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<div align="center">Homework 4</div>

1.

We know from the problem that the square matrix has all row sums of zero. By this, we can say that Ax=0 for the matrix A is nxn and a vector x populated with 1, so x=(1,1,1,...1). By Ax=0, we can say Ax=0I for I is the identity matrix of size n.

$$det(Ax) = det(0I) \Rightarrow det(Ax) = 0 \Rightarrow det(A)det(x) = 0$$

And since the determinant of the vector is 1, we can say det(A)*1=0, thus the determinant of the matrix A is 0.

So, because the determinant of the matrix is zero, we know it is singular.

2.

Prove the maximum row-sum norm. $\|A\|_1 = \max\limits_{1 \leq j \leq n} \Sigma_{i=1}^m |a_{ij}|$

For any $x \in R^n$,
$\|Ax\|_1 = \Sigma_{i=1}^m |\Sigma_{j=1}^n a_{ij} x_j| \leq \Sigma_{i=1}^m \Sigma_{j=1}^n |a_{ij}||x_j|$

$= \Sigma_{j=1}^n \Sigma_{i=1}^m |a_{ij}||x_j| = \Sigma_{j=1}^n |x_j|(\Sigma)_{i=1}^m |a_{ij}|$

$\leq \max\limits_{1 \leq j \leq n} (\Sigma_{i=1}^m |a_{ij}|) \Sigma_{j=1}^n |x_j|$

$= \max\limits_{1 \leq j \leq n} (\Sigma_{i=1}^m |a_{ij}|) \|x\|_1$

Therefore, we have $\|A\|_1 = \max\limits_{x \in R^n, x \neq 0} \frac{\|Ax\|_1}{\|x\|_1} \leq \max\limits_{1 \leq j \leq n} \Sigma_{i=1}^m |a_{ij}|$

To prove the other side of this inequality, let p be $a \leq p \leq n$ such that

$\Sigma_{i=1}^m |a_{ip}| = \max\limits_{1 \leq j \leq n} \Sigma_{i=1}^m |a_{ij}|$

We choose $x = e_p$, so that $\|Ae_p\|_1 = \Sigma_{i=1}^m |a_{ip}|$

For this particular x, $\|Ax\|_1 = \Sigma_{i=1}^m |a_{ip}| = \max_{1 \le j \le n} \Sigma_{i=1}^m |a_{ij}|$

Therefore, we have $\|A\|_1 = \max_{x \in R^n, x \ne 0} \frac{\|Ax\|_1}{\|x\|_1} \ge \max_{1 \le j \le n} \Sigma_{i=1}^m |a_{ij}|$

Hence, because we have proved $\|A\|_1 = \max_{x \in R^n, x \ne 0} \frac{\|Ax\|_1}{\|x\|_1} \ge \max_{1 \le j \le n} \Sigma_{i=1}^m |a_{ij}|$
and $\|A\|_1 = \max_{x \in R^n, x \ne 0} \frac{\|Ax\|_1}{\|x\|_1} \ge \max_{1 \le j \ge n} \Sigma_{i=1}^m |a_{ij}|$,

we can say that $\|A\|_1 = \max_{x \in R^n, x \ne 0} \frac{\|Ax\|_1}{\|x\|_1} = \max_{1 \le j \le n} \Sigma_{i=1}^m |a_{ij}|$

3.

According to the defiition of vector norms, a funcion is called a vector norm if it :

1. $\|x\| \ge 0$ for any vector $x \in R^n$ and $\|x\| = 0$ if and only if x = 0.

2. $\|ax\| = |a|\|x\|$ for any vector $x \in R^n$ and any scalar $a \in R$

3. $\|x + y\| \le \|x\| + \|y\|$

To prove 1, take the multiplication of vector x^T and matrix A, then that product and vector x. By this process, $v^T A v = B$ which, as matrix A is positive, is simililarly positive for vector v and 0 when x = 0.

To prove 2, we use the principle of scalar multipliply of vectors: $cv = (cv_1, cv_2, ..., cv_n)^T$ and scalar multipliply of matricies. To preserve magnitude and direction, we take the absolute value of the constant: $|c|$.

To prove 3, we say:
$\|x + y\| \le sup(\Sigma_{i=1}^n x_i u_i + y_i v_i)) = \|x\| + \|y\|$. The energy norm is a vector like all others, so that the triangle inequality operates on it should be simply presumed by its nature.

Therefore, the energy norm is a vector norm.

4.
a.

By solving the matrix for its determinant, we would find the determinant to be $(c * c) - (-s * s)$. Rewritten, this is $c^2 + s^2$ which is given to be 1.

The transpose of the matrix is

$$\begin{bmatrix} c & -s \\ s & c \end{bmatrix}$$

which, when multiplied by the original matrix

$$\begin{bmatrix} c & s \\ -s & c \end{bmatrix}$$

yields

$$\begin{bmatrix} (c*c)+(-s*-s) & (c*s)+(-s*c) \\ (s*c)+(c*-s) & (s*s)+(c*c) \end{bmatrix}$$

which is equal to

$$\begin{bmatrix} (c*c)+(-s*-s) & 0 \\ 0 & (s*s)+(c*c) \end{bmatrix}$$

which is $c^2 * s^2$ multiplied by the 2x2 identity matrix. Thus, we have shown that the matrix has a transpose such that the matrix multiplied by its transpose yields the identity matrix.

b.
From the matrix equation, we can assume that the determinant $= c^2 + S^2 = 1$ by orthogonality. Further, we get $c^2 + S^2 = 1$ and $a = a_1 s + a_2 c$

From $c^2 + s^2 = 1$, we can determine that $c^2 a_1 = s^2 a_1$ and $s^2 a_1 = (1 - c^2)a_1^2$

Thus, $c^2 a_2^2 = (a_1^2 - c^2 a_1^2) \Rightarrow s^2 c^2 a_2^2 = s^2 a_1^2 - s^2 c^2 a_1^2 \Rightarrow a_1^2 s^2 c^2 a_2^2 = a_1^2 s^2 a_1^2 - a_1^2 s^2 c^2 a_1^2 \Rightarrow c^2 a_2^2 c^2 a_2^2 = a_1^2 c^2 a_2^2 - a_1^2 c^2 c^2 a_2^2 \Rightarrow c = \pm \frac{a_1}{sqrta_2^2 + a_1^2}$

Therefore, because $s^2 = 1 - c^2 = 1 - (\frac{a_1}{\sqrt{a_2^2 + a_1^2}})^2 = 1 - \frac{a_1^2}{a_2^2 + a_1^2} = c^2 a_2^2$

So $s = \pm \frac{a_2}{\sqrt{a_2^2 + a_1^2}}$

Therefore, because $a = a_1 s + a_2 c$, we can state that $a = a_1(\frac{a_2}{\sqrt{a_2^2 + a_1^2}})) = \frac{a_1^2 + a_2^2}{\sqrt{a_2^2 + a_1^2}}$

So, $a = \sqrt{(a_2^2 + a_1^2)}$ and the s, c that do this job are: $s = \pm \frac{a_2}{\sqrt{a_2^2 + a_1^2}}$ and $c = \pm \frac{a_1}{\sqrt{a_2^2 + a_1^2}}$

5.
a.

By definition of the orthogonal matrix, matrix A is said to be orthogonal iff $AA^T = I$ where A^T is the transpose of the matrix A, and I is the identity matrix.
By assigning matrix B to hold A^T, we can re-write this as $AB = I$.

By taking the determinants of both sides of this equation, we get $det(AB) = det(I)$. By the properties of determinants, we can re-write this as $det(A) * det(B) = det(I) \Rightarrow det(A)det(A^T) = det(I)$

We know the determinant of a matrix is the same as the determinant of its transpose. Thus, $det(A)det(A^T) = det(I) \Rightarrow (det(A))^2 = det(I)$

The determinant of the identity matrix is 1. Thus, $(det(A))^2 = 1$
$\Rightarrow det(A) = \pm 1$

So, $det(A) = \pm 1$

b.
For projector P and vector non-zero x,
$Px = \lambda x$ for $P^2 = P$, $P^2 x = \lambda x$ by replacing P with P^2
$= P * Px = \lambda x$.

Because $Px = \lambda x$, we can say $P * Px = \lambda x$
$\Rightarrow \lambda * \lambda x = \lambda x \Rightarrow \lambda^2 = \lambda$

Therefore, the eigenvalue is 0 or 1.