

More TLA+ Examples

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Chapter 1

Introduction

1.1 Catching Problems Early

Years ago, I worked on a proprietary low power processor in an embedded system. The processor ran microcode featuring a custom instruction set. To enter (or exit) a low power state, a set of (possibly hundreds) instructions had to be executed. These instructions progressively puts the system in lower power state. For example, turn off IP A, then turn off IP B, then turn off power island to the IPs, etc. To save cost and power, the low power processor had very little debug support.

An experienced reader may start to notice some redflags.

When the system enters a low power state, it executes possibly hundreds of instruction. If the microcode attempts to access the memory interface when the power island has been shut off, the system would hang. Since the power island has been shut off, the JTAG port is also shut off, leaving the developer with *no way* of live debugging related problem. at this point the developer needs to siphon through (possibly hundreds) of instructions to catch invariant violation *manually*.

Since the proprietary low power processor was fairly simple (as far as ASIC design goes) and only had a handful of instructions, I was able to create an emulator to verify the microcode prior to deploying it on target. The emulator tracks the system state on every command execution and confirm every step along the way none of the system invariants have been violated. This included stuff such as:

- Accessing memory interface after power off leads to a hang
- Accessing certain register in certain chip revision leads to a hang
- Verify IPs are shut off in the allowed order

- ... etc

The verification algorithm was implemented using a *depth-first search* algorithm, providing *100%* microcode coverage before deploy on target.

This was a very enlightening experience. To generalize, there are a class of problem in the industry where the solution space with very high *correctness* requirement, such as lockless/waitfree data structure, distributed algorithms, OS scheduler, etc. The failure cost to these problems tend to be high, because they are possibly extremely difficult to debug or reproduce. *Why isn't there a generic way of solving this problem?*

1.2 The Generalized Problem

Fast forward to now, as I was learning about distributed algorithms, I found out about TLA+, a formalized solution to what I was looking for.

Leslie Lamport invented the TLA+ 1999, but didn't appear to have caught on until the 2010's. I'm of the opinion that TLA+ was invented ahead of its time, and recently the problem complexity finally caught up to allow the tool to more visibly demonstrate its strength.

We are also at a point in the technology curve where vertical scaling is no longer physically practical, and the industry is exploring horizontal scaling solution. This is apparent in a few regards: CPU speed has plateau in the past decade or so. Hardware vendor focuses on scaling horizontally by adding more cores. Software vendors finds it more financially attractive to buy two of the same machine than buying the same machine with twice the capability, etc. All of these motivates software or hardware solution in the horizontal scaling space.

One slight problem: *human are inherently bad at reasoning things concurrently*. Humans are fundamentally single-threaded machines. Reasoning things that execute in parallel is possible, but difficult. It is hard to enumerate all possible scenario in one's mind to ensure the design accommodates all the edge cases.

Consider a distributed system. The system is a cluster of independently operating entities and need to somehow collectively offer the correct system behaviour, while any one of the machines may receive instructions out of order, crash, recover, etc.

Consider a single producer multiple consumer lockless queue. The consumers may reserve an index in the queue in certain order, but may release them in different order. What if one reader is really slow, and another reader is super fast and possibly lapse the slow reader?

Consider a OS scheduler with locks. Assume all the processes have the same priority - can a process starve the other processes by repeatedly acquire and release the lock?

1.3 TLA+

The usual anti-pattern is to keep bandaiding the solution until bug reports stop comming in - but how does anyone know if the solution is actually *correct by design*? To solve this problem, we then must rely on tools to do the reasoning *for us*.

TLA+ is a *system specification language*, with the intent to describe the system with implementation details removed. TLA+ allows designer to describe the system as a sequence of states. The designer can expresses transition condition from one state to another, describe invariants that must hold true in every state and liveness properties that the overall system should converge to. The key innovation of TLA+ is once the system is modeled as a finite state machine, the states can be *exhaustively* explored (via breath-first-search) to ensure certain properties are held through out the entire state space (either per state or a sequence of states).

This book was motivated by the intent to solve problems. The book is designed to be example heavy with many chapter each represnting an problem that can be modelled using TLA+. The book will also contain a few chapters that discuss topics that deserve extra attention.

Chapter 2

TLA+ Primer

2.1 Design Intent

The key insight into TLA+ is modelling a system as a state machine. A blinking LED system can be described using a single variable with two states, LED being on or off. A simple digital clock can be represented by two variables, hour and minute and the number of possible states in a digital clock is $24 * 60 = 1440$. For example, 10:01 is the next state 10:00 can transition to. Extrapolating further, Assume an arbitrarily system described by N variables, each variable having K possible values such arbitrary system can have up to N^K state.

For every specification, designer can specify *safety* property (or invariants) that must be true in *every* states. For example, in any state of the digital clock hour *must* be between 0 to 23, or formally described as $hour \in 0..23$. Similarly, $minute \in 0..59$. More generic invariant examples include: in any state, only one thread has exclusive access to a critical region, all variables in the system are within allowable value, the resource allocation manager never allocates more than available resources, etc.

Designer can also specify *liveness* property. These are properties that are satisfied by a *sequence of state*. One liveness property for the digital clock could be when the clock is 10 : 00, it will eventually become 11 : 00 (10 : 00 *leads to* 11 : 00). More generic liveness property include: a distributed system eventually converges, the scheduler eventually schedules every tasks in the task queue, the resource allocation manager fairly allocates resources, etc.

TLC checks a TLA+ spec using *breath-first search* algorithm to explore *all* states in the state machine and ensure safety and liveness properties are upheld. TLA+ specifies the system using *propositional logic*.

2.2 Requirement

In this example, we will specify a *digital clock*. The digital clock has a few simple requirements:

- Two variables to represent state: hour and minute
- The clock increment one minute at a time
- The clock wraps around at midnight (ie. 23:59 transitions to 00:00)

2.3 Spec

The *Init* state of such system can be described as:

$$\begin{aligned} Init &\triangleq \\ &\wedge hour = 0 \\ &\wedge minute = 0 \end{aligned}$$

\triangleq is the *defines equal* symbol and \wedge is the *logical and* symbol. The above TLA+ syntax can be read as *Init* state is defined as both hour and minute are both 0.

The spec also always include a *Next* definition, an *action formula* describing how the system transition from one state to another. Action formula contains *primed* variables what happens to the variable in its next state. The *Next* action for the digital clock can be defined as:

$$\begin{aligned} NextHour &\triangleq \\ &\wedge minute = 59 \\ &\wedge hour' = (hour + 1) \% 24 \\ &\wedge minute' = 0 \\ NextMinute &\triangleq \\ &\wedge minute \neq 59 \\ &\wedge hour' = hour \\ &\wedge minute' = minute + 1 \\ Next &\triangleq \\ &\vee NextMinute \\ &\vee NextHour \end{aligned}$$

Here's a breakdown of what the spec does:

- *Next* can take *NextMinute* or *NextHour*
- *Next* takes *NextMinute* when *minute* is not 59, next hour is hour, next minute is minute + 1.

- *Next* takes *NextHour* when *minute* is 59, next hour is (hour + 1) modulus 24, next minute set to 0

Technically it's possible for *Next* to take both *NextMinute* and *NextHour*. This is not possible in this definition as *NextHour* and *NextMinute* are defined in a *mutually exclusively* fashion.

Finally, the spec itself is formally defined as:

$$\begin{aligned} vars &\triangleq \langle hour, minute \rangle \\ Spec &\triangleq \\ &\quad \wedge Init \\ &\quad \wedge \Box [Next]_{vars} \end{aligned}$$

$\Box [Next]_{vars}$ deserves some special attention:

- *vars* is defined to be *all* variables in the spec. Different combination of these variables constitute the states of the system (eg. 23:59 and 00:00 are both states in the system).
- $\Box [Next]_{vars}$ is a *box-action formula*, where *Next* is an action and *vars* is a state function.
- \Box operator asserts the formula is always true for every step in the behaviour.
- And steps in the behaviour is defined as $[Next]_{vars}$, where *Next* describe the action and *vars* capturing all variables representing the state.

2.3.1 Safety

Safety property describes invariant that must hold true in every state of system. A common invariant is *type safety* checks. In a digital clock, hour can only be in value between 0 to 23, and minute can only be value of 0 to 59:

$$\begin{aligned} Type_OK &\triangleq \\ &\quad \wedge hour \in 0 \dots 23 \\ &\quad \wedge minute \in 0 \dots 59 \end{aligned}$$

2.4 Liveness

Liveness property verifies certain behavioural across a sequence of state. One liveness property can be confirming the clock wraps around correctly at midnight (which involves multiple states):

$$Liveness \triangleq$$

$$\wedge \text{hour} = 23 \wedge \text{minute} = 59 \leadsto \text{hour} = 0 \wedge \text{minute} = 0$$

\leadsto is the *leads to* operator, suggesting something is eventually true. TLA+ provides a set of formulas that can be used to describe liveness property.

To verify liveness, we need to modify the spec slightly to enable *fairness* to prevent *stuttering*. In plain terms, fairness ensure *something* always happen in every step, allowing the states to transition. Without fairness the spec is allowed to *do nothing* as next step, this means liveness condition may fail because the spec permits the system to do nothing in perpetuity as next state. Fairness will be covered in more detailed in later chapter.

$$\begin{aligned} \text{Spec} &\triangleq \\ &\wedge \text{Init} \\ &\wedge \Box[\text{Next}]_{\text{vars}} \\ &\wedge \text{WF}_{\text{vars}}(\text{Next}) \end{aligned}$$

$\text{WF}_{\text{vars}}(\text{Next})$ is the fairness qualifier.

2.5 TLC Model Checker

The TLA+ spec can be verified using TLC model checker. The TLC model checker runs the spec and verifies all configured safety and liveness properties are satisfied during execution. To run TLC, we need two things:

- clock.tla - the spec itself
- clock.cfg - the corresponding configuration file

For reference, clock.tla spec is listed below:

<pre> EXTENDS <i>Naturals</i> VARIABLES <i>hour, minute</i> <i>vars</i> \triangleq $\langle \text{hour}, \text{minute} \rangle$ <i>Type_OK</i> \triangleq $\wedge \text{hour} \in 0 \dots 23$ $\wedge \text{minute} \in 0 \dots 59$ <i>Liveness</i> \triangleq $\wedge \text{hour} = 23 \wedge \text{minute} = 59 \leadsto \text{hour} = 0 \wedge \text{minute} = 0$ <i>Init</i> \triangleq $\wedge \text{hour} = 0$ $\wedge \text{minute} = 0$ <i>NextMinute</i> \triangleq $\wedge \text{minute} = 59$ $\wedge \text{hour}' = (\text{hour} + 1) \% 24$ $\wedge \text{minute}' = 0$ </pre>	MODULE <i>clock</i>
---	---------------------

$$\begin{aligned}
NextHour &\triangleq \\
&\quad \wedge minute \neq 59 \\
&\quad \wedge hour' = hour \\
&\quad \wedge minute' = minute + 1 \\
Next &\triangleq \\
&\quad \vee NextMinute \\
&\quad \vee NextHour \\
Spec &\triangleq \\
&\quad \wedge Init \\
&\quad \wedge \Box [Next]_{vars} \\
&\quad \wedge WF_{vars}(Next)
\end{aligned}$$

The corresponding clock.cfg is listed below:

```

SPECIFICATION Spec
INVARIANTS Type_OK
PROPERTIES Liveness

```

Now run TLC and one should see something like this:

```

Model checking completed. No error has been found.
...
The depth of the complete state graph search is 1440.

```

Chapter 3

Blinking LED

Let's start with a trivial specification of a blinking LED. The intent of this example is to demonstrate the core functionalities of TLA+ specification language.

TODO: briefly talk about tla+ and model checker here.

3.1 Requirement

The LED is represented by a boolean variable that can be either 0 or 1.

... that's it.

3.2 Spec

The specification language may appear alienating as it is mathematically motivated based on propositional logic. Despite the (possibly) daunting syntax, designer only need to be familiar with a handful of key operators to start realizing value using TLA+. This chapter will attempt to describe the example in exhaustive detail to reduce the learning curve.

The following describe the core portion of the blinking LED spec.

MODULE *blinking*

VARIABLES *b*
vars \triangleq $\langle b \rangle$
Init \triangleq
 $\wedge b = 0$
On \triangleq
 $\wedge b = 0$
 $\wedge b' = 1$
Off \triangleq
 $\wedge b = 1$
 $\wedge b' = 0$

$$\begin{aligned}
Next &\triangleq \\
&\vee Off \\
&\vee On \\
Spec &\triangleq \\
&\wedge Init \\
&\wedge \Box[Next]_{vars}
\end{aligned}$$

- \triangleq is the *defines equal* operator
- \wedge and \vee are the AND and OR operator. The effect of these operator follow the natural definition in English:
 - $C \triangleq A \wedge B$: C is true iff A and B are true
 - $C \triangleq A \vee B$: C is true iff A or B is true
- The ' operator represents the next state. b' represent b's next state.
- *VARIABLES* keyword defines a list of variables for the spec. In this case the spec defines a variable b which can be either 0 or 1
- *vars* is typically defined as a shorthand to refer to *all* variables in the spec.

With the above definition, we can revisit the Action definitions: *Init* defines the initial system state, where b is set to 0.

Next requires more elaboration. TLA+ specifies the system as a collection of states with transitions between them. In a simplified sense, the state is described as a collection of ANDs (eg. system is in state C if both A and B are true), the ORs then describe the states the system can possibly be in (eg. system can be in state C OR D). Revisiting the example, the blinking LED has two states:

- $On \triangleq b = 0 \wedge b' = 1$: b switches on
- $Off \triangleq 1 \wedge b' = 0$: b switches off

The system's *Next* state is defined to be one of these states:
 $Next \triangleq On \vee Off$.

$\Box[Next]_{vars}$ is a **Box-Action Formula**, where *Next* is an action and *vars* is a state function. The formula is true iff every successive pair of steps in behaviour is a $[Next]_{vars}$. Finally *Spec* is conjunction between *Init* and $\Box[Next]_{vars}$. Note **all** TLA+ specification follows very similar template. There are situation we will need to provide *fairness* description - this will be covered later.

In short: this specification describes a two-state state machine where b toggles between 0 and 1.

Note that b can technically be *anything*. b can be 0, 1, -42, a dinosaur, etc. TLA+ specifies values of b which are valid in the system.

3.3 Safety

The spec so far only defines the possible states - but the *power* of TLA+ lies in its *properties* description. Safety properties are invariants that must hold true in *every* state. An invariant in the blinking LED example is:

$$TypeOK \triangleq b \in \{0, 1\}$$

This states the only valid value of b is 0 or 1. If b is ever set to anything else, the spec is invalid.

Some example safety properties include: Only a single thread have exclusive access to critical section, number of concurrent reads cannot exceed data available to be read, etc.

3.4 Liveness

While safety properties describe invariant that must be upheld in every state, *Liveness* describe properties of a sequence of states. In the blinking LED example, a liveness property can be the if b is 0, it eventually becomes 1, and vice versa. This is described below:

$$\begin{aligned} Liveness &\triangleq \\ &\wedge b = 0 \leadsto b = 1 \\ &\wedge b = 1 \leadsto b = 0 \end{aligned}$$

It is the author's opinion liveness describes the *design essence* behind the spec. The key characteristic of a system is described by its *behaviour* across a series of states. Does a distribute algorithm eventually converge to a working state? Does a resource manager fairly allocate resources in all scenarios? Does a scheduler ensure all tasks are eventually scheduled? These are behaviours that are *cannot* be concluded by looking at a single state, but across a *sequence of state*. Liveness allows designer to express and verify these properties.

3.5 Model Checking

Since the blinking LED is trivially specified, the full specification is included below. For subsequent chapters only snippet will be included. Please refer to the accompanied material for full spec source.

TODO: install toolchain

TODO: commandline

TODO: using TLC

The following is the content of *blinking.tla*:

```

MODULE blinking

EXTENDS Naturals

VARIABLES b

vars  $\triangleq \langle b \rangle$ 

TypeOK  $\triangleq$ 
 $\wedge b \in \{0, 1\}$ 

```

$$\begin{aligned}
Liveness &\triangleq \\
&\quad \wedge b = 0 \leadsto b = 1 \\
&\quad \wedge b = 1 \leadsto b = 0 \\
Init &\triangleq \\
&\quad \wedge b = 0 \\
Next &\triangleq \\
&\quad \vee \wedge b = 0 \\
&\quad \quad \wedge b' = 1 \\
&\quad \vee \wedge b = 1 \\
&\quad \quad \wedge b' = 0 \\
Spec &\triangleq \\
&\quad \wedge Init \\
&\quad \wedge \Box [Next]_{vars} \\
&\quad \wedge WF_{vars}(Next)
\end{aligned}$$

The following is the content of *blinking.cfg*:

```

SPECIFICATION Spec
INVARIANTS TypeOK
PROPERTIES Liveness

```

3.6 Limitation

Since TLA+ exhaustively explores all possible state, a linear growth of variables leads to TLC (temporal logic checker) execution time grows *exponentially*. This means the specification must be scoped correctly to limit the state space.

Similarly, if you want to verify concurrent psuedo code implementation in PlusCal, you can likely at most verify 10s of lines of code.

Chapter 4

SPSC Lockfree Queue

Single producer single consumer (SPSC) *Lockfree* queue is a standard data exchange queue between a producer and a consumer. The SPSC lockfree queue promises data can exchange between producer and consumer in a *lockfree* fashion, suggesting all condition both producer and consumer can make progress.

Contrast to standard shared queues, a SPSC waitfree queue doesn't require the use of a *lock* (eg. mutex). The queue can be logically represented fairly simply as:

```
template <typename T, ssize_t N>
class cQueue<T> {
    ssize_t rptr = 0;
    ssize_t wptr = 0;
    std::array<T, N> buffer;
    /* TODO: API definition below... */
};
```

A real implementation need to account for memory ordering effects specific to the architecture. For example, ARM has weak memory ordering model where read/write may appear out of order between CPUs. In this chapter we will only assume *logical* execution where each command is issued sequentially (even perceived across CPUs) to focus the discussion on TLA+.

4.1 Requirement

As mentioned in earlier section, a SPSC queue is represented by an array, a pair of read write pointer. The implementation is (hopefully) descriptively trivial:

- Two executing context, reader and writer
- Writer advances wptr after writes
- Reader advances rptr after reads

- If `rtpr` equals `wptr`, queue is empty
- If $(\text{wtpr} + 1) \% N$ equals `rprr`, queue is full

A possible implementation may look like below (not accounting for memory ordering effects):

```
template <typename T, ssize_t N>
class cQueue {
    ssize_t rprr = 0;
    ssize_t wptr = 0;
    std::array<T, N> buffer;

public:
    bool read(T &v) {
        /* queue empty check */
        if (rprr == wptr) {
            return false;
        }
        /* data get */
        v = buffer[rprr];
        /* rprr update */
        rprr = (rprr + 1) % N;
        return true;
    }

    bool write(const T &v) {
        /* queue full check */
        if ((wptr + 1) % N == rprr) {
            return false;
        }
        /* data write */
        buffer[wptr] = v;
        /* wptr update */
        wptr = (wptr + 1) % N;
        return true;
    }
};
```

Since reader and writer execute in different context, the instructions in read and write can interleave in *any* way imaginable:

- queue empty check can happen before or after queue full check
- data write happens immediately before data read
- ... so on and so forth

The key observation is that `buffer[wptr]` is reserved by the producer. `buffer[wptr]` is either unused or being written to. In either case the reader is not allowed to access it. Symmetric reasoning applies to `rprr`. This provides the *safety* to the design - but how do we verify this?

This is where TLA+ can help us formally verify the design.

4.2 Spec

TLA+ specification can be written using its native formal specification language, or a C-like syntax called PlusCal (which transpiles down to its native form). In this example, I chose to implement the specification using PlusCal, since the content to be verified is pseudo implementation. While it is possible to specify SPSC in native TLA+, it is the author's opinion that it is more error prone in this case, each line is effectively an individual state needs to be modeled.

The following is a snippet of the specification written in PlusCal, hopefully intuitive to read:

```
procedure reader(i)
variable
begin
  r_chk_empty:      if rptr = wptr then
  r_early_ret:      return ;
                      end if ;
  r_read_buf:       assert buffer[rptr] ≠ 0 ;
  r_cs:              buffer[rptr] := 0 ;
  r_upd_rptr:       rptr := (rptr + 1) % N ;
                      return ;
end procedure ;

procedure writer(i)begin
  w_chk_full:       if (wptr + 1) % N = rptr then
  w_early_ret:      return ;
                      end if ;
  w_write_buf:      assert buffer[wptr] = 0 ;
  w_cs:              buffer[wptr] := wptr + 1000 ;
  w_upd_wptr:       wptr := (wptr + 1) % N ;
                      return ;
end procedure ;
```

Note each command starts with a *label*, such as *r_chk_empty*. All the actions associated with the label is assumed executed atomically. This is reflected in the generated TLA+ code:

$$\begin{aligned}
 r_chk_empty(self) \triangleq & \wedge pc[self] = \text{"r_chk_empty"} \\
 & \wedge \text{IF } rp_tr = w_ptr \\
 & \quad \text{THEN } \wedge pc' = [pc \text{ EXCEPT } ![self] = \text{"r_early_ret"}] \\
 & \quad \text{ELSE } \wedge pc' = [pc \text{ EXCEPT } ![self] = \text{"r_read_buf"}] \\
 & \wedge \text{UNCHANGED } \langle rp_tr, w_ptr, buffer, stack, i_-, i \rangle
 \end{aligned}$$

4.3 Safety

As mentioned before, safety properties need to hold true in every single state. Some safety requirement we can enforce, for example:

Reader and writer cannot access the same index at the same time:

$$\sim ((pc[100] = "w_cs") \wedge (pc[101] = "r_cs") \wedge rptr = wptr) \quad (4.1)$$

All unused index should be set to 0:

$$\forall kk \in unused : buffer[kk] = 0 \quad (4.2)$$

At any given moment, buffer[wptr] may be unused or written. buffer[rptr] may be unused or read:

$$\begin{aligned} \vee Cardinality(to_be_read) + 1 &= Cardinality(reading) \\ \vee Cardinality(to_be_read) &= Cardinality(reading) + 1 \\ \vee Cardinality(to_be_read) &= Cardinality(reading) \end{aligned}$$

4.4 Liveness

All indices are eventually used:

$$\begin{aligned} Liveness &\triangleq \\ \forall k \in 0 \dots N - 1 : \\ \Diamond(buffer[k] \neq 0) \end{aligned}$$

Unused index 0 becomes used, used index 0 becomes unused.

$$\begin{aligned} Liveness2 &\triangleq \\ \wedge (buffer[0] = 0) \rightsquigarrow buffer[0] = 1000 \\ \wedge (buffer[0] = 1000) \rightsquigarrow buffer[0] = 0 \end{aligned}$$

4.5 Configuration

Chapter 5

SPMC Lockless Queue

Chapter 6

Simple Gossip Protocol

This section the author's notes on a simple gossip protocol by Andrew Hewler:
<https://ahelwer.ca/post/2023-11-01-tla-finite-monotonic/>

6.1 Requirement

In a distributed system, a cluster of nodes collectively provide a service. A distributed database may have a collection of 10s to 100s of nodes working together to offer the service in a geo diverse fashion to be immune to partial outage. The nodes often have requirements to know about each other. In the context of distributed database, a node may need to know the key range another of its peers. The cluster needs a way to communicate this information. One such mechanism is the gossip protocol.

Gossip protocols are used to communicate cluster information in a distributed fashion, (unsurprisingly) in a distributed system. Without gossip protocol, nodes in a cluster learn about its neighbours by contacting a centralized server. This introduces a single failure point in the system. As the name suggests, gossip protocol relies on nodes to gossip with each other. The nodes in the cluster periodically select a set of neighbors to exchange what it knows about the cluster. The recency information is part of the gossip message itself, allowing the node and the peer it's talking to quickly decide who has the latest information on a node, and converge to it. Assume a N node cluster and each internal a node selects k neighbours to gossip with, the total amount of gossip propagation time is described logarithmically below:

$$propagation_time = \log_k N * gossip_interval \quad (6.1)$$

With the total number of messages exchanged:

$$messages_exchanged = \log_k N * k \quad (6.2)$$

Now let's look at how a simple gossip protocol can be described by TLA+.

6.2 Spec

6.2.1 Base

In gossip protocol, every node needs to remember every other node's current state. In programming language this is typically described as `counter[]`. The following is the equivalent in TLA+:

$$Init \triangleq counter = [n \in Node \mapsto [o \in Node \mapsto 0]]$$

This defines `counter` a collection of nodes, where each node also contains a collection of nodes initialized to 0.

The nodes can move to a new version:

$$Increment(n) \triangleq counter' = [counter \text{ EXCEPT } ![n][n] = @ + 1]$$

Note only the `n`'s version is incremented. Communicating the update is done by the gossip action defined below:

$$\begin{aligned} Gossip(n, o) &\triangleq \\ &LET \ Max(a, b) \triangleq IF \ a > b \ THEN \ a \ ELSE \ b \\ &IN \ counter' = [\\ &\quad counter \text{ EXCEPT } ![o] = [\\ &\quad \quad nn \in Node \mapsto \\ &\quad \quad \quad Max(counter[n][nn], counter[o][nn]) \\ &\quad] \\ &] \end{aligned}$$

A few things to unpack here:

- n, o are the two nodes exchanging gossip. o is the node to be updated and n is the neighbor o gossips with.
- *LET..IN* allows local definition under *LET* used under *IN*. In this case *Max* is a local macro defined to return maximum between a and b .
- $counter'$ (or referred to as counter *prime*) is what the variable will be in the next state. TLA+ doesn't provide a way to update a variable in a collection, so the convention is to assign a new array to the variable.
- $counter \text{ EXCEPT } ![o] = [...]$ return *counter* with $counter[o]$ defined in the bracket.
- where $[...]$ is a collection of nodes with with counter set to the max between the current node and neighbour.

Finally, the actual spec:

$$\begin{aligned} Next &\triangleq \vee \exists n \in Node : Increment(n) \\ &\quad \vee \exists n, o \in Node : Gossip(n, o) \end{aligned}$$

Next supports two possible next steps describe using disjunctions. The first is bumping the version of a random node, the second is select a pair of nodes to gossip. Note the *existential qualifier* on both, which basically states there

exists a node n in nodes, or there exists a pair of nodes n, o in nodes, respectively.

Finally, the actual spec definition:

$$\begin{aligned} Spec &\triangleq \wedge Init \\ &\quad \wedge \Box [Next]_{counter} \end{aligned}$$

The second conjunction formula is a **Box-Action Formula**, where $Next$ is an action and $counter$ is a state function. The formula is true iff every successive pair of steps in behaviour is a $[Next]_{counter}$. The spec defines a temporal theorem that is *always true*.

6.2.2 Finitized

There's a minor problem with the definition above. Gossip protocol, like many converging protocols, have a *monotonic increasing* requirement. On failures, the protocol bumps the version, which increases monotonically. Since TLA+ spec models the system as a graph, a monotonic increasing version number means the graph is *infinitely large*. To put the specification back into finite space, we can normalize the state:

$$\begin{aligned} GarbageCollect &\triangleq \\ &\text{LET } SetMin(s) \triangleq \text{CHOOSE } e \in s : \forall o \in s : e \leq o \text{ IN} \\ &\text{LET } Transpose \triangleq SetMin(\{counter[n][o] : n, o \in Node\}) \text{ IN} \\ &\quad \wedge counter' = [\\ &\quad \quad n \in Node \mapsto [\\ &\quad \quad \quad o \in Node \mapsto counter[n][o] - Transpose \\ &\quad \quad] \\ &\quad] \\ &\quad \wedge \text{UNCHANGED } converge \end{aligned}$$

$SetMin(s)$ implements standard TLA+ semantics to retrieve the minimum element in the set. The definition can be read as *choose an e from S such that for every o in S , o is equal or bigger than e* . $Transpose$ is then subsequently defined as the minimum value exist in counter. Finally, $counter'$ is updated such that *every* elements subtracts $Transpose$.

The increment function is now updated to:

$$\begin{aligned} Increment(n) &\triangleq \\ &\quad \wedge \neg converge \\ &\quad \wedge counter[n][n] < Divergence \\ &\quad \wedge S!Increment(n) \\ &\quad \wedge \text{UNCHANGED } converge \end{aligned}$$

The conjunction $counter[n][n] < Divergence$ limits the maximum counter value. Finally, the Next action is updated to the follow:

$$\begin{aligned} Next &\triangleq \\ &\quad \vee \exists n \in Node : Increment(n) \\ &\quad \vee \exists n, o \in Node : Gossip(n, o) \\ &\quad \vee Converge \\ &\quad \vee GarbageCollect \end{aligned}$$

Note *GarbageCollect* is a now part of possible state transition. We will discuss *Converge* later, as it is related to liveness check. Lastly:

$$\begin{aligned}
 \textit{Fairness} &\triangleq \forall n, o \in \textit{Node} : \textit{WF}_{vars}(\textit{Gossip}(n, o)) \\
 \textit{Spec} &\triangleq \\
 &\quad \wedge \textit{Init} \\
 &\quad \wedge \Box[\textit{Next}]_{vars} \\
 &\quad \wedge \textit{Fairness}
 \end{aligned}$$

The *Fairness* conjunction ensures Gossip runs between every pair of n and o .

Chapter 7

Raft Consensus Protocol

Chapter 8

Simple Scheduler

8.1 Requirement

In this section we will define a spec for a simple task scheduler. The task scheduler has the following requirements:

- Supporting N execution context (ie. CPUs)
- Supporting T number of tasks
- Tasks have identical priority and are scheduled cooperatively
- System has a single global lock
- Any task can attempt to acquire the lock, Any task attempting to acquire the lock are guaranteed to be scheduled.

Chapter 9

Simple Elevator

<https://surfingcomplexity.blog/2024/10/16/a-liveness-example-in-tla/>

Chapter 10

Miscellaneous