Thm :- Let $f(x, y, z)$ be a scalar f^n having cont. partial derivatives, then ∇f exists & its length is a vector i.e. & ~~magnitude~~ dir^n is independent does not dependent of the particular choice of cartesian co-ords in space. If at pt P the gradient is a non-zero vector, then it has the dir^n of max. increase of f at P

Proof :- We have proved earlier,

$$D_{\bar{b}} f = \nabla f \cdot \hat{b}$$

$$= \nabla f \cdot \frac{\bar{b}}{|\bar{b}|}$$

$$= |\nabla f| \frac{|\bar{b}|}{|\bar{b}|} \cos\theta$$

$$D_{\bar{b}} f = |\nabla f| \cos\theta$$

Where θ is the angle between $\bar{b}$ & ∇f & value of ∇f at P depends on P

$\therefore$ length & dir^n of ∇f depends on P on surface S & not on cartesian co-ords in space

Also, $-|\nabla f| \leq D_{\bar{b}} f \leq |\nabla f|$

$$\left\{\begin{array}{l} -1 \leq \cos\theta \leq 1 \\ -|\nabla f| \leq |\nabla f| \cos\theta \leq |\nabla f| \\ \therefore -|\nabla f| \leq D_{\bar{b}} f \leq |\nabla f| \end{array}\right.$$

$\therefore$ Max Value of $D_{\bar{b}} f = |\nabla f|$ when $\theta = 0$

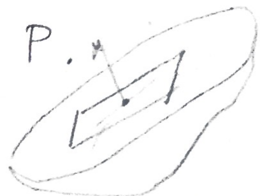
$\therefore$ Dir^{ral} deri. of f at pt. P is max in the direction of ∇f, $\nabla f \neq 0$

Ily, Mini Value of $D_{\bar{b}} f = -|\nabla f|$ when $\theta = \pi$

$\therefore$ Dir^{nal} deri. of f at P is mini in the dir^n $(-\nabla f)$.

Thm :- (For gradient as Surface Normal Vector)

Let f be a differentiable Scalar f^n that represents a surface S $f(x(t), y(t), z(t)) = d$ (const). & if gradient of f at P of S is a non-zero vector then it is a normal vector of S at P.



Proof:- Let $f(x(t), y(t), z(t)) = d$ represent a family of surfaces in space $\forall$ different values of d. And P be a fixed pt. on curve C on surface S in space

$$\therefore \ \bar{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

$$\bar{r}'(t) = x'(t)\hat{i} + y'(t)\hat{j} + z'(t)\hat{k} \quad \text{------ (1)}$$

$\therefore$ this vector (1) is tngt to S at P, as $\bar{r}'(t)$ is tngt to curve C on S at P. And therefore to all curves on S thru P we have respective tngt vectors & hence a family of tngt. vectors generates a tngt. Plane of surface S at P, and its normal is called Surface Normal of S at P. Also, any vector $\parallel$ to it is also called surface normal to S.

We have,

$$f(x(t), y(t), z(t)) = d \ \text{(const)}$$

$$\therefore \ \frac{df}{dt} = 0$$

$$\therefore \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = 0$$

$$\therefore \left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right) \circ \frac{d}{dt}\left(\hat{i}\,x(t) + \hat{j}\,y(t) + \hat{k}\,z(t)\right) = 0$$

$$\therefore \boxed{\nabla f \circ \bar{r}'(t) = 0}, \quad \nabla f \neq 0.$$

$\therefore$ ∇f & $\bar{r}'(t)$ are mutually $\perp$ to each other

$\therefore$ ∇f is the vector normal to S at P.

Hence proved.