$$y'' + xy = 0 \quad — (1)$$

about $x = 0$

$x = 0$ ordinary point -

$$y = \sum_{n=0}^{\infty} a_n x^n \quad — (2)$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

(1) $\Rightarrow$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$n \to (n+2)$ $\quad$ $n \to (n-1)$

$$\sum_{n+2=2}^{\infty} (n+2)(n+1)\, a_{n+2}\, x^{n} + \sum_{n-1=0}^{\infty} a_{n-1}\, x^{n} = 0$$

$$\sum_{\underline{n=0}}^{\infty} (n+2)(n+1)\, a_{n+2}\, x^{n} + \sum_{\underline{n=1}}^{\infty} a_{n-1}\, x^{n} = 0$$

$$2 \cdot 1 \cdot a_2 + \sum_{n=1}^{\infty} (n+2)(n+1)\, a_{n+2}\, x^{n} + \sum_{n=1}^{\infty} a_{n-1}\, x^{n} = 0$$

$$2a_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)\, a_{n+2} + a_{n-1} \right] x^{n} = 0$$

$$\underline{a_2 = 0} \quad \$ \quad (n+2)(n+1)\, a_{n+2} + a_{n-1} = 0, \quad n \geq 1$$

$$a_{n+2} = \frac{-1}{(n+1)(n+2)}\, a_{n-1} \quad n \geq 1$$

Recurrance r/n

$n=1 \Rightarrow a_3 = \frac{-1}{6} a_0$ $\qquad$ $n=2 \Rightarrow a_4 = \frac{-1}{3 \cdot 4} a_1$

$n=3 \Rightarrow a_5 = \frac{-1}{4 \cdot 5} a_2 = 0$ $\qquad$ $n=4 \Rightarrow a_6 = \frac{-1}{5 \cdot 6} a_3 = \frac{1}{6 \cdot 5 \cdot 3 \cdot 2} a_0$

Soln

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + a_1 x + \cancelto{0}{a_2 x^2} + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots$$

$$= \left(a_0 \cancel{+ a_2 x^2 + a_4 x^4 + a_6 x^6 + \cdots}\right) \cancel{+ (a_1 x +} a_3 x^3 + a_5 x^5 + \cdots)$$

$$y(x) = a_0\left[1 - \frac{x^3}{3\cdot 2} + \frac{x^6}{6\cdot 5\cdot 3\cdot 2} \overset{-}{+} \cdots\right] + a_1\left[x - \frac{x^4}{4\cdot 3} + \cdots\right]$$

$$y'' + x^2 y = 0$$

$$a_2 = 0, \; a_3 = 0 \quad \text{and} \quad a_{n+2} = \frac{-1}{(n+2)(n+1)} a_{n-2}, \; n \geq 2$$

$$y'''(x) + x^2 y''(x) + 5x\, y'(x) + 3y(x) = 0 \quad \text{— (1)}$$

$x = 0$ is an ordinary point

$$y = \sum_{n=0}^{\infty} a_n x^n \qquad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad y''' = \sum_{n=3}^{\infty} n(n-1)(n-2) a_n x^{n-3}$$

$$\sum_{n=3}^{\infty} n(n-1)(n-2) a_n x^{n-3} + x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 5x \sum_{n=1}^{\infty} n a_n x^{n-1} + 3 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=3}^{\infty} n(n-1)(n-2) a_n x^{n-3} + \sum_{n=2}^{\infty} n(n-1) a_n x^n + 5 \sum_{n=1}^{\infty} n a_n x^n + 3 \sum_{n=0}^{\infty} a_n x^n = 0$$

$, n \to n+3$

$$\sum_{n=0}^{\infty}(n+3)(n+2)(n+1)a_{n+3}x^n+\sum_{n=2}^{\infty}n(n-1)a_n x^n+5\sum_{n=1}^{\infty}n a_n x^n+3\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}(n+3)(n+2)(n+1)a_{n+3}x^n+\sum_{n=0}^{\infty}n(n-1)a_n x^n+5\sum_{n=0}^{\infty}n a_n x^n+3\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}\left[(n+3)(n+2)(n+1)\,a_{n+3}+\left(n(n-1)+5n+3\right)a_n\right]x^n=0$$

For the recurrence relation

$\underline{\underline{x^n}}$

$$a_{n+3}=-\frac{n^2+\cancel{4n+3}}{\cancel{(n+3)}(n+2)\cancel{(n+1)}}\,a_n \quad n\geq 0$$

$$a_{n+3}=-\frac{1}{n+2}\,a_n \quad n\geq 0$$

$$a_{n+3} = \frac{-1}{n+2} a_n \qquad n \geq 0$$

$$n=0 \Rightarrow a_3 = \frac{-1}{2} a_0 \qquad n=3 \Rightarrow a_6 = \frac{-1}{5} a_3 = \frac{1}{10} a_0$$

$$n=1 \Rightarrow a_4 = \frac{-1}{3} a_1 \qquad n=4 \Rightarrow a_7 = \frac{1}{18} a_1$$

$$n=2 \Rightarrow a_5 = \frac{-1}{4} a_2 \qquad n=5 \Rightarrow a_8 = \frac{1}{28} a_2$$

$$y = (a_0 + a_3 x^3 + a_6 x^6 + \cdots) + (a_1 x + a_4 x^4 + a_7 x^7 + \cdots)$$

$$+ (a_2 x^2 + a_5 x^5 + a_8 x^8 + \cdots)$$

$$y = a_0\left[1 + \frac{x^3}{2} + \frac{x^6}{10} - \cdots\right] + a_1\left[x - \frac{x^4}{3} + \frac{x^7}{18} - \cdots\right]$$

$$+ a_2\left[x^2 - \frac{x^5}{4} + \frac{x^8}{28} + \cdots\right]$$

$$\sum a_n x^n$$

Solve $y'' - xy' - y = 0, \quad x_0 = 1$ —— (1) $\Rightarrow$ $y'' - (z+1)y' + y = 0$

$y = \sum a_n z^n$

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} a_n (x-1)^n$$

$$y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}$$

$$(1) \Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} - x \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$n \to (n+2)$

$x = (x-1) + 1$ $\qquad x - 1 = z$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n - \sum_{n=1}^{\infty} n a_n (x-1)^n - \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} - \sum_{n=0} a_n (x-1)^n = 0$$

$n \to n+1$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n - \sum_{n=0}^{\infty} n a_n (x-1)^n - \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-1)^n - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - \left\{ (n+1) a_{n+1} + (n+1) a_n \right\} \right] (x-1)^n = 0$$

$$a_{n+2} = \frac{a_n + a_{n+1}}{(n+2)}, \quad n \geq 0$$

$$n=0 \Rightarrow \quad a_2 = \frac{a_0 + a_1}{2}$$

$$n=1 \Rightarrow \quad a_3 = \frac{a_1 + a_2}{3} = \frac{a_1}{3} + \frac{1}{3}\left(\frac{a_0 + a_1}{2}\right) = \frac{a_0}{6} + \frac{a_1}{2}$$

$$n=2 \Rightarrow \quad a_4 = \frac{a_2 + a_3}{4} = \frac{1}{4}\left(\frac{a_0}{2} + \frac{a_1}{2}\right) + \frac{1}{4}\left(\frac{a_0}{6} + \frac{a_1}{2}\right)$$

$$= \frac{a_0}{8} + \frac{a_0}{24} + \frac{a_1}{8} + \frac{a_1}{8}$$

$$y = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3$$

$$y = a_0\left[1 + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} + \cdots\right]$$

$$+ a_1\left[(x-1) + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{2} + \frac{(x-1)^4}{4} + \cdots\right]$$

$a_0 = 2 \quad \leftarrow \quad y(1) = 2 \qquad y'(1) = -3$

$a_1 = -3$