The Diagonalization Method

Purushothama B R

Department of Computer Science and Engineering
National Institute of Technology Goa ,INDIA

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I want to Begin With... Michael Jordan's Quote

I've missed more than 9000 shots in my career. I've lost almost 300 games. 26 times, I've been trusted to take the game winning shot and missed. I've failed over and over and over again in my life. And that is why I succeed.

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- We **can't use the counting method** to determine the relative sizes of infinite sets.

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- **How can we compare their relative size?**

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- This method compares the sizes without resorting to counting.
- We can extend this idea to infinite sets.

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Simple Way of Pairing

A correspondence is simply a way of pairing the elements of A with the elements of B.

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n	$f(n)$
1	2
2	4
3	6
$\vdots$	$\vdots$

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- So we declare these two sets to be the same size.

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- Let us try to give **correspondence** with $\mathbb{N}$ to show that Q is countable.

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- To get this list, we make an infinite matrix containing all the positive rational numbers.

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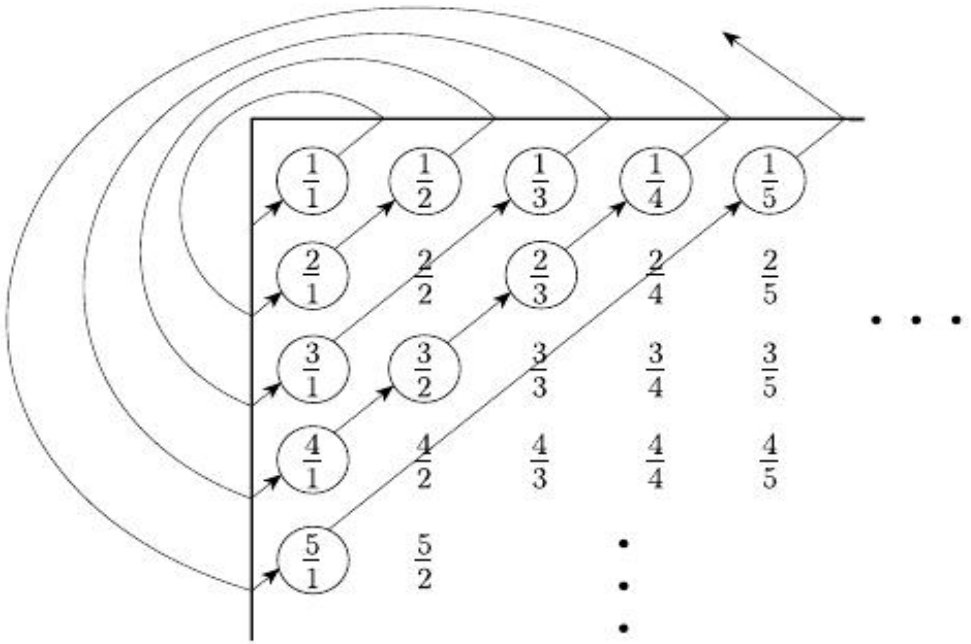

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- Continuing in this way we obtain list of all elements of Q.

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- Such sets are called **uncountable.**

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- In doing so, **he introduced the diagonalization method.**

Theorem by Cantor

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Theorem 4.17

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- Which will be our contradiction.

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- In the end, we are sure that x is different from any real number that is paired.

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n	$f(n)$
1	$3.14159...$
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3	$0.12345...$
4	$0.50000...$
$\vdots$	$\vdots$

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  - So we let x be anything different, say, 4.

Contd..

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n	$f(n)$
1	$3.\underline{1}4159\ldots$
2	$55.5\underline{5}555\ldots$
3	$0.12\underline{3}45\ldots$
4	$0.500\underline{0}0\ldots$
$\vdots$	$\vdots$

$$x = 0.4641\ldots$$

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- A slight problem arises because certain numbers, such as $0.1999\ldots$ and $0.2000\ldots$, are equal even though their decimal representations are different.
- We avoid this problem by never selecting the digits 0 or 9 when we construct x.

Corollary

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Some languages are not Turing-recognizable.

Contd..

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- Omit those strings that are not legal encodings of Turing machines, we can obtain a list of all Turing machines.

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B is uncountable.

- **This can be proved by giving similar proof as we had discussed before.**

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- Let $\Sigma^* = \{s_1, s_2, s_3, \ldots\}$
- Each language $A \in L$ has a unique sequence in B.
- The i^{th} bit of that sequence is a 1 if $s_i \in A$ and is a 0 if $s_i \in A$
- This is called characteristic sequence of A.

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- Its characteristic sequence would be

$$\begin{aligned}
\Sigma^* &= \{\ \varepsilon,\ \ 0,\ \ 1,\ \ 00,\ \ 01,\ \ 10,\ \ 11,\ \ 000, 001,\ \cdots\ \}\ ; \\
A &= \{\ \ \ \ 0,\ \ \ \ 00,\ \ 01,\ \ \ \ \ \ \ \ 000, 001,\ \cdots\ \}\ ; \\
\chi_A &= \ \ 0\ \ 1\ \ 0\ \ 1\ \ 1\ \ 0\ \ 0\ \ 1\ \ 1\ \ \cdots\ \ .
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- The function $f : L \rightarrow B$,
  - Where $f(A)$ equals the characteristic sequence of A, is one-to-one and onto.
- Therefore, as B is uncountable, L is uncountable as well.
- We have shown that the set of all languages cannot be put into a correspondence with the set of all Turing machines.
- We conclude that some languages are not recognized by any Turing machine.

We are equipped!!!

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- Note

$$A_{TM} = \{< M, w > | M \text{ is a TM and M accepts } w\}.$$

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Theorem 4.11

A_{TM} is undecidable.

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  - Furthermore, H halts and rejects M fails to accept w
- In other words, we assume that H is a TM, where

$$H(\langle M, w \rangle) = \begin{cases} \textit{accept} & \text{if } M \text{ accepts } w \\ \textit{reject} & \text{if } M \text{ does not accept } w. \end{cases}$$

Contd..

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- Now we construct a new Turing machine **D** with H as a subroutine.
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- Once D has determined this information, it does the opposite.
- That is, it rejects if M accepts and accepts if M does not accept.

Description of D

D = "On input $\langle M \rangle$, where M is a TM:

1. Run H on input $\langle M, \langle M \rangle \rangle$.
2. Output the opposite of what H outputs. That is, if H accepts, *reject*; and if H rejects, *accept*."

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- For example, a compiler is a program that translates other programs.
- A compiler for the language Python may itself be written in Python
- So, running that program on itself would make sense.

Summary

$$D(\langle M \rangle) = \begin{cases} \textit{accept} & \text{if } M \text{ does not accept } \langle M \rangle \\ \textit{reject} & \text{if } M \text{ accepts } \langle M \rangle. \end{cases}$$

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- Which is obviously a contradiction.
- Thus, neither TM D nor TM H can exist.

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- Finally, run D on itself.

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  - All their descriptions across the columns $< M_1 >, < M_2 >, \ldots,$
- The entries tell whether the machine in a given row accepts the input in a given column.
- The entry is accept if the machine accepts the input
  - But is blank if it rejects or loops on that input.

Entry i, j is accept if M_i accepts $< M_j >$

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\cdots$
M_1	*accept*		*accept*		
M_2	*accept*	*accept*	*accept*	*accept*	
M_3					$\cdots$
M_4	*accept*	*accept*			
$\vdots$		$\vdots$			

Contd..

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- In the following figure, the entries are the results of running H on inputs corresponding to Previous figure.

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\cdots$
M_1	*accept*	*reject*	*accept*	*reject*	
M_2	*accept*	*accept*	*accept*	*accept*	$\cdots$
M_3	*reject*	*reject*	*reject*	*reject*	
M_4	*accept*	*accept*	*reject*	*reject*	
$\vdots$			$\vdots$		

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M_2	*accept*	*accept*	*accept*	*accept*	$\cdots$
M_3	*reject*	*reject*	*reject*	*reject*	
M_4	*accept*	*accept*	*reject*	*reject*	
$\vdots$		$\vdots$			

- So if M_3 does not accept input M_2 ,

Contd..

- In the following figure, the entries are the results of running H on inputs corresponding to Previous figure.

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\cdots$
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M_3	*reject*	*reject*	*reject*	*reject*	
M_4	*accept*	*accept*	*reject*	*reject*	
$\vdots$		$\vdots$			

- So if M_3 does not accept input M_2 ,
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M_4	*accept*	*accept*	*reject*	*reject*	
$\vdots$		$\vdots$			

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- In the following figure, D is added to previous Figure.

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\cdots$	$\langle D \rangle$	$\cdots$
M_1	*accept*	*reject*	*accept*	*reject*		*accept*	
M_2	*accept*	*accept*	*accept*	*accept*	$\cdots$	*accept*	$\cdots$
M_3	*reject*	*reject*	*reject*	*reject*		*reject*	
M_4	*accept*	*accept*	*reject*	*reject*		*accept*	
$\vdots$		$\vdots$			$\ddots$		
D	*reject*	*reject*	*accept*	*accept*		?	
$\vdots$		$\vdots$					$\ddots$

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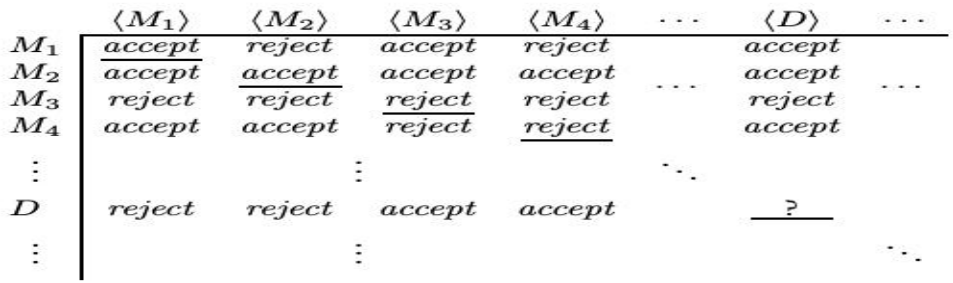

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M_1	*accept*	*reject*	*accept*	*reject*		*accept*	
M_2	*accept*	*accept*	*accept*	*accept*	$\cdots$	*accept*	$\cdots$
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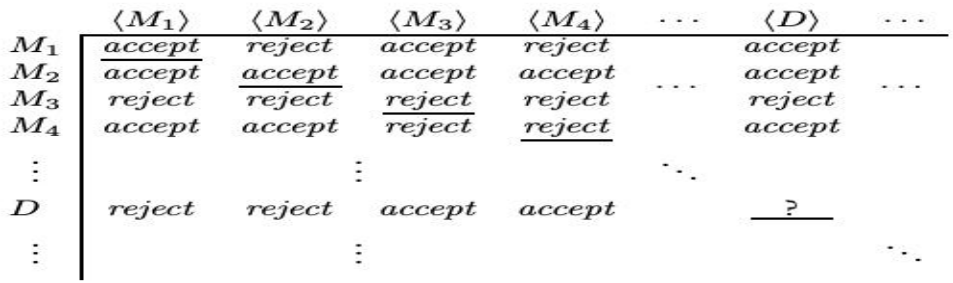

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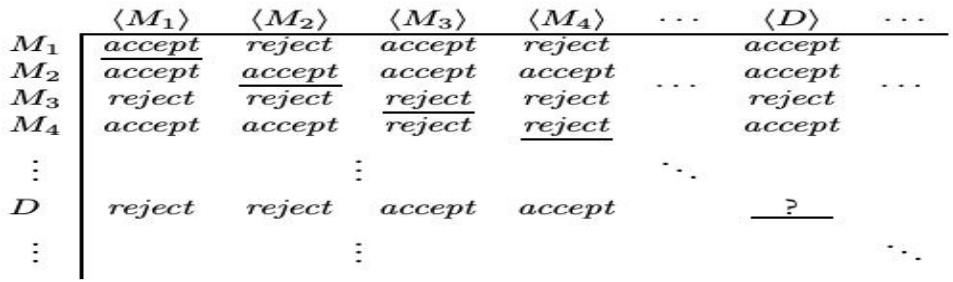

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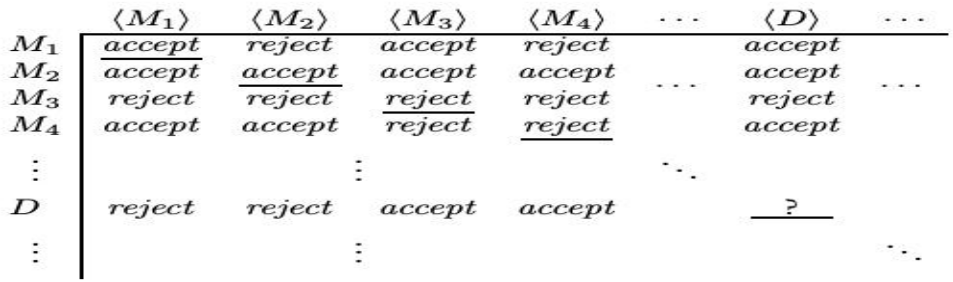

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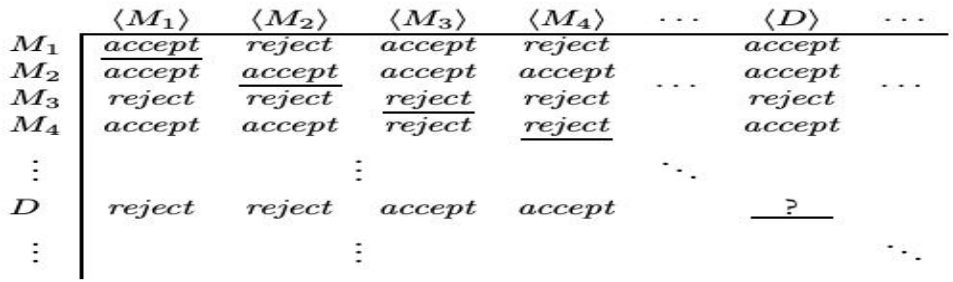

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Theorem: Condition for Decidability

Theorem 4.22

A language is decidable **iff** it is Turing-recognizable and co-Turing-recognizable.

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- First, if A is decidable, It's easy to see that
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  - Any decidable language is Turing-recognizable, and
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Other Direction

- If both A and $\overline{A}$ are Turing-recognizable,
  - let M_1 be the recognizer for A and
  - M_2 be the recognizer for $\overline{A}$.
- The following Turing machine M is a decider for A.

M = "On input w:

1. Run both M_1 and M_2 on input w in parallel.
2. If M_1 accepts, *accept*; if M_2 accepts, *reject*."

Contd..

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- Running the two machines in parallel means that M has two tapes.

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- Running the two machines in parallel means that M has two tapes.
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- In this case, M takes turns simulating one step of each machine, which continues until one of them accepts.

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- M always halts and so it is a decider.
- Furthermore, it accepts all strings in A and rejects all strings not in A.
- So M is a decider for A, and thus A is decidable.

Not a Turing Recognizable Language

Corollary

$\overline{A_{TM}}$ is not Turing-recognizable.

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- We know that $A_T M$ is Turing-recognizable.
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$\overline{A_{TM}}$ is not Turing-recognizable.

- We know that A_TM is Turing-recognizable.
- If A_TM also were Turing-recognizable, A_{TM} would be decidable.
- We have proved thatthat A_{TM} is not decidable.
- So, $\overline{A_{TM}}$ must not be Turing-recognizable.

THANK YOU