College of Engineering Pune

(An Autonomous Institute of Govt. of Maharashtra)

(MA-20004) Vector Calculus and Partial Differential Equations

Program : S.Y.B.Tech.(All Branches)
Semester- IV
Examimation : Test - 2
Date : 20/05/2022

Academic Year : 2021-22
Maximum Marks : 20
Time : 10 am - 11 am

Student MIS Number : | | | | | | | | | |

Instructions :

1. Write your MIS Number on Question Paper.
2. Writing anything on question paper is not allowed.
3. Mobile phones and programmable calculators are strictly prohibited.
4. Exchange/Sharing of stationery, calculator etc. is not allowed.
5. Figures to the right indicate the course outcomes and full marks.
6. Unless otherwise mentioned, symbols and notations have their usual standard meanings.
7. Any essential result, formula or theorem assumed for answering questions must be clearly stated.

Attempt the following.

Q.1 Solve the following questions:

(a) Plot the vector field $\vec{F}(x, y) = -y\,\hat{\imath} + x\,\hat{\jmath}$. [CO1] [1]

(b) Arjun is travelling on the curve given by $\vec{r}(t) = [2t,\ 3\sin(2t),\ 3\cos(2t)]$. Determine the position of Arjun after travelling for a distance of $\frac{\pi\sqrt{10}}{3}$ units. (Note: Starting position of Arjun is when t=0) [CO5] [3]

Q.2 Answer the following questions:

(a) State true or false, justify your answer. [CO4] [2]

i) Let $\vec{F}$, $\vec{G}$ be two vector fields and $curl(\vec{F}) = curl(\vec{G})$ then $\vec{F} = \vec{G}$.

ii) Let $(a,\, b) \in \mathbb{R}^2$, there exists a twice continuously differential vector function $\vec{F}(x,\, y)$ such that $div(curl(\vec{F})) \neq 0$ at $(a,\, b)$.

(b) Let C be curve defined by $\vec{r}(t) = [1 - 2\cos t, \sin t]$. Sketch the curve in the rectangular coordinate system. [CO2] [**2**]

Q.3 Using a suitable substitution evaluate the following double integral [CO3] [**4**]

$$\iint\limits_R e^{\left(\frac{x+y}{x-y}\right)}\, dx\, dy, \quad \text{where } R = \{(x,\, y) : x \geq 0,\, y \geq 0,\, x + y \leq 1\}.$$

Q.4 Answer the following questions:

(a) Find the directional derivative of the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & \text{if } (x,\, y) \neq (0,\, 0) \\ 0, & \text{if } (x,\, y) = (0,\, 0) \end{cases}$$

in the direction of the vector $\vec{u} = \hat{i} + \hat{j}$ at point $P(0,\ 0)$. [CO3] [**2**]

(b) Let $\vec{r} = [x, y, z]$ be a position vector and $r = \sqrt{x^2+y^2+z^2}$ be the magnitude of $\vec{r}$. Calculate $\nabla^2\left(\dfrac{1}{r}\right)$ for $r \neq 0$. [CO3] [**2**]

Q.5 Find the volume of the region that lies inside the sphere $x^2 + y^2 + z^2 = 2$ and outside the cylinder $x^2 + y^2 = 1$ using **spherical co-ordinates** [CO3] [**4**]

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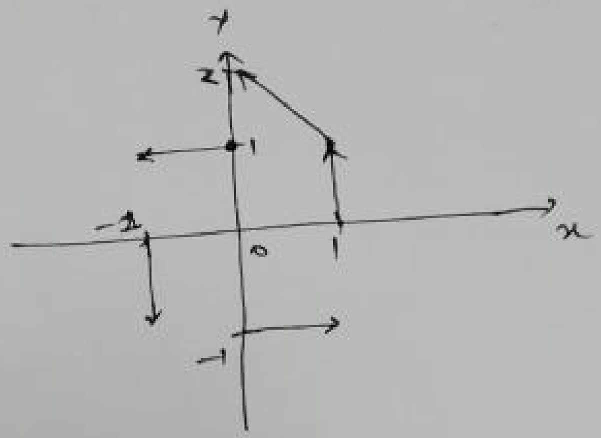
Q.1. vector field, $\bar{F} = -y\bar{i} + x\bar{j}$

soln :-

(x,y)	$\bar{F}(x,y)$
$(1,0)$	$\langle 0, 1 \rangle$
$(0,1)$	$\langle -1, 0 \rangle$
$(-1,0)$	$\langle 0, -1 \rangle$
$(0,-1)$	$\langle 1, 0 \rangle$
$(1,1)$	$\langle -1, 1 \rangle$



Q.2 — consider, $\bar{r}(t) = [2t, 3\sin 2t, 3\cos 2t]$

$\frac{1}{2}$ $\frac{1}{2}$

$s(t) = \int_0^t \sqrt{r'(\tau) r'(\tau)}\, d\tau$ --- (as Arjun is starting at $t=0$)

$\frac{1}{2}$ $\frac{1}{2}$

$r'(t) = (2, 6\cos 2t, -6\sin 2t)$

$s(t) = \int_0^t \sqrt{4 + 36\cos^2 2\tau + 36\sin^2 2\tau}\, d\tau$

$= \int_0^t \sqrt{4+36}\, d\tau = \sqrt{40}\, t = 2\sqrt{10}\, t \Rightarrow s = 2\sqrt{10}\, t$

$\Rightarrow \boxed{t = \frac{s}{2\sqrt{10}}}$

01

$\Rightarrow r\left(\frac{s}{2\sqrt{10}}\right) = \left(2\frac{s}{2\sqrt{10}}, 3\sin 2\frac{s}{2\sqrt{10}}, 3\cos 2\frac{s}{2\sqrt{10}}\right)$

$= \left(\frac{s}{\sqrt{10}}, 3\sin\frac{s}{\sqrt{10}}, 3\cos\frac{s}{\sqrt{10}}\right)$

$r(\pi/6) = \left(\frac{\pi}{3}, 3\sin\pi/3, 3\cos\pi/3\right)$

01

$= \left(\pi/3, \frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$. ✓

PMS.

Que (2) (a) State true or false, Justify your answer

$\frac{1}{2}$ (i) False, Let $\vec{H}$ be a non-zero vector field with $\operatorname{curl}(\vec{H}) = \vec{0}$

Then $\operatorname{curl}(\vec{F}+\vec{H}) = \operatorname{curl}(\vec{F}) + \operatorname{curl}(\vec{H})$

$= \operatorname{curl}(\vec{F})$

$\frac{1}{2}$ and if the statement is true

$\Rightarrow \vec{F}+\vec{H} = \vec{F}$

$\Rightarrow \vec{H} = \vec{0}$ contradiction

or

$\operatorname{curl}(\vec{F})=0$

$\operatorname{curl}(\vec{G})=0$

Here $\operatorname{curl}(\vec{F}) = \operatorname{curl}(\vec{G})$ but $\vec{F} \neq \vec{G}$.

$\frac{1}{2}$

(ii) False, for any twice continuously diff. vector function $\vec{F}$, $\operatorname{div}(\operatorname{curl}(\vec{F})) = 0$ $\frac{1}{2}$

(b) $\vec{r}(t) = [1-2\cos t, \sin t]$

Let $x = 1-2\cos t$, $y = \sin t$ $\Rightarrow \frac{(x-1)}{2} = -\cos t$, $y = \sin t$

$\Rightarrow \frac{(x-1)^2}{4} + y^2 = \cos^2 t + \sin^2 t = 1$ $\Rightarrow \boxed{\frac{(x-1)^2}{4} + y^2 = 1}$

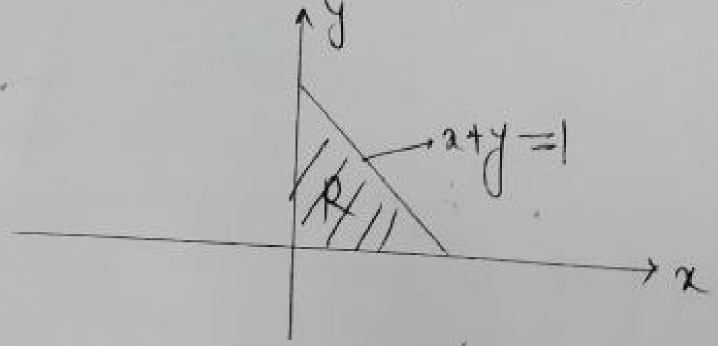
1+1

Thus our curve is ellipse with center $(1,0)$, semi-major axis $= 2$

Using a suitable substitution evaluate the following double integral

$$\iint_R e^{\left(\frac{x+y}{x-y}\right)}\,dx\,dy \text{ , where } R=\{(x,y) \mid x\geq 0,\ y\geq 0\ x+y\leq 1\}.$$

Sol^n:- $R=\{(x,y) \mid x\geq 0,\ y\geq 0,\ x+y\leq 1\}$



Step 1: Substitution

$u = x-y \ , \ v = x+y$ 1

$x = \dfrac{u+v}{2} \ , \ y = \dfrac{v-u}{2}$

— 1 M

Step 2:

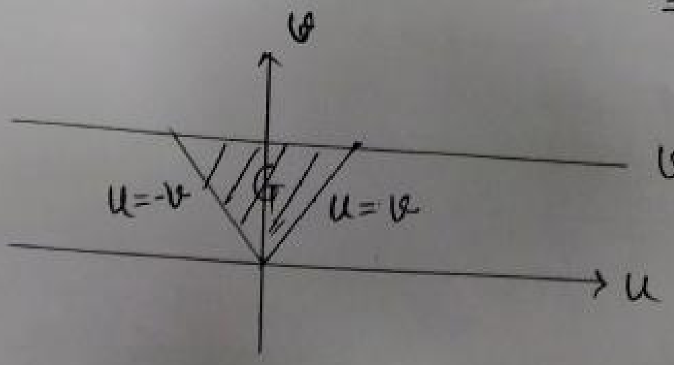
$$J(u,v) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2} \longrightarrow \frac{1}{2} \text{ M}$$

$$\Rightarrow |J(u,v)| = \left|\tfrac{1}{2}\right| = \tfrac{1}{2}.$$

$\frac{1}{2}$

Step 3: $\dfrac{\partial(x,y)}{\partial(u,v)}$

Eq^{ns} in xy-plane	Corresponding Eq^{ns} in uv-plane	
$y=0$	$v=u$	$\left.\right\}$ ½ M
$x=0$	$v=-u$	
$x+y=1$	$\cancel{\frac{u}{2}}+\frac{v}{2}+\frac{v}{2}-\cancel{\frac{u}{2}}=1 \longrightarrow$ ½ M	
	$\frac{2v}{2}=1$	
	$v=1$	



Step 4:

$$\iint_R e^{\left(\frac{x-y}{x+y}\right)} \, dx\,dy = \iint_G e^{\frac{u}{v}} \cdot \frac{1}{2} \, du\,dv$$

½M

$$= \int_0^1 \int_{-v}^{v} e^{u/v} \, \frac{1}{2} \, du\,dv$$

$$= \int_0^1 \left(\frac{v}{2} e^{\frac{u}{v}} \right)_{u=-v}^{u=v} dv$$

$$= \int_0^1 \frac{v}{2} \left(e - e^{-1} \right) dv$$

½M

$$= \frac{v^2}{4} \left(e - e^{-1} \right) \Big|_{v=0}^{1}$$

$$= \frac{1}{4} \left(e - \frac{1}{e} \right)$$

$$= \frac{e^2 - 1}{4e}.$$

Example: Find the directional derivative of the function $f: \mathbb{R}^2 \to \mathbb{R}$

defined by
$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

in the direction of vector $\vec{u} = \hat{i} + \hat{j}$ at the point $P(0,0)$.

Solution: $\vec{u} = \hat{i} + \hat{j}$ so $\hat{u} = \dfrac{\vec{u}}{\|\vec{u}\|} = \dfrac{\hat{i}+\hat{j}}{\sqrt{2}} = \dfrac{\hat{i}}{\sqrt{2}} + \dfrac{\hat{j}}{\sqrt{2}}$

$$(D_{\vec{u}}f)_{(a,b)} = \lim_{s \to 0} \frac{f(a+su_1, b+su_2) - f(a,b)}{s}$$

$\frac{1}{2}$

$$(D_{\vec{u}}f)_{(0,0)} = \lim_{s \to 0} \frac{f\left(\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\right) - f(0,0)}{s}$$

$$(D_{\vec{u}}f)_{(0,0)} = \lim_{s \to 0} \frac{\dfrac{s^2/2}{\sqrt{\frac{s^2}{2}+\frac{s^2}{2}}} - 0}{s}$$

$$(D_{\vec{u}}f)_{(0,0)} = \lim_{s \to 0} \frac{s^2/2}{s^2} = \frac{1}{2}$$ 1

Note: ~~If we use formula~~ $(D_{\vec{u}}f)_{(a,b)} = (\nabla f)_{(a,b)} \cdot \hat{a}$

$(\nabla f)_{(0,0)} = \left(\frac{\partial f}{\partial x}\right)_{(0,0)} \hat{i} + \left(\frac{\partial f}{\partial y}\right)_{(0,0)} \hat{j} = 0\hat{i} + 0\hat{j}$ then we get $(D_{\vec{u}}f)_{(0,0)} =$
If $\frac{\partial f}{\partial x}$ using definition 1

What is wrong if we use above formula for finding DD, Think on it

$\frac{1}{2}$ marks if $\left(\frac{\partial f}{\partial x}\right)_{(0,0)}$ by using $\frac{\partial (u/v)}{\partial x}$ rule

Q4(b)

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$ div(grad φ) = 0 1

grad φ = 1

$$\nabla^2\left(\frac{1}{r}\right) = \nabla^2\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$$

$$\nabla^2\left(\frac{1}{r}\right) = \frac{\partial^2}{\partial x^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial^2}{\partial z^2}\left(\frac{1}{\sqrt{x^2+y^2}}\right.$$

$$\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{-x}{(x^2+y^2+z^2)^{3/2}}$$ 1

$\frac{\partial}{\partial y}, \frac{\partial}{\partial z}$

$$\frac{\partial^2}{\partial x^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{-(x^2+y^2+z^2)^{3/2} + 3x^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{-1}{(x^2+y^2+z^2)^{3/2}} + \frac{3x^2}{(x^2+y^2+z^2)^{5/2}}$$

using symmetry of x, y and z.

$$\frac{\partial^2}{\partial y^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{-1}{(x^2+y^2+z^2)^{3/2}} + \frac{3y^2}{(x^2+y^2+z^2)^{5/2}}$$

$$\frac{\partial^2}{\partial z^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{-1}{(x^2+y^2+z^2)^{3/2}} + \frac{3z^2}{(x^2+y^2+z^2)^{5/2}}$$

1

$$\nabla^2\left(\frac{1}{r}\right) = \frac{-3}{(x^2+y^2+z^2)^{3/2}} + \frac{3(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}} = 0$$

ABD Test 2 20^{th} May 2021.

5).

$$\text{Vol} = \int_0^{2\pi} \int_{\pi/4}^{3\pi/4} \int_{\operatorname{cosec}\phi}^{\sqrt{2}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$ 2

?

$\frac{1}{2} + \frac{1}{2} + \frac{1}{2}$ $\frac{1}{2}$

$$= \frac{1}{3} \int_0^{2\pi} \int_{\pi/4}^{3\pi/4} \rho^3 \sin\phi \Big|_{\rho=\operatorname{cosec}\phi}^{\sqrt{2}} d\phi \, d\theta$$ $\frac{1}{2}$

$$= \frac{2\pi}{3} \int_{\pi/4}^{3\pi/4} (2\sqrt{2}\sin\phi - \operatorname{cosec}^2\phi) \, d\phi$$ $\frac{1}{2}$

$$= \frac{2\pi}{3} \left[-2\sqrt{2}\cos\phi + \cot\phi \right] \Big|_{\pi/4}^{3\pi/4}$$ $\frac{1}{2}$

$$= \frac{2\pi}{3} \left[-2\sqrt{2}\left(-\frac{1}{\sqrt{2}}\right) + 2\sqrt{2}\left(\frac{1}{\sqrt{2}}\right) \right]$$

$$= \frac{4\pi}{3} \text{ cu. units.}$$ $\frac{1}{2}$

$x^2+y^2=1 \leq \rho \leq x^2+y^2+z^2=1$

$$8 \int_{\phi=0}^{\pi/2} \int_{\theta=\pi/4}^{\pi/2} \int_{\operatorname{cosec}\phi}^{\sqrt{2}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$