Legendre's differential equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad — (1)$$

n is real.

Singular pts ± 1

ordinary pts $\mathbb{R} - \{\pm 1\}$

$x=0$ ordinary point

$$y(x) = \sum_{m=0}^{\infty} a_m x^m \quad — (2)$$

$$y'(x) = \sum_{m=1}^{\infty} m a_m x^{m-1}$$

$$y''(x) = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}$$

$$(1) \Rightarrow \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1) a_m x^m - 2\sum_{m=1}^{\infty} m a_m x^m + n(n+1)\sum_{m=0}^{\infty} a_m x^m = 0$$

$m \to (m+2)$

$$\sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m - \sum_{m=0}^{\infty} m(m-1) a_m x^m - 2\sum_{m=0}^{\infty} m a_m x^m + n(n+1)\sum_{m=0}^{\infty} a_m x^m = 0$$

$$\sum_{m=0}^{\infty} \left[(m+2)(m+1) a_{m+2} - \{ m^2 + m - n^2 - n \} a_m \right] x^m = 0$$

$$a_{m+2} = \frac{m^2+m-n^2-n}{(m+2)(m+1)} a_m \qquad m \geq 0$$

$$= \frac{(m^2-n^2)+(m-n)}{(m+2)(m+1)} a_m \qquad m \geq 0$$

$$a_{m+2} = \frac{(m-n)(m+n+1)}{(m+2)(m+1)} a_m \qquad m \geq 0$$

$$m=0 \Rightarrow \quad a_2 = -\frac{n(n+1)}{2!} a_0$$

$$m=1 \Rightarrow \quad a_3 = -\frac{(n-1)(n+2)}{3!} a_1$$

$$m=2 \Rightarrow \quad a_4 = -\frac{(n-2)(n+3)}{4 \cdot 3} a_2$$

$$m=3 \Rightarrow \quad a_5 = -\frac{(n-3)(n+4)}{5 \cdot 4} a_3$$

$$a_4 = \frac{(n-2)n(n+1)(n+3)}{4!} a_0$$

$$a_5 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!} a_1$$

(2) $\Rightarrow$ $$y(x) = \sum_{m=0}^{\infty} a_m x^m$$

$$= (a_0 + a_2 x^2 + a_4 x^4 + \cdots) + (a_1 x + a_3 x^3 + a_5 x^5 + \cdots)$$

$$y(x) = a_0 y_1(x) + a_1 y_2(x) \quad \text{--- (3)}$$

where

$$y_1(x) = 1 - \frac{n(n+1)}{2!} x^2 + \frac{(n-2)n(n+1)(n+3)}{4!} x^4 - \cdots$$

$$y_2(x) = x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!} x^5 - \cdots$$

Legendre's polynomials:-

$a_{m+2} = 0$, $a_{m+4} = 0$, $a_{m+6} = 0$ P_n

if n is even $n=0 \Rightarrow y_1(x) = 1$

n is odd $n=2 \Rightarrow = 1 - 3x^2$

$n=1 \Rightarrow$ x

$n=4 \Rightarrow = 1 - 10x^2 + \frac{35}{3} x^4$

$n=3 \Rightarrow$ $x - \frac{5}{3} x^3$

$n=5 \Rightarrow$ $x - \frac{14}{3} x^3 + \frac{21}{5} x^5$

$$y_1(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - \cdots$$

$$y_2(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - \cdots$$

$$y(x) = a_0 y_1(x) + a_1 y_2(x)$$

$P_n(x)$

$P_n(1) = 1$ $\quad$ $y(1) = 1$ $\quad$ $\boxed{P_n(x=1) = 1}$

either $a_0 = 0$ (or) $a_1 = 0$

$a_1 = 0$

$$y(x) = a_0 y_1(x)$$

$n=0 \Rightarrow \quad y(x) = a_0 \Rightarrow 1 = a_0 \Rightarrow P_0 = 1$

$n=2 \Rightarrow \quad y(x) = a_0(1-3x^2) \Rightarrow 1 = a_0(1-3) \Rightarrow a_0 = -\frac{1}{2} \Rightarrow P_2 = \frac{1}{2}(3x^2-1)$

$n=4 \Rightarrow \quad y(x) = a_0\left(1-10x^2+\frac{35}{3}x^4\right) \Rightarrow 1 = a_0\left(1-10+\frac{35}{3}\right) \Rightarrow a_0 = \frac{3}{8}$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$\underline{\underline{a_0 = 0}}$

$n=1 \Rightarrow y(x) = a_1 x \Rightarrow 1 = a_1 \Rightarrow P_1 = x$

$n=3 \Rightarrow y(x) = a_1\left(x - \frac{5}{3}x^3\right) \Rightarrow 1 = \left(\cancel{x}1 - \frac{5}{3}\right)a_1 \Rightarrow a_1 = -\frac{3}{2}$

$P_3 = \frac{1}{2}\left(5x^3 - 3x\right)$

$n=5 \Rightarrow y(x) = a_1\left(x - \frac{14}{3}x^3 + \frac{21}{5}x^5\right) \Rightarrow 1 = a_1\left(1 - \frac{14}{3} + \frac{21}{5}\right) \Rightarrow a_1 = \frac{15}{8}$

$P_5(x) = \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right)$

Legendre polynomials

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$

$P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)$

$P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)$

$P_5(x) = \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right)$

$$a_{m+2} = -\frac{(n-m)(n+m+1)}{(m+2)(m+1)}\, a_m$$

$$\Rightarrow a_m = -\frac{(m+2)(m+1)}{(n-m)(n+m+1)}\, a_{m+2}$$

$m = n-2$

$$a_{n-2} = -\frac{n(n-1)}{2(2n-1)}\, a_n \qquad ?$$

$$\boxed{a_n = \frac{(2n)!}{2^n (n!)^2}} \qquad P_n(1) = 1$$

$$a_{n-2} = \frac{-n(n-1)}{2(2n-1)} \cdot \frac{(2n)!}{2^n (n!)^2}$$

$$= \frac{-\cancel{n(n-1)}\; 2\cancel{n}\, \cancel{(2n-1)}\, (2n-2)!}{\cancel{2}\, \cancel{(2n-1)}\; 2^n\, \cancel{n}\, (n-1)!\; \cancel{\cdot}\; \cancel{n(n-1)}\, (n-2)!}$$

$$a_{n-2} = -\frac{(2n-2)!}{2^n (n-1)!\,(n-2)!}$$

lly

$$a_{n-4} = \frac{-(n-2)(n-3)}{4(2n-3)} a_{n-2}$$

$$= \frac{(n-2)(n-3)}{4(2n-3)} \cdot \frac{(2n-2)!}{2^n (n-1)!\,(n-2)!}$$

$$= \frac{(2n-4)!}{2^n 2!\,(n-2)!\,(n-4)!}$$

lly

$$a_{n-2l} = (-1)^l \frac{(2n-2l)!}{2^n\, l!\,(n-l)!\,(n-2l)!}$$

$\mathbb{P}_n(x)$

$$P_n(x) = \sum_{l=0}^{M} (-1)^l \frac{(2n-2l)!}{2^n\, l!\,(n-l)!\,(n-2l)!}\, x^{n-2l}$$

↳ Legendre polynomial of degree n

where $M = n/2$ (or) $(n-1)/2$ whichever is an integer.

$\underline{n=0}$

$$P_0(x) = 1$$

$\underline{n=1}$ $\quad P_1(x) = x$

$n=2$

$$P_2(x) = \sum_{l=0}^{1} (-1)^l \frac{(4-2l)!}{2^2\, l!\,(2-l)!\,(2-2l)!}\, x^{2-2l}$$

$$= \frac{4!}{2^2\, 2!\cdot 2!}\, x^2 + (-1)\frac{2!}{2^2\cdot 1\cdot 1\cdot 1}\, x^0 = \frac{3}{2}x^2 - \frac{1}{2}$$

$$P_2 = \frac{1}{2}(3x^2-1)$$