Wide-band Frequency Modulation

KEEE343 Communication Theory

Lecture #17, May 12, 2011

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Summary

- Wideband Frequency Modulation

Wide-Band Frequency Modulation

- Spectral analysis of the wide-band FM wave

$$s(t) = A_c \cos[2\pi f_c t + \beta \sin(2\pi f_m t)]$$

or

$$s(t) = \Re\left[A_c \exp[j2\pi f_c t + j\beta \sin(2\pi f_m t)]\right] = \Re[\tilde{s}(t) \exp(j2\pi f_c t)]$$

where $\tilde{s}(t) = A_c \exp\left[j\beta \sin(2\pi f_m t)\right]$ is called "complex envelope".

Note that the complex envelope is a periodic function of time with a fundamental frequency f_m which means

$$\tilde{s}(t) = \tilde{s}(t + kT_m) = \tilde{s}(t + \frac{k}{f_m})$$

where $T_m = 1/f_m$

- Then we can rewrite

$$
\begin{aligned}
\tilde{s}(t) &= \tilde{s}(t + k/f_m) \\
&= A_c \exp[j\beta \sin(2\pi f_m (t + k/f_m))] \\
&= A_c \exp[j\beta \sin(2\pi f_m t + 2k\pi)] \\
&= A_c \exp[j\beta \sin(2\pi f_m t)]
\end{aligned}
$$

- Fourier series form

$$
\tilde{s}(t) = \sum_{n=-\infty}^{\infty} c_n \exp(j2\pi n f_m t)
$$

where

$$
\begin{aligned}
c_n &= f_m \int_{-1/(2f_m)}^{1/(2f_m)} \tilde{s}(t) \exp(-j2\pi n f_m t)\, dt \\
&= f_m A_c \int_{-1/(2f_m)}^{1/(2f_m)} \exp[j\beta \sin(2\pi f_m t) - j2\pi n f_m t]\, dt
\end{aligned}
$$

- Define the new variable: $x = 2\pi f_m t$

  Then we can rewrite
$$c_n = \frac{A_c}{2\pi} \int_{-\pi}^{\pi} \exp[j(\beta \sin x - nx)]\, dx$$

- nth order Bessel function of the first kind and argument β

$$J_n(\beta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp[j(\beta \sin x - nx)]\, dx$$

- Accordingly

$$c_n = A_c J_n(\beta)$$

  which gives

$$\tilde{s}(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \exp(j2\pi n f_m t)$$

- Then the FM wave can be written as

$$\begin{aligned} s(t) &= \Re[\tilde{s}(t)\exp(j2\pi f_c t)] \\ &= \Re\left[A_c \sum_{n=-\infty}^{\infty} J_n(\beta)\exp[j2\pi n(f_c + f_m)t]\right] \\ &= A_c \sum_{n=-\infty}^{\infty} J_n(\beta)\cos[2\pi(f_c + n f_m)t] \end{aligned}$$

- Fourier transform

$$S(f) = \frac{A_c}{2}\sum_{n=-\infty}^{\infty} J_n(\beta)\left[\delta(f - f_c - n f_m) + \delta(f + f_c + n f_m)\right]$$

which shows that the spectrum consists of an infinite number of delta functions spaced at $f = f_c \pm n f_m$ for $n = 0, +1, +2, ...$

Properties of Single-Tone FM for Arbitrary Modulation Index β

1. For different values of n

$$
\begin{aligned}
J_n(\beta) &= J_{-n}(\beta), && \text{for } n \text{ even} \\
J_n(\beta) &= -J_{-n}(\beta), && \text{for } n \text{ odd}
\end{aligned}
$$

2. For small value of β

$$
\begin{aligned}
J_0(\beta) &\approx 1, \\
J_1(\beta) &\approx \frac{\beta}{2} \\
J_n(\beta) &\approx 0, \qquad n > 2
\end{aligned}
$$

6. The equality holds exactly for arbitrary β

$$\sum_{n=-\infty}^{\infty} J_n^2(\beta) = 1$$

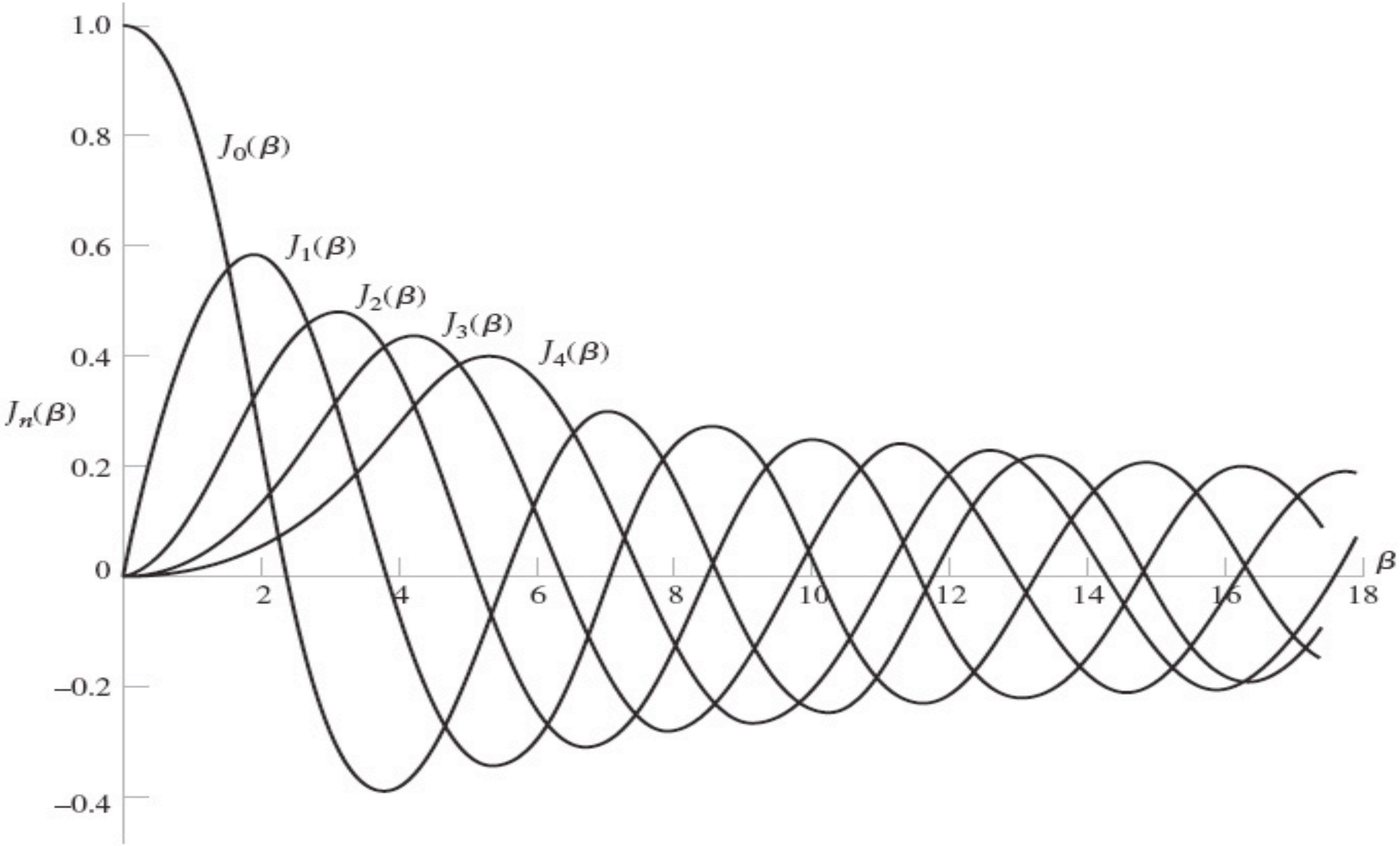

FIGURE 4.6 Plots of the Bessel function of the first kind, $J_n(\beta)$, for varying order n.

[Ref: Haykin & Moher, Textbook]

1. The spectrum of an FM wave contains a carrier component and and an infinite set of side frequencies located symmetrically on either side of the carrier at frequency separations of $f_m,\ 2f_m,\ 3f_m$...

2. The FM wave is effectively composed of a carrier and a single pair of side-frequencies at $f_c \pm f_m$.

3. The amplitude of the carrier component of an FM wave is dependent on the modulation index β .The average power of such as signal developed across a 1-ohm resistor is also constant:

$$P_{\text{av}} = \frac{1}{2} A_c^2$$

The average power of an FM wave may also be determined from

$$P_{\text{av}} = \frac{1}{2} A_c^2 \sum_{n=-\infty}^{\infty} J_n^2(\beta)$$

Transmission Bandwidth of FM Waves

- Recall the single-tone frequency modulated wave given as

$$s(t) = A_c \cos[2\pi f_c t + \beta \sin(2\pi f_m t)]$$

- and its FT is given as

$$S(f) = \frac{A_c}{2} \sum_{n=-\infty}^{\infty} J_n(\beta)[\delta(f - f_c - nf_m) + \delta(f + f_c + nf_m)]$$

where $\beta = \dfrac{k_f A_m}{f_m} = \dfrac{\Delta f}{f_m}$ for the message signal $m(t) = A_m \cos(2\pi f_m t)$

- To see the bandwidth let us consider two different cases

1. Case 1: Fix f_m and vary A_m (phase deviation is varied but the BW of message signal is fixed.)
2. Case 2: Fix A_m and vary f_m (phase deviation is fixed but the BW of message signal is varied.)

Case 1

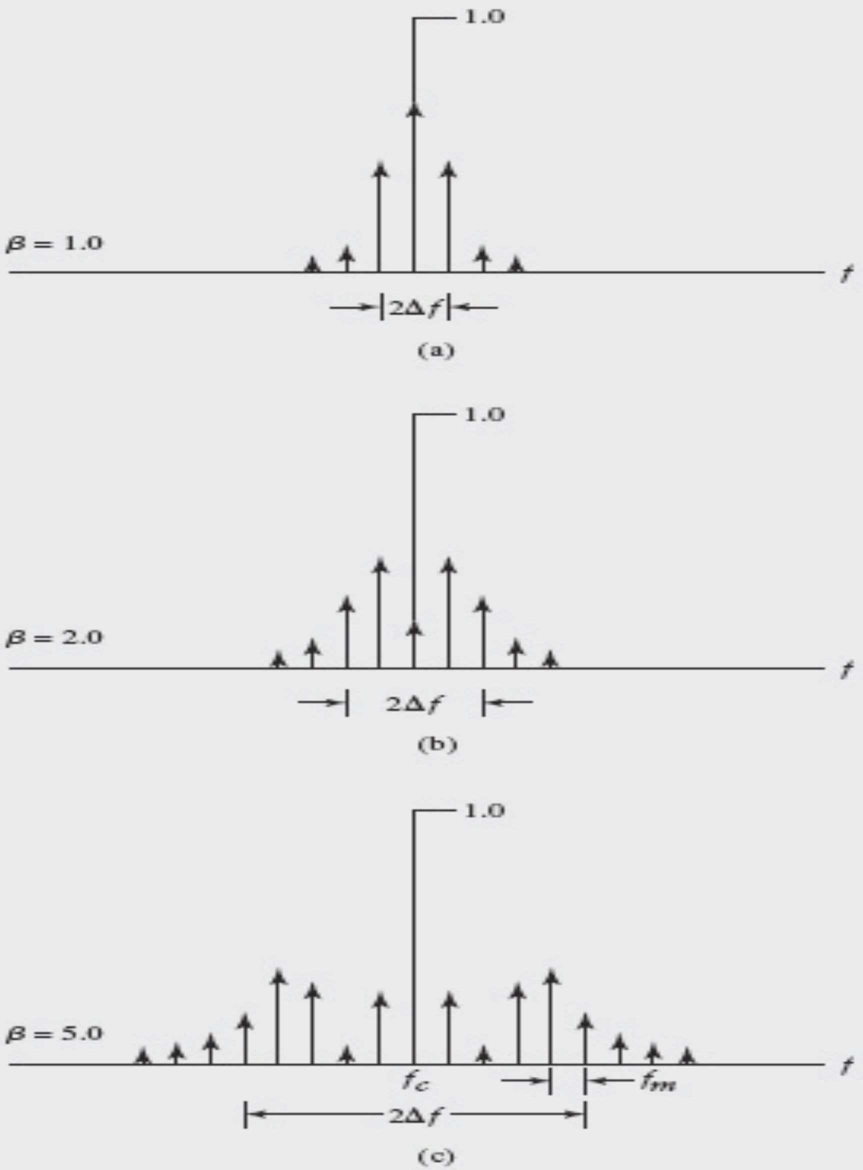

FIGURE 4.7 Discrete amplitude spectra of an FM wave, normalized with respect to the unmodulated carrier amplitude, for the case of sinusoidal modulation of fixed frequency and varying amplitude. Only the spectra for positive frequencies are shown.

[Ref: Haykin & Moher, Textbook]

Case 2

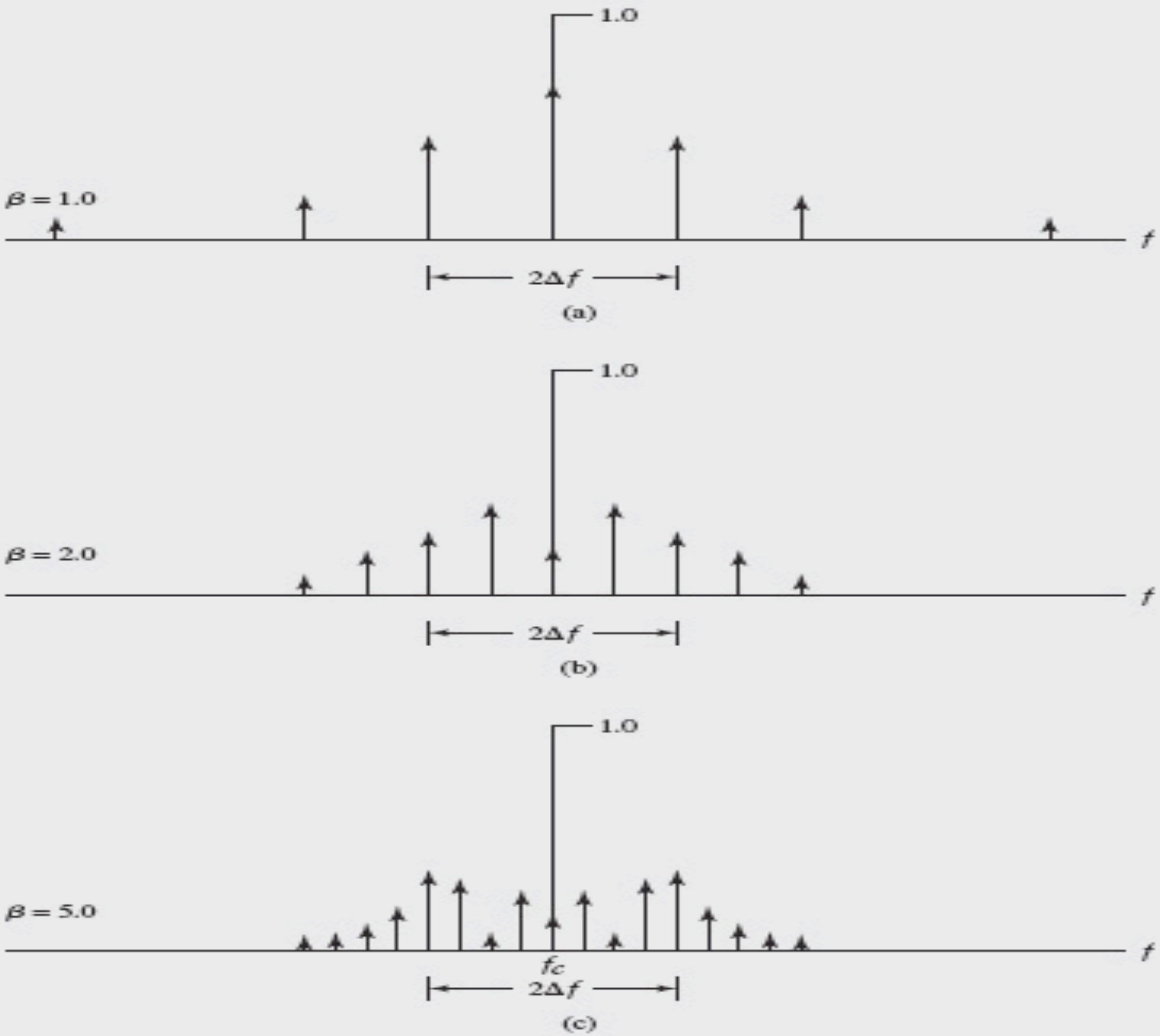

FIGURE 4.8 Discrete amplitude spectra of an FM wave, normalized with respect to the unmodulated carrier amplitude, for the case of sinusoidal modulation of varying frequency and fixed amplitude. Only the spectra for positive frequencies are shown.

[Ref: Haykin & Moher, Textbook]

Transmission Bandwidth of the FM Wave

- In theory, an FM wave contains an infinite number of side-frequencies.
- However, we find that the FM wave is effectively limited to a finite number of significant side-frequencies compatible with a specified amount of distortion.
- Observations of two limiting cases
    1. For large values of the modulation index β, the bandwidth approaches, and is only slightly greater than the total frequency excursion $2\Delta f$.
    2. For small values of the modulation index, the spectrum of the FM wave is effectively limited to one pair of side-frequencies at $f_c \pm f_m$ so that the bandwidth approaches $2 f_m$.

Carson's Rule

- Carson's rule is the approximate rule for the transmission bandwidth of an FM wave
  - Single-tone case

$$B_T \approx 2\Delta f + 2f_m = 2\Delta f \left(1 + \frac{1}{\beta}\right)$$

  - Arbitrary modulating wave

$$B_T \approx 2(\Delta f + W) = 2\Delta f \left(1 + \frac{1}{D}\right)$$

where $D = \dfrac{\Delta f}{W}$ is deviation ratio.

Universal Curve for FM Transmission Bandwidth

- Carson's rule is simple but unfortunately it does not always provide a good estimate of the transmission bandwidth, in particular, for the wideband frequency modulation.

TABLE 4.2 *Number of Significant Side-Frequencies of a Wide-Band FM Signal for Varying Modulation Index*

Modulation Index β	*Number of Significant Side-Frequencies* $2n_{max}$
0.1	2
0.3	4
0.5	4
1.0	6
2.0	8
5.0	16
10.0	28
20.0	50
30.0	70

[Ref: Haykin & Moher, Textbook]

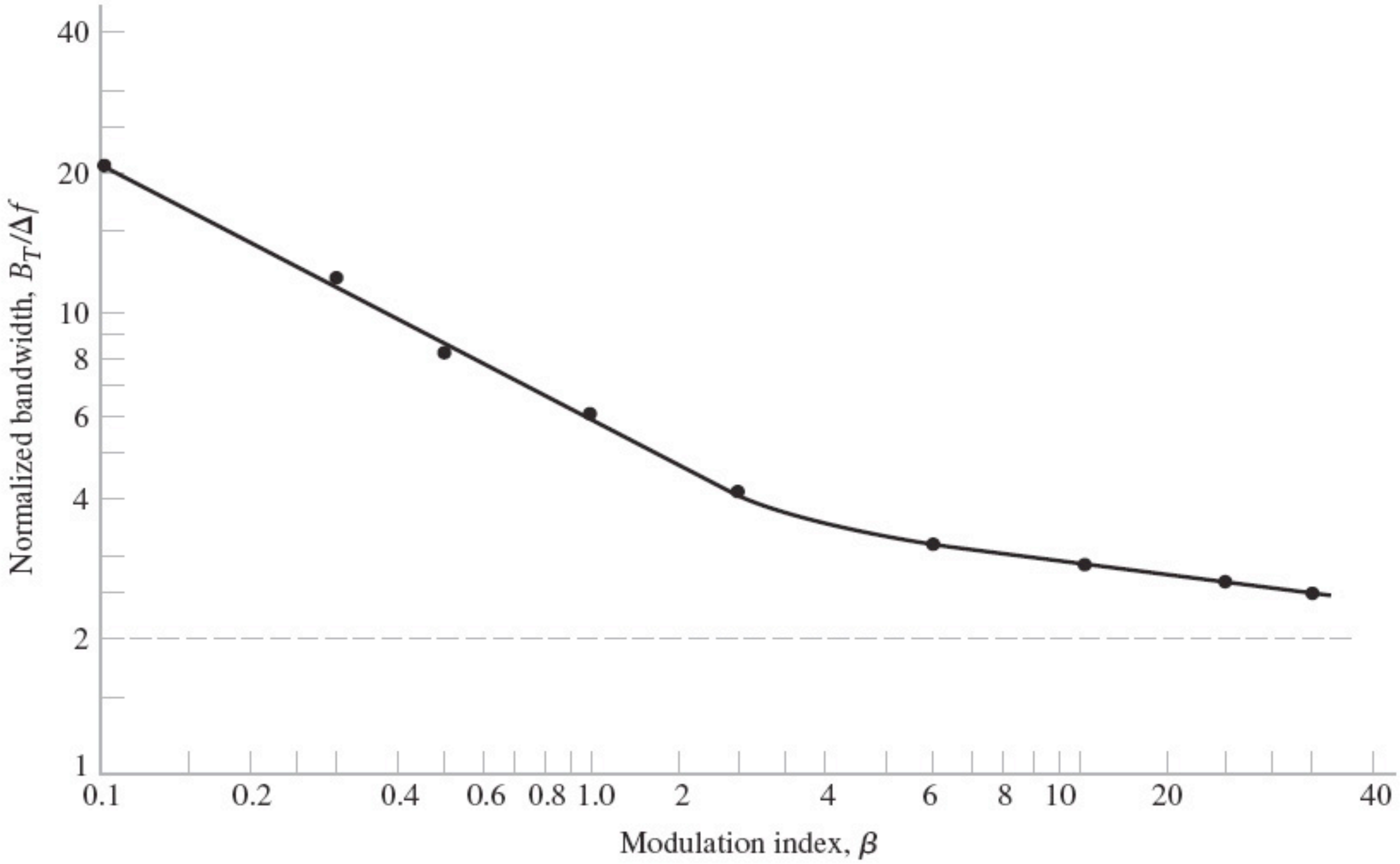

FIGURE 4.9 Universal curve for evaluating the one percent bandwidth of an FM wave.

[Ref: Haykin & Moher, Textbook]