<u>Sturm - Liouville problem</u> :- (SLP)

Any differential equation of the form

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + \left[q(x) + \lambda\, r(x)\right] y = 0 \quad \text{— (1a)} \quad \text{on } [a, b]$$

λ real constant

$$a_1 y(a) + a_2 y'(a) = 0$$
$$b_1 y(b) + b_2 y'(b) = 0 \quad \text{— (1b)}$$

where a_1, a_2, b_1, b_2 are real constant

both a_1, a_2 ~~are~~ not zero
$b_1\ b_2$

$p(x)$, $p'(x)$, $q(x)$ and $r(x)$ real valued continuous fn on $[a, b]$

$p(x) > 0$ and $r(x) > 0$

$$\text{(1a)} \Longrightarrow -\left[\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x)\right] y = \lambda\, r(x)\, y$$

S-L operator

Assume $L = -\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) - q(x)$

$$Ly = \lambda\, r(x)\, y$$

$r(x)$ weight function

$r(x) \equiv 1$

$$\boxed{Ly = \lambda y} \qquad \boxed{Ax = \lambda x}$$

λ is an eigenvalue of L corresponding to the eigenfunction y

$y \equiv 0$ is always soln of BVP

↳ trivial soln.

orthogonality

$$\int_a^b y_m(x)\, y_n(x)\, r(x)\, dx = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

$r(x)\, y_n^2(x)$

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 5y = 0 \quad ? \qquad \underline{\underline{SLP}}$$

↳ Legendre's differential eqn.

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + (q(x) + \lambda r(x))y = 0$$

$$p(x)\frac{d^2y}{dx^2} + p'(x)\frac{dy}{dx} + q(x)y + \lambda r(x)y = 0$$

$$p = 1-x^2 \qquad p' = -2x \qquad q(x) = 0 \qquad \lambda = 5$$

→ $x^2y'' + xy' + 2y = 0$? No.

→ $y'' - 2xy' + 2xy = 0$

→ $y'' + \lambda y = 0$ Yes

→ $x^2y'' + xy' + (x^2 - \nu^2)y = 0$

$\rightarrow \quad xy'' + (1-x)y' + xy = 0$ ✓

$\rightarrow \quad (1-x^2)y'' - xy' + n^2 y = 0$

$$a_2(x)\, y'' + a_1(x)\, y' + a_0(x)\, y = f(x) \qquad a_2 \neq 0.$$

$$y'' + \frac{a_1(x)}{a_2(x)} y' + \frac{a_0(x)}{a_2(x)} y = 0$$

$\mu(x)$

$$\underline{\mu(x) y'' + \mu(x) \frac{a_1(x)}{a_2(x)} y'} + \mu \frac{a_0(x)}{a_2(x)} y = 0$$

$(\mu y')'$ if $\mu(x)$ must satisfy.

$$\frac{d}{dx}\left[\mu(x)\right] = \mu(x) \frac{a_1(x)}{a_2(x)}$$

$$\boxed{\mu(x) = e^{\int \frac{a_1(x)}{a_2(x)} dx}}$$

$$\mu(x) = e^{\int \frac{a_1(x)}{a_2(x)} dx}.$$

$$\frac{\mu(x)}{a_2(x)} = \frac{e^{\int \frac{a_1(x)}{a_2(x)}}}{a_2(x)}$$ ✓

$$x^2 y'' + xy' + 2y = 0$$

$$a_2 y'' + a_1 y' + a_0 y = 0$$

$$\mu_1 = \frac{e^{\int x/x^2 dx}}{x^2} = \frac{e^{\int \frac{1}{x} dx}}{x^2} = 1/x$$

$$\underline{xy'' + y'} + \frac{2}{x} y = 0$$

$$(xy')' + \frac{2}{x} y = 0 \quad SLP.$$

$$xy'' + (1-x)y' + ny = 0$$

$$\frac{e^{\int \frac{1-x}{x}\,dx}}{x} = \frac{e^{\int\left(\frac{1}{x}-1\right)dx}}{x} = e^{-x}$$

$$\underline{e^{-x}x}\,y'' + \underline{e^{-x}(1-x)}\,y' + ne^{-x}y = 0$$

SLP

$e^{-x}x$ — ?

$$(xe^{-x}y')' + ne^{-x}y = 0$$

$$\boxed{\mu_1 = \frac{e^{\int \frac{a_1(x)}{a_2(x)}\,dx}}{a_2(x)}}$$

Determine the normalized eigen functions of the boundary value problem.

$[0, l]$

BVP $\quad y'' + \lambda y = 0, \quad \underbrace{y(0) = 0. \quad y(l)}_{BC} = 0$

IVP ?
BVP ?

$\lambda < 0 \quad \lambda = 0, \quad \lambda > 0$

IVP $\begin{cases} y'' + a_1 y' + b_1 y = f(x) \\ y(0) = 2 \\ y'(0) = 2 \end{cases}$ IC

Case 1 $\quad \lambda < 0 \quad$ when $\lambda = -K^2 < 0$

$y'' - K^2 y = 0$ ✓

$y_1 = e^{Kx}$? LI

$y_2 = e^{-Kx}$

$W(y_1, y_2) \neq 0$

$y = c_1 y_1 + c_2 y_2$

$y = e^{mx}$

$y' = m e^{mx}$

$y'' = m^2 e^{mx}$

$m^2 e^{mx} - K^2 e^{mx} = 0$

$(m^2 - K^2) e^{mx} = 0$

$m^2 - K^2 = 0 \Rightarrow m = \pm K$

$$y(x) = Ae^{kx} + Be^{-kx}$$

$y(0)=0, \quad y(l)=0$

$y(0)=0 \Rightarrow 0 = A + B.$

$y(l)=0 \Rightarrow \underline{0 = Ae^{kl} + Be^{-kl}}$

$A=0, \; B=0$

$\boxed{y \equiv 0 \text{ trivial soln}}$ $\times$

$y = e^{mx}$

$\int y'' dx = \int 0 \cdot dx$

Case (2) $\lambda = 0$ $\quad y'' = 0$

$y' = c_1$

$y = c_1 x + \underline{c_2}$

$m^2 e^{mx} = 0$

$m^2 = 0 \Rightarrow m = 0, 0$

$y = (c_1 + c_2 x) e^{0 \cdot x}$

$y = \underline{c_1 + c_2 x}$. ?

$y(0)=0 \Rightarrow 0 = c_1$

$y(l)=0 \Rightarrow 0 = c_1 + c_2 l \Rightarrow c_2 = 0$

$y \equiv 0$ trivial soln $\times$

Case (B) $\lambda > 0$ $\lambda = K^2$

$a \pm ib$

$y = e^{ax}[c_1 \cos bx + c_2 \sin bx]$

$y'' + K^2 y = 0$

$y = e^{mx}$

$m^2 + K^2 = 0$

$m = \pm iK$

$y(x) = A \cos Kx + B \sin Kx.$

$y(0) = 0, \quad y(l) = 0$

$y(0) = 0 \Rightarrow \quad 0 = A + B(0) \Rightarrow A = 0$

$y(l) = 0 \Rightarrow \quad 0 = A \cos(Kl) + B \sin(Kl)$

$B \sin(Kl) = 0$

$\sin(Kl) = 0$

$Kl = n\pi \Rightarrow K = \frac{n\pi}{l}$

$\lambda_n = K^2 = \frac{n^2\pi^2}{l^2}, \quad n = 1, 2, 3 \ldots$

$\lambda_1 = \pi^2/l^2 \quad \lambda_2 = \frac{2^2\pi^2}{l^2}, \quad \frac{3^2\pi^2}{l^2}, \quad \frac{4^2\pi^2}{l^2} \ldots$

$$\boxed{y_n(x) = B_n \sin\left(\frac{n\pi}{l}\right)x}$$

$n = 1, 2, 3, \ldots$

$$y_1 = B_1 \sin\left(\frac{\pi}{l}x\right)$$

$$y_2 = B_2 \sin\left(\frac{2\pi}{l}x\right)$$

$$y_3 = B_3 \sin\left(\frac{3\pi}{l}\right)x$$

$$\vdots$$

$$\int_0^l y_1 y_2 \, dx = 0$$

$$\int_0^l y_1^2 \, dx = \underline{1}$$

$y'' + \lambda y = 0 \qquad y(0) = 0, \; y'(\pi) = 0$

<u>Case i</u> $\quad \lambda < 0, \quad \lambda = -K^2$

$$y'' - K^2 y = 0$$

$$m^2 - K^2 = 0 \Rightarrow m = \pm K$$

$$y = c_1 e^{Kx} + c_2 e^{-Kx}$$

$y(0) = 0 \Rightarrow \quad 0 = c_1 + c_2$

$y'(\pi) = 0 \Rightarrow \quad 0 = c_1 K e^{K\pi} - c_2 K e^{-K\pi}$

$$c_1 = 0, \; c_2 = 0$$

$y \equiv 0$ trivial soln.

<u>Case ii</u> $\quad \lambda = 0 \qquad y'' = 0$

$$y = c_1 + c_2 x$$

$y(0) = 0 \Rightarrow \quad 0 = c_1 \qquad \Rightarrow y \equiv 0$ ✗

$y'(\pi) = 0 \Rightarrow \quad c_2 = 0$

case iii) $\lambda > 0$ $\lambda = k^2$

$$y'' + k^2 y = 0$$ $\boxed{y \equiv 0}$

$$y = A\cos kx + B\sin kx.$$

$y(0) = 0 \Rightarrow 0 = A$

$y'(\pi) = 0 \Rightarrow 0 = -Ak\sin k(\pi) + Bk\cos k(\pi)$

$B \neq 0$ ✗

$$B\cos(k\pi) = 0$$

$$\cos(k\pi) = 0$$

$$k\pi = \left(\frac{2n-1}{2}\right)\pi \qquad n = 0, \pm 1, \pm 2 \ldots$$

$$k = \frac{2n-1}{2}$$

$$\lambda = k^2$$

Eigen functions $y_n(x) = B\sin\left(\frac{2n-1}{2}x\right)$

③ $y'' + \lambda y = 0 \qquad y(0) - y(\pi) = 0, \quad y'(0) - y'(\pi) = 0$