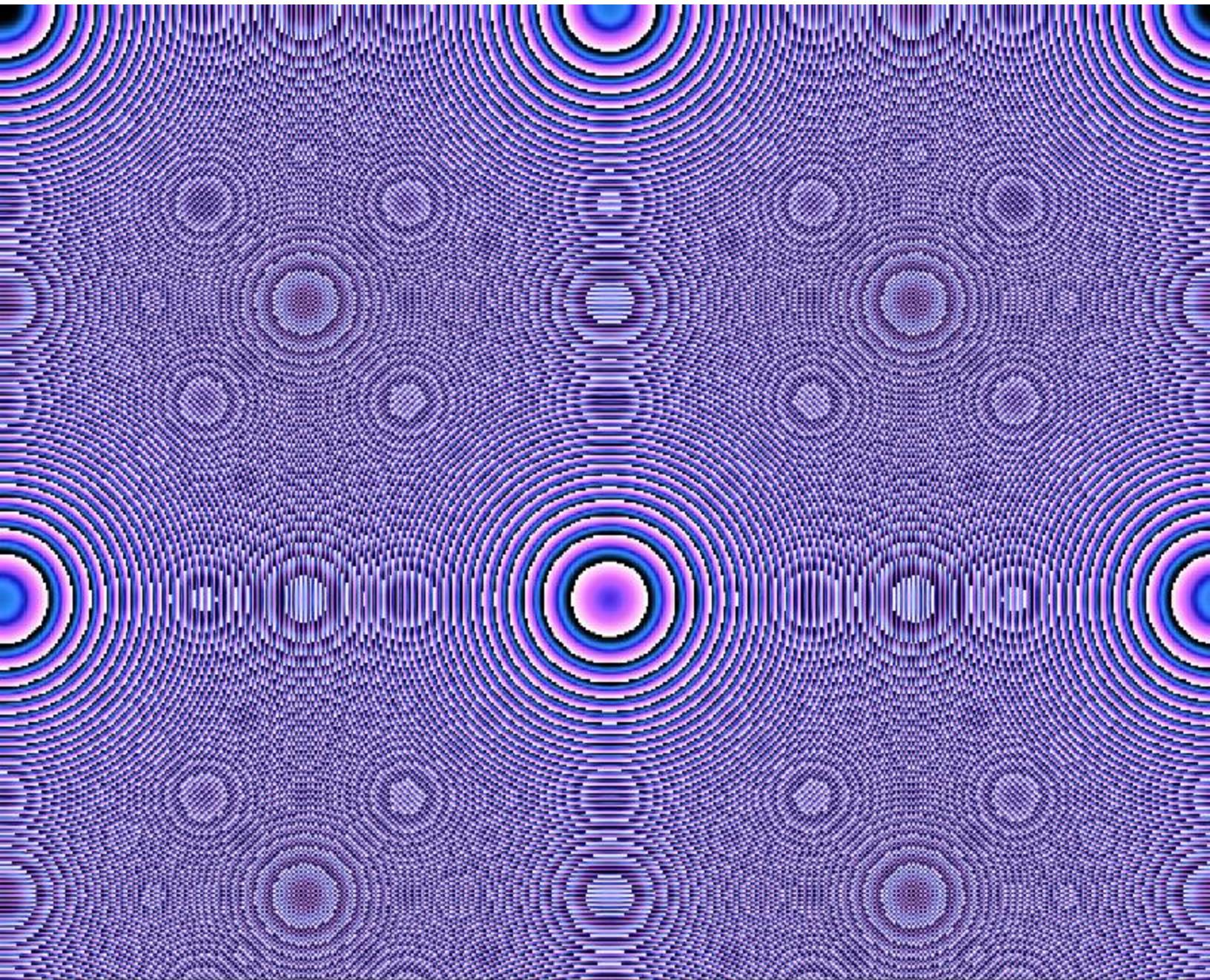


# Girls' Bulletin

December 2016/January 2017 • Volume 10 • Number 2

*To Foster and Nurture Girls' Interest in Mathematics*



An Interview with Doris Schattschneider  
Quilt-Doku!  
Mathematical Buffet  
Anna's Math Journal  
In Search of Nice Triangles, Part 8

Discovering the Quadratic Formula  
Notes from the Club  
Member's Thoughts:  
The Volume of a Regular  $n$ -Simplex

# From the Founder

Unpredictability makes math exciting. Where will understanding flow next? For me, with each new session at Girls' Angle, there's also the excitement of not knowing what math will ensue. Despite the unpredictability, here's a prediction for the spring: some member will stumble upon some math that is new to us all. We'll see if it happens!

- Ken Fan, President and Founder

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## Girls' Angle Bulletin

*The official magazine of  
Girls' Angle: A Math Club for girls  
Electronic Version (ISSN 2151-5743)*

Website: [www.girlsangle.org](http://www.girlsangle.org)  
Email: [girlsangle@gmail.com](mailto:girlsangle@gmail.com)

This magazine is published six times a year by Girls' Angle to communicate with its members and to share ideas and information about mathematics.

Girls' Angle welcomes submissions that pertain to mathematics.

The print version of the Bulletin is printed by the American Mathematical Society.

Editor: Jennifer Silva  
Executive Editor: C. Kenneth Fan

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## Girls' Angle: A Math Club for Girls

*The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.*

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On the cover:  $x^2 + y^2 \pmod{503}$ . For further explanation, see this issue's *Mathematical Buffet* on page 11.

# An Interview with Doris Schattschneider

Doris Schattschneider is Moravian College Professor Emerita of Mathematics. She received her doctoral degree in mathematics from Yale University and was the first female editor of *Mathematics Magazine*<sup>1</sup>, a publication of the Mathematical Association of America.

The interviewer was Girls' Angle summer intern Sandy Pelkowsky.

**Sandy:** What made you decide to study mathematics in college and then continue on to earn your PhD? When did you realize you wanted to study mathematics and what caused you to realize this?

**Doris:** I was always good at math, and in my freshman year at the University of Rochester, did well in math and physics. I wasn't turned on by the lab work in physics, so chose math as a major. I enjoyed math, especially working out proofs, and coming from a family of teachers (mother and grandfather), decided I wanted to teach math. I got my New York State teaching certificate while in college, but really wanted to do higher math so applied to grad school (Yale).

**Sandy:** What do you like so much about math?

**Doris:** Solving problems, discovering connections, and in particular, being able to visualize mathematical and geometrical relationships, and conveying math to others.

**Sandy:** What is your favorite topic in math?

**Doris:** Geometry, especially what is known as discrete geometry, with emphasis on tiling problems.

**Sandy:** Could you please describe a mathematical concept from this topic or give us a math problem from this topic to think about?

**Doris:** Describe or characterize particular classes of polygons that can tile the plane. This problem has been solved for triangles, quadrilaterals, and convex hexagons, and all convex polygons with 7 or more sides, but not completely solved for pentagons. That case turns out to be very difficult, and is still being pursued. The answer for triangles and quadrilaterals is "any" or "all." See if you can discover why. The answer for convex polygons with 7 or more sides is "none."

**Sandy:** Could you please explain to us one of your favorite theorems that you proved?

**Doris:** Some of my favorites are not ones I proved. However, one theorem that I proved that took several months of work and revision had to do with showing that if a tessellation by congruent polygons had each polygon surrounded in the same way, then the tessellation was isohedral, that is, for any choice of two polygons in the tessellation, there exists a transformation in the symmetry group of the tessellation that maps one of the polygons onto the other. In proving this, I used The Geometer's Sketchpad, and ended up creating an interactive table of symmetric polygons that can tile the plane isohedrally. This interactive catalog is still on line almost 20 years later, at

[mathforum.org/dynamic/one-corona/](http://mathforum.org/dynamic/one-corona/)

**Sandy:** How did you become interested in tessellations? What drew you to explore the

---

<sup>1</sup> We highly recommend reading *Mathematics Magazine* and other publications of the Mathematical Association of America.

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We're committed to producing quality math educational content and make every effort to provide this content to you for free.

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For this issue, those who do not subscribe to the print version will be missing out on the remainder of this interview with Dr. Doris SchattSchneider and some other content. We hope that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

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Thank you and best wishes,  
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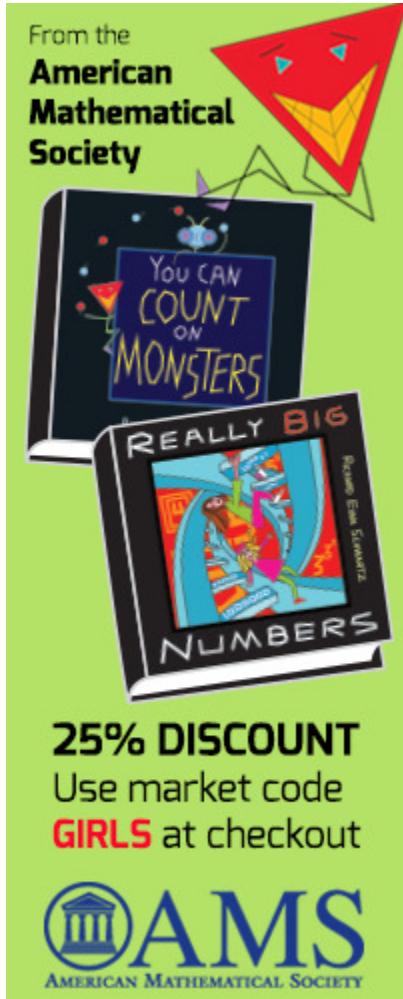


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# Quilt-doku!

Parts 1 and 2

by Beth Malmeskog | edited by Jennifer Silva

## 1. Judy's Question

One morning, I opened my email to find a note from Judy Gilmore, the mother of one of my friends. I have never met Judy in person, but had known her son Martin for years. He had told me that his mom was having a quilting problem and wanted to consult a mathematician. In making a couple of quilts of my own, no interesting math problems had come up – mostly just measuring and trying to sew in a straight line. So I was mystified by what kind of math she might need my help with. It turns out that there can be some very cool math in quilts, without any sewing at all. Her email started me on a journey through combinatorics, graph theory, and group theory, and even led me to discover some fascinating unsolved problems.

*Hi Beth,<sup>1</sup>*

*My son gave me your email address. Hopefully, he already told you I'm struggling with organizing a group project for some quilters; I can't seem to get it to work out exactly the way I want it to. Here's the scoop if you're feeling up for a puzzle. I'd certainly appreciate the help and insight.*

*I am organizing a round-robin quilt exchange. That means a group of quilters each make a small quilt block and circulate the blocks to one another until everyone in the group has had a chance to add a border to everyone else's quilt. On the first trade day, each person passes their quilt block to another person in the group and receives a block from someone else in the group, so they can add a border to that person's block. After a month or so, the blocks are passed again so the next person can add a border to the block, and so on, until everyone in the group has had a chance to sew a border on everyone else's block.*

*The last time we did this activity, we passed our project to the same person each time until we got our own quilt back. If I passed my quilt to "Martha," then every time we traded, I gave my quilt to "Martha"; that means she always had to follow my work in this game. It was a bit limiting, since part of the fun of doing these trades is getting to meet the other people in the group.*

*I'd like to arrange the trades so no one ever gets their quilt from the same person twice, but I can't quite figure it out. I tried it with 5 people in a group and, while everyone gets each quilt to sew on, the quilts are passed twice to the same person.*

*I think we will pretty much always have 5 people in a group, but it is possible that we would have groups with 4 or 6 people. Is there something that will work for that, too?*

*Judy Gilmore*

---

<sup>1</sup> The original email has been slightly edited.



Figure 1. Five quilts created by passing in a cycle. This method was unsatisfactory because each quilter always passed to the same neighbor. Notice that green always follows blue, yellow always follows green, etc.

## 2. Quilting Squares

Let  $S$  be a set of  $n$  distinct elements – for example,  $S$  may consist of  $n$  people,  $n$  colors, or the integers 0 through  $n - 1$ . An  $n$  by  $n$  array filled with elements from  $S$  such that each element appears once in each row and once in each column is called a **Latin square of order  $n$** .

The most familiar examples of Latin squares are Sudoku puzzles. In a completed Sudoku, every row and every column contains the numbers 1 through 9. A Sudoku puzzle has the additional condition that each of the marked  $3 \times 3$  boxes also contains the numbers 1 through 9.

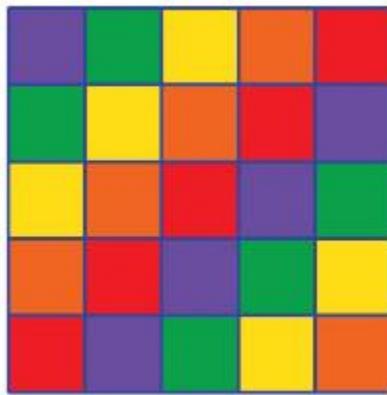
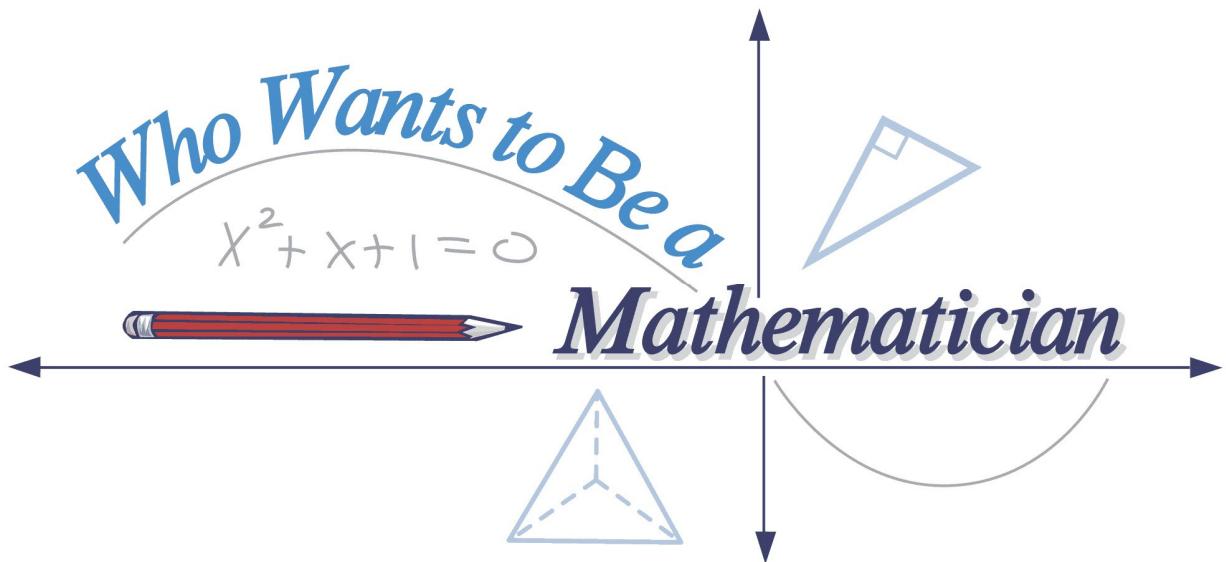


Figure 2: The Latin square of order 5 associated to the quilts in Figure 1.

A round-robin quilt exchange can also be modeled with a Latin square. Think of each row as the path of a quilt through the quilters, labeled 0 through  $n - 1$ . A quilt exchange that follows Judy's rules would make a Latin square with an additional constraint, like Sudoku, but with a different type of constraint. The condition that no quilt undergoes the same pass twice means that in the Latin square, the sequence  $(i, j)$  appears at most once in the rows of the square. There are  $n(n - 1)$  different sequences  $(i, j)$ , and  $n(n - 1)$  quilt passes must take place, so each sequence  $(i, j)$  must appear exactly once in the array. For example, if the sequence  $(1, 2)$  appears in the first row, we have to make sure that  $(1, 2)$  never appears again in any other row since we don't want person 1 to ever pass to person 2 again.

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# Mathematical Buffet

by Addie Summer

Functions in two variables, modulo a prime number

We present pictures of polynomials in 2 variables, modulo a prime number.

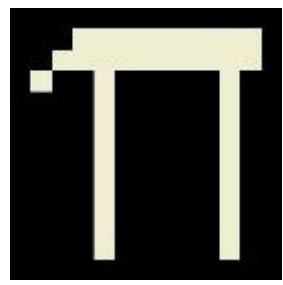
Here's an example to show how the pictures are made. At right is a multiplication table, modulo 5. It is like a regular multiplication table except that entries contain the remainders of the products after division by 5. For example,  $3 \times 4 = 12$ , which is equal to 2 modulo 5, so we place 2 in the entry whose row heading is "3" and column heading is

"4". We then assign colors to the different remainders and color the entries accordingly, as illustrated far right. The result is a picture of the polynomial  $xy$ , modulo 5.

On the next two pages, we present 12 such pictures. Each column represents the same polynomial and each row represents a different modulus. What patterns do you see? Can you explain the patterns mathematically? For a larger example, see the cover.

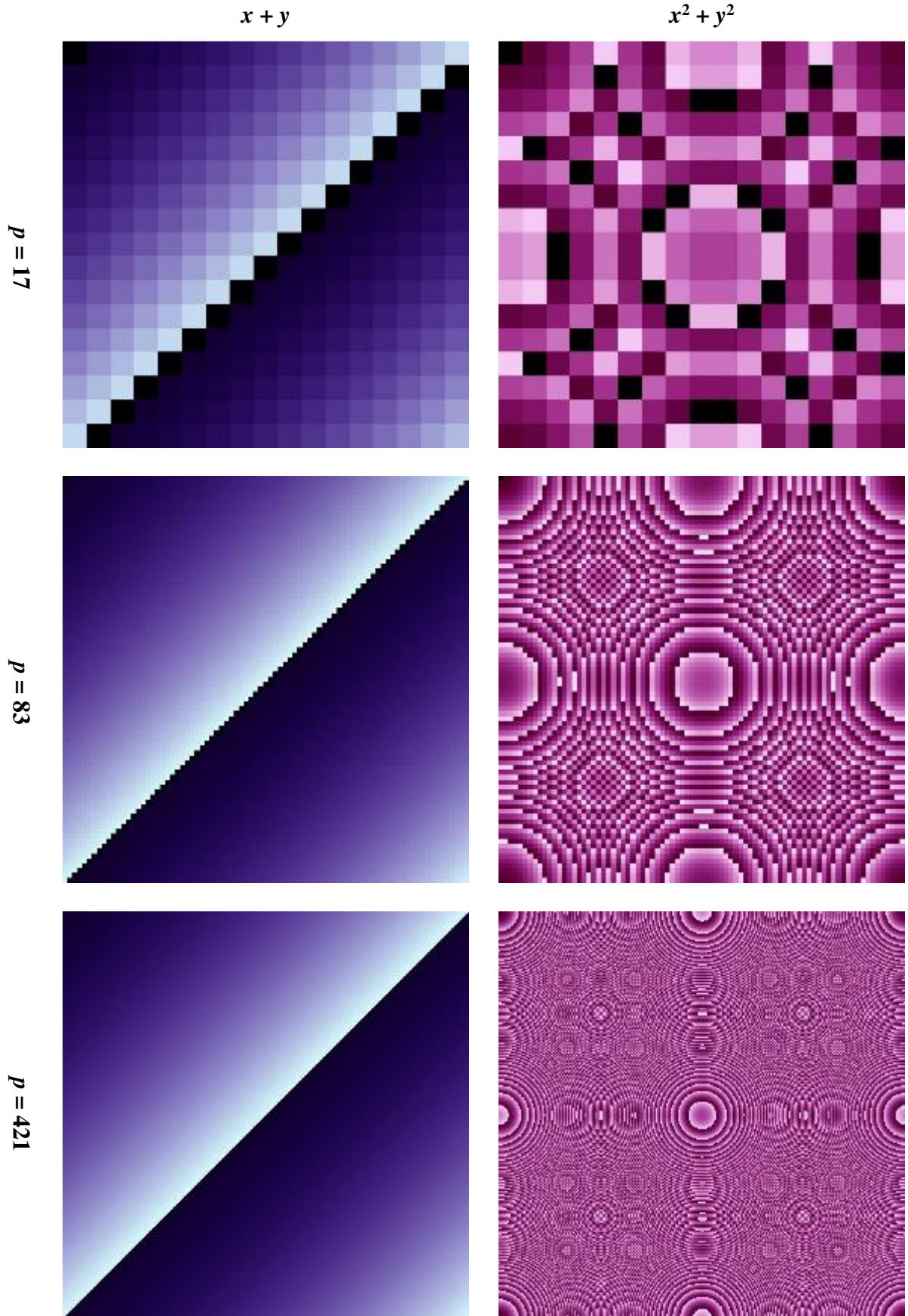
**Brief introduction to modular arithmetic** Modular arithmetic is concerned with integers, but only up to multiples of a fixed positive integer  $n$ , which is called the **modulus**. More precisely, let  $a$  and  $b$  be integers. We say that  $a$  and  $b$  are "equal modulo  $n$ " (or "congruent modulo  $n$ ") if and only if  $n$  divides evenly into  $b - a$ . Equivalently,  $a$  and  $b$  are equal modulo  $n$  if they leave the same remainder upon division by  $n$ . In symbols, we write " $a = b \pmod{n}$ ".

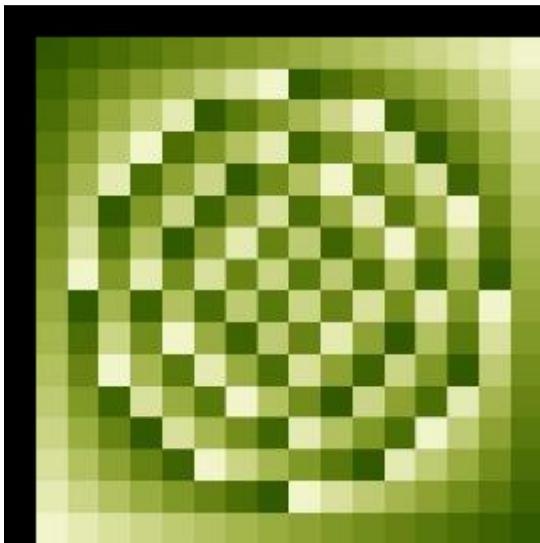
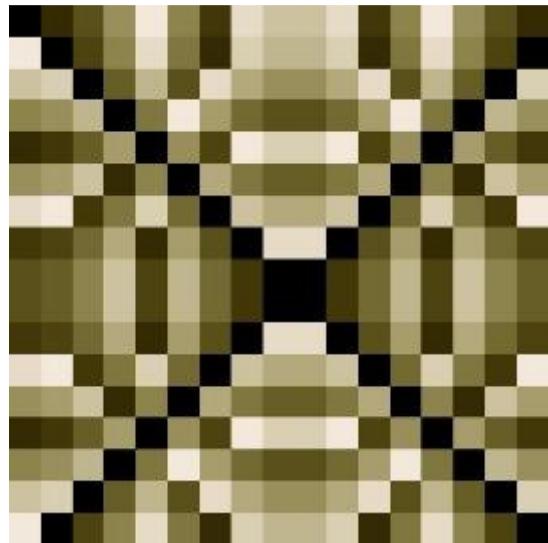
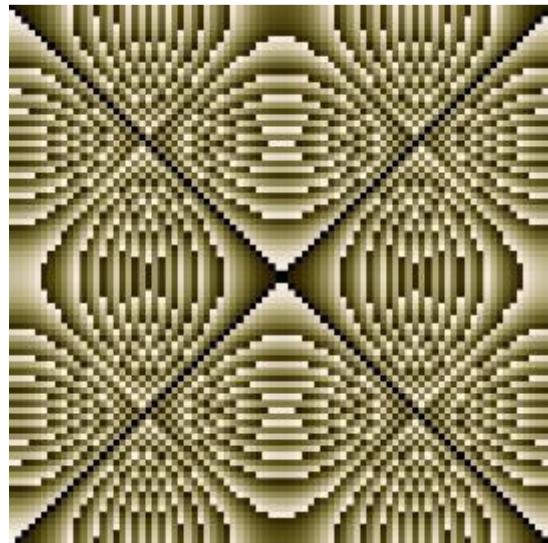
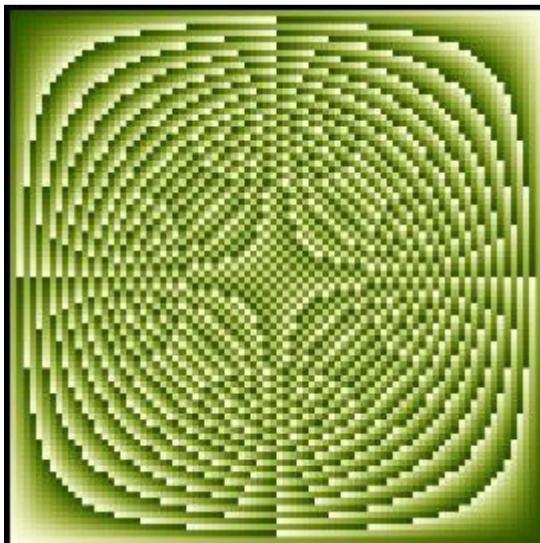
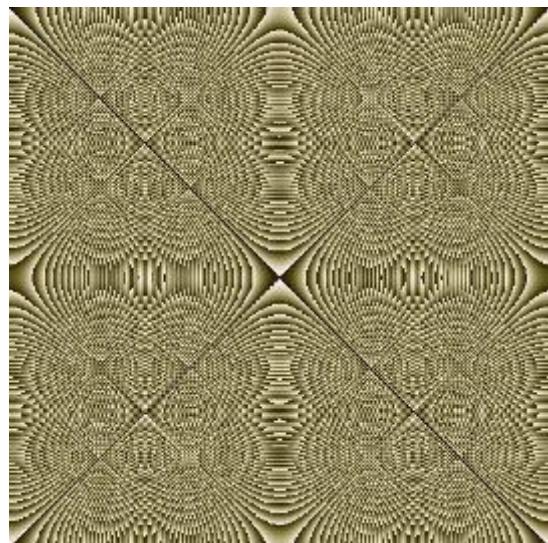
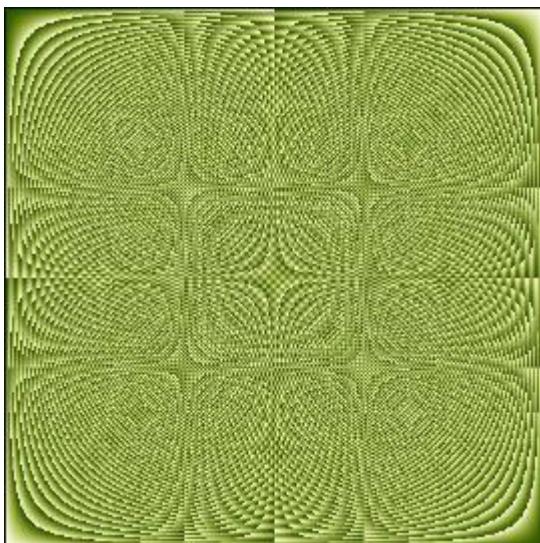
What makes modular arithmetic so useful is that arithmetic makes sense modulo  $n$ : if  $a = b \pmod{n}$  and  $c = d \pmod{n}$ , then  $a + c = b + d \pmod{n}$  and  $ac = bd \pmod{n}$ . These facts imply that polynomials with integer coefficients may be regarded as functions "modulo  $n$ ". That is, suppose  $p(x)$  is a polynomial with integer coefficients. Since  $p(x)$  can be built from numbers and variables using only addition and multiplication, if  $a = b \pmod{n}$ , then  $p(a) = p(b) \pmod{n}$ . This generalizes to polynomials (with integer coefficients) in any number of variables. For this reason, if we know the polynomial just on values of its variables restricted to the numbers 0 through  $n - 1$ , we know the polynomial everywhere, modulo  $n$ .



A function  $f(x, y)$  that satisfies  $f(a, b) = f(c, d) \pmod{n}$  whenever  $a = c \pmod{n}$  and  $b = d \pmod{n}$  is completely determined by its values on  $(x, y)$  for  $0 \leq x < n$  and  $0 \leq y < n$ . Since there are  $n$  possible remainders for each value of  $f(x, y)$ , there are only finitely many possible such functions. When  $n$  is prime, all such functions can be realized as polynomials. (Can you prove it?) Any picture you can draw with the above scheme for a prime modulus can be encoded as a polynomial. For example, the image of  $\pi$  at left is encoded by the polynomial (interpreted modulo 13):

$$\begin{aligned}
& 5xy + 3xy^2 + 11xy^3 + 9xy^4 + 2xy^5 + 7xy^6 + 4xy^7 + 9xy^8 + xy^9 + 4xy^{10} + 6xy^{11} + 10xy^{12} + 10x^2y + 4x^2y^2 + 3x^2y^3 + 10x^2y^4 + 7x^2y^5 \\
& + 8x^2y^6 + 9x^2y^7 + 10x^2y^8 + 6x^2y^9 + 5x^2y^{10} + 11x^2y^{11} + 9x^2y^{12} + 8x^3y + 4x^3y^2 + 7x^3y^3 + 6x^3y^5 + 2x^3y^6 + 10x^3y^7 + 5x^3y^8 + x^3y^9 \\
& + 10x^3y^{10} + 11x^3y^{11} + 9x^3y^{12} + 9x^4y + 3x^4y^2 + 2x^4y^3 + 9x^4y^4 + 6x^4y^5 + 7x^4y^6 + 8x^4y^7 + 9x^4y^8 + 5x^4y^9 + 4x^4y^{10} + 10x^4y^{11} + 8x^4y^{12} \\
& + 9x^5y + 7x^5y^2 + 2x^5y^3 + 6x^5y^5 + 11x^5y^6 + 8x^5y^7 + 5x^5y^9 + 8x^5y^{10} + 10x^5y^{11} + x^5y^{12} + 7x^6y + 3x^6y^2 + 7x^6y^3 + 6x^6y^4 + 3x^6y^5 \\
& + 3x^6y^7 + x^6y^8 + 6x^6y^9 + 12x^6y^{10} + 6x^6y^{11} + 10x^6y^{12} + 4x^7y + 2x^7y^2 + 10x^7y^3 + 8x^7y^4 + x^7y^5 + 6x^7y^6 + 3x^7y^7 + 8x^7y^8 + 3x^7y^{10} \\
& + 5x^7y^{11} + 9x^7y^{12} + 3x^8y + 10x^8y^2 + 9x^8y^3 + 3x^8y^4 + x^8y^6 + 2x^8y^7 + 3x^8y^8 + 12x^8y^9 + 11x^8y^{10} + 4x^8y^{11} + 2x^8y^{12} + 11x^9y + 7x^9y^2 \\
& + 10x^9y^3 + 3x^9y^4 + 9x^9y^5 + 5x^9y^6 + 8x^9y^8 + 4x^9y^9 + x^9y^{11} + 12x^9y^{12} + 4x^{10}y + 11x^{10}y^2 + 10x^{10}y^3 + 4x^{10}y^4 + x^{10}y^5 + 2x^{10}y^6 + 3x^{10}y^7 \\
& + 4x^{10}y^8 + 12x^{10}y^{10} + 5x^{10}y^{11} + 3x^{10}y^{12} + 11x^{11}y^2 + 6x^{11}y^3 + 4x^{11}y^4 + 10x^{11}y^5 + 2x^{11}y^6 + 12x^{11}y^7 + 4x^{11}y^8 + 9x^{11}y^9 + 12x^{11}y^{10} + x^{11}y^{11} \\
& + 5x^{11}y^{12} + 4x^{12}y + 3x^{12}y^2 + 2x^{12}y^3 + 5x^{12}y^4 + 4x^{12}y^5 + 2x^{12}y^6 + 12x^{12}y^7 + 7x^{12}y^8 + 4x^{12}y^9 + 6x^{12}y^{10} + 11x^{12}y^{11} + x^{12}y^{12}.
\end{aligned}$$



$xy$  $x^2 - y^2$  $p = 17$  $p = 83$  $p = 421$

# Anna's Math Journal

By Anna B.

*Mathematics is a journey of discovery. As mathematicians take this journey, they follow many wrong turns, believe many incorrect facts, and encounter many mysteries. Out of these twists and turns comes the reward of truth and understanding. However, if you look at math books, you might get the impression that mathematicians rarely err. In this column, Anna gives us a peek into her mathematical process of discovery, bravely allowing us to watch even as she stumbles.*

Anna continues counting special  $N$ -tilings of a  $1$  by  $\sqrt{2}$  rectangle.

Last time, I found a recursion relation for the number of special tilings of a  $1$  by  $\sqrt{2}$  rectangle. This time, I want to try to find a closed form formula.

$r_n = \# \text{ of special tilings of a } 1 \text{ by } \sqrt{2} \text{ rectangle}$   
that can be obtained with  $n$  cuts.

$$r_n = \sum_{k=0}^{n-1} r_k r_{n-1-k}$$

$$r_0 = 1$$

$n$	0	1	2	3	4	5	6	7	8	9	10
$r_n$	1	1	2	5	14	42	132	429	1430	4862	16796

I'll start by computing the first few values of  $r_n$ .

I defined  $r_n$  to be the number of special tilings obtainable with  $n$  cuts.

Here's the recurrence relation I found last time.

The numbers look like they have lots of factors, so I think I'll add a row for their prime factorizations.

I did numerical computations on a separate sheet of paper not shown here.

$n$	0	1	2	3	4	5	6	7	8	9	10
$r_n$	1	1	2	5	14	42	132	429	1430	4862	16796
prime factorization	1	1	2	$5$	$2 \cdot 7$	$2 \cdot 3 \cdot 7$	$2^2 \cdot 3 \cdot 11$	$3 \cdot 11 \cdot 13$	$2 \cdot 5 \cdot 11 \cdot 13$	$2 \cdot 11 \cdot 13 \cdot 17$	$2^2 \cdot 13 \cdot 17 \cdot 19$

The prime numbers in the factorization seem to grow gradually. They are always less than  $2n$ . It makes me wonder what I might find if I divide these numbers into  $(2n)!$ .

$n$	0	1	2	3	4	5	6	7	8	9	10
$r_n$	1	1	2	5	14	42	132	429	1430	4862	16796
prime factorization	1	1	2	$5$	$2 \cdot 7$	$2 \cdot 3 \cdot 7$	$2^2 \cdot 3 \cdot 11$	$3 \cdot 11 \cdot 13$	$2 \cdot 5 \cdot 11 \cdot 13$	$2 \cdot 11 \cdot 13 \cdot 17$	$2^2 \cdot 13 \cdot 17 \cdot 19$
$\frac{(2n)!}{r_n}$	1	2	$2^2 \cdot 3$	$2^4 \cdot 3^2$	$2^6 \cdot 3^5 \cdot 5$	$2^8 \cdot 3^7 \cdot 5^2$	$2^{10} \cdot 3^9 \cdot 5^2 \cdot 7$	$2^{12} \cdot 3^{11} \cdot 5^2 \cdot 7^2$	$2^{14} \cdot 3^{13} \cdot 5^2 \cdot 7^2 \cdot 11$	$2^{16} \cdot 3^{15} \cdot 5^2 \cdot 7^2 \cdot 11^2$	$2^{18} \cdot 3^{17} \cdot 5^2 \cdot 7^2 \cdot 11^2$

There are a lot of even powers. These numbers don't differ much from being perfect squares. In fact, it seems that if I divide by  $n + 1$ , I always get a perfect square.

Key:

Anna's thoughts

Anna's afterthoughts

Editor's comments

# Anna's Math Journal

$n$	0	1	2	3	4	5	6	7	8	9	10
$r_n$	1	1	2	5	14	42	132	429	1430	4862	16796
prime factorization	1	1	2	5	$2 \cdot 7$	$2 \cdot 3 \cdot 7$	$2^2 \cdot 3 \cdot 11 \cdot 13$	$3 \cdot 11 \cdot 13 \cdot 23 \cdot 11 \cdot 13 \cdot 17$	$2^2 \cdot 13 \cdot 17 \cdot 19$		
$\frac{(2n)!}{r_n}$	1	2	$2^2 \cdot 3$	$2^4 \cdot 3^2$	$2^6 \cdot 3^5 \cdot 5$	$2^8 \cdot 3^7 \cdot 5^2 \cdot 7$	$2^{10} \cdot 3^9 \cdot 7^2 \cdot 11$	$2^{12} \cdot 3^{11} \cdot 7^2 \cdot 13$	$2^{14} \cdot 3^{13} \cdot 7^2 \cdot 17$	$2^{16} \cdot 3^{15} \cdot 7^2 \cdot 19$	
$\sqrt{\frac{(2n)!}{(n+1)r_n}}$	1	1	2	$2^2 \cdot 3$	$2^3 \cdot 3^2$	$2^4 \cdot 3^3 \cdot 5$	$2^5 \cdot 3^4 \cdot 7^2$	$2^6 \cdot 3^5 \cdot 7^2 \cdot 11$	$2^7 \cdot 3^6 \cdot 7^2 \cdot 13$	$2^8 \cdot 3^7 \cdot 7^2 \cdot 17$	$2^9 \cdot 3^8 \cdot 7^2 \cdot 19$

The prime factorizations don't change much from one entry to the next. The ratios of consecutive terms are: 1, 2, 3, 4, 5, 6. Hey! These are the factorials!

It seems that the square root of  $(2n)!$  divided by the quantity  $n+1$  times  $r_n$  is  $n!$ .

$$\sqrt{\frac{(2n)!}{(n+1)r_n}} = n!$$

$$\frac{(2n)!}{(n+1)r_n} = (n!)^2$$

$$r_n = \frac{1}{n+1} \frac{(2n)!}{(n!)^2} = \frac{1}{n+1} \binom{2n}{n}$$

Solving for  $r_n$ , the table suggests that  $r_n$  is equal to  $2n$  choose  $n$  divided by the quantity  $n+1$ . Could this be true?

It would be proven if it can be shown that this formula satisfies the recurrence relation.

$$\frac{1}{n+1} \binom{2n}{n} \stackrel{?}{=} \sum_{k=0}^{n-1} \frac{1}{k+1} \binom{2k}{k} \cdot \frac{1}{n-k} \binom{2(n-1-k)}{n-1-k}$$

$$= \sum_{k=0}^{n-1} \frac{1}{k+1} \frac{(2k)!}{k! \cdot k!} \cdot \frac{1}{n-k} \frac{(2(n-1-k))!}{(n-1-k)! (n-1-k)!}$$

I did a bit of algebra to try to show this equation is true, but didn't get anywhere and produced nothing that I felt like putting in my math journal.

Can you help Anna prove or disprove the formula for  $r_n$ ?

Key:

Anna's thoughts

Anna's afterthoughts

Editor's comments

ABB 12/29/16

# In Search of Nice Triangles, Part 8

by Ken Fan | edited by Jennifer Silva

Emily: Using our factorization of  $T_n(x) - 1$  in terms of the  $p_d(x)$ , we can inductively compute  $p_n(x)$  by using polynomial long division and taking square roots of polynomials. It's a bit of work, but I'd like to compute the first several  $p_n(x)$ .

Jasmine: Okay, let's do it!

With some effort, Emily and Jasmine create this table:

$n$	$p_n(x)$
1	$x - 1$
2	$x + 1$
3	$x + 1/2$
4	$x$
5	$x^2 + x/2 - 1/4$
6	$x - 1/2$
7	$x^3 + x^2/2 - x/2 - 1/8$
8	$x^2 - 1/2$
9	$x^3 - 3x/4 + 1/8$
10	$x^2 - x/2 - 1/4$
11	$x^5 + x^4/2 - x^3 - 3x^2/8 + 3x/16 + 1/32$
12	$x^2 - 3/4$
13	$x^6 + x^5/2 - 5x^4/4 - x^3/2 + 3x^2/8 + 3x/32 - 1/64$
14	$x^3 - x^2/2 - x/2 + 1/8$
15	$x^4 - x^3/2 - x^2 + x/2 + 1/16$
16	$x^4 - x^2 + 1/8$

Emily: That wasn't so bad.

Jasmine: I thought it would be more tedious, but we only had to perform long division a few times since most of these indices have few divisors. And taking square roots of polynomials that we already know are squares of polynomials turned out to be less computation than I had expected.

Emily: You're right – when  $n$  was an odd prime number, all we had to do was divide  $T_n(x) - 1$  by  $p_1(x) = x - 1$ , then find the square root of the result.

Jasmine: But the question remains: are these *minimum* polynomials? To prove they're minimum polynomials, we'd have to show that they all have rational coefficients and cannot be factored as a product of non-constant polynomials.

Emily and Jasmine continue their investigation into nice triangles. They've been using "nice" to denote angles that measure a rational multiple of  $\pi$  radians.

They are now looking for triangles with 3 nice angles and 2 sides of integer length.

Last time, they decided to embark on a study of the minimum polynomials of the cosines of rational multiples of  $\pi$ . They defined the polynomials  $p_d(x)$ , for  $d > 1$ , to be the product of all linear factors of the form

$$x - \cos(2\pi k/d)$$

where  $1 \leq k \leq d/2$  and  $(k, d) = 1$ . They defined  $p_1(x) = x - 1$ .

They observed that, for  $n$  odd,

$$T_n(x) - 1 = 2^{n-1} p_1(x) \left( \prod_{d|n, d>1} p_d(x) \right)^2,$$

and for  $n$  even,

$$T_n(x) - 1 = 2^{n-1} p_1(x) p_2(x) \left( \prod_{d|n, d>2} p_d(x) \right)^2,$$

where  $T_n(x)$  is the  $n$ th Chebyshev polynomial of the first kind.

They conjectured that the minimum polynomial of  $\cos(2\pi k/n)$ , where  $k$  and  $n$  are relatively prime, is  $p_n(x)$ .

Try these computations yourself!

Emily: Say, I think that the way we've been computing the polynomials shows that they have rational coefficients.

Jasmine: Really?

Emily: Yes! If we have two polynomials with rational coefficients and one is a factor of the other, then the quotient must also have rational coefficients since polynomial long division only uses rational operations on the coefficients.

Rational operations are addition, subtraction, multiplication, and division. When a rational operation is performed on rational numbers (other than division by zero!), the result is a rational number.

Try some examples!  
Find square roots of  
 $x^2 + x/2 + 1/16$   
and  
 $x^4 - 4x^3 + 10x^2 - 12x + 9$ .

Jasmine: I see! And the square roots turned out to be rational polynomials every time because we were always computing square roots of rational polynomials with lead coefficient 1. So we could take the lead coefficient of the square root to be 1, and then we could find the remaining coefficients using only rational operations.

Emily: Exactly. Thus,  $p_n(x)$  has rational coefficients by induction on  $n$ . All we have to do now is show that these polynomials can't be factored as a product of non-constant polynomials.

Jasmine: Great, so let's focus on factorization!

Emily: All of the linear polynomials we found so far must be minimum polynomials since there's no way to write a linear polynomial as a product of non-constant polynomials. Therefore,  $p$  sub 1, 2, 3, 4, and 6 are minimum polynomials.

Jasmine: So let's examine the quadratic  $p_5(x) = x^2 + x/2 - 1/4$ . If it isn't a minimum polynomial, it would have to be a product of two linear factors. We know that its roots are  $\cos(2\pi/5)$  and  $\cos(4\pi/5)$ , so the linear factors would have to be  $x - \cos(2\pi/5)$  and  $x - \cos(4\pi/5)$ .

Emily: But we already know *those* cosines aren't rational, so  $p_5(x)$  is a minimum polynomial!

Jasmine: True – the only rational cosines of nice angles are 0,  $\pm 1/2$ , and  $\pm 1$ , and these are the roots of the linear polynomials that we already found. Therefore, *all* of the linear  $p_n(x)$  are in our table, and none of the nonlinear polynomials can have rational linear factors.

Emily and Jasmine found all rational cosines of nice angles in Part 1 of this series. See Volume 8, Number 6.

Emily: That also implies that all of the quadratic and cubic polynomials in our list are minimum polynomials; if a quadratic or a cubic polynomial could be factored into a product of non-constant polynomials, one of the factors would have to be linear.

Jasmine: In our table, that leaves just  $p$  sub 11, 13, 15, and 16 to check. Let's try to show that  $p_{11}(x) = x^5 + x^4/2 - x^3 - 3x^2/8 + 3x/16 + 1/32$  is a minimum polynomial.

Emily: Okay. If  $p_{11}(x) = f(x)g(x)$  where  $f(x)$  and  $g(x)$  are non-constant polynomials with rational coefficients, then one must be a quadratic and the other a cubic; we've ruled out the possibility that there are linear factors, so any non-constant factors must have degree 2 or 3. The sum of their degrees must be 5, and the only way to achieve that is as 2 + 3.

Jasmine: We know the roots of  $p_{11}(x)$ :  $\cos(2\pi/11), \cos(4\pi/11), \cos(6\pi/11), \cos(8\pi/11)$ , and  $\cos(10\pi/11)$ . The quadratic factor would have to have two of these cosines as its roots. Let's say its two roots are  $\cos(2\pi a/11)$  and  $\cos(2\pi b/11)$ , where  $1 \leq a < b \leq 5$ . Then the quadratic factor would be  $(x - \cos(2\pi a/11))(x - \cos(2\pi b/11))$ , which expands to

$$x^2 - (\cos(2\pi a/11) + \cos(2\pi b/11))x + \cos(2\pi a/11)\cos(2\pi b/11).$$

For the quadratic factor to have rational coefficients, both  $\cos(2\pi a/11) + \cos(2\pi b/11)$  and  $\cos(2\pi a/11)\cos(2\pi b/11)$  would have to be rational numbers.

Emily: How can we see that no choice of  $a$  and  $b$  allows for both  $\cos(2\pi a/11) + \cos(2\pi b/11)$  and  $\cos(2\pi a/11)\cos(2\pi b/11)$  to be rational? I don't even know how we could express the value of any of these cosines in a way that would enable us to compute their precise values.

Jasmine: I don't, either.

Emily and Jasmine are silent as they think about what they can do.

Jasmine: The set of angles  $2\pi k/n$  for those  $k$  which are relatively prime to  $n$  is a rather special set of angles. We're sort of saying that they come together in "algebraic bundles." There must be some property of these sets of angles that we can exploit to show what we hope is true.

Emily and Jasmine think some more. After several minutes of silence, Emily perks up.

Emily: If  $x$  and  $y$  are relatively prime to  $n$ , then so is  $xy$ . In fact, if we let  $A$  be the set of angles  $2\pi k/n$ , where  $k$  and  $n$  are relatively prime, then the angles in  $A$  are essentially permuted when we multiply by a number that is relatively prime to  $n$ .

Jasmine: Yes, these are standard modular arithmetic facts. Isn't it curious that we can compute the cosine of multiples of an angle by using the Chebyshev polynomials  $T_n(x)$ ? So one way to express what you're saying is that if  $m$  is relatively prime to  $n$ , then  $T_m(r)$  is a root of  $p_n(x)$  if  $r$  is a root of  $p_n(x)$ . What's more,  $p_n(x)$  divides evenly into  $p_n(T_m(x))$ . That seems rather special.

Emily: Interesting! But I still don't see how to use these observations to prove what we want.

Emily and Jasmine are silent yet again as they think.

Jasmine: I have an idea!

Emily: What is it?

Jasmine: Suppose  $r_1$  and  $r_2$  are roots of a quadratic with rational coefficients. That is, suppose that the sum  $r_1 + r_2$  and the product  $r_1 r_2$  are both rational numbers.

Emily: Okay.

Jasmine: Can't every symmetric polynomial  $s(x, y)$  with rational coefficients be expressed as a rational polynomial in the elementary symmetric functions  $x + y$  and  $xy$ ?

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# Discovering the Quadratic Formula

by Addie Summer

Addie mentioned to me that she didn't have to be taught the quadratic formula because she figured it out for herself. Here, she explains how she did it. Is there any step of her process that you don't think you could have thought of by yourself? By the way, if you don't know the quadratic formula, read the statement of the problem below and try to solve it before reading further. Even if you don't succeed, trying will enable you to understand this article much better. If you do succeed, please send us your derivation! - *Editor*

## The Problem

Let  $a$ ,  $b$ , and  $c$  be constants.

Solve for  $x$ :  $ax^2 + bx + c = 0$ .

Here's how I found the quadratic formula, which gives the solutions to the equation

$$ax^2 + bx + c = 0$$

in terms of the coefficients  $a$ ,  $b$ , and  $c$ .

First, I tried to isolate  $x$ .

I found that if there were only one term involving the variable  $x$ , I could isolate  $x$  and solve, but if both were present, I had a problem. That is, if the equation were  $ax^2 + c = 0$  or  $bx + c = 0$ , I could isolate  $x$ . In the first case, I found

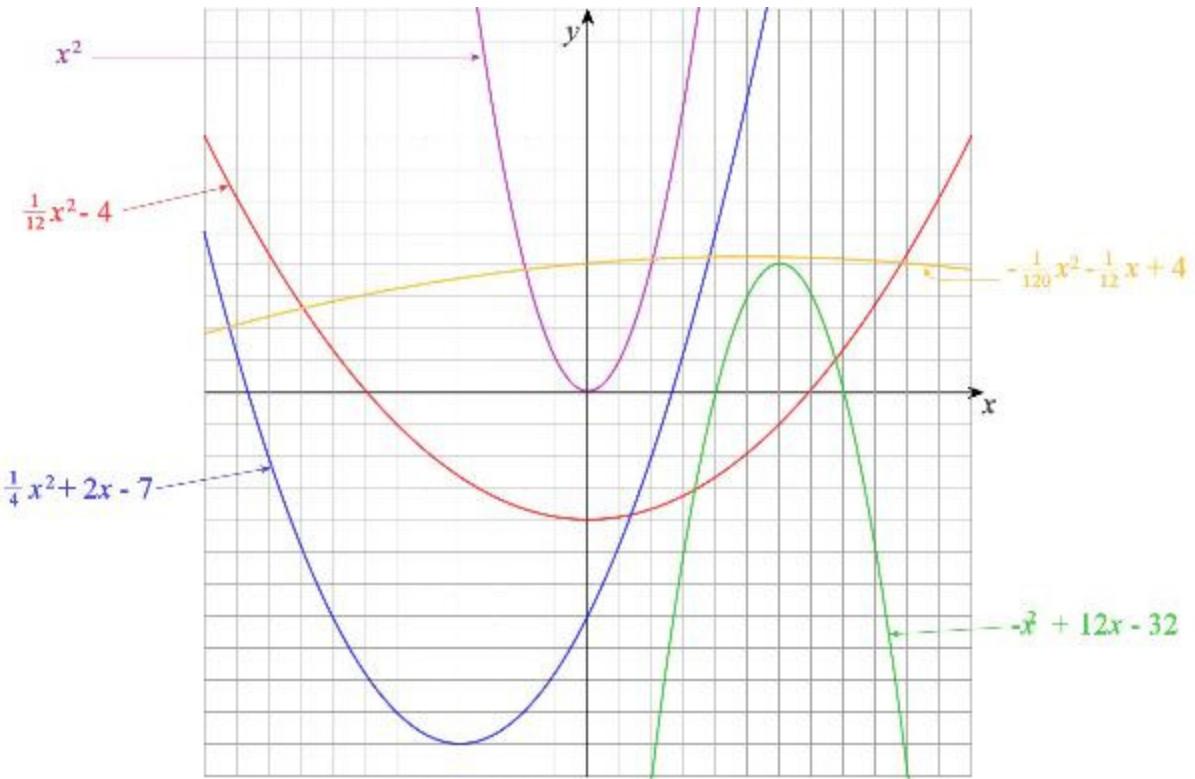
$$x = \pm \sqrt{-\frac{c}{a}}.$$

In the second case, I found  $x = -c/b$ . In this second case, the equation is a linear equation. Since I was mainly interested in the quadratic equation, I decided to assume that  $a \neq 0$ . If  $b$  was also not 0, then I ran into problems and could not immediately see how to isolate  $x$ .

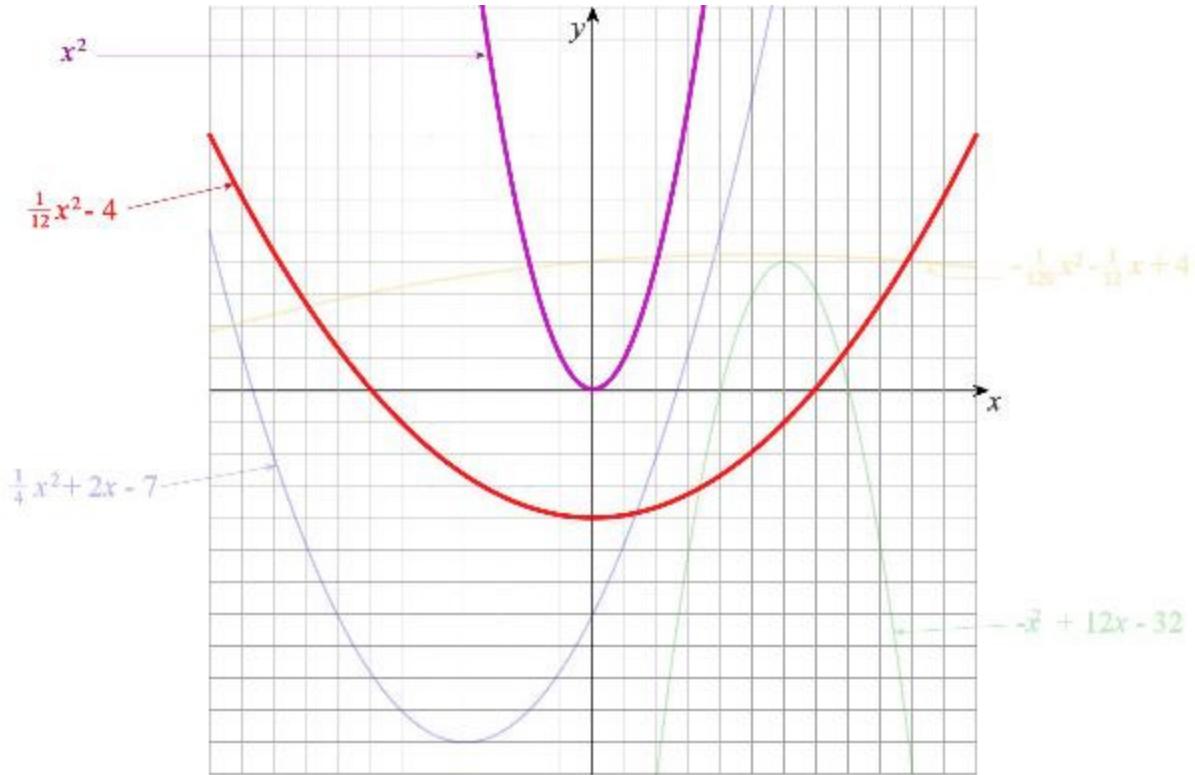
To understand quadratic equations better, I decided to draw graphs of several quadratic functions. At the top of the next page, I plotted five of the quadratics that I had graphed:

$$x^2 \quad \frac{1}{12}x^2 - 4 \quad \frac{1}{4}x^2 + 2x - 7 \quad -\frac{1}{120}x^2 - \frac{1}{12}x + 4 \quad -x^2 + 12x - 32$$

I noticed that they all had the same basic bowl or dome-like shape and they all seemed to enjoy a vertical line of mirror symmetry.



Of the quadratic functions that I had graphed, I decided to take note of which corresponded to ones where I could find the roots by isolating  $x$ , that is, those that lacked a linear term.



It looked like the ones where I could isolate  $x$  were exactly the ones whose vertical line of symmetry coincided with the  $y$ -axis.

If the graph of a function  $f(x)$  is symmetric about the  $y$ -axis, it means  $f(x) = f(-x)$ . The quadratic functions whose roots I could find by isolating  $x$  had the form  $ax^2 + c$ . Since  $ax^2 + c = a(-x)^2 + c$ , the algebra shows that these quadratic functions are symmetric about the  $y$ -axis.

Then I had an idea. Suppose quadratic functions do have a vertical line of symmetry. Could I solve for the zeroes of a quadratic by shifting sideways until its vertical line of symmetry coincided with the  $y$ -axis where I could solve by isolating  $x$ , then shifting the solutions back?

Let  $q(x) = ax^2 + bx + c$ . Shifting  $q(x)$  by  $s$  units to the left corresponds to looking at  $q(x + s)$ . So I was hoping to find  $s$  such that  $q(x + s)$  became a quadratic of the form  $Ax^2 + C$ , that is, a quadratic without a linear term. So I computed:

$$\begin{aligned} q(x + s) &= a(x + s)^2 + b(x + s) + c \\ &= a(x^2 + 2xs + s^2) + bx + bs + c \\ &= ax^2 + 2asx + as^2 + bx + bs + c \\ &= ax^2 + (2as + b)x + as^2 + bs + c \end{aligned}$$

In order for this quadratic to have no linear term, I needed the coefficient of  $x$  to be 0, that is, I needed  $2as + b = 0$ . This happens when  $s = -b/(2a)$ . With this substitution, I found

$$q(x - \frac{b}{2a}) = ax^2 + a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) + c = ax^2 + \frac{b^2}{4a} - \frac{b^2}{2a} + c = ax^2 - \frac{b^2}{4a} + c.$$

As I had hoped, I could solve the equation  $q(x - b/(2a)) = 0$  because I could isolate  $x$ . I found

$$x = \pm \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}} = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}.$$

By shifting these right by  $s = -b/(2a)$  units, I found the solutions to the original equation  $q(x) = 0$ :

$$-\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

and that's the quadratic formula!

I think it is really neat that one can see so much of the graphic intuition that went into this derivation in the quadratic formula. For example, one can read off from the formula that the axis of the quadratic  $ax^2 + bx + c$  is  $x = -b/(2a)$ . And since the quadratic is symmetric about its axis, the zeroes must be equidistant and on each side of this axis, and that corresponds, in the formula,

to adding and subtracting the same quantity,  $\frac{\sqrt{b^2 - 4ac}}{2a}$ , to  $-b/(2a)$ . Another cool thing: we can

see that the formula yields 0, 1, or 2 solutions according to whether  $b^2 - 4ac$  is less than, equal to, or greater than 0. In other words, the sign of  $b^2 - 4ac$  discriminates between 3 geometric possibilities: the graph does not intersect the  $x$ -axis, the graph kisses the  $x$ -axis at a single point, the graph crosses the  $x$ -axis twice. There are definitely other ways to derive the quadratic formula. If you come up with one of your own, please share it with us!

# Notes from the Club

These notes cover some of what happened at Girls' Angle meets. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than 5% of what happens at the club is revealed here.

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Session 19 - Meet 8  
November 3, 2016

Mentors: Bridget Bassi, Anna Ellison, Nesly Estrada,  
Hannah Larson, Isabel Macenka, Jane Wang

Several members began their first explorations into modular arithmetic. In modular arithmetic, we fix a positive integer called the **modulus**. Let  $n$  be the modulus. We say that two integers  $a$  and  $b$  are equal, modulo  $n$ , if and only if  $b - a$  is divisible by  $n$ . This equivalence is sometimes expressed " $a = b \pmod{n}$ ".

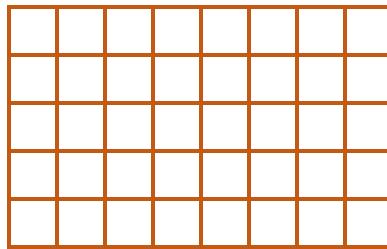
Can you prove the following properties of modular arithmetic? (In the following statements,  $n$  is a fixed positive integer and  $a, b, c$ , and  $d$  are integers.)

1. If  $a = b \pmod{n}$  and  $c = d \pmod{n}$  then  $a + c = b + d \pmod{n}$ .
2. If  $a = b \pmod{n}$  and  $c = d \pmod{n}$  then  $ac = bd \pmod{n}$ .
3. There exists an integer  $x$  that satisfies  $ax = 1 \pmod{n}$   
if and only if  $a$  and  $n$  are relatively prime.

Members made an in-depth exploration of the equation  $ax = b \pmod{n}$ , for fixed  $a$  and  $b$ . They determined the exact number of solutions such that  $0 \leq x < n$ .

For more on modular arithmetic, take a look at this issue's Math Buffet on page 11.

Some members counted shapes within shapes. For example, consider a 5 by 8 grid of unit squares (shown below). How many rectangles are formed by the lines of this grid?



Did you find 540?

In general, how many rectangles are formed by the lines in an  $n$  by  $m$  grid of unit squares? How many squares are formed? How many hexagons?

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Session 19 - Meet 9  
November 10, 2016

Mentors: Bridget Bassi, Sarah Coleman, Anna Ellison,  
Nesly Estrada, Suzanne O'Meara, Christine Soh,  
Isabel Vogt

Three members' exploration of least common multiples and greatest common divisors culminated in the beautiful observation that for any positive integers  $a$  and  $b$ , their product is equal to the product of their least common multiple and their greatest common divisor.

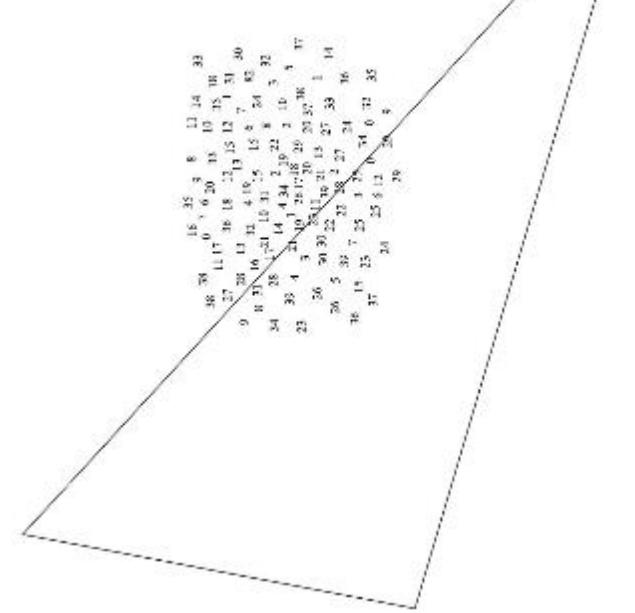
Some members studied geometric constructions. They explored the question: for which  $n$  is it possible to split a given line segment into  $n$  equal parts using compass and straightedge alone? They also constructed the Sign of the Deathly Hallows from J. K. Rowling's Harry Potter series. The Sign of the Deathly Hallows is an equilateral triangle together with its inscribed circle and one of its medians.

A number of members have been exploring permutations, combinations, the principle of inclusion/exclusion, and other fundamental counting concepts. Others extended their study of greatest common factors by looking specifically at the greatest common factors of Fibonacci numbers. We also presented a handmade slide rule, and some members spent some time to figure out how it worked.

We held our traditional end-of-session Math Collaboration. Here are four of the problems from the event:

A frog arena has 11 squares in a row. There are 3 green frogs in the leftmost 3 squares, one per square. There are 7 red frogs in the rightmost 7 squares, one per square. The frogs make

moves to swap sides, with the 3 green frogs ending up in the rightmost 3 squares and the 7 red frogs ending up in the leftmost 7 squares. For a move, a frog can move to the next space over, if it is empty, or, leap frog two spaces over if the next space over is occupied by a frog and the space two over is empty. What is the minimum number of moves that it takes for the frogs to swap sides?



In the figure at left, which number sits directly atop the center of the circumcircle of the triangle?

The greatest common factor of  $n$  and 30 is 6. The least common multiple of  $n$  and 30 is 90. What is  $n$ ?

Determine which Fibonacci numbers are divisible by 5 and prove that your claim is correct.

# Member's Thoughts

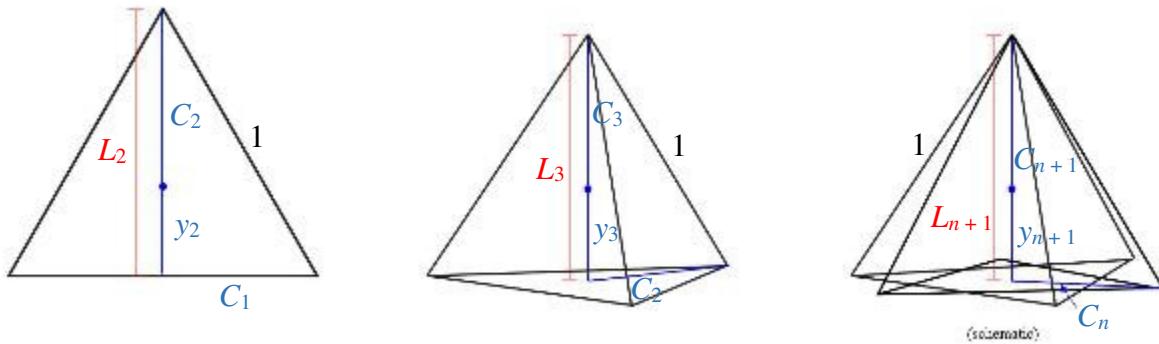
## The Volume of a Regular $n$ -Simplex by $\pi$

Let  $S_n$  be a regular  $n$ -simplex with sides of unit length.

Let  $V_n$  be the volume of  $S_n$ .

Let  $C_n$  be the distance between a vertex of  $S_n$  and its center of mass. (Here, we place unit point masses at each vertex. By symmetry, the distances between the center of mass and the vertices are all the same.)

The  $(n + 1)$ -simplex is a pyramid with base an  $n$ -simplex. The perpendicular dropped from the new vertex to the base will pass through the center of mass of  $S_{n+1}$  and meet the base at its center of mass. Let  $L_{n+1}$  be the length of this perpendicular and let  $y_{n+1} = L_{n+1} - C_{n+1}$ . This is illustrated in the figure below.



Using the technique of mass points, we see that

$$1 \cdot C_{n+1} = (n+1) \cdot y_{n+1}.$$

The factor of  $n+1$  is the total mass of the base of an  $(n+1)$ -simplex, which has  $n+1$  vertices.

Substituting  $L_{n+1} - C_{n+1}$  for  $y_{n+1}$  and solving for  $C_{n+1}$ , we find that

$$C_{n+1} = \frac{n+1}{n+2} L_{n+1}.$$

By the Pythagorean theorem,  $L_{n+1}^2 + C_n^2 = 1$ , so  $L_{n+1} = \sqrt{1 - C_n^2}$ .

Combining these last two results, we find that

$$C_{n+1} = \frac{n+1}{n+2} \sqrt{1 - C_n^2}.$$

Since  $S_1$  is a unit line segment,  $C_1 = 1/2$ .

Using our recursion formula, we can find the first few values of  $C_n$ .

<b><math>n</math></b>	<b><math>C_n</math></b>
1	1/2
2	$\sqrt{3}/3$
3	$\sqrt{6}/4$
4	$\sqrt{10}/5$
5	$\sqrt{15}/6$

The first few values of  $C_n$  appear to be fractions where the numerator is the square root of a triangular number and the denominator is equal to the index, plus 1. So we guess that

$$C_n = \frac{1}{n+1} \sqrt{\frac{n(n+1)}{2}} = \sqrt{\frac{n}{2(n+1)}}.$$

To verify that our guess is correct, we check that this formula satisfies the recursion formula we found for  $C_n$ .

$$\begin{aligned} \frac{n+1}{n+2} \sqrt{1 - C_n^2} &= \frac{n+1}{n+2} \sqrt{1 - \left( \sqrt{\frac{n}{2(n+1)}} \right)^2} \\ &= \frac{n+1}{n+2} \sqrt{1 - \frac{n}{2(n+1)}} \\ &= \frac{n+1}{n+2} \sqrt{\frac{n+2}{2(n+1)}} \\ &= \sqrt{\frac{n+1}{2(n+2)}} \\ &= C_{n+1}, \end{aligned}$$

as desired.

Using the formula for the volume of an  $(n + 1)$ -dimensional pyramid, we have

$$V_{n+1} = \frac{1}{n+1} V_n L_{n+1}.$$

Since  $C_{n+1} = \frac{n+1}{n+2} L_{n+1}$ , we find  $V_{n+1} = \frac{1}{n+1} V_n \frac{n+2}{n+1} C_{n+1} = \frac{1}{n+1} V_n \frac{n+2}{n+1} \sqrt{\frac{n+1}{2(n+2)}}$ , and

this simplifies to  $V_{n+1} = V_n \sqrt{\frac{n+2}{2(n+1)^3}}$ . Thus,  $\frac{V_{n+1}}{V_n} = \sqrt{\frac{n+2}{2(n+1)^3}}$ .

Notice that

$$\frac{V_n}{V_1} = \frac{V_n}{V_{n-1}} \cdot \frac{V_{n-1}}{V_{n-2}} \cdot \frac{V_{n-2}}{V_{n-3}} \cdots \frac{V_2}{V_1}.$$

Since  $V_1 = 1$ , we find

$$V_n = \sqrt{\frac{n+1}{2n^3}} \sqrt{\frac{n}{2(n-1)^3}} \sqrt{\frac{n-1}{2(n-2)^3}} \cdots \sqrt{\frac{3}{2(2^3)}} = \sqrt{\frac{(n+1)!}{2^n (n!)^3}} = \frac{\sqrt{n+1}}{2^{n/2} n!}.$$

We conclude that the volume of a regular  $n$ -simplex with sides of length  $s$  is

$$\frac{\sqrt{n+1}}{2^{n/2} n!} s^n.$$

Here's a table that gives the first few volumes.

<b><math>n</math></b>	<b>Volume</b>
1	$s$
2	$\frac{\sqrt{3}}{4} s^2$
3	$\frac{\sqrt{2}}{12} s^3$
4	$\frac{\sqrt{5}}{96} s^4$
5	$\frac{\sqrt{3}}{480} s^5$

The volume of the regular  $n$ -simplex with side length  $s$ .

Notice that the volume, as a fraction of the cube with side length  $s$ , tends to 0 as  $n$  tends to infinity.

# Calendar

Session 19: (all dates in 2016)

September	15	Start of the nineteenth session!
	22	Jane Kostick, woodworker
	29	
October	6	
	13	
	20	
	27	
November	3	
	10	
	17	
	24	Thanksgiving - No meet
December	1	
	8	

Session 20: (all dates in 2017)

January	26	Start of the twentieth session!
February	2	
	9	
	16	
	23	No meet
March	2	
	9	
	16	
	23	
	30	No meet
April	6	
	13	
	20	No meet
	27	
May	4	

Girls' Angle has been hosting Math Collaborations at schools and libraries. Math Collaborations are fun math events that can be adapted to a variety of group sizes and skill levels. For more information and testimonials, please visit [www.girlsangle.org/page/math\\_collaborations.html](http://www.girlsangle.org/page/math_collaborations.html).

Girls' Angle can offer custom math classes over the internet for small groups on a wide range of topics. Please inquire for pricing and possibilities. Email: [girlsangle@gmail.com](mailto:girlsangle@gmail.com).

# Girls' Angle: A Math Club for Girls

## Membership Application

**Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.**

Applicant's Name: (last) \_\_\_\_\_ (first) \_\_\_\_\_

Parents/Guardians: \_\_\_\_\_

Address (the Bulletin will be sent to this address):  
\_\_\_\_\_  
\_\_\_\_\_

Email:  
\_\_\_\_\_

Home Phone: \_\_\_\_\_ Cell Phone: \_\_\_\_\_

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don't like math, what don't you like? If you love math, what do you love? What would you like to get out of a Girls' Angle Membership?  
\_\_\_\_\_  
\_\_\_\_\_

The \$50 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- Enclosed is a check for \$50 for a 1-year Girls' Angle Membership.
- I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to [girlsangle@gmail.com](mailto:girlsangle@gmail.com).



# Girls' Angle Club Enrollment

**Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!**

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

**Who are the Girls' Angle mentors?** Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

**What is the Girls' Angle Support Network?** The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

**What is Community Outreach?** Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

**Who can join?** Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

**How do I enroll?** You can enroll by filling out and returning the Club Enrollment form.

**How do I pay?** The cost is \$20/meet for members and \$30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

**Where is Girls' Angle located?** Girls' Angle is located about 12 minutes walk from Central Square on Magazine Street in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

**When are the club hours?** Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at [www.girlsangle.org/page/calendar.html](http://www.girlsangle.org/page/calendar.html) or send us email.

**Can you describe what the activities at the club will be like?** Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

**Are donations to Girls' Angle tax deductible?** Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

**Who is the Girls' Angle director?** Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

**Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities?** Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, executive director of Science Club for Girls  
Yaim Cooper, lecturer, Harvard University  
Julia Elisenda Grigsby, assistant professor of mathematics, Boston College  
Kay Kirkpatrick, assistant professor of mathematics, University of Illinois at Urbana-Champaign  
Grace Lyo, Instructional Designer, Stanford University  
Lauren McGough, graduate student in physics, Princeton University  
Mia Minnes, SEW assistant professor of mathematics, UC San Diego  
Beth O'Sullivan, co-founder of Science Club for Girls.  
Elissa Ozanne, associate professor, The Dartmouth Institute  
Kathy Paur, Kiva Systems  
Bjorn Poonen, professor of mathematics, MIT  
Gigliola Staffilani, professor of mathematics, MIT  
Bianca Viray, assistant professor, University of Washington  
Karen Willcox, professor of aeronautics and astronautics, MIT  
Lauren Williams, associate professor of mathematics, UC Berkeley

**At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics?** We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

# Girls' Angle: Club Enrollment Form

Applicant's Name: (last) \_\_\_\_\_ (first) \_\_\_\_\_

Parents/Guardians: \_\_\_\_\_

Address: \_\_\_\_\_ Zip Code: \_\_\_\_\_

Home Phone: \_\_\_\_\_ Cell Phone: \_\_\_\_\_ Email: \_\_\_\_\_

Please fill out the information in this box.

**Emergency contact name and number:** \_\_\_\_\_

**Pick Up Info:** For safety reasons, only the following people will be allowed to pick up your daughter. Names:  
\_\_\_\_\_

**Medical Information:** Are there any medical issues or conditions, such as allergies, that you'd like us to know about?  
\_\_\_\_\_

**Photography Release:** Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

**Eligibility:** Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

**Personal Statement (optional, but strongly encouraged!):** We encourage the participant to fill out the optional personal statement on the next page.

**Permission:** I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

\_\_\_\_\_  
(Parent/Guardian Signature)

Date: \_\_\_\_\_

Participant Signature: \_\_\_\_\_

Members: Please choose one.

- Enclosed is \$216 for one session  
(12 meets)
- I will pay on a per meet basis at \$20/meet.

Nonmembers: Please choose one.

- I will pay on a per meet basis at \$30/meet.
- I'm including \$50 to become a member,  
and I have selected an item from the left.

- I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to [girlsangle@gmail.com](mailto:girlsangle@gmail.com). Also, please sign and return the Liability Waiver or bring it with you to the first meet.

**Personal Statement (optional, but strongly encouraged!):** This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

### **Girls' Angle: A Math Club for Girls Liability Waiver**

I, the undersigned parent or guardian of the following minor(s)

\_\_\_\_\_,  
do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: \_\_\_\_\_ Date: \_\_\_\_\_

Print name of applicant/parent: \_\_\_\_\_

Print name(s) of child(ren) in program: \_\_\_\_\_