Solve $xy'' + y' - xy = 0$ —— (1)

$x = 0$ singular point

$$y'' + \frac{1}{x} y' - y = 0$$

$$\lim_{x \to 0} (x - 0) \frac{1}{x} = 1$$

$$\lim_{x \to 0} (x - 0)^2 (-1) = 0 .$$

$x = 0$ RSP.

Let $y(x) = \sum_{n=0}^{\infty} a_n x^{m+n}$, $a_0 \neq 0$

$$y'(x) = \sum_{n=0}^{\infty} (m+n) a_n x^{m+n-1}$$

$$y''(x) = \sum_{n=0}^{\infty} (m+n)(m+n-1) a_n x^{m+n-2}$$

$$(1) \Rightarrow \sum_{n=0}^{\infty} \underline{(m+n)}(m+n-1) a_n x^{m+n-1} + \sum_{n=0}^{\infty} \underline{(m+n)} a_n x^{m+n-1} - \sum_{n=0}^{\infty} a_n x^{m+n+1} = 0$$

$$\sum_{n=0}^{\infty} (m+n)^2 a_n x^{m+n-1} - \sum_{n=0}^{\infty} a_n x^{m+n+1} = 0$$

$n \to (n+1)$ $\quad$ $n \to (n-1)$

$$\sum_{n=-1}^{\infty} (m+n+1)^2 a_{n+1} x^{m+n} - \sum_{n=1}^{\infty} a_{n-1} x^{m+n} = 0$$

$$\underline{\underline{m^2 a_0 x^{m-1}}} + (m+1)^2 a_1 x^m + \sum_{n=1}^{\infty} \left[(m+n+1)^2 a_{n+1} - a_{n-1} \right] x^{m+n} = 0$$

For the Indicial equation

$\underline{x^{m-1}}$

$$m^2 a_0 = 0, \quad a_0 \neq 0$$

$$m^2 = 0$$

$$m = 0, 0$$

$\underline{\underline{x^m}}$

$$(m+1)^2 a_1 \cancel{x^m} = 0 \Rightarrow a_1 = 0$$

for the recurrence relation

$\underline{x^{m+n}}$

$$a_{n+1} = \frac{a_{n-1}}{(m+n+1)^2}, \quad n \geq 1$$

$$a_{n+1} = \frac{a_{n-1}}{(m+n+1)^2} \qquad n \geq 1$$

$$n=1 \Rightarrow a_2 = \frac{a_0}{(m+2)^2}$$

$$n=2 \Rightarrow a_3 = \frac{a_1}{(m+3)^2} = 0$$

$$n=3 \Rightarrow a_4 = \frac{a_2}{(m+4)^2} = \frac{a_0}{(m+2)^2(m+4)^2}$$

$$n=4 \Rightarrow a_5 = 0$$

$$n=5 \Rightarrow a_6 = \frac{a_0}{(m+2)^2(m+4)^2(m+6)^2}$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^{m+n}$$

$$= x^m \left[a_0 + \cancelto{0}{a_1 x} + a_2 x^2 + \cancelto{0}{a_3 x^3} + a_4 x^4 + \cancelto{0}{a_5 x^5} + a_6 x^6 + \cdots \right]$$

$$(*) \text{ — } y_m(x) = a_0 x^m \left[1 + \frac{x^2}{(m+2)^2} + \frac{x^4}{(m+4)^2(m+2)^2} + \frac{x^6}{(m+6)^2(m+4)^2(m+2)^2} + \cdots \right]$$

$$y_1(x) = y_m(x) \Big|_{m=0}$$

$$y_1(x) = a_0 \left[1 + \frac{x^2}{2^2} + \frac{x^4}{4^2 \cdot 2^2} + \frac{x^6}{6^2 \cdot 4^2 \cdot 2^2} + \cdots \right]$$

$$xy_m'' + y_m' - xy_m = a_0 m^2 x^{m-1}$$

$$\left(x\frac{d^2}{dx^2} + \frac{d}{dx} - x\right) y_m = a_0 \underline{m^2} \, \underline{x^{m-1}}$$

w.r.t 'm'

$$\left(x\frac{d^2}{dx^2} + \frac{d}{dx} - x\right) \frac{\partial y_m}{\partial m} = 2a_0 m x^{m-1} + a_0 m^2 x^{m-1} \ln x .$$

at $m = 0$

$$\left(x\frac{d^2}{dx^2} + \frac{d}{dx} - x\right) \frac{\partial y_m}{\partial m} = 0$$

Hence

$$\left.\frac{\partial y_m}{\partial m}\right|_{m=0} = 0$$

$$y_2(x) = \left.\frac{\partial y_m}{\partial m}\right|_{m=0}$$

$$y'' + y \neq 0$$

$$y' + y \neq 0$$

$$y''' + y'' + y' + y \neq 0$$

$$\boxed{y \equiv 0}$$

$$y_m(x) = a_0 x^m \left[1 + \frac{x^2}{(m+2)^2} + \frac{x^4}{(m+4)^2(m+2)^2} + \frac{x^6}{(m+6)^2(m+4)^2(m+2)^2} + \cdots \right]$$

$$\frac{\partial y_m}{\partial m} = a_0 x^m \ln x \left[1 + \frac{x^2}{(m+2)^2} + \frac{x^4}{(m+4)^2(m+2)^2} + \frac{x^6}{(m+6)^2(m+4)^2(m+2)^2} + \cdots \right]$$

$$+ a_0 x^m \Bigg[0 - \frac{2x^2}{(m+2)^3} + x^4 \left\{ \frac{-2}{(m+4)^3(m+2)^2} - \frac{2}{(m+4)^2(m+2)^3} \right\}$$

$$+ x^6 \left\{ \frac{-2}{(m+6)^3(m+4)^2(m+2)^2} + \frac{1}{(m+6)^2} \left(\frac{-2}{(m+4)^3(m+2)^2} - \frac{2}{(m+4)^2(m+2)^3} \right) \right\}$$

$$+ \cdots \Bigg]$$

$$y_2(x) = \left. \frac{\partial y_m}{\partial m} \right|_{m=0}$$

$$y_2 = a_0 \ln x \left[1 + \frac{x^2}{2^2} + \frac{x^4}{4^2 \cdot 2^2} + \frac{x^6}{6^2 \cdot 4^2 \cdot 2^2} + \cdots \right] - a_0 \left[\frac{x^2}{4} + \frac{3}{128} x^4 + \cdots \right]$$

$$y_2(x) = y_1(x) \ln x \cancel{+} a_0 \left[\frac{x^2}{4} + \frac{3}{128} x^4 + \cdots \right]$$

Complete Soln: $y(x) = A\,y_1(x) + B\,y_2(x)$

Summary

$$y(x) = A\,y_1(x) + B\,y_2(x)$$

$$\underline{\underline{y_m}}$$

$$y_1(x) = y_m(x)\Big|_{m=0}$$

$$y_2 = \frac{\partial y_m}{\partial m}\Big|_{m=0}$$

Solve $x^2 y'' + x y' + (x^2 - 4) y = 0$