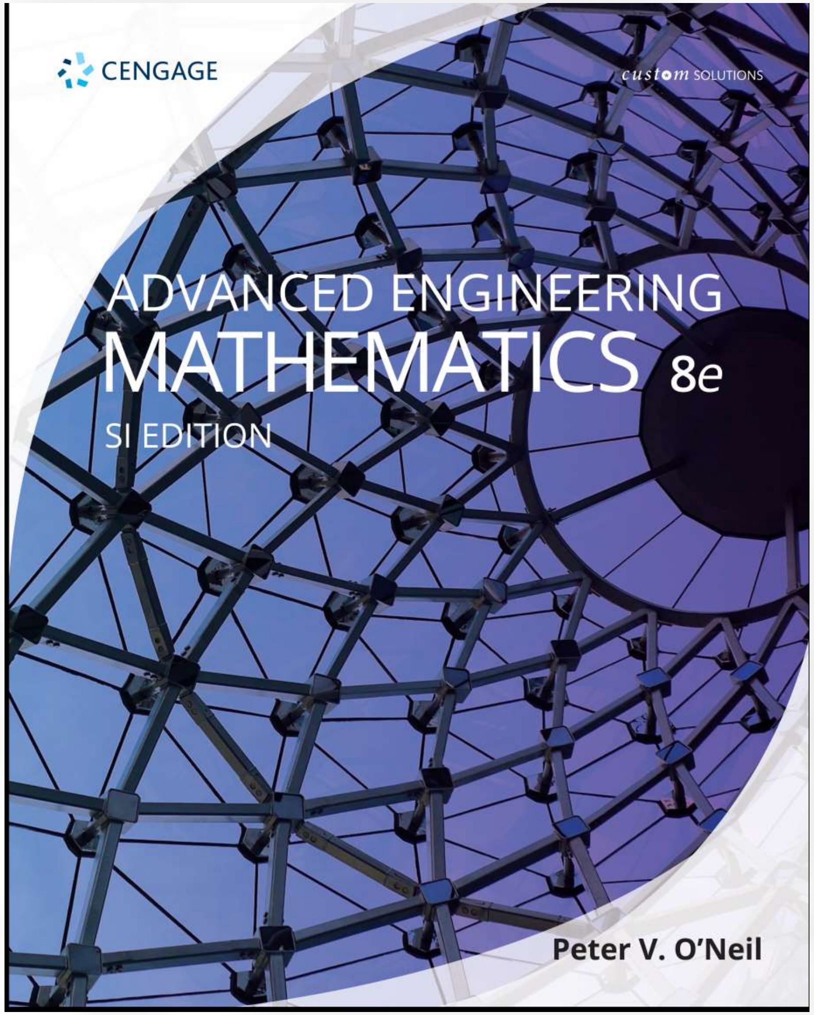


CHAPTER 13

Fourier Series

CENGAGE Learning

- In 1807, Joseph Fourier submitted a paper to the French Academy of Sciences in competition for a prize offered for the best mathematical treatment of heat conduction. In the course of this work Fourier shocked his contemporaries by asserting that “arbitrary” functions (such as might specify initial temperatures) could be expanded in series of sines and cosines. Consequences of Fourier’s work have had an enormous impact on such diverse areas as engineering, music, medicine, and the analysis of data.

Function Categories

(I)

Full range

[- L, L]

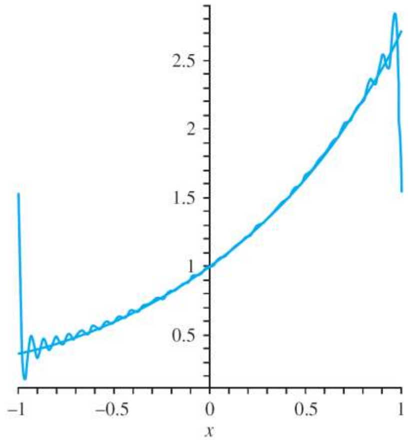


FIGURE 13.7 *Thirtieth partial sum of the Fourier series in Example 13.5.*

(II)

Half range

[0, L]

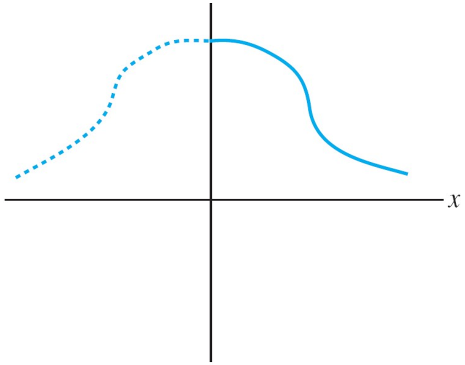


FIGURE 13.13 *Even extension of a function defined on* [0, L].

(III)

Periodic

($-\infty$, ∞)

(0, ∞)

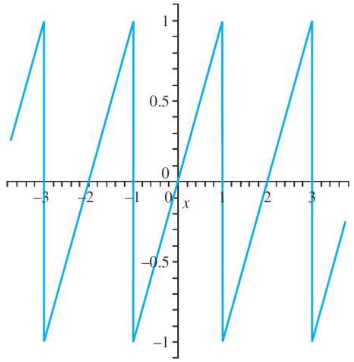
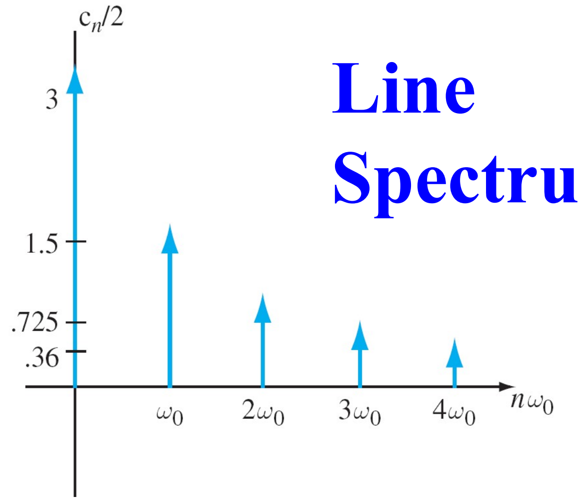


FIGURE 13.23 *Graph of f in Example 13.17.*



Line Spectrum

Finite-duration or Periodic

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}[a_n \cos(n\pi x / L) + b_n \sin(n\pi x / L)].$$

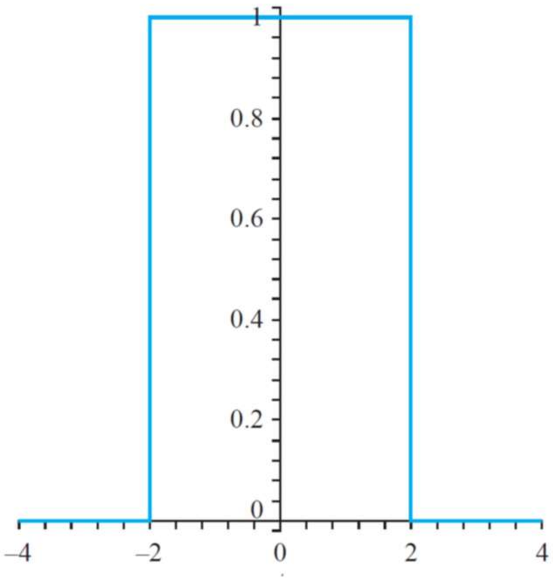
$$= \sum_{n=-\infty}^{\infty} d_n e^{in\omega_0 x},$$

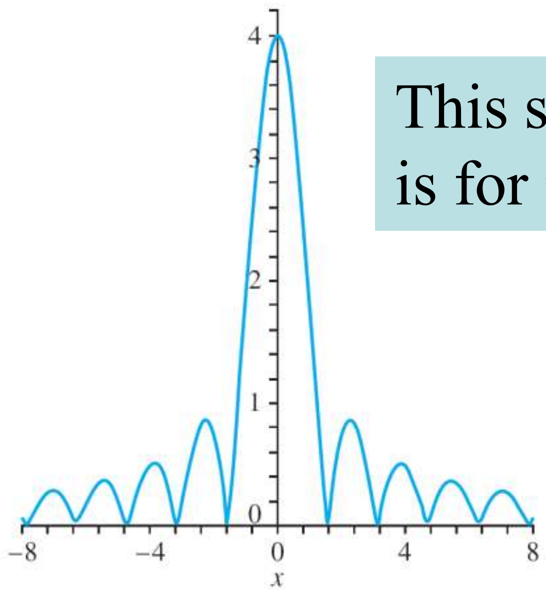
Function Categories

Finite-duration, NOT periodic   Continuous Spectrum

(IV)

Non-Perioc
(-∞, ∞)
or (0, ∞)

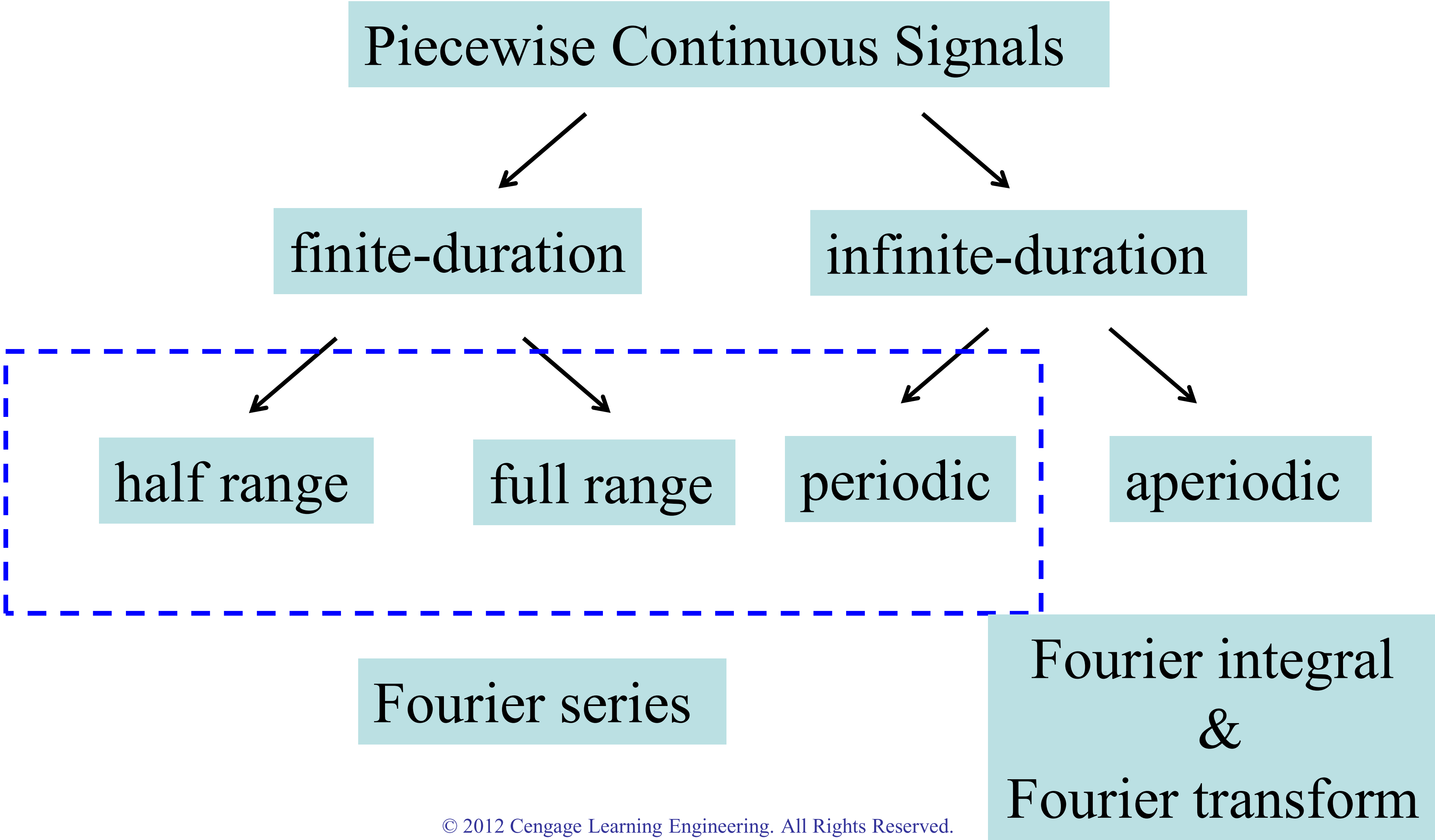




This spectrum is for indication

$$f(x) = \int_0^\infty (A_\omega \cos(\omega x) + B_\omega \sin(\omega x))d\omega$$

Tree of Fourier Analysis



Family of Fourier Series

Fourier Series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}[a_n \cos(n\pi x / L) + b_n \sin(n\pi x / L)].$$

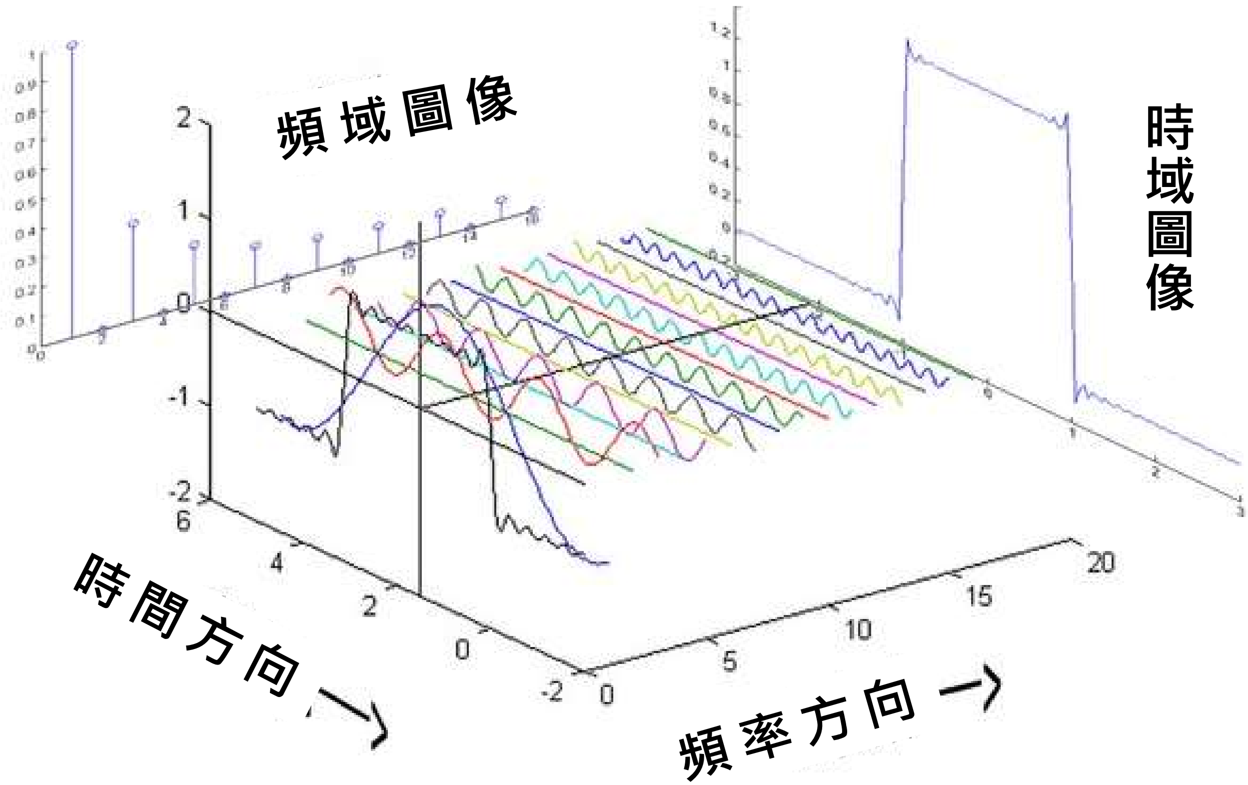
Complex Fourier Series

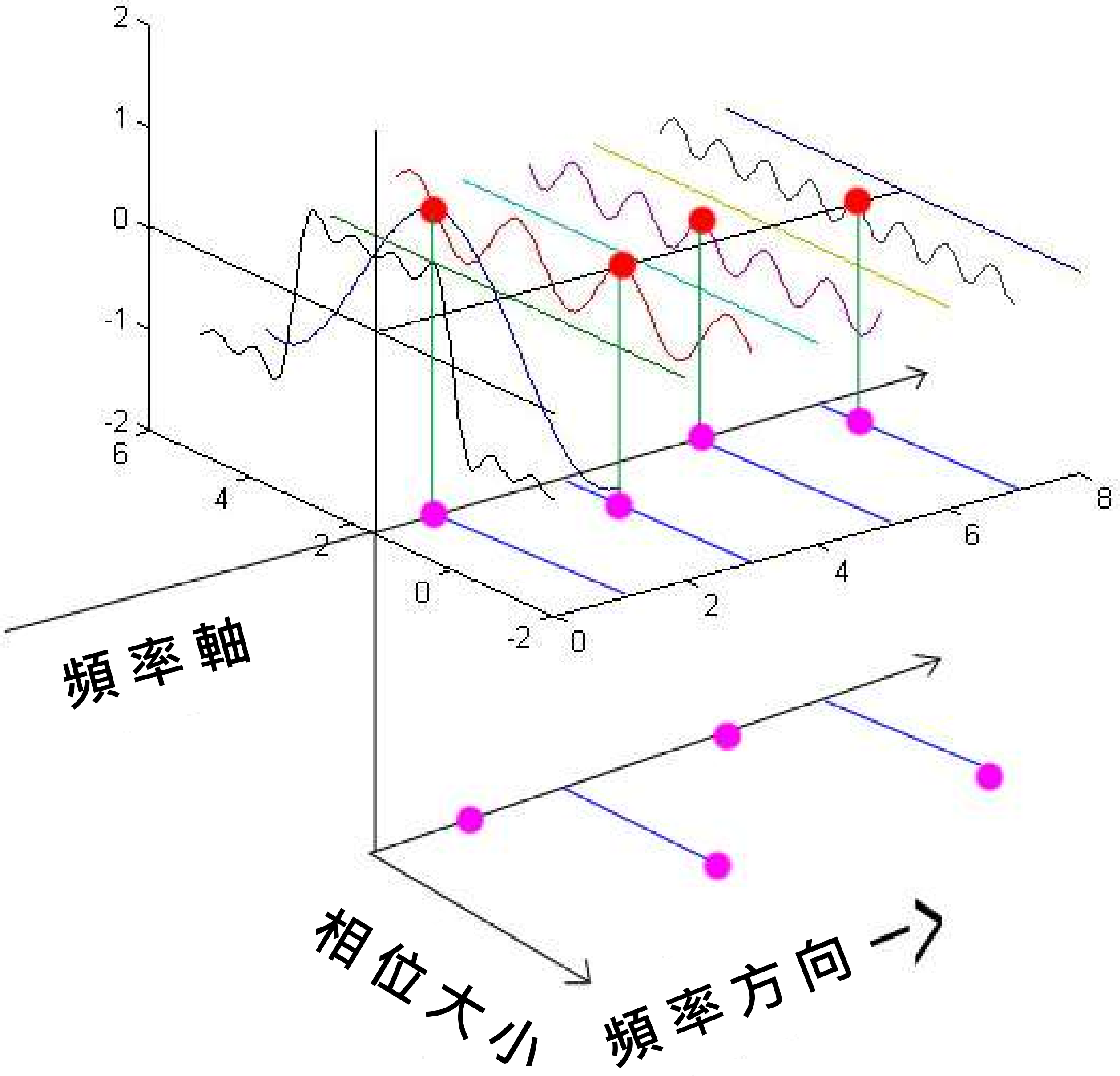
$$f(x) = \sum_{n=-\infty}^{\infty} d_n e^{in\omega_0 x},$$

Fourier Transform

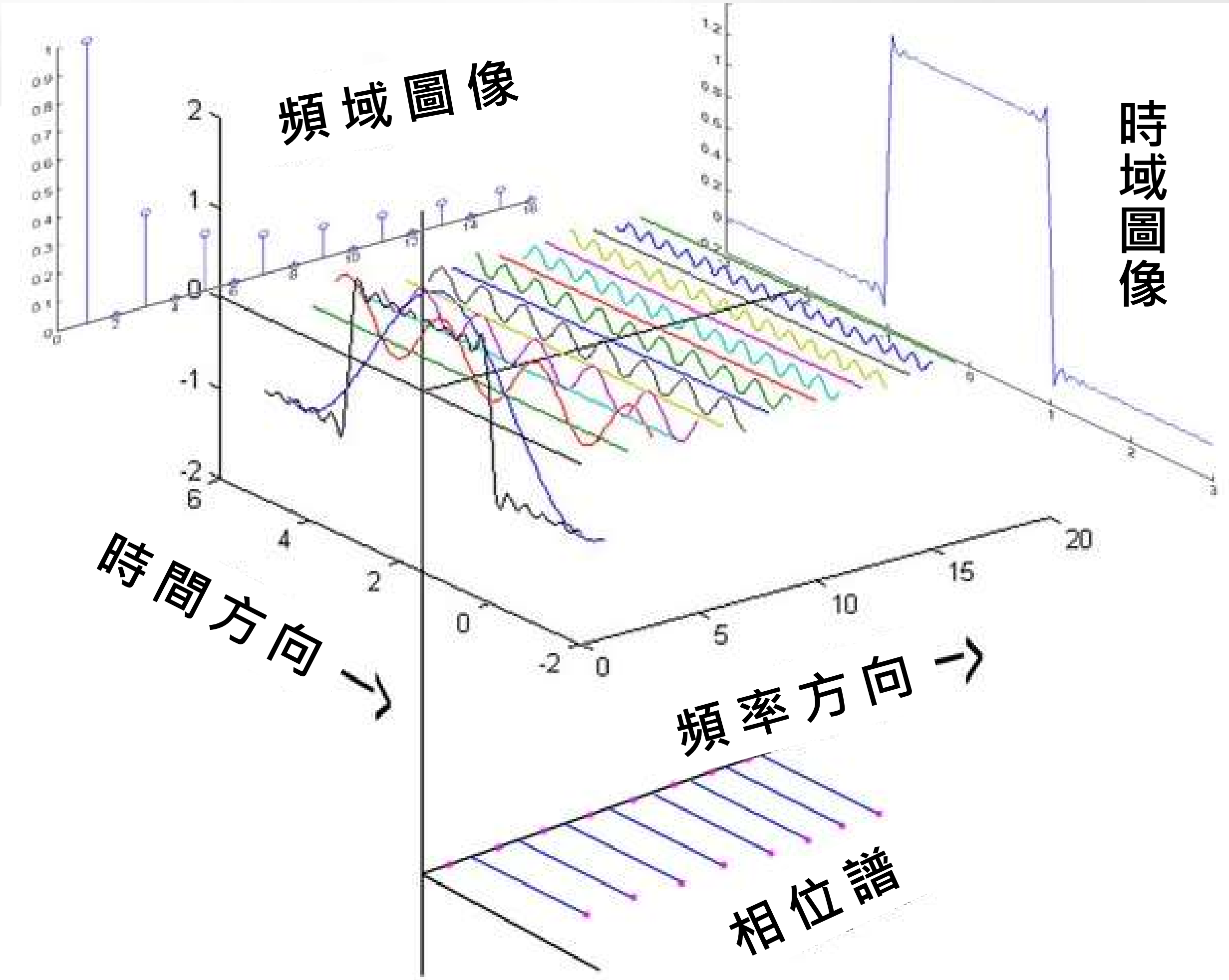
$$\mathcal{F}[f](\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt.$$

Visualization of Fourier Series





頻率軸
相位大小
頻率方向


頻域圖像
時域圖像
時間方向 →
頻率方向 →
相位譜

13.1 Fourier Series on [-*L*, *L*]

Let $f(x)$ be defined on $[-L, L]$. We want to choose numbers $a_0, a_1, a_2 \cdots$ and $b_1, b_2, \cdots$ so that

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}[a_n \cos(n\pi x / L) + b_n \sin(n\pi x / L)]. \tag{13.1}$$

The Fourier Series of a Function

- This is a decomposition of the function into a sum of terms, each representing the influence of a different fundamental frequency on the behavior of the function.

13.2 The Fourier Series of a Function

- To determine a_0, integrate equation (13.1) term by term to get

$$\int_{-L}^{L} f(x)dx = \frac{1}{2}\int_{-L}^{L} a_0 dx$$

$$+\sum_{n=1}^{\infty}\left(a_n \int_{-L}^{L} \cos\left(n\pi x / L\right)dx + b_n \int_{-L}^{L} \sin\left(n\pi x / L\right)dx\right)$$

$$= \frac{1}{2}a_0\left(2L\right) = La_0.$$

because all of the integrals in the summation are zero.

13.2 The Fourier Series of a Function

- Then

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx. \qquad (13.2)$$

The Fourier Series of a Function

- To solve for the other coefficients in the proposed equation (13.1), we will use the following three facts, which follow by routine integrations. Let *m* and *k* be integers. Then

$$\int_{-L}^{L} \cos(n\pi x / L)\sin(k\pi x / L)\ dx = 0.$$

The Fourier Series of a Function

- Furthermore, if $n \neq k$, then

$$\int_{-L}^{L} \cos(n\pi x / L)\cos(k\pi x / L)\, dx$$

$$= \int_{-L}^{L} \sin(n\pi x / L)\sin(k\pi x / L)\, dx = 0.$$

- And, if $n \neq 0$ and $n = k$, then

$$\int_{-L}^{L} \cos^2(n\pi x / L)\, dx = \int_{-L}^{L} \sin^2(n\pi x / L)\, dx = L.$$

The Fourier Series of a Function

- Now let n be any positive integer. To solve for a_n , multiply equation (13.1) by cos($k\pi x/L$) and integrate the resulting equation to get

$$\int_{-L}^{L} f\left(x\right)\cos(k\pi x / L)\ dx = \frac{1}{2}a_0\int_{-L}^{L}\cos(k\pi x / L)\ dx$$

$$+\sum_{n=1}^{\infty}\left[a_n\int_{-L}^{L}\cos(n\pi x / L)\cos(k\pi x / L)dx + b_n\int_{-L}^{L}\sin(n\pi x / L)\cos(k\pi x / L)dx\right].$$

The Fourier Series of a Function

- All of the terms on the right are zero except the coefficient of a_n , which occurs in the summation when $k = n$. The last equation reduces to

$$\int_{-L}^{L} f\left(x\right)\cos\left(k\pi x / L\right)dx = a_k \int_{-L}^{L} \cos^2\left(k\pi x / L\right)dx = a_k L$$

The Fourier Series of a Function

- Therefore

$$a_k = \frac{1}{L}\int_{-L}^{L} f(x)\cos(k\pi x / L)\, dx. \qquad (13.4)$$

This expression contains a_0 if we let $n = 0$.

The Fourier Series of a Function

- Similarly, if we multiply equation (13.1) by $\sin(k\pi x/L)$ instead of $\cos(k\pi x/L)$ and integrate, we obtain

$$b_k = \frac{1}{L}\int_{-L}^{L} f(x)\sin(k\pi x / L)dx. \qquad (13.5)$$

The Fourier Series of a Function

The numbers

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(n\pi x/L)dx \text{ for } n = 0,1,2,\cdots \quad (13.7)$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(n\pi x/L)dx \text{ for } n = 1,2,\cdots \quad (13.8)$$

are called the Fourier *coefficients of f* $f(x)$ on $[-L, L]$. When these numbers are used, the series (13.1) is called the *Fourier series of* $f(x)$ on $[-L, L]$.

EXAMPLE 1

- Let $f(x) = x - x^2$ for $-\pi \leq x \leq \pi$. Here $L = \pi$. Compute the Fourier series.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\,dx = -\frac{2}{3}\pi^2,$$

EXAMPLE 1

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(x - x^2\right)\cos\left(nx\right)dx$$

$$= \frac{4\sin(n\pi) - 4n\pi\ \cos(n\pi) - 2n^2\pi^2\sin(n\pi)}{\pi n^3}$$

$$= -\frac{4}{n^2}\cos(n\pi) = -\frac{4}{n^2}(-1)^n$$

$$= \frac{4(-1)^{n+1}}{n^2},$$

EXAMPLE 1

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\sin(nx)\ dx$$

$$= \frac{2\sin(n\pi) - 2n\pi\ \cos(n\pi)}{\pi n^2}$$

$$= -\frac{2}{n}\cos(n\pi) = -\frac{2}{n}(-1)^n$$

$$= \frac{2(-1)^{n+1}}{n}$$

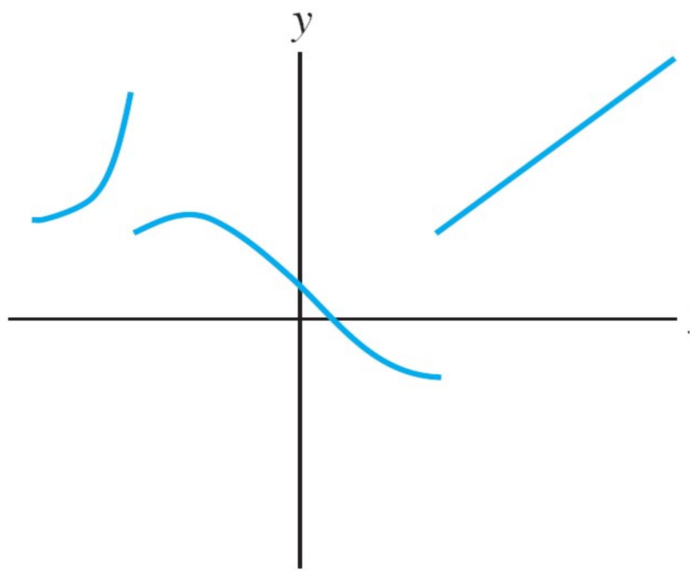
EXAMPLE 1

- We have used the facts that $\sin(n\pi) = 0$ and $\cos(n\pi) = (-1)^n$ if n is an integer.
- The Fourier series of $f(x) = x - x^2$ on $[-\pi,\pi]$ is

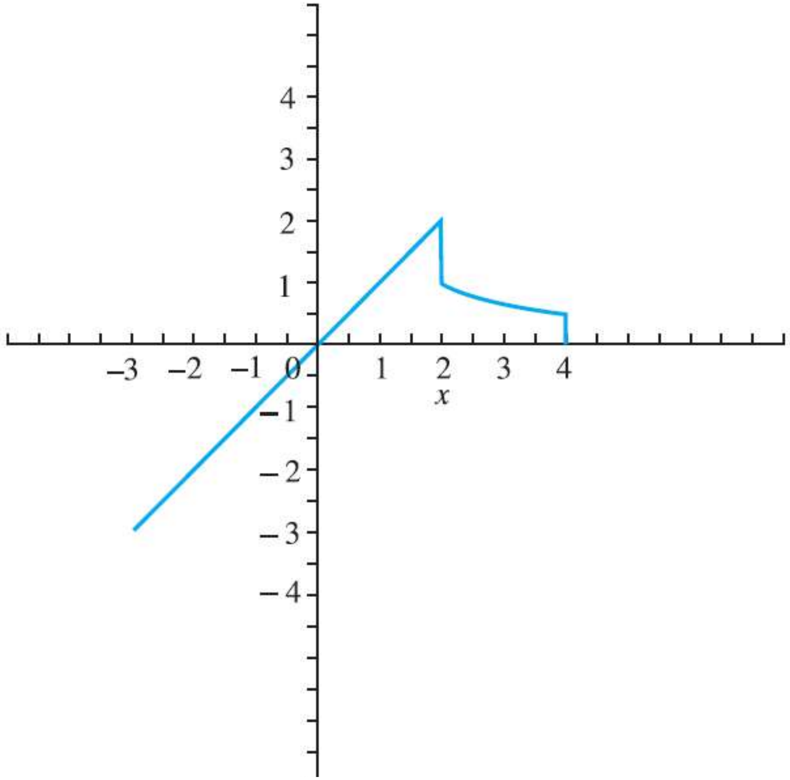
$$-\frac{1}{3}\pi^2 + \sum_{n=1}^{\infty}\left[\frac{4(-1)^{n+1}}{n^2}\cos(nx) + \frac{2(-1)^{n+1}}{n}\sin(nx)\right].$$

- This example illustrates a fundamental issue. We do not know what this Fourier series converges to.We need something that establishes a relationship between the function and its Fourier series on an interval. This will require some assumptions about the function.

- Recall that f is *piecewise continuous* on $[a, b]$ if f is continuous at all but perhaps finitely many points of this interval, and, at a point where the function is not continuous, f has finite limits at the point from within the interval. Such a function has at worst jump discontinuities, or finite gaps in the graph, at finitely many points.



Piecewise continuous function
(with jump discontinuity)



Piecewise smooth function
(with jump discontinuity or sharp point)

- If $-L < c < L$, denote the left limit of $f(x)$ at c as $f(c-)$, and the right limit of $f(x)$ at x_0 as $f(c+)$:

$$f(c-) = \lim_{h \to 0+} f(c-h) = \lim_{x \to c-} f(x)$$

$$f(c+) = \lim_{h \to 0+} f(c+h) = \lim_{x \to c+} f(x).$$

- If f is continuous at c, then these left and right limits both equal $f(c)$.
- Also, at the endpoints, we have

$$f(L-) = \lim_{x \to L-} f(x) \text{ and } f(-L+) = \lim_{x \to -L+} f(x).$$

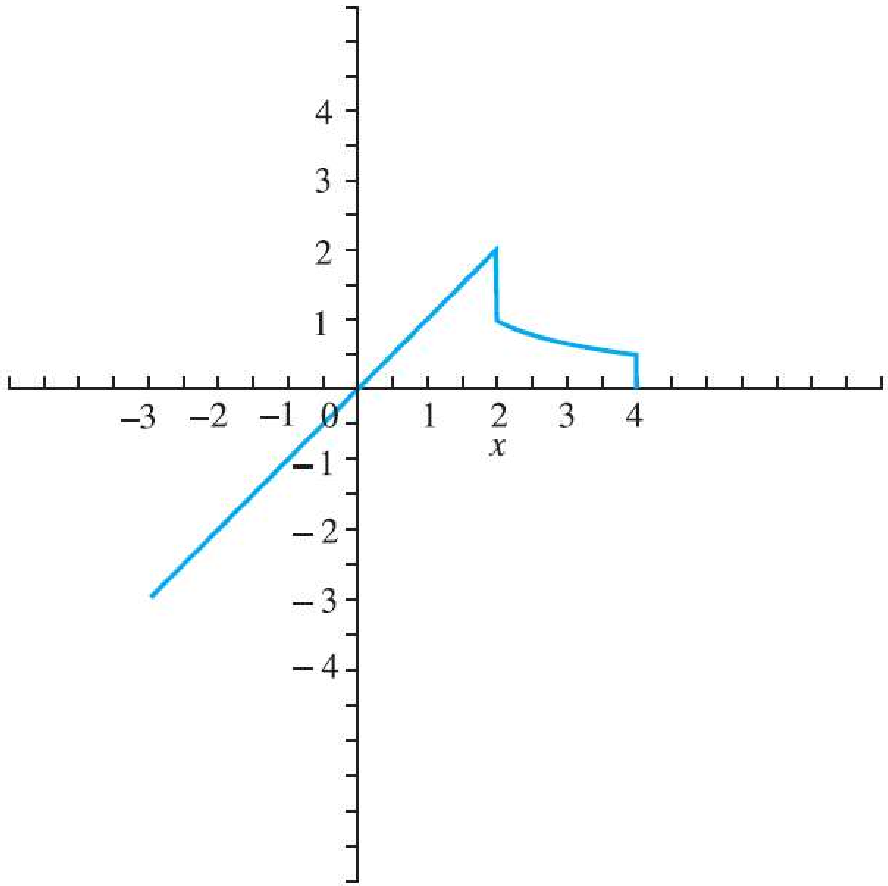
EXAMPLE 2

- Let

$$f(x) = \begin{cases} x & \text{for } -3 \le x < 2, \\ 1/x & \text{for } 2 \le x \le 4. \end{cases}$$

- f is piecewise continuous on [−3, 4], having a single discontinuity at $x = 2$. Furthermore, $f(2-)=2$ and $f(2+)=1/2$. A graph of f is shown in the nest Figure.

EXAMPLE 2



f is *piecewise smooth* on $[a, b]$ if f is piecewise continuous and f' exists and is continuous at all but perhaps finitely many points of (a, b).

EXAMPLE 3

- The function f of Example 2 is differentiable on (−3, 4) except at $x = 2$:

$$f'(x) = \begin{cases} 1 & \text{for } -3 < x < 2 \\ -1/x^2 & \text{for } 2 < x < 4. \end{cases}$$

- This derivative is itself piecewise continuous. Therefore f is piecewise smooth on [−3, 4].

Convergence of Fourier Series

- We can now state a convergence theorem.

THEOREM 13.1
Convergence of Fourier Series

- Let f be piecewise smooth on $[-L, L]$. Then, for each $x=c$ in $(-L, L)$, the Fourier series of f on $[-L, L]$ converges to

$$\frac{1}{2}(f(c-)+f(c+)).$$

- At both $-L$ and L this Fourier series converges to

$$\frac{1}{2}(f(-L+)+f(L-)).$$

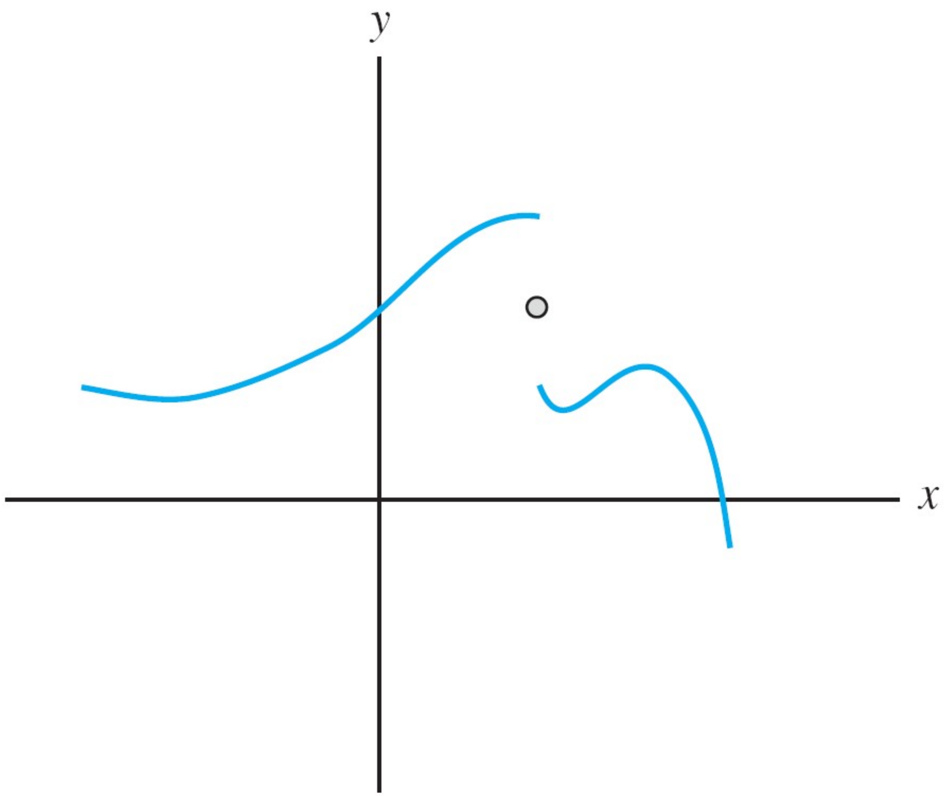


FIGURE 17.3 *Convergence of a Fourier series at a jump discontinuity.*

EXAMPLE 4

- Let $f(x) = x - x^2$ for $-\pi \leq x \leq \pi$. In Example 1 we found the Fourier series of f on $[-\pi, \pi]$. Now we can examine the relationship between this series and $f(x)$.

EXAMPLE 4

- $f'(x) = 1 - 2x$ is continuous for all x, hence f is piecewise smooth on $[-\pi,\pi]$. For $-\pi < x < \pi$, the Fourier series converges to $x - x^2$. At both π and $-\pi$, the Fourier series converges to

$$\frac{1}{2}(f(\pi-)+f(-\pi+)) = \frac{1}{2}((\pi-\pi^2)+(-\pi-(-\pi)^2))$$
$$= \frac{1}{2}(-2\pi^2) = -\pi^2.$$

EXAMPLE 4

- Figures 13.4, 13.5, and 13.6 show the fifth, tenth and twentieth partial sums of this Fourier series, together with a graph of *f* for comparison. The partial sums are seen to approach the function as more terms are included.

EXAMPLE 4

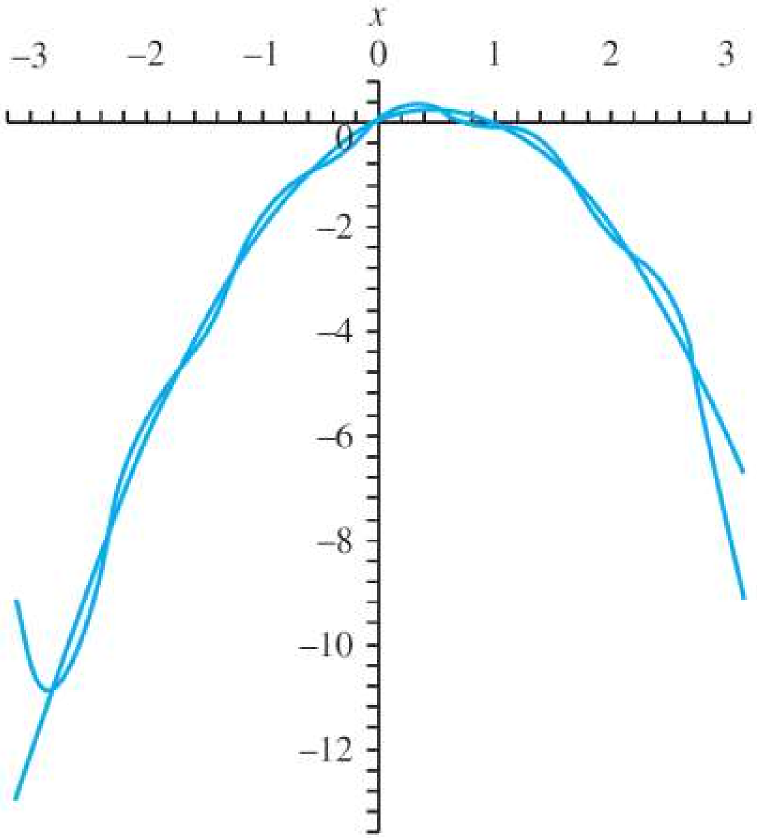


FIGURE 13.4 *Fifth partial sum of the Fourier series in Example 13.4.*

EXAMPLE 4

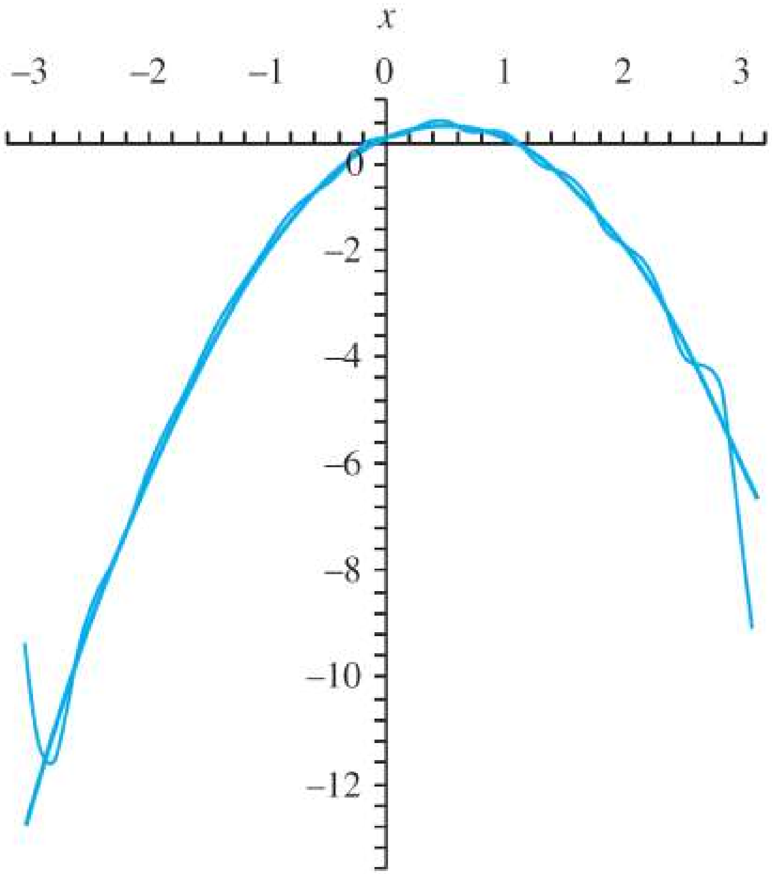


FIGURE 13.5 *Tenth partial sum in Example 13.4.*

EXAMPLE 4

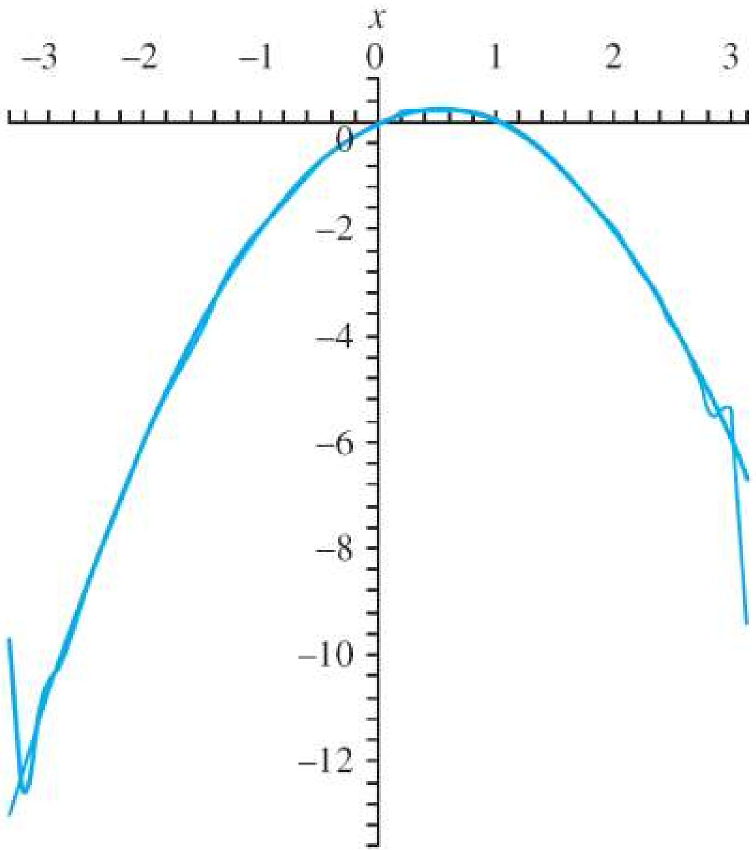


FIGURE 13.6 *Twentieth partial sum in Example 13.4.*

EXAMPLE 13.1

- Let $$f(x) = \begin{cases} x+1, & \text{for } \quad 0 \le x \le 2, \\ x-1, & \text{for } -2 \le x \le 0. \end{cases}$$

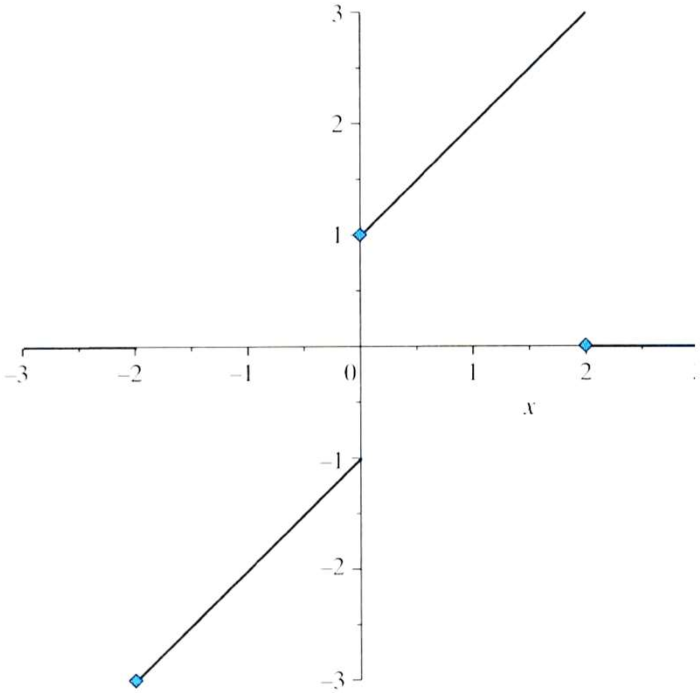
Calculate the Fourier coefficients of f on $[-1,\ 1]$.

$$f(0+) = \lim_{x \to 0+} f(x) = \lim_{x \to 0+} (x+1) = 1$$

$$f(0-) = \lim_{x \to 0-} f(x) = \lim_{x \to 0-} (x-1) = -1$$

$$f(-2+) = \lim_{x \to -2+} f(x) = \lim_{x \to -2+} (x-1) = -3$$

$$f(2-) = \lim_{x \to 2-} f(x) = \lim_{x \to 2-} (x+1) = 3$$



FIGU 13.4 · Graph of $f(x)$ in Example 17.1.

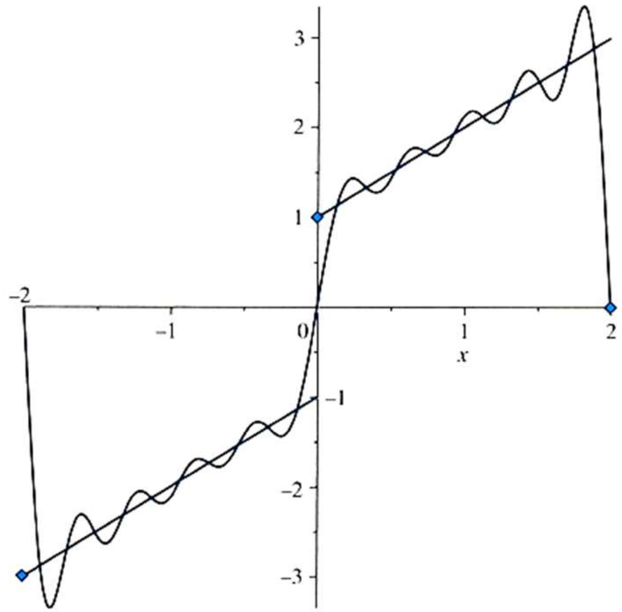


FIG 13.5 $f(x)$ and the tenth partial sum of the Fourier series in Example 17.1.

EXAMPLE 13.1

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\, dx = 0,$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right) dx = 0, \quad \text{for } n = 1, 2, \ldots,$$

and

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right) dx = \frac{2 + 6(-1)^{n+1}}{n\pi}.$$

EXAMPLE 13.1

- The Fourier series of $f(x)$ on $[-2, 2]$ is

$$\sum_{n=1}^{\infty} \frac{2+6(-1)^{n+1}}{n\pi} \sin\left(\frac{n\pi x}{2} \right).$$

- The Fourier series $f(x)$ on $[-2, 2]$ converges to

$$\begin{cases} x+1 & \text{for } 0 < x < 2 \\ x-1 & \text{for } -2 < x < 0 \\ 0 & \text{for } x = -2, 0, \text{ and } 2. \end{cases}$$

Why?

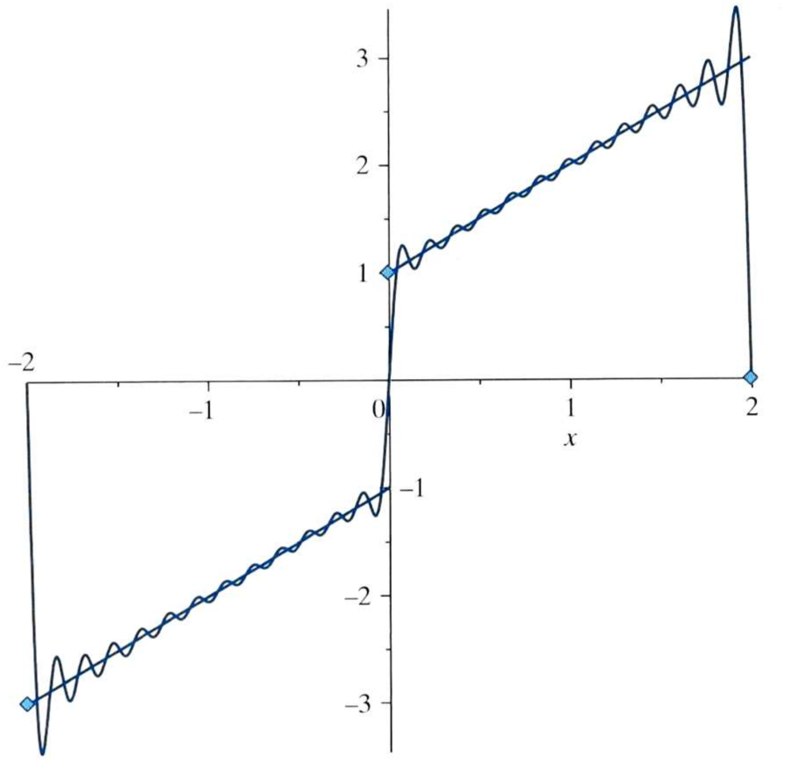


FIG 13.6 Twenty-fifth partial sum of the Fourier series in Example 17.1.

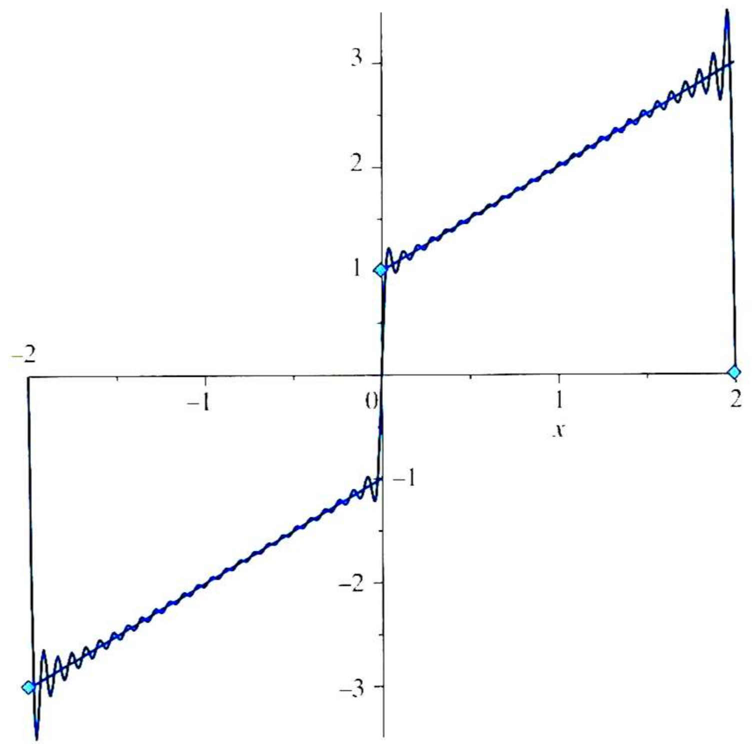


FIG 13.7 Fiftieth partial sum of the Fourier series in Example 17.1.

EXAMPLE 13.2

- Let $f(x) = e^{-x}$. The Fourier coefficients of f on $[-1, 1]$ are

$$a_0 = \int_{-1}^{1} e^{-x} dx = e - e^{-1},$$

$$a_n = \int_{-1}^{1} e^{-x} \cos(n\pi x)\ dx = \frac{(e - e^{-1})(-1)^n}{1 + n^2\pi^2},$$

and

$$b_n = \int_{-1}^{1} e^{-x} \sin(n\pi x) dx = \frac{(e\ -\ e^{-1})(-1)^n (n\pi)}{1 + n^2\pi^2}.$$

Show that $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx) + c$

Solution

$\int e^{ax} \sin bx \, dx$

$= \frac{e^{ax}}{a} \sin bx - \int \frac{e^{ax}}{a}(b \cos bx) \, dx$

$= \frac{e^{ax}}{a} \sin bx - \frac{b}{a} \int e^{ax} \cos bx \, dx$

$= \frac{e^{ax}}{a} \sin bx - \frac{b}{a}\left(\frac{e^{ax}}{a} \cos bx - \int \frac{e^{ax}}{a}(-b \sin bx) \, dx \right)$

$= \frac{e^{ax}}{a} \sin bx - \frac{b e^{ax}}{a^2} \cos bx - \frac{b^2}{a^2} \int e^{ax} \sin bx \, dx$

$\int e^{ax} \sin bx \, dx + \frac{b^2}{a^2} \int e^{ax} \sin bx \, dx$

$= \frac{e^{ax}}{a} \sin bx - \frac{b e^{ax}}{a^2} \cos bx$

$= \frac{e^{ax}}{a^2}(a \sin bx - b \cos bx)$

$\left(1 + \frac{b^2}{a^2}\right) \int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2}(a \sin bx - b \cos bx)$

$\int e^{ax} \sin bx \, dx = \frac{a^2}{a^2 + b^2} \frac{e^{ax}}{a^2}(a \sin bx - b \cos bx) + c$

$= \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx) + c$

S	D	I
+	$\sin(bx)$	e^{ax}
-	$\text{b}\cos(bx)$	$\frac{1}{a} e^{ax}$
+	$-b^2 \sin(bx)$	$\frac{1}{a^2} e^{ax}$

EXAMPLE 13.2

- The Fourier series of e^x on $[-1, 1]$ is

$$\frac{1}{2}(e-e^{-1})+(e-e^{-1})\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2\pi^2}(\cos(n\pi x)+n\pi\ \sin(n\pi x)).$$

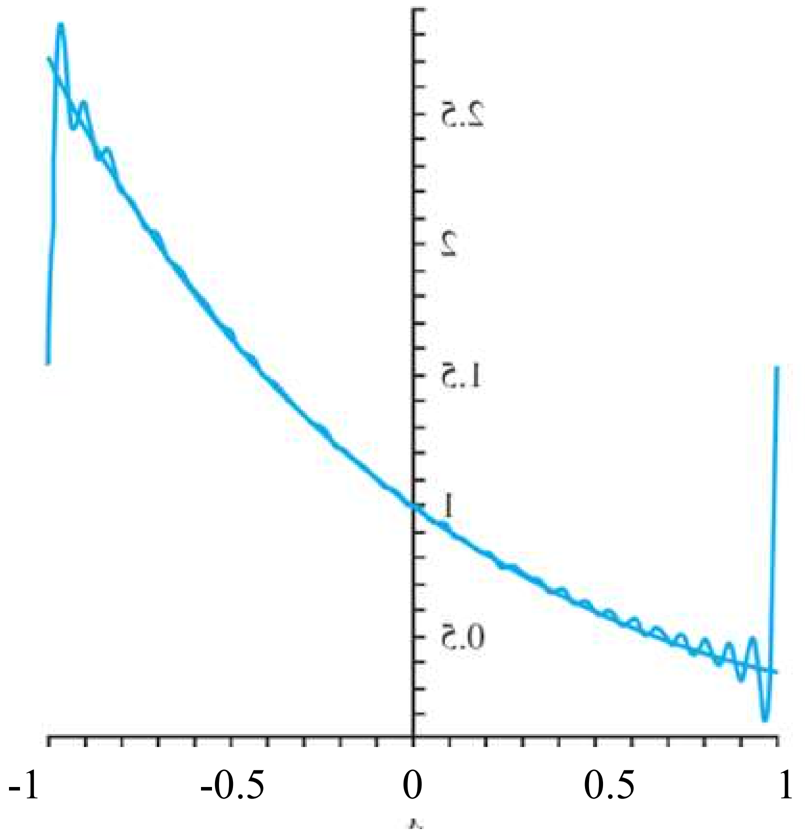
EXAMPLE 13.2

- Because e^x is continuous with a continuous derivative for all x, this series converges to

$$\begin{cases} e^{-x} & \text{for } -1 < x < 1 \\ \frac{1}{2}(e + e^{-1}) & \text{for } x = 1 \text{ and for } x = -1. \end{cases}$$

- Figure 13.8 shows the thirtieth partial sum of this series, suggesting its convergence to the function except at the endpoints −1 and 1.

EXAMPLE 13.2



13.1.1 Fourier Series o Even and Odd Functions

A function f is *even* on $[-L, L]$ if its graph on $[-L, 0]$ is the reflection across the vertical axis of the graph on $[0, L]$. This happens when $f(-x) = f(x)$ for $0 < x \leq L$. For example, x^{2n} and $\cos(n\pi x/L)$ are even on any $[-L, L]$ for any positive integer n.

A function f is *odd* on $[-L, L]$ if its graph on $[-L, 0)$ is the reflection through the origin of the graph on $(0, L]$. This means that f is odd when $f(-x) = -f(x)$ for $0 < x \leq L$. For example, x^{2n+1} and $\sin(n\pi x/L)$ are odd on $[-L, L]$ for any positive integer n.

- Figures 13.8 and 13.9 show typical even and odd functions, respectively.
- A product of two even functions is even, a product of two odd functions is even, and a product of an odd function with an even function is odd.

$$
\begin{aligned}
even \times even &= \text{even} \\
odd \times odd &= \text{even} \\
even \times odd &= \text{odd}
\end{aligned}
$$

Even and Odd Functions

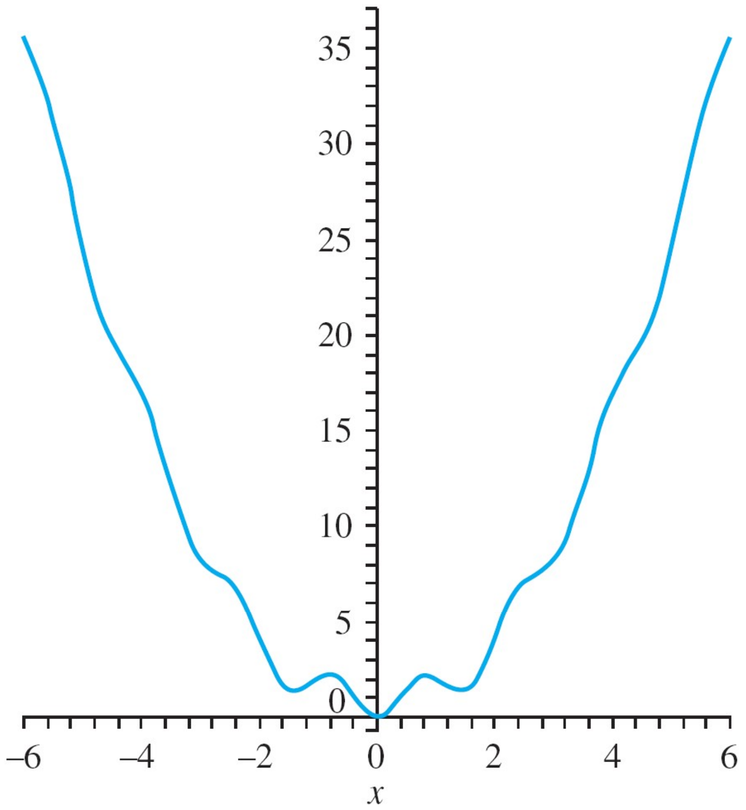


FIGURE 13.8 *A typical even function.*

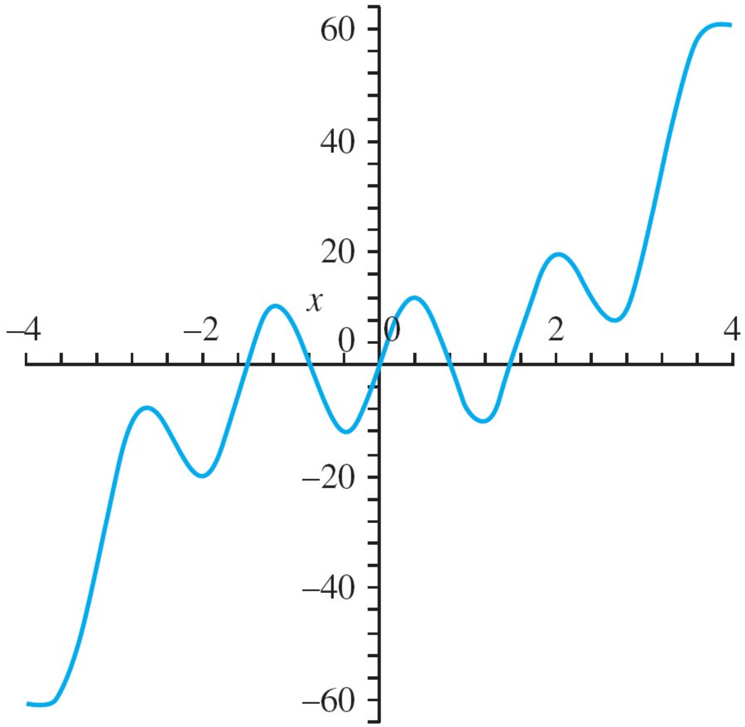


FIGURE 13.9 *A typical odd function.*

Even and Odd Functions

- If f is even on $[-L, L]$ then

$$\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx$$

and if f is odd on $[-L, L]$ then

$$\int_{-L}^{L} f(x)\,dx = 0.$$

Even and Odd Functions

- These facts are sometimes useful in computing Fourier coefficients. **If f is even, only the cosine terms and possibly the constant term will appear in the Fourier series**, because $f(x)\sin(n\pi x/L)$ is odd and the integrals defining the sine coefficients will be zero. If the function is odd then $f(x)\cos(n\pi x/L)$ is odd and the Fourier series will contain only the sine terms, since the integrals defining the constant term and the coefficients of the cosine terms will be zero.

EXAMPLE 5

- Compute the Fourier series of $f(x) = x$ on $[-\pi, \pi]$.

  Because ***x* cos(*nx*)** is an **odd** function on $[-\pi, \pi]$

  for $n = 0, 1, 2, \cdots$, each $a_n = 0$.

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\, dx = 0.$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(nx)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(nx)\, dx = 0.$$

EXAMPLE 5

- We need only compute the $b_n's$:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\ dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x\sin(nx)\ dx$$

$$= \left[\frac{2}{n^2\pi}\sin(nx) - \frac{2x}{n\pi}\cos\left(nx\right)\right]_0^{\pi}$$

$$= -\frac{2}{n}\cos\left(n\pi\right) = \frac{2}{\pi}\left(-1\right)^{n+1}.$$

EXAMPLE 5

- The Fourier series of x on $[-\pi,\pi]$ is

$$\sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1}\sin(nx).$$

- This converges to x for $-\pi < x < \pi$, and to 0 at $x = \pm\pi$. Figure 13.10 shows the twentieth partial sum of this Fourier series compared to the function.

EXAMPLE 5

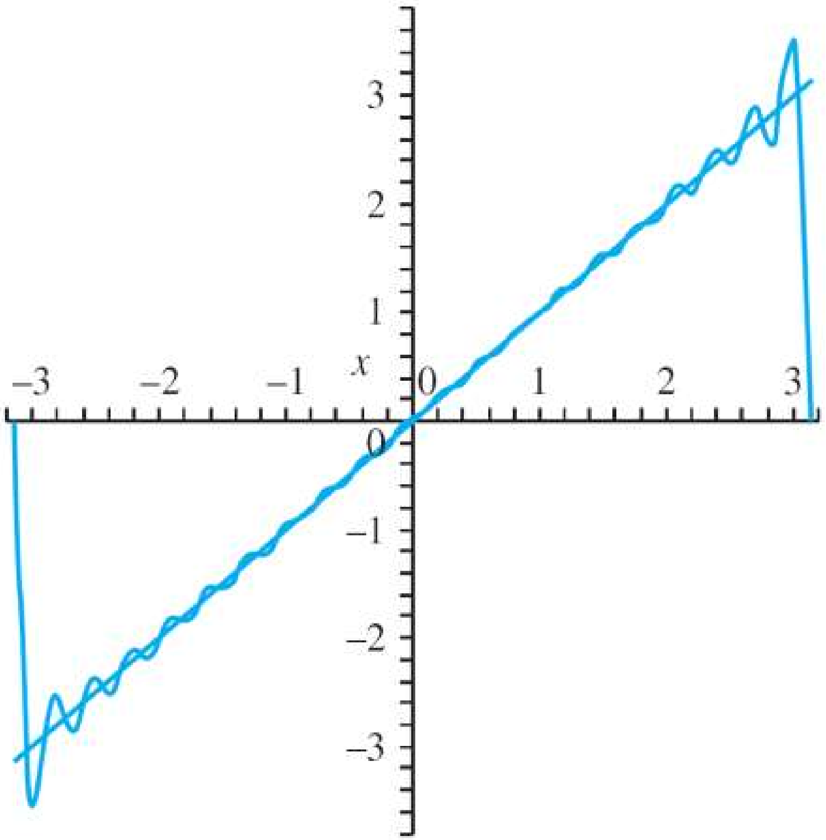


FIGURE 13.10 *Twentieth partial sum of the series in Example 13.7.*

EXAMPLE 13.3

- Write the Fourier series of x^2 on $[-3, 3]$.
- Since $x^2 \sin(n\pi x)$ is an odd function on $[-3, 3]$ for $n = 1, 2, \cdots$, each $b_n = 0$.

EXAMPLE 13.3

- Compute
  and

$$a_0 = \frac{2}{3}\int_0^3 x^2 dx = 6$$

$$a_n = \frac{2}{3}\int_0^3 x^2 \cos\left({n\pi x}/{3}\right)dx$$

$$= \frac{2}{3}\left[\frac{3(n\pi)^2\, x^2 \sin\left({n\pi x}/{3}\right) \;+\; 9(n\pi)x\cos\left({n\pi x}/{3}\right) - 27\sin\left({n\pi x}/{3}\right)}{(n\pi)^3}\right]_0^3$$

$$= \frac{2}{3}\left[\frac{27(n\pi)\cos(n\pi)}{(n\pi)^3}\right] = \frac{36(-1)^n}{(n\pi)^2}$$

EXAMPLE 13.3

- The Fourier series is

$$3+\sum_{n=1}^{\infty}\left(\frac{36(-1)^n}{n^2\pi^2}\right)\cos(n\pi x/3).$$

- This series converges to x^2 for $-3 < x < 3$. At the endpoints, the series converges to

$$\frac{1}{2}(f(-3+)+f(3-))=\frac{1}{2}(9+9)=9.$$

13.1.2 The Gibbs Phenomenon

- A.A. Michelson was a Prussian-born physicist who teamed with E.W. Morley of Case-Western Reserve University to show that the postulated “ether,” a fluid which supposedly permeated all of space, had no effect on the speed of light. Michelson also built a mechanical device for constructing a function from its Fourier coefficients. In one test, he used eighty coefficients for the series of $f(x) = x$ on $[-\pi, \pi]$ and noticed unexpected jumps in the graph near the endpoints.

13.1.2 The Gibbs Phenomenon

- At first, he thought this was a problem with his machine. It was subsequently found that this behavior is characteristic of the Fourier series of a function at a point of discontinuity. In the early twentieth century, the Yale mathematician Josiah Willard Gibbs finally explained this behavior.

13.1.2 The Gibbs Phenomenon

- To illustrate the Gibbs phenomenon, expand f in a Fourier series on $[-\pi,\pi]$, where

$$f(x) = \begin{cases} -\pi/4 & \text{for } -\pi \le x < 0 \\ 0 & \text{for } x = 0 \\ \pi/4 & \text{for } 0 < x \le \pi. \end{cases}$$

13.1.2 The Gibbs Phenomenon

- This function has a jump discontinuity at 0, but its Fourier series on $[-\pi,\pi]$ converges at 0 to

$$\frac{1}{2}(f(0+)+f(0-))=\frac{1}{2}\left(\frac{\pi}{4}-\frac{\pi}{4}\right)=0=f(0).$$

- The Fourier series therefore converges to the function on $(-\pi,\pi)$. This series is

$$\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin((2n-1)x).$$

13.1.2 The Gibbs Phenomenon

- Figure 13.12 shows the fifth and twenty-fifth partial sums of this series, compared to the function. Notice that both of these partial sums show a peak near 0, the point of discontinuity of f. Since the partial sums S_N approach the function **as $N \to \infty$, we might expect these peaks to flatten out, but they do not.** Instead they remain roughly the same height, but move closer to the y-axis as N increases. This is the Gibbs phenomenon.

13.1.2 The Gibbs Phenomenon

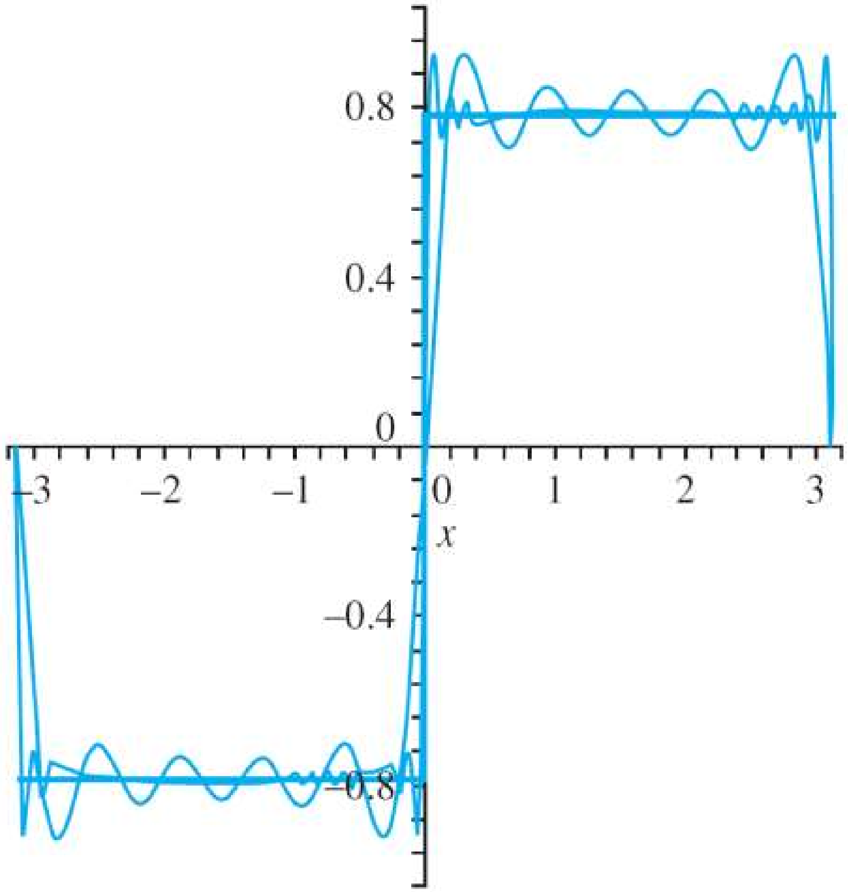


FIGURE 13.12 *The Gibbs phenomenon.*

Postscript

- We will add two comments on the ideas of this section, the first practical and the second offering a broader perspective.
  1. Writing and graphing partial sums of Fourier series are computation intensive activities. Evaluating integrals for the coefficients is most efficiently done in MAPLE using the int command, and partial sums are easily graphed using the sum command to enter the partial sum and then the plot command for the graph.

Postscript

2. Partial sums of Fourier series can be viewed from the perspective of orthogonal projections onto a subspace of a vector space (Sections 6.6 and 6.7). Let $PC[-L, L]$ be the vector space of functions that are piecewise continuous on $[-L, L]$ and let S be the subspace spanned by the functions

$$C_0(x) = 1,$$
$$C_n(x) = \cos(n\pi x / L) \text{ and}$$
$$S_n(x) = \sin(n\pi x / L) \text{ for } n = 1,\ 2, \cdots,\ N.$$

Postscript

- A dot product can be defined on $PC[-L, L]$ by

$$f \cdot g = \int_{-L}^{L} f(x)g(x)\,dx.$$

Postscript

- Using this dot product, these functions form an orthogonal basis for S. If f is piecewise continuous on $[-L, L]$, the orthogonal projection of f onto S is

$$f_S = \frac{f \cdot C_0}{C_0 \cdot C_0} C_0 + \sum_{n=1}^{N} \left(\frac{f \cdot C_n}{C_n \cdot C_n} C_n + \frac{f \cdot S_n}{S_n \cdot S_n} S_n \right).$$

Postscript

- Compare these coefficients in the orthogonal projection with the Fourier coefficients of f on $[-L, L]$. First

$$\frac{f \cdot C_0}{C_0 \cdot C_0} = \frac{\int_{-L}^{L} f(x)dx}{\int_{-L}^{L} dx}$$

$$= \frac{1}{2L}\int_{-L}^{L} f(x)dx = \frac{1}{2}a_0.$$

Postscript

- Next

$$\frac{f \cdot C_n}{C_n \cdot C_n} = \frac{\int_{-L}^{L} f(x)\cos(n\pi x / L)\,dx}{\int_{-L}^{L} \cos^2(n\pi x / L)\,dx}$$

$$= \frac{1}{L}\int_{-L}^{L} f(x)\,\cos(n\pi x / L)dx = a_n,$$

and similarly,

$$\frac{f \cdot S_n}{S_n \cdot S_n} = \frac{1}{L}\int_{-L}^{L} f(x)\sin(n\pi x / L)\,dx = b_n.$$

Postscript

- Thus, the orthogonal projection f_S of f onto S is exactly the Nth partial sum of the Fourier series of f on $[-L, L]$.

13.2 Sine and Cosine Series

- If f is piecewise continuous on $[-L, L]$, we can represent $f(x)$ at all but possibly finitely many points of $[-L, L]$ by its Fourier series. This series may contain just sine terms, just cosine terms, or both sine and cosine terms. We have no control over this.

13.2 Sine and Cosine Series

- If f is defined on the half interval $[0, L]$, we can write a Fourier cosine series (containing just cosine terms) and a Fourier sine series (containing just sine terms) for f on $[0, L]$.

Also called Half Range Expansion

13.2 Sine Series

- We can also write an expansion of f on $[0, L]$ that contains only sine terms. Now reflect the graph of f on $[0, L]$ through the origin to create an odd function h on $[-L, L]$, with $h(x) = f(x)$ for $0 < x \leq L$ (Figure 13.15).

13.2 Sine Series

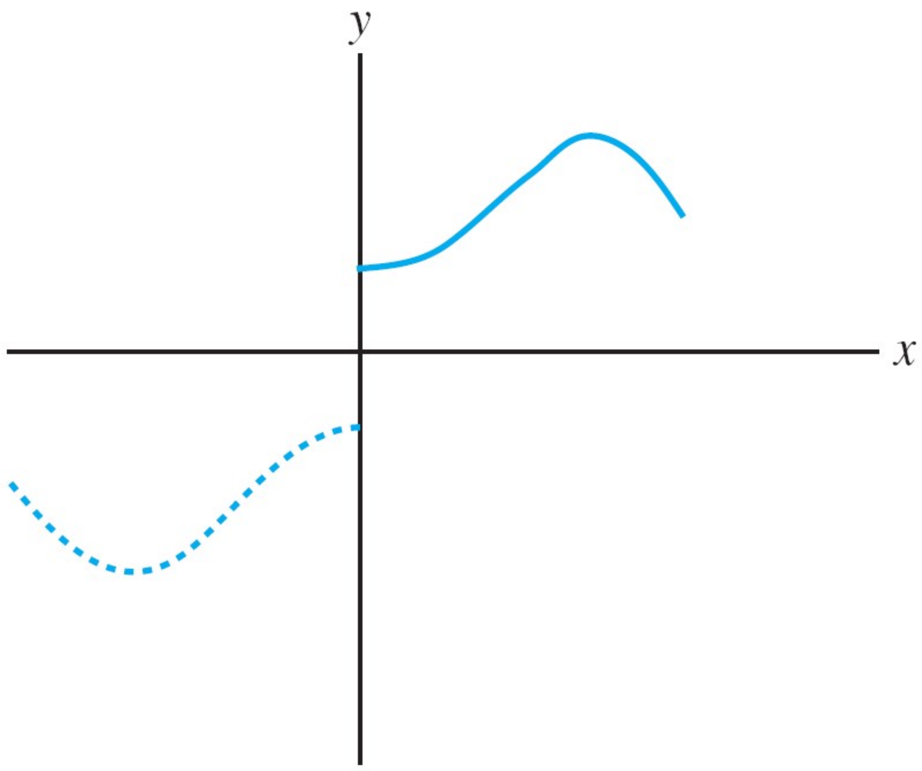


FIGURE 13.15 *Odd extension of a function defined on* $[0, L]$.

13.2 Sine Series

- The Fourier expansion of h on $[-L, L]$ has only sine terms because h is odd on $[-L, L]$. But $h(x) = f(x)$ on $[0, L]$, so this gives a sine expansion of f on $[0, L]$. This suggests the following definitions.

13.2 Sine Series

The *Fourier sine coefficients of f on* [0, L] are

$$B_n = \frac{2}{L}\int_0^L f(x)\ \sin(n\pi x / L)\ dx. \qquad (13.11)$$

for $n = 1, 2, \cdots$. With these coefficients, the series

$$\sum_{n=1}^{\infty} B_n \sin(n\pi x / L) \qquad (13.10)$$

is the *Fourier sine series for f on* [0, L].

13.2 Sine Series

- Again, we have a convergence theorem for sine series directly from the convergence theorem for Fourier series.

THEOREM 13.2
Convergence of Fourier Sine Series

- Let f be piecewise smooth on $[0, L]$. Then
  1. If $0 < x < L$, the Fourier sine series for f on $(0, L)$ converges to
  $$\frac{1}{2}(f(x+) + f(x-)).$$
  2. At $x = 0$ and $x = L$, this sine series converges to 0.

13.2 Sine Series

- Condition (2) is obvious because each sine term in the series vanishes at $x = 0$ and at $x = L$, regardless of the values of the function there.

EXAMPLE 13.4

- Let $f(x) = e^{2x}$ for $0 \le x \le 1$.We will write the Fourier sine series of f on [0, 1]. The coefficients are

$$B_n = b_n = 2\int_0^1 e^{2x} \sin(n\pi x)dx$$

$$= \left[\frac{-2n\pi e^{2x} \cos(n\pi x) \; + \; 4e^{2x} \sin(n\pi x)}{4 + n^2\pi^2} \right]_0^1$$

$$= 2\frac{n\pi(1 - (-1)^n e^2)}{4 + n^2\pi^2}.$$

EXAMPLE 13.4

- The sine expansion of e^{2x} on [0, 1] is

$$\sum_{n=1}^{\infty} 2\frac{n\pi(1-(-1)^n e^2)}{4+n^2\pi^2}\sin(n\pi x).$$

- This series converges to e^{2x} for $0< x <1$ and to 0 at $x = 0$ and $x = 1$. Figure 17.11 shows the function and the fortieth partial sum of this sine expansion.

EXAMPLE 13.4

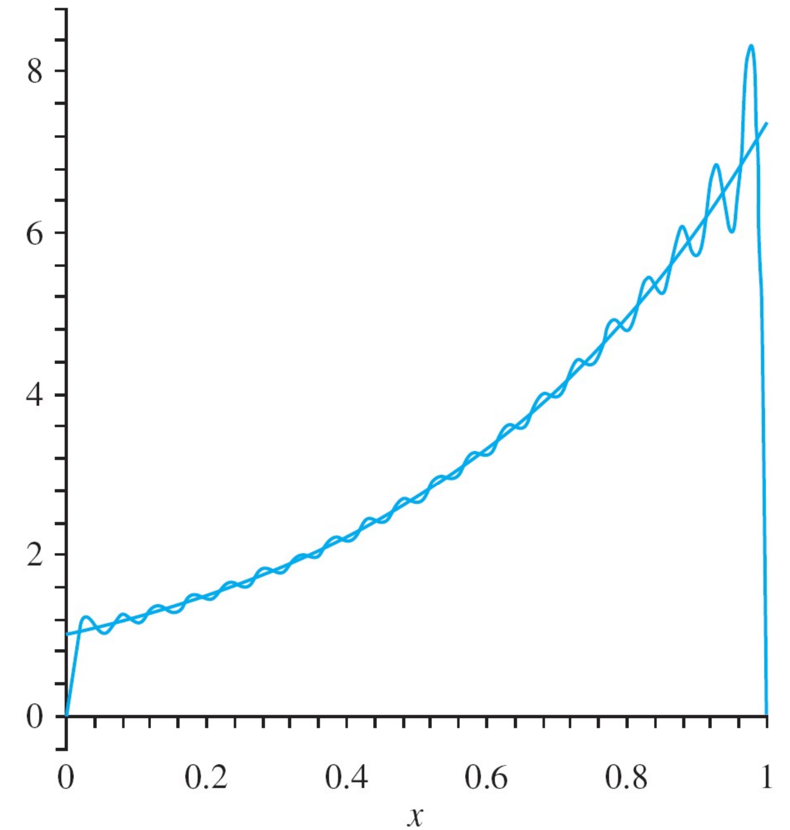
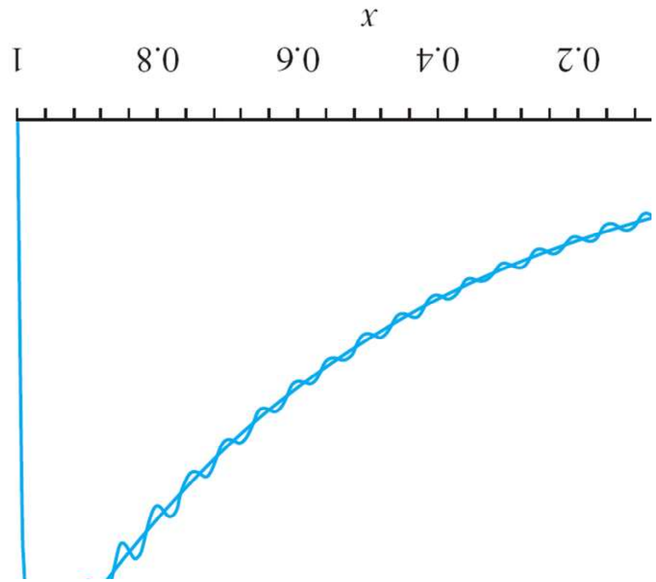


FIGURE 13.16 *Fortieth partial sum of the sine series of Example 13.10.*



13.2 Cosine Series

- Suppose $f(x)$ is defined for $0 \leq x \leq L$. To get a pure cosine series on this interval, imagine reflecting the graph of f across the vertical axis to obtain an even function g defined on $[-L, L]$.

13.2 Cosine Series

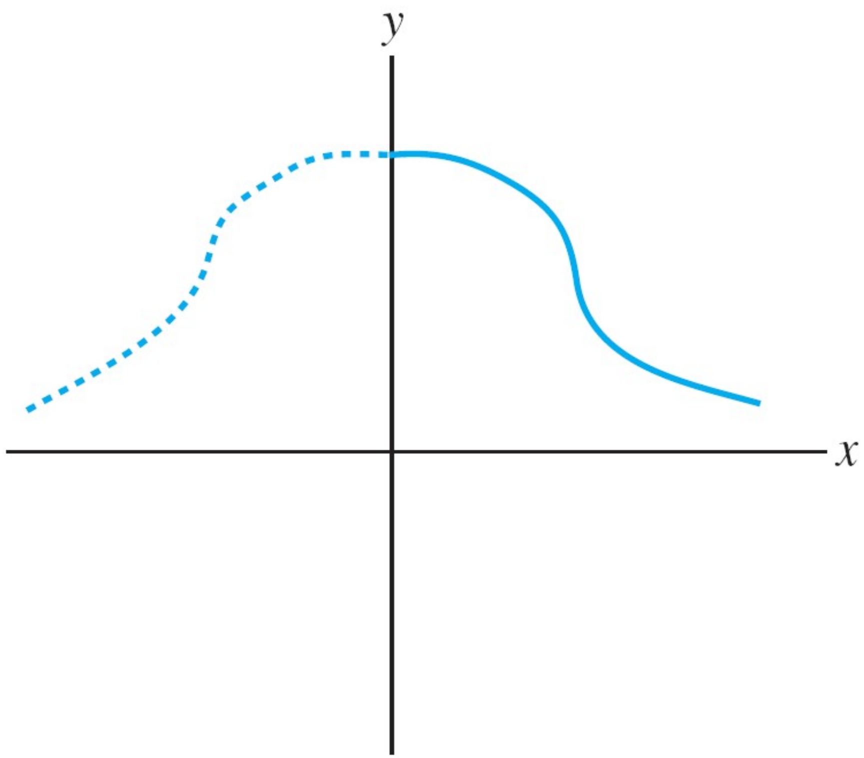


FIGURE 13.13 *Even extension of a function defined on* $[0, L]$.

13.2 Cosine Series

- Because g is even, its Fourier series on $[-L, L]$ has only cosine terms and perhaps the constant term. But $g(x) = f(x)$ for $0 \leq x \leq L$, so this gives a cosine series for f on $[0, L]$.

13.2 Cosine Series

- Furthermore, because g is even, the coefficients in the Fourier series of g on $[-L, L]$ are

$$a_n = \frac{1}{L}\int_{-L}^{L} g(x)\cos(n\pi x/L)\,dx$$
$$= \frac{2}{L}\int_{0}^{L} g(x)\cos(n\pi x/L)\,dx$$
$$= \frac{2}{L}\int_{0}^{L} f(x)\cos(n\pi x/L)\,dx.$$

for $n = 0, 1, 2, \cdots$.

13.2 Cosine Series

- Notice that we can compute a_n strictly in terms of f on $[0, L]$. The construction of g showed us how to obtain this cosine series for f, but we do not need g to compute the coefficients of this series.

13.2 Cosine Series

Based on these ideas, define the *Fourier cosine coefficients of f on* $[0, L]$ to be the numbers

$$A_n = \frac{2}{L}\int_0^L f(x)\cos(n\pi x / L)\,dx \qquad (13.13)$$

for $n = 0, 1, 2, \cdots$.

13.2 Cosine Series

Fourier cosine expansion

The *Fourier cosine series for f on* [0, *L*] is the series

$$\frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x / L) \qquad (13.12)$$

in which the $A_n's$ are the Fourier cosine coefficients of f on [0, L].

13.2 Cosine Series

- We obtain the following convergence theorem for cosine series on $[0, L]$.

THEOREM 13.3
Convergence of Fourier Cosine Series

- Let f be piecewise smooth on $[0, L]$. Then
  1. If $0 < x < L$, the Fourier cosine series for f on $(0, L)$ converges to

$$\frac{1}{2}(f(x+) + f(x-)).$$

  2. At 0 this cosine series converges to $f(0+)$.
  3. At L this cosine series converges to $f(L-)$.

EXAMPLE 13.5

- Let $f(x) = e^{2x}$ for $0 \leq x \leq 1$. We will write the cosine expansion of f on $[0, 1]$.

EXAMPLE 13.5

- The coefficients are

$$A_0 = a_0 = 2\int_0^1 e^{2x}dx = e^2 - 1$$

and for n =1, 2, $\cdots$,

$$A_n = a_n = 2\int_0^1 e^{2x}\cos(n\pi x)dx$$

$$= \left[\frac{4e^{2x}\cos(n\pi x) + 2n\pi e^{2x}\sin(n\pi x)}{4+n^2\pi^2}\right]_0^1$$

$$= 4\frac{e^2(-1)^n - 1}{4+n^2\pi^2}.$$

EXAMPLE 13.5

- The cosine series for e^{2x} on $[0, 1]$ is

$$\frac{1}{2}(e^2 - 1) + \sum_{n=1}^{\infty} 4\frac{e^2(-1)^n - 1}{4 + n^2\pi^2} \cos(n\pi x).$$

EXAMPLE 13.5

- This series converges to

$$\begin{cases} e^{2x} & \text{for } 0 < x < 1 \\ 1 & \text{for } x = 0 \\ e^2 & \text{for } x = 1. \end{cases}$$

- This Fourier cosine series converges to e^{2x} for $0 \le x \le 1$. Figure 13.12 shows e^{2x} and the fifth partial sum of this cosine series.

EXAMPLE 13.5

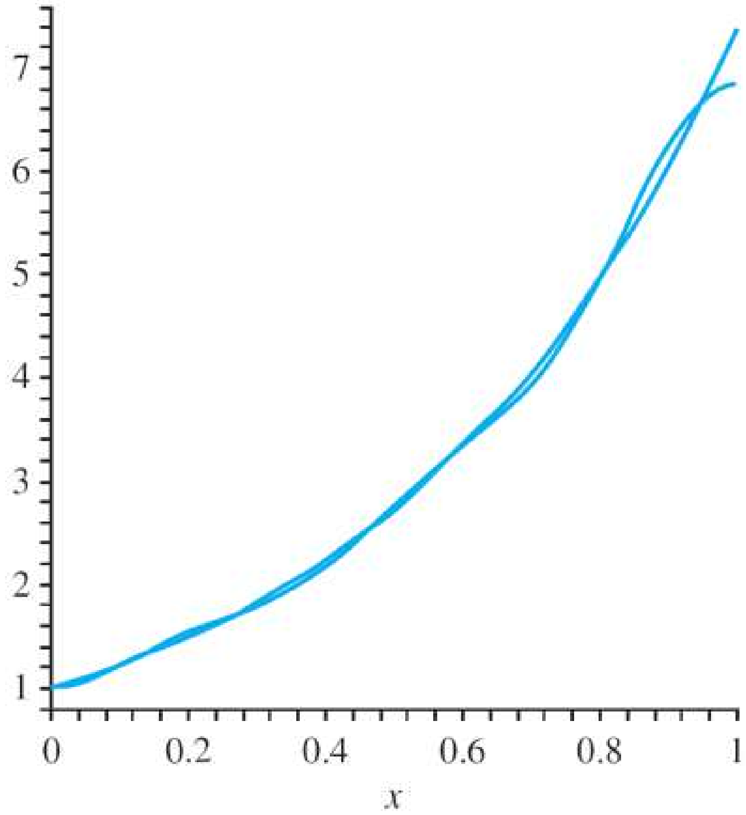


FIGURE 13.14 *Fifth partial sum of the cosine series in Example 13.9.*

13.3 Integration and Differentiation of Fourier Series

- Term by term differentiation of a Fourier series may lead to nonsense.

EXAMPLE 13.6

- Let $f(x) = x$ for $-\pi \le x \le \pi$. The Fourier series is

$$f\left(x\right) = x = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1}\sin(nx)$$

for $-\pi < x < \pi$. Differentiate this series term by term to get

$$\sum_{n=1}^{\infty} 2(-1)^{n+1}\cos(nx).$$

- This series does not converge to $f' = 1$ on $(-\pi, \pi)$.

13.3 Integration and Differentiation of Fourier Series

- However, under fairly mild conditions, we can integrate a Fourier series term by term.

THEOREM 13.4
Integration of Fourier Series

- Let f be piecewise continuous on $[-L, L]$, with Fourier series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos(n\pi x / L) + b_n \sin(n\pi x / L)).$$

THEOREM 13.4
Integration of Fourier Series

- Then, for any x with $-L \le x \le L$,

$$\int_{-L}^{x} f(t)dt = \frac{1}{2}a_0(x+L)$$

$$+\frac{L}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[a_n \sin(n\pi x/L) - b_n(\cos(n\pi x/L) - (-1)^n)\right].$$

The result of this integration
is not a Fourier series

13.3 Integration and Differentiation of Fourier Series

- The expression on the right in this equation is exactly what we obtain by integrating the Fourier series term by term, from $-L$ to x. This means that we can **integrate any piecewise continuous function f from $-L$ to x by integrating its Fourier series term by term**. This is true even if the function has jump discontinuities and its Fourier series does not converge to $f(x)$ for all x in $[-L, L]$. Notice, however, that the result of this integration is not a Fourier series.

EXAMPLE 13.7

- From Example 13.16,

$$f\left(x\right) = x = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin(nx)$$

on $(-\pi, \pi)$. Term by term differentiation results in a series that does not converge on the interval.

EXAMPLE 13.7

- However, we can integrate this series term by term:

$$\int_{-\pi}^{x} t\ dt = \frac{1}{2}(x^2 - \pi^2)$$

$$= \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \int_{-\pi}^{x} \sin(nt)\ dt$$

$$= \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \left[-\frac{1}{n}\cos(nx) + \frac{1}{n}\cos(n\pi) \right]$$

$$= \sum_{n=1}^{\infty} \frac{2}{n^2}(-1)^{n}(\cos(nx) - (-1)^{n}).$$

13.3 Integration and Differentiation of Fourier Series

- Valid term by term differentiation of a Fourier series requires **stronger conditions**.

THEOREM 13.5
Differentiation of Fourier Series

- Let f be continuous on $[-L, L]$ and suppose that $f(-L) = f(L)$. Let f' be **continuous** on $[-L, L]$. Then the Fourier series of f on $[-L, L]$ converges to $f(x)$ on $[-L, L]$:

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos(n\pi x / L) + b_n \sin(n\pi x / L))$$

for $-L \leq x \leq L$.

THEOREM 13.5
Differentiation of Fourier Series

- Further, at each x in $(-L, L)$ at which $f'(x)$ exists, the term by term derivative of the Fourier series for $f(x)$ converges to the derivative of the function $f'(x)$:

$$f'(x) = \sum_{n=1}^{\infty} \frac{n\pi}{L}(-a_n \sin(n\pi x / L) + b_n \cos(n\pi x / L)).$$

13.3 Integration and Differentiation of Fourier Series

- The idea of a proof of this theorem is to begin with the Fourier series for $f'(x)$, noting that this series converges to $f'(x)$ at each point where f'' exists. Use integration by parts to relate the Fourier coefficients of $f'(x)$ to those for $f(x)$, similar to the strategy used in proving Theorem 13.4,

EXAMPLE 13.8

- Let $f(x) = x^2$ for $-2 \le x \le 2$. By the Fourier convergence theorem,

$$x^2 = \frac{4}{3} + \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n\pi x / 2)$$

for $-2 < x < 2$.

EXAMPLE 13.8

- Only cosine terms appear in this series because x^2 is an even function. Now, $f'(x) = 2x$ is continuous and f is twice differentiable for all x. Therefore, for $-2 < x < 2$,

$$f'(x) = 2x = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi x/2).$$

- This can be verified by expanding $2x$ in a Fourier series on $[-2, 2]$.

THEOREM 13.6

- Let f be **continuous** on $[-L, L]$, and let f' be **piecewise continuous**. Suppose $f(L) = f(-L)$. Then the Fourier series for $f(x)$ on $[-L, L]$ converges **absolutely** and **uniformly** to $f(x)$ on $[-L, L]$.
- A proof is outlined in Problem 5.

Absolute Convergence

- A series to be absolutely convergent if

$$\sum_{n=0}^{\infty} \|a_n\| < \infty$$

- An integral to be absolutely convergent if

$$\int_I |f(x)| dx < \infty$$

Uniform Convergence

- A series $(f_n)_{n \in N}$ to be uniformly convergent if for every ε <0, there exists a natural number N such that for all $n \geq N$ and $x \in E$

$$|f_n(x) - f(x)| < \varepsilon$$

- For integral, if for every ε <0, there exists a natural number N such that for all $n \geq N$ and $x \in E$

$$\left| \int_a^b f_n(x)dx - \int_a^b f(x)dx \right| < \varepsilon$$

13.4 Properties of Fourier Coefficients

- Fourier coefficients, and Fourier sine and cosine coefficients, satisfy an important set of inequalities called *Bessel's inequalities*.

THEOREM 13.7
Bessel's Inequalities

- Suppose $\int_0^L g(x)\,dx$ exists.
  1. The Fourier sine coefficients B_n of $g(x)$ on $[0, L]$ satisfy

$$\sum_{n=1}^{\infty} B_n^2 \leq \frac{2}{L}\int_0^L (g(x))^2\,dx.$$

THEOREM 13.7
Bessel's Inequalities

2. The Fourier cosine coefficients A_n of $g(x)$ on $[0, L]$ satisfy

$$\frac{1}{2}A_0^2 + \sum_{n=1}^{\infty} A_n^2 \le \frac{2}{L}\int_0^L (g(x))^2\, dx.$$

3. If $\int_{-L}^{L} f(x)\, dx$ exists, then the Fourier coefficients of $g(x)$ on $[-L, L]$ satisfy

$$\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) \le \frac{1}{L}\int_{-L}^{L} (g(x))^2\, dx.$$

- In particular, the sum of the squares of the coefficients in a Fourier series (or cosine or sine series) converges.
- We will prove conclusion (1). The argument is notationally simpler than that for conclusions (2) and (3), but contains the ideas involved.

THEOREM 13.7
Bessel's Inequalities – *Proof of (1)*

- The Fourier sine series of $g(x)$ on $[0, L]$ is

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right),$$

with

$$B_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

THEOREM 13.7
Bessel's Inequalities－*Proof of (1)*

- The Nth partial sum of this sine series is

$$S_N(x) = \sum_{n=1}^{N} B_n \sin\left(\frac{n\pi x}{L}\right).$$

THEOREM 13.7
Bessel's Inequalities — *Proof of (1)*

- Then

$$0 \le \int_0^L (g(x) - S_N(x))^2 \, dx$$

$$= \int_0^L \left(g(x)\right)^2 dx - 2\int_0^L g(x) S_N(x) dx + \int_0^L (S_N(x))^2 \, dx$$

$$= \int_0^L \left(g(x)\right)^2 dx - 2\int_0^L g(x) \left(\sum_{n=1}^{N} B_n \sin\left(\frac{n\pi x}{L}\right)\right) dx$$

$$+ \int_0^L \left(\sum_{n=1}^{N} B_n \sin\left(\frac{n\pi x}{L}\right)\right)\left(\sum_{k=1}^{N} B_k \sin\left(\frac{k\pi x}{L}\right)\right) dx$$

THEOREM 13.7
Bessel's Inequalities — *Proof of (1)*

$$= \int_0^L (g(x))^2\,dx - 2\sum_{n=1}^{N} B_n \int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$+\sum_{n=1}^{N}\sum_{k=1}^{N} B_n B_k \int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{k\pi x}{L}\right)dx$$

$$= \int_0^L (g(x))^2\,dx - \sum_{n=1}^{N} B_n\left(LB_n\right) + \frac{L}{2}\sum_{n=1}^{N} B_n B_n.$$

THEOREM 13.7
Bessel's Inequalities — *Proof of (1)*

- Here we have used the fact that

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{k\pi x}{L}\right)dx = \begin{cases} 0 & \text{for } n \neq k, \\ L/2 & \text{for } n = k. \end{cases}$$

THEOREM 13.7
Bessel's Inequalities — *Proof of (1)*

- We therefore have

$$0 \le \int_0^L (g(x))^2\,dx - L\sum_{n=1}^{N} B_n^2 + \frac{L}{2}\sum_{n=1}^{N} B_n^2,$$

and this gives us

$$\sum_{n=1}^{N} B_n^2 \le \frac{2}{L}\int_0^L (g(x))^2\,dx.$$

THEOREM 13.7
Bessel's Inequalities – *Proof of (1)*

- Since this is true for all positive integers N, we can let $N \to \infty$ to obtain

$$\sum_{n=1}^{\infty} B_n^2 \leq \frac{2}{L} \int_0^L (\, g(x))^2 \, dx.$$

EXAMPLE 13.9

- We will use Bessel's inequality to derive an upper bound for the sum of a series. Let $f(x) = x^2$ on $[-\pi,\pi]$. The Fourier series is

$$f(x) = \frac{1}{3}\pi^2 + \sum_{n=1}^{\infty} 4\frac{(-1)^n}{n^2}\cos(nx)$$

for $-\pi \leq x \leq \pi$.

EXAMPLE 13.9

- Here $a_0 = 2\pi^2/3$ and $a_n = 4(-1)^n/n^2$, while each $b_n = 0$ (x^2 is an even function). By Bessel's inequality,

$$\frac{1}{2}\left(\frac{2\pi}{3}\right)^2 + \sum_{n=1}^{\infty}\left(\frac{4(-1)^n}{n^2}\right)^2 \leq \frac{1}{\pi}\int_{-\pi}^{\pi} x^4 dx = \frac{2}{5}\pi^4.$$

- Then

$$16\sum_{n=1}^{\infty}\frac{1}{n^4} \leq \left(\frac{2}{5} - \frac{2}{9}\right)\pi^4 = \frac{8\pi^4}{45}.$$

EXAMPLE 13.9

- Then

$$\sum_{n=1}^{\infty} \frac{1}{n^4} \leq \frac{\pi^4}{90},$$

which is approximately 1.0823. Infinite series are generally difficult to sum, so it is sometimes useful to be able to derive an upper bound.

- With stronger assumptions than just existence of the integral over the interval, we can derive an important equality satisfied by the Fourier coefficients of a function on $[-L, L]$ or by the Fourier sine or cosine coefficients of a function on $[0, L]$. We will state the result for $f(x)$ defined on $[-L, L]$.

THEOREM 13.8
Parseval's Theorem

- Let f be continuous on $[-L, L]$ and let f' be piecewise continuous. Suppose that $f(-L) = f(L)$. Then the Fourier coefficients of f on $[-L, L]$ satisfy

$$\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{L}\int_{-L}^{L} f(x)^2\, dx.$$

the total energy of a signal can be calculated by summing power-per-sample across time or spectral power across frequency.

THEOREM 13.8
Parseval's Theorem－*Proof*

- Begin with the fact that, from the Fourier convergence theorem,

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos(n\pi x / L) + b_n \sin(n\pi x / L)).$$

THEOREM 13.8
Parseval's Theorem－*Proof*

- Multiply this series by $f(x)$ to get

$$f(x)^2 = \frac{1}{2}a_0 f(x) + \sum_{n=1}^{\infty}(a_n f(x)\cos(n\pi x / L) + b_n f(x)\sin(n\pi x / L)).$$

- We can integrate this equation term by term (Theorem 17.4). In doing this, observe that the integrals in the series on the right are Fourier coefficients. This yields Parseval's theorem.

EXAMPLE 13.10

- Show $$\sum_{n=1}^{\infty} \frac{1}{\left(4n^2 - 1\right)^2} = \frac{\pi^2 - 8}{16}.$$

- The Fourier coefficients of $\cos(x/2)$ on $[-\pi,\pi]$ are

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(x/2)dx = \frac{4}{\pi}$$

and, for $n = 1, 2, \cdots$,,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(x/2)\cos(nx)\, dx = -\frac{4}{\pi}\frac{(-1)^n}{4n^2 - 1}.$$

EXAMPLE 13.10

- Each $b_n = 0$ because $\cos(x/2)$ is an even function. By Parseval's theorem,

$$\frac{1}{2}\left(\frac{4}{\pi}\right)^2 + \sum_{n=1}^{\infty}\left(\frac{4}{\pi}\frac{(-1)^n}{4n^2-1}\right)^2 = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos^2(x/2)\,dx = 1.$$

EXAMPLE 13.10

- After some routine manipulation, this yields

$$\sum_{n=1}^{\infty} \frac{1}{\left(4n^2 - 1\right)^2} = \frac{\pi^2 - 8}{16}.$$

- We conclude this section with sufficient conditions for a Fourier series to converge uniformly.

13.5 Phase Angle Form

A function f has period p if $f(x + p) = f(x)$ for all x. The smallest positive p for which this holds is the *fundamental period* of f. For example $\sin(x)$ has fundamental period 2π.

13.5 Phase Angle Form

- The graph of a function with fundamental period p simply repeats itself over intervals of length p. We can draw the graph for $-p/2 \leq x < p/2$, then replicate this graph on $p/2 \leq x < 3p/2$, $3p/2 \leq x < 5p/2$, $-3p/2 \leq x < -p/2$, and so on.

13.5 Phase Angle Form

- Now suppose f has fundamental period p. Its Fourier series on $[-p/2, p/2]$, with $L = p/2$, is

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos(2n\pi x / p) + b_n \sin(2n\pi x / p)),$$

where

$$a_n = \frac{2}{p}\int_{-p/2}^{p/2} f(x)\cos(2n\pi x / p)\, dx$$

and

$$b_n = \frac{2}{p}\int_{-p/2}^{p/2} f(x)\sin(2n\pi x / p)\, dx.$$

13.5 Phase Angle Form

- It is sometimes convenient to write this series in a different way. Let

$$\omega_0 = \frac{2\pi}{p}.$$

13.5 Phase Angle Form

- The Fourier series of $f(x)$ on $[-p/2, p/2]$ is

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos(n\omega_0 x) + b_n \sin(n\omega_0 x)),$$

where

$$a_n = \frac{2}{p}\int_{-p/2}^{p/2} f(x)\cos(n\omega_0 x)\, dx$$

and

$$b_n = \frac{2}{p}\int_{-p/2}^{p/2} f(x)\ \sin(n\omega_0 x)\, dx.$$

13.5 Phase Angle Form

- Now look for numbers c_n and δ_n so that

$$a_n \cos(n\omega_0 x) + b_n \sin(n\omega_0 x) = c_n \cos(n\omega_0 x + \delta_n).$$

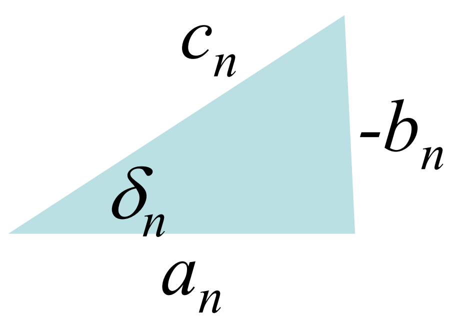
- To solve for these constants, use a trigonometric identity to write this equation as

$$a_n \cos(n\omega_0 x) + b_n \sin(n\omega_0 x)$$

$$= c_n \cos(n\omega_0 x) \cos(\delta_n) - c_n \sin(n\omega_0 x) \sin(\delta_n).$$

- One way to satisfy this equation is to put

$$c_n \cos(\delta_n) = a_n \text{ and } c_n \sin(\delta_n) = -b_n.$$



13.5 Phase Angle Form

- If we square both sides of these equations and add the results, we obtain

$$c_n^2 = a_n^2 + b_n^2,$$

so

$$c_n = \sqrt{a_n^2 + b_n^2}.$$

13.5 Phase Angle Form

- Next, divide to obtain

$$\frac{c_n \sin(\delta_n)}{c_n \cos(\delta_n)} = \tan(\delta_n) = -\frac{b_n}{a_n},$$

assuming that $a_n \neq 0$. Then

$$\delta_n = -\arctan\left(\frac{b_n}{a_n}\right).$$

13.5 Phase Angle Form

When each $a_n \neq 0$, these equations enable us to write the *phase angle* form of the Fourier series of $f(x)$ on $[-p/2, p/2]$:

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 x + \delta_n),$$

where

$$\omega_0 = 2\pi / p,\ c_n = \sqrt{a_n^2 + b_n^2},\ \text{and}\ \delta_n = -\arctan(b_n / a_n).$$

13.5 Phase Angle Form

This phase angle form is also called the *harmonic form* of the Fourier series for $f(x)$ on $[-p/2, p/2]$. The term $\cos(n\omega_0 x + \delta_n)$ is called the nth *harmonic* of f, c_n is the nth *harmonic amplitude*, and δ_n is the nth *phase angle* of f.

13.5 Phase Angle Form

- If f has fundamental period p, then in the expressions for the coefficients a_n and b_n , we can compute the integrals over any interval $[\alpha, \alpha + p]$, since any interval of length p carries all of the information about a p-periodic function.

13.5 Phase Angle Form

- This means that the Fourier coefficients of p-periodic f can be obtained as

$$a_n = \frac{2}{p}\int_{\alpha}^{\alpha+p} f(x)\cos(n\omega_0 x)\, dx$$

and

$$b_n = \frac{2}{p}\int_{\alpha}^{\alpha+p} f(x)\sin(n\omega_0 x)\, dx$$

for any number α.

EXAMPLE 13.11

- Let

$$f(x) = x^2 \text{ for } 0 \leq x < 3$$

and suppose f has fundamental period $p = 3$. A graph of f is shown in Figure 17.13.

EXAMPLE 13.11

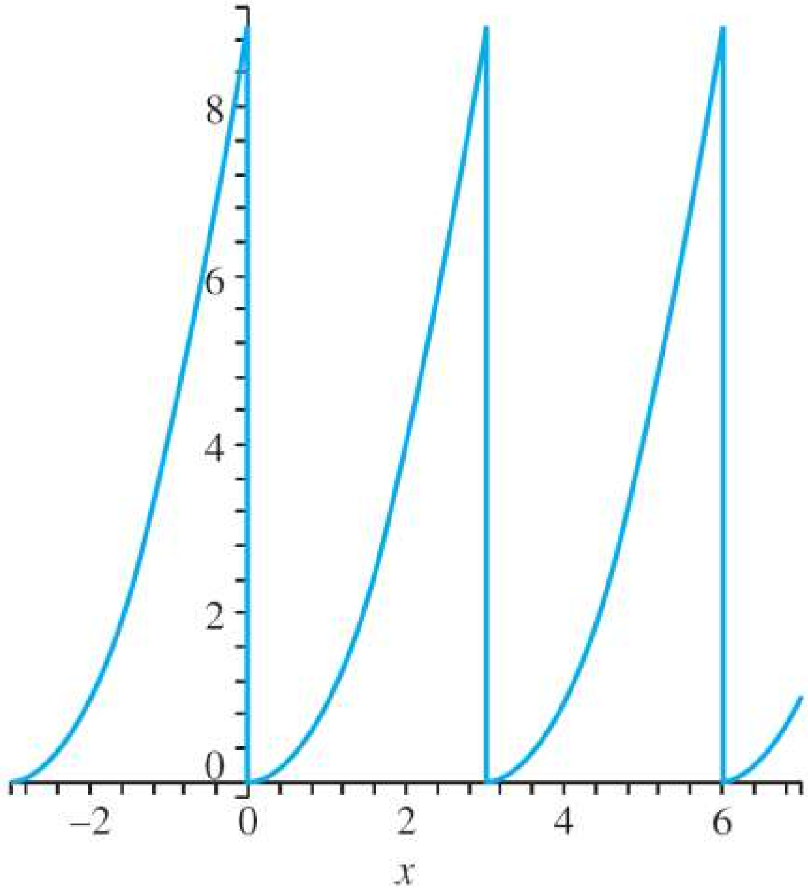


FIGURE 17.13 $f(x)$ *in Example* **17.11**

EXAMPLE 13.11

- Since f is 3-periodic, and we are given an algebraic expression for $f(x)$ only on $[0, 3)$, we will use this interval to compute the Fourier coefficients of f. That is, use $p = 3$ and $\alpha = 0$ in the preceding discussion. We also have $\omega_o = 2\pi/p = 2\pi/3$.

EXAMPLE 13.11

- The Fourier coefficients are

$$a_0 = \frac{2}{3}\int_0^3 x^2 dx = 6,$$

$$a_n = \frac{2}{3}\int_0^3 x^2 \cos\left(\frac{2n\pi x}{3}\right) dx = \frac{9}{n^2\pi^2},$$

and

$$b_n = \frac{2}{3}\int_0^3 x^2 \sin\left(\frac{2n\pi x}{3}\right) dx = -\frac{9}{n\pi}.$$

EXAMPLE 13.11

- The Fourier series of $f(x)$ is

$$3+\sum_{n=1}^{\infty}\frac{9}{n\pi}\left(\frac{1}{n\pi}\cos\left(\frac{2n\pi x}{3}\right)-\sin\left(\frac{2n\pi x}{3}\right)\right).$$

EXAMPLE 13.11

- We may think of this as the Fourier series of $f(x)$ on the symmetric interval $[-3/2, 3/2]$. However, keep in mind that $f(x)$ is not x^2 on this interval. We have $f(x) = x^2$ on $0 \le x < 3$, hence also on $[0, 3/2]$. But from Figure 17.13, $f(x) = (x+3)^2$ on $[-3/2, 0)$.

EXAMPLE 13.11

- This Fourier series converges to

$$\begin{cases} 9/4 & \text{for } x = \pm 3/2, \\ 9/2 & \text{for } x = 0, \\ (x+3)^2 & \text{for } -3/2 < x < 0, \\ x^2 & \text{for } 0 < x < 3/2. \end{cases}$$

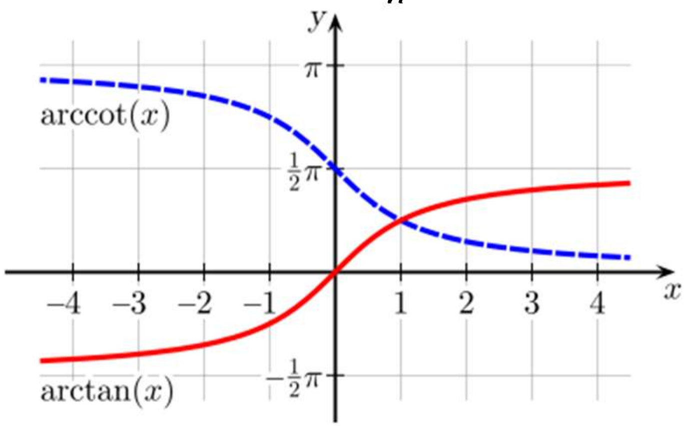
EXAMPLE 13.11

- For the phase angle form, compute

$$c_n = \sqrt{a_n^2 + b_n^2} = \frac{9}{n^2\pi^2}\sqrt{1+n^2\pi^2}$$

and

$$\delta_n = \arctan\left(-\frac{-9/n\pi}{9/n^2\pi^2}\right) = \arctan(n\pi) = 0.$$



arctan(nπ) ≠ 0

EXAMPLE 13.11

- The phase angle form of the Fourier series of $f(x)$ is

$$3+\sum_{n=1}^{\infty}\frac{9}{n^2\pi^2}\sqrt{1+n^2\pi^2}\cos\left(\frac{2n\pi x}{3}+\arctan(n\pi)\right).$$

EXAMPLE 13.11

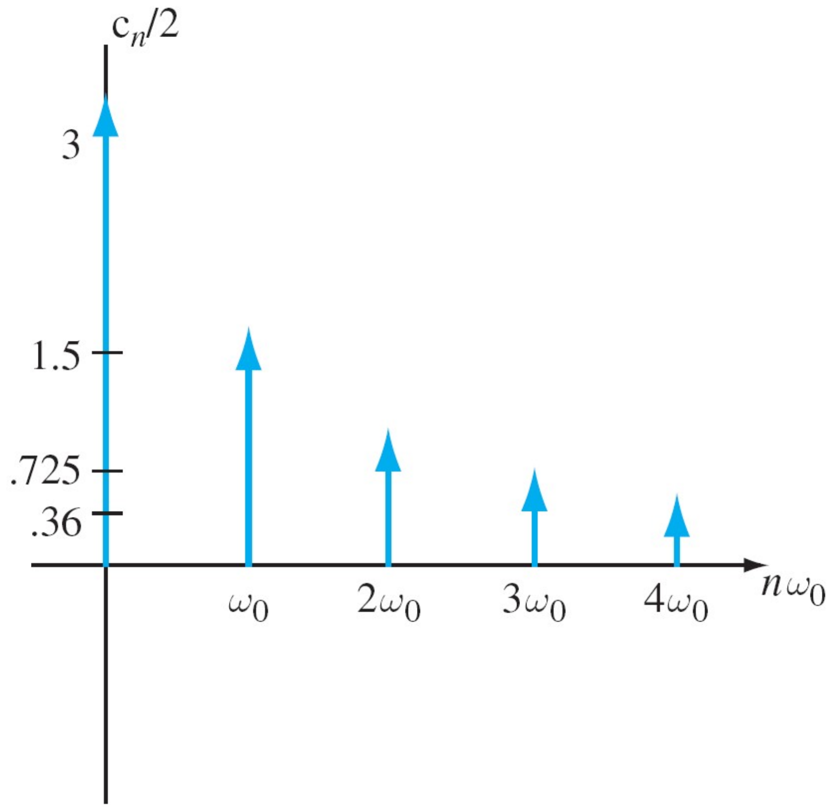


FIGURE 17.13 *Amplitude spectrum of f in Example* **17.11**

13.5 Phase Angle Form

The *amplitude spectrum* of a periodic function f is a plot of points $(n\omega_0, c_n/2)$ for $n = 1, 2, \cdots$, and also the point $(0, |c_0|/2)$. For the function of Example 17.11, this is a plot of points (0, 3) and, for nonzero integer n, points

$$\left(\frac{2n\pi}{3}, \frac{9}{2n^2\pi^2}\sqrt{1+n^2\pi^2}.\right)$$

13.5 Phase Angle Form

- The amplitude spectrum for the function of Example 13.11 is shown in Figure 13.14, with the intervals on the horizontal axis of length $\omega_0 = 2\pi/3$. This graph displays the relative effects of the harmonics in the function. This is useful in signal analysis.

13.6 Complex Fourier Series

- There is a complex form of Fourier series that is sometimes used. As preparation for this, recall that, in polar coordinates, a complex number (point in the plane) can be written

$$z = x + iy = r\cos(\theta) + ir\sin(\theta)$$

and

$$r = |z| = \sqrt{x^2 + y^2}$$

and θ is an *argument* of z.

13.6 Complex Fourier Series

- This is the angle (in radians) between the positive $x-$ axis and the line from the origin through (x, y), or this angle plus any integer multiple of 2π. Using Euler's formula, we obtain the *polar form* of z:

$$z = r[\cos(\theta) + i\ \sin(\theta)] = re^{i\theta}.$$

13.6 Complex Fourier Series

- Now

$$e^{i\theta} = \cos(\theta) + i\sin(\theta),$$

and by replacing θ with $-\theta$, we get

$$e^{-i\theta} = \cos(\theta) - i\sin(\theta).$$

17.6 Complex Fourier Series

- Solve these equations for cos(θ) and sin(θ) to obtain the complex exponential forms of the trigonometric functions:

$$\cos(\theta)=\frac{1}{2}\left(e^{i\theta}+e^{-i\theta}\right),\ \sin(\theta)=\frac{1}{2i}\left(e^{i\theta}-e^{-i\theta}\right).$$

13.6 Complex Fourier Series

- We will also use the fact that, if x is a real number, then the conjugate of e^{ix} is

$$\overline{e^{ix}} = e^{-ix}.$$

- This follows from Euler's formula, since

$$\overline{e^{ix}} = \overline{\cos(x) + i\ \sin(x)} = \cos(x) - i\ \sin(x) = e^{-ix}.$$

13.6 Complex Fourier Series

- Now let f be a piecewise smooth periodic function with fundamental period $2L$. To derive a complex Fourier expansion of $f(x)$ on $[-L, L]$, begin with the Fourier series of $f(x)$. With $\omega_0 = \pi/L$, this series is

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos(n\omega_0 x) + b_n \sin(n\omega_0 x)\right)$$

13.6 Complex Fourier Series

- Put the complex forms of $\cos(n\omega_0 x)$ and $\sin(n\omega_0 x)$ into this expansion:

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \frac{1}{2}\left(e^{in\omega_0 x} + e^{-in\omega_0 x}\right) + b_n \frac{1}{2i}\left(e^{in\omega_0 x} - e^{-in\omega_0 x}\right)\right]$$

$$= \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[\frac{1}{2}\left(a_n - ib_n\right)e^{in\omega_0 x} + \frac{1}{2}\left(a_n - ib_n\right)e^{-in\omega_0 x}\right]$$

in which we used the fact that $1/i = -i$.

13.6 Complex Fourier Series

- In this series, let

$$d_0 = \frac{1}{2} a_0$$

and, for $n = 1, 2, \cdots$,

$$d_n = \frac{1}{2}(a_n - ib_n).$$

13.6 Complex Fourier Series

- The Fourier series on $[-L, L]$ becomes

$$d_0 + \sum_{n=1}^{\infty} d_n e^{in\omega_0 x} + \sum_{n=1}^{\infty} \overline{d_n} e^{-in\omega_0 x} \qquad (13.14)$$

- Now

$$d_0 = \frac{1}{2} a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)dx$$

13.6 Complex Fourier Series

and for $n = 1, 2, \cdots,$

$$d_n = \frac{1}{2}(a_n - ib_n)$$

$$= \frac{1}{2L}\int_{-L}^{L} f(x)\cos(n\omega_0 x)dx - \frac{i}{2L}\int_{-L}^{L} f(x)\sin(n\omega_0 x)\,dx$$

$$= \frac{1}{2L}\int_{-L}^{L} f(x)\left[\cos(n\omega_0 x) - i\sin(n\omega_0 x)\right]dx$$

$$= \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\omega_0 x}dx.$$

13.6 Complex Fourier Series

- Then

$$\overline{d_n} = \frac{1}{2L}\int_{-L}^{L} f(x)\overline{e^{-in\omega_0 x}}dx = \frac{1}{2L}\int_{-L}^{L} f(x)e^{in\omega_0 x}dx = d_{-n}.$$

- With this, the expansion of equation (13.14) becomes

$$d_0 + \sum_{n=1}^{\infty} d_n e^{in\omega_0 x} + \sum_{n=1}^{\infty} d_{-n} e^{-in\omega_0 x}$$

$$= \sum_{n=-\infty}^{\infty} d_n e^{in\omega_0 x}.$$

13.6 Complex Fourier Series

This leads us to define the *complex Fourier series of f on* $[-L, L]$ to be

$$\sum_{n=-\infty}^{\infty} d_n e^{in\omega_0 x},$$

with coefficients

$$d_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\omega_0 x}\,dx$$

for $n = 0, \pm 1, \pm 2, \cdots$.

13.6 Complex Fourier Series

- Because of the periodicity of f, the integral defining the coefficients can be carried out over any interval $[\alpha, \alpha + 2L]$ of length $2L$. The Fourier convergence theorem applies to this complex Fourier expansion, since it is just the Fourier series in complex form.

EXAMPLE 13.12

- Let $f(x) = x$ for $-1 \le x < 1$ and suppose f has fundamental period 2, so $f(x + 2) = f(x)$ for all x. Figure 17.15 is part of a graph of f. Now $\omega_0 = \pi$.

EXAMPLE 13.12

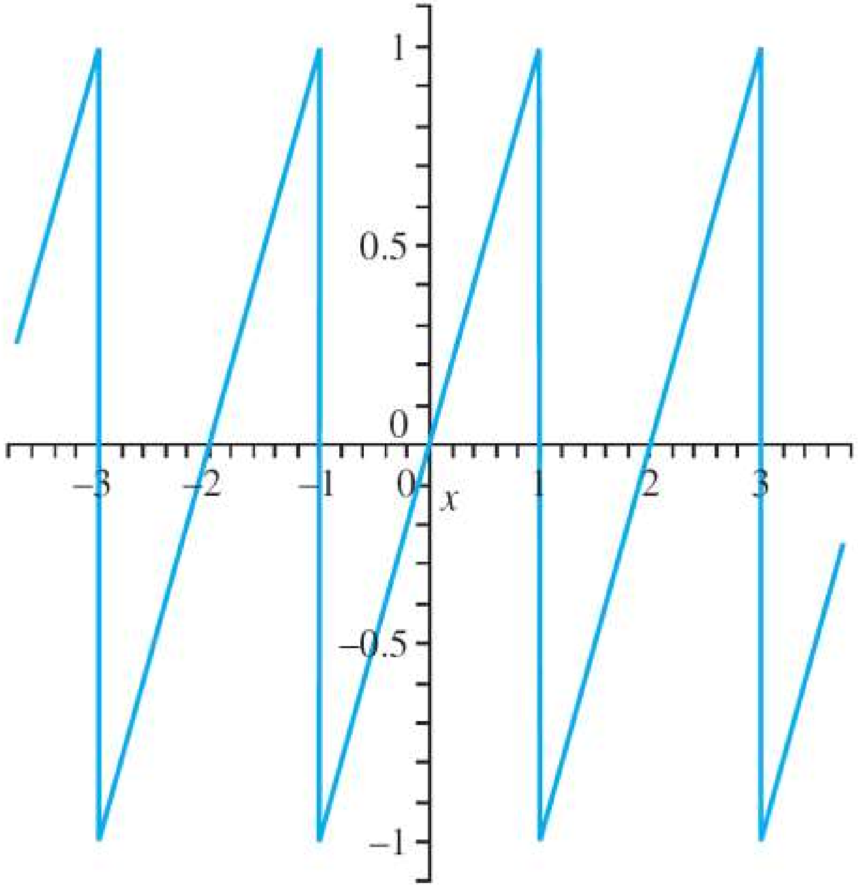


FIGURE 17.15 *Graph of f in Example* **17.12**

EXAMPLE 13.12

- Immediately $d_0 = 0$ because f is an odd function. For $n \neq 0$,

$$d_n = \frac{1}{2}\int_{-1}^{1} xe^{-in\pi x}\,dx$$

$$= \frac{1}{2n^2\pi^2}\left[in\pi e^{in\pi} - e^{in\pi} + in\pi e^{-in\pi} + e^{-in\pi}\right]$$

$$= \frac{1}{2n^2\pi^2}\left[in\pi\left(e^{in\pi} + e^{-in\pi}\right) - \left(e^{in\pi} - e^{-in\pi}\right)\right].$$

EXAMPLE 13.12

- The complex Fourier series of *f* is

$$\sum_{n=-\infty, n\neq 0}^{\infty} \frac{1}{2n^2\pi^2}\left[in\pi\left(e^{in\pi}+e^{-in\pi}\right)-\left(e^{in\pi}-e^{-in\pi}\right)\right]e^{in\pi x}.$$

EXAMPLE 13.12

- This converges to x for $-1 < x < 1$. In this example we can simplify the series. For $n \neq 0$,

$$d_n = \frac{1}{2n^2\pi^2}\left[2in\pi\,\cos(n\pi) - 2i\,\sin(n\pi)\right]$$

$$= \frac{i}{n\pi}(-1)^n.$$

EXAMPLE 13.12

- All the terms $\sin(n\pi) = 0$. In $\sum_{n=-\infty}^{-1}$, replace n with $-n$ and sum from $n = 1$ to ∞, then combine the two summations from 1 to ∞ to write

$$\sum_{n=-\infty, n\neq 0}^{\infty} \frac{i}{n\pi}(-1)^n e^{in\pi x}$$

$$= \sum_{n=1}^{\infty}\left(\frac{i}{n\pi}(-1)^n e^{in\pi x} + \frac{i}{-n\pi}(-1)^{-n} e^{-in\pi x}\right)$$

$$= \sum_{n=1}^{\infty}\frac{i}{n\pi}(-1)^n\left(e^{in\pi x} - e^{-in\pi x}\right) = \sum_{n=1}^{\infty}\frac{2}{n\pi}(-1)^{n+1}\sin\left(n\pi x\right).$$

- This is the Fourier series for $f(x) = x$ on $[-1, 1]$.

13.6 Complex Fourier Series

- The *amplitude spectrum* of a complex Fourier series of a periodic function is a graph of the points ($n\omega_0$, $|d_n|$). Sometimes this graph is also referred to as a *frequency spectrum*.