Kelvin Wave Deep Water Simulation

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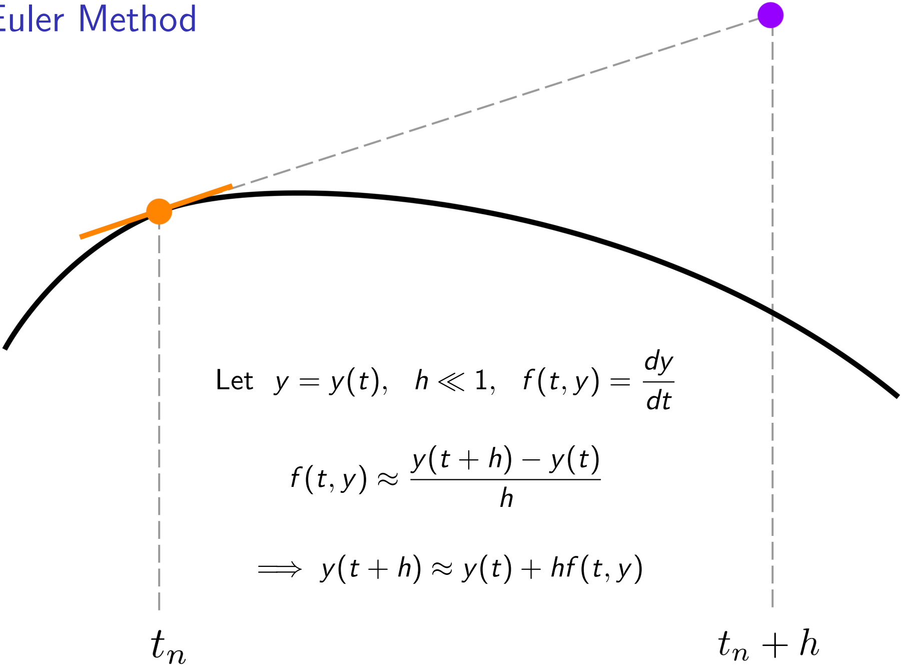
Numerical Methods

Euler Method



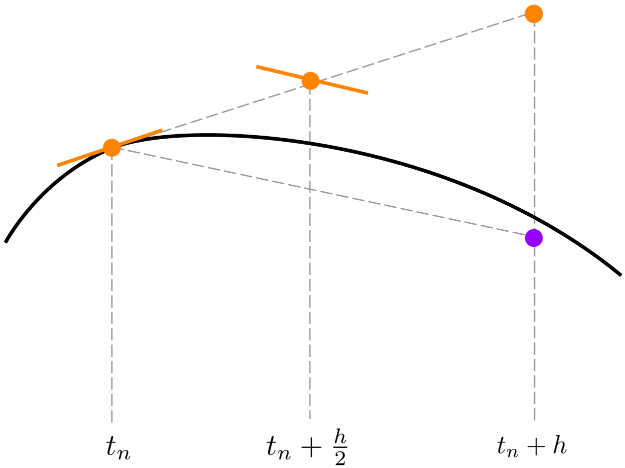
$$\text{Let} \quad y = y(t), \quad h \ll 1, \quad f(t, y) = \frac{dy}{dt}$$

$$f(t, y) \approx \frac{y(t+h) - y(t)}{h}$$

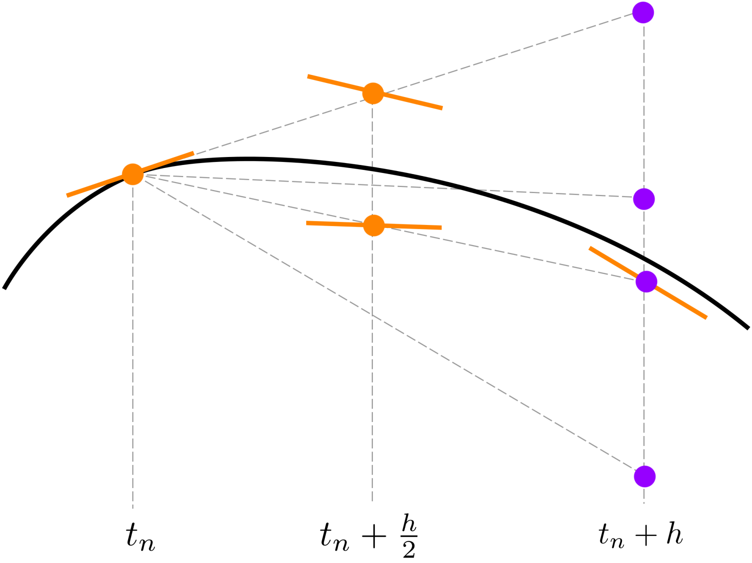
$$\implies y(t+h) \approx y(t) + hf(t, y)$$

Midpoint Method

$$y_{n+1} = y_n + hf\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}f(t_n, y_n)\right), \quad f(t, y) = \frac{dy}{dt}$$



Common RK4 Method



Common RK4 Method

Let$y = y(t), h \ll 1, f(t, y) = \frac{dy}{dt}$

Increment system as follows

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k4)$$
$$t_{n+1} = t_n + h$$

where

$$k_1 = f(t_n, y_n),$$
$$k_2 = f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1),$$
$$k_3 = f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_2),$$
$$k_4 = f(t_n + h, y_n + hk_3)$$

Derivations

Kelvin Waves

(See [1] section 3.8)

- Assume $u = 0$ and $\eta = e^{ly - \sigma t}\phi(x)$
- Governing equations become $\begin{cases} -fv = -g\eta_x \\ v_t = -g\eta_y \\ \eta_t + Hv_y = 0 \end{cases}$
- Solve and obtain $\eta = \cos(l(y - c_0 t))e^{flx/\sigma}, c_0 = \sqrt{gH}$

Deep Water Solution

(See [1] section 3.3)

- Kelvin waves assume $u = 0$, so consider planar slices parallel to the $yz-$plane
- In the plane, vorticity has one component, $\omega = w_y - v_z$
- If $\omega = 0$ then $\exists\phi$ such that $v = \phi_y$ and $w = \phi_z$
- Then $\nabla \cdot (v, z) = 0 \implies \nabla^2\phi = 0$
- Assume the surface is given by $\eta = a\cos(ky - \sigma t)$ and that the amplitude is small
- Boundary conditions for surface are $w = \frac{\partial\eta}{\partial t}$ (Las Vagas) and $p = 0$ (since water is much denser than air)
- Bernoulli gives us (assume small velocities, drop quadratic terms) $\frac{\partial\phi}{\partial t} + p + gz = 0 \implies \frac{\partial\phi}{\partial t} + g\eta = 0$ at $z = 0$
- Try separation of variables, $\phi = f(z)\sin(ky - \sigma t)$

Deep Water Solution

After solving, these equations are obtained:

$$\eta(y,t) = a\cos(ky - \sigma t)$$
$$v(x,z,t) = a\sigma e^{kz}\cos(ky - \sigma t)$$
$$w(x,z,t) = a\sigma e^{kz}\sin(ky - \sigma t)$$

The equations of motion for a particle is thus

$$\frac{dy}{dt} = v(y,z,t)$$
$$\frac{dz}{dt} = w(y,z,t)$$

where a particle path is parameterized by $(y(t), z(t))$. This is the form needed for Runge Kutta.

References

[1] Marek Stastna, Aaron Coutino and Michael Waite. (2015). *Fluid Mechanics Notes*

[2] PHYS 236, Computational Physics 1 (Fall 2012)

[3] Brian P. Flannery, Saul Teukolsky, William H. Press, and William T. Vetterling. (1988). *Numerical Recipes in C: The Art of Scientific Computing*. Cambridge University Press.