Mathematical Model of Lightning Stroke Development

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Abstract: **This paper presents a mathematical model of the lightning stroke path development. Any lightning model is a mathematical construct design to reproduce certain aspects of the physical process involved in the lightning discharge. This mathematical model resides on the underlying observed physical properties of the lightning stroke development. It uses Monte Carlo based algorithm in order to simulate stochastic and dynamic movement of the lightning stroke stepped leader descent. Influence of the lightning stroke current amplitude on the path development is also taken into account. Two examples of the lightning stroke stepped leader propagation paths are presented.**

1. INTRODUCTION

Lightning is a stochastic natural phenomenon, which originates in a thunderstorm cloud. Cumulonimbus is a typical thunderstorm cloud. Research of the lightning flash phenomenon has been carried-out, by various authors throughout the world, with the aid of high-speed photography and other methods [1], [2].

Lightning model is a mathematical construct design to reproduce certain aspects of the physical process involved in the lightning discharge. Generally speaking, lightning stroke models could be arranged in four different classes [1]. First class of models comprises the gas dynamic models, which are primarily dealing with radial evolution of a short segment of the lightning channel [1]. Second class of models comprises large amount of models based on lossy thin-wire antenna approximation of the lightning channel. These models involve numerical solution of Maxwell's equations and can be considered as electromagnetic models. Third class of models comprises the distributed-circuit models, which can be viewed as an approximation to the above mentioned electromagnetic models [1]. Finaly, forth class of models comprises various so-called engineering models [1]. Hereafter presented model of the lightning stroke development could be viewed as an simplyfied engineering model.

It has been established that lightning flash development has several distinct phases [1]. First characteristic phase is called a stepped leader [1]. Through this phase a head of the lightning stroke, initiated from the cloud, progresses downward in a series of stochastic jumps toward the earth surface. Hence, stepped leader creates a conducting path between the cloud and the earth surface. Point of lightning strike is, hence, determined by the point where stepped leader head touches the earth surface. It is not known a priori. This phase has a relatively long duration. Average speed of the stepped leader progress (including pauses between successive stochastic jumps) roughly equals to $0.001 \cdot c$, where c represents the speed of light in free space (300 m/μs).

Next distinct phase in the lightning stroke development, followed directly after the stepped leader touchdown, is called a streamer. It constitutes a major contribution to the charge transfer between the earth and cloud, and progresses from the earth surface toward the cloud, following the conducting path created by the stepped leader [1]. Thus, streamer has greater average speed, which roughly equals to $0.1 \cdot c$. This phase is accompanied with large currents, whose magnitudes exceeds hundreds of kA. Streamer phase is followed with a pause, whose duration usually approximates to several tens of milliseconds.

After this pause, a next phase of the lightning stroke development is initiated from the cloud toward the earth surface, which is usually called a dart leader [1]. Dart leader follows a pre-ionized channel, which makes it much faster then the stepped leader. When the dart leader touches the earth surface a return stroke is initiated from the earth surface toward the cloud. This return stroke is accompanied by significantly lower values of current. This process can continue through several additional dart leaders and subsequent return strokes [1]. Hence, lightning flash can be comprised of several subsequent strokes.

Figure 1 depicts a space - time diagram of the typical, just described, lightning flash development. Shape of the accompanying lightning surge current is also depicted in bottom of the Figure 1. Labels indicated in Figure 1 have following meanings:

h_0 – height of the thunderstorm cloud base,
i – lightning surge current,
t – time.

According to Figure 1, lightning surge current has a distinct impulse waveshape, which can be characterized by the peak value, duration of the current front and polarity. It has been established that a cumulative probability distribution of the lightning peak values can be approximated by a logarithmic normal distribution [3]. According to [4], approximately 90 % of all lightning strokes are negative in polarity, while the remaining 10 % of lightning strokes are positive. Hence, in accordance with IEC 62305-1 [4], following expression holds:

$$P(I) = 0.9 \cdot P_{-}(I) + 0.1 \cdot P_{+}(I) \tag{1}$$

where:

I – peak value of the lightning surge current, kA,

P(I) – probability that a lightning current peak value I will be exceeded.

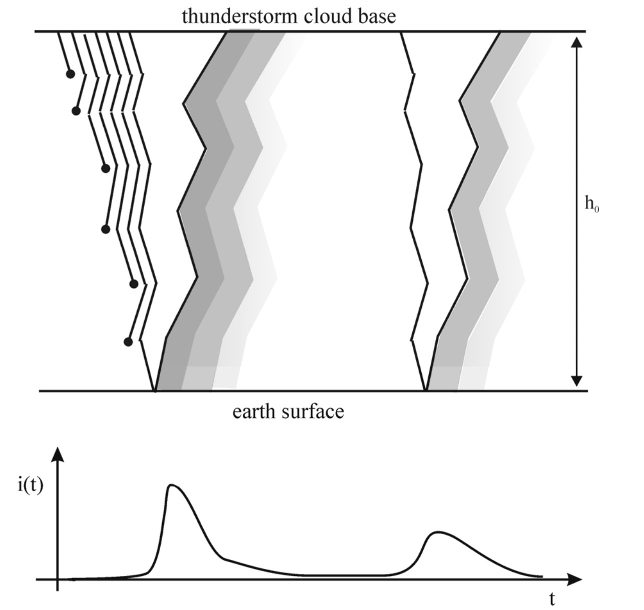

Figure 1 – Space-time diagram of the typical lightning flash development, along with lightning surge current waveshape (bottom)

Probability density function for these statistical variables, representing first negative polarity and first positive polarity lightning strikes, is according to [3] given by:

$$P_{-}(I) = P_{+}(I) = 0.5 \cdot erfc(u_0) \tag{2}$$

with:

$$erfc(u_0) = 1 - \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n \cdot u_0^{2n+1}}{n! \cdot (2n+1)} \tag{3}$$

where:

$$u_0 = \frac{\log(I) - \log(I_\mu)}{\sqrt{2} \cdot \sigma_{\log}} \tag{4}$$

Mean value (I_μ) and standard deviation ($\sigma_{\log}$) of the lightning current peak values are defined in [4], both for the first positive and first negative polarity lightning strokes. These parameters, needed in the equation (4), are briefly summarized in Table I.

Table I –Logarithmic normal distribution of lightning current parameters

Parameter	Mean (I_μ)	Standard deviation ($\sigma_{\log}$)	Lightning stroke type
I (kA)	61.1	0.576	First negative (< 20 kA)
	33.3	0.263	First negative (> 20 kA)
	33.9	0.527	First positive

2. MATHEMATICAL MODEL

Various objects, which could be struck by lightning, could be observed on the ground surface. For example, those are high voltage power lines and switchyards, telecommunication networks as well as various buildings and installations. Layout or ground-plan of these structures could be presented in Cartesian coordinate system (x, y, z), by a rectangle with sides a and b, as depicted in Figure 2. Cartesian coordinate system is chosen in such a way that plane with z = 0 coincides with the earth surface.

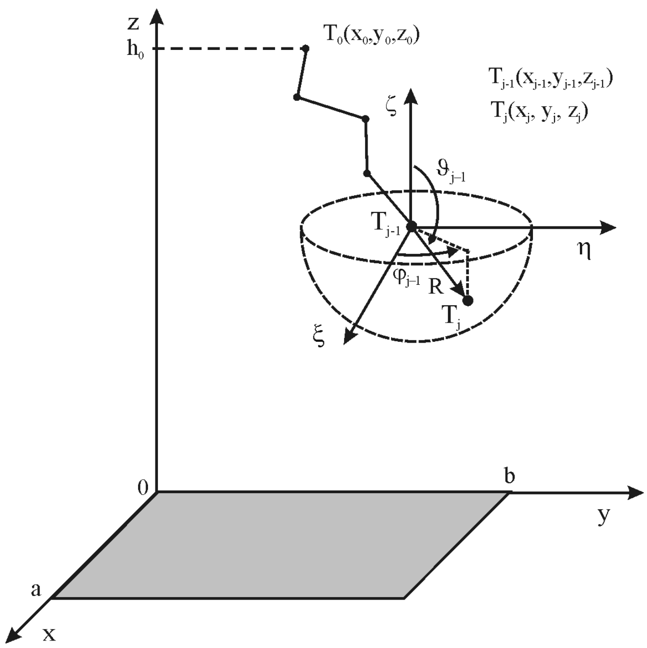

Figure 2 – Graphical depiction of a stepped leader head during j-th jump

Devious, stochastic path of the lightning stroke stepped leader is also depicted at the same figure. Stepped leader in

Figure 2 prepares to make its arbitrary j-th jump from point $T_{j-1}(x_{j-1}, y_{j-1}, z_{j-1})$ to point $T_j(x_j, y_j, z_j)$.

Due to the fact that lightning is a stochastic natural phenomenon, initiation and development of lightning stroke stepped leader stochastic path toward the earth surface is simulated by means of Monte Carlo method. Lightning stroke is initiated from the thunderstorm cloud base, which is mathematically represented by a starting point T_0 at height h_0 above the observed structure / rectangular surface (Figure 2). Coordinates x_0 and y_0 of the starting point $T_0(x_0, y_0, h_0)$ are stochastically chosen according to the following expressions:

$$x_0 = r \cdot a \tag{5}$$

$$y_0 = r \cdot b \tag{6}$$

where r represent a pseudorandom variable, drawn from the uniform distribution, $r \in [0, 1]$.

Peak value of the lightning current is also stochastically chosen. An observed interval $I \in [I_{min}, I_{max}]$ of the lightning current peak values is subdivided into an arbitrary number of clases. This is in relation to the fact that cumulative probability density function of the lightning surge peak values follows a logarithmic normal distribution. Peak value of the lightning current from the arbitrary i-th class is selected with a pseudorandom number generator, according to the following expression [5]:

$$I = {}^{i}I_{\min} + r \cdot \left({}^{i}I_{\max} - {}^{i}I_{\min}\right) \tag{7}$$

where ${}^{i}I_{min}$ and ${}^{i}I_{max}$ represent minimal and maximal peak values of the lightning current in the i-th class, respectively. They can be calculated as follows [5]:

$${}^{i}I_{\min} = I_{\min} + (i-1) \cdot \Delta I \tag{8}$$

$${}^{i}I_{\max} = {}^{i}I_{\min} + \Delta I \tag{9}$$

where span of the each class equals to:

$$\Delta I = \frac{I_{\max} - I_{\min}}{N_c} \tag{10}$$

with N_c representing a total number of lightning current peak value classes. If N is a number of statistically initiated lightning strikes, then partial number of lightning strikes N_i belonging to the i-th class, can be determined according to the expression [5]:

$$N_i = \left[P\left({}^{i}I_{\min}\right) - P\left({}^{i}I_{\max}\right) \right] \cdot N \;\; ; \; i \in [1, N_c] \tag{11}$$

Probabilities $P({}^{i}I_{min})$ and $P({}^{i}I_{max})$ are calculated according to the equations (1) – (4), from the logarithmic normal distribution.

It has been said that lightning stroke development (stepped leader descent) occurs in a number of quick jumps along a stochastic path. This jump distance is a function of the lightning current peak value. Various authors have proposed several different forms of the expression for the striking (jump) distance, presented in [6]. Strike distance (in meters) can generally be expressed as follows:

$$R = A \cdot I^{\alpha} \tag{12}$$

where I represents the peak value of lightning surge current, kA. According to CIGRE Working Group [7], IEEE Working Group [8] and IEC 62305-1 [4], parameters A and α defining the striking distance can be written as:

$$A = 10 \tag{13}$$

$$\alpha = 0.65 \tag{14}$$

It needs to be said that equation (12), with (13) and (14), constitute a convenient theoretical background for the development of electrogeometrical model, because it relates electrical and geometrical lightning stroke parameters. This has been employed in utilizing the stepped leader stochastic path development.

According to Figure 2, local sphere coordinate system has been placed in the position of the stepped leader head, after each stochastic jump. This local coordinate system, thus, travels with the stepped leader (forms in a new point after each jump). Coordinates of the stepped leader head in the global (Cartesian) coordinate system are calculated according to the following expressions [5]:

$$x_j = x_{j-1} + R \cdot \sin\vartheta_{j-1} \cdot \cos\varphi_{j-1} \tag{15}$$

$$y_j = y_{j-1} + R \cdot \sin\vartheta_{j-1} \cdot \sin\varphi_{j-1} \tag{16}$$

$$z_j = z_{j-1} + R \cdot \cos\vartheta_{j-1} \tag{17}$$

where j = 1, 2, … is a number of stepped leader jump, while φ_{j-1} and ϑ_{j-1} represent coordinates of the sphere (local) coordinate system. These coordinates are in fact defining the next position of the stepped leader head, and are stochastically determined from the following expressions:

$$\varphi_{j-1} = r \cdot 2\pi \tag{18}$$

$$\vartheta_{j-1} = (r+1) \cdot \frac{\pi}{2} \tag{19}$$

This process continues until stepped leader reaches the earth surface, which signifies a lightning stroke touchdown somewhere on the rectangular surface (Figure 2).

This can be repeated any number of times, where each new lightning strike starts at different point on the thunderstorm cloud base (represented with a plane surface at height h_0 above earth) and ends on the different (not known a priori) point on the rectangular surface (or outside of this surface) on the ground plane. It thus follows an undetermined stochastic path, which is in accordance with the observed lightning phenomena [1].

3. EXAMPLE

Following the above presented theory a computer program has been developed. This computer program allows the simulation of the lightning surge stepped leader stochastic descent from the thunderstorm cloud toward the earth surface. This will be illustrated by the following numerical example.

Two separate lightning surge stepped leader stochastic descents will be presented. First example is for the lightning surge carrying a low peak current magnitude, while second is for high value of peak current magnitude. Influence of the lightning surge peak value on the formation of stochastic path (strike distance) will be illustrated through these two selected examples.

Lightning flash is initiated on the height of 2000 meters above ground. Observed rectangular surface on the ground surface have following dimensions:

$$a = 100 \text{ m} \tag{20}$$

$$b = 500 \text{ m} \tag{21}$$

It can represent a high-voltage switchyard or some other structure. Current peak values are selected from the following interval:

$$I \in [3 - 200] \text{ kA} \tag{22}$$

subdivided into 15 classes.

Low peak value current, for the first example, is selected from the first class. This class is according to expressions (8), (9) and (10) defined by the following interval:

$$I_1 \in [3 - 16.1] \text{ kA} \tag{23}$$

Let the low peak value of the lightning surge, selected for the first example, has the following value:

$$I_{low} = 10 \text{ kA} \tag{24}$$

High peak value current, for the second example, is selected from the sixth class, which is according to the expressions (8), (9) and (10), defined by the interval:

$$I_6 \in [68.5 - 81.6] \text{ kA} \tag{25}$$

High peak value of the lightning surge, selected for the second example, has the following value:

$$I_{high} = 70 \text{ kA} \tag{26}$$

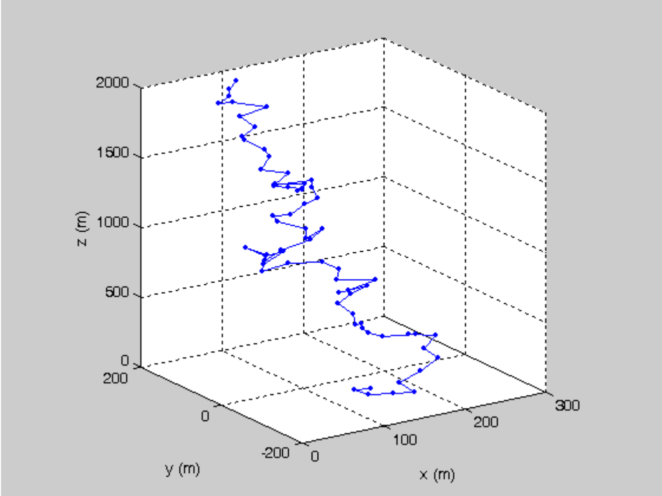

Figure 3 – Stochastic path of the low peak value current stepped leader

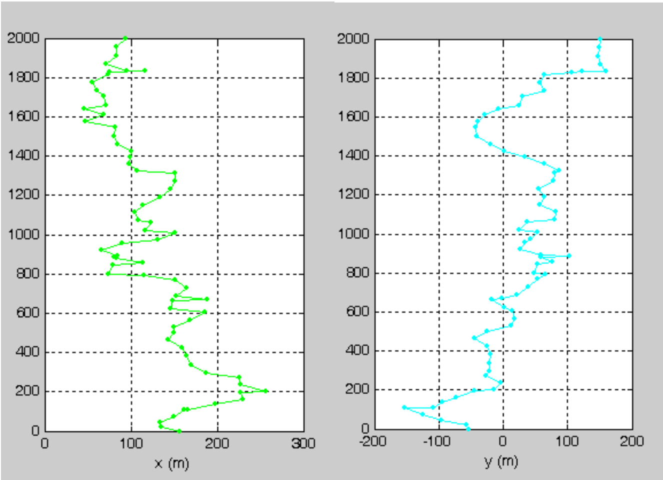

Figure 4 – Projection of the stepped leader path from Figure 3 onto the x-z (left) and y-z (right) plane

Figure 3 presents in 3D space a computed stochastic path of the low peak value current lightning surge stepped leader descent. Figure 4 presents, at the same time, a projection of the stepped leader path from Figure 3 onto the x-z plane (left) and y-z plane (right), respectively.

Figure 5 presents, also in 3D space, a computed stochastic path of the high peak value current lightning surge stepped leader descent. Figure 6 presents a projection of the stepped leader path from Figure 4 onto the x-z plane (left) and y-z plane (right), respectively.

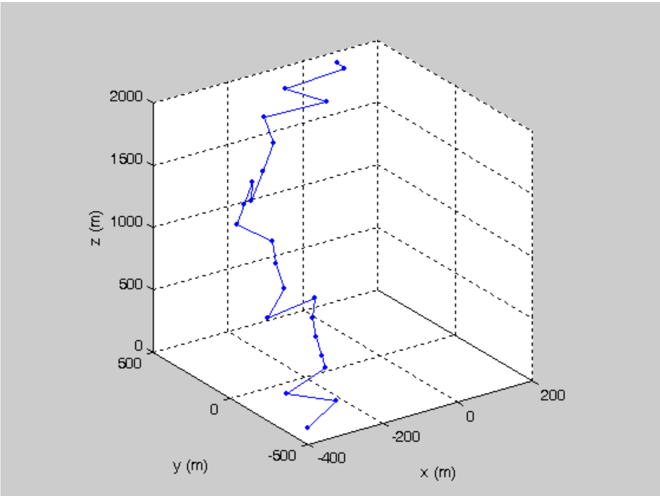

Figure 5 – Stochastic path of the high peak value current stepped leader

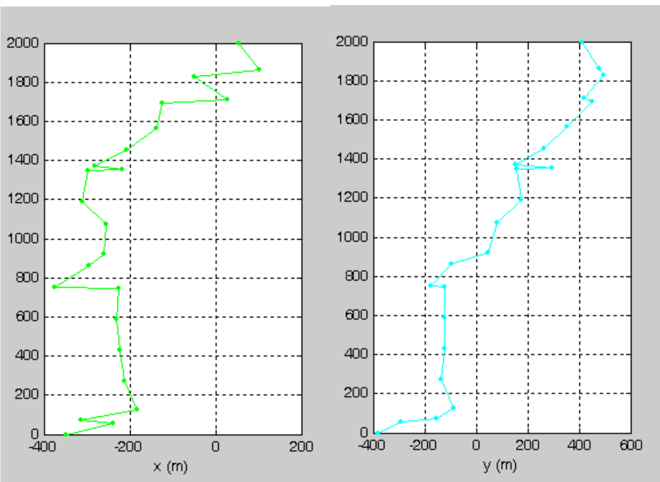

Figure 6 – Projection of the stepped leader path form Figure 5 onto the x-z (left) and y-z (right) plane

Points depicted in Figures 3, 4, 5 and 6 represent positions of the stepped leader head after each successive jump.

By comparing Figures 3 and 5, influence of the lightning surge current peak value on the propagation (jump distance) of the stochastic path of stepped leader descent can be observed. Additionally, stochastic nature of the stepped leader descent path can be nicely observed from Figures 3 and 5.

4. CONCLUSION

Mathematical model for the initiation and stochastic stepped leader path simulation has been presented in this paper. It is based on the experimentally determined lightning stroke development phases. Due to the stochastic nature of the lightning phenomenon, Monte Carlo simulation has been employed. The fact that cumulative probability density function of the lightning peak values follow a logarithmic normal distribution has been exploited as well. Electrogeometric model, along with the analytic geometry in 3D space, features prominently in the above presented mathematical model. Based on the presented theory a computer program has been developed, which can form a backbone for the efficiency analysis of lightning protection systems.

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