Math189R SP19
Homework 1
Monday, Feb 4, 2017

Feel free to work with other students, but make sure you write up the homework and code on your own (no copying homework *or* code; no pair programming). Feel free to ask students or instructors for help debugging code or whatever else, though. The starter code for problem 2 part c and d can be found under the Resource tab on course website.

Note: You need to create a Github account for submission of the coding part of the homework. Please create a repository on Github to hold all your code and include your Github account username as part of the answer to problem 2.

1 (Linear Transformation) Let $\mathbf{y} = A\mathbf{x} + \mathbf{b}$ be a random vector. show that expectation is linear:

$$\mathbb{E}[\mathbf{y}] = \mathbb{E}[A\mathbf{x} + \mathbf{b}] = A\mathbb{E}[\mathbf{x}] + \mathbf{b}.$$

Also show that

$$\text{cov}[\mathbf{y}] = \text{cov}[A\mathbf{x} + \mathbf{b}] = A\text{cov}[\mathbf{x}]A^\top = A\mathbf{\Sigma}A^\top.$$

Let $f(\mathbf{x})$ be the corresponding density function for $\mathbf{x}$ on some domain D. Then

$$\begin{aligned}
\mathbb{E}[A\mathbf{x} + \mathbf{b}] &= \int_D (A\mathbf{x} + \mathbf{b}) f(\mathbf{x}) d\mathbf{x} \\
&= \int_D A\mathbf{x} f(\mathbf{x}) d\mathbf{x} + \int_D \mathbf{b} f(\mathbf{x}) d\mathbf{x} \\
&= A \int_D \mathbf{x} f(\mathbf{x}) d\mathbf{x} + \mathbf{b} \\
&= A\mathbb{E}[\mathbf{x}] + \mathbf{b}.
\end{aligned}$$

This follows because $\mathbf{b}$ is not dependant on $\mathbf{x}$ and can thus be factored out which then eliminates the relevant integral as $\int_D f(\mathbf{x}) d\mathbf{x} = 1$ by definition. Furthermore, the simplification in the last step is because A is also not dependant on $\mathbf{x}$ and because $\int_D \mathbf{x} f(\mathbf{x}) d\mathbf{x} = \mathbb{E}[\mathbf{x}]$ by definition.

$$\text{cov}[A\mathbf{x} + \mathbf{b}] = \mathbb{E}[(A\mathbf{x} + \mathbf{b} - \mathbb{E}[A\mathbf{x} + \mathbf{b}])(A\mathbf{x} + \mathbf{b} - \mathbb{E}[A\mathbf{x} + \mathbf{b}])^\top]$$

By the previous part we can substitute $\mathbb{E}[A\mathbf{x} + \mathbf{b}] = A\mathbb{E}[\mathbf{x}] + \mathbf{b}$, giving

$$\begin{aligned}
&= \mathbb{E}[(A\mathbf{x} + \mathbf{b} - A\mathbb{E}[\mathbf{x}] - \mathbf{b})(A\mathbf{x} + \mathbf{b} - \mathbb{E}[A\mathbf{x}] - \mathbf{b})^\top] \\
&= \mathbb{E}[(A\mathbf{x} - A\mathbb{E}[\mathbf{x}])(A\mathbf{x} - \mathbb{E}[A\mathbf{x}])^\top] \\
&= \mathbb{E}[A(\mathbf{x} - \mathbb{E}[\mathbf{x}])(\mathbf{x} - \mathbb{E}[\mathbf{x}])^\top A^\top]
\end{aligned}$$

Since A and A^T are constants, we know that we can factor them out by the previous part, giving

$$
\begin{aligned}
&= A\mathbb{E}[(\mathbf{x} - \mathbb{E}[\mathbf{x}])(\mathbf{x} - \mathbb{E}[\mathbf{x}])^\top]A^\top \\
&= A\text{cov}[x]A^\top \\
&= A\Sigma A^\top
\end{aligned}
$$

$\blacksquare$

2 Given the dataset $\mathcal{D} = \{(x, y)\} = \{(0,1), (2,3), (3,6), (4,8)\}$

(a) Find the least squares estimate $y = \boldsymbol{\theta}^\top \mathbf{x}$ by hand using Cramer's Rule.

(b) Use the normal equations to find the same solution and verify it is the same as part (a).

(c) Plot the data and the optimal linear fit you found.

(d) Find randomly generate 100 points near the line with white Gaussian noise and then compute the least squares estimate (using a computer). Verify that this new line is close to the original and plot the new dataset, the old line, and the new line.

(a) Since we are using a linear regression we know that for this problem

$$\begin{pmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} b \\ m \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 6 \\ 8 \end{pmatrix}.$$

Since $X = \begin{pmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}$ and $\vec{y} = \begin{pmatrix} 1 \\ 3 \\ 6 \\ 8 \end{pmatrix}$, it follows that

$$X^\top X = \begin{pmatrix} 4 & 9 \\ 9 & 29 \end{pmatrix}$$

$$X^\top \vec{y} = \begin{pmatrix} 18 \\ 56 \end{pmatrix}$$

By Cramer's rule it then follows that

$$b = \frac{\begin{vmatrix} 18 & 9 \\ 56 & 29 \end{vmatrix}}{\begin{vmatrix} 4 & 9 \\ 9 & 29 \end{vmatrix}} = \frac{18}{35},$$

and

$$m = \frac{\begin{vmatrix} 4 & 18 \\ 9 & 56 \end{vmatrix}}{\begin{vmatrix} 4 & 9 \\ 9 & 29 \end{vmatrix}} = \frac{62}{35},$$

Meaning that $\boxed{m = \frac{62}{35}}$ and $\boxed{b = \frac{18}{35}}$.

(b) We know from the previous section that we have the following formula

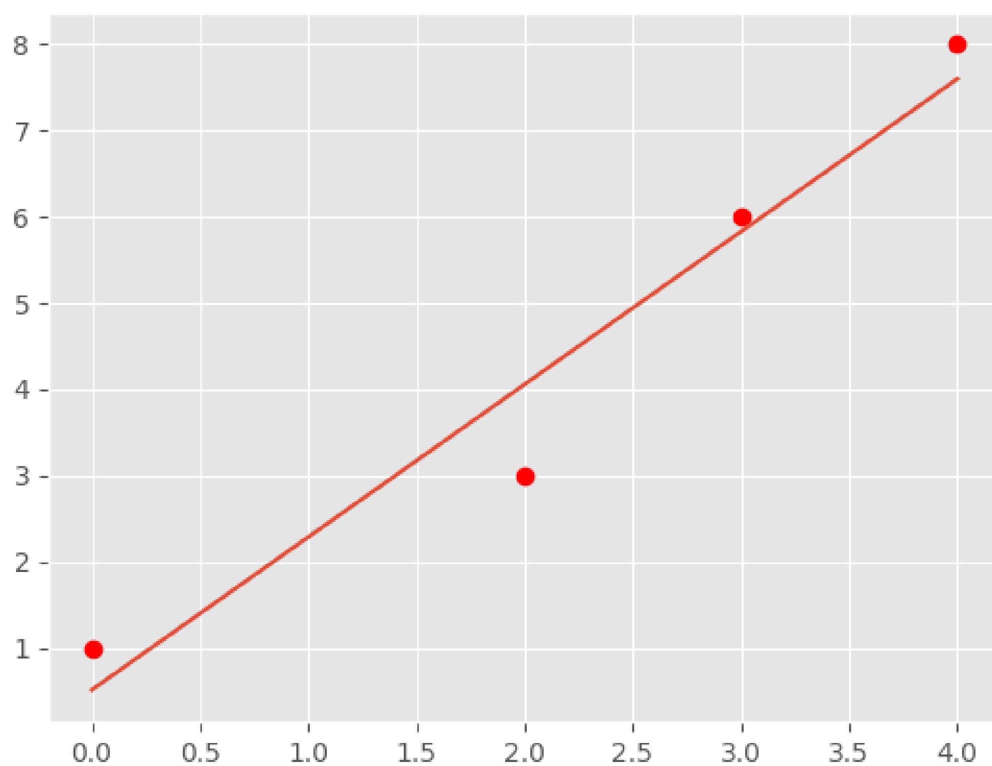
$$X^\top X\theta = X^\top \vec{y}.$$

We wish to solve for $\theta = \begin{pmatrix} m \\ b \end{pmatrix}$ which is equal to

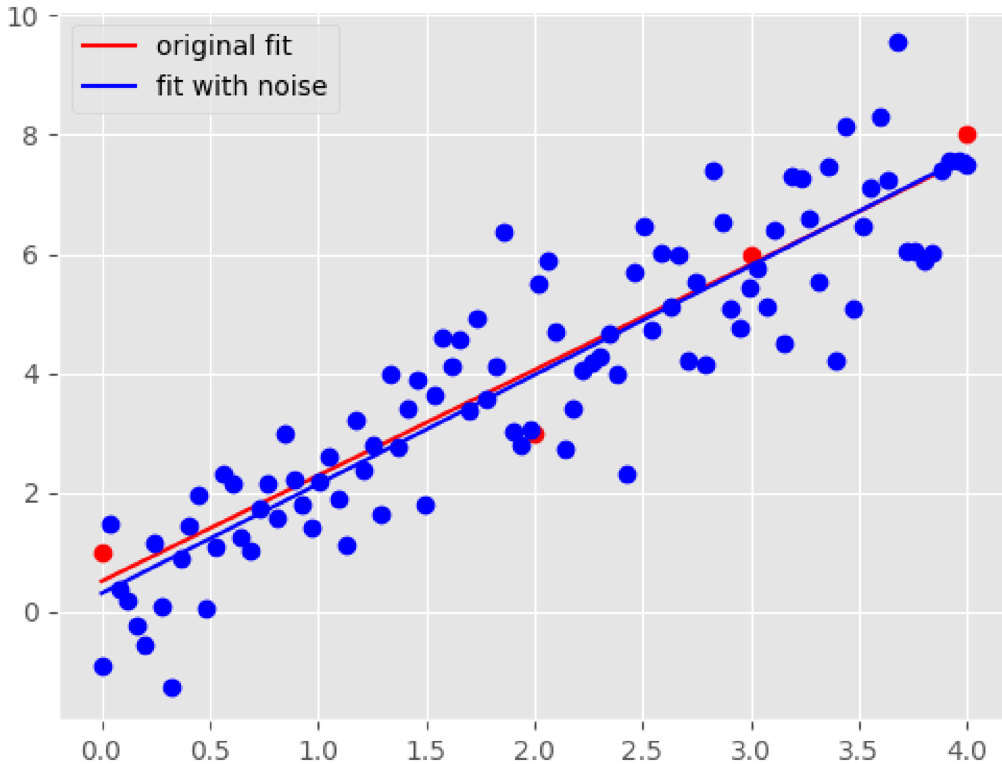
$$\begin{aligned}\theta &= (X^\top X)^{-1} X^\top \vec{y} \\ &= (\begin{pmatrix} 4 & 9 \\ 9 & 29 \end{pmatrix})^{-1} \begin{pmatrix} 18 \\ 56 \end{pmatrix} \\ &= \frac{1}{35} \begin{pmatrix} 29 & -9 \\ -9 & 4 \end{pmatrix} \begin{pmatrix} 18 \\ 56 \end{pmatrix} \\ &= \frac{1}{35} \begin{pmatrix} 18 \\ 62 \end{pmatrix}\end{aligned}$$

showing that (a) and (b) have the same solution.

(c)



(d)


original fit
fit with noise
10
8
6
4
2
0
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0

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