Game theory

Game with pure strategy.

Question 1: Find the optimum strategy of the players in the following games.

Player's (A) (strategies)	Player's B (strategies)		
	A	B	C
A	25	20	35
B	50	45	55.
C	48	40	42

Solution: Here,

Player's A strategies	Player's B (strategies)			Minimum of row pay-off
	A	B	C	
A	25	20	35	20
B	50	45 saddle point	55	45 (Maximin)
C	48	40	42	40
Maximum of column pay-off.	50	45 minimax	55	

$\therefore$ ~~Mini~~ Maximin = Minimax = 45

$\therefore$ saddle point ~~=~~ Maximin = Minimax = 45

Hence, optimal probability is

Probability of player A: $P_1 = 0, P_2 = 1, P_3 = 0$, then

$P_1 + P_2 + P_3 = 0 + 1 + 0 = 1.$

Probability of player B: $q_1 = 0, q_2 = 1, q_3 = 0$ then

$q_1 + q_2 + q_3 = 0 + 1 + 0 = 1.$

$\therefore$ Value of game = 45

Hence, a game has saddle point ~~with~~ at corresponding row 2 and column 2 cell ~~&~~ value is called game value.

Game with mixed strategy

Question: consider the following pay-off matrix with respect to player A and solved it optimally.

		Player-B	
		I	II
Player A	I	9	7
	II	5	11

Solution: Here,

		Player A		
		I	II	Minimum pay-off
Player B.	I	9	7	$\boxed{7}$ Maximin
	II	5	11	5
Maximum pay-off		$\boxed{9}$ ↑ Minimax	11	

∴ Maximin pay-off ≠ Minimax pay-off.

Hence, Maximin pay-off ≠ Minimax pay-off, then it has no saddle point. So, it is called mixed strategy. In this case, used oddments method to find the value of game.

Oddments method:

Now take 2×2 matrix and find out the oddments for both row and column.

		Player B		oddments	probability
		I	II		
Player B	I	9	7	6	$\frac{6}{6+2}$
	II	5	11	2	$\frac{2}{6+2}$
oddments		4	4		
probability		$\frac{4}{4+4}$	$\frac{4}{4+4}$		

probability of row I $(P_1) = \frac{\text{oddment of row}}{\text{sum of oddments}} = \frac{6}{6+2} = \frac{6}{8} = \frac{3}{4}$.

Probability of row II $(P_2) = \frac{2}{6+2} = \frac{2}{8} = \frac{1}{4}$

Probability of column $(q_1) = \frac{4}{4+4} = \frac{4}{8} = \frac{1}{2}$

Probability of column $(q_2) = \frac{4}{4+4} = \frac{4}{8} = \frac{1}{2}$

$\therefore$ sum of probability of player A $= \frac{3}{4} + \frac{1}{4} = 1$

sum of probability of player B $= \frac{1}{2} + \frac{1}{2} = 1$

Now,

value of game is

For row I, value game(V) $= \frac{9\times4+7\times4}{8} = \frac{64}{8} = 8$

For row II, value of game (V) $= \frac{5\times4+11\times4}{8} = \frac{64}{8} = 8$

For column I, value of game (V) $= \frac{9\times6+5\times2}{6+2} = \frac{64}{8} = 8$

For column II, value of game (V) $= \frac{7\times6+11\times2}{8} = \frac{64}{8} = 8$

$\therefore$ value of game is

$V = 8$ Ans

By formula method

probability of first row $(P_1) = \frac{A_{22}-A_{21}}{(A_{11}+A_{22})-(A_{12}+A_{21})}$

probability of second row $(P_2) = 1-P_1$

probability of column I $(q_1) = \frac{A_{22}-A_{12}}{(A_{11}+A_{22})-(A_{12}+A_{21})}$

probability of column II $(q_2) = 1-q_1$

Value of game (V) $= \frac{A_{11}\times A_{22}-A_{12}\times A_{21}}{(A_{11}+A_{22})-(A_{12}+A_{21})}$

solu

Dominance property:

In some games, it is possible to reduce the size of the pay-off matrix by eliminating row (or column) which are dominated by other rows (or column) respectively.

1. Dominance property for row: If all elements of particular row x are less than or equal to the corresponding elements of another row y then delete row x.

2. Dominance property for column: If all elements of particular column x are more than or equal to the corresponding elements of another ~~row~~ column y then delete column x.

3. A pure strategy may be dominated, if it is inferior (less than) to an average of two or more other pure strategy.

Question: The following table represents the pay-off matrix with respect to player A. solve it optimally using dominance ~~rule~~. property.

Player B

~~Row minimum~~

		1	2	3	4	5
	1	4	6	5	10	6
Player A	2	7	8	5	9	10
	3	8	9	11	10	9
	4	6	4	10	6	4

Solution: Here,

		Player B					
		1	2	3	4	5	Row minimum
Player A	1	4	6	5	10	6	4
	2	7	8	5	9	10	5
	3	8	9	11	10	9	8 maximin
	4	6	4	10	6	4	4
Column maximum		8	9	11	10	10	

minimax.

$\therefore$ Maximin = Minimax = 8 = saddlepoint.

$\therefore$ value of game (v) = 8

probability of player A = [0, 0, 1, 0]

probability of player B = [1, 0, 0, 0]

Now, using dominance probability, the value of game is same.

		Player B					
		1	2	3	4	5	Row total
Player B	1	~~4~~	~~6~~	~~5~~	~~10~~	~~6~~	31
	2	7	8	5	9	10	39
	3	8	9	11	10	9	47
	4	~~6~~	~~4~~	~~10~~	~~6~~	~~4~~	30

~~Row 4~~ Each element of row 4 is less than or equal to row 3, then row 4 is dominated by row 3, then delete row 4

Note least value is 31, then each element of row 1 is less than or equal to row 3, then row 1 is dominated by row 3, ~~then~~, so delete row 1.

Also, column reduction

	1	2	3	4	5
2	7	8	5	9	10
3	8	9	11	10	9
Column total	15	17	16	19	19

(i) Most column total is 19, each element of column 4 is more than or equal to column 1, then column 4 is dominated by column 1. so delete column 4.

(ii) Another most column total is 19, each element of column 5 is more than or equal to column 1, then column 5 is dominated by column 1, then delete column 5.

(iii) similarly, column 2 is dominated by column 1, then delete column 2

After row and column reduction, the pay-off matrix is

Player B

Player A	1	3	R.Total
2	7	5	12
3	8	11	19
C-Total	8	11	

(i) least Row total is 12, each element of row 2 is less than or equal to row 3, then row 2 dominated by row 3. so delete row 2.

(ii) Most column total is 11, each element of column 3 is more than or equal to column 1, then column 3 is dominated by column 1. So, delete column 3.

∴ optimal strategy of

Player A = [0, 0, 1, 0]

Player B = [1, 0, 0, 0]

Value of game (V) = 8

Question! The following table represents the pay-off matrix with respect to player A. Solve it optimally using dominance property with mixed strategy.

Player B

	1	2	3	4
1	5	-3	3	4
2	-4	5	4	5
3	4	-4	-3	3

Solution: Here,

Player B

	1	2	3	4	R-minimum
1	5	-3	3	4	-3 Maximin.
2	-4	5	4	5	-4
3	3	-4	-3	3	-4
C-maximum	5	5	4	5	

Minimax.

∴ Maximin = -3

Minimax = 4

∴ Maximin ≠ Minimax

∴ There is no saddle point, then it is called mixed strategy.

Now by using dominance property

Player B

		1	2	3	4	R. Total
Player A	1	5	-3	3	4	9
	2	-4	5	4	5	10
	3	~~-4~~	~~-4~~	~~-3~~	~~3~~	0

(i) least ~~Hott~~ row total is 0. each element of row 3 is less than or equal to row 1, then row 3 is dominated by row 1. So, Row 3 delete.

After row reduction, then table is

Player B

	1	2	3	4
1	5	-3	3	4
2	-4	5	4	5
C-Total	1	2	7	9

Most column total is 9. each element of column 4 is more than or equal to column 2, then column 4 is dominated by column 2. So column 4 is delete.

~~And, Another most column total is 7.~~

Now, after column reduction. No 2×2 matrix, in this case, we chaked average of any two column comparing to other column.

Player B

		1	2	3
	1	5	-3	3
player A	2	-4	5	4

Average of C_1 & C_2.

$\frac{5-3}{2} = 1$

$\frac{-4+5}{2} = \frac{1}{2}$

4.9

In order to find 2×2 matrix. using oddments method.

$$
\begin{array}{c|cc|c|c}
 & \multicolumn{2}{c|}{\text{player } B} & & \\
 & 1 & 2 & \text{oddment} & \text{probability} \\
\hline
\text{player } A \quad 1 & 5 & -3 & 9 & \frac{9}{9+8} = \frac{9}{17} \\
2 & -4 & 5 & 8 & \frac{8}{9+8} = \frac{8}{17} \\
\hline
 & 8 & 9 & & \\
\end{array}
$$

probability: $\frac{8}{8+9} = \frac{8}{17}$ $\quad \frac{9}{17}$

value of game $(v) = \dfrac{5 \times 9 + (-4) \times 8}{9+8} = \dfrac{54 - 32}{17} = \dfrac{22}{17}$

$\therefore$ probability of player A $= \cancel{\left\{ \frac{9}{17}, \frac{8}{17}, 0, 0, 0 \right]}$

$= \left\{ \frac{9}{17}, \frac{8}{17}, 0 \right\}$

Algebric method

Algebric method / equal game method:

Let $P_1, P_2, \ldots, P_m$ be the probability that the player A choose his strategy $A_1, A_2, \ldots, A_m$ respectively. Where, $P_1 + P_2 + \cdots + P_m = 1$

Also, let $q_1, q_2, \ldots, q_n$ be the probability that the player B choose his strategy $B_1, B_2, \ldots, B_n$ respectively. Where, $q_1 + q_2 + \cdots + q_n = 1$

Player A Strategy	Player B strategies				Probability
	B_1	B_2	....	B_n	
A_1	a_{11}	a_{12}	...	a_{1n}	P_1
A_2	a_{21}	a_{22}	...	a_{2n}	P_2
⋮	..	..	...	..	⋮
A_m	a_{m1}	a_{m2}	...	a_{mn}	P_m
Probability	q_1	q_2	....	q_n	1

Let V be the value of game

To find S_A

The expected gain to player A when the player B selects strategies $B_1, B_2, \ldots, B_n$ respectively. since, player A is gainer and he expects at least V, we have

$$a_{11}P_1 + a_{21}P_2 + \cdots + a_{m1}P_m \geq V$$

$$a_{12}P_1 + a_{22}P_2 + \cdots + a_{m2}P_m \geq V$$

$$\cdots$$

$$a_{1n}P_1 + a_{2n}P_2 + \cdots + a_{mn}P_m \geq V$$

To find the value of P_i's above inequality equation and are solved for given unknowns.

To find S_B:

The expected loss to player B when player A selects strategies $A_1, A_2, \ldots, A_n$ respectively. since, player B is loser player, ~~we have~~ and he expects at most V, we have

$$a_{11}q_1 + a_{12}q_2 + \cdots + a_{1n}q_n \leq V$$

$$a_{21}q_1 + a_{22}q_2 + \cdots + a_{2n}q_n \leq V$$

$$\cdots$$

$$a_{m1}q_1 + a_{m2}q_2 + \cdots + a_{mn}q_n \leq V$$

To find the values of q_i's above inequality equation and are solved for given unknows.

By substituting the values of a_i's and q_i's in any one of the above equation give the value of the game.

Example ①: Two player A and B match coins. If the coins match, then A wins two units of value, if the coin do not match, then B win 2 units of value. Determine the optimal strategies for the players and the value of the games.

Solution: Let us construct the pay-off matrix for player A.

Player B

		H	T	Row minimum		probability
Player A	H	2	−2	−2	Maximin	P_1
	T	−2	2	−2		P_2
	column maximum	$\boxed{2}$	$\boxed{2}$			
	probability	q_1	q_2			

Minimax

Since, Maximin = −2, Minimax = 2, then value of game lies between −2 to 2.

∴ Maximin ≠ Minimax, therefore, there is no saddle point. The game has no saddle point should be solved by mixed strategy. ~~Note,~~ It is 2×2 game without saddle point, we use algebraic method.

To find S_A:

$$2P_1 - 2P_2 \geq V \quad \text{——} ①$$

$$-2P_1 + 2P_2 \geq V \quad \text{——} ②$$

$$P_1 + P_2 = 1 \quad \text{——} ③$$

Equality equation of above inequality equation is

$$2P_1 - 2P_2 = V \quad \text{——} ④$$

$$-2P_1 + 2P_2 = V \quad \text{——} ⑤$$

$$P_1 + P_2 = 1 \quad \text{——} ⑥$$

solving eqn. ④ and ⑤, we have

$V = V$

or, $2P_1 - 2P_2 = -2P_1 + 2P_2$

or, $4P_1 = 4P_2$

$\therefore P_1 = P_2$

From eqn. ⑥, we get

$P_1 + P_2 = 1$

or, $P_1 + P_1 = 1$

or, $2P_1 = 1$

$\therefore P_1 = \frac{1}{2}$

$\therefore P_1 = P_2 = \frac{1}{2}$

$\therefore$ from eqn. ④, $V = 2P_1 - 2P_2 = 2\times\frac{1}{2} - 2\times\frac{1}{2} = 0$

To find SB:

$2q_1 - 2q_2 \le V$ ——— ⑦

$-2q_1 + 2q_2 \le V$ ——— ⑧

$q_1 + q_2 = 1$ ——— ⑨

equality equation of eqn. 7, 8, 9 is

$2q_1 - 2q_2 = V$ ——— ⑩

$-2q_1 + 2q_2 = V$ ——— ⑪

$q_1 + q_2 = 1$ ——— ⑫

Solving eqn. ⑩ and ⑪, we get

$V = V$

or, $2q_1 - 2q_2 = -2q_1 + 2q_2$

or, $2q_1 + 2q_1 = 2q_2 + 2q_2$

or, $4q_1 = 4q_2$

$\therefore q_1 = q_2$

From eqn. ⑫, we have

$q_1 + q_2 = 1$

or, $q_1 + q_1 = 1$

$\therefore 2q_1 = 1$

$\therefore q_1 = \frac{1}{2}$

$\therefore q_1 = q_2 = \frac{1}{2}$.

$\therefore$ From eqn. (6), $V = 2q_1 - 2q_2 = 2\times\frac{1}{2} - 2\times\frac{1}{2} = 0$

Hence,

the optimal strategy for player A and B is

~~the~~ probability of player A $= [\frac{1}{2}, \frac{1}{2}]$

probability of player B $= [\frac{1}{2}, \frac{1}{2}]$

And, value of game $(V) = 0$. Ans

Example (8): solve the following 2×2 game.

	Player A	
Player B	2	5
	7	3

Solution: Here,

	Player A		R.Min	Probability
Player B	2	5	2	p_1
	7	3	$\boxed{3}$	p_2
C. Max	9	$\boxed{8}$		
Probability	q_1	q_2		

since, Maximin $= 3$

Minimax $= 8$

$\therefore$ Maximin $\neq$ Minimax, there is no saddle point.

Since, a game has no saddle point is called mixed strategy. Now we need algebraic method.

~~For~~ To find S_A:

$$2q_1 + 5q_2 = V \quad \text{——— (1)}$$

$$7q_1 + 3q_2 = V \quad \text{——— (2)}$$

$$q_1 + q_2 = 1 \quad \text{——— (3)}$$

(4.1)

solving eqn. ① and ②

$$2q_1 + 5q_2 = 7q_1 + 3q_2$$

or, $2q_1 - 7q_2 = 3q_2 - 5q_2$

or, $-5q_1 = -2q_2$

or, $5q_1 = 2q_2$

or, $q_1 = \frac{2q_2}{5}$

$\therefore q_1 = \frac{2q_2}{5}$ —— ④

From eqn. ③, we get

$$q_1 + q_2 = 1$$

or, $\frac{2q_2}{5} + q_2 = 1$

or, $\frac{2q_2 + 5q_2}{5} = 1$

or, $2q_2 + 5q_2 = 5$

or, $7q_2 = 5$

$\therefore q_2 = \frac{5}{7}$

From eqn. ④, we get

$$q_1 = \frac{2}{5} \times \frac{5}{7} = \frac{2}{7}$$

From eqn. ①, we get

$$v = 2q_1 + 5q_2$$

$$= 2 \times \frac{2}{7} + 5 \times \frac{5}{7}$$

$$= \frac{4}{7} + \frac{25}{7}$$

$$= \frac{29}{7}$$

To find S_B

$2p_1 + 7p_2 = v$ —— (5)

$5p_1 + 3p_2 = v$ —— (6)

$p_1 + p_2 = 1$ —— (7)

Solving eqn. (5) and (6), we get

$$2p_1 + 7p_2 = 5p_1 + 3p_2$$

or, $2p_1 - 5p_1 = 3p_2 - 7p_2$

or, $-3p_1 = -4p_2$

or, $p_1 = \frac{4}{3}p_2$ —— (8)

From eqn. (7), we get

$p_1 + p_2 = 1$

or, $\frac{4}{3}p_2 + p_2 = 1$

or, $\frac{4p_2 + 3p_2}{3} = 1$

or, $7p_2 = 3$

$\therefore p_2 = \frac{3}{7}$

From eqn. (8), we have

$$p_1 = \frac{4}{3} \times \frac{3}{7} = \frac{4}{7}$$

From eqn. (5), we get

$$v = 2p_1 + 7p_2$$

$$= 2 \times \frac{4}{7} + 7 \times \frac{3}{7}$$

$$= \frac{8}{7} + \frac{21}{7}$$

$$= \frac{29}{7}$$

Hence, optimum strategies is

probability of player A $= \left(\frac{2}{7}, \frac{5}{7}\right)$

probability of player B $= \left(\frac{4}{7}, \frac{3}{7}\right)$

$\therefore$ value of game $(v) = \frac{29}{7}$

Arithmetic method

Arithmetic method provides an easy technique for obtaining the optimum strategy for each player in 2x2 matrix without saddle point.

① Find the difference of two numbers in column I, and put into under the column II (neglecting the negative sign if occurs).

② Find the difference of two numbers in column II, and put it under the column I. (neglecting the negative sign if occurs)

③ Proceed same for each row.

Question: Two players A and B without showing each other, put on a table a coin, with head or tail up. A wins Rs 8 when both the coin show head and Rs 1 when both are tails. B wins Rs 3 when the coin do not match. Given the choice of ~~beining~~ being matching player (A) on non-matching player (B). Which one should you choose and what would be your strategy?

Solution:

step(1): The pay-off matrix for A:—

Player B

Player A	H	T
H	8	-3
T	-3	1

step(2): Apply minimax criteria to find saddle point.

Player B

Player A	H	T	R.min
H	8	-3	-3
T	-3	1	-3
C.max	8	1	