A string is streched and fastened to two points l apart. Motion is started by displacing the string in the form $y = A\sin\left(\frac{\pi x}{l}\right)$ from which it is released at time ~~to~~ $t=0$. Show that the displacement of any point at distance x from one end at time t is given by

$$y(x,t) = A\sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{\pi c t}{l}\right)$$

The eqn of string is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

$u(x,0) = f(x)$

$\frac{\partial u}{\partial t} = g(x)$

$y(0,t) = 0$

$y(l,t) = 0$

$y(x,0) = A\sin\left(\frac{\pi x}{l}\right)$

$\frac{\partial y}{\partial t} = 0$

$y($

$$y(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{l}\right) \left[A_n \cos\left(\frac{n\pi ct}{l}\right) + B_n \sin\left(\frac{n\pi ct}{l}\right) \right]$$

$$A_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$B_n = \frac{2}{n\pi c} \int_0^l g(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$B_n = 0$ since $g(x) = 0$

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi ct}{l}\right)$$

→ Solution of the Heat equation:-

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < L \quad t > 0$$

$$u(0,t) = u(L,t) = 0 \qquad t \geq 0$$

$$u(x,0) = f(x)$$

$$u(x,t) = X(x)\,T(t)$$

$$\frac{\partial u}{\partial t} = XT' \qquad \frac{\partial^2 u}{\partial x^2} = X''T$$

$$XT' = c^2 X''T \quad \Rightarrow \quad \frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T} = k.$$

$$X'' - kX = 0 \quad \text{and} \quad T' - kc^2 T = 0$$

Case 1 $\quad k = 0 \qquad X'' = 0 \quad$ and $\quad T' = 0$

$$X = c_1 x + c_2 \qquad T = c_3.$$

$$u(x,t) = c_3(c_1 x + c_2) \quad \Rightarrow \quad u(x,t) = 0.$$

Case 2 $\quad K > 0$ i.e $K = \lambda^2$

$$X'' - \lambda^2 X = 0 \qquad T' - \lambda^2 c^2 T = 0$$

$$X(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x} \qquad T = c_3 e^{c^2 \lambda^2 t} \quad ?$$

$u = X(x)\,T(t)$
$= 0$ ✗

$$u(x,t) = 0 \quad ✗$$

Case 3 $\quad K < 0$ i.e $K = -\lambda^2$

$$X'' + \lambda^2 X = 0 \qquad T' + \lambda^2 c^2 T = 0$$

$$X(x) = c_1 \cos(\lambda x) + c_2 \sin(\lambda x) \quad ✓ \quad \text{and} \quad T = c_3 e^{-c^2 \lambda^2 t}$$

$u\, X(x)\,T(t) = 0$
$X(0)\,T(t) = 0$
$X(0) = 0$

$\cancel{u(x,0)} = 0 \qquad u(0,t) = 0$

$\cancel{u(x,L)} = 0 \qquad u(L,t) = 0$

$$u(x,t) = \left[c_1 \cos(\lambda x) + c_2 \sin(\lambda x)\right] c_3 e^{-c^2 \lambda^2 t} \quad !$$

$$X(0) = 0 \qquad X(L) = 0$$

$$X(0) = 0 = c_1(1) + c_2(0) \Rightarrow \underline{c_1 = 0}$$

$$X(x) = c_2 \sin(\lambda x)$$

$$X(L) = 0$$

$$X(L) = 0 = c_2 \sin(\lambda L)$$

$$\underline{c_2} \sin \underline{(\lambda L)} = 0 \quad \rightarrow \quad c_2 = 0 \ \times$$

$$\sin(\lambda L) = 0$$

$$\lambda L = n\pi$$

$$\lambda = \frac{n\pi}{L}$$

$$X(x) = c_2 \sin\left(\frac{n\pi x}{L}\right) \quad \checkmark$$

$$T(t) = c_3 e^{-c^2 \lambda^2 t} = c_3 e^{-\left(\frac{n\pi c}{L}\right)^2 t}$$

$$u_n(x,t) = A_n e^{-\left(\frac{n\pi c}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, 3 \ldots$$

Superposition principle

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi c}{L}\right)^2 t}$$

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

$$A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi c}{L}\right)^2 t} \quad \text{where } A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx}$$

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$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad c^2 = 1 \qquad L = 1$$

I.C $\quad u(x,0) = 3\sin(\pi x) \qquad 0 < x < 1 \qquad t > 0$

B.C $\quad u(0,t) = 0$ and $u(1,t) = 0$

General soln of heat eqn

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)\, e^{-n^2\pi^2 t}$$

$$A_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx$$

$$A_n = 2\int_0^1 3\sin(\pi x)\sin(n\pi x)\,dx$$

$$= 6\int_0^1 \sin(\pi x)\sin(n\pi x)\,dx$$

$$= 3\int_0^1 \left[\cos(n-1)\pi x - \cos(n+1)\pi x\right]dx.$$

$$= 3\left[\frac{\sin(n-1)\pi x}{n-1} - \frac{\sin(n+1)\pi x}{n+1}\right]_0^1$$

$= 0$ ✓ $\qquad n-1 \neq 0 \quad \cancel{n=1}$

$\underline{\underline{n=1}}$

$$A_1 = 2\int_0^1 3\sin(\pi x)\sin(\pi x)\,dx$$

$$A_1 = \cancel{6}\int_0^1 \frac{1-\cos 2\pi x}{\cancel{2}}\,dx$$

$$A_1 = 3\left[x - \frac{\sin 2\pi x}{2\pi}\right]_0^1$$

$$\underline{\underline{A_1 = 3}}$$

$$\boxed{u(x,t) = \cancel{\sum_{n=1}^{\infty}} 3\sin(n\pi x)\, e^{-n^2\pi^2 t}}$$