Turing Machines and Decidability

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I want to Begin With... J K Rowling's Quote

It is impossible to live without failing at something, unless you live so cautiously that you might has well not have lived at all, in which case you have failed by default.

Investigate the Power of Algorithms

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- The unsolvability of certain problems may come as a surprise.

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    - Help you gain an important perspective on computation.

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  - Starting with examples where decidability is possible helps you to appreciate the undecidable examples.

DECIDABLE PROBLEMS CONCERNING REGULAR LANGUAGES

ALgorithms for Regular Languages

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  - Whether two finite automata are equivalent.

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Proof Idea

We simply need to present a TM M that decides A_{DFA}.

Algorithm

$M =$ "On input $\langle B, w \rangle$, where B is a DFA and w is a string:

1. Simulate B on input w.
2. If the simulation ends in an accept state, *accept*. If it ends in a nonaccepting state, *reject*."

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  - If not, M rejects.

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    - N first converts the NFA it receives as input to a DFA before passing it to M .

Algorithm

$N =$ "On input $\langle B, w\rangle$, where B is an NFA and w is a string:

1. Convert NFA B to an equivalent DFA C, using the procedure for this conversion given in Theorem 1.39.
2. Run TM M from Theorem 4.1 on input $\langle C, w\rangle$.
3. If M accepts, *accept*; otherwise, *reject*."

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$P =$ "On input $\langle R, w\rangle$, where R is a regular expression and w is a string:

1. Convert regular expression R to an equivalent NFA A by using the procedure for this conversion given in Theorem 1.54.
2. Run TM N on input $\langle A, w\rangle$.
3. If N accepts, *accept*; if N rejects, *reject*."

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- In the preceding three theorems we had to determine whether a finite automaton accepts a particular string.
- Now we must determine whether or not a finite automaton accepts any strings at all.

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- We can design a TM T that uses a marking algorithm that I had discussed earlier.

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$T =$ "On input $\langle A \rangle$, where A is a DFA:

1. Mark the start state of A.
2. Repeat until no new states get marked:
3. Mark any state that has a transition coming into it from any state that is already marked.
4. If no accept state is marked, *accept*; otherwise, *reject*."

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- These constructions are algorithms that can be carried out by Turing machines.

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$F =$ "On input $\langle A, B\rangle$, where A and B are DFAs:

1. Construct DFA C as described.
2. Run TM T from Theorem 4.4 on input $\langle C\rangle$.
3. If T accepts, *accept*. If T rejects, *reject*."

DECIDABLE PROBLEMS CONCERNING CONTEXT-FREE LANGUAGES

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  - To determine whether a CFG generates a particular string w
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Focus

- I describe algorithms
  - To determine whether a CFG generates a particular string w
  - To determine whether the language of a CFG is empty.
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- We know how to can convert G to Chomsky normal form.

Proof

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$S =$ "On input $\langle G, w \rangle$, where G is a CFG and w is a string:

1. Convert G to an equivalent grammar in Chomsky normal form.
2. List all derivations with $2n - 1$ steps, where n is the length of w; except if $n = 0$, then instead list all derivations with one step.
3. If any of these derivations generate w, *accept*; if not, *reject*."

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- More efficient algorithm for recognizing general context-free languages exist.
- Even greater efficiency is possible for recognizing deterministic context-free languages.

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  - We need to take a different approach

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- The algorithm does so by solving a more general problem.
- It determines for each variable,
  - whether that variable is capable of generating a string of terminals.
- When the algorithm has determined that a variable can generate some string of terminals,
  - The algorithm keeps track of this information by placing a mark on that variable.

Proof Idea

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  - The algorithm knows that this variable can be marked, too.
- The algorithm continues in this way until it cannot mark any additional variables.

Proof

$R =$ "On input $\langle G \rangle$, where G is a CFG:

1. Mark all terminal symbols in G.
2. Repeat until no new variables get marked:
3. Mark any variable A where G has a rule $A \rightarrow U_1 U_2 \cdots U_k$ and each symbol $U_1, \ldots, U_k$ has already been marked.
4. If the start variable is not marked, *accept*; otherwise, *reject*."

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  $EQ_{CFG} = \{< G, H > | G \text{ and } H \text{ are CFGs and } L(G) = L(H).$

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  - complementation or
  - intersection.
- In fact, EQ_{CFG} **is not decidable.**
- I will prove it after going through the technique.

CFLs are Decidable

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Theorem 4.9

Every context-free language is decidable.

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  - And we know that **any nondeterministic TM can be converted into an equivalent deterministic TM.**
  - Yet, there is a difficulty!!!

Contd..

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- A different Idea is necessary!!!

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M_G = "On input w:

1. Run TM S on input $\langle G, w \rangle$.
2. If this machine accepts, *accept*; if it rejects, *reject*."

Relationships

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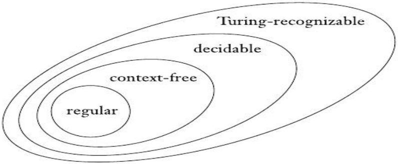

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- Computers appear to be so powerful that you may believe that all problems will eventually yield to them.
- The theorem that I present strates that computers are limited in a fundamental way.

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- **No!**
- Even some **ordinary problems turn out to be computationally unsolvable.**

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Challenge

You need to verify that the program performs as specified (i.e., that it is correct).

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- However, you will be disappointed.

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- We aim to help you develop a feeling for the types of problems that are unsolvable .
- To learn **techniques for proving unsolvability.**

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- Note: A_{DFA} and $A_C FG$ is decidable!!!. I have proved!!!

Theorem

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Theorem 4.11

A_{TM} is undecidable.

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- It shows that recognizers are more powerful than deciders.
- Requiring a TM to halt on all inputs restricts the kinds of languages that it can recognize.

Turing machine U to recognize A_{TM}

U = "On input $\langle M, w\rangle$, where M is a TM and w is a string:

1. Simulate M on input w.
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- However, I will discuss that algorithm has no way to determine this.

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  - It is capable of **simulating any other Turing machine** from the description of that machine.
- The UTM played an important early role in the development of **stored-program computers.**

THANK YOU