Comments on Sigma 7/4/2005 - Breen Sweeney's "A Probability Paradox Revisited"

Andy Mitchell

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I'll briefly steal a bit of Breen's contribution to restate the scenario :-

> You start with an initial stake A (e.g. £100). You make a bet with a 50% chance of winning (e.g. toss a coin, heads you win, tails you lose). When winning you receive a certain proportion of you total current amount, call it x, and for losing your money is reduced by that proportion. So for winning you will have $A(1+x)$ and for losing you'll have $A(1-x)$.

In Breen's contribution, he pondered the approach of finding the long term result by looking at a large number $m+n$ of games with n wins and m losses, but realised that there was a flaw in his reasoning.

The offending step is the line :

> Now in the limit as n and m get larger, we know that the wins must be equal to the losses, hence n and m will be equal ...

This is essentially the original paradox where only the situation of an equal number of wins and losses is considered, but this time repeated n-fold.

A waffly argument which might help could start by just looking at the number of heads and tails after $2n$ tosses (assuming we're using the coin method of generating our 50% of course!).
The probability of n wins and n losses (i.e. $m=n$ in the argument) is:-

$$p_{n,n} = \binom{2n}{n}\frac{1}{2^{2n}} = \frac{(2n)!}{(n!)^2}\frac{1}{2^{2n}} \tag{1}$$

Using Stirling's formula ($k! \approx \sqrt{2\pi k}\cdot k^k e^{-k}$) :-

$$p_{n,n} \approx \frac{\sqrt{2\pi 2n}\cdot 2^{2n}n^{2n}e^{-2n}}{2\pi n\cdot n^{2n}e^{-2n}2^{2n}} = \frac{1}{\sqrt{\pi n}} \tag{2}$$

so the actual chance of having equal wins and losses after $2n$ throws tends to 0 as $n\to\infty$.

(By $\approx$, I mean that the ratio of each side tends to 1 as $n\to\infty$.)

Also, the probability of $n+k$ wins and $n-k$ losses (or $n-k$ wins and $n+k$ losses) after $2n$ throws is (for $0 < k \ll n$) :-

$$p_{n-k,n+k} = p_{n+k,n-k} = \binom{2n}{n+k}\frac{1}{2^{2n}} \tag{3}$$

$$= p_{n,n}\frac{(n-k+1)(n-k+2)\cdots n}{(n+1)(n+2)\cdots(n+k)} \tag{4}$$

$$\approx p_{n,n} \tag{5}$$

So, not only does exactly n wins and n losses get progressively more unlikely as n increases, there is a widening range of 'surrounding' possibilities each with similar likelihood of occurring, but symmetrical about $p_{n,n}$. To the expected overall takings, however, the contributions from the "$n+k$ wins, $n-k$ losses" add more than the "$n-k$ wins, $n+k$ losses" subtract (provided $x > 0$), so the question is whether these are enough to boost the expected 'take home' amount from $A(1-x^2)^n$ to something non-vanishing. My waffle doesn't have sufficient power to answer this, but the actual expected takings after $2n$ games (as Anthony Robin also demonstrated for n games in his original article) can be seen to be :-

$$A\sum_{i=0}^{2n}\binom{2n}{i}\frac{(1-x)^i(1+x)^{2n-i}}{2^{2n}} \tag{6}$$

$$= A\left(\frac{1}{2}(1-x)+\frac{1}{2}(1+x)\right)^{2n} = A \tag{7}$$

(using the binomial theorem as Breen suspected).

As an aside, can anybody find a neat/elegant demonstration of :-

$$\lim_{n\to\infty}\binom{2n}{n}\frac{\sqrt{\pi n}}{2^{2n}} = 1 \tag{8}$$

without using Stirling's formula? I haven't, but it feels like there might be!