A small addition to Anthony Robin's article on "Egyptian Fractions" from SIGMA 14/05/2014

Andy Mitchell

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Anthony mentions the approach of constructing Egyptian fractions for p/q by initially subtracting the largest possible $1/a_1$, then the largest possible $1/a_2$ from the remainder etc. and demonstrated that this would produce a maximum of p unit fractions in the Egyptian fraction representation.
He also mentions a couple of examples in which the maximum number of unit fractions are required :-

$$\frac{4}{25} = \frac{1}{7} + \frac{1}{59} + \frac{1}{5163} + \frac{1}{53307975}$$

$$\frac{5}{241} = \frac{1}{49} + \frac{1}{2953} + \frac{1}{11623993} + \frac{1}{202675808272081} + \frac{1}{82154966517482471538039869041}$$

I wondered if, for each value of p, there was an Egyptian fraction representation which requires the maximum number of unit fractions.
After finding several other examples, I soon found amongst them :-

$$\frac{4}{49} = \frac{1}{13} + \frac{1}{213} + \frac{1}{67841} + \frac{1}{9204734721}$$

$$\frac{4}{73} = \frac{1}{19} + \frac{1}{463} + \frac{1}{321091} + \frac{1}{206198539471}$$

$$\frac{5}{121} = \frac{1}{25} + \frac{1}{757} + \frac{1}{763309} + \frac{1}{873960180913} + \frac{1}{1527612795642093418846225}$$

which suggests that the fraction $\frac{n}{kn!+1}$ might require the maximum of n unit fractions.
Using Anthony's method (and noting that $\frac{n}{kn!+1} = \frac{1}{k(n-1)!+1/n}$) it is clear that the largest unit fraction to be subtracted is $\frac{1}{k(n-1)!+1}$.

Now

$$\frac{n}{kn!+1} - \frac{1}{k(n-1)!+1} = \frac{nk(n-1)!+n-kn!-1}{(kn!+1)(k(n-1)!+1)}$$

$$= \frac{n-1}{(kn(n-1)!+1)(k(n-1)!+1)}$$

$$= \frac{n-1}{[k^2n(n-1)!+kn+k](n-1)!+1}$$

$$= \frac{n-1}{k'(n-1)!+1}$$

where $k' = k^2n(n-1)! + kn + k$.
This is of the same form as $\frac{n}{kn!+1}$ with n replaced with $n-1$ (and note that one unit fraction has been subtracted).
Repeating this process, it's hopefully clear that $\frac{n}{kn!+1}$ will require the full set of n unit fractions in its representation as an Egyptian fraction.