Solution of Laplace equation in two dimension:—

$$u(0,y) = u(l,y) = 0$$
$$u(x,0) = 0$$
$$u(x,a) = f(x)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$u = X(x)\,Y(y).$$

$$\frac{\partial^2 u}{\partial x^2} = X''Y \quad \& \quad \frac{\partial^2 u}{\partial y^2} = XY''$$

$$X''Y + XY'' = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = K$$

case i $\quad K = 0$

$$X'' - KX = 0 \Rightarrow Y'' + YK = 0$$

$$X'' = 0 \qquad Y'' = 0$$

$$X = c_1 x + c_2 \qquad Y = c_3 y + c_4$$

$$u(x,y) = XY = (c_1 x + c_2)(c_3 y + c_4)$$

Case ii $\quad K = \lambda^2 > 0$

$$X'' - \lambda^2 X = 0 \quad \text{and} \quad Y'' + \lambda^2 Y = 0$$

$$m^2 - \lambda^2 = 0$$

$$m = \pm \lambda$$

$$\therefore X = c_1 e^{\lambda x} + c_2 e^{-\lambda x} \qquad Y = c_3 \cos \lambda y + c_4 \sin \lambda y.$$

$$u(x,y) = XY = (c_1 e^{\lambda x} + c_2 e^{-\lambda x})(c_3 \cos \lambda y + c_4 \sin \lambda y)$$

Case iii $\quad K = -\lambda^2 < 0$

$$X'' + \lambda^2 X = 0 \quad \text{and} \quad Y'' - \lambda^2 Y = 0$$

$$X = c_1 \cos \lambda x + c_2 \sin \lambda x \qquad Y = c_3 e^{\lambda y} + c_4 e^{-\lambda y}$$

Soln $\quad u(x,y) = XY = (c_1 \cos \lambda x + c_2 \sin \lambda x)(c_3 e^{\lambda y} + c_4 e^{-\lambda y})$

$$u(x,y) = \sum_{n=0}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right) \left[B_n e^{\lambda y} \quad e^{-\lambda y} \right]$$

(1)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$u(0,y) = u(l,y) = 0 \Rightarrow X(0) = 0, \cancel{X}(l) = 0$$

$$u(x,0) = 0$$

$$u(x,a) = \sin\left(\frac{n\pi x}{l}\right)$$

soln

$$u = XY$$

$$X''Y + XY'' = 0$$

$$\frac{X''}{X} = \frac{-Y''}{Y} = K$$

case i $K = 0$

$$X'' = 0 \qquad Y'' = 0$$

$$\boxed{u(x,y) = 0}$$

trivial soln.

$$X = c_1 x + c_2 \qquad Y = c_3 y + c_4$$

$$0 = c_1(0) + c_2 \qquad 0 = c_1 l + c_4$$

$$c_2 = 0$$

$$0 = c_1 l + \cancel{c_2}^{0} \Rightarrow c_1 = 0$$

Case ii $K = \lambda^2$

$$X'' - \lambda^2 X = 0 \quad \text{and} \quad Y'' + \lambda^2 Y = 0$$

$$X = c_1 e^{\lambda x} + c_2 e^{-\lambda x} \qquad Y = c_3 \cos\lambda x + c_4 \sin\lambda x .$$

$$X(0) = 0, \quad X(l) = 0$$

$$0 = c_1 + c_2 .$$
$$0 = c_1 e^{\lambda l} + c_2 e^{-\lambda l} \quad \Rightarrow \quad c_1 = 0, \; c_2 = 0 .$$

$$u(x,y) = 0$$

Case iii $K = -\lambda^2$

$$X'' + \lambda^2 X = 0 \qquad Y'' - \lambda^2 Y = 0$$

$$X = c_1 \cos\lambda x + c_2 \sin\lambda x \qquad Y = c_3 e^{\lambda y} + c_4 e^{-\lambda y} .$$

$$X(0) = 0$$

$$0 = c_1(1) + \Rightarrow c_1 = 0$$

$$X(l) = 0$$

$$0 = \cancel{c_1 \cos\lambda l} + c_2 \sin\lambda l \Rightarrow c_2 \sin\lambda l = 0$$

$$\sin \lambda l = 0$$

$$\lambda l = n\pi$$

$$\lambda = \frac{n\pi}{l}$$

$$u(x,y) = XY$$

$$= c_2 \sin \lambda x \left(c_3 e^{\lambda y} + c_4 e^{-\lambda y} \right)$$

$$= \sin \lambda x \left(A e^{\lambda y} + B e^{-\lambda y} \right)$$

$$u(x,y) = \sin\left(\frac{n\pi x}{l}\right) \left[A e^{\frac{n\pi y}{l}} + B e^{-\frac{n\pi y}{l}} \right]$$

$$u(x,y) = \sum_{n=0}^{\infty} \sin\left(\frac{n\pi x}{l}\right) \left[A_n e^{\frac{n\pi y}{l}} + B_n e^{-\frac{n\pi y}{l}} \right]$$

$$u(x,0) = 0$$

$$\Rightarrow A_n + B_n = 0 \Rightarrow B_n = -A_n$$

$$u(x,y) = \sum_{n=0}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right)\left[e^{\frac{n\pi y}{l}} - e^{-n\pi y/l}\right]$$

$$u(x,a) = \sin\left(\frac{n\pi x}{l}\right) = \sum_{n=0}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right)\left(e^{\frac{n\pi a}{l}} - e^{\frac{-n\pi a}{l}}\right)$$

when

$$A_n\left(e^{\frac{n\pi a}{l}} - e^{-n\pi a/l}\right) = \frac{2}{\cancel{a}\,l}\int_0^{\cancel{x}\,l} \sin^2\left(\frac{n\pi x}{l}\right) dx.$$

$$\frac{e^{\theta} - e^{-\theta}}{2} = \sinh\theta$$

$$A_n\, \cancel{2}\sinh\left(\frac{n\pi a}{l}\right) = \frac{\cancel{2}}{a}\int_0^{a} \frac{1-\cos\left(\frac{2n\pi x}{l}\right)}{2}\, dx$$

$$A_n = \frac{1}{2a\sin\left(\frac{n\pi a}{l}\right)}\int_0^{a}\left[\frac{1-\cos\left(\frac{2n\pi x}{l}\right)}{2}\right] dx$$

$$A_n = \frac{1}{2a\sinh\left(\frac{n\pi a}{l}\right)}\left[\frac{1}{2}x - \sin\left(\frac{2n\pi x}{l}\right)\cdot\frac{l}{2n\pi}\right]_0^{a}$$

$$A_n = \frac{1}{2a\sinh\left(\frac{n\pi a}{l}\right)}\left[\frac{a}{2} - \frac{l}{2n\pi}\sin\left(\frac{2n\pi a}{l}\right)\right]$$

$$u(x,y) = \sum_{n=0}^{\infty} 2A_n \sin\left(\frac{n\pi x}{l}\right) \sinh\left(\frac{n\pi y}{l}\right)$$

where

$$A_n = \frac{1}{2a\sin\left(\frac{n\pi a}{l}\right)}\left[\frac{l}{2} - \underbrace{\frac{l}{2n\pi}\sin\left(\frac{2n\pi l}{l}\right)}_{\to 0}\right]$$

$$A_n = \frac{1}{2\cancel{a}\sin\left(\frac{n\pi a}{l}\right)} \cdot \frac{l}{2}$$

$$A_n = \frac{1}{4\sin\left(\frac{n\pi a}{l}\right)}$$

$$u(0,y) = u(l,y) = 0$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$u(x,0) = 0$$

$$u(x,a) = f(x)$$

$$u(x,y) = \sum_{n=0}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right)\left[e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}}\right]$$

$$\boxed{u(x,y) = \sum_{n=0}^{\infty} 2A_n \sin\left(\frac{n\pi x}{l}\right)\sinh\left(\frac{n\pi y}{l}\right)}$$

$$u(x,a) = f(x)$$

$$u(x,a) = f(x) = \sum_{n=0}^{\infty} 2A_n \sin\left(\frac{n\pi x}{l}\right)\sinh\left(\frac{n\pi a}{l}\right)$$

$$2A_n \sinh\left(\frac{n\pi a}{l}\right) = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$\boxed{A_n = \frac{1}{l\sinh\left(\frac{n\pi a}{l}\right)}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx}$$