 - y^2 = 0$

Exercise 4.2

1) Show that lines represented by $3x^2 - 4xy - 3y^2 = 0$ are perpendicular to each other.
2) Show that lines represented by $x^2 + 6xy + gy^2 = 0$ are coincident.
3) Find the value of k if lines represented by $kx^2 + 4xy - 4y^2 = 0$ are perpendicular to each other.
4) Find the measure of the acute angle between the lines represented by:
   i) $3x^2 - 4\sqrt{3}xy + 3y^2 = 0$
   ii) $4x^2 + 5xy + y^2 = 0$
   iii) $2x^2 + 7xy + 3y^2 = 0$
   iv) $(a^2 - 3b^2)x^2 + 8abxy + (b^2 - 3a^2)y^2 = 0$
5) Find the combined equation of lines passing through the origin each of which making an angle of 30° with the line $3x + 2y - 11 = 0$
6) If the angle between lines represented by $ax^2 + 2hxy + by^2 = 0$ is equal to the angle between lines represented by $2x^2 - 5xy + 3y^2 = 0$ then show that $100(h^2 - ab) = (a + b)^2$.
7) Find the combined equation of lines passing through the origin and each of which making angle 60° with the Y- axis.

4.4 General Second Degree Equation in x and y:

Equation of the form $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, where at least one of a,b,h is not zero, is called a general second degree equation in x and y.

Theorem 4.5 : The combined equation of two lines is a general second degree equation in x and y.

Proof: Let $u \equiv a_1x + b_1y + c_1$ and $v \equiv a_2x + b_2y + c_2$. Equations $u = 0$ and $v = 0$ represent lines. Their combined equation is $uv = 0$.

$\therefore\ (a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = 0$

$a_1a_2x^2 + a_1b_2xy + a_1c_2x + b_1a_2xy + b_1b_2y^2 + b_1c_2y + c_1a_2x + c_1b_2y + c_1c_2 = 0$

Writing $a_1a_2 = a$, $b_1b_2 = b$, $a_1b_2 + a_2b_1 = 2h$, $a_1c_2 + a_2c_1 = 2g$, $b_1c_2 + b_2c_1 = 2f$, $c_1c_2 = c$,

we get, $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, which is the general equation of degree two in x and y.

Remark : The converse of the above theorem is **not true**. Every general second degree equation in x and y **need not** represent a pair of lines. For example $x^2 + y^2 = 25$ is a general second degree equation in x and y but it does not represent a pair of lines. It represents a circle.

Equation $x^2 + y^2 - 4x + 6y + 13 = 0$ is also a general second degree equation which does not represent a pair of lines. How to identify that whether the given equation represents a pair of lines or not?

4.4.1 The necessary conditions for a general second degree equation.

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ to represent a pair of lines are:

i) $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ ii) $h^2 - ab \geq 0$

Remarks :

If equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of lines then

1) These lines are parallel to the line represented by $ax^2 + 2hxy + by^2 = 0$

2) The acute angle between them is given by $\tan\theta = \left|\dfrac{2\sqrt{h^2 - ab}}{a+b}\right|$

3) Condition for lines to be perpendicular to each other is $a + b = 0$.

4) Condition for lines to be parallel to each other is $h^2 - ab = 0$.

5) Condition for lines to intersect each other is $h^2 - ab > 0$ and the co-ordinates of their point of intersection are $\left(\dfrac{hf - bg}{ab - h^2}, \dfrac{gh - af}{ab - h^2}\right)$

6) The expression $abc + 2fgh - af^2 - bg^2 - ch^2$ is the expansion of the determinant $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

7) The joint equation of the bisector of the angle between the lines represented by $ax^2+2hxy+by^2 = 0$ is $hx^2-(a-b)xy - hy^2 = 0$. Here coefficient of x^2 + coefficient of $y^2 = 0$. Hence bisectors are perpendicular to each other

Solved Examples

Ex.1) Show that equation $x^2 - 6xy + 5y^2 + 10x - 14y + 9 = 0$ represents a pair of lines. Find the acute angle between them. Also find the point of their intersection.

Solution: We have $x^2 - 6xy + 5y^2 = (x - 5y)(x - y)$

Suppose $x^2 - 6xy + 5y^2 + 10x - 14y + 9 = (x - 5y + c)(x - y + k)$

$\therefore x^2 - 6xy + 5y^2 + 10x - 14y + 9 = x^2 - 6xy + 5y^2 + (c + k)x - (c + 5k)y + ck$

$\therefore c + k = 10$, $c + 5k = 14$ and $ck = 9$

We observe that $c = 9$ and $k = 1$ satisfy all three equations.

$\therefore$ Given general equation can be factorized as $(x - 5y + 9)(x - y + 1) = 0$

$\therefore$ Given equation represents a pair of intersecting lines.

The acute angle between them is given by

$$\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right| = \left|\frac{2\sqrt{(-3)^2 - (1)(5)}}{1+5}\right| = \frac{2}{3}$$

$$\therefore \theta = \tan^{-1}\left(\frac{2}{3}\right)$$

Their point of intersection is given by

$$\left(\frac{hf-bg}{ab-h^2},\frac{gh-af}{ab-h^2}\right)=\left(\frac{21-25}{5-9},\frac{-15+7}{5-9}\right)=(1,2)$$

Remark :

Note that condition $abc+2fgh-af^2-bg^2-ch^2=0$ is not sufficient for equation to represent a pair of lines. We can't use this condition to show that given equation represents a pair of lines.

Ex.2) Find the value of k if the equation $2x^2+4xy-2y^2+4x+8y+k=0$ represents a pair of lines.

Solution: Comparing given equation with $ax^2+2hxy+by^2+2gx+2fy+c=0$ we get

$a=2,\ b=-2,\ c=k,\ f=4,\ g=2,\ h=2.$

As given equation represents a pair of lines, it must satisfy the necessary condition.

$\therefore\ abc+2fgh-af^2-bg^2-ch^2=0$

$\therefore\ (2)(-2)(k)+2(4)(2)(2)-2(4)^2-(-2)(2)^2-(k)(2)^2=0$

$\therefore -4k+32-32+8-4k=0$

$\therefore\ 8k=8$

$\therefore\ k=1.$

Ex.3) Find p and q if the equation $2x^2+4xy-py^2+4x+qy+1=0$ represents a pair of perpendicular lines.

Solution: Comparing given equation with $ax^2+2hxy+by^2+2gx+2fy+c=0$ we get

$a=2,\ b=-p,\ c=1,\ f=\frac{q}{2},\ g=2,\ h=2$

As lines are perpendicular to each other, a + b= 0

$\therefore\ 2+(-p)=0$

$\therefore\ p=2$

As given equation represents a pair of lines, it must satisfy the necessary condition.

$\therefore\ abc+2fgh-af^2-bg^2-ch^2=0$

$$\therefore\ (2)(-p)(1)+2\left(\frac{q}{2}\right)(2)(2)-2\left(\frac{q}{2}\right)^2-(-p)(2)^2-1(2)^2=0$$

$$\therefore\ -2p+4q-\frac{q^2}{2}+4p-4=0$$

$$\therefore\ 2p+4q-\frac{q^2}{2}-4=0 \qquad ...(1)$$

substituting $p=2$ in (1), we get

$$\therefore\ 2(2)+4q-\frac{q^2}{2}-4=0$$

$$\therefore\ 4q-\frac{q^2}{2}=0\ \therefore\ 8q-q^2=0$$

$\therefore\ q(8-q)=0$

$\therefore\ q=0$ or $q=8$

Ex.4) Δ OAB is formed by lines $x^2 - 4xy + y^2 = 0$ and the line $x + y - 2 = 0$. Find the equation of the median of the triangle drawn from O.

Solution : Let the co-ordinates of A and B be (x_1, y_1) and (x_2, y_2) respectively.

The midpoint of segment AB is

$$P\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

The co-ordinates of A and B can be obtained by solving equations $x + y - 2 = 0$ and $x^2 - 4xy + y^2 = 0$ simultaneously.

put $y = 2 - x$ in $x^2 - 4xy + y^2 = 0$.

$x^2 - 4x(2 - x) + (2 - x)^2 = 0$

$\therefore\ 6x^2 - 12x + 4 = 0$

$\therefore\ 3x^2 - 6x + 2 = 0$

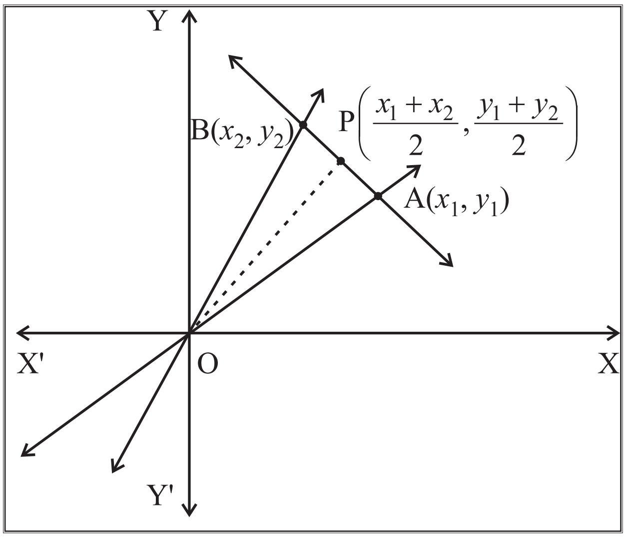


Figure 4.5

x_1 and x_2 are roots of this equation.

$$x_1 + x_2 = -\frac{-6}{3} = 2$$

$$\therefore\ \frac{x_1+x_2}{2} = 1$$

The x co-ordinate of P is 1.

As P lies on the line $x + y - 2 = 0$

$\therefore\ 1 + y - 2 = 0 \quad \therefore\ y = 1$

$\therefore$ Co-ordinates of P are (1,1).

The equation of the median OP is $\dfrac{y-0}{1-0} = \dfrac{x-0}{1-0}$

$\therefore\ y = x \quad \therefore\ x - y = 0.$

Exercise 4.3

1) Find the joint equation of the pair of lines:
   i) Through the point (2, –1) and parallel to lines represented by $2x^2 + 3xy - 9y^2 = 0$
   ii) Through the point (2, –3) and parallel to lines represented by $x^2 + xy - y^2 = 0$
2) Show that equation $x^2 + 2xy + 2y^2 + 2x + 2y + 1 = 0$ does not represent a pair of lines.
3) Show that equation $2x^2 - xy - 3y^2 - 6x + 19y - 20 = 0$ represents a pair of lines.
4) Show the equation $2x^2 + xy - y^2 + x + 4y - 3 = 0$ represents a pair of lines. Also find the acute angle between them.
5) Find the separate equation of the lines represented by the following equations :
   i) $(x - 2)^2 - 3(x - 2)(y + 1) + 2(y + 1)^2 = 0$
   ii) $10(x + 1)^2 + (x + 1)(y - 2) - 3(y - 2)^2 = 0$
6) Find the value of k if the following equations represent a pair of lines :
   i) $3x^2 + 10xy + 3y^2 + 16y + k = 0$
   ii) $kxy + 10x + 6y + 4 = 0$
   iii) $x^2 + 3xy + 2y^2 + x - y + k = 0$

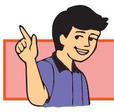
7) Find p and q if the equation $px^2 - 8xy + 3y^2 + 14x + 2y + q = 0$ represents a pair of perpendicular lines.

8) Find p and q if the equation $2x^2 + 8xy + py^2 + qx + 2y - 15 = 0$ represents a pair of parallel lines.

9) Equations of pairs of opposite sides of a parallelogram are $x^2 - 7x + 6 = 0$ and $y^2 - 14y + 40 = 0$. Find the joint equation of its diagonals.

10) ΔOAB is formed by lines $x^2 - 4xy + y^2 = 0$ and the line $2x + 3y - 1 = 0$. Find the equation of the median of the triangle drawn from O.

11) Find the co-ordinates of the points of intersection of the lines represented by $x^2 - y^2 - 2x + 1 = 0$.



Let's remember!

- An equation which represents two lines is called the combined equation of those two lines.
- The equation $uv = 0$ represents the combined equation of lines $u = 0$ and $v = 0$.
- The sum of the indices of all variables in a term is called the degree of the term.
- An equation in which the degree of every term is same, is called a homogeneous equation.
- The combined equation of a pair of lines passing through the origin is a homogeneous equation of degree two in x and y.
- Every homogeneous equation of degree two need not represents a pair of lines.
- A homogeneous equation of degree two in x and y, $ax^2 + 2hxy + by^2 = 0$ represents a pair of lines passing through the origin if $h^2 - ab \geq 0$.
- If $h^2 - ab > 0$ then lines are distinct.
- If $h^2 - ab = 0$ then lines are coincident.
- Slopes of these lines are $m_1 = \dfrac{-h - \sqrt{h^2 - ab}}{b}$ and $m_2 = \dfrac{-h + \sqrt{h^2 - ab}}{b}$
- Their sum is, $m_1 + m_2 = -\dfrac{2h}{b}$ and product is, $m_1 m_2 = \dfrac{a}{b}$
- The quadratic equation in m whose roots are m_1 and m_2 is given by $bm^2 + 2hm + a = 0$, called the **auxiliary** equation.
- Lines represented by $ax^2 + 2hxy + by^2 = 0$ are perpendicular to each other if and only if $a + b = 0$.
- Lines represented by $ax^2 + 2hxy + by^2 = 0$ are coincident if and only if $h^2 - ab = 0$.
- The acute angle θ between the lines represented by $ax^2 + 2hxy + by^2 = 0$ is given by

  $$\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right|$$

- Equation of the form $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is called a general second degree equation in x and y.
- The combined equation of two lines is a general second degree equation in x and y.
- The necessary conditions for a general second degree equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ to represent a pair of lines are:
  i) $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ ii) $h^2 - ab \geq 0$

- The expression $abc + 2fgh - af^2 - bg^2 - ch^2$ is the expansion of the determinant $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

- If equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of lines then
  1) These lines are parallel to the lines represented by $ax^2 + 2hxy + by^2 = 0$.

  2) The acute angle between them is given by $\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right|$

  3) Condition for lines to be perpendicular to each other is $a + b = 0$.
  4) Condition for lines to be parallel to each other is $h^2 - ab = 0$.
  5) Condition for lines to intersect each other is $h^2 - ab \geq 0$ and the co-ordinates of their point of intersection are $\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$

MISCELLANEOUS EXERCISE 4

I : Choose correct alternatives.

1) If the equation $4x^2 + hxy + y^2 = 0$ represents two coincident lines, then $h =$ __________.

A) ± 2 B) ± 3
C) ± 4 D) ± 5

2) If the lines represented by $kx^2 - 3xy + 6y^2 = 0$ are perpendicular to each other then __________.

A) $k = 6$ B) $k = -6$
C) $k = 3$ D) $k = -3$

3) Auxiliary equation of $2x^2 + 3xy - 9y^2 = 0$ is __________.

A) $2m^2 + 3m - 9 = 0$ B) $9m^2 - 3m - 2 = 0$
C) $2m^2 - 3m + 9 = 0$ D) $-9m^2 - 3m + 2 = 0$

4) The difference between the slopes of the lines represented by $3x^2 - 4xy + y^2 = 0$ is __________.

A) 2 B) 1
C) 3 D) 4

5) If the two lines $ax^2 + 2hxy + by^2 = 0$ make angles α and β with X-axis, then $\tan(\alpha + \beta) =$ ______.

A) $\frac{h}{a+b}$ B) $\frac{h}{a-b}$

C) $\frac{2h}{a+b}$ D) $\frac{2h}{a-b}$

6) If the slope of one of the two lines $\frac{x^2}{a} + \frac{2xy}{h} + \frac{y^2}{b} = 0$ is twice that of the other, then $ab:h^2 =$ ____.

A) 1 : 2 B) 2 : 1

C) 8 : 9 D) 9 : 8

7) The joint equation of the lines through the origin and perpendicular to the pair of lines $3x^2 + 4xy - 5y^2 = 0$ is _________.

A) $5x^2 + 4xy - 3y^2 = 0$ B) $3x^2 + 4xy - 5y^2 = 0$

C) $3x^2 - 4xy + 5y^2 = 0$ D) $5x^2 + 4xy + 3y^2 = 0$

8) If acute angle between lines $ax^2 + 2hxy + by^2 = 0$ is, $\frac{\pi}{4}$ then $4h^2 =$ _________.

A) $a^2 + 4ab + b^2$ B) $a^2 + 6ab + b^2$

C) $(a + 2b)(a + 3b)$ D) $(a - 2b)(2a + b)$

9) If the equation $3x^2 - 8xy + qy^2 + 2x + 14y + p = 1$ represents a pair of perpendicular lines then the values of p and q are respectively _________.

A) -3 and -7 B) -7 and -3

C) 3 and 7 D) -7 and 3

10) The area of triangle formed by the lines $x^2 + 4xy + y^2 = 0$ and $x - y - 4 = 0$ is _________.

A) $\frac{4}{\sqrt{3}}$ Sq. units B) $\frac{8}{\sqrt{3}}$ Sq. units

C) $\frac{16}{\sqrt{3}}$ Sq. units D) $\frac{15}{\sqrt{3}}$ Sq. units

11) The combined equation of the co-ordinate axes is _________.

A) $x + y = 0$ B) $x\,y = k$

C) $xy = 0$ D) $x - y = k$

12) If $h^2 = ab$, then slope of lines $ax^2 + 2hxy + by^2 = 0$ are in the ratio _________.

A) 1 : 2 B) 2 : 1

C) 2 : 3 D) 1 : 1

13) If slope of one of the lines $ax^2 + 2hxy + by^2 = 0$ is 5 times the slope of the other, then $5h^2 =$ _________.

A) ab B) $2\,ab$

C) $7\,ab$ D) $9\,ab$

14) If distance between lines $(x - 2y)^2 + k(x - 2y) = 0$ is 3 units, then $k =$

A) $\pm\ 3$ B) $\pm\ 5\sqrt{5}$

C) 0 D) $\pm\ 3\ \sqrt{5}$

II. Solve the following.

1) Find the joint equation of lines:

i) $x - y = 0$ and $x + y = 0$

ii) $x + y - 3 = 0$ and $2x + y - 1 = 0$

iii) Passing through the origin and having slopes 2 and 3.

iv) Passing through the origin and having inclinations 60° and 120°.

v) Passing through (1,2) amd parallel to the co-ordinate axes.

vi) Passing through (3,2) and parallel to the line $x = 2$ and $y = 3$.

vii)Passing through $(-1,2)$ and perpendicular to the lines $x + 2y + 3 = 0$ and $3x - 4y - 5 = 0$.

viii) Passing through the origin and having slopes $1 + \sqrt{3}$ and $1 - \sqrt{3}$

ix) Which are at a distance of 9 units from the Y - axis.

x) Passing through the point (3,2), one of which is parallel to the line $x - 2y = 2$ and other is perpendicular to the line $y = 3$.

xi) Passing through the origin and perpendicular to the lines $x + 2y = 19$ and $3x + y = 18$.

2) Show that each of the following equation represents a pair of lines.

i) $x^2 + 2xy - y^2 = 0$

ii) $4x^2 + 4xy + y^2 = 0$

iii) $x^2 - y^2 = 0$

iv) $x^2 + 7xy - 2y^2 = 0$

v) $x^2 - 2\sqrt{3}xy - y^2 = 0$

3) Find the separate equations of lines represented by the following equations:

i) $6x^2 - 5xy - 6y^2 = 0$

ii) $x^2 - 4y^2 = 0$

iii) $3x^2 - y^2 = 0$

iv) $2x^2 + 2xy - y^2 = 0$

4) Find the joint equation of the pair of lines through the origin and perpendicular to the lines given by :

i) $x^2 + 4xy - 5y^2 = 0$

ii) $2x^2 - 3xy - 9y^2 = 0$

iii) $x^2 + xy - y^2 = 0$

5) Find k if

i) The sum of the slopes of the lines given by $3x^2 + kxy - y^2 = 0$ is zero.

ii) The sum of slopes of the lines given by $2x^2 + kxy - 3y^2 = 0$ is equal to their product.

iii) The slope of one of the lines given by $3x^2 - 4xy + ky^2 = 0$ is 1.

iv) One of the lines given by $3x^2 - kxy + 5y^2 = 0$ is perpendicular to the $5x + 3y = 0$.

v) The slope of one of the lines given by $3x^2 + 4xy + ky^2 = 0$ is three times the other.

vi) The slopes of lines given by $kx^2 + 5xy + y^2 = 0$ differ by 1.

vii) One of the lines given by $6x^2 + kxy + y^2 = 0$ is $2x + y = 0$.

6) Find the joint equation of the pair of lines which bisect angle between the lines given by $x^2 + 3xy + 2y^2 = 0$

7) Find the joint equation of the pair of lies through the origin and making equilateral triangle with the line $x = 3$.

8) Show that the lines $x^2 - 4xy + y^2 = 0$ and $x + y = 10$ contain the sides of an equilateral triangle. Find the area of the triangle.

9) If the slope of one of the lines represented by $ax^2 + 2hxy + by^2 = 0$ is three times the other then prove that $3h^2 = 4ab$.

10) Find the combined equation of the bisectors of the angles between the lines represented by $5x^2 + 6xy - y^2 = 0$.

11) Find a if the sum of slope of lines represented by $ax^2 + 8xy + 5y^2 = 0$ is twice their product.

12) If line $4x - 5y = 0$ coincides with one of the lines given by $ax^2 + 2hxy + by^2 = 0$ then show that $25a + 40h + 16b = 0$.

13) Show that the following equations represent a pair of lines, find the acute angle between them.

i) $9x^2 - 6xy + y^2 + 18x - 6y + 8 = 0$

ii) $2x^2 + xy - y^2 + x + 4y - 3 = 0$

iii) $(x-3)^2 + (x-3)(y-4) - 2(y-4)^2 = 0$

14) Find the combined equation of pair of lines through the origin each of which makes angle of 60° with the Y-axis.

15) If lines represented by $ax^2 + 2hxy + by^2 = 0$ make angles of equal measures with the co-ordinate axes then show that $a = \pm b$.

16) Show that the combined equation of a pair of lines through the origin and each making an angle of α with the line $x + y = 0$ is $x^2 + 2(\sec 2\alpha)\, xy + y^2 = 0$.

17) Show that the line $3x + 4y + 5 = 0$ and the lines $(3x + 4y)^2 - 3(4x - 3y)^2 = 0$ form an equilateral triangle.

18) Show that lines $x^2 - 4xy + y^2 = 0$ and $x + y = \sqrt{6}$ form an equilateral triangle. Find its area and perimeter.

19) If the slope of one of the lines given by $ax^2 + 2hxy + by^2 = 0$ is square of the other then show that $a^2b + ab^2 + 8h^3 = 6abh$.

20) Prove that the product of lengths of perpendiculars drawn from P (x_1, y_1) to the lines repersented by $ax^2 + 2hxy + by^2 = 0$ is $\left|\dfrac{ax_1^2 + 2hx_1y_1 + by_1^2}{\sqrt{(a-b)^2 + 4h^2}}\right|$

21) Show that the difference between the slopes of lines given by $(\tan^2\theta + \cos^2\theta)x^2 - 2xy\tan\theta + (\sin^2\theta)y^2 = 0$ is two.

22) Find the condition that the equation $ay^2 + bxy + ex + dy = 0$ may represent a pair of lines.

23) If the lines given by $ax^2 + 2hxy + by^2 = 0$ form an equilateral triangle with the line $lx + my = 1$ then show that $(3a + b)(a + 3b) = 4h^2$.

24) If line $x + 2 = 0$ coincides with one of the lines represented by the equation $x^2 + 2xy + 4y + k = 0$ then show that $k = -4$.

25) Prove that the combined equation of the pair of lines passing through the origin and perpendicular to the lines represented by $ax^2 + 2hxy + by^2 = 0$ is $bx^2 - 2hxy + ay^2 = 0$

26) If equation $ax^2 - y^2 + 2y + c = 1$ represents a pair of perpendicular lines then find a and c.

✦✦✦✦✦

26) $\frac{\pi}{4}$

27) $\frac{1}{\sqrt{3}}$

28) 0

29) $\frac{1}{6}$

♦♦♦♦♦

4. Pair of Straight Lines

Exercise 4.1

1) (i) $6x^2 + xy - y^2 = 0$

(ii) $x^2 - xy - 6y^2 + x + 7y - 2 = 0$

(iii) $xy - 3x - 2y + 6 = 0$

(iv) $6x^2 - 7xy - 3y^2 - 3x + 32y - 45 = 0$

(v) $3x^2 + 11xy + 6y^2 - 16x - 13y + 5 = 0$

2) (i) $y = 0,\ 7x + 3y = 0$

(ii) $\sqrt{5}\,x - 3y = 0,\ \sqrt{5}\,x + 3y = 0,$

(iii) $x = 0,\ x - 4y = 0$

(iv) $3x + 2y = 0,\ x - 4y = 0$

(v) $3x + \sqrt{3}\,y = 0,\ x = \sqrt{3}\,y = 0$

(vi) $(\operatorname{cosec}\alpha - \cot\alpha)x + y - 0,\ (\operatorname{cosec}\alpha + \cot\alpha)x - y = 0$

(vii) $(\sec\alpha - \tan\alpha)x + y = 0,\ (\sec\alpha + \tan\alpha)x - y = 0$

3) (i) $3x^2 + 8xy + 5y^2 = 0$

(ii) $x^2 + 2xy - 5y^2 = 0$

(iii) $x^2 - xy = 0$

(iv) $4xy + 3y^2 = 0$

4) (i) -2 (ii) ± 2 (iii) 12

5) (i) $25a + 16b = 40h$ (ii) $9a + 6h + b = 0$

6) $ap^2 + 2hpq + bq^2 = 0$

7) $3x^2 - y^2 = 0$

Exercise 4.2

3) k = 4 4) i) 30° ii) $\tan^{-1}\left(\frac{3}{5}\right)$ iii) 45° iv) 60°

5) $23x^2 + 48xy + 3y^2 = 0$ 7) $x^2 - 3y^2 = 0$

Exercise 4.3

1) (i) $2x^2 + 3xy - 9y^2 - 5x - 24y - 7 = 0$ (ii) $x^2 + xy - y^2 - x - 8y - 11 = 0$

2) $h^2 - ab = -1 < 0$

3) $2x - 3y + 4 = 0$ and $x + y - 5 = 0$ are separate equations of lines.

4) $2x - y + 3 = 0$ and $x + y - 1 = 0$ are separate equations. $\theta = \tan^{-1}(3)$.

5) (i) $x - y - 3 = 0, x - 2y - 4 = 0$ (ii) $2x - y + 4 = 0, 5x + 3y - 1 = 0$

6) (i) -12 (ii) 15 (iii) -6

7) $p = -3, q = -8$

8) $p = 8, q = 1$

9) $36x^2 - 25xy - 252x + 350y - 784 = 0$

10) $7x - 8y = 0$

11) $(1, 0)$

Miscellaneous exercise - 4

I.

1	2	3	4	5	6	7	8	9	10	11	12	13	14
B	B	B	A	D	D	A	B	B	B	C	C	D	D

II. 1) (i) $x^2 - y^2 = 0$ (ii) $2x^2 + 3xy + y^2 - 7x - 4y + 3 = 0$ (iii) $6x^2 - 5xy + y^2 = 0$

(iv) $3x^2 - y^2 = 0$ (v) $xy - 2x - y + 2 = 0$ (vi) $xy - 2x - 3y + 6 = 0$

(vii) $8x^2 + 2xy - 3y^2 + 12x + 14y - 8 = 0$ (viii) $2x^2 + 2xy - y^2 = 0$

(ix) $x^2 - 81 = 0$ (x) $x^2 - 2xy - 2x + 6y - 3 = 0$ (xi) $2x^2 - 7xy + 3y^2 = 0$

3) (i) $2x - 3y = 0, 3x + 2y = 0$ (ii) $x - 2y = 0, x + 2y = 0$

(iii) $\sqrt{3}\,x + y = 0, \sqrt{3}\,x - y = 0$ (iv) $(\sqrt{3} - 1)x + y = 0, (\sqrt{3} + 1)x - y = 0$

4) (i) $5x^2 + 4xy - y^2 = 0$ (ii) $9x^2 - 3xy - 2y^2 = 0$ (iii) $x^2 + xy - y^2 = 0$

5) (i) 0 (ii) -1 (iii) 1 (iv) 8 (v) 1 (vi) 6 (vii) 5

6) $3x^2 + 2xy - 3y^2 = 0$

7) $x^2 - 3y^2 = 0$

8) $\dfrac{50}{\sqrt{3}}$

10) $x^2 - 2xy - y^2 = 0$

11) -4

13) (i) $0°$ (ii) $\tan^{-1}(3)$ (iii) $\tan^{-1}(3)$

14) $x^2 - 3y^2 = 0$

18) Area $= \sqrt{3}$ sq. unit, Perimeter = 6 unit

22) $e = 0$ or $bd = ae$ **26)** $a = 1, c = 0$.

♦♦♦♦♦