$$x^2 y'' + xy' + (x^2-4)y = 0 \quad — (1)$$

$x=0$ singular point

$$(1) \Rightarrow \quad y'' + \frac{1}{x} y' + \frac{(x^2-4)}{x^2} y = 0$$

$$\lim_{x \to 0} (x-0) \cdot \frac{1}{x} = 1$$

$$\lim_{x \to 0} (x-0)^2 \frac{x^2-4}{x^2} = -4$$

$x=0$ RSP

$$y(x) = \sum_{n=0}^{\infty} a_n x^{m+n}$$

$$y'(x) = \sum_{n=0}^{\infty} (m+n) a_n x^{m+n-1}$$

$$y''(x) = \sum_{n=0}^{\infty} (m+n)(m+n-1) a_n x^{m+n-2}$$

$$① \Rightarrow \sum_{n=0}^{\infty} (m+n)(m+n-1)\, a_n x^{m+n} + \sum_{n=0}^{\infty} (m+n)\, a_n x^{m+n} + \sum_{n=0}^{\infty} a_n x^{m+n+2} - 4 \sum_{n=0}^{\infty} a_n x^{m+n} = 0$$

$$(n \to n-2)$$

$$\Rightarrow \sum_{n=0}^{\infty} \left[(m+n)^2 - 4\right] a_n x^{m+n} + \sum_{n=2}^{\infty} a_{n-2}\, x^{m+n} = 0$$

$$(m^2-4)\, a_0 x^m + \left[(m+1)^2 - 4\right] a_1 x^{m+1} + \sum_{n=2}^{\infty} \left[\left\{(m+n)^2 - 4\right\} a_n + a_{n-2}\right] x^{m+n} = 0$$

x^m

$$(m^2-4)\, a_0 = 0, \quad a_0 \neq 0$$

$$m^2 - 4 = 0 \Rightarrow m = \pm 2 \quad \text{— Indicial roots}$$

$$2 \qquad -2$$

x^{m+1}

$$\left[(m+1)^2 - 4\right] a_1 = 0$$

$$a_1 = 0$$

x^{m+n}

$$a_n = \frac{-a_{n-2}}{\left[(m+n)^2 - 4\right]}, \quad n \geq 2$$

$$a_n = \frac{-1}{(m+n+2)(m+n-2)} a_{n-2} \qquad n \geq 2$$

$$n=2 \Rightarrow a_2 = \frac{-1}{(m+4)m} a_0 \qquad n=3 \Rightarrow a_3 = 0$$

$$n=4 \Rightarrow a_4 = \frac{1}{(m+6)(m+4)(m+2)m} a_0$$

$$y_m(x) = a_0 x^m \left[1 - \frac{x^2}{(m+4)m} + \frac{x^4}{(m+6)(m+4)(m+2)m} - \cdots \right] \quad \text{—} (2)$$

$$y_1(x) = y_m(x)\Big|_{m=LIR=2}$$

$$= a_0 x^2 \left[1 - \frac{x^2}{6\cdot 2} + \frac{x^4}{8\cdot 6\cdot 4\cdot 2} - \cdots \right]$$

Remark $y_m(x)$ becomes infty at $m = m_0$. $a_0 = b_0(m - m_0)$, $b_0 \neq 0$

Well defined.

$$a_0 = b_0(m+2) \longleftarrow$$

② $\Rightarrow$

$$y_m(x) = b_0 x^m \left[(m+2) - \frac{m+2}{(m+4)m} x^2 + \frac{1}{(m+6)(m+4)m} x^4 - \cdots \right]$$

$$y_2(x) = \left.\frac{\partial y_m}{\partial m}\right|_{m=-2}$$

$$y_2(x) = b_0 x^m \underline{\underline{\ln x}} \left[(m+2) - \frac{m+2}{(m+4)m} x^2 + \frac{1}{(m+6)(m+4)m} x^4 - \cdots \right]$$

$$+ b_0 x^m \left[1 - \left\{ \frac{1}{(m+4)m} + (m+2)\left(\frac{-1}{(m+4)^2 m} - \frac{1}{(m+4)m^2} \right) \right\} x^2 \right.$$

$$\left. + \left\{ \frac{-1}{(m+6)^2(m+4)m} + \frac{1}{m+6}\left(\frac{-1}{(m+4)^2 m} - \frac{1}{(m+4)m^2} \right) \right\} x^4 - \cdots \right]$$

$$y_2(x) = b_0 x^{-2} \ln x \left[\frac{-x^4}{4 \cdot 2^2} + \frac{x^6}{2^3 \cdot 4 \cdot 6} - \cdots \right]$$

$$+ b_0 x^{-2} \left[1 + \frac{x^2}{2^2} + \cdots \right]$$

Complete soln

$$\boxed{y(x) = A\,y_1(x) + B\,y_2(x)}$$

x^{m+n}

Solve $x^2y'' + 6xy' + (6+x^2)y = 0$

$x=0$ RSP

$$\sum_{n=0}^{\infty}(m+n)(m+n-1)a_n x^{m+n} + 6\sum_{n=0}^{\infty}(m+n)a_n x^{m+n}$$

$$+6\sum_{n=0}^{\infty}a_n x^{m+n} + \sum_{n=0}^{\infty}a_n x^{m+n+2} = 0$$

$n \to n-2$

$$\sum_{n=0}^{\infty}\left[(m+n)(m+n+5)+6\right]a_n x^{m+n} + \sum_{n=2}^{\infty}a_{n-2}x^{m+n} = 0$$

$$\left[m(m+5)+6\right]a_0 x^m + \left[(m+1)(m+6)+6\right]a_1 x^{m+1}$$

$$+\sum_{n=2}^{\infty}\left[\left\{(m+n)(m+n+5)+6\right\}a_n + a_{n-2}\right]x^{m+n} = 0$$

$\underline{\underline{x^m}}$ $\quad [m(m+5)+6]a_0 = 0$

m_1, m_2

$m^2+5m+6=0 \Rightarrow m = -2, -3$

$\underline{x^{m+1}}$ $\quad [(m+1)(m+6)+6]\, a_1 = 0$

$m=-2 \Rightarrow 2a_1 = 0 \Rightarrow a_1 = 0$

$m=-3 \Rightarrow 0=0$

Complete soln at $m=-3$

$$a_n = \frac{-1}{[(m+n)(m+n+5)+6]}\, a_{n-2}, \quad n \geq 2$$

$\underline{m=-3}$

$$a_n = \frac{-a_{n-2}}{(n-3)(n+2)+6}, \quad n \geq 2$$

$n=2$ $\quad a_2 = \dfrac{-a_0}{2}$ $\qquad n=3 \Rightarrow a_3 = \dfrac{-a_1}{6}$

$n=4$ $\quad a_4 = \dfrac{-a_2}{12} = \dfrac{a_0}{4!}$ $\qquad n=5 \Rightarrow a_5 = \dfrac{-a_3}{20} = \dfrac{a_1}{5!}$

$$y(x) = x^m\left[a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots\right]$$

$$= a_0 x^{-3}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right]$$

$$+ x^{-3} a_1\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right]$$

$$\boxed{y(x) = \left(a_0 \cos x + a_1 (\sin x)\right) / x^3}$$

$$a_n = \frac{-a_{n-2}}{\left[(n-2)(n+3)+6\right]} \quad n \geq 2$$

$n \Rightarrow 2 \Rightarrow \quad a_2 = \frac{-a_0}{6}$ $\qquad n=3 \Rightarrow a_3 = 0$ $\qquad y = a_0 x^{-3} \sin x$

$n=4 \Rightarrow \quad a_4 = -\frac{a_2}{20} = \frac{a_0}{120} = \frac{a_0}{5!}$

$$y(x) = x^{-3} a_0\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right]$$