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APPLIED MATHEMATICS AND COMPUTATION
ELSEVIER Applied Mathematics and Computation 160 (2005) 579–587
www.elsevier.com/locate/amc

Using rationalized Haar wavelet for solving linear integral equations

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Abstract

The main purpose of this paper is to demonstrate that using rationalized Haar wavelet for solving linear Fredholm integral equation of the second kind. We convert the integral equation to a system of linear equations, and by using numerical examples we show our estimation have a good degree of accuracy.

Keywords: Fredholm integral equations; Operational matrix; Product operation; Rationalized Haar functions

1. Introduction

In recent years, many different basic functions have used to the solution of integral equations, such as orthonormal bases and wavelets.

The orthogonal set of Haar functions is group of square wave with magnitude of $+2^{i/2}$, $-2^{i/2}$ and 0, $i = 0, 1, 2, \dots$ [5]. Just these zeros make the Haar transform faster than other square functions such as Walsh function. Lynch and Reis [2] have rationalized the Haar transform by deleting the irrational numbers and introducing the integral powers of two. This modification results in what is called the rationalized Haar (RH) transform. The RH transform preserves all the properties of the original Haar transform and can be efficiently implemented using digital pipline architecture [6]. The corresponding functions

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0096-3003/$ - see front matter
doi:10.1016/j.amc.2003.11.036

are known as RH functions. The RH functions are composed of only three amplitude +1, −1 and 0.

2. Properties of rationalized Haar functions

2.1. Rationalized Haar functions

The RH functions $\mathrm{RH}(r,t)$, $r = 1, 2, 3, \ldots$ are composed of three values +1, −1 and 0 and can be defined on the interval $[0, 1)$ as [4]

$$\mathrm{RH}(r,t) = \begin{cases} 1, & J_1 \leqslant t < J_{\left(\frac{1}{2}\right)}, \\ -1, & J_{\left(\frac{1}{2}\right)} \leqslant t < J_0, \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

where

$$J_u = \frac{j-u}{2^i}, \quad u = 0, \frac{1}{2}, 1.$$

The value of r is defined by two parameters i and j as

$$r = 2^i + j - 1, \quad i = 0, 1, 2, 3, \ldots, \ \ j = 1, 2, 3, \ldots, 2^i.$$

$\mathrm{RH}(0,t)$ is defined for $i = j = 0$ an given by

$$\mathrm{RH}(0,t) = 1, \quad 0 \leqslant t < 1. \tag{2}$$

A set of the first eight RH functions is shown in Figs. 1–8, where $r = 0, 1, 2, \ldots, 7$.

The orthogonality property is given by

$$\int_0^1 \mathrm{RH}(r,t)\mathrm{RH}(v,t)\,\mathrm{d}t = \begin{cases} 2^{-i} & \text{for } r = v \\ 0 & \text{for } r \neq v, \end{cases}$$

where $v = 2^n + m - 1$, $n = 0, 1, 2, 3, \ldots$, $m = 1, 2, 3, \ldots, 2^n$.

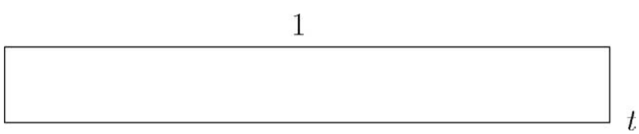

Fig. 1. $\mathrm{RH}(0,t)$ obtained for $i = 0$ and $j = 0$.

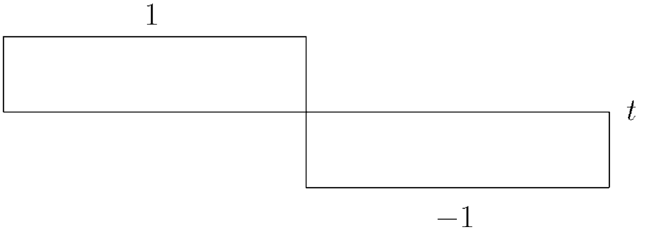

Fig. 2. RH$(1,t)$ obtained for $i=0$ and $j=1$.

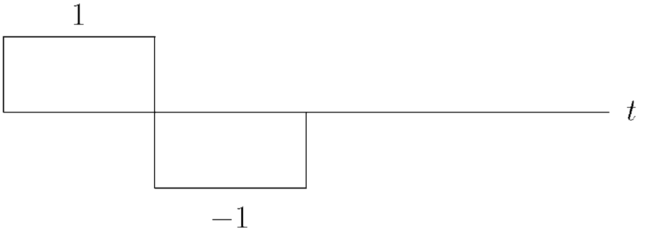

Fig. 3. RH$(2,t)$ obtained for $i=1$ and $j=1$.

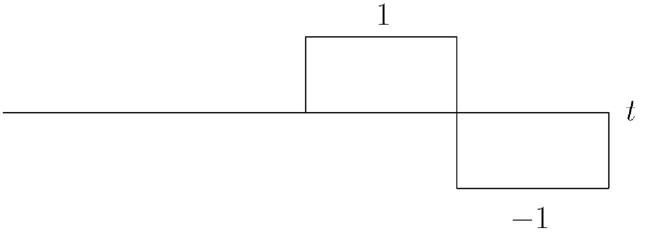

Fig. 4. RH$(3,t)$ obtained for $i=1$ and $j=2$.

Fig. 5. RH$(4,t)$ obtained for $i=2$ and $j=1$.

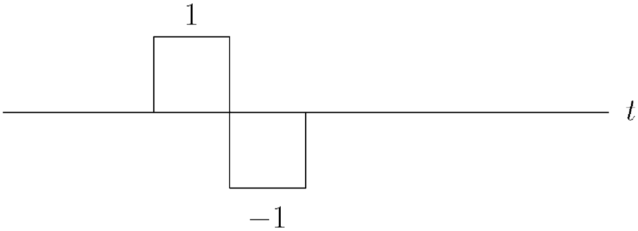

Fig. 6. $\mathrm{RH}(5,t)$ obtained for $i=2$ and $j=2$.

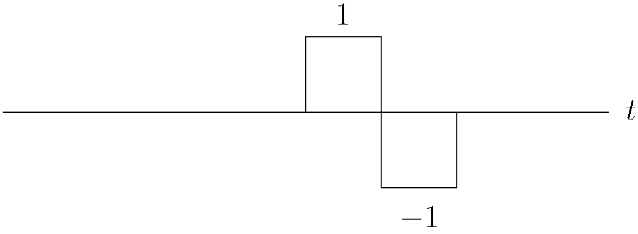

Fig. 7. $\mathrm{RH}(6,t)$ obtained for $i=2$ and $j=3$.

Fig. 8. $\mathrm{RH}(7,t)$ obtained for $i=2$ and $j=4$.

2.2. Function approximation

A function $f(t)$ defined over the interval $[0,1)$ may be expanded in RH functions as

$$f(t)=\sum_{r=0}^{+\infty}a_r\mathrm{RH}(r,t), \tag{3}$$

where a_r, $r=0,1,2,\ldots$, are given by

$$a_r=2^i\int_0^1 f(t)\mathrm{RH}(r,t)\,\mathrm{d}t$$

with $r = 2^i + j - 1$, $i = 0, 1, 2, \ldots$, $j = 1, 2, 3, \ldots, 2^i$ and $r = 0$ for $i = j = 0$. The series in Eq. (3) contains an infinite number of terms. If we let $i = 0, 1, 2, \ldots, \alpha$ then the infinite series in Eq. (3) is truncated up to its first k terms as

$$f(t) = \sum_{r=0}^{k-1} a_r \mathrm{RH}(r, t) = A^{\mathrm{T}} \phi(t), \tag{4}$$

where $k = 2^{\alpha+1}$, $\alpha = 0, 1, 2, \ldots$.

The RH functions coefficient vector A and RH function vector $\phi(t)$ are defined as

$$\begin{aligned} A &= [a_0, a_1, \ldots, a_{k-1}]^{\mathrm{T}}, \\ \phi(t) &= [\phi_0(t), \phi_1(t), \ldots, \phi_{k-1}(t)]^{\mathrm{T}}, \end{aligned} \tag{5}$$

where $\phi_r(t) = \mathrm{RH}(r, t)$, $r = 0, 1, 2, 3, \ldots, k - 1$.

Now, let $k(t, s)$ be a function of two independent variable defined for $t \in [0, 1)$ and $s \in [0, 1)$. Then $k(t, s)$ can be expanded in RH functions as

$$k(t, s) = \sum_{v=0}^{k-1} \sum_{r=0}^{k-1} h_{vr} \phi_r(t) \phi_v(t),$$

where h_{vr} for $v = 0, 1, 2, \ldots, k - 1$, $r = 0, 1, 2, \ldots, k - 1$, is

$$h_{vr} = 2^{i+n} \int_0^1 \int_0^1 k(t, s) \phi_r(t) \phi_v(s) \, \mathrm{d}t \, \mathrm{d}s$$

for i, $n = 0, 1, 2, 3, \ldots, \alpha$. Hence we have

$$k(t, s) = \phi^{\mathrm{T}}(t) H \phi(s), \tag{6}$$

where

$$H = (h_{vr})^{\mathrm{T}}_{k \times k}. \tag{7}$$

The first four RH functions can be expressed as

$$\begin{aligned} \phi_0(t) &= [\,1 \quad 1 \quad 1 \quad 1\,], \\ \phi_1(t) &= [\,1 \quad 1 \quad -1 \quad -1\,], \\ \phi_2(t) &= [\,1 \quad -1 \quad 0 \quad 0\,], \\ \phi_3(t) &= [\,0 \quad 0 \quad 1 \quad -1\,], \end{aligned}$$

which can be written in matrix form as

$$\widehat{\Phi}_{4\times 4} = \begin{bmatrix} \phi_0(t) \\ \phi_1(t) \\ \phi_2(t) \\ \phi_3(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}. \tag{8}$$

In Eq. (8), the row denotes the order of the Haar function. The matrix $\widehat{\Phi}_{k\times k}$ can be expressed as

$$\widehat{\Phi}_{k\times k} = \left[\,\phi\left(\tfrac{1}{2k}\right), \phi\left(\tfrac{3}{2k}\right), \ldots, \phi\left(\tfrac{2k-1}{2k}\right)\,\right], \tag{9}$$

and using Eq. (4) we get

$$\left[\,f\left(\tfrac{1}{2k}\right), f\left(\tfrac{3}{2k}\right), \ldots, f\left(\tfrac{2k-1}{2k}\right)\,\right] = A^{\mathrm{T}}\widehat{\Phi}_{k\times k}. \tag{10}$$

From Eqs. (7) and (10) we have

$$H = \left(\widehat{\Phi}^{-1}_{k\times k}\right)^{\mathrm{T}} H \widehat{\Phi}^{-1}_{k\times k}, \tag{11}$$

where

$$\widehat{H} = (\hat{h}_{lp})_{k\times k}, \quad \hat{h}_{lp} = k\left(\frac{2l-1}{2k}, \frac{2p-1}{2k}\right), \quad p, l = 1, 2, 3, \ldots, k.$$

2.3. Operation matrix of integration

The integration of the $\phi(t)$ defined in Eq. (5) is given by

$$\int_0^t \phi(t')\,\mathrm{d}t' = P\phi(t),$$

where $P = P_{k\times k}$ is the $k \times k$ operational matrix for integration and is given in [3] as

$$P_{k\times k} = \frac{1}{2k}\begin{bmatrix} 2kP_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)} & -\widehat{\Phi}_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)} \\ \widehat{\Phi}^{-1}_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)} & 0 \end{bmatrix},$$

where $\widehat{\Phi}_{1\times 1} = [1]$, $P_{1\times 1} = [\frac{1}{2}]$, $\widehat{\Phi}_{k\times k}$ is given by Eq. (9), and

$$\widehat{\Phi}^{-1}_{k\times k} = \left(\frac{1}{k}\right)\widehat{\Phi}^{\mathrm{T}}_{k\times k} \cdot \mathrm{diag}\left(1, 1, 2, 2, \underbrace{2^2, \ldots, 2^2}_{2^2}, \underbrace{2^3, \ldots, 2^3}_{2^3}, \ldots, \underbrace{\frac{k}{2}, \ldots, \frac{k}{2}}_{\frac{k}{2}},\right).$$

Also, the integration of cross of two RH function vector is

$$\int_0^1 \phi(t)\phi^{\mathrm{T}}(t)\,\mathrm{d}t = D,$$

where D is diagonal matrix given by

$$D = \operatorname{diag}\left(1, 1, \frac{1}{2}, \frac{1}{2}, \underbrace{\frac{1}{2^2}, \ldots, \frac{1}{2^2}}_{2^2}, \underbrace{\frac{1}{2^3}, \ldots, \frac{1}{2^3}}_{2^3}, \ldots, \underbrace{\frac{1}{2^\alpha}, \ldots, \frac{1}{2^\alpha}}_{2^\alpha}\right). \tag{12}$$

3. Fredholm integral equation of the second kind

Consider the following integral equation:

$$y(t) = \int_0^1 k(t,s)y(s)\,\mathrm{d}s + x(t), \tag{13}$$

where $x(t) \in L^2[0,1)$, $k(t,s) \in L^2([0,1) \times [0,1))$ and $y(t)$ is an unknown function [1].

Let approximate $x(t)$, $y(t)$ and $k(t,s)$ by (4) and (6) as follows:

$$x(t) \simeq X^{\mathrm{T}}\phi(t), \quad y(t) \simeq Y^{\mathrm{T}}\phi(t), \quad k(t,s) \simeq \phi^{\mathrm{T}}(t)H\phi(s).$$

With substituting in (13)

$$Y^{\mathrm{T}}\phi(t) = \int_0^1 \phi^{\mathrm{T}}(t)H\phi(s)Y^{\mathrm{T}}\phi(s)\,\mathrm{d}s + X^{\mathrm{T}}\phi(t),$$

$$Y^{\mathrm{T}}\phi(t) = \phi^{\mathrm{T}}(t)H\left(\int_0^1 \phi(s)\phi^{\mathrm{T}}(s)\,\mathrm{d}s\right)Y + X^{\mathrm{T}}\phi(t)$$

with Eq. (12) we have

$$\phi^{\mathrm{T}}(t)Y = \phi^{\mathrm{T}}(t)HDY + \phi^{\mathrm{T}}(t)X,$$

then

$$(I - HD)Y = X.$$

By solving this linear system we can find vector Y, so

$$y(t) \simeq \phi^{\mathrm{T}}(t)Y.$$

4. Numerical examples

Consider the Fredholm integral equations

Example 1. Consider the equation

$$y(t) = \mathrm{e}^t - \frac{\mathrm{e}^{t+1} - 1}{t+1} + \int_0^1 \mathrm{e}^{ts} y(s)\,\mathrm{d}s, \quad 0 \leqslant t \leqslant 1.$$

The solution for $y(t)$ is obtained the method in Section 3. The computational results for $k = 16$ and $k = 32$ together with the exact solution $y(t) = \mathrm{e}^t$ are given in Table 1.

Example 2. Consider the equation

$$y(t) = t + \int_0^1 k(t,s) y(s)\,\mathrm{d}s, \quad 0 \leqslant t \leqslant 1,$$

where

$$k(t,s) = \begin{cases} t & t \leqslant s, \\ s & s \leqslant t. \end{cases}$$

Table 1

t	Approximate for $k = 16$	Approximate for $k = 32$	Exact solution
0	1.03443	1.01642	1
0.1	1.10102	1.11627	1.10517
0.2	1.24734	1.22593	1.2214
0.3	1.32762	1.34637	1.34986
0.4	1.50412	1.47864	1.49182
0.5	1.60097	1.62391	1.64872
0.6	1.81379	1.84004	1.82212
0.7	2.05492	2.02082	2.01375
0.8	2.18727	2.21936	2.22554
0.9	2.47808	2.43742	2.4596
1	2.63769	2.6769	2.71828

Table 2

t	Approximate for $k = 16$	Approximate for $k = 32$	Exact solution
0	0.05785	0.02892	0
0.1	0.17333	0.20205	0.18477
0.2	0.40181	0.37341	0.3677
0.3	0.51391	0.54148	0.54695
0.4	0.73168	0.7048	0.72074
0.5	0.83648	0.86192	0.88732
0.6	1.03589	1.05943	1.04505
0.7	1.21972	1.19686	1.19233
0.8	1.30378	1.32378	1.32769
0.9	1.45749	1.43908	1.44979
1	1.52595	1.54173	1.55741

The solution for $y(t)$ is obtained the method in Section 3. The computational results for $k=16$ and $k=32$ together with the exact solution $y(t)=\sec(1)\cdot\sin(t)$ are given in Table 2.

5. Conclusion

The rationalized Haar functions are used to solve the Fredholm integral equations. The same approach can be used to solve other problems. The numerical examples shows that the accuracy improves with increasing the k, then for better results, using the larger k is recommended.

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