Sturm - Liouville problem :- (SLP)

Any differential equation of the form

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + \left[q(x) + \lambda\, r(x)\right] y = 0 \quad \text{--- (1a) on } [a, b]$$

λ real constant

$$a_1 y(a) + a_2 y'(a) = 0$$
$$b_1 y(b) + b_2 y'(b) = 0 \quad \text{--- (1b)}$$

where a_1, a_2, b_1, b_2 are real constant

both a_1, a_2 are not zero

$b_1 \; b_2$

$p(x), p'(x), q(x)$ and $r(x)$ real valued continuous fn on $[a, b]$

$p(x) > 0$ and $r(x) > 0$

$$(1a) \Longrightarrow \; -\left[\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x)\right] y = \lambda\, r(x)\, y$$

S-L operator

Assm $L = -\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) - q(x)$

$$Ly = \lambda\, r(x)\, y$$

$r(x)$ weight function

$r(x) \equiv 1$

$$\boxed{Ly = \lambda y} \qquad \boxed{Ax = \lambda x}$$

λ is an eigenvalue of L corresponding to the eigenfunction y

$y \equiv 0$ is always soln of BVP

↳ trivial soln.

orthogonality

$$\int_a^b y_m(x)\, y_n(x)\, r(x)\, dx = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

$r(x)\, y_n^2(x)$

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 5y = 0 \quad ? \qquad \text{SLP}$$

$\hookrightarrow$ Legendre's differential eqn.

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + (q(x) + \lambda r(x))y = 0$$

$$p(x)\frac{d^2y}{dx^2} + p'(x)\frac{dy}{dx} + q(x)y + \lambda r(x)y = 0$$

$$p = 1-x^2 \qquad p' = -2x \qquad q(x) = 0 \qquad \lambda = 5$$

$\rightarrow$ $x^2y'' + xy' + 2y = 0$? No.

$\rightarrow$ $y'' - 2xy' + 2ny = 0$

$\rightarrow$ $y'' + \lambda y = 0$ Yes

$\rightarrow$ $x^2y'' + xy' + (x^2 - \nu^2)y = 0$

$\rightarrow \quad xy'' + (1-x)y' + xy = 0$

$\rightarrow \quad (1-x^2)y'' - xy' + n^2 y = 0$

$a_2(x)\,y'' + a_1(x)\,y' + a_0(x)\,y = \cancel{f(x)}$ $\qquad a_2 \neq 0.$

$$y'' + \frac{a_1(x)}{a_2(x)}\,y' + \frac{a_0(x)}{a_2(x)}\,y = 0$$

$\mu(x)$

$$\underline{\mu(x)y'' + \mu(x)\frac{a_1(x)}{a_2(x)}\,y'} + \mu\,\frac{a_0(x)}{a_2(x)}\,y = 0$$

$(\mu y')'$ if $\mu(x)$ must satisfy.

$$\frac{d}{dx}\left[\mu(x)\right] = \mu(x)\,\frac{a_1(x)}{a_2(x)}$$

$$\boxed{\mu(x) = e^{\int \frac{a_1(x)}{a_2(x)}\,dx}}$$

$$\mu(x) = e^{\int \frac{a_1(x)}{a_2(x)} dx}.$$

$$\frac{\mu(x)}{a_2(x)} = \frac{e^{\int \frac{a_1(x)}{a_2(x)}}}{a_2(x)}$$

$$x^2 y'' + xy' + 2y = 0$$

$$a_2 y'' + a_1 y' + a_0 y = 0$$

$$\mu_1 = \frac{e^{\int x/x^2 dx}}{x^2} = \frac{e^{\int \frac{1}{x} dx}}{x^2} = 1/x$$

$$\underline{xy'' + y'} + \frac{2}{x} y = 0$$

$$(xy')' + \frac{2}{x} y = 0 \qquad SLP.$$

$$xy'' + (1-x)y' + \cancel{x}y = 0$$

$$\frac{e^{\int \frac{1-x}{x}\,dx}}{x} = \frac{e^{\int\left(\frac{1}{x}-1\right)dx}}{x} = e^{-x}$$

$$\underline{e^{-x}x}\,y'' + \underline{e^{-x}(1-x)}\,y' + \cancel{x}e^{-x}y = 0$$

SLP .

$e^{-x}x$ — ?

$$(xe^{-x}y')' + ne^{-x}y = 0$$

$$\boxed{\mu_1 = \frac{e^{\int \frac{a_1(x)}{a_2(x)}dx}}{a_2(x)}}$$