空气动力学方程组间断有限元方法

1　方程简介

考虑二维 Euler 方程组，将其写成双曲守恒律形式：

$$\begin{cases} \boldsymbol{u_t} + \nabla \cdot \boldsymbol{f}(\boldsymbol{u}) = 0 \\ \boldsymbol{u}(x, y, 0) = \boldsymbol{u_0}(x, y) \end{cases} \qquad (x, y, t) \in \Omega \times (0, T) \tag{1}$$

其中，$\boldsymbol{u}$ 为守恒量，$\boldsymbol{f}(u) = (f_1(u), f_2(u))$ 为通量，

$$\boldsymbol{u} = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \rho \\ \rho v_1 \\ \rho v_2 \\ E \end{bmatrix} \quad f_1(\boldsymbol{u}) = \begin{bmatrix} \rho v_1 \\ \rho v_1^2 + p \\ \rho v_1 v_2 \\ (E+p) v_1 \end{bmatrix}, \quad f_2(\boldsymbol{u}) = \begin{bmatrix} \rho v_2 \\ \rho v_1 v_2 \\ \rho v_2^2 + p \\ (E+p) v_2 \end{bmatrix} \tag{2}$$

其中，ρ 表示密度，v_1 为 x 方向的速度，v_2 为 y 方向的速度，E 为能量，压力

$$p = (\gamma - 1)\left(E - \frac{1}{2}\rho\left(v_1^2 + v_2^2\right)\right) = (\gamma - 1)\left(u_3 - \frac{1}{2u_0}\left(u_1^2 + u_2^2\right)\right) \tag{3}$$

绝热指数 γ 在计算中一般取为常数 1.4。

下面给出二维 Euler 方程的一个有光滑解的例子，常用来测试算法的精度。在该例子中，计算区域 $\Omega = [0, 2] \times [0, 2]$，给定初始条件：

$$\begin{cases} \rho(x, y, 0) = 1 + 0.2 \sin(\pi(x + y)) \\ v_1(x, y, 0) = 0.7, \quad v_2(x, y, 0) = 0.3 \\ p(x, y, 0) = 1 \end{cases} \tag{4}$$

取周期边界条件, 则密度函数有精确解：

$$\rho(x, y, t) = 1 + 0.2 \sin(\pi(x + y - t))$$

该算例一般可以计算到 $T = 2$。

2 基函数与外法向量

设 $\mathcal{T}_h=\{K\}$ 为区域 Ω 的一个三角形剖分，在 $\mathcal{T}_h$ 中的任意一个三角形 K 上，设 $\{(x_i,y_i)\}_{i=0}^{2}$ 为三角形的三个顶点的坐标，且顶点顺序按逆时针方向排列，如图1所示。三角形三个顶点所对的三条边的中点分别记为 $m_i=\left(\hat{x}_i^K,\hat{y}_i^K\right)$，$i=0,1,2$，其中 m_i 为第 i 个顶点所对的边的中点，即

$$\begin{aligned}\hat{x}_0^K&=\frac{x_1^K+x_2^K}{2},\quad \hat{x}_1^K=\frac{x_0^K+x_2^K}{2},\quad \hat{x}_2^K=\frac{x_0^K+x_1^K}{2},\\ \hat{y}_0^K&=\frac{y_1^K+y_2^K}{2},\quad \hat{y}_1^K=\frac{y_0^K+y_2^K}{2},\quad \hat{y}_2^K=\frac{y_0^K+y_1^K}{2}.\end{aligned}\tag{5}$$

如图1所示。定义三个基函数

$$\begin{aligned}\varphi_0^K(x,y)&=\frac{\left(\hat{y}_1^K-\hat{y}_2^K\right)\left(x-\hat{x}_1^K\right)+\left(\hat{x}_2^K-\hat{x}_1^K\right)\left(y-\hat{y}_1^K\right)}{\left(\hat{y}_1^K-\hat{y}_2^K\right)\left(\hat{x}_0^K-\hat{x}_1^K\right)+\left(\hat{x}_2^K-\hat{x}_1^K\right)\left(\hat{y}_0^K-\hat{y}_1^K\right)}\\ \varphi_1^K(x,y)&=\frac{\left(\hat{y}_2^K-\hat{y}_0^K\right)\left(x-\hat{x}_2^K\right)+\left(\hat{x}_0^K-\hat{x}_2^K\right)\left(y-\hat{y}_2^K\right)}{\left(\hat{y}_2^K-\hat{y}_0^K\right)\left(\hat{x}_1^K-\hat{x}_2^K\right)+\left(\hat{x}_0^K-\hat{x}_2^K\right)\left(\hat{y}_1^K-\hat{y}_2^K\right)}\\ \varphi_2^K(x,y)&=\frac{\left(\hat{y}_0^K-\hat{y}_1^K\right)\left(x-\hat{x}_0^K\right)+\left(\hat{x}_1^K-\hat{x}_0^K\right)\left(y-\hat{y}_0^K\right)}{\left(\hat{y}_0^K-\hat{y}_1^K\right)\left(\hat{x}_2^K-\hat{x}_0^K\right)+\left(\hat{x}_1^K-\hat{x}_0^K\right)\left(\hat{y}_2^K-\hat{y}_0^K\right)}\end{aligned}\tag{6}$$

显然，它们满足

$$\varphi_i^K(\hat{x}_j^K,\hat{y}_j^K)=\delta_{ij}=\begin{cases}0, & i\neq j\\ 1, & i=j\end{cases}\tag{7}$$

利用这些基函数, 我们可以在 K 上定义 $\boldsymbol{u}$ 的近似

$$\boldsymbol{u}_h^K(x,y,t)=\sum_{i=0}^{2}\boldsymbol{u}_i^K(t)\varphi_i^K(x,y)\quad (x,y)\in K\tag{8}$$

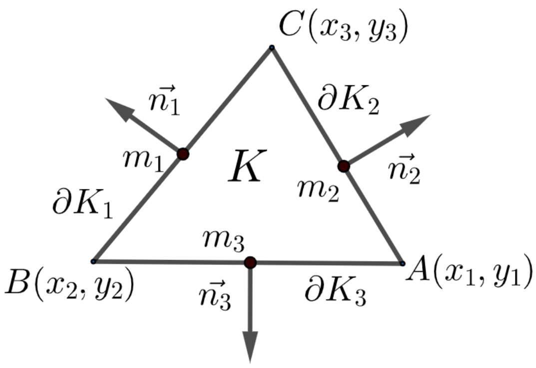

Figure 1: 三角形单元

通过计算可知，基函数 $\{\varphi_i^K(x,y)\}_{i=0}^2$ 满足：

$$\int_K \varphi_i^K(x,y)\varphi_j^K(x,y)dxdy = \begin{cases} 0, & i \neq j \quad (\text{相互正交}) \\ \dfrac{|K|}{3}\displaystyle\sum_{m=1}^{3}(\varphi_j^K)^2\left(\hat{x}_m^K, \hat{y}_m^K\right) = \dfrac{|K|}{3}, & i = j \end{cases} \tag{9}$$

其中

$$|K| = \frac{1}{2}\begin{vmatrix} 1 & x_0^K & y_0^K \\ 1 & x_1^K & y_1^K \\ 1 & x_2^K & y_2^K \end{vmatrix} = \frac{1}{2}\left|x_0^K\left(y_1^K - y_2^K\right) + x_1^K\left(y_2^K - y_0^K\right) + x_2^K\left(y_0^K - y_1^K\right)\right|$$

为三角形 K 的面积。

通过变换

$$\begin{cases} x = x(\xi,\eta) = x_0\xi + x_1\eta + x_2(1-\xi-\eta) \\ y = y(\xi,\eta) = y_0\xi + y_1\eta + y_2(1-\xi-\eta) \end{cases} \tag{10}$$

可将三角形 K 变换到标准三角形单元 $\widehat{K} = \{(\xi,\eta)|\xi+\eta \leq 1, 0 \leq \xi,\eta \leq 1\}$，那么基函数在参考坐标 ξ,η 下的表达式为

$$\begin{aligned} \hat{\varphi}_0(\xi,\eta) =& \varphi_0^K(x(\xi,\eta),y(\xi,\eta)) = 1-2\xi, \quad \hat{\varphi}_1(\xi,\eta) = \varphi_1^K(x(\xi,\eta),y(\xi,\eta)) = 1-2\eta, \\ \hat{\varphi}_2(\xi,\eta) =& \varphi_2^K(x(\xi,\eta),y(\xi,\eta)) = 2\xi+2\eta-1. \end{aligned} \tag{11}$$

利用变换将 K 上的积分转换到参考单元上得

$$\int_K \varphi_i^K(x,y)\varphi_j^K(x,y)dxdy = 2|K|\int_0^1\int_0^{1-\xi}\hat{\varphi}_i(\xi,\eta)\hat{\varphi}_j(\xi,\eta)d\eta d\xi \tag{12}$$

经过简单计算我们有

$$\int_0^1\int_0^{1-\xi}\hat{\varphi}_0(\xi,\eta)\hat{\varphi}_0(\xi,\eta)d\eta d\xi = \frac{1}{6}, \quad \int_0^1\int_0^{1-\xi}\hat{\varphi}_0(\xi,\eta)\hat{\varphi}_1(\xi,\eta)d\eta d\xi = 0. \tag{13}$$

类似我们还可以验证得

$$\int_0^1\int_0^{1-\xi}\hat{\varphi}_i(\xi,\eta)\hat{\varphi}_j(\xi,\eta)d\eta d\xi = \begin{cases} \dfrac{1}{6}, & i = j \\ 0, & i \neq j \end{cases} \tag{14}$$

将上述结论代入(12)可得(9)。

设 $\boldsymbol{n}_0$，$\boldsymbol{n}_1$，$\boldsymbol{n}_2$ 分别表示为单元 K 三个顶点所对的边上的单位外法向量，如图 1 所示。如果顶点 $A(x_0^K, y_0^K)$，$B(x_1^K, y_1^K)$，$C(x_2^K, y_2^K)$ 按逆时针方向排列，则有：

$$\begin{aligned} \boldsymbol{BC} &= \left(x_2^K - x_1^K, y_2^K - y_1^K\right), & \boldsymbol{n_0} &= \left(y_2^K - y_1^K, x_1^K - x_2^K\right)/|BC| \\ \boldsymbol{CA} &= \left(x_0^K - x_2^K, y_0^K - y_2^K\right), & \boldsymbol{n_1} &= \left(y_0^K - y_2^K, x_2^K - x_0^K\right)/|CA| \\ \boldsymbol{AB} &= \left(x_1^K - x_0^K, y_1^K - y_0^K\right), & \boldsymbol{n_2} &= \left(y_1^K - y_0^K, x_0^K - x_1^K\right)/|AB| \end{aligned} \tag{15}$$

3 DG 格式的空间离散

为了求得在任意三角形单元 K 上的近似解

$$\boldsymbol{u}_h^K(x,y,t)=\sum_{i=0}^{2}\boldsymbol{u}_j^K(t)\varphi_j^K(x,y) \tag{16}$$

我们要求近似解 $\boldsymbol{u}_h^K(x,y,t)$ 在 K 上满足方程

$$\int_K \frac{d}{dt}\boldsymbol{u}_h^K(x,y,t)\varphi_j^K dxdy=\int_K \boldsymbol{f}(\boldsymbol{u}_h^K(x,y,t))\cdot\nabla\varphi_j^K dxdy-\int_{\partial K}\hat{\boldsymbol{f}}(x,y,t)\varphi_j^K ds \tag{17}$$

其中 $j=0,1,2$，$\hat{\boldsymbol{f}}(x,y,t)$ 为人为定义的数值通量，将在后面给出具体的定义。

将表达式(8)代入上式，并应用性质(9)得，系数 $\boldsymbol{u}_j^K(t)$ 满足常微分方程组：

$$\frac{d}{dt}\boldsymbol{u}_j^K(t)=\frac{1}{\omega_j}\left[\int_K \boldsymbol{f}\left(\boldsymbol{u}_h^K(x,y,t)\right)\cdot\nabla\varphi_j^K dxdy-\int_{\partial K}\hat{\boldsymbol{f}}\left(x,y,t\right)\varphi_j^K ds\right] \tag{18}$$

其中，$\omega_j=\int_K(\varphi_j^K(x,y))^2dxdy=\frac{|K|}{3}$

方程右端的积分项可以用数值求积公式来进行计算：

$$\begin{aligned}\int_K \boldsymbol{f}\left(\boldsymbol{u}_h^K(x,y,t)\right)\cdot\nabla\varphi_j^K(x,y)dxdy\approx|K|\sum_{m=0}^{2}\frac{1}{3}\boldsymbol{f}\left(\boldsymbol{u}_h^K\left(\hat{x}_m^K,\hat{y}_m^K,t\right)\right)\cdot\nabla\varphi_j^K\left(\hat{x}_m^K,\hat{y}_m^K\right)\\ =|K|\sum_{m=0}^{2}\frac{1}{3}\boldsymbol{f}\left(\boldsymbol{u}_m^K(t)\right)\cdot\nabla\varphi_j^K\left(\hat{x}_m^K,\hat{y}_m^K\right)\\ \int_{\partial K}\hat{\boldsymbol{f}}\left(x,y,t\right)\varphi_j^K(x,y)ds=\sum_{i=0}^{2}\int_{\partial K_i}\hat{\boldsymbol{f}}(x,y,t)\varphi_j^K(x,y)ds\\ \approx\sum_{i=0}^{2}\left(|\partial K_i|\sum_{m=0}^{1}\frac{1}{2}\hat{\boldsymbol{f}}\left(\bar{x}_m^{\partial K_i},\bar{y}_m^{\partial K_i},t\right)\varphi_j^K\left(\bar{x}_m^{\partial K_i},\bar{y}_m^{\partial K_i}\right)\right)\end{aligned} \tag{19}$$

其中

$$\begin{aligned}\left(\bar{x}_0^{\partial K_i},\bar{y}_0^{\partial K_i}\right)&=\left(\hat{x}_i^K-\frac{\Delta x_i^K}{2\sqrt{3}},\hat{y}_i^K-\frac{\Delta y_i^K}{2\sqrt{3}}\right)\\ \left(\bar{x}_1^{\partial K_i},\bar{y}_1^{\partial K_i}\right)&=\left(\hat{x}_i^K+\frac{\Delta x_i^K}{2\sqrt{3}},\hat{y}_i^K+\frac{\Delta y_i^K}{2\sqrt{3}}\right)\\ \Delta x_0^K=x_2^K-x_1^K,&\quad\Delta y_0^K=y_2^K-y_1^K,\\ \Delta x_1^K=x_0^K-x_2^K,&\quad\Delta y_1^K=y_0^K-y_2^K,\\ \Delta x_2^K=x_1^K-x_0^K,&\quad\Delta y_2^K=y_1^K-y_0^K.\end{aligned} \tag{20}$$

为边 ∂K_i 上的高斯求积节点，$|\partial K_i|$ 表示边的长度，如图2。

定义数值通量

$$\hat{\boldsymbol{f}}(x,y,t)=\hat{\boldsymbol{f}}\left(\boldsymbol{u}_h^K(x,y,t),\boldsymbol{u}_h^{K'}(x,y,t)\right) \tag{21}$$

表示在单元边界点 (x,y) 处的数值通量。如图3所示，$\boldsymbol{u}_h^K(x,y,t)$ 与 $\boldsymbol{u}_h^{K'}(x,y,t)$ 分别表示单元 K 与 K' 上的数值解在 (x,y) 处的取值。

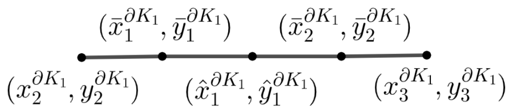

Figure 2: 边界 ∂K_1 上的高斯积分点

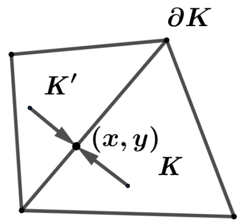

Figure 3: 数值解在边界处的左右极限

实际计算中，常用 LF 通量：

$$\hat{\boldsymbol{f}}(x,y,t)=\frac{1}{2}\left[\boldsymbol{f}(\boldsymbol{u}_h^K(x,y,t))\cdot\boldsymbol{n}+\boldsymbol{f}(\boldsymbol{u}_h^{K'}(x,y,t))\cdot\boldsymbol{n}-\alpha_K(x,y,t)(\boldsymbol{u}_h^{K'}(x,y,t)-\boldsymbol{u}_h^K(x,y,t))\right] \tag{22}$$

其中 $\boldsymbol{n}=(n_x,n_y)$ 表示在三角形单元 K 的边上的单位外法向量

$$\alpha_K(x,y,t)=\max\left\{\lambda\left(\boldsymbol{u}_h^K(x,y,t)\right),\lambda\left(\boldsymbol{u}_h^{K'}(x,y,t)\right)\right\} \tag{23}$$

$\lambda(\boldsymbol{u})$ 为 $\dfrac{\partial(f(\boldsymbol{u})\cdot\boldsymbol{n})}{\partial\boldsymbol{u}}$ 的谱半径，Jacobian 矩阵定义为

$$\frac{\partial(\boldsymbol{f}(\boldsymbol{u})\cdot\boldsymbol{n})}{\partial\boldsymbol{u}}=n_x\frac{\partial f_1(\boldsymbol{u})}{\partial\boldsymbol{u}}+n_y\frac{\partial f_2(\boldsymbol{u})}{\partial\boldsymbol{u}} \tag{24}$$

其中

$$\begin{aligned}
\frac{\partial f_1(\boldsymbol{u})}{\partial\boldsymbol{u}}&=\begin{bmatrix}
\frac{\partial(\rho v_1)}{\partial\rho} & \frac{\partial(\rho v_1)}{\partial(\rho v_1)} & \frac{\partial(\rho v_1)}{\partial(\rho v_2)} & \frac{\partial(\rho v_1)}{\partial E}\\
\frac{\partial(\rho v_1^2+p)}{\partial\rho} & \frac{\partial(\rho v_1^2+p)}{\partial(\rho v_1)} & \frac{\partial(\rho v_1^2+p)}{\partial(\rho v_2)} & \frac{\partial(\rho v_1^2+p)}{\partial E}\\
\frac{\partial(\rho v_1v_2)}{\partial\rho} & \frac{\partial(\rho v_1v_2)}{\partial(\rho v_1)} & \frac{\partial(\rho v_1v_2)}{\partial(\rho v_2)} & \frac{\partial(\rho v_1v_2)}{\partial E}\\
\frac{\partial((E+p)v_1)}{\partial\rho} & \frac{\partial((E+p)v_1)}{\partial(\rho v_1)} & \frac{\partial((E+p)v_1)}{\partial(\rho v_2)} & \frac{\partial((E+p)v_1)}{\partial E}
\end{bmatrix}\\
\frac{\partial f_2(\boldsymbol{u})}{\partial\boldsymbol{u}}&=\begin{bmatrix}
\frac{\partial(\rho v_2)}{\partial\rho} & \frac{\partial(\rho v_2)}{\partial(\rho v_1)} & \frac{\partial(\rho v_2)}{\partial(\rho v_2)} & \frac{\partial(\rho v_2)}{\partial E}\\
\frac{\partial(\rho v_1v_2)}{\partial\rho} & \frac{\partial(\rho v_1v_2)}{\partial(\rho v_1)} & \frac{\partial(\rho v_1v_2)}{\partial(\rho v_2)} & \frac{\partial(\rho v_1v_2)}{\partial E}\\
\frac{\partial(\rho v_2^2+p)}{\partial\rho} & \frac{\partial(\rho v_2^2+p)}{\partial(\rho v_1)} & \frac{\partial(\rho v_2^2+p)}{\partial(\rho v_2)} & \frac{\partial(\rho v_2^2+p)}{\partial E}\\
\frac{\partial((E+p)v_2)}{\partial\rho} & \frac{\partial((E+p)v_2)}{\partial(\rho v_1)} & \frac{\partial((E+p)v_2)}{\partial(\rho v_2)} & \frac{\partial((E+p)v_2)}{\partial E}
\end{bmatrix}
\end{aligned} \tag{25}$$

通过计算可得它的谱半径为：

$$\lambda(\boldsymbol{u})=|\boldsymbol{v}\cdot\boldsymbol{n}|+c_s=\left|\frac{u_1}{u_0}n_x+\frac{u_2}{u_0}n_y\right|+c_s \tag{26}$$

其中 $c_s=\sqrt{\dfrac{\gamma p}{\rho}}$ 为声速。

4 时间离散

关于系数函数 $\boldsymbol{u}_j^K(t), \forall K \in \mathcal{T}_h, j=0,1,2$ 的常微分方程组(18)可以将其简写为：

$$\frac{d\boldsymbol{u}_j^K(t)}{dt} = \boldsymbol{L}_j^K\left(\boldsymbol{u}_h^K(x,y,t), \boldsymbol{u}_h^{K'}(x,y,t), \boldsymbol{\beta}_h(x,y,t)\right) \quad j=0,1,2 \quad \forall K \in \mathcal{T}_h \tag{27}$$

其中，

$$\begin{aligned}
&\boldsymbol{L}_j^K\left(\boldsymbol{u}_h^K(x,y,t), \boldsymbol{u}_h^{K'}(x,y,t), \boldsymbol{\beta}_h(x,y,t)\right) \\
=&\frac{|K|}{3\omega_j}\sum_{m=0}^{2}\boldsymbol{f}\left(\boldsymbol{u}_m^K(t)\right)\cdot\nabla\varphi_j^K\left(\hat{x}_m^K, \hat{y}_m^K\right) \\
&-\sum_{i=0}^{2}\frac{|\partial K_i|}{2\omega_j}\sum_{m=0}^{1}\hat{\boldsymbol{f}}\left(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i}, t\right)\varphi_j^K\left(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i}\right)
\end{aligned} \tag{28}$$

$\boldsymbol{\beta}_h(x,y,t)$ 表示数值解在计算区域的边界 $\partial\Omega$ 上的取值，由具体的边界条件给出，在紧邻边界的网格上计算数值通量时需要用到。

将要计算的时间区间 $[0,T]$ 划分成剖分 $0=t^0<t^1<\cdots<t^n<\cdots<t^N=T$，记 $\boldsymbol{u}_j^{K,n}$ 为系数函数 $\boldsymbol{u}_j^K(t)$ 在任意 t^n 时刻的近似，结合空间离散近似(16)可得在 t^n 时刻解 $\boldsymbol{u}(x,y,t)$ 在单元 K 上的全离散近似为

$$\boldsymbol{u}(x,y,t^n) \approx \sum_{j=0}^{2}\boldsymbol{u}_j^{K,n}\varphi_j(x,y) \tag{29}$$

为了求得所有的系数 $\boldsymbol{u}_j^{K,n}$，我们采用二阶 TVD Runge-kutta 方法进行离散常微分方程组(27)，记第 n 步的时间步长 $\Delta t^n = t^{n+1}-t^n$，那么 $\boldsymbol{u}_j^{K,n}$ 的计算格式为

$$\begin{cases}
\boldsymbol{u}_j^{K,n+\frac{1}{2}} = \boldsymbol{u}_j^{K,n} + \Delta t^n \boldsymbol{L}_j^K\left(\boldsymbol{u}_h^{K,n}, \boldsymbol{u}_h^{K',n}, \boldsymbol{\beta}_h\left(t^n\right)\right) \quad j=0,1,2 \\
\boldsymbol{u}_j^{K,n+1} = \dfrac{1}{2}\left(\boldsymbol{u}_j^{K,n} + \boldsymbol{u}_j^{K,n+\frac{1}{2}}\right) + \dfrac{1}{2}\Delta t^n \boldsymbol{L}_j^K\left(\boldsymbol{u}_h^{K,n+\frac{1}{2}}, \boldsymbol{u}_h^{K',n+\frac{1}{2}}, \boldsymbol{\beta}_h\left(t^{n+1}\right)\right)
\end{cases} \tag{30}$$

在上述推进计算中需给初始值 $\boldsymbol{u}_j^{K,0}$，根据初始条件 $\boldsymbol{u}(x,y,t=0)=\boldsymbol{u}_0(x,y)$，可设定初始值

$$\boldsymbol{u}_j^{K,0} = \boldsymbol{u}_0^K\left(\hat{x}_j^K, \hat{y}_j^K\right) \quad j=0,1,2 \tag{31}$$

其中 $\left(\hat{x}_j^K, \hat{y}_j^K\right)$ 表示单元 K 第 j 号顶点所对边的中点。由于使用的是时间显格式，时间步长 $\Delta t^n = t^{n+1}-t^n$ 需满足限制条件

$$\Delta t^n = \frac{CFL}{\max\limits_{K\in\mathcal{T}_h}\left[\left(\|\boldsymbol{v}\|+c_s\right)\cdot\frac{\text{perimeter }(K)}{|K|}\right]} \tag{32}$$

其中，$\dfrac{\text{perimeter }(K)}{|K|}$ 为三角单元 K 的周长与面积之比，CFL 条件数取 0.3，

$$c_s + \|\boldsymbol{v}\| \triangleq \sqrt{v_1^2\left(\bar{\boldsymbol{u}}_h^{K,n}\right) + v_2^2\left(\bar{\boldsymbol{u}}_h^{K,n}\right)} + c_s\left(\bar{\boldsymbol{u}}_h^{K,n}\right) \quad (v_1, v_2 \text{ 表示速度}) \tag{33}$$

为 t^n 时的单元均值。

下面我们将每个单元 K 上的计算进行向量化。为此，我们记

$$\boldsymbol{u}^{K,n} = \begin{bmatrix} \boldsymbol{u}_0^{K,n} & \boldsymbol{u}_1^{K,n} & \boldsymbol{u}_2^{K,n} \end{bmatrix}_{4\times 3}, \quad \boldsymbol{L}^{K,n} = \begin{bmatrix} \boldsymbol{L}_0^{K,n} & \boldsymbol{L}_1^{K,n} & \boldsymbol{L}_2^{K,n} \end{bmatrix}_{4\times 3} \tag{34}$$

我们考虑 $\boldsymbol{L}_j^K\left(\boldsymbol{u}_h^{K,n}, \boldsymbol{u}_h^{K',n}, \boldsymbol{\beta}_h\left(t^n\right)\right)$ 的计算。我们有

$$\boldsymbol{u}_h^{K,n}\left(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i}\right) = \sum_{j=0}^{2} \boldsymbol{u}_j^{K,n} \varphi_j^K\left(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i}\right) = \boldsymbol{u}^{K,n} \begin{bmatrix} \varphi_0^K\left(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i}\right) \\ \varphi_1^K\left(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i}\right) \\ \varphi_2^K\left(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i}\right) \end{bmatrix} \tag{35}$$

$$\begin{aligned} \hat{\boldsymbol{f}}\left(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i}, t^n\right) =& \frac{1}{2}\left[\boldsymbol{f}\left(\boldsymbol{u}_h^{K,n}\left(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i}\right)\right) + \boldsymbol{f}\left(\boldsymbol{u}_h^{K',n}\left(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i}\right)\right)\right] \cdot \boldsymbol{n} \\ &- \frac{\alpha_K(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i}, t^n)}{2}\left[\boldsymbol{u}_h^{K',n}\left(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i}\right) - \boldsymbol{u}_h^{K,n}\left(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i}\right)\right] \end{aligned} \tag{36}$$

且根据定义(23)以及谱半径表达式(26)可得

$$\alpha_K(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i}, t^n) = \max\left\{\lambda\left(\boldsymbol{u}_h^{K,n}(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i})\right), \lambda\left(\boldsymbol{u}_h^{K',n}(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i})\right)\right\} \tag{37}$$

$$\begin{aligned} \lambda(\boldsymbol{u}_h^{K,n}(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i})) =& \left|\frac{u_1^{K,n}(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i})}{u_0^{K,n}(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i})} n_x + \frac{u_2^{K,n}(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i})}{u_0^{K,n}(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i})} n_y\right| \\ &+ \sqrt{\frac{\gamma(\gamma-1)\left(u_3^{K,n}(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i}) - \frac{1}{2u_0^{K,n}(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i})}\left((u_1^{K,n}(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i}))^2 + (u_2^{K,n}(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i}))^2\right)\right)}{u_0^{K,n}(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i})}} \end{aligned} \tag{38}$$

5 边界条件处理

在实际计算中，经常会出现如下几类边界条件：

1. 周期边界：此时计算区域一般为矩形，网格点在边界处为一致分布，如图4：

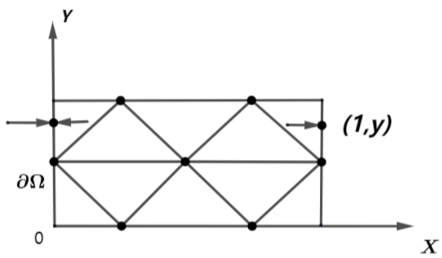

Figure 4: 周期边界条件

在左边界点 $(0, y)$ 处计算数值通量时，需要用到边界外侧的极限值 $\boldsymbol{u}_h^{\text{out}}(0, y, t)$，它由在点 $(1, y)$ 处的单元内部的值 $\boldsymbol{u}_h^{\text{in}}(1, y, t)$ 给出。

2. 入流与出流边界：

$$\boldsymbol{u}_h^{\text{out}}(x,y,t)=\boldsymbol{u}_h^{\text{in}}(x,y,t)$$

3. 反射边界：在反射边界处，密度 ρ 与能量 E 取值大小相等，速度的法向分量大小相反，切向分量大小相等。设向量 $\boldsymbol{n}=(n_1,n_2)$ 为单元 K 在 $\partial\Omega$ 处的单位外法向量，切向量 $\boldsymbol{\tau}=(-n_2,n_1)$ 有：

$$\begin{gathered}\begin{cases}\rho^{\text{out}}=\rho^{\text{in}}\\ E^{\text{out}}=E^{\text{in}}\end{cases},\quad \begin{cases}\left((\rho\boldsymbol{v}_1)^{\text{out}},(\rho\boldsymbol{v}_2)^{\text{out}}\right)\cdot\boldsymbol{n}=\left((\rho\boldsymbol{v}_1)^{\text{in}},(\rho\boldsymbol{v}_2)^{\text{in}}\right)\cdot(-\boldsymbol{n})\\ \left((\rho\boldsymbol{v}_1)^{\text{out}},(\rho\boldsymbol{v}_2)^{\text{out}}\right)\cdot\boldsymbol{\tau}=\left((\rho\boldsymbol{v}_1)^{\text{in}},(\rho\boldsymbol{v}_2)^{\text{in}}\right)\cdot(\boldsymbol{\tau})\end{cases}\\ \begin{bmatrix}(\rho\boldsymbol{v}_1)^{\text{out}}\\(\rho\boldsymbol{v}_2)^{\text{out}}\end{bmatrix}=\begin{bmatrix}n_1&-n_2\\n_2&n_1\end{bmatrix}\begin{bmatrix}-(\rho\boldsymbol{v})^{\text{in}}\cdot\boldsymbol{n}\\(\rho\boldsymbol{v})^{\text{in}}\cdot\boldsymbol{\tau}\end{bmatrix}\end{gathered}\tag{39}$$

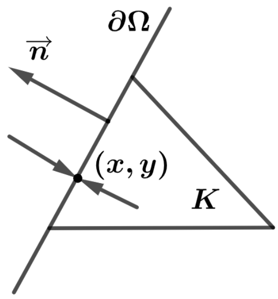

Figure 5: 入流与出流边界条件

在处理边界条件时有两种方法：

- 将靠近 $\partial\Omega$ 的单元 K 沿着边界处翻转过来，设置一个虚拟单元 K'，进行编号。跟区域 Ω 内部的单元 K 一样，在 K' 上定义数值解。这样做的好处是：计算数值通量时，在区域 Ω 的内部单元与边界单元处可以统一处理。

- 不设置虚拟单元，在区域边界 $\partial\Omega$ 处计算数值通量时需用到数值解在边界外侧的极限值，它可以由边界条件直接给出。