Derivative :-

The derivative of a function $f(x)$ with respect to the variable x is the function $f'(x)$ whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Set $z = x+h$

$$\Rightarrow \quad f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

$f(x) = |x|$ at $x=0$

16 week.
1 mid
1 End
14 ~~lec~~ $\times 3 =$
10 45 /42
$\boxed{25}$

$\lim_{x \to c} f(x) = f(c)$

Theorem Every differentiable function is continuous ?

Proof Let $f(x)$ be a function

Given that $f(x)$ has a derivative at $x = c$

i.e $f'(c)$ exist.

We have to prove that, $f(x)$ is a continuous at $x = c$

i.e $\lim_{h \to 0} f(c+h) = f(c)$

Consider

$$f(c+h) = f(c+h) - f(c) + f(c)$$

$$f(c+h) = f(c) + \frac{f(c+h)-f(c)}{h} h$$

$$\lim_{h\to 0} f(c+h) = \lim_{h\to 0}\left[f(c) + \frac{f(c+h)-f(c)}{h} h \right]$$

$$= \lim_{h\to 0} f(c) + \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} \cdot \lim_{h\to 0} h$$

$$= f(c) + f'(c) \times 0$$

$$= f(c)$$

$$\Rightarrow \lim_{h\to 0} f(c+h) = f(c)$$

$\therefore$ $f(x)$ is a ~~cou~~ continuous at $x = c$

The converse of above theorem statement need not be true.

i.e Every continuous function need not be differentiable

Ex: $|x|$ at $x=0$

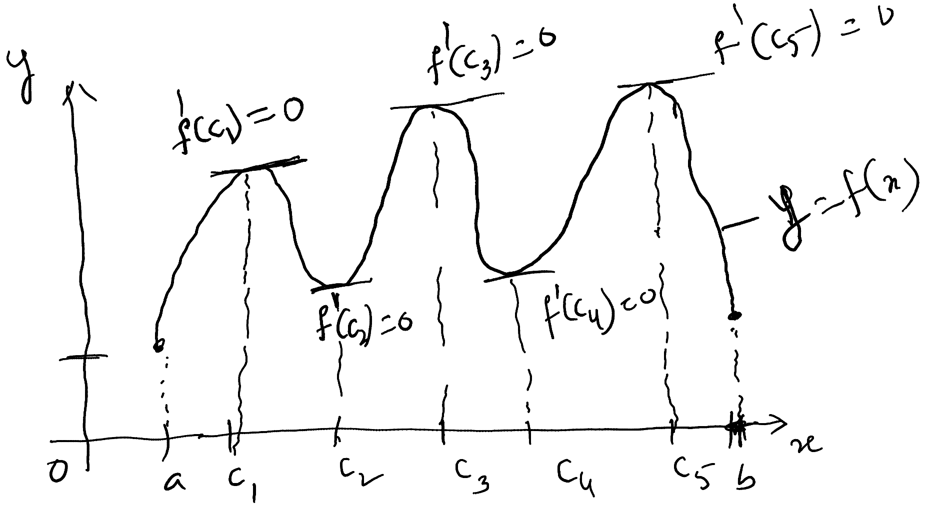
$\sin(x^2 y)$

$x^2 y + x^3 + 3xy^2 = 0 \rightarrow \frac{dy}{dx}$.

Chapter 3

Chapter 4
Application of derivatives

Rolle's theorem:-



$f \rightarrow [a\ b]$

$f(a) = f(b)$

$\exists c \in (a,b) \ni$

$f'(c) = 0$

Suppose that $y = f(x)$ be a function

i) $f(x)$ is continuous over the closed interval $[a\ b]$

ii) $f(x)$ is differentiable in the open interval $(a\ b)$

iii) $f(a) = f(b)$

if there exist at least one point $c \in (a,b)$ such that

$$f'(c) = 0$$

Verify the Rolle's theorem for the following function

$f(x) = (x-a)^m (x-b)^n$ where m, n are the integers in $[a, b]$

Ans

$$f(x) = (x-a)^m (x-b)^n$$

Every polynomial is continuous and differentiable everywhere.

(i) $\therefore$ $f(x) = (x-a)^m (x-b)^n$ is continuous in $[a, b]$

(ii) $$f'(x) = m(x-a)^{m-1}(x-b)^n + n(x-a)^m (x-b)^{n-1}$$

differentiable on (a, b)

(iii) $f(a) = 0 = f(b)$

$$f'(c) = 0$$

$$m(c-a)^{m-1}(c-b)^n + n(c-a)^m (c-b)^{n-1} = 0$$

$$(c-a)^{m-1}(c-b)^{n-1}\left[m(c-b) + n(c-a)\right] = 0$$

$$m(c-b) + n(c-a) = 0$$

$$(m+n)\,c = mb + na$$

$$\boxed{c = \frac{mb+na}{m+n} \in (a, b)}$$

$\therefore$ Rolle's theorem Verified.

Verify Rolle's theorem.

<u>Ex:-</u> $f(x) = (x+2)^3 (x-3)^4$ in $[-2, 3]$

$a = -2,\ b = 3,\ m = 3,\ n = 4$

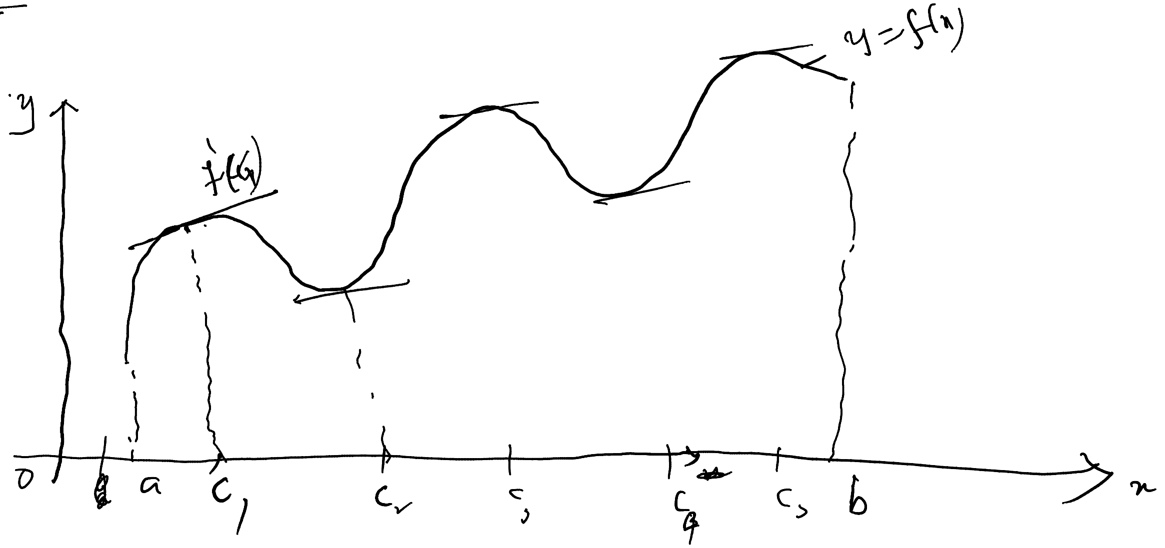
$$c = \frac{9-8}{7} = \frac{1}{7} \in (-2\ 3) \checkmark$$

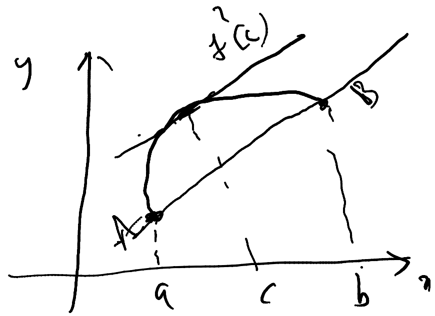
Verified

Mean value theorem :-

$f(x)$ continuous

$f(x)$ differentiable





$A(a, f(a)) \qquad B(b, f(b))$

$$y - f(a) = \frac{f(b) - f(a)}{b-a}(x-a)$$

$$f'(c) = \frac{f(b) - f(a)}{b-a}.$$

Suppose $f(x)$ be a function

(i) $f(x)$ is a continuous on $[a\ b]$

(ii) $f(x)$ is a differentiable on (a, b)

then ~~the~~ there exist atleast one $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Cauchy's mean value theorem:—

Suppose $f(x)$ and $g(x)$ are two functions

(i) $f(x)$ and $g(x)$ are continuous on $[a\ b]$

(ii) $f(x)$ and $g(x)$ are differentiable on (a, b)

(iii) $g'(x) \neq 0$

then there exist a number $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Note

set $g(x) = x$

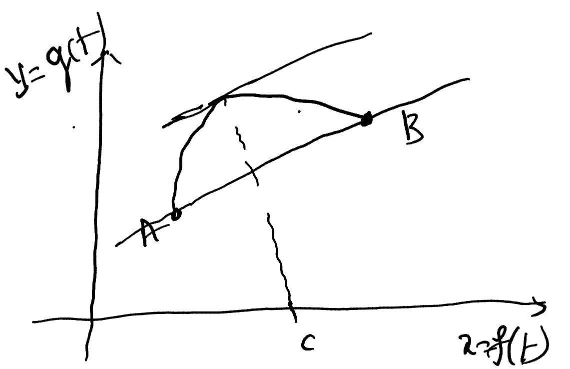
$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

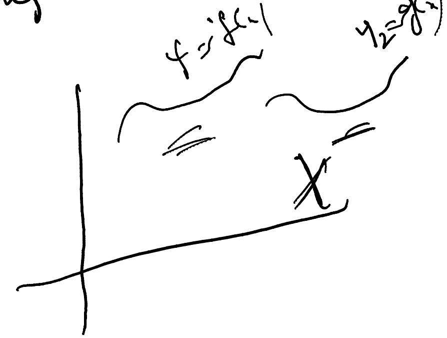
$f(x)$ and $g(x)$

Note $g(x) = x$

$$\frac{f'(c)}{1} = \frac{f(b)-f(a)}{b-a}$$

Cauchy's mean value theorem is reduce to Legrange mean value theorem by setting $g(x) = x$





$x = f(t)$, $y = g(t)$ } parametric representation

$A = (f(a), g(a))$

$B = (f(b), g(b))$

$$\frac{g(b)-g(a)}{f(b)-f(a)}$$

$$\frac{g'(t)}{f'(t)} \ ?$$

$$\Rightarrow \frac{g'(c)}{f'(c)} = \frac{g(b)-g(a)}{f(b)-f(a)}$$

① Verify Lagrange mean value theorem for the function

$$f(x) = (x-4)(x-6)(x-8) \quad \text{in } [4\ 10]$$

(a) $c = 8$?

(b) $c = 5$

(c) $c = 4\ 8$

(d)

Soln

$f(x) = (x-4)(x-6)(x-8)$ is continuous on $[4\ 10]$

$f'(x) = (x-6)(x-8) + (x-4)(x-8) + (x-4)(x-6)$

differentiable on $(4, 10)$

$f'(x) = 3x^2 - 36x + 104$

$c \Rightarrow \quad f'(c) = \dfrac{f(10) - f(4)}{10-4}$

$f'(c) = \dfrac{48 - 0}{6} = 8$

$\Rightarrow \quad 3c^2 - 36c + 104 = 8$

$\Rightarrow \quad c = 4, 8 \in (4\ 10)$

H.W

② Verify the cauchy mean value theorem for the functions $f(x)$ and $f'(x)$ in (a, b) given that $f(x) = \log x$.

Increasing and decreasing functions :-

Suppose that f is continuous on $[a\ b]$ and differentiable on (a, b)

(i) If $f'(x) > 0$ at each point $x \in (a, b)$ then f is increasing on $[a\ b]$

(ii) If $f'(x) < 0$ at each point $x \in (a, b)$ then f is decreasing on $[a, b]$

Proof

x_1, x_2

f on $[x_1\ x_2]$

$c \in (x_1, x_2)$

$\exists\ f'(c) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

$x_1 < x_2$ then $f(x_2) > f(x_1)$ function is increasing

$\Rightarrow$ Sign of (rhs) > 0

$\Rightarrow f'(c) \geq 0$ $\therefore$ function is increasing

Thy f is decreasing if $f'(x) < 0$

$f(x)$

$f'(x) \geq 0$ & $f > 0$

I.D I.- $f'(x) < 0$

$f'(x) \geq 0$ - D, I

$A \quad x_1 \quad x_2 \quad b$

$x_1 < x_2$

$f'(x) = \lim\limits_{h \to 0} \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

$f(x_2) > f(x_1)$

$f(x_2) - f(x_1) > 0$

$\lim\limits_{h \to 0} \dfrac{f(x_2) - f(x_1)}{x_2 - x_1} > 0$

$f'(x) > 0$

1. Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the open intervals on which f is increasing / decreasing.

Soln

$$f(x) = x^3 - 12x - 5$$

$$f'(x) = 3x^2 - 12$$

$f'(x) > 0 \ \forall x$

$f'(x) < 0 \ \forall (x)$

$$\Rightarrow f'(x) = 0$$

$$3x^2 - 12 = 0$$

$$x = \pm 2$$

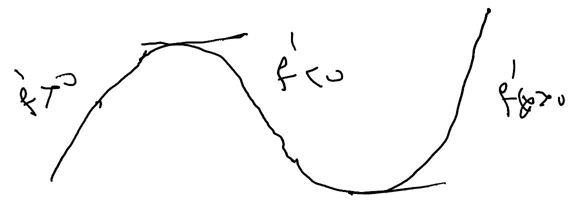
$f' > 0$ $f' < 0$ $f' > 0$

Increa... -2 decre... 2 Increasin

$(-\infty \ \ -2)$ ✓ $\Rightarrow f'(x) > 0$ Increasin

$(-2 \ \ 2)$ ✓ $\Rightarrow f'(x) < 0$ Decrea

$(2 \ \ \infty)$ ✓ $\Rightarrow f'(x) > 0$ Increasing



$x = -2$ local maxim pt

$x = 2$ local mini pt.

First Derivative Test for Local Extrema:—

Suppose that 'c' is a Critical point of a Continuous function $f(x)$, and $f(x)$ is differentiable at every point in some interval containing 'c' except possibly at 'c' itself. Moving across this interval from left to right.

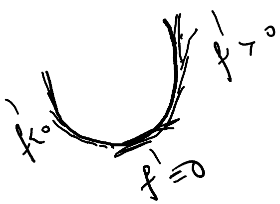
$f'(x) = 0$

① If $f'(x)$ changes from negative to positive at c then f has local minimum at 'c'

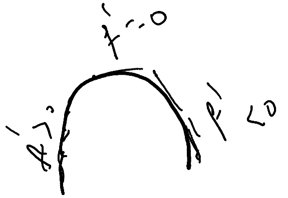
② If $f'(x)$ changes from positive to negative at c then f has a local maximum at c

③ If $f'(x)$ does not change sign at 'c' then f has no local extremum at c

Def The graph of differentiable function $y = f(x)$ is



(i) Concave up on an open interval I if f' is increasing on I



(ii) Concave down on an open interval I if f' is decreasing on I

Second derivative Test for Concavity:—

Let $y = f(x)$ be twice-differentiable on an interval I

(i) If $f'' > 0$ on I, the graph of $f(x)$ over I is concave up

(ii) If $f'' < 0$ on I, the graph of $f(x)$ over I is Concave Down.

Point of inflection:- neither maxima nor minima ✓
saddle point

if $f''(x) > 0$ then f has local minimum

if $f(x) < 0$ then f has Local maximum

<u>Taylor Series</u>

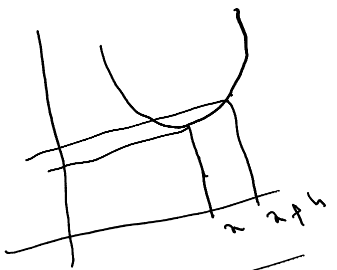
$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \cdots$$

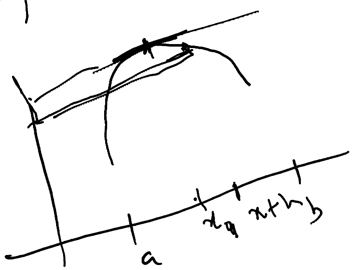
$A < 0$ max

f has max if $f(x) > f(x+h)$? $\Rightarrow$ $f(x+h) - f(x) < 0$

min if $f(x) < f(x+h)$ $\Rightarrow$ $A > 0$ Min

Set $A = f(x+h) - f(x)$





$f'(x) = 0$

$$f(x+h) = f(x) + \cancelto{0}{h f'(x)} + \frac{h^2}{2!} f''(x)$$

$$f(x+h) - f(x) = \frac{h^2}{2!} f''(x)$$

$$A = \frac{h^2}{2!} f''(x)$$

$$\operatorname{sign}(\Delta) = \operatorname{sign}(f''(x))$$

(i) $f''(x) > 0$ if $\Delta > 0$

$\therefore$ f has local minima.

(ii) $f''(x) < 0$ if $\Delta < 0$

$\therefore$ f has local maxima.

Ex:-

① Determine the concavity and find the inflection points of the function

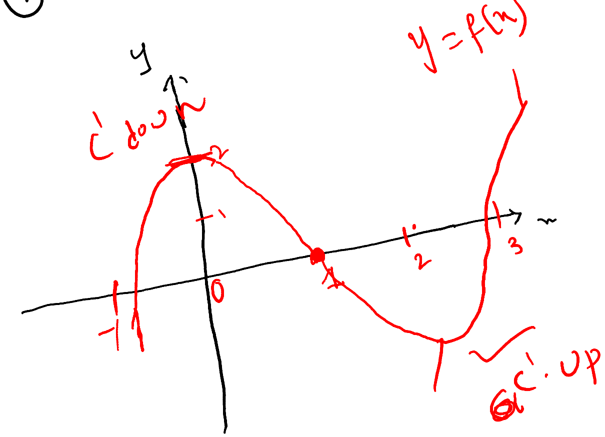
$f(x) = x^3 - 3x^2 + 2$.

$f'(x) = 0$

$3x^2 - 6x = 0$

$x^2 - 2x = 0$

$x = 0, 2$



$x = 1$

$f''(x) = 0$ — point

$x \neq 1$

- ✓ Concave up if $x > 1$
- ✓ concave down if $x < 1$

(2) Sketch a graph of the function

$$f(x) = x^4 - 4x^3 + 10$$

using the following steps

(i) Identify where the extrema of f occur

(ii) Find the intervals on which f is increasing / decreasing

(iii) Find where the graph of f is concave up and concave down

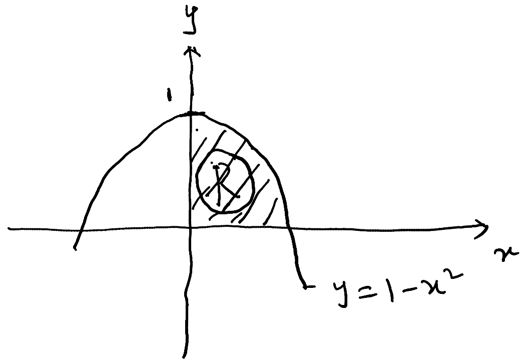
(iv) sketch the general shape of the graph 'f'

(v) plot some specific points such as local maximum and minimum points, points of inflection, and intercepts then sketch the curve.

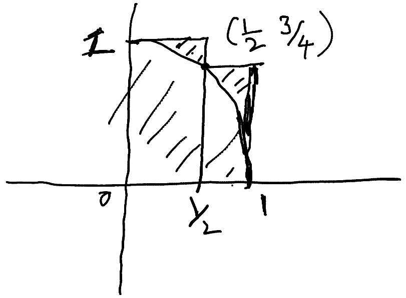
Try yourself !

Integrals :-

$y = 1 - x^2$

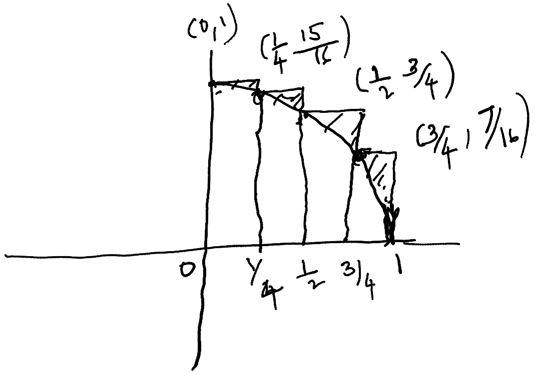


Area of the $y = 1 - x^2$ x-axis



$A \approx 1 \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{2}$

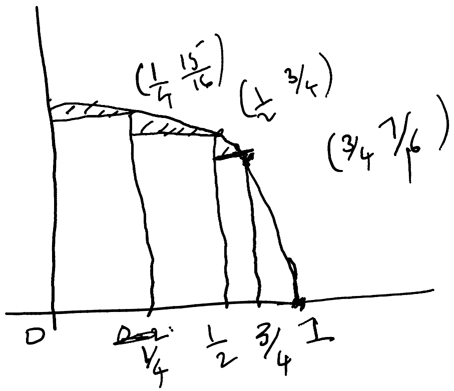
≈ 0.875



$A \approx 1 \cdot \frac{1}{4} + \frac{15}{16} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} + \frac{7}{16} \cdot \frac{1}{4}$

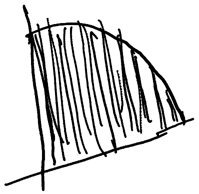
≈ 0.78125

Lower Sum

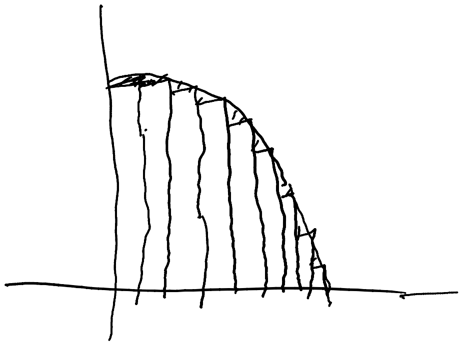


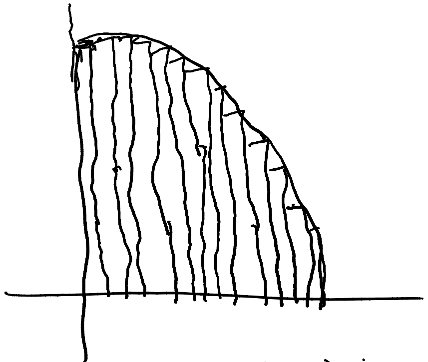
$$A \simeq \frac{15}{16}\cdot\frac{1}{4} + \frac{3}{4}\cdot\frac{1}{4} + \frac{7}{16}\cdot\frac{1}{4} + 0\cdot\frac{1}{4}$$

$$\simeq 0.53125$$



$$0.53125 < A < 0.78125$$





$$f(c_1)\,\Delta x + f(c_2)\,\Delta x + f(c_3)\,\Delta x + \dots + f(c_n)\,\Delta x$$

$n \to \infty$

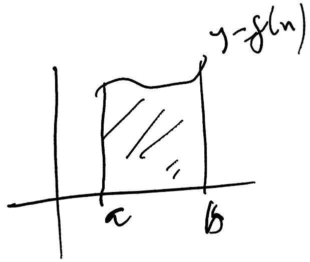
Sum $\quad S \quad \int$

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

b — upper limit of integration

Integral sign $\quad \int_a^b \underbrace{f(x)}\,dx$ — x var. of integration

a — lower limit $\qquad$ $f(x)$ — Integrand



Def: If $y = f(x)$ is non-negative and integrable over a closed interval $[a\ b]$, then the area under the curve $y = f(x)$ over $[a\ b]$ is the integral of f from a to b

$$A = \int_a^b f(x)\,dx$$

Mean value theorem for Definite integrals: —

<u>Theorem</u>:— If $f(x)$ is continuous on $[a\,b]$ then at some point c in $[a\,b]$

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx .$$

Avg value.

<u>Theorem</u>:- The fundamental theorem of calculus part-I

If f is continuous on $[a\,b]$ then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a\,b]$ and differentiable on $(a\,b)$ and its derivative is $f(x)$

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

F.T

<u>part-II</u> If f is continuous over $[a,b]$ and F is any antiderivative of f on $[a\,b]$ then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Substitution in Definite Integrals: —

If g' is continuous on the interval $[a\ b]$ and f is continuous on the range of $g(x) = u$ then

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

$F \to f$

Proof

$$\int_a^b f(g(x))\, g'(x)\, dx = F(g(x)) \Big|_a^b$$

$$= F(g(b)) - F(g(a))$$

$$= F(u) \Big|_{u=g(a)}^{g(b)}$$

$$= \int_{u=g(a)}^{g(b)} f(u)\, du .$$

Rules satisfied by definite integrals

1. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

2. $\int_a^a f(x)\,dx = 0$

3. $\int_a^b K\, f(x)\,dx = K \int_a^b f(x)\,dx$.

4. $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$.

5. $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$.

6 Max_Min inequality

If $f(x)$ has maximum value $\max f$ and minimum value $\min(f)$ on $[a\ b]$ then

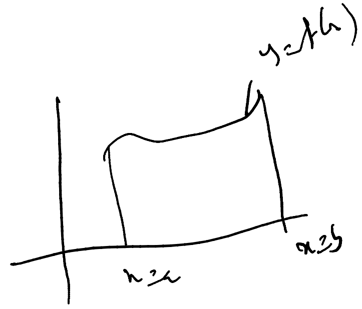
$$\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le (\max f) \cdot (b-a)$$

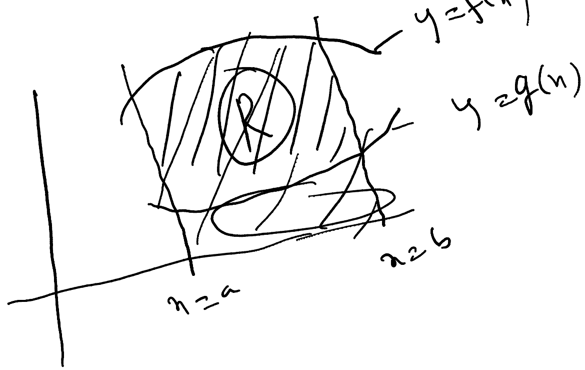
7. If $f(x) \geq g(x)$ on $[a\ b]$

$$\Rightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

8 If $f(x) \geq 0$ on $[a, b]$ then

$$\int_a^b f(x) \geq 0.$$





$$\int_a^b [f(x) - g(x)]\,dx$$

Area under the curve $y = f(x)$ and $x = a$, $x = b$

$$A = \int_a^b f(x)\,dx$$

Area between Curves :-

Def

Required Area of R

$$\int_a^b f(x)\,dx$$

Area under the curve $y = g(x)$ $x = a$ & $x = b$

$$\int_a^b g(x)\,dx$$

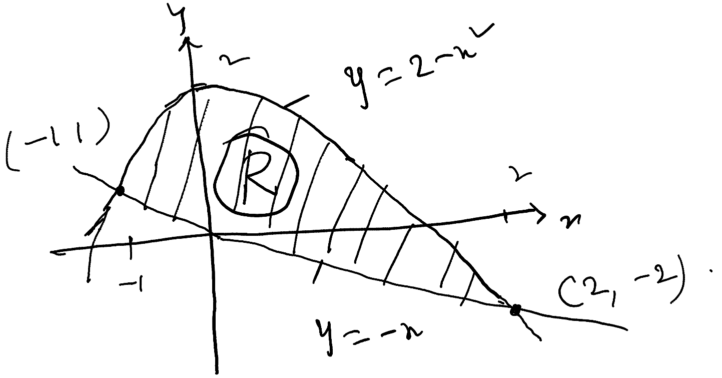
$$A_R = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

$$= \int_a^b [f(x) - g(x)]\,dx \quad \text{—?}$$

Def If $f(x)$ and $g(x)$ are continuous with $f(x) \geq g(x)$ throughout $[a\ b]$ then the area of the region between the curves $y=f(x)$ and $y=g(x)$ from a to b is

$$A = \int_a^b \left[f(x)-g(x)\right] dx.$$

Ex:- Find the area of the region enclosed by the parabola $y=2-x^2$ and the line $y=-x$



4.5
1.5

$y=2-x^2$

$y=-x$

$2-x^2=-x$

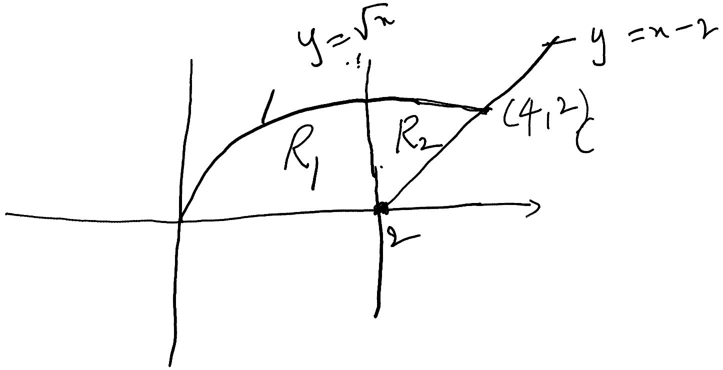
$x=-1, 2$

$$\text{Area} = \int_a^b \left[f(x)-g(x)\right] dx$$

$$= \int_{-1}^{2} \left[2-x^2-(-x)\right] dx = 2x - \frac{x^3}{3} + \frac{x^2}{2} \Big|_{-1}^{2}$$

$$= 9/2 = 4.5$$

(2) Find the area of the region in the first quadrant that is bounded below $y=\sqrt{x}$ and above by x-axis and the line $y=x-2$.



(a) $\frac{16}{3}$

+3 - (b) $10/3$

(c)

$$\text{Area} = \int_{R_1} f(x)\,dx + \int_{R_2} f(x)\,dx$$

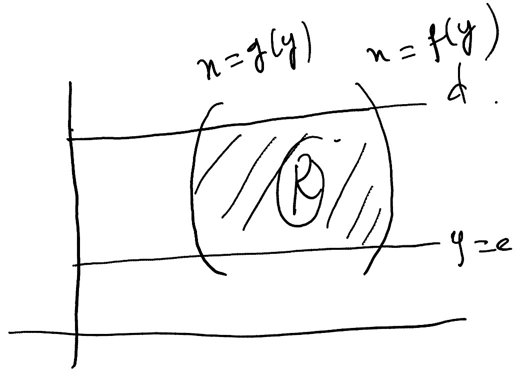
$\int_{R_1} f(x)\,dx$: Region R_1

$$\int_{R_1} f(x)dx = \int_0^2 (\sqrt{x}-0)\,dx =$$

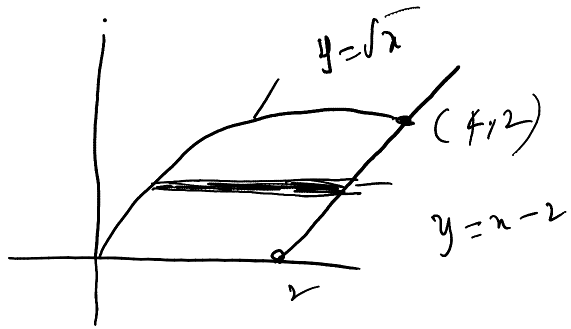
$$\int_{R_2} f(x)dx = \int_2^4 \left[\sqrt{x} - (x-2)\right] dx$$

$$\text{Area} = \int_{R} f(x)dx + \int_{R_2} f(x)dx = 10/3 \text{ //}$$

Integration w.r.t y



$$\text{Area} = \int_{y=c}^{d} \left[f(y) - g(y)\right] dy$$



$$\int_{y=0}^{2} \left[(y+2) - y^2\right] dy$$

$$= \frac{y^2}{2} + 2y - \frac{y^3}{3} \Big|_0^2$$

$$= \frac{10}{3}$$