

空气动力学方程组间断有限元方法

1 方程简介

考虑二维 Euler 方程组，将其写成双曲守恒律形式：

$$\begin{cases} \mathbf{u}_t + \nabla \cdot \mathbf{f}(\mathbf{u}) = 0 \\ \mathbf{u}(x, y, 0) = \mathbf{u}_0(x, y) \end{cases} \quad (x, y, t) \in \Omega \times (0, T) \quad (1)$$

其中， \mathbf{u} 为守恒量， $\mathbf{f}(\mathbf{u}) = (f_1(\mathbf{u}), f_2(\mathbf{u}))$ 为通量，

$$\mathbf{u} = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \rho \\ \rho v_1 \\ \rho v_2 \\ E \end{bmatrix} \quad \mathbf{f}_1(\mathbf{u}) = \begin{bmatrix} \rho v_1 \\ \rho v_1^2 + p \\ \rho v_1 v_2 \\ (E + p)v_1 \end{bmatrix}, \quad \mathbf{f}_2(\mathbf{u}) = \begin{bmatrix} \rho v_2 \\ \rho v_1 v_2 \\ \rho v_2^2 + p \\ (E + p)v_2 \end{bmatrix} \quad (2)$$

其中， ρ 表示密度， v_1 为 x 方向的速度， v_2 为 y 方向的速度， E 为能量，压力

$$p = (\gamma - 1) \left(E - \frac{1}{2} \rho (v_1^2 + v_2^2) \right) = (\gamma - 1) \left(u_3 - \frac{1}{2u_0} (u_1^2 + u_2^2) \right) \quad (3)$$

绝热指数 γ 在计算中一般取为常数 1.4。

下面给出二维 Euler 方程的一个有光滑解的例子，常用来测试算法的精度。在该例子中，计算区域 $\Omega = [0, 2] \times [0, 2]$ ，给定初始条件：

$$\begin{cases} \rho(x, y, 0) = 1 + 0.2 \sin(\pi(x + y)) \\ v_1(x, y, 0) = 0.7, \quad v_2(x, y, 0) = 0.3 \\ p(x, y, 0) = 1 \end{cases} \quad (4)$$

取周期边界条件，则密度函数有精确解：

$$\rho(x, y, t) = 1 + 0.2 \sin(\pi(x + y - t))$$

该算例一般可以计算到 $T = 2$ 。

2 基函数与外法向量

设 $\mathcal{T}_h = \{K\}$ 为区域 Ω 的一个三角形剖分, 在 \mathcal{T}_h 中的任意一个三角形 K 上, 设 $\{(x_i, y_i)\}_{i=0}^2$ 为三角形的三个顶点的坐标, 且顶点顺序按逆时针方向排列, 如图1所示。三角形三个顶点所对的两条边的中点分别记为 $m_i = (\hat{x}_i^K, \hat{y}_i^K)$, $i = 0, 1, 2$, 其中 m_i 为第 i 个顶点所对的边的中点, 即

$$\begin{aligned} \hat{x}_0^K &= \frac{x_1^K + x_2^K}{2}, & \hat{x}_1^K &= \frac{x_0^K + x_2^K}{2}, & \hat{x}_2^K &= \frac{x_0^K + x_1^K}{2}, \\ \hat{y}_0^K &= \frac{y_1^K + y_2^K}{2}, & \hat{y}_1^K &= \frac{y_0^K + y_2^K}{2}, & \hat{y}_2^K &= \frac{y_0^K + y_1^K}{2}. \end{aligned} \quad (5)$$

如图1所示。定义三个基函数

$$\begin{aligned} \varphi_0^K(x, y) &= \frac{(\hat{y}_1^K - \hat{y}_2^K)(x - \hat{x}_1^K) + (\hat{x}_2^K - \hat{x}_1^K)(y - \hat{y}_1^K)}{(\hat{y}_1^K - \hat{y}_2^K)(\hat{x}_0^K - \hat{x}_1^K) + (\hat{x}_2^K - \hat{x}_1^K)(\hat{y}_0^K - \hat{y}_1^K)} \\ \varphi_1^K(x, y) &= \frac{(\hat{y}_2^K - \hat{y}_0^K)(x - \hat{x}_2^K) + (\hat{x}_0^K - \hat{x}_2^K)(y - \hat{y}_2^K)}{(\hat{y}_2^K - \hat{y}_0^K)(\hat{x}_1^K - \hat{x}_2^K) + (\hat{x}_0^K - \hat{x}_2^K)(\hat{y}_1^K - \hat{y}_2^K)} \\ \varphi_2^K(x, y) &= \frac{(\hat{y}_0^K - \hat{y}_1^K)(x - \hat{x}_0^K) + (\hat{x}_1^K - \hat{x}_0^K)(y - \hat{y}_0^K)}{(\hat{y}_0^K - \hat{y}_1^K)(\hat{x}_2^K - \hat{x}_0^K) + (\hat{x}_1^K - \hat{x}_0^K)(\hat{y}_2^K - \hat{y}_0^K)} \end{aligned} \quad (6)$$

显然, 它们满足

$$\varphi_i^K(\hat{x}_j^K, \hat{y}_j^K) = \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \quad (7)$$

利用这些基函数, 我们可以在 K 上定义 \mathbf{u} 的近似

$$\mathbf{u}_h^K(x, y, t) = \sum_{i=0}^2 \mathbf{u}_i^K(t) \varphi_i^K(x, y) \quad (x, y) \in K \quad (8)$$

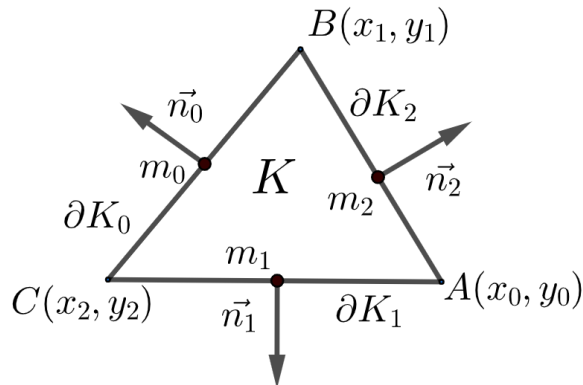


Figure 1: 三角形单元

通过计算可知，基函数 $\{\varphi_i^K(x, y)\}_{i=0}^2$ 满足：

$$\int_K \varphi_i^K(x, y) \varphi_j^K(x, y) dx dy = \begin{cases} 0, & i \neq j \quad (\text{相互正交}) \\ \frac{|K|}{3} \sum_{m=1}^3 (\varphi_j^K)^2(\hat{x}_m^K, \hat{y}_m^K) = \frac{|K|}{3}, & i = j \end{cases} \quad (9)$$

其中

$$|K| = \frac{1}{2} \begin{vmatrix} 1 & x_0^K & y_0^K \\ 1 & x_1^K & y_1^K \\ 1 & x_2^K & y_2^K \end{vmatrix} = \frac{1}{2} |x_0^K (y_1^K - y_2^K) + x_1^K (y_2^K - y_0^K) + x_2^K (y_0^K - y_1^K)|$$

为三角形 K 的面积。

通过变换

$$\begin{cases} x = x(\xi, \eta) = x_0\xi + x_1\eta + x_2(1 - \xi - \eta) \\ y = y(\xi, \eta) = y_0\xi + y_1\eta + y_2(1 - \xi - \eta) \end{cases} \quad (10)$$

可将三角形 K 变换到标准三角形单元 $\hat{K} = \{(\xi, \eta) | \xi + \eta \leq 1, 0 \leq \xi, \eta \leq 1\}$ ，那么基函数在参考坐标 ξ, η 下的表达式为

$$\begin{aligned} \hat{\varphi}_0(\xi, \eta) &= \varphi_0^K(x(\xi, \eta), y(\xi, \eta)) = 1 - 2\xi, & \hat{\varphi}_1(\xi, \eta) &= \varphi_1^K(x(\xi, \eta), y(\xi, \eta)) = 1 - 2\eta, \\ \hat{\varphi}_2(\xi, \eta) &= \varphi_2^K(x(\xi, \eta), y(\xi, \eta)) = 2\xi + 2\eta - 1. \end{aligned} \quad (11)$$

利用变换将 K 上的积分转换到参考单元上得

$$\int_K \varphi_i^K(x, y) \varphi_j^K(x, y) dx dy = 2|K| \int_0^1 \int_0^{1-\xi} \hat{\varphi}_i(\xi, \eta) \hat{\varphi}_j(\xi, \eta) d\eta d\xi \quad (12)$$

经过简单计算我们有

$$\int_0^1 \int_0^{1-\xi} \hat{\varphi}_0(\xi, \eta) \hat{\varphi}_0(\xi, \eta) d\eta d\xi = \frac{1}{6}, \quad \int_0^1 \int_0^{1-\xi} \hat{\varphi}_0(\xi, \eta) \hat{\varphi}_1(\xi, \eta) d\eta d\xi = 0. \quad (13)$$

类似我们还可以验证得

$$\int_0^1 \int_0^{1-\xi} \hat{\varphi}_i(\xi, \eta) \hat{\varphi}_j(\xi, \eta) d\eta d\xi = \begin{cases} \frac{1}{6}, & i = j \\ 0, & i \neq j \end{cases} \quad (14)$$

将上述结论代入(12)可得(9)。

设 $\mathbf{n}_0, \mathbf{n}_1, \mathbf{n}_2$ 分别表示为单元 K 三个顶点所对的边上的单位外法向量，如图 1 所示。如果顶点 $A(x_0^K, y_0^K), B(x_1^K, y_1^K), C(x_2^K, y_2^K)$ 按逆时针方向排列，则有：

$$\begin{aligned} \mathbf{BC} &= (x_2^K - x_1^K, y_2^K - y_1^K), & \mathbf{n}_0 &= (y_2^K - y_1^K, x_1^K - x_2^K) / |\mathbf{BC}| \\ \mathbf{CA} &= (x_0^K - x_2^K, y_0^K - y_2^K), & \mathbf{n}_1 &= (y_0^K - y_2^K, x_2^K - x_0^K) / |\mathbf{CA}| \\ \mathbf{AB} &= (x_1^K - x_0^K, y_1^K - y_0^K), & \mathbf{n}_2 &= (y_1^K - y_0^K, x_0^K - x_1^K) / |\mathbf{AB}| \end{aligned} \quad (15)$$

3 DG 格式的空间离散

为了求得在任意三角形单元 K 上的近似解

$$\mathbf{u}_h^K(x, y, t) = \sum_{i=0}^2 \mathbf{u}_j^K(t) \varphi_j^K(x, y) \quad (16)$$

我们要求近似解 $\mathbf{u}_h^K(x, y, t)$ 在 K 上满足方程

$$\int_K \frac{d}{dt} \mathbf{u}_h^K(x, y, t) \varphi_j^K dx dy = \int_K \mathbf{f}(\mathbf{u}_h^K(x, y, t)) \cdot \nabla \varphi_j^K dx dy - \int_{\partial K} \hat{\mathbf{f}}(x, y, t) \varphi_j^K ds \quad (17)$$

其中 $j = 0, 1, 2$, $\hat{\mathbf{f}}(x, y, t)$ 为人为定义的数值通量, 将在后面给出具体的定义。

将表达式(8)代入上式, 并应用性质(9)得, 系数 $\mathbf{u}_j^K(t)$ 满足常微分方程组:

$$\frac{d}{dt} \mathbf{u}_j^K(t) = \frac{1}{\omega_j} \left[\int_K \mathbf{f}(\mathbf{u}_h^K(x, y, t)) \cdot \nabla \varphi_j^K dx dy - \int_{\partial K} \hat{\mathbf{f}}(x, y, t) \varphi_j^K ds \right] \quad (18)$$

其中, $\omega_j = \int_K (\varphi_j^K(x, y))^2 dx dy = \frac{|K|}{3}$

方程右端的积分项可以用数值求积公式来进行计算:

$$\begin{aligned} \int_K \mathbf{f}(\mathbf{u}_h^K(x, y, t)) \cdot \nabla \varphi_j^K(x, y) dx dy &\approx |K| \sum_{m=0}^2 \frac{1}{3} \mathbf{f}(\mathbf{u}_h^K(\hat{x}_m^K, \hat{y}_m^K, t)) \cdot \nabla \varphi_j^K(\hat{x}_m^K, \hat{y}_m^K) \\ &= |K| \sum_{m=0}^2 \frac{1}{3} \mathbf{f}(\mathbf{u}_m^K(t)) \cdot \nabla \varphi_j^K(\hat{x}_m^K, \hat{y}_m^K) \\ \int_{\partial K} \hat{\mathbf{f}}(x, y, t) \varphi_j^K(x, y) ds &= \sum_{i=0}^2 \int_{\partial K_i} \hat{\mathbf{f}}(x, y, t) \varphi_j^K(x, y) ds \\ &\approx \sum_{i=0}^2 \left(|\partial K_i| \sum_{m=0}^1 \frac{1}{2} \hat{\mathbf{f}}(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i}, t) \varphi_j^K(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i}) \right) \end{aligned} \quad (19)$$

其中

$$\begin{aligned} (\bar{x}_0^{\partial K_i}, \bar{y}_0^{\partial K_i}) &= \left(\hat{x}_i^K - \frac{\Delta x_i^K}{2\sqrt{3}}, \hat{y}_i^K - \frac{\Delta y_i^K}{2\sqrt{3}} \right) \\ (\bar{x}_1^{\partial K_i}, \bar{y}_1^{\partial K_i}) &= \left(\hat{x}_i^K + \frac{\Delta x_i^K}{2\sqrt{3}}, \hat{y}_i^K + \frac{\Delta y_i^K}{2\sqrt{3}} \right) \\ \Delta x_0^K &= x_2^K - x_1^K, \quad \Delta y_0^K = y_2^K - y_1^K, \\ \Delta x_1^K &= x_0^K - x_2^K, \quad \Delta y_1^K = y_0^K - y_2^K, \\ \Delta x_2^K &= x_1^K - x_0^K, \quad \Delta y_2^K = y_1^K - y_0^K. \end{aligned} \quad (20)$$

为边 ∂K_i 上的高斯求积节点, $|\partial K_i|$ 表示边的长度, 如图2。

定义数值通量

$$\hat{\mathbf{f}}(x, y, t) = \hat{\mathbf{f}}(\mathbf{u}_h^K(x, y, t), \mathbf{u}_h^{K'}(x, y, t)) \quad (21)$$

表示在单元边界点 (x, y) 处的数值通量。如图3所示, $\mathbf{u}_h^K(x, y, t)$ 与 $\mathbf{u}_h^{K'}(x, y, t)$ 分别表示单元 K 与 K' 上的数值解在 (x, y) 处的取值。

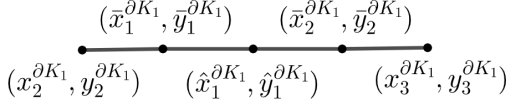


Figure 2: 边界 ∂K_1 上的高斯积分点

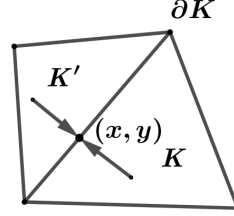


Figure 3: 数值解在边界处的左右极限

实际计算中, 常用 LF 通量:

$$\hat{\mathbf{f}}(x, y, t) = \frac{1}{2} \left[\mathbf{f}(\mathbf{u}_h^K(x, y, t)) \cdot \mathbf{n} + \mathbf{f}(\mathbf{u}_h^{K'}(x, y, t)) \cdot \mathbf{n} - \alpha_K(x, y, t)(\mathbf{u}_h^{K'}(x, y, t) - \mathbf{u}_h^K(x, y, t)) \right] \quad (22)$$

其中 $\mathbf{n} = (n_x, n_y)$ 表示在三角形单元 K 的边上的单位外法向量

$$\alpha_K(x, y, t) = \max \left\{ \lambda(\mathbf{u}_h^K(x, y, t)), \lambda(\mathbf{u}_h^{K'}(x, y, t)) \right\} \quad (23)$$

$\lambda(\mathbf{u})$ 为 $\frac{\partial(\mathbf{f}(\mathbf{u}) \cdot \mathbf{n})}{\partial \mathbf{u}}$ 的谱半径, Jacobian 矩阵定义为

$$\frac{\partial(\mathbf{f}(\mathbf{u}) \cdot \mathbf{n})}{\partial \mathbf{u}} = n_x \frac{\partial f_1(\mathbf{u})}{\partial \mathbf{u}} + n_y \frac{\partial f_2(\mathbf{u})}{\partial \mathbf{u}} \quad (24)$$

其中

$$\begin{aligned} \frac{\partial f_1(\mathbf{u})}{\partial \mathbf{u}} &= \begin{bmatrix} \frac{\partial(\rho v_1)}{\partial \rho} & \frac{\partial(\rho v_1)}{\partial(\rho v_1)} & \frac{\partial(\rho v_1)}{\partial(\rho v_2)} & \frac{\partial(\rho v_1)}{\partial E} \\ \frac{\partial(\rho v_1^2 + p)}{\partial \rho} & \frac{\partial(\rho v_1^2 + p)}{\partial(\rho v_1)} & \frac{\partial(\rho v_1^2 + p)}{\partial(\rho v_2)} & \frac{\partial(\rho v_1^2 + p)}{\partial E} \\ \frac{\partial(\rho v_1 v_2)}{\partial \rho} & \frac{\partial(\rho v_1 v_2)}{\partial(\rho v_1)} & \frac{\partial(\rho v_1 v_2)}{\partial(\rho v_2)} & \frac{\partial(\rho v_1 v_2)}{\partial E} \\ \frac{\partial((E+p)v_1)}{\partial \rho} & \frac{\partial((E+p)v_1)}{\partial(\rho v_1)} & \frac{\partial((E+p)v_1)}{\partial(\rho v_2)} & \frac{\partial((E+p)v_1)}{\partial E} \end{bmatrix} \\ \frac{\partial f_2(\mathbf{u})}{\partial \mathbf{u}} &= \begin{bmatrix} \frac{\partial(\rho v_2)}{\partial \rho} & \frac{\partial(\rho v_2)}{\partial(\rho v_1)} & \frac{\partial(\rho v_2)}{\partial(\rho v_2)} & \frac{\partial(\rho v_2)}{\partial E} \\ \frac{\partial(\rho v_1 v_2)}{\partial \rho} & \frac{\partial(\rho v_1 v_2)}{\partial(\rho v_1)} & \frac{\partial(\rho v_1 v_2)}{\partial(\rho v_2)} & \frac{\partial(\rho v_1 v_2)}{\partial E} \\ \frac{\partial(\rho v_2^2 + p)}{\partial \rho} & \frac{\partial(\rho v_2^2 + p)}{\partial(\rho v_1)} & \frac{\partial(\rho v_2^2 + p)}{\partial(\rho v_2)} & \frac{\partial(\rho v_2^2 + p)}{\partial E} \\ \frac{\partial((E+p)v_2)}{\partial \rho} & \frac{\partial((E+p)v_2)}{\partial(\rho v_1)} & \frac{\partial((E+p)v_2)}{\partial(\rho v_2)} & \frac{\partial((E+p)v_2)}{\partial E} \end{bmatrix} \end{aligned} \quad (25)$$

通过计算可得它的谱半径为:

$$\lambda(\mathbf{u}) = |\mathbf{v} \cdot \mathbf{n}| + c_s = \left| \frac{u_1}{u_0} n_x + \frac{u_2}{u_0} n_y \right| + c_s \quad (26)$$

其中 $c_s = \sqrt{\frac{\gamma p}{\rho}}$ 为声速。

4 时间离散

关于系数函数 $\mathbf{u}_j^K(t)$, $\forall K \in \mathcal{T}_h, j = 0, 1, 2$ 的常微分方程组(18)可以将其简写为:

$$\frac{d\mathbf{u}_j^K(t)}{dt} = \mathbf{L}_j^K(\mathbf{u}_h^K(x, y, t), \mathbf{u}_h^{K'}(x, y, t), \beta_h(x, y, t)) \quad j = 0, 1, 2 \quad \forall K \in \mathcal{T}_h \quad (27)$$

其中,

$$\begin{aligned} & \mathbf{L}_j^K(\mathbf{u}_h^K(x, y, t), \mathbf{u}_h^{K'}(x, y, t), \beta_h(x, y, t)) \\ &= \frac{|K|}{3\omega_j} \sum_{m=0}^2 \mathbf{f}(\mathbf{u}_m^K(t)) \cdot \nabla \varphi_j^K(\hat{x}_m^K, \hat{y}_m^K) \\ & - \sum_{i=0}^2 \frac{|\partial K_i|}{2\omega_j} \sum_{m=0}^1 \hat{\mathbf{f}}(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i}, t) \varphi_j^K(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i}) \end{aligned} \quad (28)$$

$\beta_h(x, y, t)$ 表示数值解在计算区域的边界 $\partial\Omega$ 上的取值, 由具体的边界条件给出, 在紧邻边界的网格上计算数值通量时需要用到。

将要计算的时间区间 $[0, T]$ 划分成剖分 $0 = t^0 < t^1 < \dots < t^n < \dots < t^N = T$, 记 $\mathbf{u}_j^{K,n}$ 为系数函数 $\mathbf{u}_j^K(t)$ 在任意 t^n 时刻的近似, 结合空间离散近似(16)可得在 t^n 时刻解 $\mathbf{u}(x, y, t)$ 在单元 K 上的全离散近似为

$$\mathbf{u}(x, y, t^n) \approx \sum_{j=0}^2 \mathbf{u}_j^{K,n} \varphi_j(x, y) \quad (29)$$

为了求得所有的系数 $\mathbf{u}_j^{K,n}$, 我们采用二阶 TVD Runge-kutta 方法进行离散常微分方程组(27), 记第 n 步的时间步长 $\Delta t^n = t^{n+1} - t^n$, 那么 $\mathbf{u}_j^{K,n}$ 的计算格式为

$$\begin{cases} \mathbf{u}_j^{K,n+\frac{1}{2}} = \mathbf{u}_j^{K,n} + \Delta t^n \mathbf{L}_j^K(\mathbf{u}_h^{K,n}, \mathbf{u}_h^{K',n}, \beta_h(t^n)) & j = 0, 1, 2 \\ \mathbf{u}_j^{K,n+1} = \frac{1}{2}(\mathbf{u}_j^{K,n} + \mathbf{u}_j^{K,n+\frac{1}{2}}) + \frac{1}{2}\Delta t^n \mathbf{L}_j^K(\mathbf{u}_h^{K,n+\frac{1}{2}}, \mathbf{u}_h^{K',n+\frac{1}{2}}, \beta_h(t^{n+1})) \end{cases} \quad (30)$$

在上述推进计算中需给初始值 $\mathbf{u}_j^{K,0}$, 根据初始条件 $\mathbf{u}(x, y, t = 0) = \mathbf{u}_0(x, y)$, 可设定初始值

$$\mathbf{u}_j^{K,0} = \mathbf{u}_0^K(\hat{x}_j^K, \hat{y}_j^K) \quad j = 0, 1, 2 \quad (31)$$

其中 $(\hat{x}_j^K, \hat{y}_j^K)$ 表示单元 K 第 j 号顶点所对边的中点。由于使用的是时间显格式, 时间步长 $\Delta t^n = t^{n+1} - t^n$ 需满足限制条件

$$\Delta t^n = \frac{CFL}{\max_{K \in \mathcal{T}_h} \left[(\|\mathbf{v}\| + c_s) \cdot \frac{\text{perimeter}(K)}{|K|} \right]} \quad (32)$$

其中, $\frac{\text{perimeter}(K)}{|K|}$ 为三角单元 K 的周长与面积之比, CFL 条件数取 0.3,

$$c_s + \|\mathbf{v}\| \triangleq \sqrt{v_1^2(\bar{\mathbf{u}}_h^{K,n}) + v_2^2(\bar{\mathbf{u}}_h^{K,n})} + c_s(\bar{\mathbf{u}}_h^{K,n}) \quad (v_1, v_2 \text{ 表示速度}) \quad (33)$$

为 t^n 时的单元均值。

下面我们将每个单元 K 上的计算进行向量化。为此，我们记

$$\mathbf{u}^{K,n} = \begin{bmatrix} \mathbf{u}_0^{K,n} & \mathbf{u}_1^{K,n} & \mathbf{u}_2^{K,n} \end{bmatrix}_{4 \times 3}, \quad \mathbf{L}^{K,n} = \begin{bmatrix} \mathbf{L}_0^{K,n} & \mathbf{L}_1^{K,n} & \mathbf{L}_2^{K,n} \end{bmatrix}_{4 \times 3} \quad (34)$$

我们考虑 $\mathbf{L}_j^K (\mathbf{u}_h^{K,n}, \mathbf{u}_h^{K',n}, \beta_h(t^n))$ 的计算。我们有

$$\mathbf{u}_h^{K,n}(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i}) = \sum_{j=0}^2 \mathbf{u}_j^{K,n} \varphi_j^K(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i}) = \mathbf{u}^{K,n} \begin{bmatrix} \varphi_0^K(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i}) \\ \varphi_1^K(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i}) \\ \varphi_2^K(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i}) \end{bmatrix} \quad (35)$$

$$\begin{aligned} \hat{\mathbf{f}}(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i}, t^n) &= \frac{1}{2} \left[\mathbf{f}(\mathbf{u}_h^{K,n}(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i})) + \mathbf{f}(\mathbf{u}_h^{K',n}(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i})) \right] \cdot \mathbf{n} \\ &\quad - \frac{\alpha_K(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i}, t^n)}{2} \left[\mathbf{u}_h^{K',n}(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i}) - \mathbf{u}_h^{K,n}(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i}) \right] \end{aligned} \quad (36)$$

且根据定义(23)以及谱半径表达式(26)可得

$$\alpha_K(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i}, t^n) = \max \left\{ \lambda \left(\mathbf{u}_h^{K,n}(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i}) \right), \lambda \left(\mathbf{u}_h^{K',n}(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i}) \right) \right\} \quad (37)$$

$$\begin{aligned} \lambda(\mathbf{u}_h^{K,n}(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i})) &= \left| \frac{u_1^{K,n}(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i})}{u_0^{K,n}(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i})} n_x + \frac{u_2^{K,n}(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i})}{u_0^{K,n}(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i})} n_y \right| \\ &\quad + \sqrt{\frac{\gamma(\gamma-1) \left(u_3^{K,n}(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i}) - \frac{1}{2u_0^{K,n}(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i})} \left((u_1^{K,n}(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i}))^2 + (u_2^{K,n}(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i}))^2 \right) \right)}{u_0^{K,n}(\bar{x}_m^{\partial K_i}, \bar{y}_m^{\partial K_i})}} \right| \end{aligned} \quad (38)$$

5 边界条件处理

在实际计算中，经常会出现如下几类边界条件：

1. 周期边界：此时计算区域一般为矩形，网格点在边界处为一致分布，如图4：

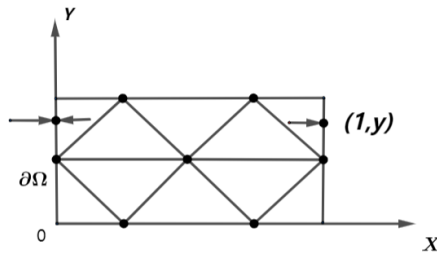


Figure 4: 周期边界条件

在左边界点 $(0, y)$ 处计算数值通量时，需要用到边界外侧的极限值 $\mathbf{u}_h^{\text{out}}(0, y, t)$ ，它由在点 $(1, y)$ 处的单元内部的值 $\mathbf{u}_h^{\text{in}}(1, y, t)$ 给出。

2. 入流与出流边界:

$$\mathbf{u}_h^{\text{out}}(x, y, t) = \mathbf{u}_h^{\text{in}}(x, y, t)$$

3. 反射边界: 在反射边界处, 密度 ρ 与能量 E 取值大小相等, 速度的法向分量大小相反, 切向分量大小相等。设向量 $\mathbf{n} = (n_1, n_2)$ 为单元 K 在 $\partial\Omega$ 处的单位外法向量, 切向量 $\boldsymbol{\tau} = (-n_2, n_1)$ 有:

$$\begin{cases} \rho^{\text{out}} = \rho^{\text{in}} \\ E^{\text{out}} = E^{\text{in}} \end{cases}, \quad \begin{cases} ((\rho \mathbf{v}_1)^{\text{out}}, (\rho \mathbf{v}_2)^{\text{out}}) \cdot \mathbf{n} = ((\rho \mathbf{v}_1)^{\text{in}}, (\rho \mathbf{v}_2)^{\text{in}}) \cdot (-\mathbf{n}) \\ ((\rho \mathbf{v}_1)^{\text{out}}, (\rho \mathbf{v}_2)^{\text{out}}) \cdot \boldsymbol{\tau} = ((\rho \mathbf{v}_1)^{\text{in}}, (\rho \mathbf{v}_2)^{\text{in}}) \cdot (\boldsymbol{\tau}) \end{cases} \quad (39)$$

$$\begin{bmatrix} (\rho \mathbf{v}_1)^{\text{out}} \\ (\rho \mathbf{v}_2)^{\text{out}} \end{bmatrix} = \begin{bmatrix} n_1 & -n_2 \\ n_2 & n_1 \end{bmatrix} \begin{bmatrix} -(\rho \mathbf{v})^{\text{in}} \cdot \mathbf{n} \\ (\rho \mathbf{v})^{\text{in}} \cdot \boldsymbol{\tau} \end{bmatrix}$$

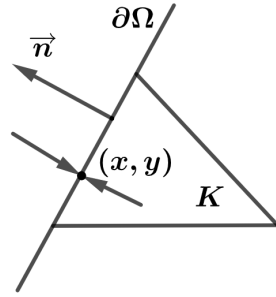


Figure 5: 入流与出流边界条件

在处理边界条件时有两种方法:

- 将靠近 $\partial\Omega$ 的单元 K 沿着边界处翻转过来, 设置一个虚拟单元 K' , 进行编号。跟区域 Ω 内部的单元 K 一样, 在 K' 上定义数值解。这样做的好处是: 计算数值通量时, 在区域 Ω 的内部单元与边界单元处可以统一处理。
- 不设置虚拟单元, 在区域边界 $\partial\Omega$ 处计算数值通量时需用到数值解在边界外侧的极限值, 它可以由边界条件直接给出。