

# 空气动力学方程组间断有限元方法

## 1 方程简介

考虑三维 Euler 方程组, 将其写成双曲守恒律形式:

$$\begin{cases} \mathbf{u}_t + \nabla \cdot \mathbf{f}(\mathbf{u}) = 0 \\ \mathbf{u}(x, y, z, 0) = \mathbf{u}_0(x, y, z) \end{cases} \quad (x, y, z, t) \in \Omega \times (0, T) \quad (1)$$

其中,  $\mathbf{u}$  为守恒量,  $\mathbf{f}(\mathbf{u}) = (f_1(\mathbf{u}), f_2(\mathbf{u}), f_3(\mathbf{u}))$  为通量,

$$\mathbf{u} = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} \rho \\ \rho v_1 \\ \rho v_2 \\ \rho v_3 \\ E \end{bmatrix} \quad \mathbf{f}_1(\mathbf{u}) = \begin{bmatrix} \rho v_1 \\ \rho v_1^2 + p \\ \rho v_2 v_1 \\ \rho v_3 v_1 \\ (E + p)v_1 \end{bmatrix}, \quad \mathbf{f}_2(\mathbf{u}) = \begin{bmatrix} \rho v_2 \\ \rho v_1 v_2 \\ \rho v_2^2 + p \\ \rho v_3 v_2 \\ (E + p)v_2 \end{bmatrix}, \quad \mathbf{f}_3(\mathbf{u}) = \begin{bmatrix} \rho v_3 \\ \rho v_1 v_3 \\ \rho v_2 v_3 \\ \rho v_3^2 + p \\ (E + p)v_3 \end{bmatrix} \quad (2)$$

其中,  $\rho$  表示密度,  $\mathbf{v} = (v_1, v_2, v_3)$  为速度,  $E$  为能量, 压力

$$p = (\gamma - 1) \left( E - \frac{1}{2} \rho \|\mathbf{v}\|^2 \right) = (\gamma - 1) \left( u_4 - \frac{1}{2u_0} (u_1^2 + u_2^2 + u_3^2) \right) \quad (3)$$

绝热指数  $\gamma$  在计算中一般取为常数 1.4。

下面给出二维 Euler 方程的一个有光滑解的例子, 常用来测试算法的精度。在该例子中, 计算区域  $\Omega = [0, 2]^3$ , 给定初始条件:

$$\begin{cases} \rho(x, y, z, 0) = 1 + 0.2 \sin(\pi(x + y + z)) \\ v_1(x, y, z, 0) = 0.4, \quad v_2(x, y, z, 0) = 0.3 \\ v_3(x, y, z, 0) = 0.3, \quad p(x, y, z, 0) = 1 \end{cases} \quad (4)$$

取周期边界条件, 则密度函数有精确解:

$$\rho(x, y, z, t) = 1 + 0.2 \sin(\pi(x + y + z - t))$$

该算例一般可以计算到  $T = 2$ 。

## 2 基函数与外法向量

设  $\mathcal{T}_h = \{K\}$  为区域  $\Omega$  的一个四面体剖分, 在  $\mathcal{T}_h$  中的任意一个四面体  $K$  上, 设  $\{(x_i, y_i, z_i)\}_{i=0}^3$  为四面体的四个顶点的坐标, 如图1所示。

设四面体单元  $K$  的体积为  $V$ ,  $P$  为四面体内一点, 四面体  $PA_1A_2A_3$ ,  $PA_0A_2A_3$ ,  $PA_0A_1A_3$  的体积分别为  $V_0$ ,  $V_1$ ,  $V_2$ , 则有

$$\begin{aligned} 6V &= \begin{vmatrix} 1 & x_0 & y_0 & z_0 \\ 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \end{vmatrix}, & 6V_0 &= \begin{vmatrix} 1 & x & y & z \\ 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \end{vmatrix}, \\ 6V_1 &= \begin{vmatrix} 1 & x_0 & y_0 & z_0 \\ 1 & x & y & z \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \end{vmatrix}, & 6V_2 &= \begin{vmatrix} 1 & x_0 & y_0 & z_0 \\ 1 & x_1 & y_1 & z_1 \\ 1 & x & y & z \\ 1 & x_3 & y_3 & z_3 \end{vmatrix}. \end{aligned} \quad (5)$$

令:  $\eta_0 = V_0/V$ ,  $\eta_1 = V_1/V$ ,  $\eta_2 = V_2/V$ , 其中

$$\begin{aligned} \nabla \eta_0 &= \frac{1}{6V} \left( - \begin{vmatrix} 1 & y_1 & z_1 \\ 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \end{vmatrix}, \begin{vmatrix} 1 & x_1 & z_1 \\ 1 & x_2 & z_2 \\ 1 & x_3 & z_3 \end{vmatrix}, - \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} \right) \\ \nabla \eta_1 &= \frac{1}{6V} \left( \begin{vmatrix} 1 & y_0 & z_0 \\ 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \end{vmatrix}, - \begin{vmatrix} 1 & x_0 & z_0 \\ 1 & x_2 & z_2 \\ 1 & x_3 & z_3 \end{vmatrix}, \begin{vmatrix} 1 & x_0 & y_0 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} \right) \\ \nabla \eta_2 &= \frac{1}{6V} \left( - \begin{vmatrix} 1 & y_0 & z_0 \\ 1 & y_1 & z_1 \\ 1 & y_3 & z_3 \end{vmatrix}, \begin{vmatrix} 1 & x_0 & z_0 \\ 1 & x_1 & z_1 \\ 1 & x_3 & z_3 \end{vmatrix}, - \begin{vmatrix} 1 & x_0 & y_0 \\ 1 & x_1 & y_1 \\ 1 & x_3 & y_3 \end{vmatrix} \right) \end{aligned} \quad (6)$$

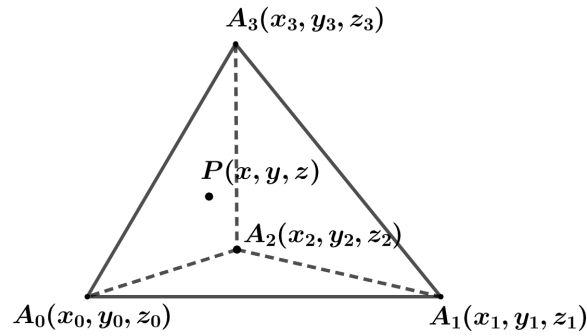


Figure 1: 四面体单元

通过体积坐标变换:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \eta_0 \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \eta_1 \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \eta_2 \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} + (1 - \eta_0 - \eta_1 - \eta_2) \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} \quad (7)$$

可将单元  $K$  变换到标准四面体单元:

$$\hat{K} = \{\boldsymbol{\eta} = (\eta_0, \eta_1, \eta_2) \mid \eta_0 + \eta_1 + \eta_2 \leq 1, 0 \leq \eta_0, \eta_1, \eta_2 \leq 1\} \quad (8)$$

在参考单元  $\hat{K}$  上考虑一次多项式空间, 可以得到四个相互正交的基函数如下:

$$\begin{aligned} \hat{\varphi}_0 &= 1, & \hat{\varphi}_1 &= -\frac{1}{4} + \eta_0, & \hat{\varphi}_2 &= -\frac{1}{3} + \frac{1}{3}\eta_0 + \eta_1, \\ \hat{\varphi}_3 &= -\frac{1}{2} + \frac{1}{2}\eta_0 + \frac{1}{2}\eta_1 + \eta_2. \end{aligned} \quad (9)$$

在参考单元上定义  $L^2$  内积  $(\hat{\varphi}_i, \hat{\varphi}_j) = \int_{\hat{K}} \hat{\varphi}_i \hat{\varphi}_j d\boldsymbol{\eta}$ , 经过简单计算有:

$$\begin{aligned} (\hat{\varphi}_i, \hat{\varphi}_j) &= 0 \quad (i \neq j), & (\hat{\varphi}_0, \hat{\varphi}_0) &= \frac{1}{6}, & (\hat{\varphi}_1, \hat{\varphi}_1) &= \frac{1}{160}, \\ (\hat{\varphi}_2, \hat{\varphi}_2) &= \frac{1}{180}, & (\hat{\varphi}_3, \hat{\varphi}_3) &= \frac{1}{240}. \end{aligned} \quad (10)$$

利用这些基函数, 我们可以在单元  $K$  上定义  $\mathbf{u}$  的近似

$$\mathbf{u}_h^K(\mathbf{X}, t) = \sum_{j=0}^3 \mathbf{u}_j^K(t) \varphi_j^K(\mathbf{X}) \quad \mathbf{X} = (x, y, z) \in K \quad (11)$$

其中,  $\varphi_j^K(\mathbf{X}) = \hat{\varphi}_j(\boldsymbol{\eta}(\mathbf{X}))$ , 因为

$$\int_K (\varphi_j^K)^2 d\mathbf{X} = \int_{\hat{K}} \hat{\varphi}_j^2 |J| d\boldsymbol{\eta} = |J| (\hat{\varphi}_j, \hat{\varphi}_j) \quad (12)$$

$|J| = 6|K|$  为 Jacobian 矩阵行列式的绝对值, 所以  $\varphi_j^K(\mathbf{X})$  也是单元  $K$  上的正交基函数。

设  $\mathbf{n}_0, \mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$  分别表示为单元  $K$  四个顶点  $A_0, A_1, A_2, A_3$  所对的面上的单位外法向量, 下面以  $\mathbf{n}_0$  为例来说明它的计算方法:

$$\begin{aligned} \mathbf{A}_1 \mathbf{A}_2 &= (x_2 - x_1, y_2 - y_1, z_2 - z_1) & \mathbf{A}_1 \mathbf{A}_3 &= (x_3 - x_1, y_3 - y_1, z_3 - z_1) \\ \mathbf{A}_1 \mathbf{A}_2 \times \mathbf{A}_1 \mathbf{A}_3 &= \left( \begin{vmatrix} y_2 - y_1 & z_2 - z_1 \\ y_3 - y_1 & z_3 - z_1 \end{vmatrix}, - \begin{vmatrix} x_2 - x_1 & z_2 - z_1 \\ x_3 - x_1 & z_3 - z_1 \end{vmatrix}, \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix} \right) \\ \mathbf{A}_1 \mathbf{A}_0 &= (x_0 - x_1, y_0 - y_1, z_0 - z_1) \\ \mathbf{n}_0 &= \begin{cases} \frac{\mathbf{A}_1 \mathbf{A}_2 \times \mathbf{A}_1 \mathbf{A}_3}{|\mathbf{A}_1 \mathbf{A}_2 \times \mathbf{A}_1 \mathbf{A}_3|} & (\mathbf{A}_1 \mathbf{A}_2 \times \mathbf{A}_1 \mathbf{A}_3) \cdot \mathbf{A}_1 \mathbf{A}_0 < 0, \\ -\frac{\mathbf{A}_1 \mathbf{A}_2 \times \mathbf{A}_1 \mathbf{A}_3}{|\mathbf{A}_1 \mathbf{A}_2 \times \mathbf{A}_1 \mathbf{A}_3|} & (\mathbf{A}_1 \mathbf{A}_2 \times \mathbf{A}_1 \mathbf{A}_3) \cdot \mathbf{A}_1 \mathbf{A}_0 > 0. \end{cases} \end{aligned} \quad (13)$$

### 3 DG 格式的空间离散

为了求得在任意三角形单元  $K$  上的近似解

$$\mathbf{u}_h^K(\mathbf{X}, t) = \sum_{j=0}^3 \mathbf{u}_j^K(t) \varphi_j^K(\mathbf{X}) \quad (14)$$

我们要求近似解  $\mathbf{u}_h^K(\mathbf{X}, t)$  在  $K$  上满足方程

$$\int_K \frac{d}{dt} \mathbf{u}_h^K(\mathbf{X}, t) \varphi_j^K d\mathbf{X} = \int_K \mathbf{f}(\mathbf{u}_h^K(\mathbf{X}, t)) \cdot \nabla \varphi_j^K d\mathbf{X} - \int_{\partial K} \hat{\mathbf{f}}(\mathbf{X}, t) \varphi_j^K ds \quad (15)$$

其中  $j = 0, 1, 2, 3$ ,  $\hat{\mathbf{f}}(\mathbf{X}, t)$  为人为定义数值通量, 将在后面给出具体的定义。将表达式(11)代入上式, 并应用性质(12)得, 系数  $\mathbf{u}_j^K(t)$  满足常微分方程组:

$$\frac{d}{dt} \mathbf{u}_j^K(t) = \frac{1}{a_j} \left[ \int_K \mathbf{f}(\mathbf{u}_h^K(\mathbf{X}, t)) \cdot \nabla \varphi_j^K d\mathbf{X} - \int_{\partial K} \hat{\mathbf{f}}(\mathbf{X}, t) \varphi_j^K ds \right] \quad (16)$$

其中,  $a_j = \int_K (\varphi_j^K(\mathbf{X}))^2 d\mathbf{X} = \int_{\hat{K}} \hat{\varphi}_j^2(\boldsymbol{\eta}) |J| d\boldsymbol{\eta} = |J| (\hat{\varphi}_j, \hat{\varphi}_j)$ 。

方程右端的积分项可以用数值求积公式来进行计算:

$$\int_K \mathbf{f}(\mathbf{u}_h^K(\mathbf{X}, t)) \cdot \nabla \varphi_j^K(\mathbf{X}) d\mathbf{X} \approx |K| \sum_{m=0}^3 \frac{1}{4} \mathbf{f}(\mathbf{u}_h^K(\hat{\mathbf{X}}_m^K, t)) \cdot \nabla \varphi_j^K(\hat{\mathbf{X}}_m^K) \quad (17)$$

其中,  $\mathbf{u}_h^K(\hat{\mathbf{X}}_m^K, t) = \sum_{j=0}^3 \mathbf{u}_j^K(t) \varphi_j^K(\hat{\mathbf{X}}_m^K) = \sum_{j=0}^3 \mathbf{u}_j^K(t) \hat{\varphi}_j(\hat{\boldsymbol{\eta}}_m)$

$\hat{\boldsymbol{\eta}}_m$  为标准单元内的求积节点, 有  $\hat{\boldsymbol{\eta}}_0 = (\alpha, \beta, \beta)$ ,  $\hat{\boldsymbol{\eta}}_1 = (\beta, \alpha, \beta)$ ,  $\hat{\boldsymbol{\eta}}_2 = (\beta, \beta, \alpha)$ ,  $\hat{\boldsymbol{\eta}}_3 = (\beta, \beta, \beta)$ , 参数  $\alpha = 0.58541020$ ,  $\beta = 0.13819660$ 。

$$\nabla \varphi_j^K(\hat{\mathbf{X}}_m^K) = \left( \frac{\partial \varphi_j}{\partial x}, \frac{\partial \varphi_j}{\partial y}, \frac{\partial \varphi_j}{\partial z} \right) = \left( \frac{\partial \hat{\varphi}_j}{\partial \eta_0}, \frac{\partial \hat{\varphi}_j}{\partial \eta_1}, \frac{\partial \hat{\varphi}_j}{\partial \eta_2} \right) \begin{pmatrix} \frac{\partial \eta_0}{\partial x} & \frac{\partial \eta_0}{\partial y} & \frac{\partial \eta_0}{\partial z} \\ \frac{\partial \eta_1}{\partial x} & \frac{\partial \eta_1}{\partial y} & \frac{\partial \eta_1}{\partial z} \\ \frac{\partial \eta_2}{\partial x} & \frac{\partial \eta_2}{\partial y} & \frac{\partial \eta_2}{\partial z} \end{pmatrix} \quad (18)$$

$$\begin{aligned} \int_{\partial K} \hat{\mathbf{f}}(\mathbf{X}, t) \varphi_j^K(\mathbf{X}) ds &= \sum_{i=0}^3 \int_{\partial K_i} \hat{\mathbf{f}}(\mathbf{X}, t) \varphi_j^K(\mathbf{X}) ds \\ &\approx \sum_{i=0}^3 \left( |\partial K_i| \sum_{m=0}^2 \frac{1}{3} \hat{\mathbf{f}}(\bar{\mathbf{X}}_m^{\partial K_i}, t) \varphi_j^K(\bar{\mathbf{X}}_m^{\partial K_i}) \right) = \sum_{i=0}^3 \left( |\partial K_i| \sum_{m=0}^2 \frac{1}{3} \hat{\mathbf{f}}(\bar{\mathbf{X}}_m^{\partial K_i}, t) \hat{\varphi}_j(\bar{\boldsymbol{\eta}}_m^{\partial K_i}) \right) \end{aligned} \quad (19)$$

其中,  $\bar{\boldsymbol{\eta}}_m^{\partial K_i}$  为标准单元边界  $\partial K_i$  上的求积节点,  $|\partial K_i|$  表示面积, 如图2。

$$\begin{aligned} \bar{\mathbf{X}}_0^{\partial K_3} &= \frac{2}{3} \mathbf{X}_0 + \frac{1}{6} \mathbf{X}_1 + \frac{1}{6} \mathbf{X}_2 & \bar{\boldsymbol{\eta}}_0^{\partial K_3} &= \left( \frac{2}{3}, \frac{1}{6}, \frac{1}{6} \right) \\ \bar{\mathbf{X}}_1^{\partial K_3} &= \frac{1}{6} \mathbf{X}_0 + \frac{2}{3} \mathbf{X}_1 + \frac{1}{6} \mathbf{X}_2 & \bar{\boldsymbol{\eta}}_1^{\partial K_3} &= \left( \frac{1}{6}, \frac{2}{3}, \frac{1}{6} \right) \\ \bar{\mathbf{X}}_2^{\partial K_3} &= \frac{1}{6} \mathbf{X}_0 + \frac{1}{6} \mathbf{X}_1 + \frac{2}{3} \mathbf{X}_2 & \bar{\boldsymbol{\eta}}_2^{\partial K_3} &= \left( \frac{1}{6}, \frac{1}{6}, \frac{2}{3} \right) \end{aligned} \quad (20)$$

类似的，可以得到  $\partial K_2, \partial K_1, \partial K_0$  上求积节点的体积坐标：

$$\begin{aligned}\bar{\eta}_0^{\partial K_2} &= \left(\frac{2}{3}, \frac{1}{6}, 0\right) & \bar{\eta}_1^{\partial K_2} &= \left(\frac{1}{6}, \frac{2}{3}, 0\right) & \bar{\eta}_2^{\partial K_2} &= \left(\frac{1}{6}, \frac{1}{6}, 0\right) \\ \bar{\eta}_0^{\partial K_1} &= \left(\frac{2}{3}, 0, \frac{1}{6}\right) & \bar{\eta}_1^{\partial K_1} &= \left(\frac{1}{6}, 0, \frac{2}{3}\right) & \bar{\eta}_2^{\partial K_1} &= \left(\frac{1}{6}, 0, \frac{1}{6}\right) \\ \bar{\eta}_0^{\partial K_0} &= \left(0, \frac{2}{3}, \frac{1}{6}\right) & \bar{\eta}_1^{\partial K_0} &= \left(0, \frac{1}{6}, \frac{2}{3}\right) & \bar{\eta}_2^{\partial K_0} &= \left(0, \frac{1}{6}, \frac{1}{6}\right)\end{aligned}\quad (21)$$

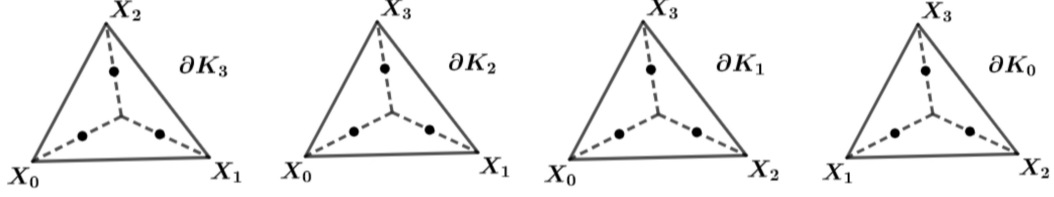


Figure 2: 边界处的求积节点

定义单元边界点  $\mathbf{X}$  处的数值通量

$$\hat{\mathbf{f}}(\mathbf{X}, t) = \hat{\mathbf{f}}\left(\mathbf{u}_h^K(\mathbf{X}, t), \mathbf{u}_h^{K'}(\mathbf{X}, t)\right) \quad (22)$$

其中  $\mathbf{u}_h^K(\mathbf{X}, t)$  与  $\mathbf{u}_h^{K'}(\mathbf{X}, t)$  分别表示相邻单元  $K$  与  $K'$  上的数值解在  $\mathbf{X}$  处的取值，有

$$\mathbf{u}_h^K(\bar{\mathbf{X}}_m, t) = \sum_{j=0}^3 \mathbf{u}_j^K(t) \varphi_j^K(\bar{\mathbf{X}}_m) = \sum_{j=0}^3 \mathbf{u}_j^K(t) \hat{\varphi}_j(\bar{\eta}_m) \quad (23)$$

在计算  $\mathbf{u}_h^{K'}(\bar{\mathbf{X}}_m, t)$  时，需要用到点  $\bar{\mathbf{X}}_m$  在单元  $K'$  上对应的体积坐标。

实际计算中，常用  $LF$  通量：

$$\hat{\mathbf{f}}(\mathbf{X}, t) = \frac{1}{2} \left[ \mathbf{f}(\mathbf{u}_h^K(\mathbf{X}, t)) \cdot \mathbf{n} + \mathbf{f}(\mathbf{u}_h^{K'}(\mathbf{X}, t)) \cdot \mathbf{n} - \alpha_K(\mathbf{X}, t)(\mathbf{u}_h^{K'}(\mathbf{X}, t) - \mathbf{u}_h^K(\mathbf{X}, t)) \right] \quad (24)$$

其中  $\mathbf{n} = (n_x, n_y, n_z)$  表示在四面体单元  $K$  的面上的单位外法向量

$$\alpha_K(\mathbf{X}, t) = \max \left\{ \lambda(\mathbf{u}_h^K(\mathbf{X}, t)), \lambda(\mathbf{u}_h^{K'}(\mathbf{X}, t)) \right\} \quad (25)$$

$\lambda(\mathbf{u})$  为  $\frac{\partial(\mathbf{f}(\mathbf{u}) \cdot \mathbf{n})}{\partial \mathbf{u}}$  的谱半径，有

$$\lambda(\mathbf{u}) = |\mathbf{v} \cdot \mathbf{n}| + c_s = \left| \frac{u_1}{u_0} n_x + \frac{u_2}{u_0} n_y + \frac{u_3}{u_0} n_z \right| + c_s \quad (26)$$

其中  $c_s = \sqrt{\frac{\gamma p}{\rho}}$  为声速。

## 4 时间离散

关于系数函数  $\mathbf{u}_j^K(t)$ ,  $\forall K \in \mathcal{T}_h, j = 0, 1, 2, 3$  的常微分方程组(16)可以将其简写为:

$$\frac{d\mathbf{u}_j^K(t)}{dt} = \mathbf{L}_j^K(\mathbf{u}_h^K(\mathbf{X}, t), \mathbf{u}_h^{K'}(\mathbf{X}, t), \beta_h(\mathbf{X}, t)) \quad j = 0, 1, 2, 3 \quad \forall K \in \mathcal{T}_h \quad (27)$$

其中,

$$\begin{aligned} & \mathbf{L}_j^K(\mathbf{u}_h^K(\mathbf{X}, t), \mathbf{u}_h^{K'}(\mathbf{X}, t), \beta_h(\mathbf{X}, t)) \\ &= \frac{|K|}{4a_j} \sum_{m=0}^3 \mathbf{f}(\mathbf{u}_h^K(\hat{\mathbf{X}}_m^K, t)) \cdot \nabla \varphi_j^K(\hat{\mathbf{X}}_m^K) \\ & - \sum_{i=0}^3 \frac{|\partial K_i|}{3a_j} \sum_{m=0}^2 \hat{\mathbf{f}}(\bar{\mathbf{X}}_m^{\partial K_i}, t) \varphi_j^K(\bar{\mathbf{X}}_m^{\partial K_i}) \end{aligned} \quad (28)$$

$\beta_h(\mathbf{X}, t)$  表示数值解在计算区域的边界  $\partial\Omega$  上的取值, 由具体的边界条件给出, 在紧邻边界的网格上计算数值通量时需要用到。

将要计算的时间区间  $[0, T]$  划分成剖分  $0 = t^0 < t^1 < \dots < t^n < \dots < t^N = T$ , 记  $\mathbf{u}_j^{K,n}$  为系数函数  $\mathbf{u}_j^K(t)$  在任意  $t^n$  时刻的近似, 结合空间离散近似(14)可得在  $t^n$  时刻解  $\mathbf{u}(\mathbf{X}, t)$  在单元  $K$  上的全离散近似为

$$\mathbf{u}(\mathbf{X}, t^n) \approx \sum_{j=0}^3 \mathbf{u}_j^{K,n} \varphi_j(\mathbf{X}) \quad (29)$$

为了求得所有的系数  $\mathbf{u}_j^{K,n}$ , 我们采用二阶 TVD Runge-kutta 方法进行离散常微分方程组(27), 记第  $n$  步的时间步长  $\Delta t^n = t^{n+1} - t^n$ , 那么  $\mathbf{u}_j^{K,n}$  的计算格式为

$$\begin{cases} \mathbf{u}_j^{K,n+\frac{1}{2}} = \mathbf{u}_j^{K,n} + \Delta t^n \mathbf{L}_j^K(\mathbf{u}_h^{K,n}, \mathbf{u}_h^{K',n}, \beta_h(t^n)) & j = 0, 1, 2 \\ \mathbf{u}_j^{K,n+1} = \frac{1}{2}(\mathbf{u}_j^{K,n} + \mathbf{u}_j^{K,n+\frac{1}{2}}) + \frac{1}{2} \Delta t^n \mathbf{L}_j^K(\mathbf{u}_h^{K,n+\frac{1}{2}}, \mathbf{u}_h^{K',n+\frac{1}{2}}, \beta_h(t^{n+1})) \end{cases} \quad (30)$$

在上述推进计算中需给初始值  $\mathbf{u}_j^{K,0}$ , 根据初始条件  $\mathbf{u}(\mathbf{X}, t = 0) = \mathbf{u}_0(\mathbf{X})$ , 可设定初始值

$$\begin{aligned} \mathbf{u}_j^{K,0} &= \frac{1}{a_j} \int_K \mathbf{u}_0(\mathbf{X}) \varphi_j^K(\mathbf{X}) d\mathbf{X} \\ &\approx \frac{|K|}{4a_j} \sum_{m=0}^3 \mathbf{u}_0(\hat{\mathbf{X}}_m^K) \varphi_j^K(\hat{\mathbf{X}}_m^K) \\ &= \frac{|K|}{4a_j} \sum_{m=0}^3 \mathbf{u}_0(\mathbf{X}(\hat{\boldsymbol{\eta}}_m)) \hat{\varphi}_j(\hat{\boldsymbol{\eta}}_m) \quad j = 0, 1, 2, 3 \end{aligned} \quad (31)$$

由于使用的是时间显格式, 时间步长  $\Delta t^n = t^{n+1} - t^n$  需满足限制条件

$$\Delta t^n = \frac{CFL}{\max_{K \in \mathcal{T}_h} \left[ (\|\mathbf{v}\| + c_s) \cdot \frac{\text{surfacearea}(K)}{|K|} \right]} \quad (32)$$

其中,  $\frac{\text{surfacearea}(K)}{|K|}$  为四面体单元  $K$  的表面积与体积之比, CFL 条件数取 0.3,

$$c_s + \|\mathbf{v}\| \triangleq \sqrt{v_1^2 \left( \bar{\mathbf{u}}_h^{K,n} \right) + v_2^2 \left( \bar{\mathbf{u}}_h^{K,n} \right) + v_3^2 \left( \bar{\mathbf{u}}_h^{K,n} \right) + c_s \left( \bar{\mathbf{u}}_h^{K,n} \right)} \quad (v_1, v_2, v_3 \text{ 表示速度}) \quad (33)$$

其中,  $\bar{\mathbf{u}}_h^{K,n} = \frac{1}{|K|} \int_K \mathbf{u}_h^{K,n}(\mathbf{X}) d\mathbf{X} = \mathbf{u}_0^{K,n}$  为  $t^n$  时的单元均值。

下面我们将每个单元  $K$  上的计算进行向量化。为此, 我们记

$$\mathbf{u}^{K,n} = \begin{bmatrix} \mathbf{u}_0^{K,n} & \mathbf{u}_1^{K,n} & \mathbf{u}_2^{K,n} & \mathbf{u}_3^{K,n} \end{bmatrix}_{4 \times 4}, \quad \mathbf{L}^{K,n} = \begin{bmatrix} \mathbf{L}_0^{K,n} & \mathbf{L}_1^{K,n} & \mathbf{L}_2^{K,n} & \mathbf{L}_3^{K,n} \end{bmatrix}_{4 \times 4} \quad (34)$$

我们考虑  $\mathbf{L}_j^K \left( \mathbf{u}_h^{K,n}, \mathbf{u}_h^{K',n}, \beta_h(t^n) \right)$  的计算。我们有

$$\mathbf{u}_h^{K,n} \left( \bar{\mathbf{X}}_m^{\partial K_i} \right) = \sum_{j=0}^3 \mathbf{u}_j^{K,n} \varphi_j^K \left( \bar{\mathbf{X}}_m^{\partial K_i} \right) = \mathbf{u}^{K,n} \begin{bmatrix} \varphi_0^K \left( \bar{\mathbf{X}}_m^{\partial K_i} \right) \\ \varphi_1^K \left( \bar{\mathbf{X}}_m^{\partial K_i} \right) \\ \varphi_2^K \left( \bar{\mathbf{X}}_m^{\partial K_i} \right) \\ \varphi_3^K \left( \bar{\mathbf{X}}_m^{\partial K_i} \right) \end{bmatrix} = \mathbf{u}^{K,n} \begin{bmatrix} \hat{\varphi}_0 \left( \bar{\boldsymbol{\eta}}_m^{\partial K_i} \right) \\ \hat{\varphi}_1 \left( \bar{\boldsymbol{\eta}}_m^{\partial K_i} \right) \\ \hat{\varphi}_2 \left( \bar{\boldsymbol{\eta}}_m^{\partial K_i} \right) \\ \hat{\varphi}_3 \left( \bar{\boldsymbol{\eta}}_m^{\partial K_i} \right) \end{bmatrix} \quad (35)$$

$$\begin{aligned} \hat{\mathbf{f}} \left( \bar{\mathbf{X}}_m^{\partial K_i}, t^n \right) &= \frac{1}{2} \left[ \mathbf{f} \left( \mathbf{u}_h^{K,n} \left( \bar{\mathbf{X}}_m^{\partial K_i} \right) \right) + \mathbf{f} \left( \mathbf{u}_h^{K',n} \left( \bar{\mathbf{X}}_m^{\partial K_i} \right) \right) \right] \cdot \mathbf{n} \\ &\quad - \frac{\alpha_K \left( \bar{\mathbf{X}}_m^{\partial K_i}, t^n \right)}{2} \left[ \mathbf{u}_h^{K',n} \left( \bar{\mathbf{X}}_m^{\partial K_i} \right) - \mathbf{u}_h^{K,n} \left( \bar{\mathbf{X}}_m^{\partial K_i} \right) \right] \end{aligned} \quad (36)$$

且根据定义(25)以及谱半径表达式(26)可得

$$\alpha_K \left( \bar{\mathbf{X}}_m^{\partial K_i}, t^n \right) = \max \left\{ \lambda \left( \mathbf{u}_h^{K,n} \left( \bar{\mathbf{X}}_m^{\partial K_i} \right) \right), \lambda \left( \mathbf{u}_h^{K',n} \left( \bar{\mathbf{X}}_m^{\partial K_i} \right) \right) \right\} \quad (37)$$

$$\begin{aligned} \lambda \left( \mathbf{u}_h^{K,n} \left( \bar{\mathbf{X}}_m^{\partial K_i} \right) \right) &= \left| \frac{u_1^{K,n} \left( \bar{\mathbf{X}}_m^{\partial K_i} \right)}{u_0^{K,n} \left( \bar{\mathbf{X}}_m^{\partial K_i} \right)} n_x + \frac{u_2^{K,n} \left( \bar{\mathbf{X}}_m^{\partial K_i} \right)}{u_0^{K,n} \left( \bar{\mathbf{X}}_m^{\partial K_i} \right)} n_y + \frac{u_3^{K,n} \left( \bar{\mathbf{X}}_m^{\partial K_i} \right)}{u_0^{K,n} \left( \bar{\mathbf{X}}_m^{\partial K_i} \right)} n_z \right| \\ &+ \sqrt{\frac{\gamma(\gamma-1) \left( u_4^{K,n} \left( \bar{\mathbf{X}}_m^{\partial K_i} \right) - \frac{1}{2u_0^{K,n} \left( \bar{\mathbf{X}}_m^{\partial K_i} \right)} \left( (u_1^{K,n} \left( \bar{\mathbf{X}}_m^{\partial K_i} \right))^2 + (u_2^{K,n} \left( \bar{\mathbf{X}}_m^{\partial K_i} \right))^2 + (u_3^{K,n} \left( \bar{\mathbf{X}}_m^{\partial K_i} \right))^2 \right) \right)}{u_0^{K,n} \left( \bar{\mathbf{X}}_m^{\partial K_i} \right)}} \end{aligned} \quad (38)$$

## 5 边界条件处理

边界条件处理流体力学方程计算中边界条件通常有如下四种:

1. 周期边界: 在我们的数值例子中, 当计算边界上的数值流时, 对应单元直接赋值即可;
2. 入流边界: 密度、速度、压力这些物理量都已经指定好, 直接使用这些物理量计算;
3. 出流边界: 令边界外部的物理量值等于边界内部的值;

4. 反射边界，我们把边界内部的法向速度改变符号后赋给边界外部，而其他方向的速度、密度与压力复制给边界外部.

在处理边界条件时有两种方法：

- 将靠近  $\partial\Omega$  的单元  $K$  沿着边界处翻转过来，设置一个虚拟单元  $K'$ ，进行编号。跟区域  $\Omega$  内部的单元  $K$  一样，在  $K'$  上定义数值解。这样做的好处是：计算数值通量时，在区域  $\Omega$  的内部单元与边界单元处可以统一处理。
- 不设置虚拟单元，在区域边界  $\partial\Omega$  处计算数值通量时需用到数值解在边界外侧的极限值，它可以由边界条件直接给出。