

Derivative :-

The derivative of a function $f(x)$ with respect to the variable x is the function $f'(x)$ whose value at x is

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$\text{Set } z = x + h$$

$$\Rightarrow f'(x) = \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x}$$

Theorem Every differentiable function is continuous ?

Proof

Let $f(x)$ be a function

Given that $f(x)$ has a derivative at $x=c$

i.e. $f'(c)$ exist.

We have to prove that, $f(x)$ is a continuous at $x=c$

i.e. $\lim_{h \rightarrow 0} f(c+h) = f(c)$

$$f(x) = |x| \text{ at } x=0$$

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$\lim_{x \rightarrow c} f(x) = f(c)$

Consider

$$f(c+h) = f(c+h) - f(c) + f(c)$$

$$f(c+h) = f(c) + \frac{f(c+h) - f(c)}{h} h$$

$$\lim_{h \rightarrow 0} f(c+h) = \lim_{h \rightarrow 0} \left\{ f(c) + \frac{f(c+h) - f(c)}{h} h \right\}$$

$$= \lim_{h \rightarrow 0} f(c) + \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h} \cdot \lim_{h \rightarrow 0} h$$

$$= f(c) + f'(c) \times 0$$

$$= f(c)$$

$$\Rightarrow \lim_{h \rightarrow 0} f(c+h) = f(c)$$

continuous at $x=c$

$\therefore f(x)$ is a ~~curve~~

The converse of above theorem statement need not be true.

i.e. Every continuous function need not be differentiable
 $\text{ex: } |x|$ at $x=0$

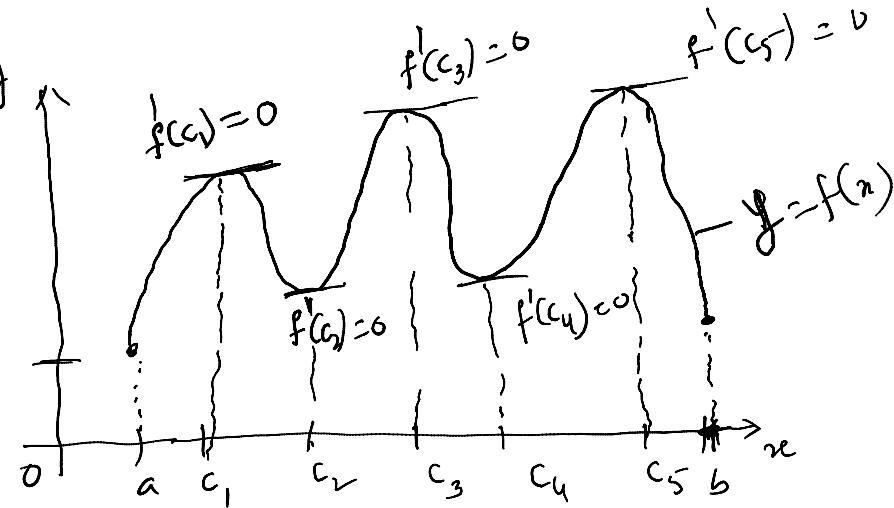
$\sin(\tilde{x}y)$

$$x^2y + x^3 + 3xy^2 = 0 \rightarrow \frac{dy}{dx}.$$

Chapter 4
Application of derivatives

Chapter 3

Roll's theorem:-



$$\begin{aligned} f &\rightarrow [a, b] \\ f(a) &= f(b) \\ \exists c \in (a, b) \text{ such that} \\ f'(c) &= 0 \end{aligned}$$

Suppose that $y = f(x)$ be a function

(i) $f(x)$ is continuous over the closed interval $[a, b]$

(ii) $f(x)$ is differentiable in the open interval (a, b)

(iii) $f(a) = f(b)$

If there exist at least one point $c \in (a, b)$ such that

$$f'(c) = 0$$

Verify the Rolle's theorem for the following function

$$f(x) = (x-a)^m (x-b)^n \text{ where } m, n \text{ are two integers in } [a, b]$$

Ans

$$f(x) = (x-a)^m (x-b)^n$$

Every polynomial is continuous and differentiable everywhere.

(i) $\therefore f(x) = (x-a)^m (x-b)^n$ is continuous in $[a, b]$

(ii) $f'(x) = m(x-a)^{m-1} (x-b)^n + n(x-a)^m (x-b)^{n-1}$
differentiable on (a, b)

(iii) $f(a) = 0 = f(b)$

$$\begin{aligned} f'(c) &= 0 \\ m(c-a)^{m-1} (c-b)^n + n(c-a)^m (c-b)^{n-1} &= 0 \\ (c-a)^{m-1} (c-b)^{n-1} \left[m(c-b) + n(c-a) \right] &= 0 \end{aligned}$$

$$m(c-b) + n(c-a) = 0$$

$$\begin{aligned} (m+n)c &= mb+na \\ c &= \frac{mb+na}{m+n} \in (a, b) \end{aligned}$$

\therefore Rolle's theorem Verified.

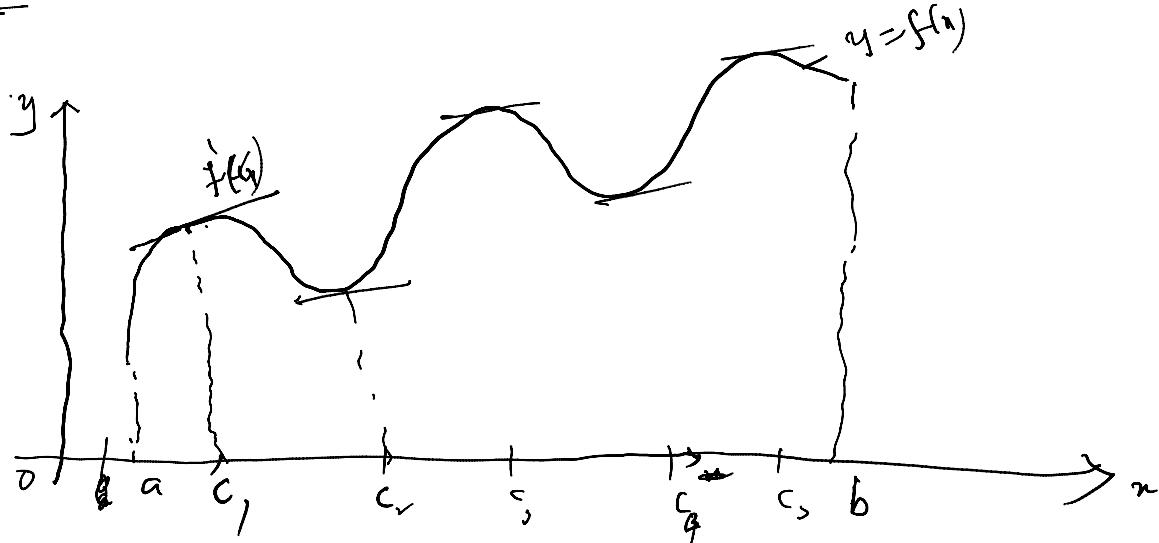
Verify Rolle's theorem -

Ex:- $f(x) = (x+2)^3 (x-3)^4$ in $[-2, 3]$
 $a = -2, b = 3, m = 3, n = 4$

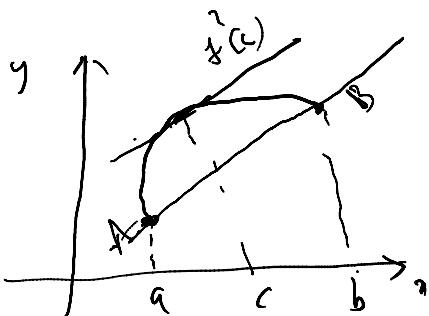
$$c = \frac{9-8}{7} = \frac{1}{7} \in (-2, 3) \quad \checkmark$$

Verified
=

Mean Value Theorem :-



$f(x)$ continuous
 $f(x)$ differentiable



$$\begin{aligned} & A(a, f(a)) \quad B(b, f(b)) \\ & y - f(a) = \frac{f(b) - f(a)}{b-a} (x - a) \end{aligned}$$

$$f(c) = \frac{f(b) - f(a)}{b-a}.$$

Suppose $f(x)$ be a function

(i) $f(x)$ is continuous on $[a, b]$

(ii) $f(x)$ is differentiable on (a, b)

then there exist at least one $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Cauchy's mean value theorem:

Suppose $f(x)$ and $g(x)$ are two functions

(i) $f(x)$ and $g(x)$ are continuous on $[a, b]$

(ii) $f(x)$ and $g(x)$ are differentiable on (a, b)

(iii) $g'(c) \neq 0$

then there exist a number $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Note

Set $g(x) = x$

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)} \quad \checkmark$$

$f(x)$ and $g(x)$

Note $g(x) = x$

$$\frac{f(x)}{x} = \frac{f(b) - f(a)}{b - a}$$

Cauchy's mean value theorem is reduced to Lagrange mean value

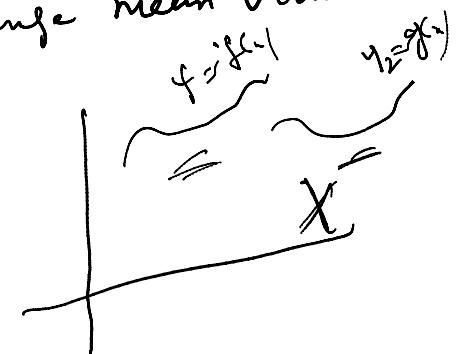
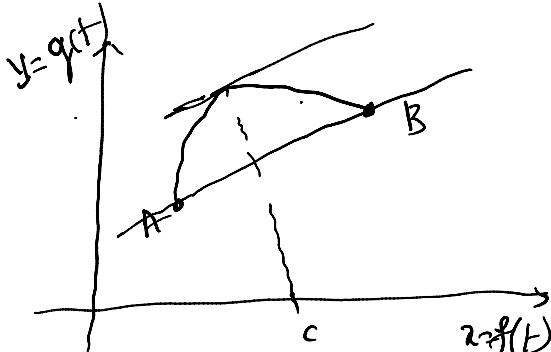
Theorem by setting $g(x) = x$

$x = f(t)$ by parametric representation
 $y = g(t)$

$$A = (g(a), g(a))$$

$$B = (f(b), g(b))$$

$$\frac{g(b) - g(a)}{f(b) - f(a)}$$



$$\frac{g'(t)}{f'(t)} ?$$

$$\Rightarrow \frac{g'(c)}{f'(c)} = \frac{g(b) - g(a)}{f(b) - f(a)}$$

① Verify Lagrange mean value theorem for the function

$$f(x) = (x-4)(x-6)(x-8) \quad \text{in } [4, 10]$$

Solution

$f(x) = (x-4)(x-6)(x-8)$ is continuous on $[4, 10]$

$$f'(x) = (x-6)(x-8) + (x-4)(x-8) + (x-4)(x-6)$$

differentiable on $(4, 10)$

$$\Rightarrow f'(c) = \frac{f(10) - f(4)}{10 - 4}$$

$$f'(x) = 3x^2 - 36x + 104$$

$$f'(c) = \frac{48 - 0}{6} = 8$$

$$\Rightarrow 3c^2 - 36c + 104 = 8$$

$$\Rightarrow c = 4, 8 \in (4, 10)$$

② Verify the Cauchy mean value theorem for the functions $f(x)$ and $\frac{1}{f(x)}$ in (a, b) given that $f(x) = \log x$.

(a) $c = \underline{\underline{8}}$?

(b) $c = 5$

(c) $c = 4 \frac{9}{8}$

(d)

Increasing and decreasing functions:-

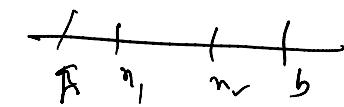
Suppose that f is continuous on $[a, b]$ and differentiable on (a, b)

(i) If $f'(x) > 0$ at each point $x \in (a, b)$ then f is increasing on $[a, b]$

(ii) If $f'(x) < 0$ at each point $x \in (a, b)$ then f is decreasing on $[a, b]$

$$f'(x) > 0 \Leftrightarrow f' \geq 0$$

$$\text{Q.E.D. } \frac{f'(x)}{f'(x)} < 0 \quad \text{if } f'(x) > 0 \rightarrow \text{D.Q.}$$



Proof

$$\left\{ \begin{array}{l} x_1, x_2 \\ f \text{ on } [x_1, x_2] \\ c \in (x_1, x_2) \end{array} \right.$$

$$\exists f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$x_1 < x_2$ then $f(x_2) > f(x_1)$ function is increasing

$$\Rightarrow \text{say } f'(rhs) > 0$$

If f is decreasing
if $f'(x) < 0$

$\Rightarrow f'(c) \leq 0 \therefore$ function is decreasing

$$\left\{ \begin{array}{l} x_1 < x_2 \\ f'(x) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \\ \lim_{\Delta x \rightarrow 0} \frac{f(x_2) - f(x_1)}{x_2 - x_1} \\ \sqrt{f(x_2) - f(x_1)} > 0 \\ f(x_2) - f(x_1) > 0 \end{array} \right.$$

$$\lim_{\Delta x \rightarrow 0} \frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0$$

$$\sqrt{f'(x)} > 0$$

① Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the open intervals on which f is increasing/decreasing.

Solution

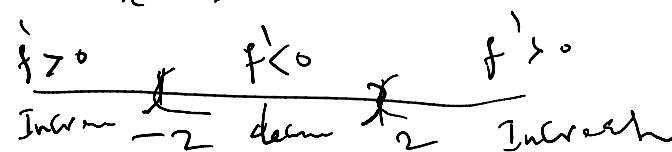
$$f(x) = x^3 - 12x - 5$$

$$f'(x) = 3x^2 - 12$$

$$\Rightarrow f'(x) = 0$$

$$3x^2 - 12 = 0$$

$$x = \pm 2$$



$$(-\infty, -2) \Rightarrow f'(x) > 0 \quad \text{Increasing}$$

$$(-2, 2) \Rightarrow f'(x) < 0 \quad \text{Decreasing}$$

$$(-2, \infty) \Rightarrow f'(x) > 0 \quad \text{Increasing}$$

= ✓

$$\begin{aligned} f'(x) > 0 &\Rightarrow x > 2 \\ f'(x) < 0 &\Rightarrow x < 2 \end{aligned}$$



$x = -2$ local maximum pt
 $x = 2$ local minimum pt

First Derivative Test for Local Extrema:-

Suppose that 'c' is a critical point of a (continuous) function $f(x)$, and $f(x)$ is differentiable at every point in some interval containing 'c' except possibly at 'c' itself. Moving across this interval from left to right.

① If $f'(x)$ changes from negative to positive at 'c'

then f has local minimum at 'c'

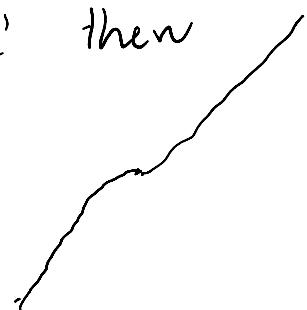
② If $f'(x)$ changes from positive to negative at 'c'

then f has a local maximum at 'c'

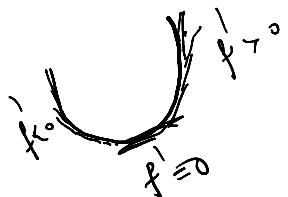
③ If $f'(x)$ does not change sign at 'c' then

f has no local extremum at 'c'

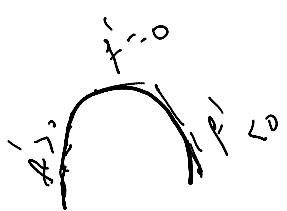
$$f'(x)=0$$



Def The graph of differentiable function $y=f(x)$ is



(i) Concave up on an open interval I if f' is increasing on I



(ii) Concave down on an open interval I if f' is decreasing on I

Second derivative Test for Concavity:-

Let $y=f(x)$ be twice-differentiable on an

interval I

(i) If $f'' > 0$ on I , the graph of $f(x)$ over I is concave up

(ii) If $f'' < 0$ on I , the graph of $f(x)$ over I is concave down.

Point of inflection:- neither maxima nor minima ✓
Saddle point

if $f''(x) > 0$ ^{then} f has local minimum

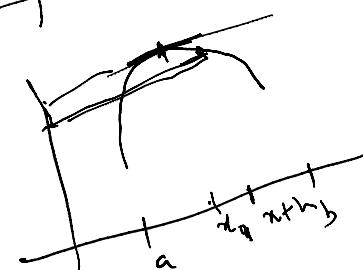
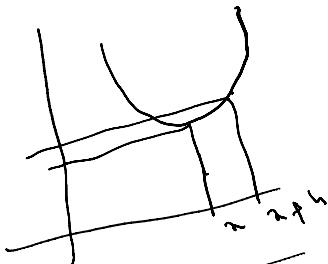
if $f''(x) < 0$ then f has local maximum

Taylor Series

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots$$

$\Delta < 0$ man
f has max. if $f(x) > f(x+h)$? $\Rightarrow f(x+h) - f(x) < 0$
min if $f(x) < f(x+h)$ $\Rightarrow \Delta > 0$ min

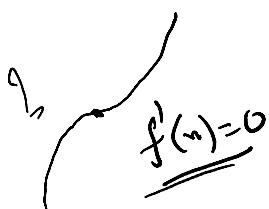
$$\text{set } A = f(x+h) - f(x)$$



$$f(x+h) = f(x) + h f'(x) + \cancel{\frac{h^2}{2!} f''(x)}$$

$$f(x+h) - f(x) = \frac{h^2}{2!} f''(x)$$

$$A = \frac{h^2}{2!} f''(x)$$

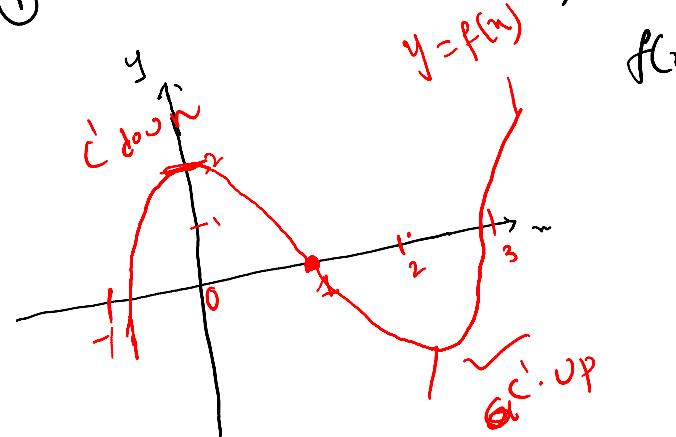


$$\text{Sign } (\Delta) = \text{Sign } (f''(x))$$

(i) $f''(x) > 0$ if $\Delta > 0$
 $\therefore f$ has local minima.

(ii) $f''(x) < 0$ if $\Delta < 0$
 $\therefore f$ has local maxima.

Eg: Determine the concavity and find the inflection points of the function



$$f(x) = x^3 - 3x^2 + 2$$

$$f'(x) = 0$$

$$3x^2 - 6x = 0$$

$$x^2 - 2x = 0$$

$$x = 0, 2$$

$$x = 1$$

$$f''(x) = 0 \text{ - point}$$

$x \neq 1$

- ✓ concave up if $x > 1$
- ✓ concave down if $x < 1$

② Sketch a graph of the function

$$f(x) = x^4 - 4x^3 + 10$$

using the following steps

- (i) Identify where the extrema of f occur
- (ii) Find the intervals on which f is increasing / decreasing
- (iii) Find where the graph of f is concave up and concave down
- (iv) sketch the general shape of the graph f'
- (v) plot some specific points such as local maximum and minimum points, points of inflection, and intercepts
then sketch the curve.

Try yourself!