Bessels equation

$$x^2y'' + xy' + (x^2-\nu^2)y = 0$$

Indicial equation $m^2 - \nu^2 = 0$

$$m = \pm\nu$$

Recurrence relation

$$a_n = -\frac{a_{n-2}}{(m+n)^2-\nu^2} \qquad n \geq 2$$

$$a_0 = \frac{1}{2^\nu \Gamma(\nu+1)}$$

Case i ν neither zero nor integer

$$y(x) = A J_\nu(x) + B J_{-\nu}(x)$$

where $J_\nu(x) = \sum_{k=0}^{\infty} (-1)^k \left(\frac{x}{2}\right)^{\nu+2k} \cdot \frac{1}{k!\,\Gamma(\nu+k+1)}$

case ii when $\nu = 0$

$$x^2 y'' + xy' + (x^2 - \nu^2) y = 0$$

$$x^2 y'' + xy' + x^2 y = 0$$

Indicial roots : 0, 0

Complete soln

$$y(x) = A\, y_1(x) + B\, y_2(x)$$

where $y_1(x) = y_m(x)\Big|_{m=0}$

$$y_2(x) = \frac{\partial y}{\partial m}\Big|_{m=0}.$$

$$y_m(x) = a_0 x^m \left[1 - \frac{x^2}{(m+2)^2} + \frac{x^4}{(m+4)^2 (m+2)^2} + \cdots \right]$$

$$y_1(x) = y_m(x)\Big|_{m=0}$$

$$y_1(x) = a_0 \left[1 - \frac{x^2}{2^2} + \frac{x^4}{4^2 \cdot 2^2} - \frac{x^6}{6^2 \cdot 4^2 \cdot 2^2} + \cdots \right]$$

$$\frac{\partial y}{\partial m} = a_0 x^m \log x \left[1 - \frac{x^2}{(m+2)^2} + \frac{x^4}{(m+4)^2(m+2)^2} - \cdots \right]$$

$$+ a_0 x^m \left[0 + \frac{x^2 \cdot 2}{(m+2)^3} + x^4 \left\{ \frac{-2}{(m+4)^3 (m+2)^2} - \frac{2}{(m+4)^2 (m+2)^3} \right\} - - - \right]$$

$$y_2(x) = \left. \frac{\partial y}{\partial m} \right|_{m=0} .$$

$$y_2(x) = a_0 \log x \left[1 - \frac{x^2}{2^2} + \frac{x^4}{4^2 \cdot 2^2} - - - \right] + a_0 \left[\frac{x^2}{2^2} - \frac{2}{4^2 \cdot 2^2} \cdot \frac{3}{4} x^4 + - - \right]$$

Case ii ν integer

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \quad (*)$$

$$y(x) = A y_1(x) + B y_2(x)$$

$$y_1(x) = J_\nu(x)$$

Assume $y = u(x)\, y_1$

$$y(x) = u J_\nu$$

$$y' = u' J_\nu + u J_\nu'$$

$$y'' = u'' J_\nu + u' J_\nu' + u J_\nu'' + u' J_\nu'$$

$$= u'' J_\nu + 2u' J_\nu' + u J_\nu''$$

$$(*) \Rightarrow x^2\left[u'' J_\nu + 2u' J_\nu' + u J_\nu''\right] + x\left[u' J_\nu + u J_\nu'\right] + (x^2 - \nu^2)\, u J_\nu = 0$$

$$x^2 u'' J_\nu + 2x^2 u' J_\nu' + x u' J_\nu + \left[x^2 J_\nu'' + x J_\nu' + \cancel{(x^2 - \nu^2) J_\nu}\right] u = 0$$

J_ν

$$x^2 u'' J_\nu + 2x^2 u' J_\nu' + x u' J_\nu = 0$$

divide $- x^2 u' J_\nu$

$$\int \frac{u''}{u'} + 2\int \frac{J_\nu'}{J_\nu} + \int \frac{1}{x} = 0 \Rightarrow \log u' + 2\log J_\nu + \log x = \log B$$

$$\log(u' J_\nu^2 x) = \log B$$

$$u' J_\nu^2 x = B.$$

$$u' = \frac{B}{J_\nu^2 x}$$

Integrate w.r.t x

$$u(x) = \int \frac{B}{x[J_\nu(x)]^2} dx + \cancel{B}\cancel{C}\ A$$

Complete soln

$$y(x) = u(x)\, J_\nu(x)$$

$$= \left[\int \frac{B}{x J_\nu^2(x)} dx + A\right] J_\nu(x)$$

$$y(x) = A J_\nu(x) + B J_\nu(x) \cdot \int \frac{dx}{x[J_\nu(x)]^2}$$

$$\boxed{y(x) = A J_\nu(x) + B Y_\nu(x)}$$

Where

$$Y_\nu(x) = J_\nu(x) \int \frac{dx}{x[J_\nu(x)]^2}$$ — Bessels function of Second Kind of order ν.

y_1 and y_2

$y_2 = (c_1 x + c_2) y_1$

$\boxed{y_2 = (c_1 x + c_2) e^{-x}}$

$= c_1 x e^{-x} + c_2 e^{-x}$

$y = (c_1 x + c_2) e^{-x}$

$= c_1 x e^{-x} + c_2 e^{-x}$

$e^{-x} + (-e^{-x})$

$y'' + 2y' + y = 0$

$y = e^{mx}$

$m^2 + 2m + 1 = 0$

$(m+1)^2 = 0$

$m = -1, -1$

$y_1 = e^{-x}$

$y_2 = e^{-x} x$

$c_1 e^{-x} + c_2 e^{-x}$

$(c_1 + c_2) e^{-x}$

$c e^{-x}$

Assume

$\boxed{y_2 = u(x) y_1}$

$y_2'' + 2y_2' + y_2 = 0$

$(u y_1)'' + 2(u y_1)' + u y_1 = 0$

$(u'' y_1 + 2u' y_1' + u y_1'') + 2(u' y_1 + u y_1') + u y_1$

$u'' y_1 + 2u'(y_1' + y_1) + \cancel{(y_1'' + 2y_1' + y_1)} u = 0$

$u'' y_1 + 2 \cancel{(y_1 + y_1')} u' = 0$

$y_2 = x y_1$

$y_2 = u(x) y_1$

$u'' y_1 = 0$

$u'' = 0$

$u = c_1 x + c_2$