Wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{— (*)} \qquad \text{for all } 0 < x < l \text{ and } t > 0$$

Boundary conditions

$$u(0,t) = u(l,t) = 0 \quad \forall\ t > 0$$

Initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x) \quad \forall\ 0 \le x < l$$

$$u(x,t) = X(x)\,T(t)$$

$$\frac{\partial^2 u}{\partial t^2} = X T'' \qquad \frac{\partial^2 u}{\partial x^2} = X'' T$$

$$(*) \Rightarrow \quad X T'' = c^2 X'' T$$

$$\frac{X''}{X} = \frac{1}{c^2} \frac{T''}{T} = K$$

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = k.$$

Case i $\quad k=0$

$X''=0$ and $T''=0$

$X = c_1 x + c_2 \qquad T = c_3 t + c_4.$

$$u(x,t) = X(x)\,T(t)$$
$$= (c_1 x + c_2)(c_3 t + c_4)$$

$u(0,t)=0 \Rightarrow c_2(c_3 t + c_4) = 0 \Rightarrow c_2 = 0$

$u(l,t)=0 \Rightarrow (c_1 l + c_2)(c_3 t + c_4) = 0 \Rightarrow c_1 = 0$

$u(x,t) = 0$ trivial soln

$\Downarrow$

Case ii $\quad K > 0 \quad K = \lambda^2$

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = \lambda^2$$

$$X'' - \lambda^2 X = 0 \qquad T'' - c^2\lambda^2 T = 0$$

$$X(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x} \quad \text{and} \quad T(t) = c_3 e^{c\lambda t} + c_4 e^{-c\lambda t}$$

$$u(x,t) = X(x)T(t)$$

$$u(x,t) = \left(c_1 e^{\lambda x} + c_2 e^{-\lambda x}\right)\left(c_3 e^{c\lambda t} + c_4 e^{-c\lambda t}\right)$$

$$u(0,t) = 0 \Rightarrow (c_1 + c_2)\left(c_3 e^{c\lambda t} + c_4 e^{-c\lambda t}\right) = 0$$

$$u(l,t) = 0 \Rightarrow \left(c_1 e^{\lambda l} + c_2 e^{-\lambda l}\right)\left(c_3 e^{c\lambda t} + c_4 e^{-c\lambda t}\right) = 0$$

$$c_1 + c_2 = 0$$

$$c_1 e^{\lambda l} + c_2 e^{-\lambda l} = 0$$

$$c_1 = 0 \qquad c_2 = 0$$

$$u(x,t) = 0 \qquad \times$$

Case iii) $K \leq 0 \Rightarrow K = -\lambda^2$

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = -\lambda^2$$

$$X'' + \lambda^2 X = 0 \quad \text{and} \quad T'' + c^2\lambda^2 T = 0$$

$$X(x) = c_1 \cos\lambda x + c_2 \sin\lambda x) \qquad T(t) = c_3 \cos c\lambda t + c_4 \sin c\lambda t$$

$$u(x,t) = X(x)\,T(t)$$

$$u(x,t) = (c_1 \cos\lambda x + c_2 \sin\lambda x)\,(c_3 \cos(c\lambda t) + c_4 \sin(c\lambda t))$$

$$u(\cancel{x},\cancel{t}) = 0 \Rightarrow c_1\,(c_3 \cos(c\lambda t) + c_4 \sin(c\lambda t)) = 0$$

$$c_1 = 0$$

$$u(l,t) = 0 \Rightarrow (c_1 \cancel{\cos\lambda l} + c_2 \sin\lambda l)\,(c_3 \cos(c\lambda t) + c_4 \sin(c\lambda t)) = 0$$

$$c_2 \sin\lambda l\,(c_3 \cos(c\lambda t) + c_4 \sin(c\lambda t)) = 0$$

$$c_2 \sin\lambda l = 0$$

$$\sin\lambda l = 0$$

$$\lambda l = n\pi \Rightarrow \lambda_n = \frac{n\pi}{l}$$

$$n = 1, 2, 3 \ldots$$

$$u_n(x,t) = \sin\frac{n\pi x}{l}\left(A_n\cos\frac{n\pi ct}{l} + B_n\sin\frac{n\pi ct}{l}\right)$$

u_1 u_2 u_3 u_4

$$u = c_1u_1 + c_2u_2 + c_3u_3 + c_4u_4 + \dots + c_nu_n$$

$$= \sum_{n=1}^{\infty} c_n u_n$$

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{l}\right)\left[A_n\cos\left(\frac{n\pi ct}{l}\right) + B_n\sin\left(\frac{n\pi ct}{l}\right)\right] \quad (1)$$

A_n, B_n

$$u(x,0) = f(x)$$

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n\sin\left(\frac{n\pi x}{l}\right)$$

Half-range Fourier Sine Series

$$A_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)$$

$$u_t(x,0) = g(x)$$

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{l}\right)\left[A_n \cos\left(\frac{n\pi c t}{l}\right) + B_n \sin\left(\frac{n\pi c t}{l}\right)\right]$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$$

$$B_n \frac{n\pi c}{l}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} B_n\left(\frac{n\pi c}{l}\right)\sin\frac{n\pi x}{l}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$B_n \frac{n\pi c}{l} = \frac{2}{l}\int_0^l g(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$B_n = \frac{2}{n\pi c}\int_0^l g(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < l$$

$$u(0,t) = u(l,t) = 0$$

$$u(x,0) = f(x) \quad \text{and} \quad u_t(x,0) = g(x)$$

Soln

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{l}\right)\left[A_n \cos\left(\frac{n\pi c t}{l}\right) + B_n \sin\left(\frac{n\pi c t}{l}\right)\right]$$

where $A_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$

$$B_n = \frac{2}{n\pi c}\int_0^l g(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

Find the solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{for all} \quad 0 < x < 2 \qquad t > 0$$

B.C $\qquad u(0,t) = u(2,t) = 0$

$$u(x,0) = \sin^3\left(\frac{\pi x}{2}\right)$$

$$u_t(x,0) = 0$$

$\underline{\underline{l=2}}$ $\qquad f(x) = \sin^3\left(\frac{\pi x}{2}\right) \qquad g(x) = 0$

$$B_n = 0$$

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{2}\right) \cos\left(\frac{n\pi c t}{2}\right)$$

$$t = 0$$

$$u(x,0) = \sin^3\left(\frac{\pi x}{2}\right) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{2}\right)$$

$$\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{2}\right) = \sin^3\left(\frac{\pi x}{2}\right)$$

$\sin(3\theta) = 3\sin\theta - 4\sin^3\theta$

$\sin^3\theta = 3/4 \sin\theta - \frac{1}{4}\sin 3\theta$

$$\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{2}\right) = \frac{3}{4}\sin\left(\frac{\pi x}{2}\right) - \frac{1}{4}\sin\left(\frac{3\pi x}{2}\right)$$

$A_2 = A_4 = A_5 = A_6 = \cdots = 0$

$A_1 = \frac{3}{4} \quad A_3 = -\frac{1}{4}$

$$\boxed{u(x,t) = \frac{3}{4}\sin\left(\frac{\pi x}{2}\right)\cos\left(\frac{\pi c t}{2}\right) - \frac{1}{4}\sin\left(\frac{3\pi x}{2}\right)\cos\left(\frac{3\pi c t}{2}\right)}$$