# Induction

- Recall the induction principle:

  If $dom_p = \mathbb{N}$ such that,

  a) $p(1)$ is true

  b) $p(k)$ is true $\Rightarrow p(k+1)$ is true

  Then $p(n)$ is true $\forall\, n \in \mathbb{N}$

*Exercise:*

1) Prove that $1 + 2 \ldots + n = \frac{n(n+1)}{2}\ \forall\, n \in \mathbb{N}$

1) $p(n): \dfrac{n(n+1)}{2} \iff \displaystyle\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$

a) $p(1): \dfrac{1(1+1)}{2} = 1 \quad \checkmark$

b) Suppose $1 + 2 + \ldots + k = \dfrac{k(k+1)}{2} \iff \displaystyle\sum_{j=1}^{k} j = \frac{k(k+1)}{2}$

Prove that $1 + 2 + \ldots + k \quad \dfrac{(k+1)(k+2)}{2}$

$$\sum_{j=1}^{k+1} j = \frac{(k+1)(k+2)}{2} = \sum_{j=1}^{k} j = \frac{k(k+1)}{2} + (k+1) \quad \text{(By the supposition)}$$

$$= \frac{k^2 + k + 2k + 2}{2}$$

$$= \frac{k^2 + 3k + 2}{2}$$

$$= \frac{(k+1)(k+2)}{2} \quad \square$$

2) Prove that $n^3 > 2n - 2 \; \forall n \in \mathbb{N}$

$p(1)$: $1^3 > 2(1) - 2 \iff 1 > 0$ ✓

$p(k)$: Suppose $k^3 > 2k - 2$, prove that $(k+1)^3 > 2(k+1) - 2$

$$\begin{aligned} (k+1)^3 &= \underline{k^3} + 3k^2 + 3k + 1 \\ &> \underline{(2k-2)} + 3k^2 + 3k + 1 \quad \text{(By the suppositian)} \\ &= 3k^2 + 5k - 1 \end{aligned}$$

Substituted

$$\begin{aligned} (k+1)^3 > 2(k+1) - 2 &\iff 3k^2 + 5k - 1 > (2k+2) - 2 \\ &\iff 3k^2 + 5k - 1 > 2k \\ &\iff 3k^2 + 3k - 1 > 0 \\ &\iff 3k(k+1) > 1 \\ &\iff k(k+1) > \tfrac{1}{3} \quad \square \; ✓ \end{aligned}$$

$\therefore n^3 > 2n - 2 \; \forall n \in \mathbb{N}$

3) Prove that $(n + 1)! \geq 2^n \; \forall n \in \mathbb{N}$

$(n+1)! \geq 2^n$

a) $p(1)$: $(1+1)! \geq 2^1 \iff 2 \geq 2$ ✓

b) Suppose $(k+1)! \geq 2^k$ is true.

c) Prove that $((k+1)+1)! \geq 2^{k+1}$

$$\begin{aligned} (k+2)! &= (k+1)!\,(k+2) \quad \leftarrow \text{Substitute case } k \\ &\geq 2^k (k+2) \quad \text{(By the suppositian)} \\ &= k2^k + 2^1 \cdot 2^k \\ &\geq \underline{k2^k} + 2^{k+1} \\ &> 2^{k+1} \end{aligned}$$

from case $k$, already supposed to true

adding any number makes the sign $>$

$\therefore (n+1)! \geq 2^n \; \forall n \in \mathbb{N}$

4) Prove that $6 \mid (3n^2 + 3n)\ \forall n \in \mathbb{N}$ (a|b: "a divides b")

$p(n): 6 \mid (3n^2 + 3n)\ \forall n \in \mathbb{N}$

$p(1): 6 \mid 3(1)^2 + 3(1) = 6 \mid 6$ ✓

$p(k):$ Suppose $6 \mid (3k^2 + 3k)$

$p(k+1):$ Prove $6 \mid [3(k+1)^2 + 3(k+1)]$

$3(k+1)^2 + 3(k+1) = 3(k^2 + 2k + 1) + 3k + 3$

$= 3k^2 + 3k + 6k + 6$

$= (3k^2 + 3k) + 6(k+1)$

$(3k^2 + 3k) + (k+1) =$ (By the Suppositon)

? How did you go from * to

$6\left(\frac{3k^2+3k}{6} + k+1\right)$

Supposition, but why the 6

## Sigma Notation

We use capital sigma $\sum$ to shorten notation of long sums:

$$\sum_{i=0}^{k} a_i = a_1 + a_2 + a_3 + \cdots + a_k$$

*Exercise:*

Expand

a) $\sum_{i=2}^{6} 2i^2$

b) $\sum_{i=1}^{10} \underline{2}$

constant

a) $2 \cdot 2^2 + 2 \cdot 3^2 + 2 \cdot 4^2 + 2 \cdot 5^2 + 2 \cdot 6^2 = 180$

b) $2+2+2+2+2+2+2+2+2+2 = 20$

*Exercise:*

Simplify

a) $1 + 3 + 5 + \cdots + 13$
b) $1 + 4 + 9 + \cdots + m^2$

a) $\sum_{i=0}^{13} 2i + 1$ b) $\sum_{i=1}^{m} i^2$

- By the laws of addition, we have the following properties:

1)
$$\sum_{i=m}^{n} (a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i$$

2)
$$\sum_{i=m}^{n} k a_i = k \sum_{i=m}^{n} a_i$$

## Generalised Mathematical Induction

- Let $p(n)$ be defined for all $n \in \mathbb{N}$, and let $a \in \mathbb{N}$. If
  a) $p(a)$ is true, and
  b) For all $k \in \mathbb{N}, k \geq a,\ p(k)$ is true $\Rightarrow p(k+1)$ is true
  Then $p(n)$ is true for all $n \in \mathbb{N} \ni n \geq a$

*Exercise:*

Is $2^n > 2n + 1\ \forall\, n \in \mathbb{N}$?

If we tried $p(1)$: $2^1 > 2(1) + 1 \iff 2 \not> 3$

Prove that $2^n > 2n+1\ \forall n \geq 3$

$p(3)$: $2^3 > 2(3) + 1 \iff 8 > 7$ ✓

$p(k)$: $2^k > 2k+1$, $k \geq 3$, prove that $2^{k+1} > 2(k+1) + 1$

$p(k+1)$: $2^{k+1} > 2(k+1) + 1$

$2^{k+1} = 2 \cdot 2^k > 2\underline{(2k+1)} = 4k+2$ (By the supposition)

$4k+2 > 2(k+1)+1 \iff 4k+2 > 2k+3$

$\iff 2k+2 > 3$

$\iff 2k > 1$ $\square$

$\therefore 2^n > 2n+1\ \forall n \geq 3$

*Exercise:*

Prove that.

$$\sum_{i=1}^{n-1} \frac{i}{i+1} < \frac{n^2}{n+1} \;\forall\, n \geq 2$$

$P(2)$: $\sum_{i=1}^{2-1} \frac{i}{i+1} < \frac{k^2}{k+1} \iff \frac{1}{1+1} < \frac{2}{2+1} \iff \frac{1}{2} < \frac{4}{3}$ ✓

$P(k)$: Suppose $\sum_{i=1}^{k-1} \frac{i}{i+1} < \frac{k^2}{k+1}$. Prove $\sum_{i=1}^{k} \frac{i}{i+1} < \frac{(k+1)^2}{k+2}$

$$\sum_{i=1}^{k} \frac{i}{i+1} = \sum_{i=1}^{k-1} \frac{i}{i+1} + \frac{k}{k+1}$$

$$< \frac{k^2}{k+1} + \frac{k}{k+1} = \frac{k(k+1)}{k+1} = k = \frac{k(k+2)}{k+2} = \frac{k^2+2k}{k+2}$$

$$< \frac{k^2+2k+1}{k+2} = \frac{(k+1)^2}{k+2}$$ ✓ $\square$

$\therefore \sum_{i=1}^{n-1} \frac{i}{i+1} < \frac{n^2}{n+1} \;\forall\, n \geqslant 2$

## Recursive Sequences

- A sequence of numbers $a_1, a_2, a_3, \ldots$ is defined recursively if each $a_n$ for $n \geq n_0$ is defined in terms of some or all of $a_1, a_2, \ldots, a_{n_0}$.

*Exercise:*

Let $a_1 = 1, a_2 = 4, a_n = 5a_{n-1} - 6a_{n-2} \forall\, n \geq 3$. Find $a_3$ and $a_4$.

$$a_3 = 5a_2 - 6a_1 = 5 \cdot 4 - 6 \cdot 1 = 14$$

$$a_4 = 5a_3 - 6a_2 = 5 \cdot 14 - 6 \cdot 4 = 46$$

*Exercise:*

The Fibonacci numbers are the numbers in the famous sequence:

$$1, 1, 2, 3, 5, 8, 13, \ldots$$

This sequence is defined by:

$$f_1 = f_2 = 1, f_n = f_{n-2} + f_{n-1} \;\forall\, n \geq 3$$

Can we show that $f_n < 2^n \;\forall\, n \in \mathbb{N}$?

- Using induction

## Strong Mathematical Induction

- Let $p(n)$ be defined for all $n \in \mathbb{N}$m let $a \in \mathbb{N}$ if:
  a) $p(1), p(2), \ldots, p(a)$ are true, AND
  b) For all $k \in \mathbb{N}, k \geq a, p(k)$ is true $\Rightarrow p(k+1)$ is true,
  Then $p(n)$ is true for all $n \in \mathbb{N}$.

*Exercise:*
For the Fibonacci sequence $f_1 = f_2 = 1, f_n = f_{n-2} + f_{n-1}\ \forall\, n \geq 3$, prove that $f_n < 2^2\ \forall\, n \in \mathbb{N}$

a) $p(1): f_1: 1 < 2^1 = 2$ ✓
$p(2): f_2: 1 < 2^2 = 4$ ✓
$p(3): f_3: 2 < 2^3 = 8$ ✓

b) Suppose for $k \geq 3$, $f_1 < 2^1, f_2 < 2^2, \ldots, f_k < 2^k$.
Prove $f_{k+1} < 2^{k+1}$

c) $f_{k+1} = f_{k-1} + f_k < 2^{k-1} + 2^k < 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$ $\square$

$\therefore f_n < 2^n\ \forall n \in \mathbb{N}$

*Exercise:*
Let $a_1 = 2, a_2 = 4, a_n = 5a_{n-1} - 6a_{n-2}\ \forall\, n \geq 3$. Prove that $a_n = 2^n\ \forall\, n \in \mathbb{N}$.

① $a_1 = 2^1, a_2 = 2^2, a_3 = 5 \cdot 4 - 6 \cdot 2 = 8 = 2^3$. ✓

② Suppose for $k \geq 3$, $a_i = 2^i$ for $i = 1, 2, \ldots, k$. Prove $a_{k+1} = 2^{k+1}$

$a_{k+1} = 5a_k - 6a_{k-1} = 5 \cdot 2^k - 6 \cdot 2^{k-1}$
$= 5 \cdot 2^k - 3 \cdot 2^k = 2 \cdot 2^k = 2^{k+1}$ ✓

$\therefore a_n = 2^n\ \forall n \in \mathbb{N}$