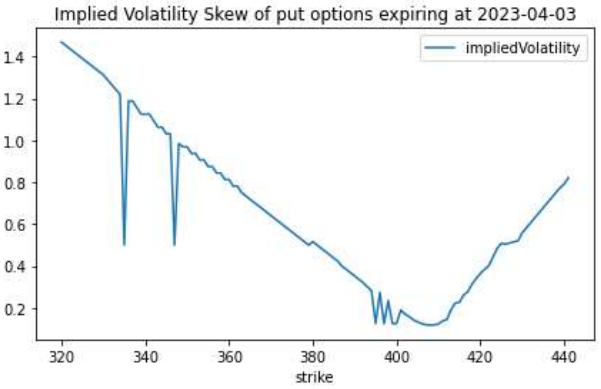
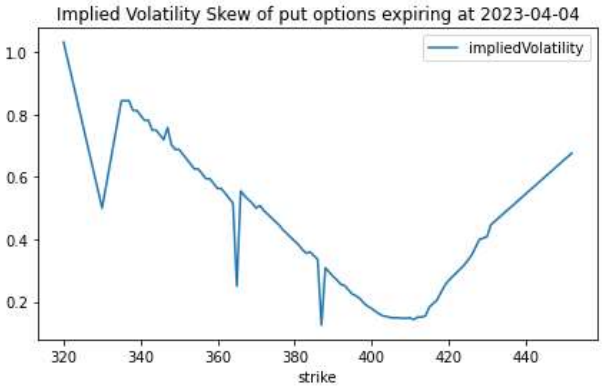
Monte Carlo for American Options

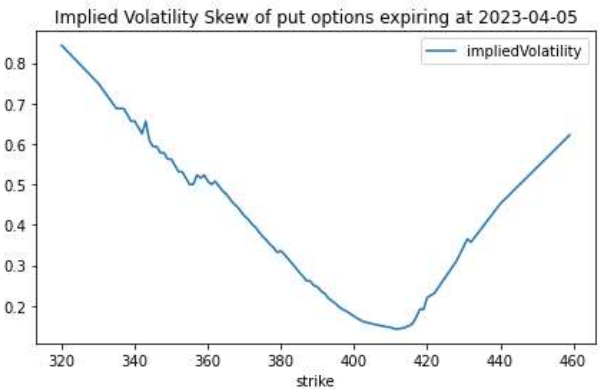
Initially, I use python programming to collect options chain data for SPDR S&P 500 ETF Trust (SPY), an actively traded American option in real time from Yahoo! Finance. The following snapshot shows all option contracts and their strike, last price, bid price, ask price, volume, implied volatility, and expirations.

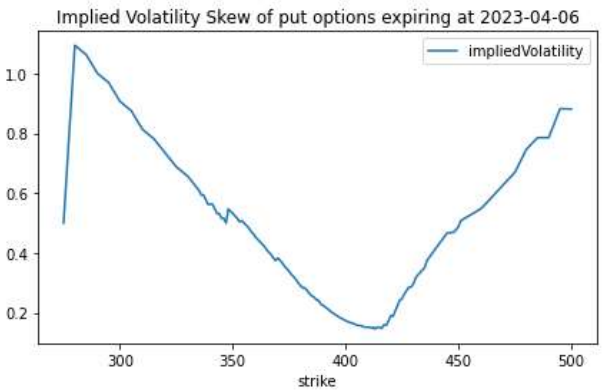
```
       contractSymbol             lastTradeDate  strike  lastPrice     bid  \
0   SPY230403C00320000 2023-03-31 19:18:39+00:00   320.0      90.65   88.85
1   SPY230403C00334000 2023-03-22 13:53:24+00:00   334.0      64.94   74.85
2   SPY230403C00339000 2023-03-30 14:22:28+00:00   339.0      64.41   69.98
3   SPY230403C00340000 2023-03-29 14:38:08+00:00   340.0      59.30   68.89
4   SPY230403C00342000 2023-03-30 18:55:44+00:00   342.0      61.25   66.91
..                 ...                       ...     ...        ...     ...
76  SPY251219P00570000 2023-02-14 14:38:15+00:00   570.0     159.75  177.50
77  SPY251219P00600000 2023-03-06 14:30:10+00:00   600.0     195.00  188.50
78  SPY251219P00610000 2023-02-07 17:03:46+00:00   610.0     199.44  209.00
79  SPY251219P00630000 2023-02-02 20:15:08+00:00   630.0     215.27  223.50
80  SPY251219P00650000 2023-02-23 20:39:26+00:00   650.0     248.00  249.69

       ask  change  percentChange  volume  openInterest  impliedVolatility  \
0    88.99    2.25       2.545249     1.0           1.0           0.000010
1    74.99    0.00       0.000000     NaN           2.0           0.000010
2    70.15    0.00       0.000000     NaN           1.0           0.000010
3    69.07    0.00       0.000000     NaN         320.0           0.000010
4    67.03    0.00       0.000000     NaN          10.0           0.000010
..     ...     ...            ...     ...           ...                ...
76  182.50    0.00       0.000000     NaN           0.0           0.242783
77  193.50    0.00       0.000000     2.0           2.0           0.146569
78  214.00    0.00       0.000000     1.0           0.0           0.226051
79  228.50    0.00       0.000000     NaN           0.0           0.203286
80  259.69    0.00       0.000000     1.0           0.0           0.278266

    inTheMoney contractSize currency optionType          expiration  \
0         True      REGULAR      USD       call 2023-04-03 23:59:59
1         True      REGULAR      USD       call 2023-04-03 23:59:59
2         True      REGULAR      USD       call 2023-04-03 23:59:59
3         True      REGULAR      USD       call 2023-04-03 23:59:59
4         True      REGULAR      USD       call 2023-04-03 23:59:59
..         ...          ...      ...        ...                 ...
76        True      REGULAR      USD        put 2025-12-19 23:59:59
77        True      REGULAR      USD        put 2025-12-19 23:59:59
78        True      REGULAR      USD        put 2025-12-19 23:59:59
79        True      REGULAR      USD        put 2025-12-19 23:59:59
80        True      REGULAR      USD        put 2025-12-19 23:59:59
```

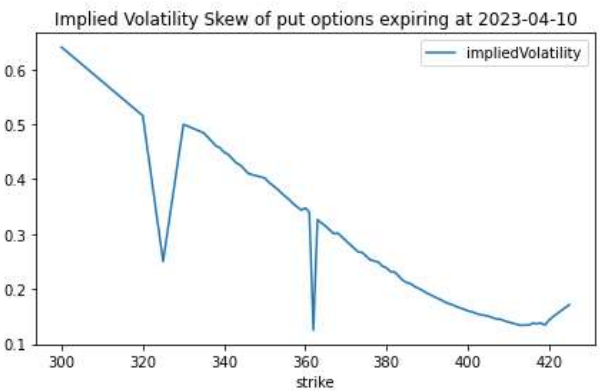
Now, we can plot the implied volatilities of the put options against the strike price for different expiration periods -

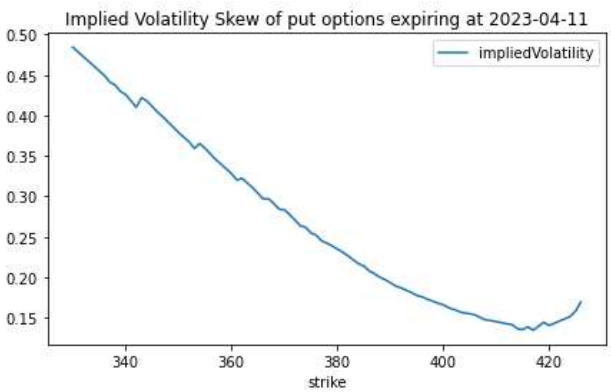

Implied Volatility Skew of put options expiring at 2023-04-03
impliedVolatility
strike

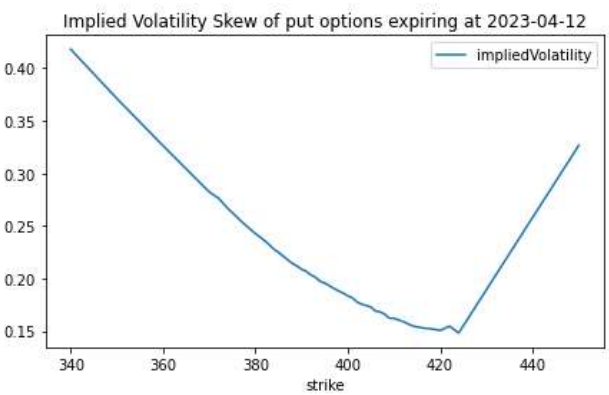

Implied Volatility Skew of put options expiring at 2023-04-04
impliedVolatility
strike

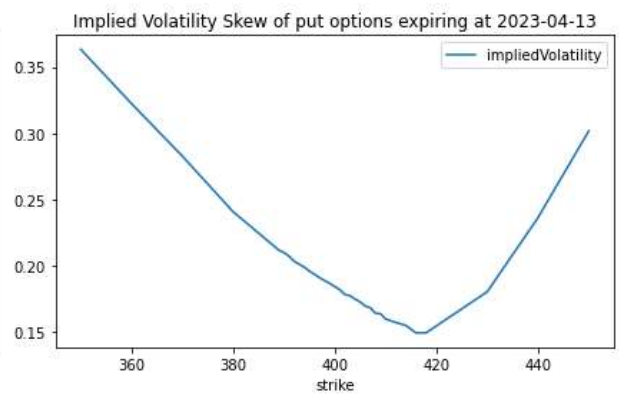

Implied Volatility Skew of put options expiring at 2023-04-05
impliedVolatility
strike

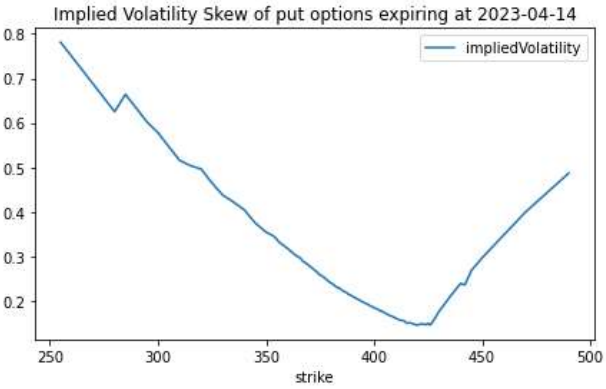

Implied Volatility Skew of put options expiring at 2023-04-06
impliedVolatility
strike

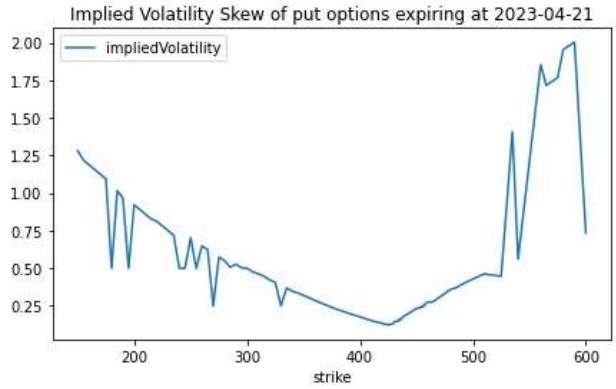

Implied Volatility Skew of put options expiring at 2023-04-10
impliedVolatility
strike

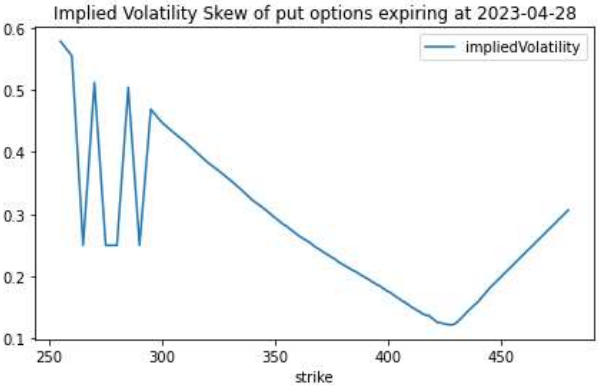

Implied Volatility Skew of put options expiring at 2023-04-11
impliedVolatility
strike

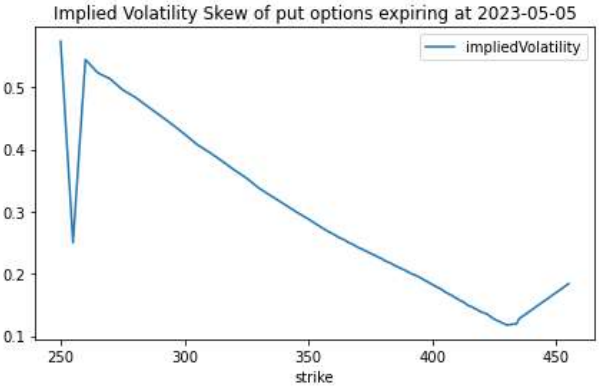

Implied Volatility Skew of put options expiring at 2023-04-12
impliedVolatility
strike

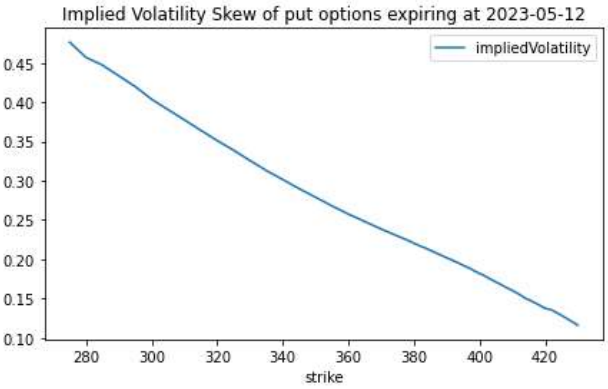

Implied Volatility Skew of put options expiring at 2023-04-13
impliedVolatility
strike

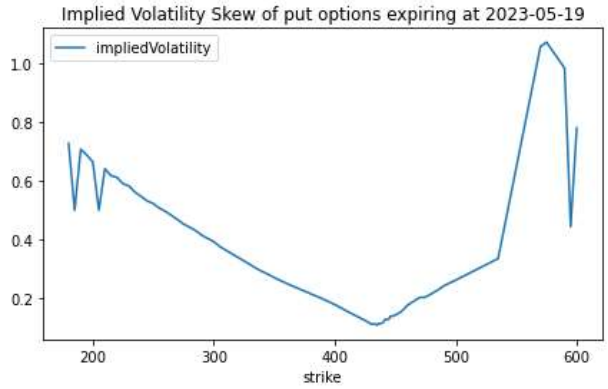

Implied Volatility Skew of put options expiring at 2023-04-14
impliedVolatility
strike

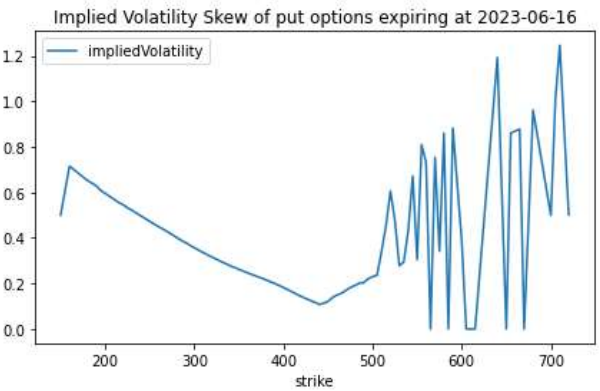

Implied Volatility Skew of put options expiring at 2023-04-21
impliedVolatility
strike

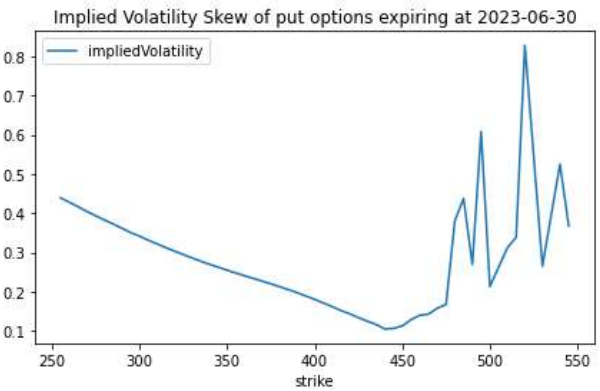

Implied Volatility Skew of put options expiring at 2023-04-28
impliedVolatility
strike

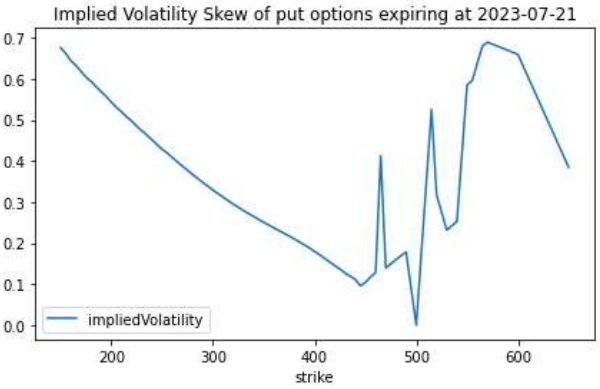

Implied Volatility Skew of put options expiring at 2023-05-05
impliedVolatility
strike

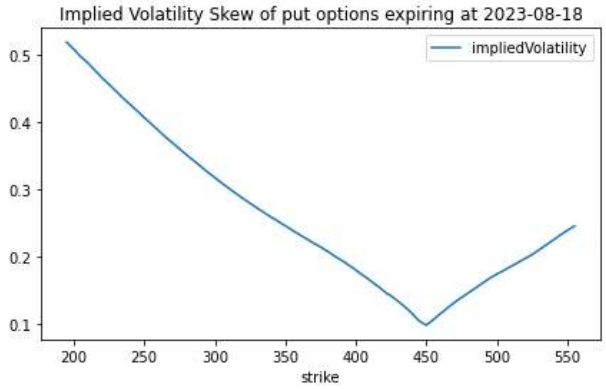

Implied Volatility Skew of put options expiring at 2023-05-12
impliedVolatility
strike


Implied Volatility Skew of put options expiring at 2023-05-19
impliedVolatility
strike


Implied Volatility Skew of put options expiring at 2023-06-16
impliedVolatility
strike


Implied Volatility Skew of put options expiring at 2023-06-30
impliedVolatility
strike


Implied Volatility Skew of put options expiring at 2023-07-21
impliedVolatility
strike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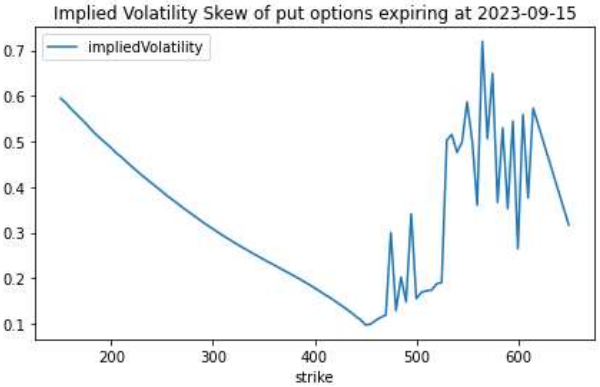
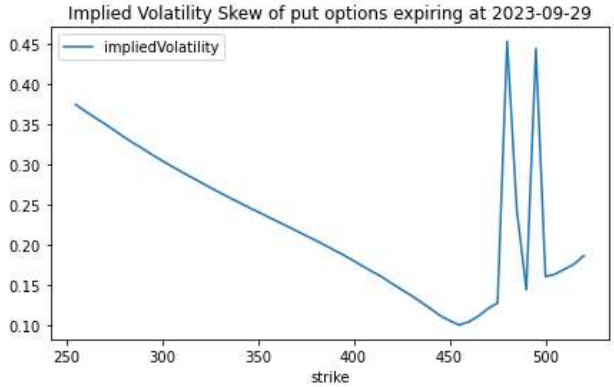
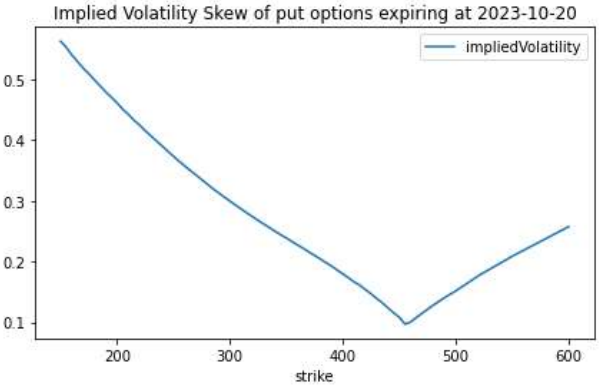
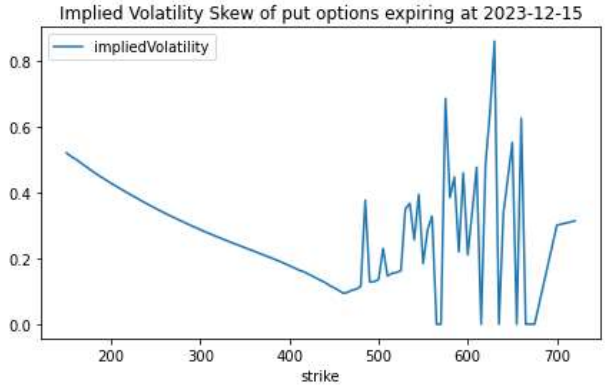
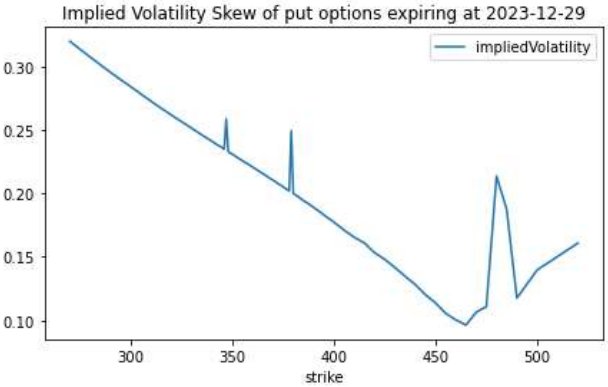
impliedVolatility
strike

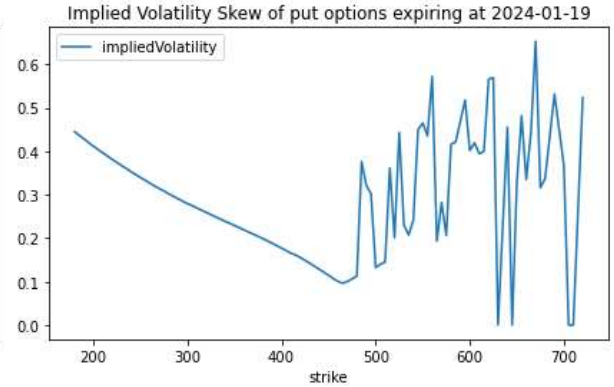

Implied Volatility Skew of put options expiring at 2023-09-15
impliedVolatility
strike

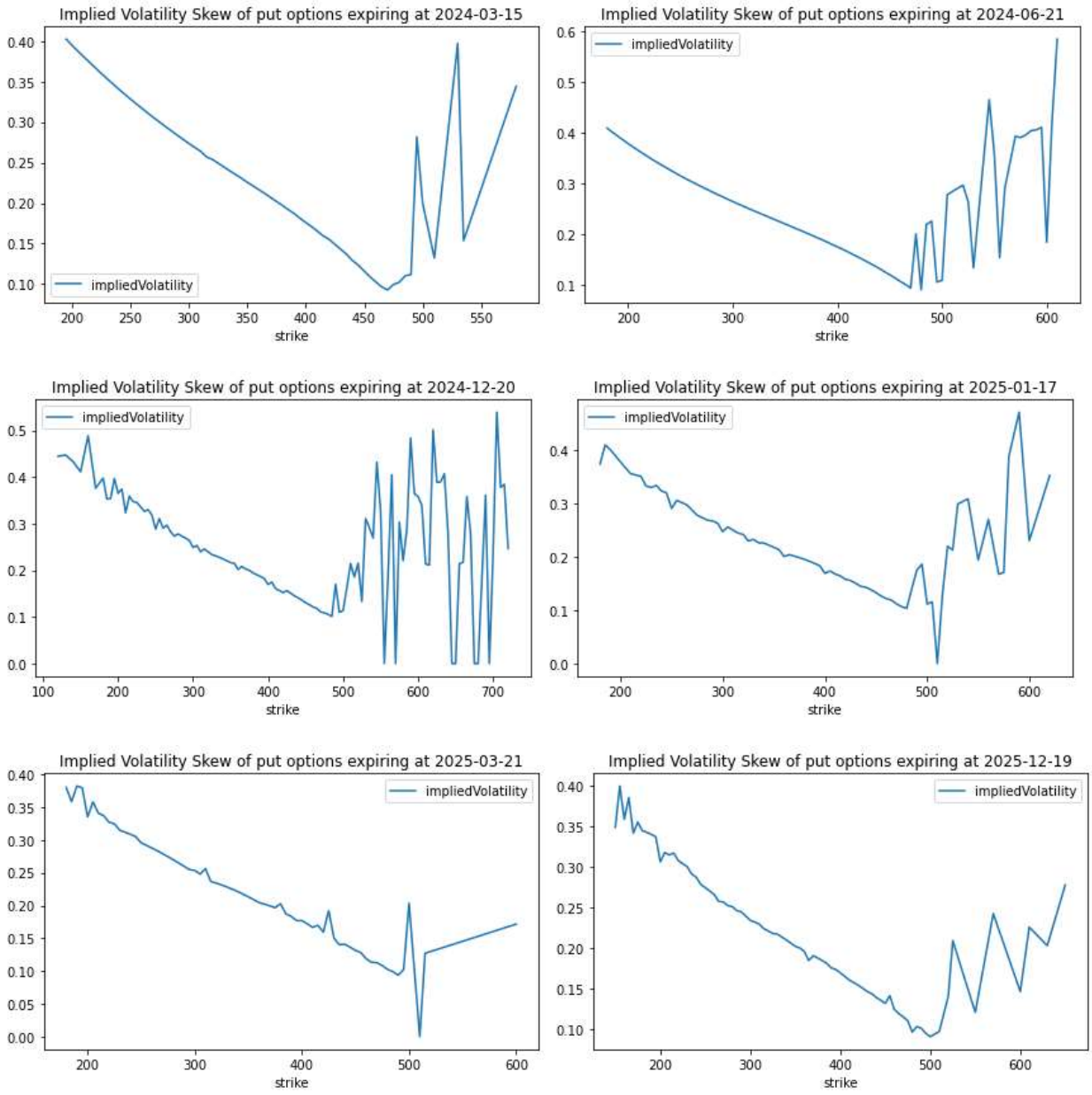

Implied Volatility Skew of put options expiring at 2023-09-29
impliedVolatility
strike


Implied Volatility Skew of put options expiring at 2023-10-20
impliedVolatility
strike


Implied Volatility Skew of put options expiring at 2023-12-15
impliedVolatility
strike


Implied Volatility Skew of put options expiring at 2023-12-29
impliedVolatility
strike


Implied Volatility Skew of put options expiring at 2024-01-19
impliedVolatility
strike



We observe that the implied volatilities for all expiration periods decrease as the strike increase from $200 to $500. With the strike price beyond $500, the implied volatilities increase with high fluctuations. The following snapshot extracts all details for the put options -

	contractSymbol	lastTradeDate	strike	lastPrice	bid	ask	change	percentChange	volume	openInterest	impliedVolatility	inTheMoney	contractSize	currency	optionType	expiration	daysToExpiration
0	SPY230403P00320000	2023-04-03 13:48:57+00:00	320.0	0.01	0.00	0.01	0.0	0.0	1.0	616.0	1.468753	False	REGULAR	USD	put	2023-04-03	1
1	SPY230403P00330000	2023-03-31 13:57:02+00:00	330.0	0.01	0.00	0.01	0.0	0.0	1111.0	1512.0	1.312503	False	REGULAR	USD	put	2023-04-03	1
2	SPY230403P00334000	2023-03-31 13:57:02+00:00	334.0	0.01	0.00	0.01	0.0	0.0	736.0	1022.0	1.218754	False	REGULAR	USD	put	2023-04-03	1
3	SPY230403P00335000	2023-03-31 13:59:27+00:00	335.0	0.01	0.00	0.00	0.0	0.0	944.0	1464.0	0.500005	False	REGULAR	USD	put	2023-04-03	1
4	SPY230403P00336000	2023-03-31 20:02:35+00:00	336.0	0.01	0.00	0.01	0.0	0.0	2.0	904.0	1.187504	False	REGULAR	USD	put	2023-04-03	1
..	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
76	SPY251219P00570000	2023-02-14 14:38:15+00:00	570.0	159.75	177.50	182.50	0.0	0.0	NaN	0.0	0.242783	True	REGULAR	USD	put	2025-12-19	992
77	SPY251219P00600000	2023-03-06 14:30:10+00:00	600.0	195.00	188.50	193.50	0.0	0.0	2.0	2.0	0.146569	True	REGULAR	USD	put	2025-12-19	992
78	SPY251219P00610000	2023-02-07 17:03:46+00:00	610.0	199.44	209.00	214.00	0.0	0.0	1.0	0.0	0.226051	True	REGULAR	USD	put	2025-12-19	992
79	SPY251219P00630000	2023-02-02 20:15:08+00:00	630.0	215.27	223.50	228.50	0.0	0.0	NaN	0.0	0.203286	True	REGULAR	USD	put	2025-12-19	992
80	SPY251219P00650000	2023-02-23 20:39:26+00:00	650.0	248.00	249.69	259.69	0.0	0.0	1.0	0.0	0.278266	True	REGULAR	USD	put	2025-12-19	992

[3060 rows x 17 columns]

Now, we can **price the put options chain under the assumption that they are European options**. That is, we use **the standard Black-Scholes model (BSM) with a dividend yield if one is paid**. The following shows the option price implied by the BSM -

	Contract Symbol	Strike Price ($)	Bid Price ($)	Ask Price ($)	\
0	SPY251219P00150000	150.0	1.52	2.27	
1	SPY251219P00155000	155.0	0.53	5.00	
2	SPY251219P00160000	160.0	1.50	3.52	
3	SPY251219P00165000	165.0	0.50	5.50	
4	SPY251219P00170000	170.0	0.54	3.70	
..	...	...	...	...	
76	SPY251219P00570000	570.0	177.50	182.50	
77	SPY251219P00600000	600.0	188.50	193.50	
78	SPY251219P00610000	610.0	209.00	214.00	
79	SPY251219P00630000	630.0	223.50	228.50	
80	SPY251219P00650000	650.0	249.69	259.69	

	Implied Volatility	Expiration	Last Price	Black-Scholes Put Price	\
0	0.35	2025-12-19	2.20	35.09	
1	0.40	2025-12-19	3.00	48.73	
2	0.36	2025-12-19	2.97	41.75	
3	0.39	2025-12-19	0.11	50.47	
4	0.34	2025-12-19	3.29	42.17	
..	...	...	...	...	
76	0.24	2025-12-19	159.75	236.19	
77	0.15	2025-12-19	195.00	196.38	
78	0.23	2025-12-19	199.44	254.41	
79	0.20	2025-12-19	215.27	254.48	
80	0.28	2025-12-19	248.00	317.30	

We observe that the BSM put option price significant greater than the last trading price of the put option contracts because it is assumed that the volatility remains constant over time. Now, we can **price the American put options chain using the Cox, Rubenstein, and Ross (CRR) binomial tree options pricing model with $\geq$ 200 steps per options series tree**. The following snapshot shows the put option priced implied in the CRR pricing method -

	Contract Symbol	Strike Price ($)	Bid Price ($)	Ask Price ($)	\
0	SPY251219P00150000	150.0	1.52	2.27	
1	SPY251219P00155000	155.0	0.53	5.00	
2	SPY251219P00160000	160.0	1.50	3.52	
3	SPY251219P00165000	165.0	0.50	5.50	
4	SPY251219P00170000	170.0	0.54	3.70	
..	...	...	...	...	
76	SPY251219P00570000	570.0	177.50	182.50	
77	SPY251219P00600000	600.0	188.50	193.50	
78	SPY251219P00610000	610.0	209.00	214.00	
79	SPY251219P00630000	630.0	223.50	228.50	
80	SPY251219P00650000	650.0	249.69	259.69	

	Implied Volatility	Expiration	Last Price	Black-Scholes Put Price	\
0	0.35	2025-12-19	2.20	35.09	
1	0.40	2025-12-19	3.00	48.73	
2	0.36	2025-12-19	2.97	41.75	
3	0.39	2025-12-19	0.11	50.47	
4	0.34	2025-12-19	3.29	42.17	
..	...	...	...	...	
76	0.24	2025-12-19	159.75	236.19	
77	0.15	2025-12-19	195.00	196.38	
78	0.23	2025-12-19	199.44	254.41	
79	0.20	2025-12-19	215.27	254.48	
80	0.28	2025-12-19	248.00	317.30	

	CRR Option price
0	4.09
1	8.10
2	5.90
3	8.68
4	6.03
..	...
76	174.24
77	190.79
78	204.93
79	221.00
80	0.00

Compared the put option price implied in the BSM method, the put price implied in the CRR methods looks very rational if we compare the implied put prices with the last trading price. Now, we can **price these American options chain using Monte Carlo simulations with $\geq$ 500 simulations per option series**. The following snapshot presents the implied put price in the Monte Carlo simulations -

	Contract Symbol	Strike Price ($)	Bid Price ($)	Ask Price ($)	\
0	SPY251219P00150000	150.0	1.52	2.27	
1	SPY251219P00155000	155.0	0.53	5.00	
2	SPY251219P00160000	160.0	1.50	3.52	
3	SPY251219P00165000	165.0	0.50	5.50	
4	SPY251219P00170000	170.0	0.54	3.70	
..	...	...	...	...	
76	SPY251219P00570000	570.0	177.50	182.50	
77	SPY251219P00600000	600.0	188.50	193.50	
78	SPY251219P00610000	610.0	209.00	214.00	
79	SPY251219P00630000	630.0	223.50	228.50	
80	SPY251219P00650000	650.0	249.69	259.69	

	Implied Volatility	Expiration	Last Price	Black-Scholes Put Price	\
0	0.35	2025-12-19	2.20	35.09	
1	0.40	2025-12-19	3.00	48.73	
2	0.36	2025-12-19	2.97	41.75	
3	0.39	2025-12-19	0.11	50.47	
4	0.34	2025-12-19	3.29	42.17	
..	...	...	...	...	
76	0.24	2025-12-19	159.75	236.19	
77	0.15	2025-12-19	195.00	196.38	
78	0.23	2025-12-19	199.44	254.41	
79	0.20	2025-12-19	215.27	254.48	
80	0.28	2025-12-19	248.00	317.30	

	CRR Option price	Monte Carlo Simulation Price
0	4.09	0.00
1	8.10	0.00
2	5.90	0.00
3	8.68	0.00
4	6.03	0.00
..	...	...
76	174.24	166.23
77	190.79	194.45
78	204.93	206.30
79	221.00	225.56
80	0.00	0.00

We can observe that the Monte Carlo put prices are 0 for the first few contracts as the option payoffs are 0. However, as the options go in the money, we get very rational pricing for the put options. Now, we can **create a table showing the root mean squared error for each of the pricing methods for the options chain**. The following table compares RMSE of each of the three option pricing methods -

	Contract Symbol	RMSE of BSM	RMSE of CRR	RMSE of Monte Carlo
0	SPY251219P00150000	5.558738	4.274088	3.153059
1	SPY251219P00155000	4.043182	3.828532	3.153059
2	SPY251219P00160000	4.818738	4.072977	3.153059
3	SPY251219P00165000	3.849849	3.764088	3.153059
4	SPY251219P00170000	4.772071	4.058532	3.153059
..	...	...	...	...
76	SPY251219P00570000	16.785706	14.631468	15.316941
77	SPY251219P00600000	12.362373	16.470357	18.452497
78	SPY251219P00610000	18.810151	18.041468	19.769163
79	SPY251219P00630000	18.817929	19.827023	21.909163
80	SPY251219P00650000	25.797929	4.728532	3.153059

From the table, we observe that the contract prices the American put options in the Cox, Rubenstein, and Ross binomial tree options pricing model seem to be more accurate than those in BSM and Monte Carlo pricing method because of two reasons –

1) RMSE of CRR is smaller and more narrowly spread out than that of BSM and Monte Carlo.
2) The option prices in CRR are much closer to the last trading price than those in other pricing models.

Now, we can **vary the number of steps in the CRR model and the number of simulations used for the Monte Carlo method** to assess whether **the accuracy of the model-implied option price varies with the number of steps and the number of simulations used**. The following snapshot shows the modified implied price of CRR and Monte Carlo when we increase the number of steps from 200 to 500 and the number of simulations from 500 to 1000 -

	Contract Symbol	Strike Price ($)	Bid Price ($)	Ask Price ($)	\
0	SPY251219P00150000	150.0	1.52	2.27	
1	SPY251219P00155000	155.0	0.53	5.00	
2	SPY251219P00160000	160.0	1.50	3.52	
3	SPY251219P00165000	165.0	0.50	5.50	
4	SPY251219P00170000	170.0	0.54	3.70	
..	...	...	...	...	
76	SPY251219P00570000	570.0	177.50	182.50	
77	SPY251219P00600000	600.0	188.50	193.50	
78	SPY251219P00610000	610.0	209.00	214.00	
79	SPY251219P00630000	630.0	223.50	228.50	
80	SPY251219P00650000	650.0	249.69	259.69	

	Implied Volatility	Expiration	Last Price	Black-Scholes Put Price	\
0	0.35	2025-12-19	2.20	35.09	
1	0.40	2025-12-19	3.00	48.73	
2	0.36	2025-12-19	2.97	41.75	
3	0.39	2025-12-19	0.11	50.47	
4	0.34	2025-12-19	3.29	42.17	
..	...	...	...	...	
76	0.24	2025-12-19	159.75	236.19	
77	0.15	2025-12-19	195.00	196.38	
78	0.23	2025-12-19	199.44	254.41	
79	0.20	2025-12-19	215.27	254.48	
80	0.28	2025-12-19	248.00	317.30	

	CRR Option price	Monte Carlo Simulation Price
0	4.09	0.00
1	8.09	0.00
2	5.89	0.00
3	8.66	0.00
4	6.06	0.00
..	...	...
76	174.26	166.48
77	190.79	194.03
78	204.91	205.86
79	220.99	225.37
80	0.00	0.00

To assess whether the accuracy of the model-implied option price varies with the number of steps and the number of simulations used, we calculate the RMSE for the modified option prices in the following -

	Contract Symbol	RMSE of BSM	RMSE of CRR	RMSE of Monte Carlo	\
0	SPY251219P00150000	5.558738	4.274088	3.153059	
1	SPY251219P00155000	4.043182	3.828532	3.153059	
2	SPY251219P00160000	4.818738	4.072977	3.153059	
3	SPY251219P00165000	3.849849	3.764088	3.153059	
4	SPY251219P00170000	4.772071	4.058532	3.153059	
..	...	...	...	...	
76	SPY251219P00570000	16.785706	14.631468	15.316941	
77	SPY251219P00600000	12.362373	16.470357	18.452497	
78	SPY251219P00610000	18.810151	18.041468	19.769163	
79	SPY251219P00630000	18.817929	19.827023	21.909163	
80	SPY251219P00650000	25.797929	4.728532	3.153059	

	Modified RMSE of CRR	Modified RMSE of Monte Carlo
0	4.273484	3.151770
1	3.829040	3.151770
2	4.073484	3.151770
3	3.765706	3.151770
4	4.054595	3.151770
..	...	...
76	14.634294	15.346008
77	16.470960	18.407119
78	18.039849	19.721564
79	19.826516	21.889342
80	4.727929	3.151770

We do not observe any significant changes in RMSE of both CRR and Monte Carlo put option prices when we change the number of steps from 200 to 500 and the number of simulations from 500 to 1000. However, the accuracy of the model-implied option price will significantly increase if we bring a massive change in the number of steps (i.e., from 200 to 200,000) and in the number of simulations (i.e., from 500 to 500,000).