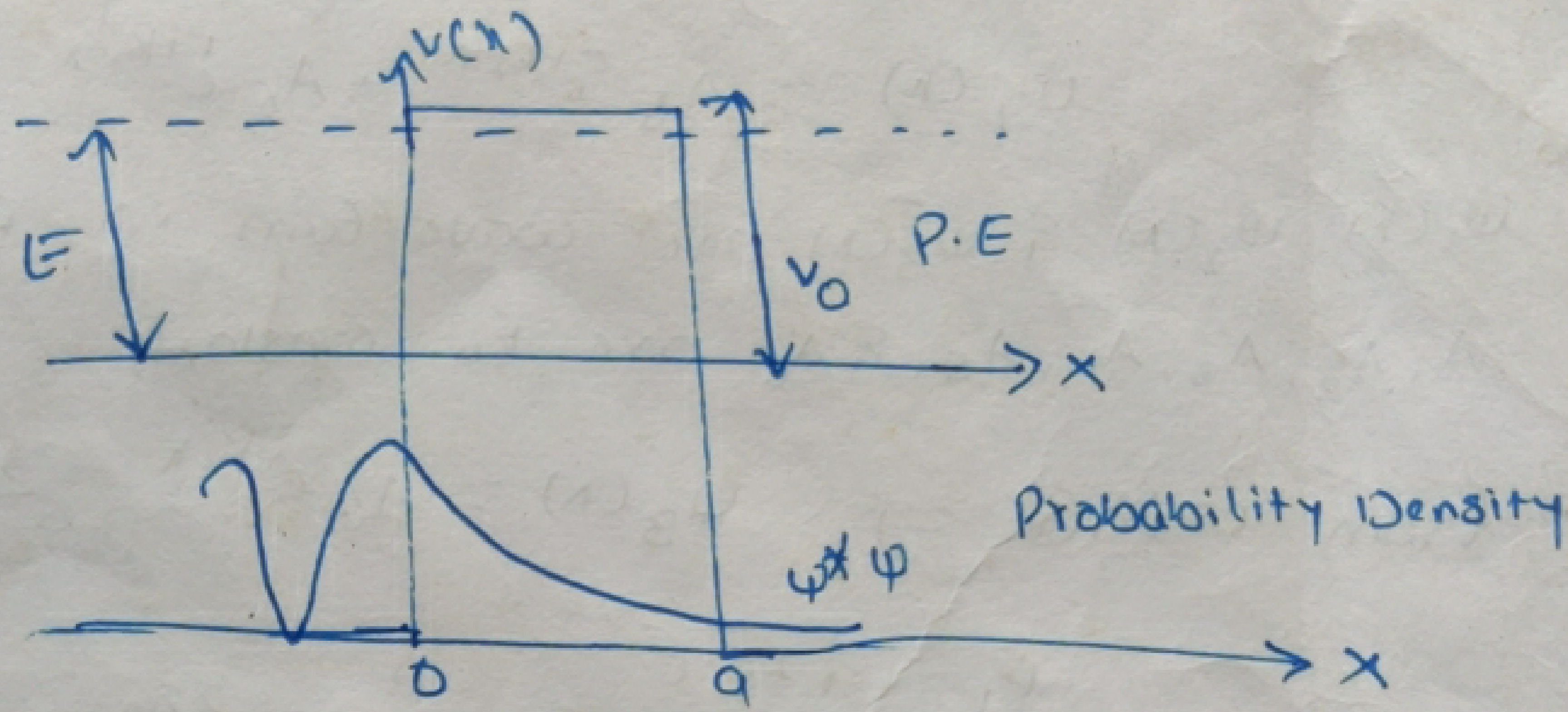
Potential Barrier: Tunneling

Ex: an α-Particle which is trying to escape coloumb barrier.



$V(x) = 0$ for $x<0$, Reg-I

$V(x) = V_0$ for $0<x<a$, Reg-II

$V(x) = 0$ for $x>a$. Reg-III

The Potential Barrier is considered b/w $x=0$ and $x=a$

The Particle incident on the barrier has energy E which is less than barrier height V_0, i.e $E<V_0$

C.M $\rightarrow$ $E<V_0$ Particle can never penetrate the barrier and appear in Reg-III. i.e the Particle is always reflected from the barrier. $T=0$

Q.M $\rightarrow$ This is not true and there is some Probability to penetrate the barrier. This means a fraction of a particle incident from the left will cross the barrier and appear in Reg-III. The Probability of crossing the barrier is called tunneling effect.

S.E for Reg-I & III are given by

$$\frac{d^2\psi_1(x)}{dx^2} + \frac{2mE}{\hbar^2}\psi_1(x) = 0 \qquad \text{—} (1)$$

$$\frac{d^2\psi_3(x)}{dx^2} + \frac{2mE}{\hbar^2}\psi_3(x) = 0 \qquad \text{—} (2)$$

Reg-II

$$\frac{d^2\psi_2(x)}{dx^2} + \frac{2m(E-V_0)}{\hbar^2}\psi_2(x) = 0 \quad -③$$

$$\rightarrow \frac{d^2\psi_2(x)}{dx^2} - \frac{2m(V_0-E)}{\hbar^2}\psi_2(x) = 0$$

The solution of the above eqn's are

$$\psi_1(x) = A_1 e^{ik_1x} + A_2 e^{-ik_1x} \quad -④$$

$$\psi_3(x) = A_3 e^{ik_1x} + A_4 e^{-ik_1x} \quad -⑤$$

$$\psi_2(x) = A_5 e^{-ik_2x} + A_6 e^{+ik_2x} \quad -⑥$$

$\psi_1(x)$, $\psi_2(x)$ & $\psi_3(x)$ are wave fun's in Reg-I, II, & III
$A_1, A_2, A_3, A_4, A_5, \& A_6$ are the constants

from eqn⑤ $\Rightarrow$ $\psi_3(x) = A_3 e^{ik_1x}$ $\quad -⑦$

$$k_1 = \frac{\sqrt{2mE}}{\hbar} = \frac{p}{\hbar} = \frac{2\pi}{\lambda} \quad -⑧$$

$$k_2 = \frac{\sqrt{2m(V_0-E)}}{\hbar} \quad -⑨$$

We shall now apply boundary condition to wave fun's ψ_1, ψ_2 & ψ_3
the boundary conditions at $x=0$

$$\psi_1(0) = \psi_2(0) \quad -⑩$$

$$\frac{d\psi_1(0)}{dx} = \frac{d\psi_2(0)}{dx} \quad -⑪$$

The boundary conditions at $x=a$ are given by

$$\psi_2(a) = \psi_3(a) \quad -⑫$$

$$\frac{d\psi_2(a)}{dx} = \frac{d\psi_3(a)}{dx} \quad -⑬$$

Using these boundary condition

$$A_1 + A_2 = A_5 + A_6 \quad -⑭$$

$$ik_1A_1 - ik_1A_2 = -k_2A_5 + k_2A_6 \quad -⑮$$

$$A_5 e^{-k_2a} + A_6 e^{k_2a} = A_3 e^{ik_1a} \quad -⑯$$

$$-k_2A_5 e^{-k_2a} + k_2A_6 e^{k_2a} = ik_1A_3 e^{ik_1a} \quad -⑰$$

Solving Equins (14) & (15)

$$A_1 = \frac{(ik_1 - k_2)}{2ik_1} A_5 + \frac{(ik_1 + k_2)}{2ik_1} A_6 \quad -(18)$$

Solving Equns (18) & (17)

$$A_6 = \frac{(k_2 + ik_1) e^{ik_1 a}}{2k_2 e^{k_2 a}} A_3 \quad -(19)$$

$$\& \quad A_5 = \frac{(k_2 - ik_1) e^{ik_1 a}}{2k_2 e^{-k_2 a}} A_3 \quad -(20)$$

sub eqns (19) & (20) in eqn (18)

$$\frac{A_1}{A_3} = \left[\frac{1}{2} + \frac{i}{4}\left(\frac{k_2}{k_1} - \frac{k_1}{k_2}\right)\right] e^{(ik_1 + k_2)a} + \left[\frac{1}{2} - \frac{i}{4}\left(\frac{k_2}{k_1} - \frac{k_1}{k_2}\right)\right] e^{(ik_1 - k_2)a} \quad -(21)$$

Since the height of Potential barrier is much heigher than the energy of the incident particles $(E < V_0)$. $\therefore$ $\frac{k_2}{k_1} \gg \frac{k_1}{k_2}$ and we approximate

$$\frac{k_2}{k_1} - \frac{k_1}{k_2} \approx \frac{k_2}{k_1} \quad -(22)$$

Further, since the Potential barrier is wide enough so that for ψ_2 we get severely weakend b/w $x=0$ & $x=a$. $\therefore k_2 a \gg 1$, i.e

$$e^{k_2 a} \gg e^{-k_2 a} \quad -(23)$$

eqn (21), therefore approximated by

$$\frac{A_1}{A_3} = \left[\frac{1}{2} + \frac{ik_2}{4k_1}\right] e^{(ik_1 + k_2)a} \quad -(24)$$

The complex conjugate of eqn (24) is given by

$$\frac{A_1^*}{A_3^*} = \left[\frac{1}{2} - \frac{ik_2}{4k_1}\right] e^{(-ik_1 + k_2)a} \quad -(25)$$

multiplying eqn (24) & (25)

$$\frac{A_1 A_1^*}{A_3 A_3^*} = \left[\frac{1}{4} + \frac{k_2^2}{16 k_1^2}\right] e^{2k_2 a} \quad -(26)$$

Since the coefficient A_1 is related to the w.f ψ_1. i.e of the incident particle and A_3 is related to the w.f ψ_3. i.e the transmitted particle

The transmission probability

$$T = \frac{A_3 A_3^*}{A_1 A_1^*} = \frac{|A_3|^2}{|A_1|^2} = \left[\frac{16}{4 + \left(\frac{k_1}{k_2}\right)^2}\right] e^{-2k_2 a} \quad -(27)$$

From the defination of k_1 & k_2

$$\left(\frac{k_2}{k_1}\right)^2 = \frac{2m(V_0 - E)/\hbar^2}{2mE/\hbar^2} = \frac{V_0}{E} - 1 \quad -(28)$$

eqn (27) shows that the quantity in the bracket varies slowly with E and V_0 than the variation of exponential term. So the approximated transmission probability is given by

$$T = e^{-k_2 a}$$