Social Networks and Graph Theory

In this video, Vera Vertesi proves that if you choose any six people in the world (*any* six at all!), it is either true that three of the people all know each other or that three of the people all do not know each other. This is a pretty surprising fact. Yet it has a reasonably straightforward proof!

Vera explains how this type of problem can be understood more easily by relating it to a mathematical question about objects called *graphs*. A graph (in this context) is a finite collection of points, called *vertices*, along with a finite collection of lines connecting the points, called *edges*. Graphs are used in a wide variety of real-world situations. For example, any computer network (even the entire internet itself!) can be modeled using a graph. Just represent each computer by a vertex, and connect two vertices by an edge any time there is a connection between the two corresponding computers in the network.

As Vera shows in this video, graphs can also be used to model social networks. In fact, it turns out that the original statement:

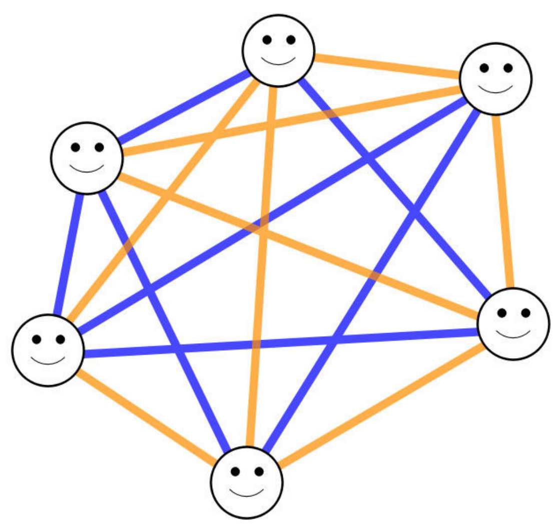
> *Among any six people it is either true that three of the people know each other or that three of the people do not know each other*,

is completely equivalent to the following statement about graphs:

> *Any graph with 6 vertices and a single edge (colored either orange or blue) connecting each pair of vertices, contains at least one all-blue triangle or one all-orange triangle.*

It takes a little thought to understand why these two questions are the same, but the point is to think of the 6 random people as vertices of a graph. We can then represent relationships between people using colored edges. If two people know each other, we draw an orange line between their corresponding vertices. If they do not know each other, we draw a blue line between their corresponding vertices.

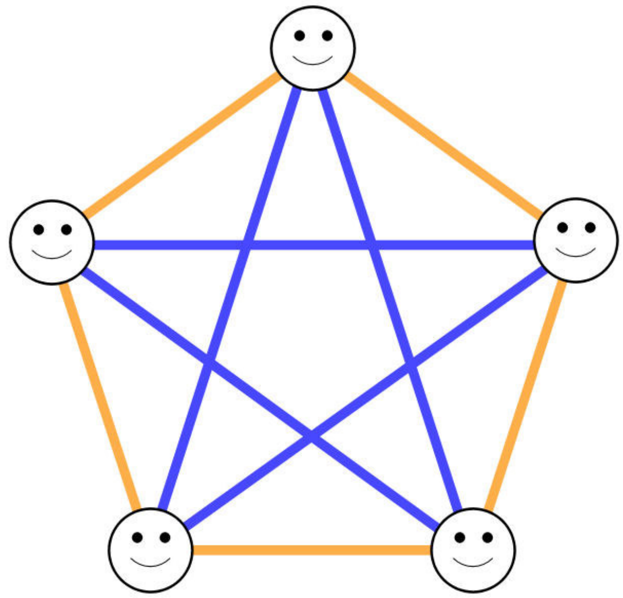
Any time three of the six people all know each other, we will see a triangle in our graph (whose vertices correspond to the people in question) with all-orange edges. Similarly, if three of the six people all do not know each other, we will see an all-blue triangle in our graph.



Since this graph has six vertices and a single edge, colored either orange or blue, connecting each pair of vertices, there must be blue triangle or an orange triangle. Can you find it?

Vera goes on to show that the original statement is *not* true if we replace the number "six" with the number "five." In other words, it is possible that five people have the property that no three of them all know each other and no three of them all do not know each other.

This leads her to define the so-called *Ramsey numbers*. Given a pair of positive whole numbers r and s, the Ramsey number $R(r, s)$ is the smallest whole number with the property that if you choose any $R(r, s)$ people, then either r will all know each other or s will all not know each other. She makes a few claims during this discussion without explaining why they are true. Don't be discouraged by this! Instead, think of it as an opportunity to ponder some of these questions for yourself. I collect some of them here. Feel free to add your own!



But with 5 vertices, the edges can be colored orange and blue in such a way that there is neither a blue triangle nor an orange one. Can you find one in this graph?

1. Why is $R(r, s) \le R(r-1, s) + R(r, s-1)$?

2. Why is $R(r, s)$ definable for every pair (r, s) of positive integers? In other words, why can we always find *some* number of people such that either r of them know each other or s of them do not know each other? And why do we know that there is a smallest number with this property?

3. Why is the table of Ramsey numbers symmetric in r and s? In other words, why does $R(r, s) = R(s, r)$?

4. Why is $R(1, s) = 1$ and why is $R(2, s) = s$?

For more about the Ramsey numbers (and other related questions in graph theory), visit the Wikipedia page en.wikipedia.org/wiki/Ramsey's_theorem.

See you next time!