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MATH 4441

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Homework 6

1. a.

L:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

U:

$$\begin{bmatrix} 5 & 6 & 7 & 8 \\ 0 & 4 & 3 & 2 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

P=

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

1. b.

$$\begin{bmatrix} 5 & 6 & 7 & 8 \\ 0 & 4 & 3 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} 26 \\ 9 \\ 1 \\ -3 \end{bmatrix} \Rightarrow \begin{bmatrix} 5 & 6 & 7 & 8 \\ 0 & 4 & 3 & 2 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 26 \\ 9 \\ -3 \\ 1 \end{bmatrix}$$

Yields, through substitution, $x = (1, 1, 1, 1)^T$

2.0

See code

3.

Prove that $\frac{||x-x*||}{||x*||} \leq k(A)\frac{||r||}{||b||}$:

For Ax*=b, we are adding a small perterbation (by approximation) to x*, being x: $Ax = b + \delta b$

Thus, the error of $\frac{||x-x^*||}{||x^*||} = \frac{||\delta x^*||}{||x^*||}$

From the equations $Ax^* = b$ and $A(x^* + \delta x^*) = Ax = b + \delta b$, and by subtracting Ax* to clear Ax* and b, we get $\delta x^* = A^{-1}\delta b$, which can be normed to get $||\delta x^*|| = ||A^{-1}\delta b||$

Therefore, $||\delta x^*|| \leq ||A^{-1}||\,||\delta b||$, so $\frac{||\delta x^*||}{||x^*||} \leq \frac{||A^{-1}||\,||\delta b||}{||x^*||}$.

Then, the equation b=Ax* implies that $||b|| = ||Ax^*|| \leq ||A||\,||x^*||$, or equivalently, $\frac{1}{||x^*|||} \leq ||A||\,\frac{1}{||b||}$
.

Therefore, $\frac{||\delta x^*||}{||x^*||} \leq \frac{||A^{-1}||\,||\delta b||}{||x^*||} \leq \frac{||A||\,||A^{-1}||\,||\delta b||}{||b||}$

As $k(A) = ||A||\,||A^{-1}||$, we can re-write $\frac{||A||\,||A^{-1}||\,||\delta b||}{||b||}$ to be $k(A)\frac{||\delta b||}{||b||}$

Then, $\delta b = A\delta x = A(x^* - x) = (Ax^* - Ax) = b - Ax$
Therefore, $k(A)\frac{||\delta b||}{||b||} = k(A)\frac{||b-Ax||}{||b||}$ and since r=b-Ax, we can say it is $k(A)\frac{||r||}{||b||}$

So, $\frac{||x-x*||}{||x*||} \leq k(A)\frac{||r||}{||b||}$ when the residual r = (b - Ax) and $k(A) = ||A||\,||A^{-1}||$

4.

Prove that if A is orthogonal, then $k(A) = ||A||_2||A^{-1}||_2 = 1$:

By the norm equvalency, $||A||\,||A^{-1}|| \geq ||AA^{-1}||$

Orthogonal matrix multiplication by its inverse yields the identity matrix, thus $||AA^{-1}|| = ||I|| = 1$

So, $||A||_2||A^{-1}||_2 \geq 1$

By l_2 norm and by preposition 9, $||A||_2 = \sqrt{p(A^TA)}$

Because $A^TA = I$ by principle of orthogonal matrixes, $||A||_2 = \sqrt{p(I)}$

By preposition, $p(A) \leq ||A^k||^{1/k}$ in the case where A = I, $p(I) = 1$

So, $||A||_2 \leq 1$ and $||A^T||_2 \leq 1$ by the same principle, then $||A||_2\,||A^{-1}||_2 \leq 1$.

Therefore, as $||A||_2\,||A^{-1}||_2 \leq 1$ and $||A||_2||A^{-1}||_2 \geq 1$, we can say that $k(A) = ||A||_2||A^{-1}||_2 = 1$ if A is orthogonal.