# Vector Integral Calculus

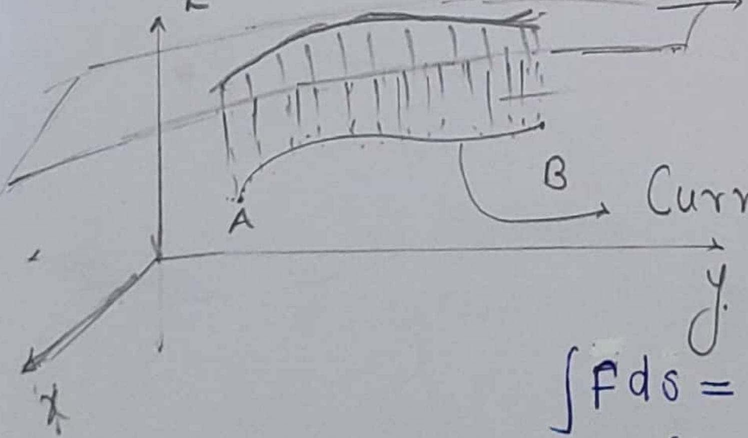
## Line Integrals :-

- Applications
  - → To find Curved surface area
  - → Area enclosed by a closed curve.
  - → Work done
    - While moving an object
    - When a charged particle travels along a curve
  - → Mass of the wire
  - → First moments and hence center of the mass of the wire.

### What is the line integral ?

It is a integral where function to be integrated along a curve C in space (or in plane).

It is also called curve integral.




$\int_C F\,ds$ = Curved surface area.

(i.e area above this curve and below the surface).

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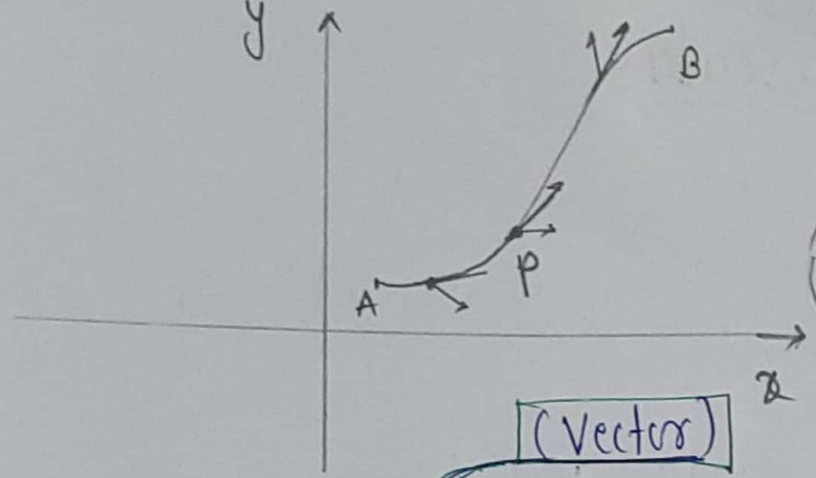
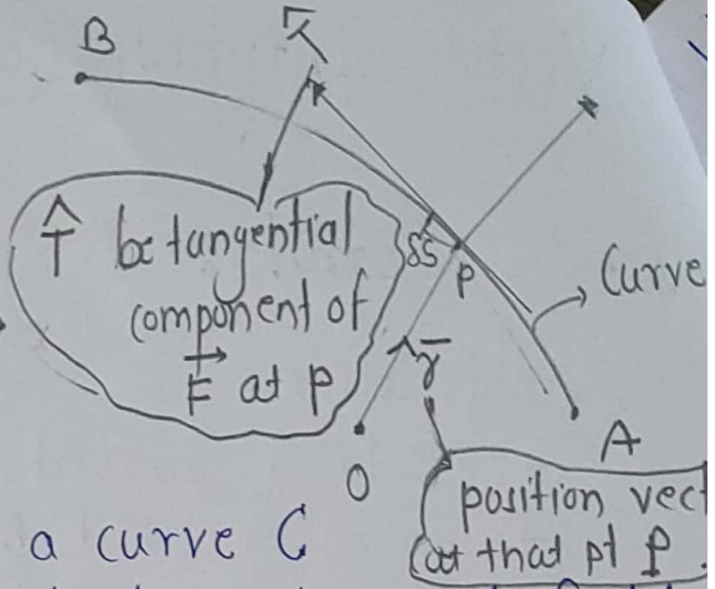



Consider a field $\vec{F}$ and a curve $C$

que: what is the work done by that field if I move some particals along that curve.

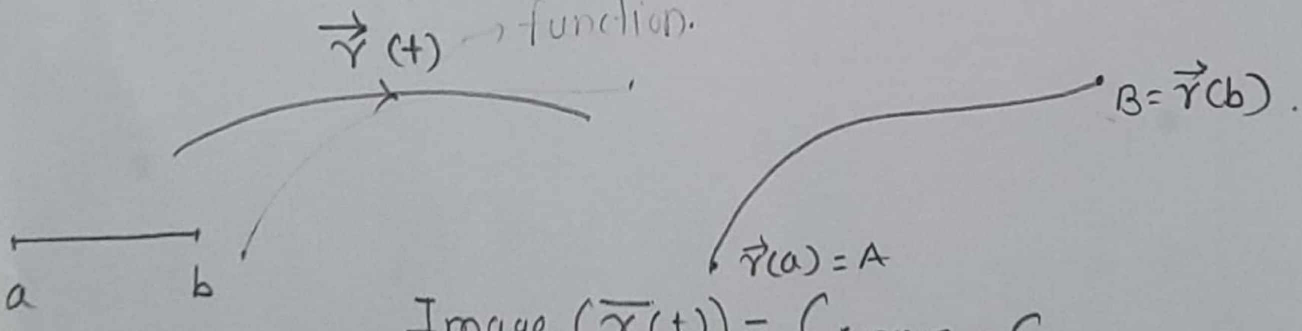
$\therefore$ Work done done along the segment $\delta s$ $=$ Force $\cdot$ Displacement

$$= \vec{F} \cdot \hat{T} \, \delta s$$

$\therefore$ Total work done along $C = \int_C \vec{F} \cdot \hat{T} \, ds.$

Let $\vec{r}(t) = [x(t), y(t), z(t)] = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$, $a \leq t \leq b$.

be parametric represention of curve $C$.




Image$(\vec{r}(t)) =$ Curve - C

We also assume that our paramatrized cure is smooth curve, that is.

- $\vec{r}(t)$ is differentiable
- $\vec{r}'(t)$ is continuous
- $\vec{r}'(t) \neq \vec{0}$ for $a \le t \le b$.

In this case $\vec{r}'(t)$ is nonzero tangent vector pointing in the forward direction.

Thus line integral of $\vec{F}$ along C is

$$\int_C \vec{F} \cdot \hat{T}\, ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\, dt = \int_a^b \vec{F}(\vec{r}(t))\, \vec{r}'(t) dt$$

$\frac{d\vec{r}}{ds}$ $\qquad$ $\frac{ds}{dt} dt$

$\boxed{\text{or}}$

$$\int_C \vec{F}(\vec{r}) \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\, \vec{r}'(t)\, dt.$$

$\hat{T} = \frac{d\vec{r}}{\cancel{ds}} \cdot \cancel{ds}$

In terms of components,

$$\vec{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$$

$$d\vec{r} = dx\, \hat{i} + dy\, \hat{j} + dz\, \hat{k}.$$

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$\therefore \int_C \vec{F}\, d\vec{r} = \int_C (F_1\hat{i} + F_2\hat{j} + F_3\hat{k}) \cdot (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$

$= \int_C (F_1\,dx + F_2\,dy + F_3\,dz)$ integrand is scalar (not a vector)

$= \int_a^b (F_1 x' + F_2 y' + F_3 z')\,dt$, integrand is scalar not a vector,

where $' = \frac{d}{dt}$.

$\int_a^b F \frac{dx}{dt}dt + F_2 \frac{dy}{dt}dt + F_3 \frac{dz}{dt}dt$

$\int_a^b F_1 x'\,dt + F_2 y'\,dt + F_3 z'\,dt$

$\int_a^b (F_1 x' + F_2 y' + F_3 z')\,dt.$

$\boxed{\text{Imp}}$

* If $f$ - Scalar field $f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$, then line integral along curve $C$ is

$$\int_C f\,ds = \int_{t=a}^{b} f(\vec{r}(t)) \cdot |\vec{r}\,'(t)|\,dt,$$

where $\vec{r}(t)$ is paramatric representation of $C$.

* If $C$ is a closed curve then.

$$\oint_C \vec{F}\cdot d\vec{r} = \int_a^b F(r(t)) \cdot \vec{r}\,'(t)\,dt.$$

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Evaluate the line integral.

$\int_C y\,dx + z^2 dy + x\,dz$ along the cure

$\vec{r}(t) = [2t, t^2, \sqrt{t}]$ on the interval $0 \le t \le 4$

$\rightarrow$ As $\vec{r}(t) = [2t, t^2, \sqrt{t}]$

$x(t) = 2t \qquad dx = 2\,dt$

$y(t) = t^2 \qquad dy = 2t\,dt$

$z(t) = \sqrt{t} \qquad dz = \frac{1}{2} t^{-1/2} dt$

$$\int_0^4 \left[t^2(2) + t\,(2t) + 2t\left(\tfrac{1}{2}(t)^{-1/2}\right)\right] dt$$

$$= \int_0^4 \left(2t^2 + 2t^2 + t^{1/2}\right) dt$$

$$= \int_0^4 \left(4t^2 + t^{1/2}\right) dt$$

$$= \frac{4}{3}t^3 + \frac{2}{3}t^{3/2} \Big|_0^4$$

$$= \left(\frac{4}{3}(4)^3 + \frac{2}{3}(4)^{3/2}\right) - (0)$$

$$= \frac{256}{3} + \frac{16}{3} = \frac{273}{3}.$$

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(3) Calculate

$$\int_C x^2 dx + xy\, dy + dz$$

where $C$ is the cure paramatrized by $\vec{r}(t) = [t, t^2, 1]$, moving from $t=0$ to $t=1$.

→ Here $x(t) = t, \rightarrow dx = 1\,dt$

$y(t) = t^2 \rightarrow dy = 2t\,dt$

$z(t) = 1 \rightarrow dz = 0\,dt$

So we have

$$\int_0^1 t^2 1\,dt + t t^2\, 2t\,dt + 1 \cdot 0$$

$$= \int_0^1 (t^2 + 2t^4)\,dt$$

$$= \left(\frac{t^3}{3} + 2\frac{t^5}{5}\right)\Big|_0^1$$

$$= \frac{1}{3} + \frac{2}{5}$$

$$= \frac{5+6}{15} = \frac{11}{15}.$$

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Example Consider $\vec{F} = x^2\hat{i} + xy\hat{j}$.

① Find $\int_{C_1} \vec{F}(\vec{r})\cdot d\vec{r}$ where $C_1 : y^2 = x$ joining $(0,0)$ and $(1,1)$.

② Find $\int_{C_2} \vec{F}(\vec{r})\cdot d\vec{r}$ where $C_2$ is the curve $y=x$ joining the same points.

→ ① We have $\vec{F} = x^2\hat{i} + xy\hat{j}$.

$C_1 : y^2 = x$ corresponding paramatrization

$\vec{r}(t) = [t^2, t]$, $0 \le t \le 1$.

$\vec{r}'(t) = [2t, 1]$.

$$\therefore \int_{C_1} \vec{F}(\vec{r})\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot \vec{r}'(t)\,dt$$

$$= \int_0^1 \vec{F}(\vec{r}(t))\cdot \vec{r}'(t)\,dt. \quad \text{—(1)}$$

$\vec{F}(x,y) = x^2\hat{i} + xy\hat{j}$

$\vec{F}(\vec{r}(t)) = \vec{F}(t^2, t) = [t^4, t^2\cdot t] = [t^4, t^3]$.

$\therefore \vec{F}(\vec{r}(t))\cdot\vec{r}'(t) = [t^4, t^3]\cdot[2t, 1]$

$= 2t^5 + t^3$

$\therefore$ By (1)

$$\int_{C_1} \vec{F}\,d\vec{r} = \int_0^1 (2t^5 + t^3)\,dt = \left(\frac{2t^6}{6} + \frac{t^4}{4}\right)_0^1$$

$$= \left(\frac{1}{3} + \frac{1}{4}\right) = \frac{4+3}{12} = \frac{7}{12}$$

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(ii) $C_2 : y = x, \quad \vec{r}(t) = [t, t], \quad 0 \le t \le 1$

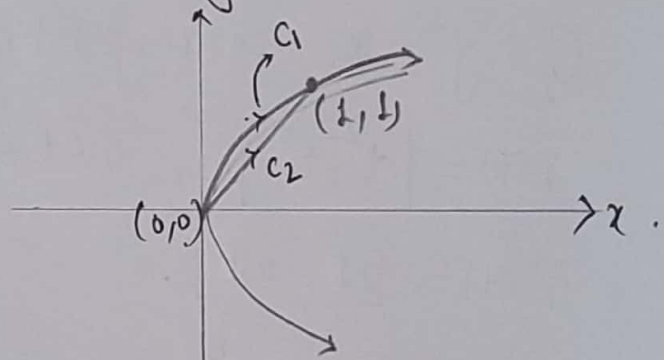
$\vec{r}'(t) = [1, 1].$

$$\int_{C_2} \vec{F}(\vec{r}) \cdot d\vec{r} = \int_0^1 F(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$$

$$= \int_0^1 [t^2, t^2] \cdot [1, 1]\, dt$$

$$= \int_0^1 (t^2 + t^2)\, dt$$

$$= 2\int_0^1 t^2\, dt = 2\left(\frac{t^3}{3}\right)_0^1 = \frac{2}{3}.$$



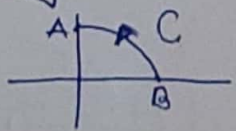

Thus we get different values for different paths.

H.W: ① Find the line integral $\vec{F}(\vec{r}) = [z, x, y]$ and $C$ is the helix.

where $C : \vec{r}(t) = [\cos t, \sin t, 3t], \quad 0 \le t \le 2\pi.$

Ans = 21.99

② Find the line integral $\vec{F}(\vec{r}) = [-y, -xy]$ and $C$ is the Circular arc in




Ans $\approx$ 0.4521.

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③ Compute the line integral $\int_C z\,ds$ $f = z$ (Scalar field) (Scalar line integral)

where $C: \vec{r}(t) = [\cos t, \sin t, t]$ on $0 \le t \le 1$

$\rightarrow$ $$\int_C f\,ds = \int_{t=a}^{b} f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt.$$

Here $\vec{r}(t) = [\cos t, \sin t, t]$

$\vec{r}\,'(t) = [-\sin t, \cos t, 1]$

$$|\vec{r}\,'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} = \sqrt{2}$$

$$\therefore \int_C z\,dz = \int_0^1 t \cdot \sqrt{2}\,dt$$

$$= \sqrt{2}\int_0^1 t\,dt = \sqrt{2}\left(\frac{t^2}{2}\right)^2 = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}.$$

④ Evalute $\int_C f(x,y,z)\,ds$ where

$f(x,y,z) = x^2 + y^2 - 1 + z$, and $C$ is the helix paramatrized by $\vec{r}(t) = (\cos t, \sin t, t)$, $0 \le t \le 3\pi$.

$\rightarrow$ $$\int_C f(x,y,z)\,ds = \int_{t=a}^{b} f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt.$$

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Here,

$$\vec{r}(t) = [\cos t, \sin t, t] \quad , \quad 0 \le t \le 3\pi$$

See that

$$f(\vec{r}(t)) = \cos^2 t + \sin^2 t - 1 + t$$
$$= 1 - 1 + t = t$$

Also see that

$$\vec{r}'(t) = [-\sin t, \cos t, 1]$$

$$|\vec{r}'(t)| = \sqrt{(-\sin t)^2 + \cos^2 t + 1} = \sqrt{2}$$

$$\therefore \int_0^{3\pi} \sqrt{2}\, t \, dt = \sqrt{2}\left[\frac{1}{2}t^2\right]_0^{3\pi} = \frac{\sqrt{2}}{2} 9\pi^2.$$

(5) Evaluate $\int_C xy + 4z \, ds$ where $C$ is the line segment from $P(2,-1,3)$ to $Q(4,2,2)$.

→ First find $\vec{r}(t)$.

$$\frac{x-2}{4-2} = \frac{y+1}{1+2} = \frac{z-3}{2-3} = t$$

$$x - 2 = 2t \Rightarrow x = 2t + 2$$
$$y + 1 = 3t \Rightarrow y = 3t - 1$$
$$z - 3 = -t \Rightarrow z = 3 - t$$

$$\therefore \vec{r}(t) = [2t+2, 3t-1, 3-t] \quad , \quad 0 \le t \le 1$$

$$\vec{r}'(t) = [2, 3, -1]$$

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$F(\vec{r}(t)) = (2t+2)(3t-1) + 4(3-t)$

$= 6t^2 \underline{-2t} + 6t - 2 + 12 \underline{-4t}$

$= 12t^2 - 6t + 10$

$|\vec{r}'(t)| = \sqrt{2^2 + 3^2 + (-1)^2} = \sqrt{4+9+1} = \sqrt{14}$

$\therefore \int_C f\,ds = \int_0^1 (12t^2 - 6t + 10)\sqrt{14}\,dt$

$= \sqrt{14}\left(\frac{12t^3}{3} - \frac{6t^2}{2} + 10t\right)_0^1$

$= \sqrt{14}\,(4 - 3 + 10)$

$= \sqrt{14}\cdot(11) \cdot \cancel{/}\ \underline{12\sqrt{14}}$

Home work.

Find $\int_C f\,ds$.

- $f = x^2 + y^2, \quad C: \vec{r} = [t, 4t, 0], \quad 0 \le t \le 1.$
- $f = 1 - \sinh^2 x, \quad C: \vec{r} = [t, \cosh t], \quad 0 \le t \le 2.$

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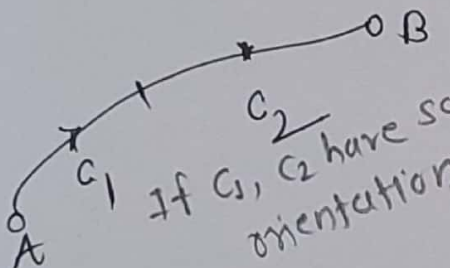
# Simple general properties of the line integral

(1) $\int_C k\vec{F}\cdot d\vec{r} = k\int_C \vec{F}\cdot d\vec{r}$ (k- constant)

(2) $\int_C (\vec{F}+\vec{G})\cdot d\vec{r} = \int_C \vec{F}\cdot d\vec{r} + \int_C \vec{G}\cdot d\vec{r}$

(3) $\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\, d\vec{r} + \int_{C_2} \vec{F}\cdot dr$

(Where $\vec{F}$ and $\vec{G}$ are vector fields).




If $C_1$, $C_2$ have same orientation as C

If the senie of integration along C is reversed, the value of integral is multiple by $-1$.

<u>Note</u> ① Any representation of C that gives the same positive direction on C also yeild the same value of the line integral.

② If the path of integration C is a Closed cure then

$$\oint_C \vec{F}(\vec{r})\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot \vec{r}'(t)\,dt.$$

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Show path dependence with example (simple). of your choice involving two paths.

→ Let $\vec{F} = [0, xy, 0]$,

$C_1 : \vec{r}_1(t) = [t, t, 0]$, $C_2 : \vec{r}_2(t) = [t, t^2, 0]$, $0 \le t \le 1$.

$\vec{r}_1'(t) = [1, 1, 0]$, $\vec{r}_2'(t) = [1, 2t, 0]$.

$$\int_{C_1} \vec{F}\, d\vec{r} = \int_{C_1} \vec{F}(\vec{r}_1(t)) \cdot \vec{r}_1'(t)\, dt$$

$$= \int_0^1 [0, t^2, 0] \cdot [1, 1, 0]\, dt$$

$$= \int_0^1 t^2\, dt = \left(\frac{t^3}{3}\right)_0^1 = \frac{1}{3}.$$

$$\int_{C_2} \vec{F}\, d\vec{r} = \int_{C_2} \vec{F}(\vec{r}_2(t)) \cdot \vec{r}_2'(t)\, dt$$

$$= \int_0^1 [0, t^3, 0] \cdot [1, 2t, 0]\, dt$$

$$= \int_0^1 2t^4\, dt = \frac{2}{5}.$$

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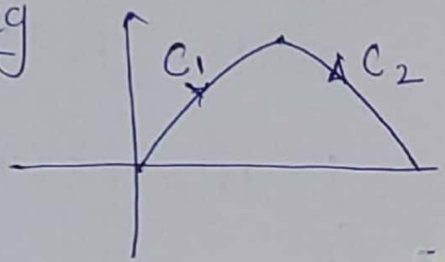
* Line integral over piecewise smooth curves :-

If $\vec{F}$ - continuous,
C - piecewise smooth curve
$\vec{F}\cdot\vec{r}'$ - piecewise const

piecewise smooth curve = It consists of finitely many smooth curves.

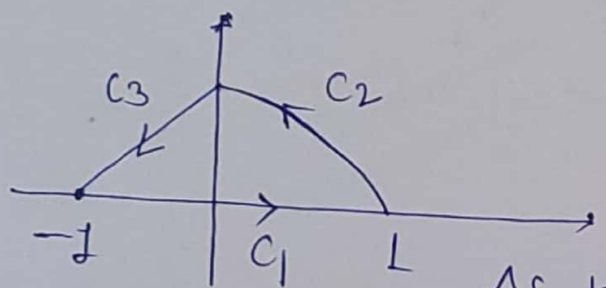
For. eg




Curve $C = C_1 \cup C_2$ piecewise smooth curve.

**Example ①**

Let $\vec{F} = [y, x]$, $C : C_1 \cup C_2 \cup C_3$




Then. Find $\int_C \vec{F}\cdot d\vec{r}$.

As we know,

$$\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F} d\vec{r} + \int_{C_2} \vec{F} d\vec{r} + \int_{C_3} \vec{F} d\vec{r}.$$

For along Curve $C_1 : \vec{r}_1(t) = (t, 0) \; -1 \le t \le 1$

we have $\vec{r}_1'(t) = (1, 0)$

$\vec{F}(\vec{r}(t)) = [0, t]$.

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_{-1}^{1} [0, t]\cdot[1, 0] = 0$$

For along Curve $C_2$: arc of circle $x^2+y^2=1$ in $1^{st}$ qud.

$C_2$: $\vec{\gamma}(t) = [\cos t, \sin t], \quad 0 \le t \le \pi/2$

$F(\vec{\gamma}(t)) = [\sin t, \cos t]$

$\vec{\gamma}'(t) = [-\sin t, \cos t].$

$$\therefore \int_{C_2} \vec{F}\cdot d\vec{\gamma} = \int_0^{\pi/2} F(\vec{\gamma}(t)) \cdot \vec{\gamma}'(t)\, dt$$

$$= \int_0^{\pi/2} [\sin t, \cos t] \cdot [-\sin t, \cos t]\, dt$$

$$= \int_0^{\pi/2} [-\sin^2 t + \cos^2 t]\, dt = \text{Do it}$$

$C_3$: line joing $(0,1)$ to $(-1,0)$

$$\frac{x-0}{-1-0} = \frac{y-1}{0-1} = t$$

$$\Rightarrow \frac{x}{-1} = t \Rightarrow x(t) = -t$$

and $y-1 = -t \Rightarrow y(t) = 1-t$

$\vec{\gamma}(t) = [-t, 1-t], \quad \vec{\gamma}(0) = [0, 1)$

$\vec{\gamma}(1) = (-1, 0)$

$\vec{\gamma}'(t) = [-1, 1]$

$\vec{F}(\vec{\gamma}(t)) = [1-t, -t]$

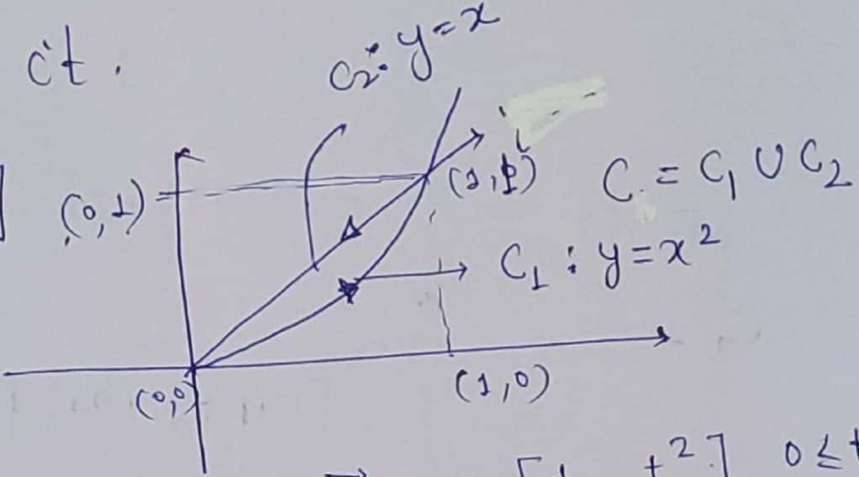
$\vec{F}(\vec{\gamma}(t)) \cdot \vec{\gamma}'(t) = [1-t, -t] \cdot [-1, 1] = -1 + \cancel{t} - \cancel{t} = -1$

$$\therefore \int_{C_3} \vec{F}\cdot d\vec{r} = \int_0^1 -1\, dt = (-1)(1) = -1.$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} + \int_{C_3} \vec{F}\cdot d\vec{r}.$$

$= $ Do it.

(2) $\vec{F} = [-y, -x]$




$C = C_1 \cup C_2$

Find $\int_C \vec{F}\cdot d\vec{r}$.

Along curve $C_1 : y = x^2$; $C_1 : \vec{r}(t) = [t, t^2]$, $0 \le t \le 1$

$$\vec{r}\,'(t) = [1, 2t].$$

$$\vec{F}(\vec{r}(t)) = [-t^2, -t].$$

$$\vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t) = [-t^2, -t]\cdot[1, 2t]$$

$$= -t^2 - 2t^2 = -3t^2.$$

$$\therefore \int_{C_1} \vec{F}\cdot d\vec{r} = \int_0^1 -3t^2\, dt = -1.$$

Along curve $C_2 : y = x$, $C_2 : \vec{r}(t) = [t, t]$, $1 \le t \le 0$
($\because$ (1,1) to (0,0))

$$\vec{r}\,'(t) = [1, 1]$$

$$\vec{F}(\vec{r}(t)) = [-t, -t]$$

$$\vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t) = [-t, -t]\cdot[1, 1] = -2t$$

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$$\therefore \int_{C_2} \vec{F}\cdot d\vec{r} = \int_1^0 -2t\,dt = -2\left(\frac{t^2}{2}\right)_1^0$$

$$= -2\left(0 - \frac{1}{2}\right) = 1.$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r}$$

$$= -1 + 1$$

$$= 0.$$

H.W (Do it)

① If $\vec{F} = (2x + y^2)\hat{i} + (3y - 4x)\hat{j}$
evaluate $\oint_C \vec{F}\cdot d\vec{r}$ around a tringular
regain ABC in xy-plane with
A(0,0), B(2,0), C(2,1).

(a) In the counterclockwise direction (ans $= -\frac{14}{3}$)

(b) What is the value in the opposite direction (ans $= \frac{14}{3}$)

② Evaluate $\int_C x^{-1}(y+z)\,ds$, where
C is arc of circle $x^2+y^2 = 4$, $z=0$ from
(2,0,0) to $(\sqrt{2}, \sqrt{2}, 0)$ in counter-clockwise direction.
(ans $= \ln 2$)

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