Mahindra University, Hyderabad
MA 3115: Computational Methods for PDE

Problem sheet-III

Problem 1. Solve the BVP (Boundary Value Problem)

$$\frac{d^2y}{dx^2} = y + x, y(0) = 0, y(1) = 0$$

using finite difference method with $h = \dfrac{1}{4}$. Then compare with exact values.

Problem 2. Solve the following boundary value problem using finite difference method, with $\Delta x = \Delta y = \frac{1}{3}$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < 1, 0 < y < 1$$

$$u(x,0) = u(0,y) = 0, u(x,1) = x, u(1,y) = y, 0 \leq x \leq 1, 0 \leq y \leq 1$$

Problem 3. Solve the following BVP

$$\nabla^2 u = 0,\ 0 \leq r \leq 1,\ 0 \leq \theta \leq \pi$$

$$u(1,\theta) = \frac{4}{\pi}(\pi\theta - \theta^2),\ u(r,0) = u(r,\pi) = 0,\ u(0,\theta) < \infty.$$

with $\Delta r = \frac{1}{2}$ and $\Delta\theta = \frac{\pi}{4}$.

Problem 4. Perform two iterations to solve the following one dimensional heat equation using FTCS, BTCS and Crank−Nicolson schemes, with $\Delta x = \frac{1}{4}$.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\ 0 < x < 1, t > 0.$$

$$u(0,t) = 0, u_x(1,t) = 1,\ t \geq 0 \text{ and } u(x,0) = x,\ 0 \leq x \leq 1.$$

Problem 5. Discuss the stability and consistency of the following DuFort-Frankel scheme for solving 1D heat equation

$$\frac{U_{i,j+1} - U_{i,j-1}}{2\Delta t} = \frac{U_{i+1,j} - U_{i,j+1} - U_{i,j-1} + U_{i-1,j}}{(\Delta x)^2}$$

Problem 6. The equation

$$\alpha\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} - f(x,t) = 0, \alpha \in \mathbb{R}$$

is approximated at the point $(i\Delta x, j\Delta t)$ in the $xt-$ plane by the following difference scheme,

$$\frac{\alpha(U_{i,j+1} - 0.5(U_{i+1,j} + U_{i-1,j}))}{\Delta t} + \frac{U_{i+1,j} - U_{i-1,j}}{2\Delta x} - f_{i,j} = 0.$$

Then investigate the consistency of this scheme for (a) $\Delta t = r\Delta x$ and (b) $\Delta t = r(\Delta x)^2, r \in \mathbb{R}^+$.

Problem 7. Discuss the consistency of the following numerical scheme for solving 1D wave equation

$$\frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{(\Delta t)^2} = c^2 \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{(\Delta x)^2}.$$

Problem 8. Solve the following PDE

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},\ 0 < x < 1, t > 0.$$

$$u(x,0) = \sin(\pi x),\ \left(\frac{\partial u}{\partial t}\right)_{(x,0)} = 0,\ 0 \le x \le 1$$

$$u(0,t) = u(1,t) = 0,\ t \ge 0.$$

Problem 9. Discuss the stability of Lax-Wendroff explicit scheme to solve the following PDE,

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0,\ a > 0$$

Problem 10. Use Lax-Wendroff explicit scheme to solve the following PDE,

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0,\ u(0,t) = t, t > 0$$

$$u(x,0) = x(x-1),\ 0 \le x \le 1;\ u(x,0) = 1 - x,\ x \ge 1.$$

PROJECT (Weightage 30 %) (Date of submission: December 10, 2023)

Problem 11. Write a MATLAB program to solve the following BVPs: Problem 1, Problem 2, Problem 3, Problem 4, Problem 8 and Problem 10.

Practice Problems

Problem 12. Find values of a, b, c and d such that the following finite difference scheme gives the best possible discritization to first order derivative for a fixed step size h.

$$\frac{du}{dx} \approx a\ u(x+h) + b\ u(x) + c\ u(x-h) + d\ u(x-2h)$$

Problem 13. Find constants A, B, C, D and E such that the following rule is exact for the highest degree polynomial and find truncation error

$$\frac{d^2u}{dx^2} \approx A\ u(x-2h) + B\ u(x-h) + C\ u(x) + D\ u(x+h) + E\ u(x+2h),$$

where $h = \Delta x$.

Problem 14. Perform two iterations to solve the following one dimensional heat equation using FTCS and Crank−Nicolson scheme, with $\Delta x = \frac{1}{4}$.

$$\frac{\partial u}{\partial t} = 4\ \frac{\partial^2 u}{\partial x^2},\ 0 < x < 1, t > 0.$$

$$u(0,t) = u(1,t) = 0,\ t \ge 0 \text{ and } u(x,0) = x(1-x),\ 0 < x < 1.$$

Problem 15. Discuss the stability of following numerical scheme for solving 1D heat equation

$$\frac{U_{i,j+1} - U_{i,j-1}}{2\Delta t} = \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{(\Delta x)^2}$$

Problem 16. Discuss the stability and consistency of the following implicit scheme for solving 1D heat equation

$$\frac{U_{i,j} - U_{i,j-1}}{\Delta t} = \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{(\Delta x)^2}$$

Problem 17. Discuss the stability of the following numerical scheme for solving 1D wave equation

$$\frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{(\Delta t)^2} = \frac{c^2}{2}\left[\frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{(\Delta x)^2} + \frac{U_{i+1,j+1} - 2U_{i,j+1} + U_{i-1,j+1}}{(\Delta x)^2}\right].$$

Problem 18. Solve the following PDEs

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},\ 0 < x < 1, t > 0.$$

$$u(x,0) = 0, \left(\frac{\partial u}{\partial t}\right)_{(x,0)} = \sin^3(\pi x),\ 0 \le x \le 1$$

$$u(0,t) = u(1,t) = 0,\ t \ge 0.$$