Triple Integrals in Rectangular Co-ordinates :-

Recall

If $f(x,y) \geq 0$ and the limit exists

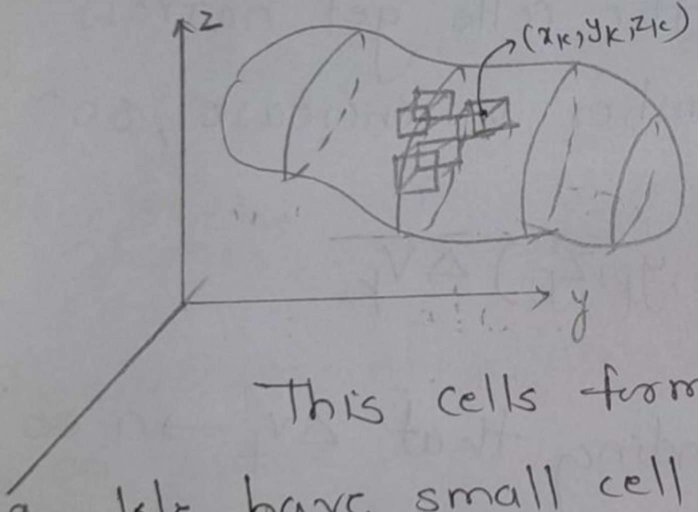
$$\lim_{n\to\infty} \sum_{K=1}^{n} f(x_K, y_K)\, \Delta A_K = \iint_R f(x,y)\, dA$$

= Volume under the surface.

where R is closed and bounded region in plane i.e $R \subseteq \mathbb{R}^2$.

— x — x — x — x —

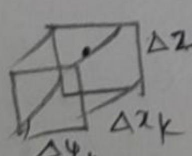
If $f(x,y,z)$ is defined and continuous function on a closed bounded region D in space i.e. $D \subseteq \mathbb{R}^3$.



First we do partition D into small cells / littles cubes / little rectangular boxes.

This cells form a partition of D.

k^{th} have small cell / little rectangular box of volume $\Delta V_K = \Delta x_K \cdot \Delta y_K \Delta z_K$.



To form a sum over D, we choose a point (x_K, y_K, z_K) in k^{th} cell, multiplying the value of f at that point by the volume ΔV_K and adding together the products.

$$S_n = \sum_{k=1}^{n} f(x_k, y_k, z_k)\, \Delta V_k .$$

But we are interested in what happens to these sums as the $\Delta V_k \to 0$ for each cell.

In this case, the norm of partition is defined as largest value amoung $\Delta x_k, \Delta y_k\ \Delta z_k$ and all approches to zero (i.e $\Delta V_k = \Delta x_k \Delta y_k \Delta z_k \longrightarrow 0$).

As $\|p\| \longrightarrow 0$ and the cells get narrow and short, their number n increase, so whave

$$\lim_{n \to \infty} \sum_{k=1}^{n} f(x_k, y_k, z_k)\, \Delta V_k$$

with the understanding that $\Delta V_k \longrightarrow 0$ as $n \longrightarrow \infty$ and $\|p\| \to 0$.

To from a limit, we repet the process again. When a single limiting value is attained no matter how the partitions and points (x_k, y_k, z_k) are choosen, we say f is integrable over D.

i.e $\lim_{n \to \infty} \sum_{k=1}^{n} f(x_k, y_k, z_k) \Delta V_k = \iiint_D f(x,y,z) \, dV$.

= Volume for fun of three varibles.

— x — ^ — x — ^ —

Volume of a region in space :-

If $f(x,y,z) = 1 \ \forall \ x \in D$, then the sum reduce to

$$S_n = \sum_{k=1}^{n} f(x_k, y_k, z_k) \Delta V_k = \sum_{k=1}^{n} 1 \cdot \Delta V_k = \sum_{k=1}^{n} \Delta V_k.$$

So $\lim_{n \to \infty} S_n = \lim_{n \to \infty} \sum_{k=1}^{n} \Delta V_k = \iiint_D dV$.

The volum of a closed, bounded region D in space is

$$V = \iiint_D dV.$$

— x — x —

Average value of a function in space:-

The average value of a function f over a region D in space is defined by

$$\text{Average value of } f \text{ on } D = \frac{\iiint\limits_D f(x,y,z)\,dv}{\text{Volume of } D}$$

$$= \frac{\iiint\limits_D f(x,y,z)\,dv}{\iiint\limits_D 1 \cdot dv.}$$

Properties of Triple Integrals :-

If $f(x,y,z)$ and $g(x,y,z)$ are continuous functions then.

(1) Constant Multiple : $\iiint\limits_D k f\,dv = k\iiint\limits_D f\,dv$ (any number k)

(2) Sum and difference : $\iiint\limits_D (f \pm g)\,dv = \iiint\limits_D f\,dv + \iiint\limits_D g\,dv$.

(3) Dominition

(a) $\iiint\limits_D f\,dv \geqslant 0$ if $f \geqslant 0$ on D.

(b) $\iiint\limits_D f\,dv \geqslant \iiint\limits_D g\,dv$ if $f \geqslant g$ on D

(4) Additivity $\iiint\limits_D f\,dv = \iiint\limits_{D_1} f\,dv + \iiint\limits_{D_2} f\,dv$

if D is the union of two nonoverlapping regions D_1 and D_2.

Finding Limits of Integration :-

We evalute a triple integral by applying a three-dimensional version of Fubini's Theorem to evaluate it by three repeated single integration.

To evaluate

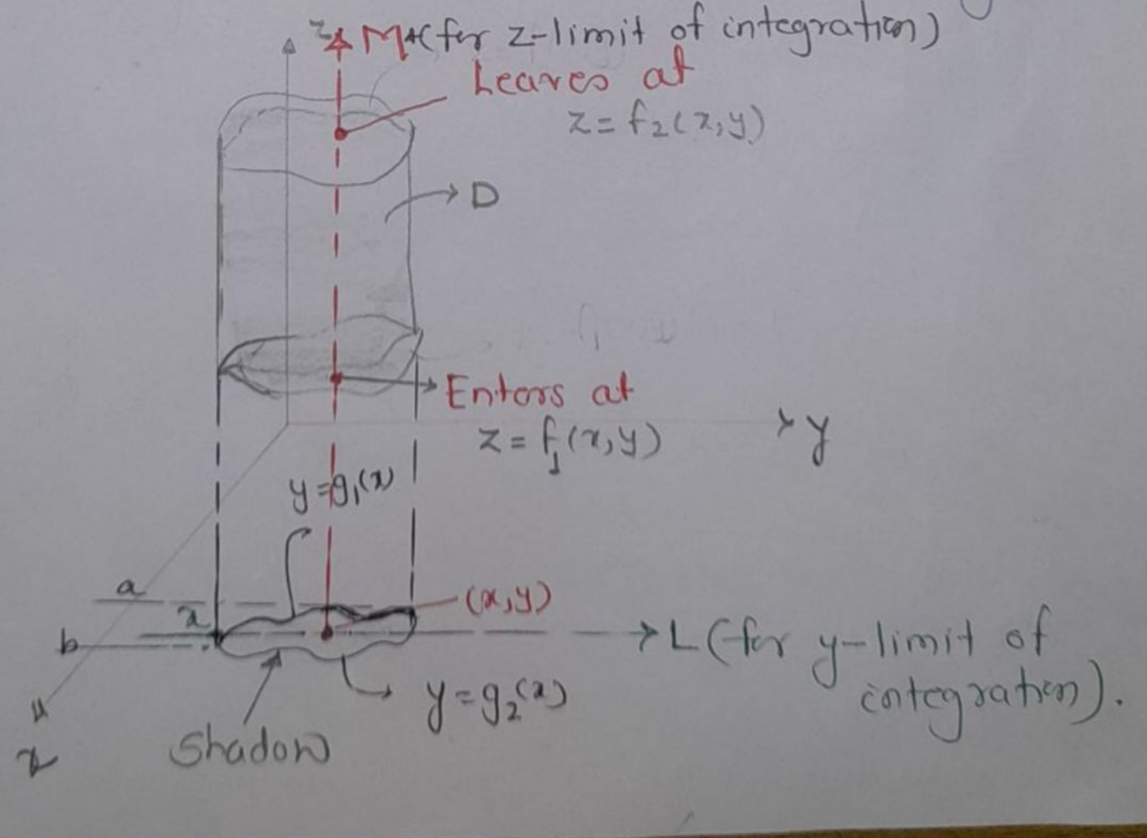
$$\iiint_D f(x,y,z)\,dV$$

$D = \{(x,y,z) \mid f_1(x,y) \le z \le f_2(x,y),\ g_1(x) \le y \le g_2(x),\ a \le x \le b\}$

over a region D, integrate first with respect to z, then with respect to y, finally with x.

ie $dz\,dy\,dx$, for shadow (as)

(1) Sketch : Sketch the region D. in xyz-plane
Draw it's shadow R in xy-plane



(2) To find the z-limits of integration, draw a line M passing through a typical point (x,y) in R parallel to the z-axis. Look at, M enters D at $z = f_1(x,y)$ and leves at $z = f_2(x,y)$. These are the z-limits of integration.

So here., $f_1(x,y) \leq z \leq f_2(x,y)$.

(3). Find the y-limits of integration; look at Shadow in xy-plane, for that region R draw a $\parallel^{er}$ line $\overset{L}{\text{to}}$ y-axies, look at L enters R at $y = g_1(x)$ and leaves at $y = g_2(x)$.

So, here $g_1(x) \leq y \leq g_2(x)$.

(4) To find the x-limits of integration:

Choose x-limts that include all lines through R $\parallel^{er}$ to y-axies.
(Here $x = a$ and $y = b$.)

. So, $a \leq x \leq b$

- Follow similar procedure if you change the order of integration.
- The "shadow" of region D lies in the plane of last two variables.

with respect to which the iterated integration take place.

- The above procedure applies whenever a solid region D is bounded above and below bay a surface and when 'shadow' region R is bounded by a lower and upper curve.
- It does not apply to complicated regions.

Note:- Similar to double integrals, the order of integration is immaterial if the limit of integration are constants

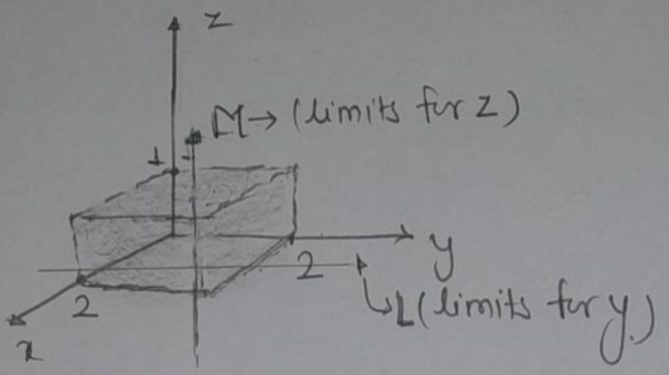
$$\int_a^b \int_c^d \int_e^f f(x,y,z)\, dx\, dy\, dz = \int_e^f \int_c^d \int_a^b f(x,y,z)\, dz\, dy\, dx$$

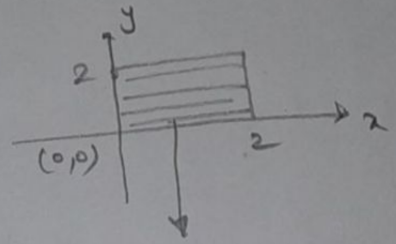
where a, b, c, d, e, f are constants.

Example (1) Evaluate $\iiint_D x^2yz\, dv$ where

D: $0 \le x \le 2$, $0 \le y \le 2$ and $0 \le z \le 1$.

(Integration order not given in Ex).





Shadow

or

This is a projection of region in xy plane.

$$\iiint_D x^2yz\, dv$$

$$= \int_{x=0}^{2}\int_{y=0}^{2}\int_{z=0}^{z=1} x^2yz\, dz\, dy\, dx$$

$$= \int_0^2\int_0^2 x^2y\left(\frac{z^2}{2}\right)_0^1 dy\,dx$$

$$= \frac{1}{2}\int_0^2\int_0^2 x^2y\, dy\,dx$$

$$= \frac{1}{2}\int_0^2 x^2\left(\frac{y^2}{2}\right)_0^2 dx$$

$$= \frac{1}{2}\int_0^2 x^2\ 2\, dx$$

$$= \frac{2}{2}\left(\frac{x^3}{3}\right)_0^2$$

$$= \frac{8}{3} \text{ cu. units.}$$

H.W: (1) Evaluate $\iiint_D (x+y+z)\,dx\,dy\,dz$ where

$D: 0 \le x \le 1,\ 1 \le y \le 2,\ 2 \le z \le 3$. Ans $= \frac{9}{2}$ cub. units

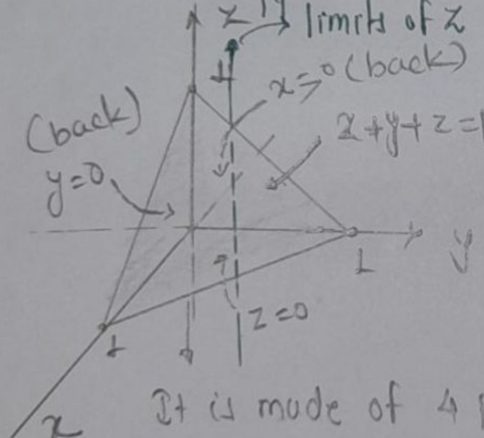
(2) Evaluate $\int_0^{\log 2}\int_0^{x}\int_0^{x+y} e^{x+y+z}\,dx\,dy\,dz$, Ans $= \frac{5}{8}$.

Example (2)

Evaluate $I = \iiint_D \frac{dV}{(x+y+z+1)^3}$

D is the region bounded by $x=0, y=0, z=0$ and $x+y+z=1$.

(In Ex. limits of integrations are not given).



It is made of 4 planes.

yz-plane
xz-plane
xy-plane
x+y+z=1-plane.

Here

$x=0 \to$ yz plane in 3D dim
$y=0 \to$ xz — " — " —
$z=0 \to$ xy — " — 3D ...

$x+y+z=1$

To find x intercept
put $y=z=0 \Rightarrow x=1$
y-intercept
put $x=z=0 \Rightarrow y=1$
z-intercept
put $x=y=0 \Rightarrow z=1$

To plot a unique plane we need 3 pts.

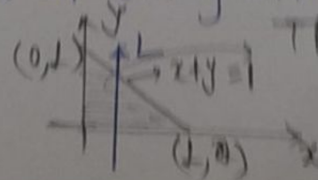
[First w.r.to z, then w.r.to y and [finally w.r.to x]

Draw a line M.

Look at line M, enters in D at $z=0$ and leaves at $z=1-x-y$.

$\therefore\ 0 \le z \le 1-x-y$.

To find the y-limits of integration draw shadow in xy-plane.



This is a projection of region in xy-plane.

Now we are in 2-dim. space you know how to find the limits of y and x. Draw a line L, look at that line., $0 \le y \le 1-x$ and $0 \le x \le 1$.

$$I = \int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} \frac{1}{(x+y+z+1)^3} \, dz \, dy \, dx.$$

$$= \int_{0}^{1} \int_{0}^{1-x} \left[-\frac{1}{2} (x+y+z+1)^{-2} \right]_0^{1-x-y} dy \, dx$$

$$= \int_{0}^{1} \int_{0}^{1-x} -\frac{1}{2} \left[\left(\cancel{x} + \cancel{y} + 1 - \cancel{x} - \cancel{y} + 1 \right)^{-2} - (x+y+1)^{-2} \right] dy \, dx$$

$$= -\frac{1}{2} \int_{0}^{1} \int_{0}^{1-x} \left(\frac{1}{4} - \frac{1}{(x+y+1)^2} \right) dy \, dx$$

$$= -\frac{1}{2} \int_{0}^{1} \left\{ \frac{1}{4} [y]_0^{1-x} + \left[\frac{1}{(x+y+1)} \right]_0^{1-x} \right\} dx$$

$$= -\frac{1}{2} \int_{0}^{1} \left\{ \frac{1}{4} (1-x) + \left(\frac{1}{\cancel{x}+1-\cancel{x}+1} - \frac{1}{x+1} \right) dx \right\}$$

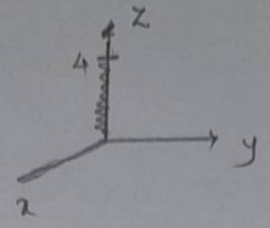
$$= -\frac{1}{2} \int_{0}^{1} \left\{ \frac{1}{4} (1-x) + \left(\frac{1}{2} - \frac{1}{(x+1)} \right) dx \right\}$$

$$= \frac{1}{2} \left[\log 2 - \frac{5}{8} \right]$$

(3) Change the order of integration

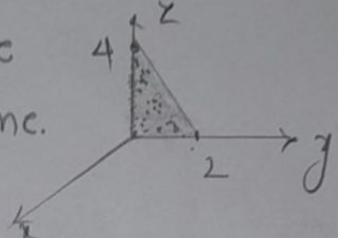
$$\int_0^4 \int_0^{2-\frac{1}{2}z} \int_0^{1-\frac{y}{2}-\frac{z}{4}} f(x,y,z)\, dx\, dy\, dz \ \underline{\underline{\text{to}}}\ \underline{dz\, dy\, dx}.$$

$\rightarrow$ Here, $0 \le z \le 4$.



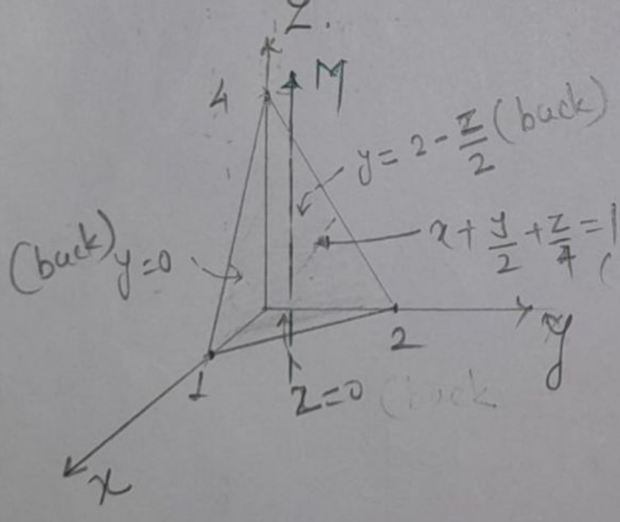
$0 \le y \le 2 - \frac{z}{2}$ → line in yz plane
→ plane in xyz plane.

when you put $z = 0$
from $y = 2 - \frac{z}{2}$ gives
$y = 2$.



$0 \le x \le 1 - \frac{y}{2} - \frac{z}{4}$, so $x = 1 - \frac{y}{2} - \frac{4}{2}$ → plane in xyz $\underline{\underline{\text{dim}}}$

To plot a unique we need therce points; How to find that points? 4, 2, is there to cal. last point put $y = 0 = z$ in $x = 1 - \frac{y}{2} - \frac{z}{4} \Rightarrow x = 1$.



or

$x + \frac{y}{2} + \frac{z}{4} = 1$

Suppose I am on z axis, $x = y = 0$

$\therefore \frac{z}{4} = 1 \Rightarrow z = 4$

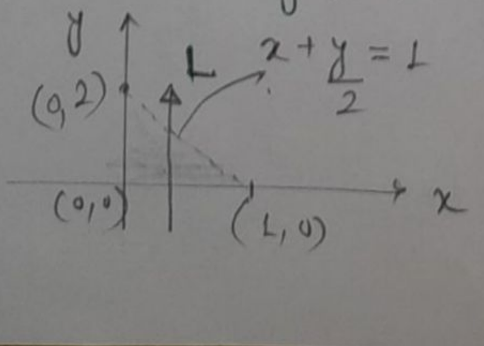
on y axis, $x = z = 0$

$\frac{y}{2} = 1 \Rightarrow y = 2$

on x axis, $y = z = 0$

$x = 1$

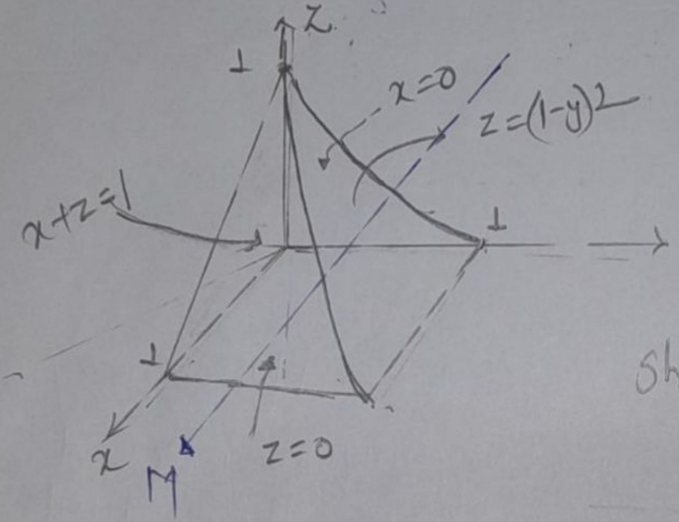
Shadow in xy-plane



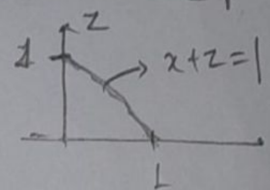
$$\int_0^1 \int_{y=0}^{2-2x} \int_{z=0}^{z=4-4x-2y} f(x,y,z)\,dz\,dy\,dx.$$

Example (4) Let D with $x; y, z \geq 0$, $y \leq 1$, $x+z \leq 1$, $z \leq (1-y)^2$

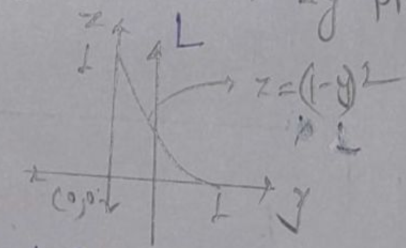
Find $\iiint_D 30\,dx\,dz\,dy$. (Limits Not Given).



for $x+z \leq 1$



Shadow on zy-plane



$$\iiint_D 30\,dx\,dz\,dy = \int_{x=0}^{1}\int_{z=0}^{(1-y)^2}\int_{x=0}^{x=1-z} 30\,dx\,dz\,dy.$$

$$= \int_0^1 \int_0^{(1-y)^2} 30x \Big|_{x=0}^{1-z} dz\,dy$$

$$= \int_0^1 \int_0^{(1-y)^2} (30 - 30z)\,dz\,dy$$

$$= \int_0^1 (30z - 15z^2)\Big|_{z=0}^{(1-y)^2} dy$$

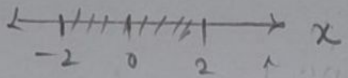
$$= \int_0^1 \left[30(1-y)^2 - 15(1-y)^4\right] dy$$

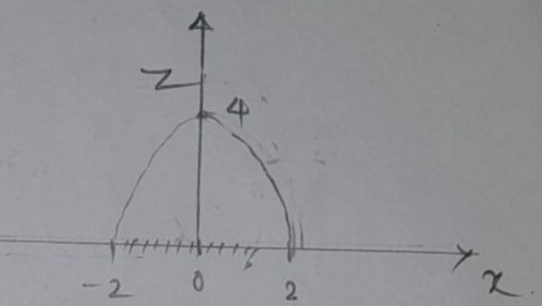
$=$ Do it.

Example (5) Rewrite the order of integration

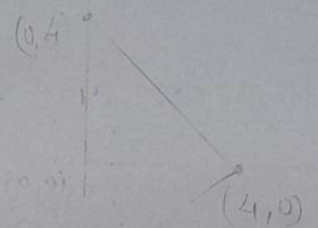
$$\int_{-2}^{2}\int_{0}^{4-x^2}\int_{0}^{4-x} f(x,y,z)\,dy\,dz\,dx \text{ to } dx\,dy\,dz.$$

Sol^n $-2 \le x \le +2$





$0 \le z \le 4-x^2$

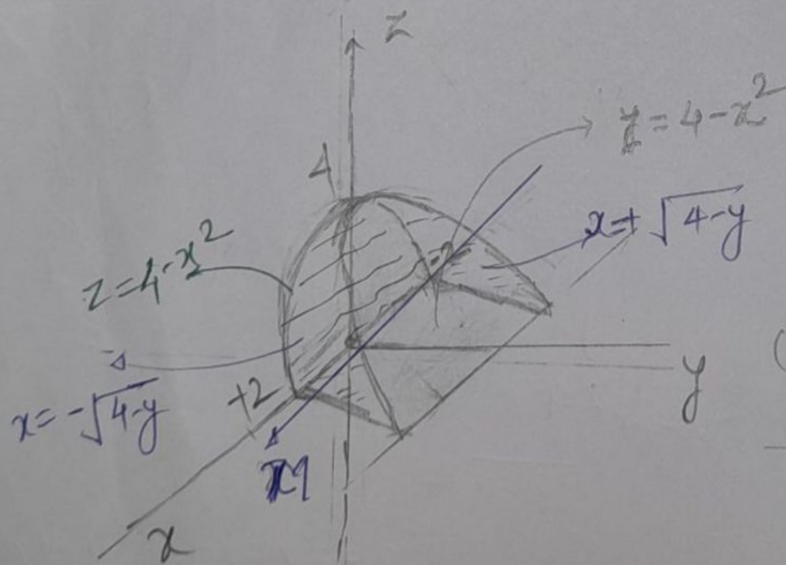


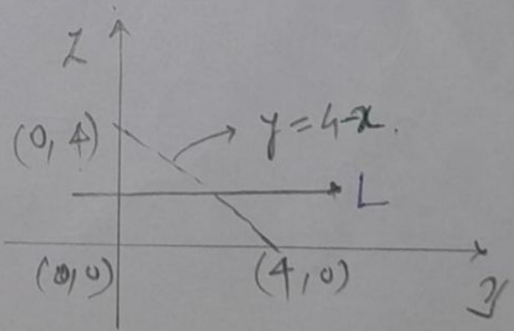
$0 \le y \le 4-x$

M line enters $y = 4-x^2$

$\Rightarrow x^2 = 4-y$

$\Rightarrow x = \pm\sqrt{4-y}$.





Shadow

$$\int_{0}^{4}\int_{y=0}^{y=4-x}\int_{x=-\sqrt{4-y}}^{x=\sqrt{4-y}} dx\,dy\,dz.$$

Example (6) Sketch the solid bounded by $x^2+y^2=4$, $z=1$, and $z=9-3x-2y$.

Calculate the volume.

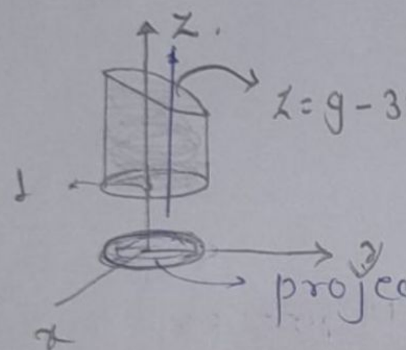
$\rightarrow$ Let D be solid bounded by

$x^2+y^2=4 \longrightarrow$ cylinder in 3D dim

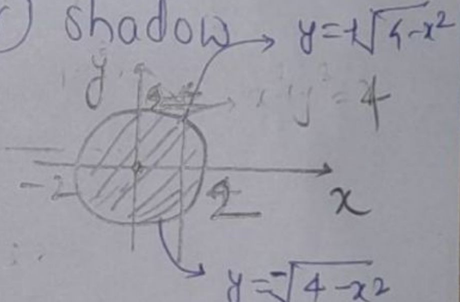
$z=1 \longrightarrow$ const. plane in 3D dim

$z=9-3x-2y \rightarrow$ plane.

x and y postive,



projection in xy-plane (or) shadow



We know,

$$\text{Vol}(D) = \iiint_D dV$$

$$= \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{z=1}^{z=9-3x-2y} dz\, dy\, dx.$$

$$= \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (8-3x-2y)\, dy\, dx.$$

Try to write the limits of (integration) y and x in polar co-ordinates. For that look shadow.

$$\mathrm{Vol}(D) = \int_0^{2\pi}\int_0^{2}\left(8 - 3r\cos\theta - 2r\sin\theta\right) r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[4r^2 - r^3\cos\theta - \frac{2}{3}r^3\sin\theta\right]_{r=0}^{r=2} d\theta$$

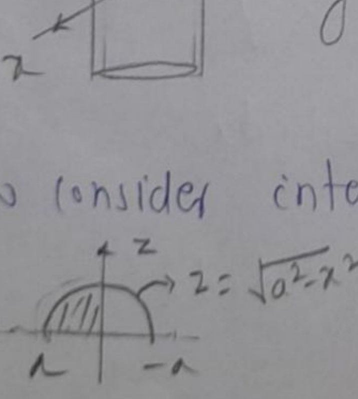
$$= \int_0^{2\pi}\left[16 - 8\cos\theta - \frac{16}{3}\sin\theta\right] d\theta$$

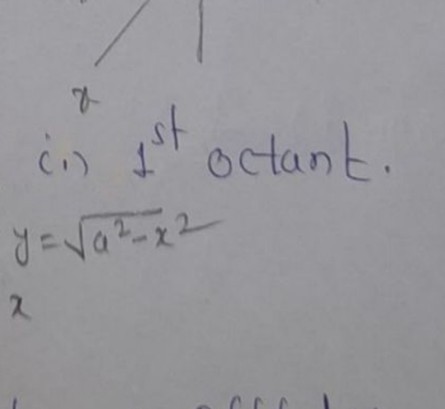
$$= 32\pi \ \text{cu. units}$$

Example (7) Find the volume common to the cylinders $x^2+y^2 \le a^2$ and $x^2+z^2 \le a^2$ with $a > 0$.

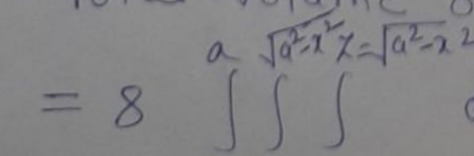
→ Consider the volume of two intersecting cylinders

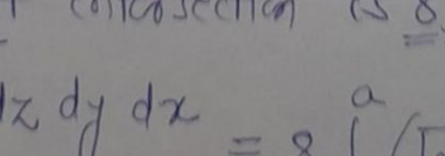
$x^2+y^2 \le a^2$ and $x^2+z^2 \le a^2$





Now consider intersection S_1 in 1^{st} octant.





Total volume of intersection is $8\iiint_{S_1} dV$

$$= 8\int_{-a}^{a}\int_{y=0}^{\sqrt{a^2-x^2}}\int_{z=0}^{\sqrt{a^2-x^2}} dz\,dy\,dx = 8\int_{-a}^{a}\left(\sqrt{a^2-x^2}\right)^2 dx$$

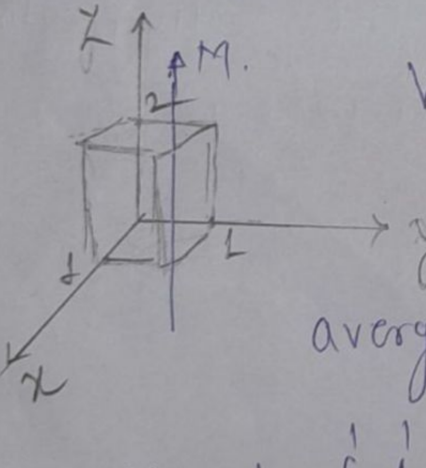
$$= 8\int_0^a a^2-x^2\,dx$$

$$= 8\cdot\frac{2}{3}a^3$$

$$= \frac{16}{3}a^3 \text{ Cu. Units.}$$

Example(8) Find the average value $F(x,y,z)$ over the given region

$F(x,y,z) = x+y-z$ over the rectangular solid in first octant bounded by the co-ordinate planes and the planes $x=1$, $y=1$ and $z=2$.



$$V = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{z=2} 1\cdot dz\,dy\,dx = \frac{1}{2}.$$

$$\text{averge}(f) = \frac{\iiint_D f(x,y,z)\,dA}{V}$$

$$\therefore \text{averge}(f) = \frac{1}{2}\int_0^1\int_0^1\int_0^2 (x+y-z)\,dz\,dy\,dx$$

$$= \frac{1}{2}\int_0^1\int_0^1 \left(xz+yz-\frac{z^2}{2}\right)_0^2 dy\,dx$$

$$= \frac{1}{2}\int_0^1\int_0^1 (2x+2y-2)\,dy\,dx = \frac{1}{2}\int_0^1 (2x-1)\,dx$$

$$= 0.$$

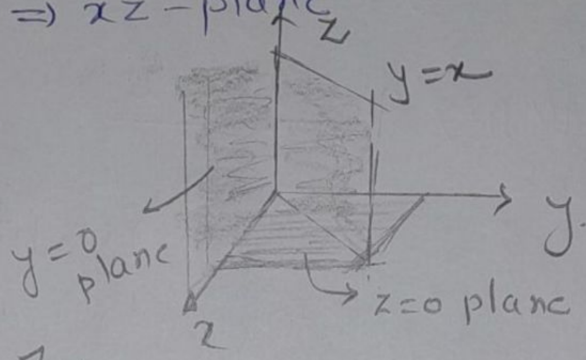
Example

(9) Let D be the solid $(x, y, z \geq 0)$ bounded by $z = 0$, $y = 0$, $y = x$ and $z = 2 - \frac{1}{2}x^2$. Write $\iiint\limits_{D} f(x,y,z)\,dV$ as an iterated integrals.

$\rightarrow$ Here $z = 0 \Rightarrow xy$-plane

$y = 0 \Rightarrow xz$-plane

$y = x$



$z = 2 - \frac{x^2}{2}$

$\downarrow$

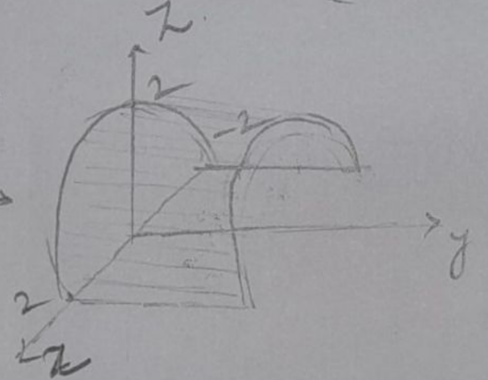
$z = 0$

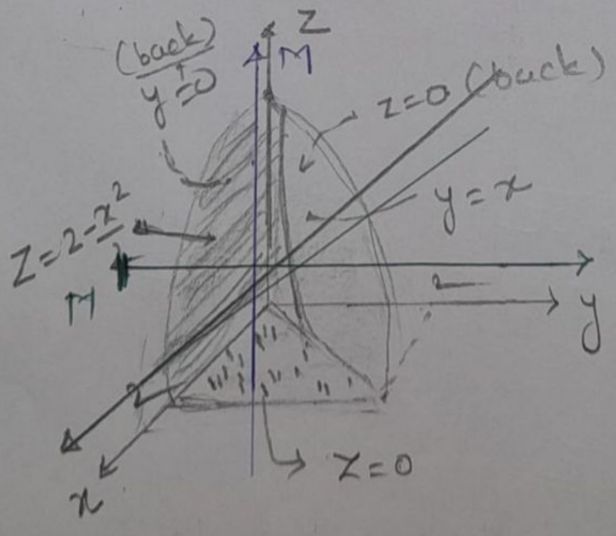
$2 - \frac{x^2}{2} = 0$ in 3D

$\Downarrow$

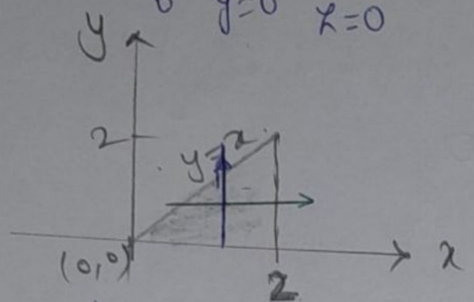
$4 = x^2$

$\pm 2 = x$





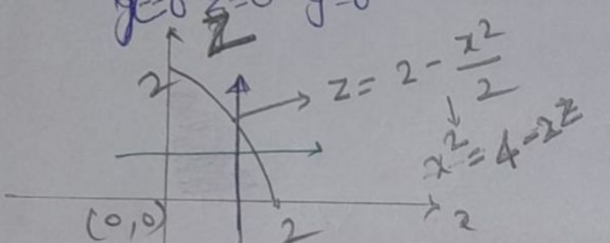
(1) $$\int_{0}^{2}\int_{y=0}^{y=x}\int_{z=0}^{z=\frac{x^2}{2}} f(x,y,z)\,dz\,\underline{dy\,dx}.$$



$$\int_{0}^{2}\int_{x=y}^{x=2}\int_{z=0}^{2-\frac{x^2}{2}} f(x,y,z)\,dz\,\underline{dx\,dy}$$

<u>shadow</u> in xy plane

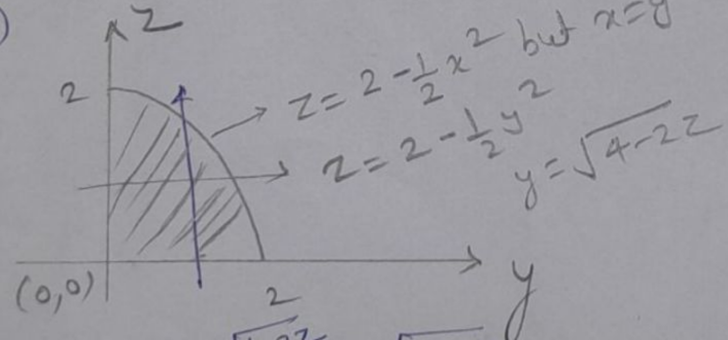
(3) $$\int_{x=0}^{2}\int_{z=0}^{z=2-\frac{x^2}{2}}\int_{y=0}^{x} f(x,y,z)\,dy\,\underline{dz\,dx}$$



$$\int_{0}^{2}\int_{x=0}^{\sqrt{4-2z}}\int_{y=0}^{y=x} f(x,y,z)\,dy\,dx\,dz.$$

shadow in xz plane.

(4)



$$\int_{x=0}^{2}\int_{y=0}^{y=\sqrt{4-2z}}\int_{x=y}^{x=\sqrt{4-2z}} f(x,y,z)\,dx\,dy\,dz$$

$$\int_{y=0}^{2}\int_{z=0}^{z=2-\frac{x^2}{2}}\int_{x=y}^{x=\sqrt{4-2z}} f(x,y,z)\,dx\,dy\,dz.$$