Recurrence relations of Bessels functions

If $J_n(x)$ is a Bessels function of order ν then

(i) $$\frac{d}{dx}\left[x^{\nu} J_{\nu}(x)\right] = x^{\nu} J_{\nu-1}(x) \qquad \nu \geq 0.$$

(ii) $$\frac{d}{dx}\left[x^{-\nu} J_{\nu}(x)\right] = -x^{-\nu} J_{\nu+1}(x)$$

$$J_{\nu}(x) = \sum_{k=0}^{\infty} (-1)^k \left(\frac{x}{2}\right)^{\nu+2k} \frac{1}{k!\,\Gamma(\nu+k+1)}$$

$$x^{\nu} J_{\nu}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2\nu+2k}}{2^{\nu+2k}} \cdot \frac{1}{k!\,\Gamma(\nu+k+1)}$$

$$\frac{d}{dx}\left[x^{\nu} J_{\nu}(x)\right] = \sum_{k=0}^{\infty} (-1)^k (2\nu+2k) \frac{x^{2\nu+2k-1}}{2^{\nu+2k}} \cdot \frac{1}{k!\,\Gamma(\nu+k+1)}$$

$$\Gamma(\nu+k+1) = (\nu+k)\,\Gamma(\nu+k)$$

$$= \sum_{k=0}^{\infty} (-1)^k x^{\nu} \cdot \frac{x^{(\nu-1)+2k}}{2^{\nu-1+2k}} \cdot \frac{1}{k!\,\Gamma(\nu+k)}$$

$$= x^{\nu} \sum_{k=0}^{\infty} (-1)^k \cdot \left(\frac{x}{2}\right)^{(\nu-1)+2k} \cdot \frac{1}{k!\,\Gamma(\nu-1)+k+1}$$

$$\frac{d}{dx}\left[x^{\nu} J_{\nu}(x)\right] = x^{\nu} J_{\nu-1}(x)$$

(iii) $2 J_{\nu}'(x) = J_{\nu-1}(x) - J_{\nu+1}(x)$ ✓

(iv) $2\nu J_{\nu}(x) = x\left[J_{\nu-1}(x) + J_{\nu+1}(x)\right]$ ✓

(v) $x J_{\nu}'(x) = \nu J_{\nu}(x) - x J_{\nu+1}(x)$

$$\frac{d}{dx}\left[x^{\nu} J_{\nu}(x)\right] = x^{\nu} J_{\nu-1}$$

$$x^{\nu} J_{\nu}'(x) + \nu x^{\nu-1} J_{\nu}(x) = x^{\nu} J_{\nu-1} \qquad \text{— (1)}$$

$$\frac{d}{dx}\left[x^{-\nu} J_{\nu}(x)\right] = -x^{-\nu} J_{\nu+1}$$

$$x^{-\nu} J_{\nu}' - \nu x^{-\nu-1} J_{\nu} = -x^{-\nu} J_{\nu+1} \qquad \text{— (2)}$$

$$① \Rightarrow \quad J_\nu'(x) + \frac{\nu}{x} J_\nu = J_{\nu-1} \quad — ③$$

$$② \Rightarrow \quad J_\nu'(x) - \frac{\nu}{x} J_\nu = -J_{\nu+1} \quad — ④$$

Add ③ & ④

$$\boxed{2 J_\nu'(x) = J_{\nu-1} - J_{\nu+1}}$$

$$2 \frac{\nu}{x} J_\nu = J_{\nu-1} + J_{\nu+1}$$

$$\boxed{2\nu J_\nu(x) = x \left[J_{\nu-1}(x) + J_{\nu+1}(x) \right]}$$

prove $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$ ✓

$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$.

$J_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x \right)$ — ?

$J_{-3/2}(x) = -\sqrt{\frac{2}{\pi x}} \left(\frac{\cos x}{x} + \sin x \right)$ ✓

Proof

$$J_\nu(x) = \sum_{k=0}^{\infty} (-1)^k \left(\frac{x}{2}\right)^{\nu+2k} \frac{1}{k!\, \Gamma(\nu+k+1)}$$

$$J_\nu(x) = \frac{x^\nu}{2^\nu \Gamma(\nu+1)} \left[1 - \frac{x^2}{4(\nu+1)} + \frac{x^4}{4^2 \cdot 2!\, (\nu+1)(\nu+2)} - \cdots \right]$$

$\nu = 1/2$

$\Gamma(3/2) = \Gamma(\frac{1}{2}+1) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi}$

$$= \frac{\sqrt{x}}{\sqrt{2}\,\Gamma(3/2)} \left[1 - \frac{x^2}{4 \cdot 3/2} + \frac{x^4}{4^2 \cdot 2! \cdot 3/2 \cdot 5/2} - \cdots \right]$$

$$= \sqrt{\frac{2 x \times x}{\pi x}} \left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots \right]$$

$$\sqrt{\frac{2}{\pi x}} \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right] = \sqrt{\frac{2}{\pi x}} \sin x$$

$$J_\nu(x) = \frac{x^\nu}{2^\nu \Gamma(\nu+1)} \left[1 - \frac{x^2}{4(\nu+1)} + \frac{x^4}{4^2 \cdot 2! \, (\nu+1)(\nu+2)} - \cdots \right]$$

$$\nu = -1/2$$

$$J_{-1/2}(x) = \frac{x^{-1/2}}{2^{-1/2} \, \Gamma 1/2} \left[1 - \frac{x^2}{4 \cdot \frac{1}{2}} + \frac{x^4}{4^2 \cdot 2! \cdot \frac{1}{2} \cdot \frac{3}{2}} - \cdots \right]$$

$$= \sqrt{\frac{2}{\pi x}} \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \right]$$

$$\boxed{J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x}$$

③

$$\frac{2\nu}{x} J_\nu(x) = J_{\nu-1}(x) + J_{\nu+1}(x)$$

$$\nu = 1/2$$

$$\frac{1}{x} J_{1/2}(x) = J_{-1/2}(x) + J_{3/2}(x)$$

$$J_{3/2}(x) = \frac{1}{x} J_{1/2}(x) - J_{-1/2}(x)$$

$$= \frac{1}{x} \sqrt{\frac{2}{\pi x}} \sin x - \sqrt{\frac{2}{\pi x}} \cos x$$

$$= \sqrt{\frac{2}{\pi x}} \left[\frac{\sin x}{x} - \cos x \right]$$