# Methods of Proof



## Arguments

Consider the sequence of statements.

If x is a pig, then x is pink.

Peppa is a pig.

Therefore, Peppa is pink.

* **Argument:** a sequence of statements, all but the final of which are called assumptions/premises/ hypotheses, and final of which is called the conclusion.
* The word “therefore” is normally placed just before the conclusion.
* The logical form of the above argument is:

If p, then q.

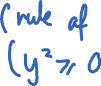
p.

Therefore, q.

* An argument is **valid** the conclusion is true whenever all the assumptions are true (no matter what particular statements are substituted for the variables).
* **Proof:** a valid argument used to establish a result.
* **Note:** The assumptions in an argument or a proofcan be axioms, previouslyproved theorems, or may follow from previous statements by a mathematical or logical rule.

Exercise:

Prove that if x ∈ ℝ and n ∈ ℕ is even, then xn ≥ 0.



* A proof should be complete (contain all necessary statements).
* A proof should be concise (not contain extra or unneeded statements).

## Testing Validity

* To test an argument for validity, follow these steps:
* Identify the assumptions and conclusion.
* Construct a truth table of all the assumptions and the conclusion.
* If the conclusion is true in every case where all the assumptions are true, the argument is valid.
* If there is a row of all true assumptions and false conclusion, the argument is invalid.

Exercise:

Is the argument valid?

P => Q ∨ ~R,

Q => P ∧ R,

.:. P => R.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| ***p*** | ***q*** | ***r*** | ***p => q* ∨ ~*r*** | ***q => p ∧ r*** | ***p => r*** |
| T | T | T |  |  |  |
| T | T | F |  |  |  |
| T | F | T |  |  |  |
| T | F | F |  |  |  |
| F | T | T |  |  |  |
| F | T | F |  |  |  |
| F | F | T |  |  |  |
| F | F | F |  |  |  |



Exercise:

Test the validity:

1. p ∨ (q ∨ r),  
   ~r,  
   ∴ p ∨ q



|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| ***p*** | ***q*** | ***r*** | ***~r*** | ***p ∨ (q* ∨ *r)*** | ***p ∨ q*** |
| T | T | T | F |  |  |
| T | T | F | T |  |  |
| T | F | T | F |  |  |
| T | F | F | T |  |  |
| F | T | T | F |  |  |
| F | T | F | T |  |  |
| F | F | T | F |  |  |
| F | F | F | T |  |  |



1. p => q,  
   p,  
   ∴ q.



|  |  |  |
| --- | --- | --- |
| ***p*** | ***q*** | ***p => q*** |
| T | T |  |
| T | F |  |
| F | T |  |
| F | F |  |



* An argument consisting of 2 premises and a conclusion is called a **syllogism**.
* The most famous syllogism is the **modus ponens**, Latin for “method of affirming.”  
    
  If p, then q,  
  p,  
  Therefore, q.

Exercise:

1. Is the statement “n ∈ ℕ is even => n2 is even” true? Prove it.
2. Let n = 9866. Is it true or false to say n2 is even?



## Principles of Mathematical Induction

* If p(n) is a statement with domp = ℕ such that

1. p(1) is true, and
2. p(k) true => p(k + 1) true,

Then p(n) is true for all n ∈ ℕ.

Exercise:

Prove that 4n – 1 is a multiple of 3 ∀ n ∈ ℕ.



## The Law of Syllogisms

* Is the following a tautology?

[([p => q) ∧ (q => r)] => p => r



|  |
| --- |
| **Law of Syllogism** |
| If p => q and q => r, then p => r |

Exercise:

Suppose these 2 statements are true.

1. If it rains today, then I’ll drive to school.
2. If I drive to school today, then I’ll go over my gas budget.

Then by the law of syllogism, we can infer another truth:

1. If it rains today, then I’ll go over my gas budget.

## Proving ∃ Statements

* How do we prove a statement of the form?

∃ x ∈ D ∋ p(x)

* We need to find at least one x ∈ D that makes p(x) true.

Exercise:

Prove that there exists an even number that can be written in two ways as the sum of two prime numbers.



Exercise:

Prove ∃ x ∈ ℝ ∋ x + 5 = 0



Exercise:

Prove there is a month of the year whose name has 3 letters.



## Proving ∀ Statements

* How do we prove a statement of the form:

∀ x ∈ D, p(x)

* There are two options:

1. **Method of Exhaustion**
2. **Generalized Proof**

* The method of exhaustion checks that p(x) is true for every x ∈ D.
* This is dine when D is small, but becomes a lot of work for D large.
* If D is infinite, this method fails to be of any use.

Exercise:

Prove that every even number between 4 and 16 can be written as the sum of 2 primes.



Exercise:

Prove that every even n ∈ ℕ can be written as the sum of 2 primes.



* The generalized proof is constructed so that it applies to every possible situation.

It takes as many nonspecific elements of D as needed and proves the statement, so that the proof is valid for all elements of D.

Exercise:

Prove that if a, b ∈ ℤ, then 10a + 8b is divisible by 2.



## Disproving ∃ Statements

* To disprove a statement means to prove its negation.
* Recall the negation of an existential statement:

**~(∃ x ∈ D ∋ p(x)) ≡ ∀ x ∈ D ∋ ~p(x)**

* To disprove an ∃ statement, we must prove a ∀ statement, via method of exhaustion or generalized proof.

Exercise:

Disprove the statement “there exists an even prime number larger than 2.”



* This is an example of proof by contradiction.

## Disproving ∀ Statements

* To disprove a ∀ statement, we must prove an ∃ statement.

**~(∀ x ∈ D, p(x)) ≡ ∃ x ∈ D ∋ ~p(x)**

* We must find one x ∈ D such that p(x) is false (a counterexample).

Exercise:

Disprove the statement “∀x **∈** ℝ, x < 0 ∨ x > 0.”



Exercise:

Disprove the statement “∀ a, b **∈** ℝ, if a2 = b2, then a = b.



Exercise:

Prove or disprove: “∀ x **∈** ℝ, ∃ y **∈** ℝ ∋ x + y = 0.”



## Generalized Proof 1: Direct Proof

* A **direct proof** works in a straightforward manner from assumptions to solution.
* We often rewrite assumptions in logical notation.

Exercise:

Prove that if 3x – 9 = 15, then x = 8.



Exercise:

Prove that the sum of any two even numbers is even.



Exercise:

Prove that if a, b are perfect squares, then ab is a perfect square.



Exercise:

Prove that ∀ x ∈ ℝ, -x2 + 2x + 1 ≤ 2.



## Generalised Proof 2: Proof by Contradiction

Exercise:

Prove that p => q ≡ ~q => ~p



|  |  |  |  |
| --- | --- | --- | --- |
| ***p*** | ***q*** | ***p => q*** | ***~q => ~p*** |
| T | T |  |  |
| T | F |  |  |
| F | T |  |  |
| F | F |  |  |

* To prove p => q, one may instead prove ~q => ~p.
* That is, assume that the negation of the conclusion is true, and show that one of the assumptions (or some other well-known truth) is false.

Exercise:

Prove “∀ n ∈ ℕ, if n2 is even, then n is even” by contradiction.



Exercise:

Prove by contradiction that y ∈ ℝ \ ℚ => y + 7 ∈ ℝ \ ℚ.



## Generalized Proof 3: Proof by Cases

* How do we prove “if x ≠0 or y ≠ 0, then x2 + y2 > 0”?
* We need to split the problem into cases, proving the conclusion first if x ≠ 0, the if y ≠ 0.
* Any statement of the form:

**(p ∨ q) => r**

Can be done this way, because of the logical equivalence

**(p ∨ q) => r ≡ (p => r) ∧ (q => r)**

Exercise:

Prove “x ≠ 0 or y ≠ 0 => x2 + y2 > 0.”



Exercise:

Prove that ∀ m ∈ ℕ, m2 + m + 1 is odd.

