Girls' Angle Bulletin

To Foster and Nurture Girls' Interest in Mathematics

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From the Director

We're all set to begin session 6! All past members are warmly welcome back and we hope to meet some new faces. Please help spread the word!

Some exciting news: Dr. Kay Kirkpatrick, a Courant instructor and PIRE fellow at NYU has joined our advisory board. Kay not only brings mathematical expertise to Girls' Angle, she also brings with her extensive experience teaching children math.

Also, thanks to Science House, the Girls' Angle Women in Mathematics videos are now being produced. Conceived by Girls' Angle director Elisenda Grigsby, these videos will showcase the diversity of women in mathematics today. Lauren McGough has constructed a webpage for these videos. Look for the first video on September 30 (see the cover).

Finally, I'd like to give a special Thank You to Connie Chow and the Science Club for Girls for holding our treasury during our nascent stage while we were awaiting 501(c)(3) status.

All my best,
Ken Fan
Founder and Director

Girls' Angle Donors

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**Girls' Angle:
A Math Club for Girls**

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and to empower girls to be able to tackle any field no matter the level of mathematical sophistication required.

On the cover: Ina Petkova explains a proof of the Pythagorean theorem in one of the upcoming Girls' Angle Women in Mathematics videos.

Knot or Not a Knot?

by Allison Henrich
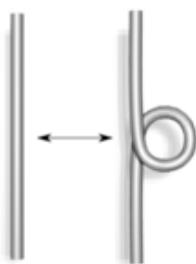
In the last issue of the Bulletin, we discussed the mathematical definition of a knot and learned some things about knots and their diagrams. We said that a "knot" is a knotted circle. You can think of it as a knotted piece of rope with the ends glued together. Recall that we say two knots are the same if we can change one knot into the other by pulling, bending and stretching the rope without breaking it.

In many cases, people find it easier to think about and communicate ideas about knots by drawing pictures (also known as "diagrams") of them rather than making them out of rope. For instance, if you wanted to tell your friend in California about a cool knot you're learning about, it would be easier to e-mail her a picture of the knot rather than send her a knotted piece of rope in the mail. This brings up a question we discussed in the last Bulletin. Given two knot diagrams, how can we tell if they represent the same knot or two different knots?
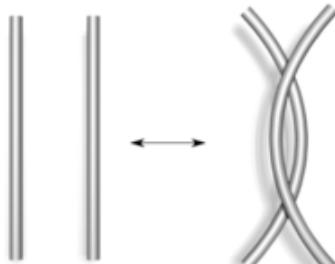
Figure 1: A diagram of the trefoil knot

We looked at a partial answer to this question. We know that two knot diagrams are equivalent exactly when they can be related by a sequence of Reidemeister moves, see Figure 2.
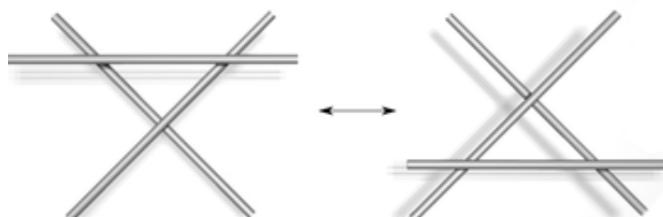
Reidemeister Move I
en.wikipedia.org/wiki/File:Reidemeister_move_1.png

Reidemeister Move II
en.wikipedia.org/wiki/File:Reidemeister_move_2.png

Reidemeister Move III
en.wikipedia.org/wiki/File:Reidemeister_move_3.png

Figure 2: Reidemeister moves I, II and III.

The problem is what if you have two diagrams of knots that you can't see how to relate to each other by a sequence of Reidemeister moves? If they are, in fact, diagrams of different knots, how can you prove it? (Indeed, a failure to prove something is not a proof that the opposite statement is true!) There are many partial answers to this question that people have thought of and there are many more answers yet to be found. Can you think of a way you might be able to prove that two knot diagrams represent different knots? For instance, how might you be able to show that the trefoil knot from Figure 1 is different from the unknot? (Recall that the unknot is the knot that can be drawn as a circle with no crossings.)

While I encourage you to think of your own way to solve this problem, I offer one solution. We introduce the notion of tricolorability. Let's say you have a knot diagram and three colors you can use to color the strands of the knot. A "strand" in a knot will refer to a piece of a knot that passes from one undercrossing to the next undercrossing in the diagram. Suppose you want to color your diagram so that (1) you use at least two colors, and (2) at each crossing, the three strands involved in the crossing are either all the same color or all different colors. See Figure 3 to get an idea of what this looks like.
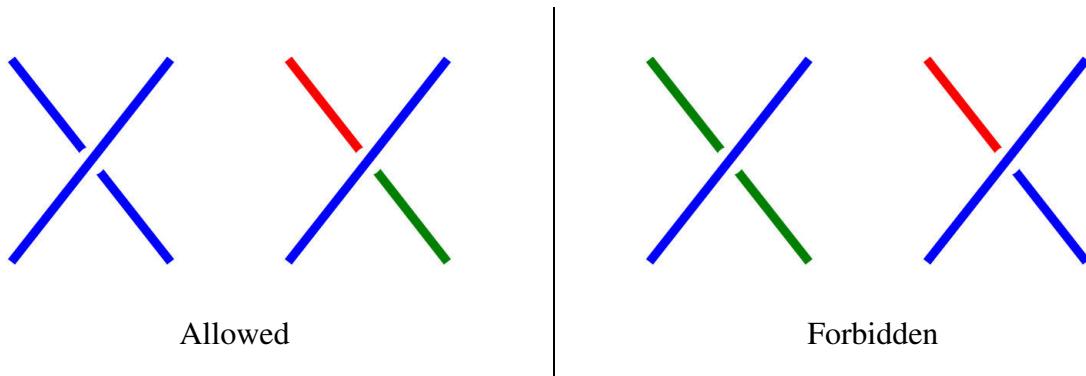
Allowed   Forbidden

Figure 3: Allowed and forbidden colorings at a crossing

Let's look at some examples of colorings of knots to make sure we get the idea. Notice that the unknot can't be colored with all three colors, so it is not tricolorable. On the other hand, the knot on the right (which is secretly a trefoil) is tricolorable because we can see it has a valid coloring that uses all three colors.

Not Tricolorable   Tricolorable
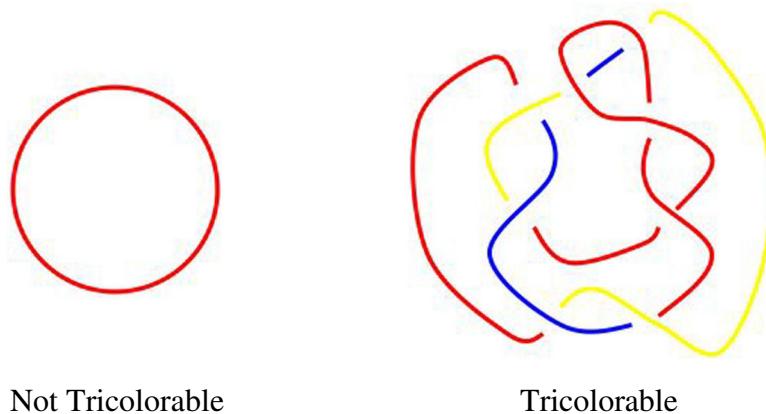
Figure 4: Examples of allowable colorings of knots

Now, using the diagram of the trefoil from Figure 1, show that the trefoil is tricolorable.

What is really cool about tricolorability is that if a diagram of a knot is tricolorable and you perform a Reidemeister move on the diagram, then the result is also tricolorable. Can you prove that this is true? For instance, can you take your tricoloring of the trefoil diagram and extend it to a tricoloring of the diagram after you've performed a Reidemeister 2 move? How about after a Reidemeister 1 move? Use Figure 5 as your canvas.

Figure 5: Three diagrams of the trefoil (tricolor them!)

Now that we know that any two diagrams of the same knot must either both be tricolorable or both non-tricolorable, can you show that the following two knot diagrams represent different knots? The knot on the left is called the figure-eight knot and the one on the right is called the granny knot.
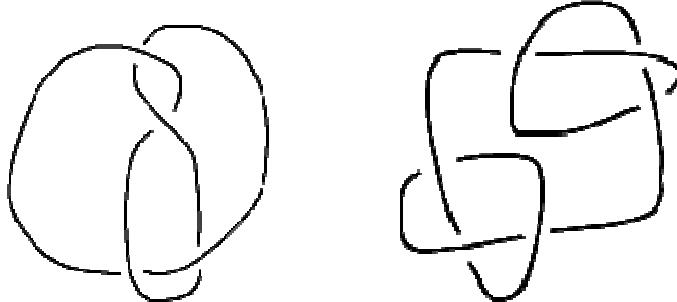
Figure 6: Is either of these knots tricolorable?

Now, to get more practice with tricoloring, make several of your own knot diagrams and classify each of your knots as tricolorable or non-tricolorable. Notice that you have the tools you need to prove in some cases that different diagrams of knots represent knots that are actually different.

Of course, this is not the end of the story. There are many examples of pairs of knots that are both non-tricolorable or both tricolorable yet distinct from one another. That is why we need many ways to tell knots apart. So, as food for future thought, I'll pose our core question again:

Can you think of any other ways you might tell two knots apart?

Good luck and have fun!

An Interview with Rebecca Goldin, Part II

This is the second part of the interview with Rebecca Goldin. Dr. Goldin is associate professor of mathematics at George Mason University and also the director of research at STATS.

Ken: You're also a mother of four handsome boys. I think that there is a feeling "out there" that, especially for women, it is hard to have a family and a career in academia. You are having both. Did you ever feel that your family life and your career were at odds? Is it difficult to be a mathematician and raise a family?

Rebecca: This is a difficult issue. I am in an incredibly lucky position at this point in my life, with four healthy kids and tenure in a place I like. When I had the first two kids, I was a postdoc[1] and I knew that I was risking not being able to get a tenure-track[2] job. This is for two reasons: on the one hand, pregnancy and early childcare is exhausting and time consuming (even when you get daycare) and I was afraid I wouldn't be able to do as much math. And on the other hand, I had much less flexibility because of the kids– living separately from my husband for a year, for example, was not a possibility. At that time, I had to ask myself some hard questions, like whether I would resent my family if I didn't manage to stay in mathematics– it's competitive. But I didn't have to think too hard about it, either– I love kids and I love mathematics, and I knew that if I didn't stay in the university setting, I would be happy doing other fulfilling jobs involving math, such as teaching in a high school. When I realized that the "worst case" scenario with kids was really a great life, I stopped worrying and I never looked back. The second pair of kids (one of whom I had while on tenure track and the other after getting tenure) came along when I was much more established and felt fairly confident that I would be able to get tenure.

I know other women who felt they wouldn't be happy doing any other job than being a math professor– some of them delayed children until they were more established, and several have intimated that they would have been fine with not having kids. Others had kids while on tenure track– and were pretty stressed out for a few years. Still others had kids in grad school and found it too demanding to finish, and still others decided to wait for children and then had trouble having them. The decision about how to balance children and an academic career is extremely difficult, and boils down to your own personal "imperatives" as much as anything else. But if you do have kids while trying to manage any type of career, the most important thing you can do for yourself is to get a lot of help, with everything from housework to laundry to daycare.

Ken: There aren't very many women in mathematics today. Do you think there is gender bias against women in the field of mathematics?

Rebecca: To the extent that gender bias exists today, it's much more subtle than it used to be. I remember when I was in college 15 years ago, and the chair at the time solicited input from the students to make the math department better. I suggested to the chair that women were dropping out of the math program partially because they were too intimidated to ask questions, he stood up

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[1] A "postdoc" (short for postdoctoral position) is a temporary job held by people who have doctoral degrees. A typical postdoc position in mathematics lasts two or three years.
[2] A tenured position in academia can be held until retirement. In order to obtain tenure, one must be hired into a "tenure-track" position, which is a job that could lead to tenure depending on the performance of the employee.

and started screaming at me that women should "JUST RAISE THEIR HANDS." This was in front of about ten people– I shrunk in my chair and a (male) colleague re-expressed my ideas and had a conversion with the chair about them. Little did I know that there was a lot of politics going on behind the scenes– about why there were no female faculty. But even then, men would not make explicit comments about how women can't do math– at least not in front of them– as they did in the 70s and into the 80s.[3]

Nowadays, there is little overt sexism, and I think much less overall sentiment that women aren't as good as men. Very consistently, faculty will note that their best students are girls– but this doesn't always extend into the graduate world. There may be bias behind important decisions such as who gets into graduate school or who gets a job, but it's much harder to put your finger on it. For example, if a school emphasizes the math achievement test as an admission criterion, is that discrimination if boys do better than girls on that test? A priori, the answer seems no– but if other equally important skills such as taking and doing well in advanced mathematics courses are de-emphasized, and women do better in these other fields, then it seems that there is some inherent bias in the system– even without someone intending to discriminate.

The place where I see a bigger problem actually has to do with evaluation, promotion, salary, and general career direction with women versus men. Many women still feel that the service burden of an academic job is larger for them than for their male counterparts. There is also evidence that women are paid less, given less lab space, and promoted less frequently for the same level of accomplishments than men. This could be because women are less aggressive on average than their male counterparts about getting higher salaries, but I do think there's a component of (perhaps subconscious) sexism. Not too long ago, I felt underpaid by my department and a colleague commented "well, your husband is here, so you're stuck" (my husband is a professor in another department of the same university). I don't know if that comment would have been made to him if he were discussing his salary! Finally, many, many women find it difficult to balance their rights for family leave with the sentiment that they will be less respected if they take it in an academic environment– I hear many, many stories from female colleagues about inappropriate comments made by male colleagues, possible more out of ignorance than out of sexism.

However, there are some people and organizations specifically interested in promoting women. Last year, I was awarded the Ruth Michler Award for mathematics by the Association for Women in Math. I spent the semester at Cornell thanks to this award, and it has had a snowball effect. More visibility has led to better negotiations at my own job, several prominent speaking opportunities, and generally a more prominent place within the mathematical community.

Ken: Do you have any advice for the girls at Girls' Angle?

Rebecca: Do what you love– and do all that you love. Don't worry about what you think you can or can't do, or who you think is better than you. Trust your heart– and leave the judgment to others.

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[3] Lately, however, the concept that perhaps women are innately inferior to men at doing mathematics is making a comeback. Some argue that there may be a very slight difference in average ability, and that this slight difference is magnified at the tail end of the bell curve where professional mathematicians reside. Girls' Angle, of course, does not believe these arguments are sound. Also, recent work of Hyde and Mertz which appeared in the Proceedings of the National Academy of Science counters such notions.

7

Anna's Math Journal

By Anna Boatwright

Mathematics is a journey of discovery. As mathematicians take this journey, they follow many wrong turns, believe many incorrect facts, and encounter many mysteries. Out of these twists and turns comes the reward of truth and understanding. However, if you look at math books, you might get the impression that mathematicians rarely err. In this column, Anna Boatwright gives us a peek into her mathematical process of discovery, bravely allowing us to watch even as she stumbles.

Here, Anna finds beloved factors of 15!. (A "beloved" number is a square of a perfect square.)

> My first reaction is to try a simpler example. I'll try 4! first instead of 15!.

Simpler example:

$$4! = 1 \times 2 \times 3 \times 4 = 24$$

factors of 24 are 1, 2, 3, 4, 6, 8, 12, 24

No non-trivial beloved factors.

> It turns out 4! is almost too simple to be of help. I think I'll try to tackle 15! directly.

I will tackle 15!

Want to find

$$15! = 15 \times 14 \times \cdots \times a^4 \times \cdots$$

for some integer "a".

> *Please note that the term "beloved number" is not a standard term in mathematics. These so-called "beloved numbers" are the same thing as perfect fourth powers, and that's what mathematicians would really call them.*

> The first non-trivial beloved number is four squared, which is sixteen. I'll look for a factor of 16 in 15!.

Note: $12 = 6 \times 2$

$$\Rightarrow 15! = 15 \times 14 \times 13 \times 6 \times 2 \times 11 \times 10 \cdots \times 2 \times 1$$

Thus, 4 is a factor of 15!, And 4 is a perfect number.

> Ooops! I meant "perfect square"!

$$15! = 15 \times 14 \times 13 \times 6 \times 2^2 \times 11 \times 10 \times 9 \times 4 \times 2 \times 7 \times \cdots \times 3 \times 1$$

Thus, $2^2 \times 4 = 16$ is a factor of 15!

And 16 is a <u>beloved</u> number.

$$15! = 15 \times 14 \times 13 \times (3 \times 3 \times 2) \times 11 \times 10 \times (3 \times 3) \times 8 \times \cdots \times 3 \times 2 \times 1$$

Thus 3^4 is a factor of 15!

$$= 81$$

> In a similar way, I'll try to find the next beloved factor, which is 81.

> Great! I managed to find two...actually three...beloved factors of 15!: 1, 16, and 81. Maybe I should make a list of beloved numbers.

Key:

> Anna's thoughts

> Anna's afterthoughts

> *Editor's comments*

Integers	Perfect Square	Beloved Number
1	1	1
2	4	16
3	9	81
4	16	256
5	25	625
6	36	1296
7	49	2401
8	64	4096
9	81	6561
10	100	10,000
$\vdots$	$\vdots$	$\vdots$
a	a^2	$(a^2)^2 = a^4$

> Here I begin a list of beloved numbers.

> The numbers get big pretty fast. I'm not sure it would be very efficient to go through this list and check to see if each is a factor of 15!.

> I think I'll look for only those beloved factors that are powers of 2.

$15! = 15 \times 7 \times 2 \times 13 \times 6 \times 2 \times 11 \times 5 \times 2 \times 9 \times 4 \times 2 \times 7 \times 3 \times 2 \times 5 \times 2 \times 2 \times 3 \times 2 \times 1$

> **Ooops!**
> Actually, there are eleven factors of 2 in 15!. For some reason, I forgot the factor of 2 in 6 and the two factors of 2 in this 4. I missed 3 factors of 2. But that won't change my conclusion that 4 and 256 are the only powers of 2 that are beloved factors of 15!.

There are eight 2's that are factors of 15!.

So 2^4 is a factor of 15!

And $2^4 \times 2^4 = 2 \times 2$

$2^8 = 2^4 \times 2^4 = \underbrace{2 \times 2} \times \underbrace{2 \times 2} \times \underbrace{2 \times 2} \times \underbrace{2 \times 2}$

$\qquad = 2^2 \times 2^2 \times 2^2 \times 2^2$

$\qquad = (2^2)^4$

$\qquad = 4^4 \qquad$ is a factor of 15!

~~is a factor~~

Thus, $2^4 = 16$ is a beloved factor of 15!

and $4^4 = 256$ is a beloved factor of 15!

Keep Thinking!...
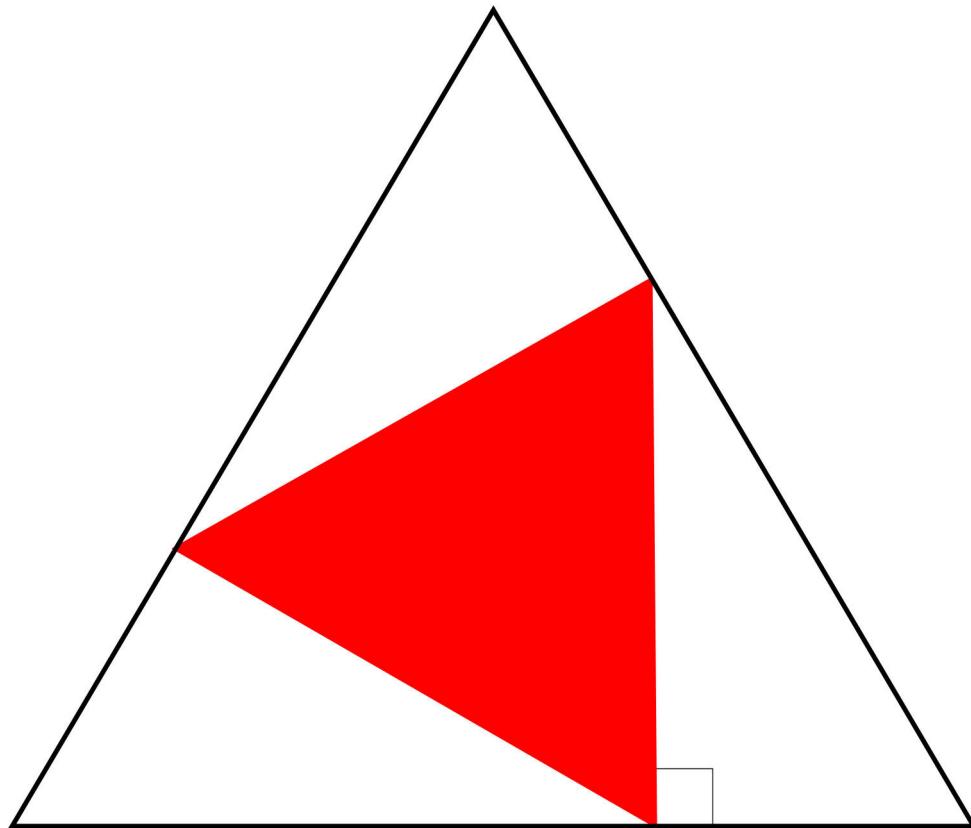
> **Can you help Anna finish this problem?**

> There's got to be an easier way to do this. I know there are other beloved factors of 15!. I already found the beloved factor 81. Hmmm. I'll have to come back to this later.

Key:
- Anna's thoughts
- Anna's afterthoughts
- Editor's comments

ABB 8.23.09

An Equilateral Triangle Problem

The figure below shows a red equilateral triangle inscribed in a black equilateral triangle. The horizontal edge of the black triangle is perpendicular to the vertical edge of the red triangle.

What is the ratio of the area of the black triangle to the area of the red triangle?

When you get a math problem, try to get into the habit of playing with it beyond finding the solution. Can you modify this problem and come up with a new math problem?

For example, what would the answer be if the red equilateral triangle were inscribed at different angles with respect to the black triangle? What would analogous problems and their answers be for other polygons, such as isosceles triangles or squares?

Send your questions and answers to girlsangle@gmail.com.

The Euclidean Algorithm

by Doris Dobi

Recall that the greatest common divisor (sometimes called the "highest common factor") of two integers a and b (which we'll denote by "$GCD(a, b)$") is the greatest positive integer which divides them both. For example, $GCD(3, 4) = 1$ and $GCD(-34, 119) = 17$. For small integers a and b, finding their greatest common divisor is easy enough since we can just exhaustively check all of the positive integer factors of the smaller of the two numbers. But, what if I wanted to find $GCD(234125345234511113, 13998992200982823)$? Since in general it is laborious to factor an integer, as the numbers grow so does the amount of computation we have to do if we go through the factors exhaustively.

The famous Greek mathematician Euclid came up with an efficient method to find the greatest common factor of two integers. He described this method in his famous book *Elements*, which appeared around 300 B.C. At the heart of Euclid's method lies the following observation: $GCD(a, b) = GCD(a, b - a) = GCD(a, b - 2a) = \ldots = GCD(a, b - ka)$ for any integer k. Can you see why this is true? Try working out some examples before reading further. For example, try computing $GCD(14, 392)$ and $GCD(14, 378)$. Can you prove this observation is true?

Let's prove this observation. Let $g = GCD(a, b)$. This means that g divides both a and b. So we can write $a = ga_1$ and $b = gb_1$, where a_1 and b_1 are integers. Then $b - ka = g(b_1 - ka_1)$. This shows that g divides evenly into $b - ka$ for all integers k. So, to show that $g = GCD(a, b - ka)$, we have to show that g is the *greatest* positive integer that divides both a and $b - ka$. To see this, we can go in reverse: if some integer d divides both a and $b - ka$, then, using similar reasoning, d must divide b. But since g is the greatest common divisor of a and b, we would have to have $d \leq g$. Hence, g must be the greatest common divisor of a and $b - ka$.

Notice also that $GCD(a, b) = GCD(b, a)$, so we also know that $GCD(a, b) = GCD(a - kb, b)$ for any integer k.

Let's see how this observation can help us find $GCD(98, 214)$. By what we discussed above, we may write $GCD(98, 214) = GCD(98, 214 - 2 \cdot 98) = GCD(98, 18)$. This reduces our problem to one involving smaller numbers. Rather than directly compute the greatest common divisor of 98 and 18, we can reduce even further!

$$GCD(98, 18) = GCD(98 - 5 \cdot 18, 18) = GCD(8, 18)$$

And further yet:

$$GCD(8, 18) = GCD(8, 18 - 2 \cdot 8) = GCD(8, 2)$$

And why not still further:

$$GCD(8, 2) = GCD(8 - 4 \cdot 2, 2) = GCD(0, 2) = 2. \text{ Thus, } GCD(98, 214) = 2.$$

Notice that we never had to consider what the factors of the original numbers 98 and 214 are.

We can re-express the method described above in the form of a series of divisions with remainders:

$$214 = 2 \cdot 98 + 18$$
$$98 = 5 \cdot 18 + 8$$
$$18 = 2 \cdot 8 + 2$$
$$8 = 4 \cdot 2 + 0$$

What do you notice? First of all, notice that the last nonzero remainder is the sought-after greatest common divisor. Also, at each step we express the quotient as a multiple of the previous remainder plus whatever is left over. The reason this works is precisely the same argument given above. In general, if we wish to find GCD(a, b) we can proceed as follows:

$$a = q_0 \cdot b + r_0$$
$$b = q_1 \cdot r_0 + r_1$$
$$r_0 = q_2 \cdot r_1 + r_2$$
$$r_1 = q_3 \cdot r_2 + r_3$$
$$.$$
$$.$$
$$.$$

In the case that $a < b$, the first step of the algorithm just swaps the numbers so that the initial quotient q_0 will equal zero and r_0 would equal a.

When you divide, the remainders are always smaller than the divisor. Hence, r_k is smaller than its predecessor r_{k-1} for all $k > 0$ until…one finally gets a remainder of zero. When a remainder of zero is reached, we stop, and the penultimate remainder is the greatest common divisor of a and b.

This process of dividing by successive remainders until one arrives at a remainder of zero to find the greatest common divisor of two numbers is called the **Euclidean algorithm**. Notice that the algorithm must terminate because the remainders are all nonnegative and get smaller and smaller. Eventually, one has to get to zero.

I conclude by mentioning some further generalizations of the Euclidean algorithm described above. This material is quite advanced, so don't worry if you don't understand most of it. But often it is helpful to be exposed to things even if they are too advanced. This article describes the original Euclidean algorithm, which is used to compute the greatest common divisor of pairs of natural numbers. But, the algorithm can be generalized to apply to geometric lengths (real numbers), and, it was further generalized in the nineteenth century to other types of numbers, such as **Gaussian integers** (which are numbers of the form $m + n\sqrt{-1}$, where m and n are standard integers). The algorithm can even be applied to polynomials in one variable. Can you figure out how? Here's a hint: the Euclidean algorithm consists of repeated application of division with remainders. Can you figure out how to divide one polynomial into another?
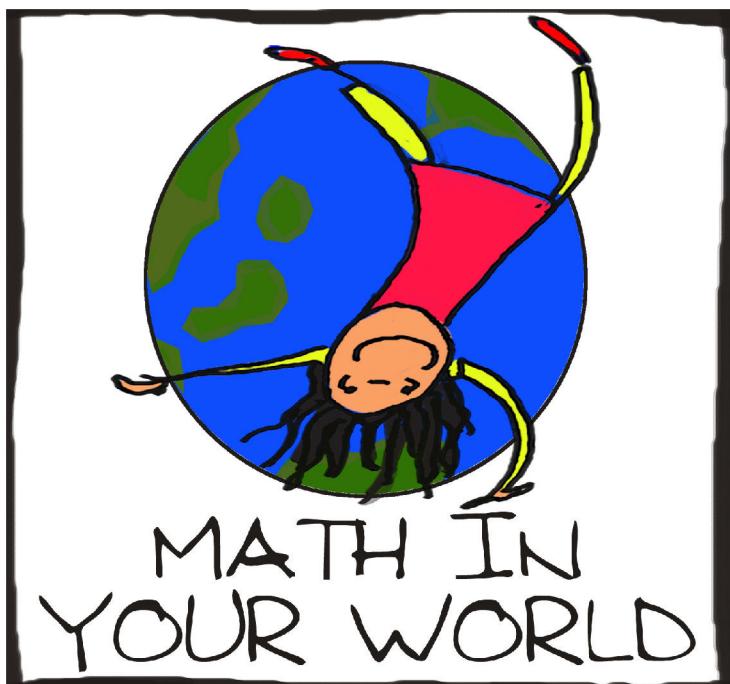
> *Try this!*   Take two consecutive Fibonacci numbers and apply the Euclidean algorithm to them. What happens? What do you conclude is the greatest common divisor of consecutive Fibonacci numbers?

Population + Elevation?

By Katy Bold

When you enter a new city or town, there is usually a sign welcoming you. It usually tells the city's population and sometimes also the elevation and the year the town was established. The sign welcoming visitors to Gold Hill, CO does more than just tell you those three numbers – it also gives you their sum!

"Why" you might ask, "would someone add population, elevation, and date of establishment?" I do not know, but it is a great example of someone forgetting to remember units.

Though the example from Gold Hill may incite a few chuckles, it is a harmless error in units. There have been mistakes with units with much graver consequences. In 1999, NASA lost the Mars Climate Orbiter spacecraft due to a mistake with English and metric units. One set of scientists used the metric system, while another set used the English system. In 1983, a commercial Canadian airplane ran out of fuel because of an improper conversion between pounds and kilograms. Fortunately, the pilots landed the plane and everyone survived.

Most of the United States uses the English system, while most of the rest of the world uses the metric system.

Photo courtesy of Prof. Joseph Smyth

	Metric	English
Length	Meters	Feet/ yards/ miles
Temperature	Celsius	Fahrenheit
Volume	Liters	Cups
Mass	Grams	Pounds

You may come across units and conversion of units in every day life, for example in cooking, calculating gas mileage, converting currency when traveling, or when talking about the temperature with a friend from another country. It is not mathematically difficult to convert between units, as long as you keep track of what quantities you are dealing with.

When converting units:
- Start with the known quantity
- Multiply by conversion factors so that units cancel.

Let's convert between mL (milliliters) and cups. If you have a 330 mL juice, how many cups is that? This is the conversion factor for cups and mL: 1 cup = 240 mL.

So we can compute $330 \text{ mL} \times \dfrac{1 \text{ cup}}{240 \text{ mL}} = 1.375$ cups.

If you forget whether to multiply or divide by 240, just remember that units should cancel, like this:

$$330 \; \cancel{\text{mL}} \times \dfrac{1 \text{ cup}}{240 \; \cancel{\text{mL}}} = 1.375 \text{ cups.}$$

Most people need a calculator (or at least pencil and paper) to compute conversions. In day-to-day life, though, you may not need an exact conversion, and there are some shortcuts that can help you do the math in your head. The shortcuts come from rounding the conversion factors to numbers that are easier to work with.

For example, to convert between Celsius, C, and Fahrenheit, F, the exact conversions are $F = \dfrac{9}{5}C + 32$ and $C = \dfrac{5}{9}(F - 32)$. A shortcut that gives an approximate conversion is to replace 32 by 30 and $\dfrac{9}{5}$ by 2: $F \approx 2C + 30$ and $C \approx \dfrac{1}{2}(F - 30)$.

	Exact Conversion	Shortcuts
Temperature	Fahrenheit = $\dfrac{9}{5}$Celsius + 32	$F \approx 2C + 30$
Length	inches = 2.54 centimeters miles = (0.6213…) kilometers	$I \approx 2.5C$ $M \approx 0.6K$
Volume	1 liter = (4.2267…) cups	
Mass	1 kilogram = (2.204…) pounds	

Come up with a good shortcut!

Try it Out
Estimate the conversions using a shortcut of your choice, then find the exact conversions. How close (or different) is the answer from your shortcut? Why?

Running races
A common race distance is the 5 K (5 kilometers). How many miles is the race?
If the average running stride is 4 feet, on average how many steps would someone take during the race?

The length of a marathon race is 26.2 miles. About how many kilometers is this?

At the beach
My Italian friend loves to go to the beach when it is 28 degrees outside. What! That would be so cold! Oh, wait, my friend is Italian so probably thinks in degrees Celsius. What temperature is that in degrees Fahrenheit? Would you like to go to the beach at that temperature?

Fibonacci and the Angry Pigeon

Using the Pigeonhole Principle to Study the Fibonacci Sequence

by Lauren McGough

Even though the pigeonhole principle might seem like an "obvious" statement, it can lead to non-obvious results! Here, we discuss the solution to the last problem from my Summer Fun problem set from the last issue.

In order to tackle this problem, let's ask: Is there a way we can tell if the last n digits of the Fibonacci sequence are periodic after a point in a finite amount of time? As humans with a finite amount of computation time, we can't really just sit down and compute all of the Fibonacci numbers, and then check to see if they start repeating at some point. There are infinitely many Fibonacci numbers and so going through this process would take an infinite amount of time. Moreover, it could be that the last n digits of the Fibonacci numbers don't start repeating until many, many Fibonacci numbers have gone by, so even if we were to just start computing, it might take a very long time to reach a point where we see repetitions.

One place we might start when thinking about the digits of the Fibonacci numbers is to remember that the Fibonacci numbers are not random numbers, and thus neither are their digits, in some sense. What I mean by that is: we must remember that the Fibonacci sequence is generated by a *recurrence relation* – each Fibonacci number after the first two is the sum of the previous two Fibonacci numbers. But then, notice that if each Fibonacci number is the sum of the previous two Fibonacci numbers, the last n digits of each Fibonacci number are the last n digits of the sum of the last n digits of the previous two Fibonacci numbers (where, if a number doesn't have n digits, we just fill in the missing digits with zeros). (By "last n digits", I mean the rightmost n digits.)

It turns out that this realization and a pigeonhole principle argument, is just what we need to show that the last n digits of the Fibonacci numbers are eventually periodic.

Let's start by considering the periodicity of just the very last digit (i.e. the case $n = 1$). Focus your attention on pairs of consecutive last digits: (1, 1), (1, 2), (2, 3), (3, 5), (5, 8), (8, 3), (3, 1), etc. If, in this sequence of pairs, a pair shows up that has appeared before, then it must be the case that the last digits of the Fibonacci sequence are eventually periodic. The reason is that each Fibonacci number is determined only by the prior two, so if a pair appears again, the digits that follow each of the two identical pairs will be indistinguishable. This means that the pattern of digits starting from the first of the identical pairs to just before the second will repeat over and over and over.

So, what we need to do is show that there are two identical pairs in our modified Fibonacci sequence. Can you sniff an application of the pigeonhole principle? After all, the pigeonhole principle concludes that at least two things must be in the same place… So, to this end, we calculate an upper bound on the number of possible pairs of last digits. Since there are ten options for the first number in the pair (the last digit can be 0, 1, 2, 3, 4, 5, 6, 7, 8 or 9), and also ten options for the second number in the pair, there are $10 \times 10 = 100$ possible pairs of last digits.

By the pigeonhole principle, given 101 pairs of last digits, at least two of the pairs must be equal.

So let's consider the first 101 pairs of last digits of Fibonacci numbers! By the preceding argument, at least two of the pairs are equal. But wait! That's just what we needed! Thus the last digit of the Fibonacci sequence is eventually periodic.

For the next question, we note that we can rephrase our argument for the last digit to work for the last two digits and even the last n digits of the Fibonacci sequence without too much work. We simply pair off the Fibonacci numbers as before only this time, we consider pairs consisting of the last n digits of numbers in the Fibonacci sequence. For each pair of last n digits there are $(10^n)^2 = 10^{2n}$ possible values. Thus, if we take the first $10^{2n} + 1$ pairs of Fibonacci numbers, at least two of these pairs will be equal, and thus we conclude by the same reasoning that the last n digits of the Fibonacci sequence will eventually be periodic.

Do you understand why it was necessary to consider *pairs* of Fibonacci numbers, and how the recurrence relation combined with the pigeonhole principle guarantee eventual periodicity?

Try your hand at the following challenge: the argument above shows that the last n digits of the Fibonacci sequence are *eventually* periodic. But are they also actually periodic? (To understand the difference, a sequence that starts out with the numbers 12, 13, 24, 52, 63, 74, would have a periodic last digit if the last digits after this point continued to be 2, 3, 4, 2, 3, 4, ... forever, but would only have an eventually periodic last digit (not a periodic last digit) if after this point the last digits went 3, 4, 3, 4, 3, 4, 3, 4... forever.) Can you prove or disprove the periodicity (not just the eventual periodicity) of the last digit of the Fibonacci numbers? How about the last two digits? Or the last n digits?

And for more practice with the pigeonhole principle, try this:

Given a collection of $N+2$ integers, show that there exist two of them whose sum, or else whose difference, is divisible by 2N.

Summer Fun!

In the last issue, we invited members to submit solutions to a number of Summer Fun problem sets.

In this issue, solutions to all of the problems are provided. These solutions will sometimes be rather terse and, in some cases, are more of a hint than a solution. We prefer not to give completely detailed solutions before we know that most of the members have spent time thinking about the problems. The reason is that *doing* mathematics is very important if you want to learn mathematics really well. If you haven't tried to solve these problems yourself, you won't gain as much when you read these solutions.

Solutions that are especially curt will be indicated in red. Please do not get frustrated if you read a solution and have difficulty understanding it. If you run into difficulties, we are here to help! Just ask!

If you haven't thought about the problems, we urge you to do so *before* reading the solutions. Even if you cannot solve a problem, you will benefit from trying. Even more, it will become easier to read other people's solutions after you've tried to solve the problems yourself.

With mathematics, don't be passive! Get active!

Move that pencil and move your mind! Your mind may just end up somewhere no one has been before.

Also, the solutions presented are *not* definitive. Try to improve them or find different solutions.

Summer Fun!

The Coin Flipping Game!

by Maria Monks

1. One coin is placed heads-up in each of the squares of a 1×10 grid as shown below.
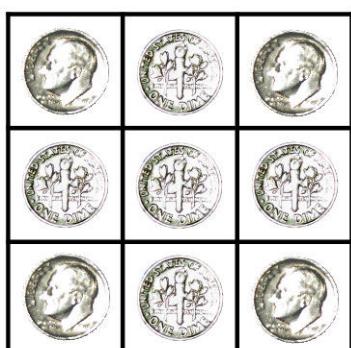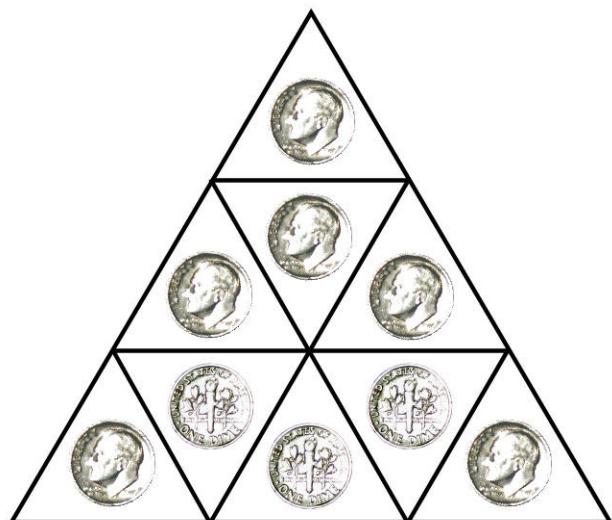
You play a game where one move consists of first choosing one of the coins and then flipping over that coin along with any coin that is right next to it. For example, you can form the following pattern by choosing the first and sixth coins in the row of coins above, then flipping according to the rules:

You can make as many moves as you like. The goal is to eventually get the row to be all tails up. Can you do this? What if you are not allowed to choose the first or tenth coin, that is, you must always flip three coins? Can you do it if there were only 9 coins in a row?

2. Now coins are placed heads up in each of the triangles of a triangular grid with side length 3 units (each of the small triangles have side length 1 unit). A move consists of flipping a coin and all of its "neighbors," the coins that are in a triangle sharing an edge with the triangle containing the chosen coin. (In the grid shown at right, the coin in the bottom middle triangle was chosen and flipped together with its neighbors.) Can the grid be flipped to all tails in this manner? What if you start with a triangular grid that has side length 2? How about 4?

3. Now, suppose there is a coin in each of the squares of a 3×3 grid, starting heads up. As before, you are allowed to make moves consisting of choosing a coin and flipping over it and all its "neighbors," the coins in the squares sharing an edge with it. For instance, if you choose the coin in the middle to flip first, you will get the pattern shown. Can you make a sequence of moves that makes all the coins tails up? What if you start with a 4×4 grid? A 5×5?

Summer Fun!

Solutions (Maria Monks)
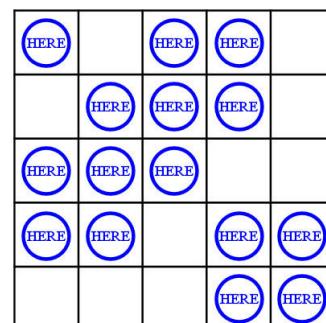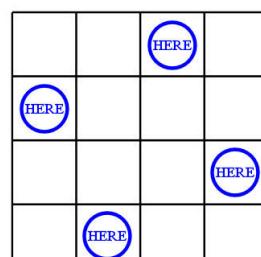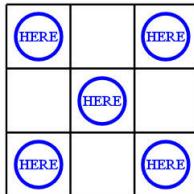
1. For a sequence of 10 coins starting heads up, we can make a move on the first, fourth, seventh, and tenth coins in the row to make the sequence show all tails. For a sequence of 9 coins, we can make a move on the second, fifth, and eighth coins to result in a sequence of all tails.

The second question is more difficult. If you play around with it for a while, it certainly seems that you can't get the coins to be all tails in this way. But how can we be certain that we're not missing some move? A nice way to simplify the problem is to look at it from the point of view of each coin. Pick a particular coin, say the fourth coin. This coin can only be flipped when a move is made on either the third, fourth, or fifth coin. Furthermore, suppose we made a sequence of moves that includes a move on the third coin six different times. This flipped the fourth coin six times, which is even. So from each coin's point of view, it is the same as flipping it zero times! Similarly, if we made a move on a particular coin an odd number of times, it is the same as flipping it one time. Therefore, we only have to consider sequences of flips in which each coin is chosen either 0 or 1 times.

With this in mind, let's look at the endpoints. For the first coin to be flipped, the second coin must be flipped exactly once. But then the second and third coin shows tails after this flip, and so the third coin must be chosen for a move 0 times. Continuing along the row to the right, it follows that the fourth coin is flipped 0 times, the fifth coin flipped 1 time, the sixth 0 times, etc, flipping every third coin. But this leaves the 10th coin unflipped, and so it is indeed impossible to get the coins to show all tails.

2. For a triangle of any size, we can always make moves on precisely those coins which are in triangles that are in the same orientation as the entire grid, that is, we can scale the smaller triangle up without rotating to obtain the larger triangle. This will flip each of the chosen coins once and all of the other coins exactly 3 times, so that all of the coins then show tails. However, this is not the only way to solve the triangular grid problem. For the triangular grid of side length 2, we can simply choose the middle coin to make a move, and we are done in one move. Bonus Question: Can you see a way to solve the triangular grid of side length 4 with only four flips?

3. For the 3×3 grid, we can make a move on the center square and the four corners to win. For the 4×4 and 5×5, we can make a move on the squares as shown on the grids below.

The Pigeonhole Principle
by Lauren McGough

In this problem set, when I say "number", I mean an integer, that is, a counting number, the negative of a counting number, or zero.

1. Suppose you have a drawer that contains exactly six black socks, eight blue socks and nothing else. Imagine pulling socks out of the drawer with eyes closed. What is the minimum number of socks you need to pull from the drawer in order for you to be 100% sure that you have pulled out two socks of the same color? What if you start out with a drawer that contains four red socks as well as six black socks and eight blue socks – does your answer change? What if you start out with a drawer that contains socks of m different colors with four socks of each color?

2. This spring at Girls' Angle, we spent a lot of time discussing numbers written in different bases: we discovered that the *binary* number system, for example, is a way of representing numbers using only the digits 1 and 0, and that the *ternary* number system is a way of representing numbers using only the digits 0, 1 and 2. Can you show that, given any four ternary numbers, at least two of them must share the same last digit? What if you are given any nine binary numbers – what is the maximum number of them that is always guaranteed to share the same last digit? Suppose we write numbers in base N. Can you prove that, given $N + 1$ numbers written in base N, at least two of them must share the same last digit? Given $N^2 + 1$ numbers written in base N, can you prove that at least two of them must share the same last two digits? More generally, can you prove that, given $N^m + 1$ numbers written in base N, at least two of them must share the same last m digits?

3. Suppose that there are sixteen girls at the first meeting of Girls' Angle, and suppose that every girl shakes hands with some number of other girls (for example, Girl 1 might shake hands with five others and Girl 2 might shake hands with three others). Show that at least two of the girls shook hands with the same number of people.

4. You are given a collection of N integers. Show that there exists some pair of numbers in your collection whose difference is divisible by $N - 1$. Can you show that there is some (nonempty) subcollection whose sum is divisible by N? (A subcollection might contain only one integer, in which case, the sum of the integers in that subcollection is just the integer itself.)

5. The Fibonacci sequence is a famous sequence of numbers that is formed as follows: The first two numbers in the sequence are both 1, then each subsequent number is the sum of the two numbers that precede it (so the first few Fibonacci numbers are 1, 1, 2, 3, 5, ...). Prove that the last digit of the Fibonacci sequence is eventually periodic (it eventually repeats – for example, the sequence 1, 2, 3, 7, 9, 5, 8, 3, 9, 2, 9, 5, 2, 4, 9, 5, 2, 4, 9, 5, 2, 4, ... would be eventually periodic if it just continued repeating "9, 5, 2, 4" forever because even though it didn't start out periodic, it eventually became a repeating sequence). Can you also show that the last two digits of the Fibonacci sequence are eventually periodic? Can you extend this even further to show that the last n digits of the Fibonacci sequence are periodic?

Summer Fun!

Solutions (Lauren McGough)

1. If we have m different designs and at least one pair of at least one of the designs, we need to pull out at least $m + 1$ socks. If we pull out m or fewer socks, then we could get m (or fewer) socks of different designs, so the minimum to be guaranteed to have a pair has to be greater than m. But if we pull $m + 1$ socks then we will have more socks than we have designs, so at least two of the socks must have the same design, so we must have pulled out a pair. Using this fact, assuming we have at least one pair in each collection, if we have 2 different colors of socks, we need to pull three to be guaranteed a pair; and if we have 3 different colors of socks, we need to pull four to be guaranteed a pair.

2. In base N there are N options for each digit. This means there are N^m possibilities for the last m digits of any number (where if a number has fewer than m digits, we just fill in the unused digits with zeros), and thus if we have $N^m + 1$ numbers written in base N, at least two of these share the same last m digits in the same order. Moreover, this is a minimum, since there are N^m distinct possibilities for the last m digits of a number written in base N, so some collections of any fewer than $N^m + 1$ numbers will have distinct last m digits, and we are not guaranteed that two of them have the same last m digits. (Note that by "the same last m digits," we mean the same values in the same order.) It follows that if we have four ternary numbers, at least two of them will have the same last digit, and we can see that since $9 = 4 \times 2 + 1$, if we have nine binary numbers, at least five of them will have the same last digit (do you see why?). The other statements follow by substituting different values of m in the fact shown above.

3. We note that each girl could have shaken hands with as few as 0 girls or as many as 15 other girls; this gives us 16 options for the number of girls to have shaken hands with and we know we have 16 girls. However, if one girl shook hands with 0 girls, no girl shook hands with 15 other girls, since that girl would have had to have shaken hands with all the other girls. Similarly, if one girl shook hands with 15 girls, no girl shook hands with 0 girls. Thus, there are really only 15 possible options for the number of girls each girl could have shaken hands with: 1, 2, 3, ..., 14, and either 0 or 15 but not both. We then have 16 girls and only 15 options for the number of girls each girl shook hands with, leaving us to conclude that at least two girls must have shaken hands with the same number of girls.

4. Let's call the N integers a_1, a_2, a_3, …, a_N. For the first question, note that there are only N - 1 possible remainders when you divide a number by N - 1. By the pigeonhole principle, at least two of the N integers a_1, a_2, a_3, …, a_N must therefore leave the same remainder when you divide them by N - 1. Their difference must then be divisible by N - 1.

Here's one way to solve the second question: Consider the N sums a_1, $a_1 + a_2$, $a_1 + a_2 + a_3$, …, $a_1 + a_2 + a_3 + . . . + a_N$. If any of these sums are divisible by N, then we are done, so let us assume that none of these sums are divisible by N. In that case, when we divide each by N, the remainder will be a number between 1 and N - 1, inclusive. Because there are N sums, by the pigeonhole principle, at least two must leave the same remainder. Suppose the two sums that leave the same remainder are $a_1 + a_2 + a_3 + . . . + a_s$ and $a_1 + a_2 + a_3 + . . . + a_t$ where $1 \leq s < t \leq N$. Then, their difference $(a_1 + a_2 + a_3 + . . . + a_t) - (a_1 + a_2 + a_3 + . . . + a_s) = a_{s+1} + . . . + a_t$ must be divisible by N, and we're done!

5. See page 15.

"Distance" Between Numbers

by Elisenda Grigsby

We all know instinctively what we mean by numbers being "close" or "far." For example, 4 is closer to 5 than it is to 6, and it's *much* closer to 6 than it is to 1000. But how do we actually quantify what our instincts tell us? That's easy! If we want to know the distance between two numbers, x and y, —let's denote this distance by $d(x, y)$— we just take the absolute value of their difference: $d(x, y) = |x - y|$.

Recall that the absolute value of zero or a positive number is the number itself, and the absolute value of a negative number is -1 times that number. For example, $|5| = 5$, and $|-5| = 5$.

Using this notion of distance, we compute that $d(4, 5) = |4 - 5| = |-1| = 1$, $d(4, 6) = 2$, and $d(4, 1000) = 996$. In other words, the definition of "distance" I defined above matches our instincts. The larger $d(x, y)$ is, the "farther apart" x and y are.

I would like to tell you about a *completely different* (but, in a certain sense, just as valid!) notion of distance between numbers that is completely counter to our intuition about distance. Let's define this new notion of distance as follows. Suppose x and y are whole numbers, and suppose a is the maximum number of times we can divide $|x - y|$ by 2 and still get a whole number. Let's define the "2-adic" distance between x and y, denoted $\Delta_2(x, y)$, as follows: $\Delta_2(x, y) = \dfrac{1}{2^a}$. Since it is not clear what to do when $x - y = 0$, we also assert: $\Delta_2(x, x) = 0$.

So, for example, $\Delta_2(6, 14) = 1/2^3 = 1/8$, since $6 - 14 = -8$, and we can divide $|-8| = 8$ by 2 three times and still get a whole number.

1. Compute the following 2-adic distances: $\Delta_2(10, 6)$, $\Delta_2(90, 10)$ and $\Delta_2(194, 2)$.

2. Show that the 2-adic distance function satisfies the so-called "triangle inequality": If x, y and z are numbers, then $\Delta_2(x, y) + \Delta_2(y, z)$ is greater than or equal to $\Delta_2(x, z)$. Show that the "normal" distance function I defined at the beginning also satisfies the triangle inequality.[1]

3. Make a list of whole numbers, x_1, x_2, x_3, x_4, etc. that satisfy the inequalities
$$\Delta_2(0, x_1) > \Delta_2(0, x_2) > \Delta_2(0, x_3) > \Delta_2(0, x_4) > \dots$$

4. Suppose p is any prime number and let x and y again represent whole numbers. Define the "right" notion of the "p-adic" distance between x and y (in particular, you should be able to show that it satisfies the triangle inequality and the distance between two numbers should be 0 exactly when the two numbers are equal).

5. Suppose p is any prime number and x and y are fractions.[2] How should we define the p-adic distance between x and y?

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[1] Every "self-respecting" distance function should satisfy the triangle inequality, which roughly says, "It is always shorter to go directly from x to z, without stopping at any other point, y, along the way."

[2] A fraction is any number that can be written as a quotient of two whole numbers.

Summer Fun!

<u>Solutions</u> (Eli Grigsby)

1. Note that asking, "How many times can we divide a number, N, by 2 and still get a whole number?" is the same thing as asking, "How many times can we multiply 2 by itself, and still get a number that evenly divides N?" In other words, "What is the largest power of 2 that evenly divides N?" We compute:

a. $|10 - 6| = 4$. Now $2 \times 2 = 2^2 = 4$ evenly divides 4, but $2 \times 2 \times 2 = 2^3 = 8$ does not. Therefore, $\Delta_2(10, 6) = 1/2^2 = 1/4$.
b. $|90 - 10| = 80$. Now $2^4 = 16$ evenly divides 80, but $2^5 = 32$ does not. So, $\Delta_2(90, 10) = 1/16$.
c. $|194 - 2| = 192$. Now $2^6 = 64$ evenly divides 192, but $2^7 = 128$ does not. So, $\Delta_2(194, 2) = 1/64$.

2. First let's show that the "normal" distance function satisfies the triangle inequality. Note that if $x = z$, then $d(x, z) = 0$ and there is nothing to prove. So assume $x \neq z$. Now, either y is between x and z (and possibly equal to one or the other) or it isn't. If y is between x and z (that is, $x \geq y \geq z$ or $z \geq y \geq x$), then $d(x, z) = d(x, y) + d(y, z)$. If y is not between x and z, then we must either have $d(x, y) > d(x, z)$ (that is, x is farther from y than it is from z) or $d(z, y) > d(z, x)$ (that is, z is farther from y than it is from x). In both cases, the fact that a length (and, hence, the distance between two numbers) is always nonnegative allows us to conclude that $d(x, z)$ is always less than or equal to $d(x, y) + d(y, z)$.

To show that the 2-adic distance function satisfies the triangle inequality, we will again consider two cases: either y is between x and z, or it isn't.

In the first case, $|x- z| = |x - y| + |y - z|$, so if 2^j is the largest power of 2 dividing $|x - y|$, and 2^k is the largest power of 2 dividing $|y - z|$, then $|x- z|$ is divisible by 2^m, where m is the smaller of j and k. Here, we are using the fact that if a, b, and c are whole numbers, and a evenly divides both b and c, then a evenly divides $b + c$. Furthermore, 2^{m+1} does not divide $|x- z|$. (Why? Hint: If a evenly divides both b and c, then a evenly divides $b - c$.) Therefore, 2^m is the largest power of 2 that divides $|x- z|$, so $\Delta_2(x, z) = 1/2^m$. But this is less than $\Delta_2(x, y) + \Delta_2(y, z) = 1/2^j + 1/2^k$, which proves the triangle inequality in this case.

In the second case, either $|y - z| = |y - x| + |x - z|$ or $|x - y| = |x - z| + |y - z|$. Using the same type of argument as we used in case 1, we can again conclude that if 2^j is the largest power of 2 dividing $|x - y|$, and 2^k is the largest power of 2 dividing $|y - z|$, then $|x - z|$ is divisible by 2^m, where m is the smaller of j and k. Furthermore, 2^{m+1} does not divide $|x - z|$. So, again, we have $\Delta_2(x, z) = 1/2^m$, and $\Delta_2(x, z) \leq \Delta_2(x, y) + \Delta_2(y, z) = 1/2^j + 1/2^k$, as desired.

3. There are many such lists, but here is a simple one. Just let x_1 be 2, x_2 be $2^2 = 4$, x_3 be $2^3 = 8$, and so on (in general, define x_n to be 2^n). Then

$$\Delta_2(0, x_1) = 1/2 > \Delta_2(0, x_2) = 1/4 > \Delta_2(0, x_3) = 1/8 > \Delta_2(0, x_4) = 1/16 > ..., \text{ as desired.}$$

Any list satisfying the property that, for each n, x_n is divisible by a higher power of 2 than x_{n-1} will satisfy the desired property. Can you think of other lists?

4. Just replace "2" by "p" in all of the definitions and in the proof of the triangle inequality!

Summer Fun!

5. To find the "right" definition of the *p*-adic distance between two fractions, we should use the fact that a fraction is a ratio of two whole numbers. We also want the definition to match the definition we have already made for the *p*-adic distance between whole numbers.

Suppose that *x* and *y* are fractions. Start by computing $|x - y|$. The result will itself be a fraction, a ratio of two whole numbers: *a/b*. We know that the definition of the *p*-adic distance should involve powers of *p*, since it should match the original definition when *a/b* is a whole number. But *a* and *b* are themselves whole numbers! Why not see how many powers of *p* divide *a*, and how many powers of *p* divide *b*, and record the answer somehow?

We should be careful, though. Remember that a fraction does not have a unique expression as a ratio of two whole numbers. For example, 27/18 represents the same fraction as 3/2. Furthermore, 27 is divisible by 3^3 and 18 is divisible by 3^2, while 3 is only divisible by 3^1 and 2 is divisible by 3^0. Note, however, that the *difference* of the two exponents is well-defined for the fraction: 3 - 2 = 1 - 0 = 1.

This suggests that we define the *p*-adic distance in terms of this difference, as follows: If $|x - y| = a/b$, then $\Delta_p(x, y) = 1/p^{j-k}$ where *j* is the largest power of *p* dividing *a* and *k* is the largest power of *p* dividing *b*.

So, in the above example, $\Delta_3(3/2, 0) = 1/3$.

Making sure that the triangle inequality works for this extension of the definition of the *p*-adic distance is a good exercise, using essentially the same ideas we used in the solution to problem 2, taking care to add and subtract fractions correctly. We leave this verification to the patient reader.

Do what you love– and do all that you love. Don't worry about what you think you can or can't do, or who you think is better than you. Trust your heart– and leave the judgment to others.

\- Rebecca Goldin

Summer Fun!

Math and Tarot Cards

by Gregg Musiker
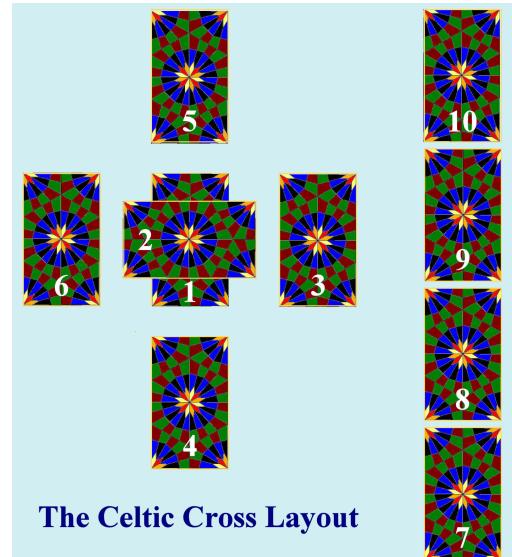
A Tarot Deck consists of 78 cards. 56 of the cards are known as minor arcana and come in four suits (Wands, Cups, Swords, and Pentacles). Each suit consists of 14 cards (A, 2, 3, 4, 5, 6, 7, 8, 9, 10, Page, Knight, Queen and King). The remaining 22 cards are known as major arcana and include the Fool, Magician, High Priestess, Empress, Emperor, Hierophant, Lovers, Chariot, Strength, Hermit, Wheel of Fortune, Justice, Hanged Man, Death, Temperance, Devil, Tower, Star, Moon, Sun, Judgment, and World. Major arcana do not have suits.

Glenda is a Tarot card reader and recently has been seeing some eerie coincidences among the cards. She believes the cards may be telling her something. She is of the impression that she used to be strong-willed and until very recently was a star, but that she is presently being quite foolish. Foolish enough, she will be taken in by a cunning Emperor in the near future, with whom she will fall in love. However, it appears that this will lead to her downfall and she will be left alone after her fortunes turn. Her only hope is that a wise priestess or empress can allay her fear and help steer her towards a better final outcome.

Help set her mind at ease by computing probabilities! In all of these problems, ten cards are freshly dealt in the ten-card Celtic Cross layout (pictured at right).

1. What is the probability that the Fool card is in Position 1 (the present) and the Emperor card is in Position 6 (the immediate future) in the same reading?

2. How probable is it that Positions 3 and 4 (the distant and recent past) include the Strength card and the Star Card, in either order, in the same reading?

3. What is the probability that the Judgment or Justice card is in Position 2 (the immediate challenge) and the Lovers or Chariot card is in Position 5 (the best outcome) in the same reading?

The Celtic Cross Layout

4. How likely is it that at least one of the cards dealt in Positions 7-10 is the Wheel of Fortune, Hermit, or Magician card?

5. What is the probability that the High Priestess and the Empress both appear in the ten card spread?

6. How likely is it that at least one of the cards dealt in Positions 1-6 is the Sun, Moon, or World?

7. Glenda does seven readings in a row, and, after finishing, notes that the High Priestess appeared three times (possibly more) in the seven readings. What is the probability of this event? Should Glenda be surprised? Explain.

Same question, except now Glenda does fourteen readings? Twenty-one readings?

Solutions (Gregg Musiker)

1. We start with a deck of 78 cards. If we deal off the top card and place it in position 1, there are therefore 78 possibilities it could be. Only 1 of these is the Fool card. Therefore the probability that we deal the Fool card in position 1 is 1/78.

We then deal out cards into positions 2-5 and for the purposes of this question, do not care what we get. We then deal a card into position 6. If we assume that we have already dealt the Fool card into position 1, there are 77 cards remaining in the deck. Only one of these is the Emperor card. Therefore there is a probability of 1/77 that the Emperor card is in position 6 (if we assume the Fool card is in position 1). This is called **conditional probability**. The probability of both of these events happening is the product $1/78 \times 1/77 = 1/6006$.

2. This problem is similar to the last. It does not matter that we are asking about positions 3 and 4 instead of positions 1 and 6. However, it *does* matter that we will accept either order (Strength in position 3 and Star in position 4, or, Star in position 3 and Strength in position 4).

The probability of having either Strength or Star in position 3 is 2/78. Then the probability of having either Strength or Star in position 4 (assuming that we already succeeded in obtaining Strength or Star in position 3) is 1/77. Therefore the total probability is $2/78 \times 1/77 = 1/3003$.
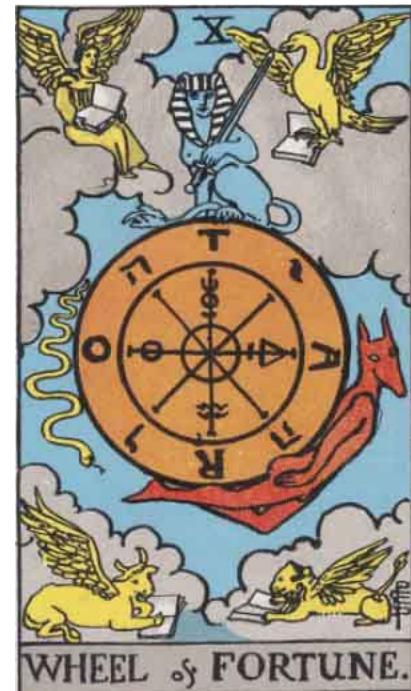
3. The probability of Judgement or Justice in position 2 is 2/78. The probability of Lovers or Chariot in position 5 (assuming Judgement or Justice in position 2) is 2/77. Note that since there is no overlapping of cards this time, the probability of success is higher. The total probability is therefore $2/78 \times 2/77 = 2/3003$.

4. One way to solve this problem is to apply the approach above: We first calculate the probability of Wheel of Fortune, Hermit, or Magician in position 7 (which is 3/78). If such an event occurs, it doesn't matter what happens in positions 8-10, we are guaranteed a layout where "at least one of cards dealt in positions 7-10 is Wheel of Fortune, Hermit, or the Magician card."

There are other ways to succeed. There is a probability of 75/78 that the card dealt in position 7 isn't any of the three desired cards. In this case, there is one less undesirable card out of the deck and the probability of dealing one of the three special cards into position 8 is 3/77. Even if this fails, there is still a possibility of success, if the correct cards shows up in the 9th or 10th spot.

The Wheel of Fortune from the Rider-Waite Tarot Deck

Taking into account all of these possibilities, we add up the probabilities since the scenarios that we have described are independent from one another:

$$\frac{3}{78} + \frac{75}{78}\frac{3}{77} + \frac{75}{78}\frac{74}{77}\frac{3}{76} + \frac{75}{78}\frac{74}{77}\frac{73}{76}\frac{3}{75} = \frac{2813}{19019} \approx 14.8\%$$

Note that this probability includes the possibility that two or all three of these cards show up.

A second, and perhaps easier, approach to this problem starts by calculating the probability that none of these three cards shows up in positions 7-10. There are then 75 out of 78 possible cards for position 7 and 74 out of 77 possible cards for position 8 and 73 out of 76 possible cards for position 9 and 72 out of 75 possible cards for position 10. Thus the probability that none of the three cards show up is $\frac{75\times74\times73\times72}{78\times77\times76\times75} = \frac{16206}{19019}$. If we subtract this quantity from 1, we get the answer to the question because the event where none of the three cards appear and the event where at least one of the three cards appear are mutually exclusive.

5. This can be answered using the exhaustive method similar to the first approach for solving problem 4. We present an alternative solution where we break down the question by characterizing the ten card layouts that have both of these cards, and reducing this to an earlier question.

Since we want both the High Priestess and Empress cards to appear, we need to specify which positions they appear in.

There are $10 \times 9 = 90$ choices since there are 10 slots for the High Priestess card, and 9 slots remaining for the Empress, assuming that the High Priestess has already taken up a slot. Once specific positions have been fixed, the problem reduces to problem 1. Thus the probability of both of these cards appearing somewhere in the spread is 90 times more likely than the probability of the High Priestess in position 1 and the Empress in position 6. Thus the probability is $90/6006 = 15/1001$.

6. This problem is similar to problem 4. We first calculate the probability that positions 1-6 do not contain the Sun, Moon, or World: $\frac{75}{78}\times\frac{74}{77}\times\frac{73}{76}\times\frac{72}{75}\times\frac{71}{74}\times\frac{70}{73} = \frac{2130}{2717}$. The answer is

$1-\frac{2130}{2717} = \frac{587}{2717} \approx 21.6\%$.

7. This problem has multiple parts. First, we need to calculate the probability that the High Priestess appears as one of the ten cards in the layout. Since we are dealing out 10 cards out of the 78 card deck, the probability is 10/78.

Each reading is an independent event, so to calculate the probability that the High Priestess appears in at least three of seven readings, we use the **binomial distribution**. This is just a fancy way of saying that we must calculate the probability that the first three readings contain the High Priestess and the final four do not. We then multiply this probability by the number of ways these three readings can be distributed among the seven (such as the first, third, and sixth readings containing the High Priestess and the second, fourth, fifth, and seventh missing this card). This kind of computation arises frequently and so there is a name and symbol for the number of ways to pick out b things from a set of a things: "a **choose** b", or, in symbols, $\binom{a}{b}$.

Here we need $\binom{7}{3}$, which is $\dfrac{7 \times 6 \times 5}{3 \times 2 \times 1}$ (can you see why?).

We then do a similar calculation for the probability that the High Priestess card appears exactly four times, five times, six times, or seven times.

Thus the probability is:

$$\left(\frac{10}{78}\right)^3 (1-\frac{10}{78})^4 \binom{7}{3} + \left(\frac{10}{78}\right)^4 (1-\frac{10}{78})^3 \binom{7}{4} + \left(\frac{10}{78}\right)^5 (1-\frac{10}{78})^2 \binom{7}{5} + \left(\frac{10}{78}\right)^6 (1-\frac{10}{78})^1 \binom{7}{6} + \left(\frac{10}{78}\right)^7 (1-\frac{10}{78})^0 \binom{7}{7}$$

which is approximately 4.9%.

This is a pretty low probability. However, please see below for a more advanced answer to "whether Glenda should be surprised?"

To calculate the probability that the High Priestess appears in *at least* three out of fourteen readings, we calculate the exact probability that this card does not appear in any reading, the probability the card appears in exactly one reading, and the probability it appears in exactly two readings. The answer is then 1 minus the sum of these three values:

$$1 - \left(\frac{10}{78}\right)^0 (1-\frac{10}{78})^{14} \binom{14}{0} - \left(\frac{10}{78}\right)^1 (1-\frac{10}{78})^{13} \binom{14}{1} - \left(\frac{10}{78}\right)^2 (1-\frac{10}{78})^{12} \binom{14}{2} \approx 26.4\%.$$

Similarly, for 21 readings, the probability is

$$1 - \left(\frac{10}{78}\right)^0 (1-\frac{10}{78})^{21} \binom{21}{0} - \left(\frac{10}{78}\right)^1 (1-\frac{10}{78})^{20} \binom{21}{1} - \left(\frac{10}{78}\right)^2 (1-\frac{10}{78})^{19} \binom{21}{2} \approx 51.6\%,$$

which means this will happen more than half the time.

We now discuss a second, more complicated, answer to the question "Should Glenda be surprised after seeing the High Priestess appear in three out of seven readings?"

The crucial observation is that as Glenda began the first reading, she was not looking specifically for how many times the High Priestess would appear, but only noticed the pattern after all seven of the readings. In fact, if any of the 22 major arcana cards had shown up at least 3 out of 7 times, it might have looked like more than mere coincidence.

So, as an extra challenge problem, think about how you would calculate the probability that at least one of the 22 cards appears at least three out of seven times. This is laborious to calculate directly, but running computer simulations, the probability turns out to be around 84%!!! Taking this into account, Glenda should not be as surprised or spooked by the Tarot cards.
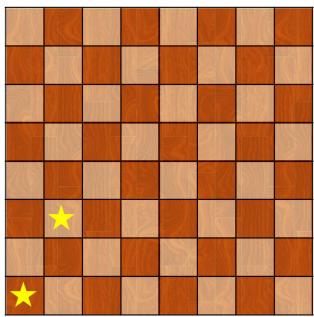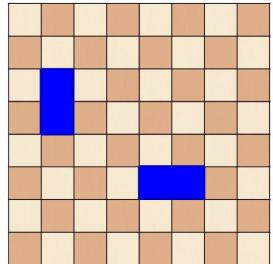
Summer Fun!

Chess Road Trip
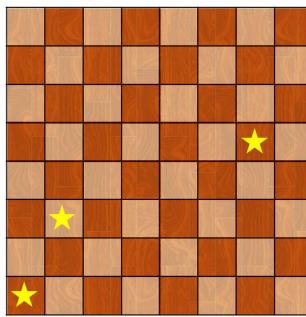
by Grace Lyo

Taylor Walker and her older sister Casi are on a road trip to the Grand Canyon with their parents. They brought a chess set and dominos to pass the many hours they will be spending in the car. After playing dominos for a couple hours, they get bored and decide to play chess, only to discover that they forgot to bring the pieces! They decide to invent games and puzzles of their own instead.
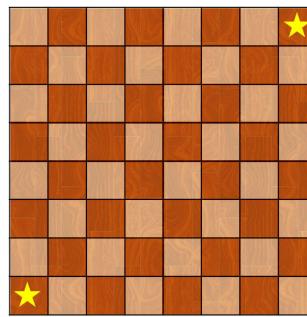
Dominos Casi comes up with a puzzle for Taylor first. Each domino is exactly the size of two squares on the chess board, so it can be placed on the board horizontally or vertically as in the diagram at right. Casi asks Taylor if in each of the scenarios below she can arrange the dominos so that every square except those marked with a yellow star is covered (see pictures (a) through (c)). Will Taylor be able to find domino tilings that work?
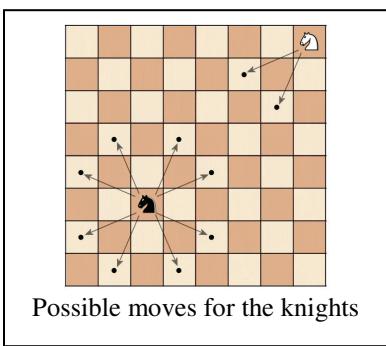

| a. | b. | c. |
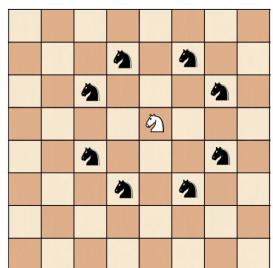
(d) After solving (a), (b), and (c), Taylor asks Casi if there is any quick and easy way to determine whether a board minus a few squares can be tiled. Is there? If so, prove that your solution is correct.

Possible moves for the knights
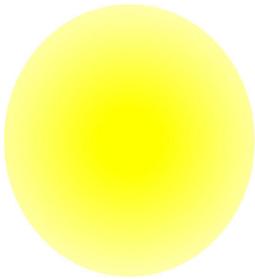
Horses Taylor, having solved Casi's domino-tiling puzzle now gives Casi a puzzle of her own. A "knight" is a chess piece that looks like a horse. It moves in a very special way. In one turn, it can either move horizontally two squares and then vertically one square, or vertically two squares and horizontally one square.

Two chess knights are "attacking" each other if they can move to each other's squares in one turn. At right, the black knights are attacking the white knight and vice versa, but none of the black knights are attacking each other.

(a) What is the maximum number of knights that can be placed on the chessboard in such a way that no two knights are attacking one another?

(b) Challenge question: Prove that your answer is correct. Note that coming up with a configuration and then showing that no more knights can be added to that configuration is not a valid proof!
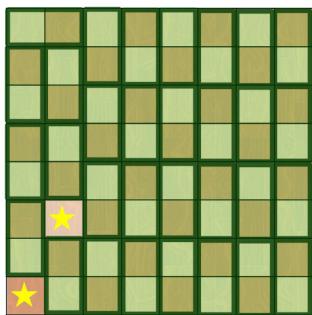
Solutions (Ken Fan)

Dominos

a. Yes

b. No. Removal of 3 squares results in an odd number of squares and any tiling by 2×1 dominos involves an even number of squares.

c. No. Every domino must cover one dark and one light square. This means that any tiling by dominos must cover a region with an equal number of dark and light squares. Removal of two dark squares from a chess board results in a board with more light squares than dark.

d. There's no quick way to determine this in general. However, there are a few ways to quickly see that some cases can or cannot be tiled. For example, removal of an odd number of squares will result in a board that cannot be tiled. Also, there is this result of Gomory:

One can always tile a board that results from deleting any one dark and any one light square from a chessboard using two by one dominos.

Can you reprove this result? If you'd like to read a proof, see page 66 of the book *Mathematical Gems* by Ross Honsberger.

Horses

a. 32.

b. Observe that knights always move to a square of different color. Thus, if knights are placed on all of the light squares, they will all be attacking empty dark squares and not each other.

To see that more than 32 knights is impossible, we'll use the fact that a knight can travel about a chess board in such a way as to visit every one of the 64 squares exactly once before returning to its starting location. Such a circuit through the 64 squares of a chessboard is known as a **knight's tour**. Finding a knight's tour is a famous puzzle by itself; can you find one?

If we now place more than 32 knights on a chess board and imagine how these knights are arranged with respect to a knight's tour, we can see that some two knights must be adjacent. After all, if the next square in the knight's tour after each knight is empty, there would be more than 32 empty squares along with the more than 32 squares occupied by knights. But two knights adjacent to each other on a knight's tour must be attacking each other.

Cars and Goats

by Kay Kirkpatrick

You're a contestant on a game show, and you are presented with three doors.[1] Behind two of the doors there are goats; behind the third, a car. The game-show host knows what is behind each door and invites you to choose a door. Once you have chosen one of the doors (but not opened it), the host must open a different door to reveal a goat. Then you are invited to switch to the other unopened door if you wish. You get to take (or drive) home what is behind the door of your choice, and you want the car. Should you switch?

1. First, play this game with a partner and three cards, the ace of spades to represent the car and two red cards to represent the goats. One of you will be the host and will place the three cards face down, knowing what they are. The other will be the contestant and will select (but not look at) one card. Then, remembering the remaining two cards, the host will turn over one of them to reveal a red card. Then the contestant will decide whether to switch. After that, look to see whether you won the ace. Take turns being the contestant and the host, and play this game 10 or 15 times, recording whether you switched and whether you won. How often did you win when switching? When not switching? Do you see a pattern?

2. Next, imagine playing the game with 10 doors instead of three. In this variation, the host would invite you to choose one, and then would open 8 of the remaining doors, revealing 8 goats. Would switching increase your chances of winning the car? Why? (You can play this variation with a partner, the ace of spades, and 9 red cards.)

3. Let's analyze the 10-door variation of the game by cases. You will initially choose either the door with the car behind it or one of the others. You have one chance in 10 of initially choosing the door with the car behind it, or as we say, that happens with probability 1/10. Does switching result in winning in this case? On the other hand, with what probability do you initially choose a door with a goat behind it? Which doors will the host open then? And does switching result in winning in this case? Now look over your analysis and summarize it by answering this question: Is the strategy of always switching a good one?

4. We can analyze the original three-door problem similarly. With what probability do you initially choose the door with the car behind it? With what probability do you initially choose a door with a goat behind it? Which doors could or would the host open? Is the strategy of always switching a good one?

5. What if, in the 10-door variation, the host were to open only 7 doors, revealing 7 goats? Would switching increase your chances of winning? What about only 6? Only 1?

6. What if, in the three-door problem, the host doesn't know what's behind the doors and just opens one of the two remaining doors at random? If the car is revealed, then the game is over with no prize. If a goat is revealed, then you are invited to switch. How would this variation affect your strategy?

Goat image from en.wikipedia.org/wiki/File:Irish_Goat.jpg

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[1] The game described here is often referred to by the name of the host who popularized it. We'll reveal that host with the solutions.

Solutions (Kay Kirkpatrick)

This is called the Monty Hall problem, after the host for the game show *Let's Make a Deal*. The short answer is you should always switch- it doubles your chances of winning! Here's why.

- If you play the game long enough, you'll find yourself winning about twice as often when you switch than when you don't switch.

- With 10 doors, the crux of the problem becomes clearer. If you play this version of the game with cards, you'll find yourself winning much more often (nine times as often) when you switch than when you don't.

- One time in ten you will select the car at the beginning. In this case, Monty Hall can open any 8 of the 9 remaining doors, because all 9 have goats behind them. Thus switching will result in losing this one time in ten, or, 10% of the time. What about the other nine times out of ten? With probability $\frac{9}{10}$ the car is behind one of the doors that you did not select. Monty Hall can't open the door with the car behind it, so he must open the other 8 doors. Switching in this case will result in winning 90% of the time. In short, the strategy of always switching is nine times better than not switching: it increases your chances of winning to 90% from 10%.

- The probability that you originally chose the door with the car is $\frac{1}{3}$. The probability that the car is behind one of the other two doors is $\frac{2}{3}$. Once one of those doors is opened, the $\frac{2}{3}$ probability remains with the other door, twice the probability that you chose the car originally. So the strategy of always switching doubles your chances of getting the car. Another way of thinking about this is in terms of the following diagram.

Choose:	Probability	Monty opens	Switch to:	Win on switching?	Win on staying?
Car	$\frac{1}{3}$	Goat A	Goat B	No	Yes
		Goat B	Goat A		
Goat A	$\frac{1}{3}$	Goat B	Car	Yes	No
Goat B	$\frac{1}{3}$	Goat A	Car	Yes	No
Proportion of wins on strategy:				$\frac{2}{3}$	$\frac{1}{3}$

You will win on staying only one-third of the time, that is, only when you choose the car with your first pick. You will win on switching two-thirds of the time, when you choose a goat at first. This is twice as good as the staying strategy.

Summer Fun!
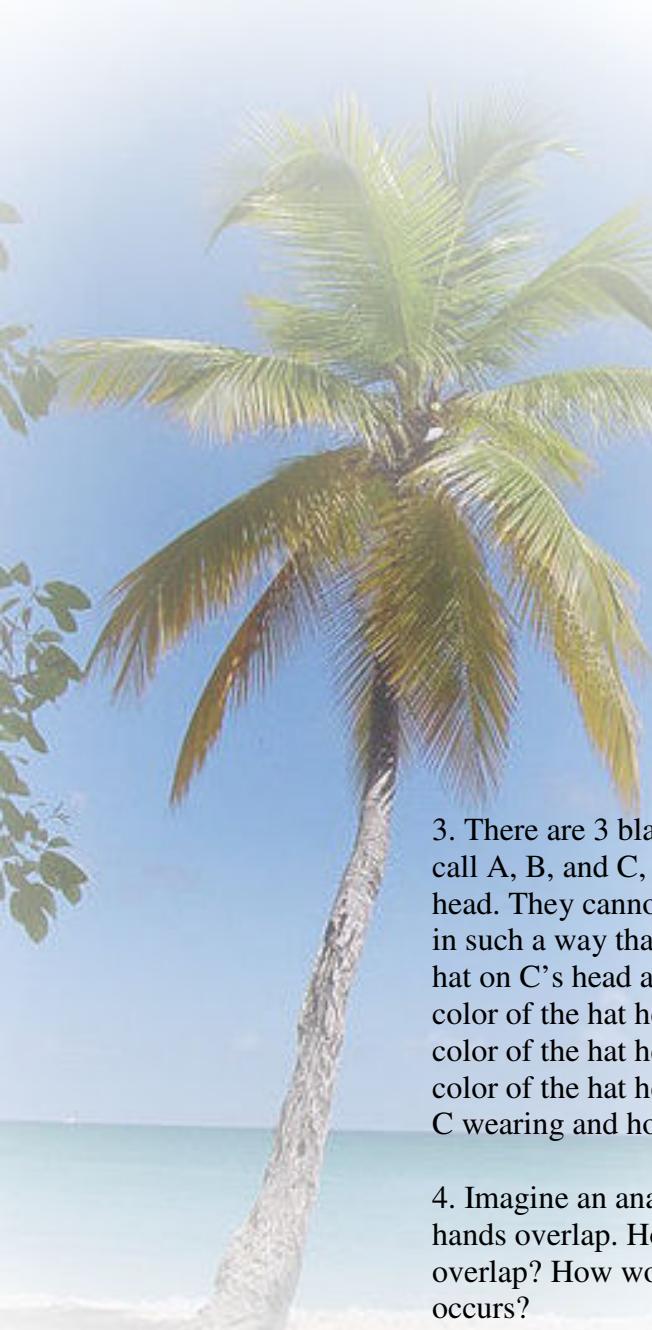
- If Monty opens only 7 of the 9 remaining doors in the 10-door variation, switching still gives you an advantage: there's still a 10% chance that the car is behind the door you originally selected, but a 90% chance that the car is behind one of the other two doors that he leaves open. So switching to either of those two doors increases your chances of winning from 10% to $\dfrac{90\%}{2} = 45\%$. Similarly, if he opens only 6 of the remaining doors, there's a 90% chance that the car is behind one of the three unopened doors (besides your original choice), so switching to any of those three increases your chances of winning from 10% to 30%. If he opens only one of the 9 remaining doors, switching to any of the other 8 increases your chances of winning from 10% to $\dfrac{90\%}{8} = 11.25\%$. So each door that Monty opens gives you useful information about where the car is not.

- The following table illustrates the breakdown of the modified game.

Choose:	Probability	Monty opens	Switch to:	Win on switching?	Win on staying?
Car	$\dfrac{1}{3}$	Goat A	Goat B	No	Yes
		Goat B	Goat A		
Goat A	$\dfrac{1}{3}$	Goat B	Car	Yes	No
		Car	Game over	-	-
Goat B	$\dfrac{1}{3}$	Goat A	Car	Yes	No
		Car	Game over	-	-
			Proportion of wins on strategy:	$\dfrac{1}{6} + \dfrac{1}{6} = \dfrac{1}{3}$	$\dfrac{1}{3}$

With probability $\dfrac{1}{3}$, you choose the car at first, and this part is the same as before. But if you choose a goat at first (probability $\dfrac{2}{3}$), Monty Hall has an equal chance of revealing the car (probability $\dfrac{1}{3}$ - game over) as the other goat (probability $\dfrac{1}{3}$). In other words, you have an equal chance of winning regardless of whether your strategy is switching or staying.

A Potpourri of Problems

by Doris Dobi

If none of the other problems caught your fancy, perhaps one of these miscellaneous ones will!

1. Each natural number can be decomposed into a product of primes. For example, $24 = 2 \times 12$, $154 = 2 \times 7 \times 11$ and $23 = 23$. This is called decomposing a number into its prime factorization. Let us call a number "spunky" if its prime factorization consists of exactly three consecutive primes. The first spunky number is $30 = 2 \times 3 \times 5$. Find the fourth spunky number.

2. An analog clock reads 3:15. What is the angle between the minute hand and hour hand?

3. There are 3 black hats and 2 white hats in a box. Three men, which we'll call A, B, and C, each reach into the box and place one of the hats on his own head. They cannot see what color hat they have chosen. The men are situated in such a way that A can see the hats on B and C's heads, B can only see the hat on C's head and C cannot see any hats. When A is asked if he knows the color of the hat he is wearing, he says no. When B is asked if he knows the color of the hat he is wearing he says no. When C is asked if he knows the color of the hat he is wearing he says yes and he is correct. What color hat is C wearing and how can he know?

4. Imagine an analog clock set to 12 o'clock. Note that the hour and minute hands overlap. How many times each day do both the hour and minute hands overlap? How would you determine the exact times of the day that this occurs?

5. Alba, Ada and Antea are best friends. One of the girls always tells the truth, one always tells lies, and one answers yes or no randomly. The girls know each other very well so that each girl knows which girl is which. You may ask three yes or no questions to determine who is who. If you ask the same question to more than one person you must count each time you ask as one of the three questions asked. What three questions should you ask?

6. On a deserted island there live five people and a monkey. One day everybody gathers coconuts and puts them together in a community pile to be divided the next day. During the night one person decides to take his share himself. He divides the coconuts into five equal piles, with one coconut left over. He gives the extra coconut to the monkey, hides his pile, and puts the other four piles back into a single pile. The other four islanders then do the same thing, one at a time, each giving one coconut to the monkey to make the piles divide equally. What is the smallest possible number of coconuts in the original pile?

Summer Fun!

<u>Solutions</u> (Doris Dobi)

1. For this problem note that the fist few primes are 2, 3, 5, 7, 11, 13, etc. So, the first *spunky* number is the product of the first three primes, namely, it is $2 \times 3 \times 5 = 30$. The second spunky number is $3 \times 5 \times 7 = 105$, the third is $5 \times 7 \times 11 = 385$, so the fourth is $7 \times 11 \times 13 = 1001$.

2. Draw a circle with 12 equally spaced dashes representing the hours. Since a circle consists of 360 degrees, it must be the case that between any two such dashes (i.e., between any two hours) there are $360 \div 12 = 30$ degrees. Now, if it were exactly 3 o'clock the hour hand would be pointing exactly at the 3, but since 15 minutes have passed (which is a quarter of an hour) the hour hand is a quarter of the way between 3 and 4 and the minute hand will be pointing at the 3. Hence, there are $30 \div 4 = 7.5$ degrees between the minute and the hour hand.

3. First note that since A can see both the colors of the hats B and C are wearing, they cannot both be wearing white hats otherwise A would conclude that he must be wearing a black hat. Thus, B and C deduce that one of them must be wearing a black hat. Now, since B doesn't know the color of the hat he is wearing, it must be the case that C is wearing a black hat (otherwise, from what we already concluded, B would know that if C were wearing a white hat then he must be wearing a black hat). Hence, C concludes that he is wearing a black hat and he is right.

4. We show that such an event occurs 11 times a day. Suppose such a time occurs at x o'clock and y minutes, where x is an integer from 1 to 12, inclusive and y is a real number that satisfies $0 \leq y < 60$. The hour hand and minute hand overlap when the angle between them is 0 degrees (or some integral multiple of 360 degrees). The angle made by the minute hand with the ray pointing straight up is given by $6y$, because there are 60 minutes in 360 degrees, so each minute must count for 6 degrees. Similarly, the angle measured by the hour hand with the ray pointing straight up is given by $30x + 30\left(\dfrac{y}{60}\right)$ degrees. (Check this!) So what we need is that $6y$ and $30x + 30\left(\dfrac{y}{60}\right)$ differ by a multiple of 360 degrees. Because of the range restrictions on x and y, we know that $0 \leq 6y < 360$ and $30 \leq 30x + 30\left(\dfrac{y}{60}\right) < 390$. This means that we need only look for cases where $6y$ and $30x + 30\left(\dfrac{y}{60}\right)$ are exactly equal or the latter is exactly 360 greater than the former. In the first case, the equation becomes $6y = 30x + 30\left(\dfrac{y}{60}\right)$ which simplifies to $5.5y = 30x$. Because y is a real number, there is a solution for each of the values of x from 1 to 10. (When $x = 11$, then $y = 60$, but we are restricting y to be less than 60.) In the second case, the equation becomes $6y = 30x + 30\left(\dfrac{y}{60}\right) - 360$, which simplifies to $5.5y = 30x - 360$. The only solution to this (within our restriction on the range) is $x = 12$ and $y = 0$. So, there are 11 such times and they are at 12 o'clock sharp and then at x hours and $\dfrac{60}{11}x$ minutes, where x can be any integer from 1 to 11, inclusive.

5. First, for simplicity let Alba, Ada, and Antea be A, B, and C respectively. Note that there are six possible scenarios. If we let the first position in a triplet correspond to the truth teller, the second position correspond to the liar, and the third position correspond to the random person, then we can write, for instance, (A, B, C), to mean that Alba is the truth teller, Ada is the liar, and Antea is the random person. We thus have the six scenarios:

1	2	3	4	5	6
(A, B, C)	(A, C, B)	(B, A, C)	(B, C, A)	(C, A, B)	(C, B, A)

Follow these steps which will determine in no more than 3 questions which scenario we're in:

Step	Action
1	Ask A, "Is B more likely to tell the truth than C?" If yes go to Step 2, if no go to Step 5.
2	Ask C, "Are you the random girl?" If yes go to step 3, if no go to Step 4.
3	Ask C, "Is A the truth girl?" If yes then we're in scenario 4, if no we're in scenario 2.
4	Ask C, "Is A the lying girl?" If yes we're in scenario 5, if no we're in scenario 6.
5	Ask B, "Are you the random girl?" If yes go to step 6, if no go to step 7.
6	Ask B, "Is A the truth girl?" If yes we're in scenario 6, if no we're in scenario 1.
7	Ask B, "Is A the lying girl?" If yes we're in scenario 3, if no we're in scenario 6.

6. This is a version of a classic problem. From the conditions, it must be that: The original pile must have a number of coconuts such that you can subtract one and then divide by 5 and get an integer. Possibilities are 6, 11, 16, 21, 26, etc. (Can you see why there cannot be just 1 coconut?) After the first person is finished the remaining pile will have 4, 8, 12, 16, 20, etc. coconuts if there were originally 6, 11, 16, 21, 26, etc. coconuts. The remaining pile after the first person must have a number of coconuts such that you can again subtract one then divide by 5 and get an integer. Of the possible remaining amounts found, the only that also have this property are 16, 36, 56, 76, 96, etc., and these will leave 12, 28, 44, 60, 76, etc. coconuts after the second person is done with the pile. The number of coconuts remaining after the second person must again be such that you can subtract one then divide evenly by 5. Possibilities are 76, 156, 236, 316, 396, etc., and these will leave 60, 124, 188, 252, 316, etc. coconuts after the third person is done with the pile. Of these numbers, only 316, 636, 956, 1276, 1596, etc. have the property that if you subtract one the result is evenly divisible by 5, and these pile sizes will leave 252, 508, 764, 1020, 1276, etc. coconuts after the fourth person is done. The remaining pile after the fourth person must again have a number of coconuts that is one more than a multiple of 5. The smallest possibility is 1276, and this will leave 1020 coconuts after the fifth person is done with the pile. So the fifth person will hide 1276 - 1020 - 1 = 255 coconuts and leave behind a pile of 1020 coconuts. The fourth person will leave behind a pile of 1020 + 255 + 1 = 1276 coconuts and will hide $1276 \div 4 = 319$ coconuts. The third person will leave behind a pile of 1276 + 319 + 1 = 1596 coconuts and will hide $1596 \div 4 = 399$ coconuts. The second person will leave behind a pile of 399 + 1596 + 1 = 1996 coconuts and will hide $1996 \div 4 = 499$ coconuts. The first person will leave behind a pile of 499 + 1996 + 1 = 2496 coconuts and will hide $2496 \div 4 = 624$ coconuts.

We conclude that the original pile must have 624 + 2496 + 1 = 3121 coconuts.

Calendar

Session 5: (all dates in 2009)

September	10	Start of fifth session!
	17	Tanya Khovanova, mathematician
	24	
October	1	
	8	
	15	
	22	Jane Kostick, wood worker
	29	No meet
November	5	
	12	JJ Gonson, Cuisine En Locale
	19	
	26	Thanksgiving - No meet
December	3	
	10	

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Girls' Angle: A Math Club for Girls

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

What is Girls' Angle? Girls' Angle is a math club for girls and a supportive community for all girls and women engaged in the study, use and creation of mathematics. Our primary mission is to foster and nurture girls' interest and ability in mathematics and empower them to be able to tackle any field, no matter the level of mathematical sophistication required. We offer a comprehensive approach to math education and use a four component strategy to achieve our mission: Girls' Angle mentors, the Girls' Angle Support Network, the Girls' Angle Bulletin and Community Outreach.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society. They write articles for the Bulletin, take part in interviews and visit the club.

What is the Girls' Angle Bulletin? The Girls' Angle Bulletin is a bimonthly (6 issues per year) electronic publication that features interviews, articles and information of mathematical interest as well as a comic strip that involves mathematics.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-11. We aim to overcome math anxiety and build solid foundations, so we welcome *all girls* regardless of perceived mathematical ability. There is no entrance test.

In what ways can a girl participate? There are 2 ways: **membership** and **active subscription** to the Girls' Angle Bulletin. **Membership** is granted per session and includes access to the club and extends the member's subscription to the Girls' Angle Bulletin to one year from the start of the current or upcoming session. You can also pay per session. If you pay per session and come for three meets, you will get a subscription to the Bulletin. **Active subscriptions** to the Girls' Angle Bulletin allow the subscriber to ask and receive answers to math questions through email.   Please note that we will not answer email questions if we think that we are doing the asker's homework! We currently operate in 12 meet sessions, but girls are welcome to join at any time. The program is individually focused so the concept of "catching up with the group" doesn't apply. Note that you can receive the Girls' Angle Bulletin free of charge. Just send us email with your request.

Where is Girls' Angle located? Girls' Angle is located about 10 minutes walk from Central Square on Magazine Street in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes. Girls' Angle is a registered 501(c)(3) corporation. Please make donations out to **Girls' Angle** and send checks to Ken Fan, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was an assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science and worked in the mathematics educational publishing industry. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum. These experiences and the enthusiasm of the girls of Science Club for Girls have motivated him to create Girls' Angle.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:
Connie Chow, executive director of Science Club for Girls
Yaim Cooper, graduate student in mathematics, Princeton
Julia Elisenda Grigsby, assistant professor of mathematics, Boston College
Kay Kirkpatrick, Courant Instructor/PIRE fellow, NYU
Grace Lyo, Moore Instructor, MIT
Lauren McGough, MIT '12
Mia Minnes, Moore Instructor, MIT
Beth O'Sullivan, co-founder of Science Club for Girls.
Elissa Ozanne, Senior Research Scientist, Harvard Medical School.
Kathy Paur, Ph.D., Harvard
Katrin Wehrheim, associate professor of mathematics, MIT
Lauren Williams, assistant professor of mathematics, UC Berkeley

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: A Math Club for Girls
Membership Application

Applicant's Name: (last) _____ (first) _____

Applying For: □ Membership (Access to club, premium subscription)
                       □ Active Subscription (interact with mentors through email)

Parents/Guardians: _____

Address: _____ Zip Code: _____

Home Phone: _____ Cell Phone: _____ Email: _____

Emergency contact name and number: _____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. They will have to sign her out. Names: _____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about? _____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes?   **Yes**   **No**

Eligibility: For now, girls who are roughly in grades 5-11 are welcome. Although we will work hard to include every girl no matter her needs and to communicate with you any issues that may arise, Girls' Angle has the discretion to dismiss any girl whose actions are disruptive to club activities.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

_____   Date: _____
(Parent/Guardian Signature)

Membership-Applicant Signature: _____

    □ Enclosed is a check for (indicate one) (prorate as necessary)
          □ $216 for a 12 session membership   □ $50 for a one year active subscription
          □ I am making a tax free charitable donation.

    □ I will pay on a per session basis at $20/session. (Note: You still must return this form.)

Please make check payable to: **Girls' Angle**. Mail to: Ken Fan, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Paying on a per session basis comes with a one year subscription to the Bulletin, but not the math question email service. Also, please sign and return the Liability Waiver.

Girls' Angle: A Math Club for Girls
Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

_____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: _____Date: _____

Print name of applicant/parent: _____

Print name(s) of child(ren) in program: _____

Girls'
Angle

A Math Club for Girls