CHAPTER 15

Graph Theory - II

15.1. Graph Colouring

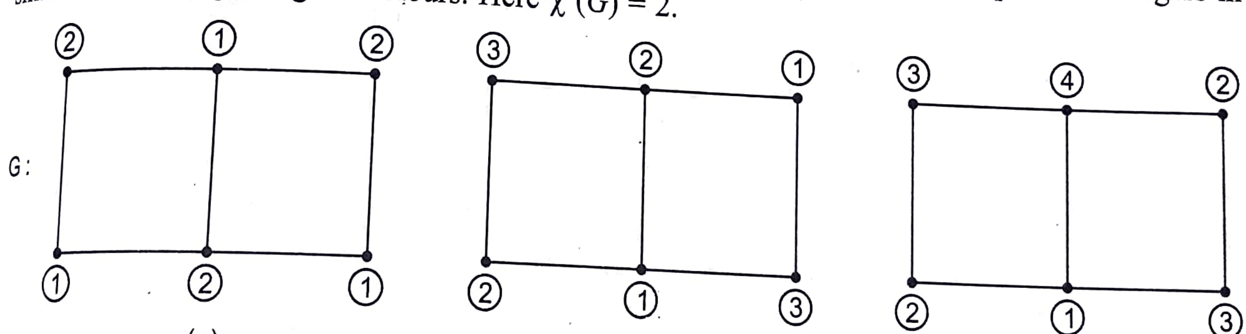
Graph colouring is an assignment of colours to elements of a graph subject to certain constraints. The starting point of graph colouring is the vertex colouring. The other colouring problems like edge colouring and region colouring can be transformed into vertex colouring. An edge colouring of a graph is just a vertex colouring of its line graph. The region colouring of a planar graph is the problem of colouring of vertex of its planar dual graph. In this chapter we present the basic results concerning vertex colouring and edge colouring of graphs and their applications.

15.2. Vertex Colouring

The assign of colours to the vertices of G, one colour to each vertex, so that the adjacent vertices are assigned different colours is called the **proper colouring** of G or simply **vertex colouring** of G. A graph in which every vertex has been assigned a colour according to a proper colouring is called a **properly coloured graph.** The n-colouring of G is a proper colouring of G using n-colours. If G has n colouring, then G is said to be ***n*-colourable.**

Chromatic Number: The chromatic number of a graph G is the minimum number of colours needed for a proper colouring *i.e.* the minimum number of colours needed to assign colours to each vertex of G such that no two adjacent vertices are of same colour. It is denoted by $\chi(G)$. Thus a graph G is K chromatic if $\chi(G) = K$.

The graph of Fig. 15.1 of n colouring for $n = 2,3,4$ are displayed, with positive integers in small circles designating the colours. Here $\chi(G) = 2$.



(a) (b) (c)

Fig. 15.1

In graph colouring, we consider the colouring of simple connected graphs only. This is so because the colouring of vertices of one component of a disconnected graph has no effect on the colouring of the other components. Self loops are not considered for colouring. Parallel edges between two vertices can be replaced by a single edge without affecting the adjacency of vertices.

Note: (*i*) A graph consisting of only isolated vertices is 1-chromatic. The chromatic number of a null graph is also 1.

(*ii*) $\chi(G) \leq n$, where n is the number of vertices of G.

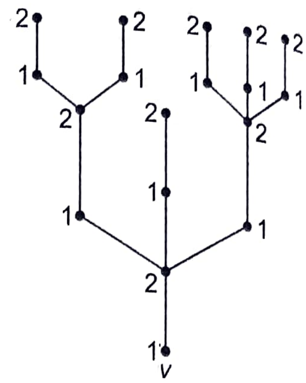
(*iii*) A simple connected graph with one or more edges is at least 2 *i.e.*. $\chi(G) \geq 2$.

(*iv*) Chromatic number of a graph having a triangle is at least 3 (because three colours are required to colour a triangle)

(*v*) If deg (v) = d for a vertex v in G, then at most d colours are required for proper colouring of the vertices adjacent to v.

(*vi*) Every k-chromatic graph has at least k vertices, say $v_1, v_2, \ldots, v_k$ such that deg $(v_i) \geq k - 1$, $i = 1, 2, \ldots, k$.

(*vii*) If H is a subgraph of G, then $\chi(H) \leq \chi(G)$.



Theorem 15.1. Every tree with two or more vertices is 2-chromatic.

Proof: Choose any vertex v in the given tree T. Consider T as a rooted tree at vertex v and paint v with colour 1. Paint all vertices adjacent to v using colour 2. Next, paint the vertices adjacent to these vertices (that just have been coloured with 2) using colours 1. Continue this colouring process till every vertex in T has been painted.

Thus, all vertices at odd distances from v have colour 2, while v and vertices at even distances from v have colour 1.

Now along any path in T, the vertices are of alternating colours. We know that there is one and only one path between any two vertices in a tree, so no two adjacent vertices of T have the same colour. Thus T can be properly coloured using two colours. Hence $x(T) = 2$ *i.e.* T is 2-chromatic.

Note: The converse of the theorem is not true *i.e.* not every 2-chromatic is tree. The graph C_{2n} is 2-chromatic but it is not a tree.

Chromatic Number of a *Bi*-Partite Graph ($K_{m,n}$)

Theorem 15.2. The chromatic number of a non-null graph is 2 if and only if the graph is bipartite.

Proof: Let G be a bipartite graph. So its vertex set V can be partitioned in two sets $V_1 = \{v_1, v_2, \ldots v_m\}$ and $V_2 = \{u_1, u_2, \ldots u_n\}$ such that every edge of G joins some v_i to some u_j. Since no two of the vertices in V_i are adjacent, we can assign a colour-1 to each v_i. Since u_i is adjacent to a v_j and since no two of u_is are adjacent we can assign colour-2 to each u_i. Thus the graph is coloured only with colour-1 and colour-2, Hence $\chi(G) = 2$.

Conversely, let G be a 2 chromatic graph *i.e.* $\chi(G) = 2$. Let V_1 denote the set of all vertices for which first colour is assigned and V_2 be the set of all vertices for which second colour is assigned. Thus V_1 and V_2 are disjoint and $V = V_1 \cup V_2$. Hence the two colour classes form a partition of G. Otherwise at least two vertices in V_1 or V_2 have the same colour. Therefore, G is bipartite graph.

Note: The chromatic number of a complete bipartite graph is 2 *i.e.* $\chi(K_{n,n}) = 2$.

Chromatic Number of a Complete Graph (K_n)

Theorem 15.3. The chromatic number of a complete graph with n vertices (K_n) is n.

Proof: Let $v_1, v_2, \ldots, v_n$ are n vertices of a complete graph K_n. Let a colour C_1 be assigned to v_1 and since v_2 is adjacent to v_1, a different colour C_2 is required to be assigned to v_2. Next, since v_3 is adjacent to both v_1 and v_2, so another colour C_3 (different from C_1, C_2) is required to be assigned to v_3. In this way the different colours $C_1, C_2, \ldots, C_n$ are required to be assigned to the vertices $v_1, v_2, \ldots, v_n$ respectively because no two vertices can be assigned the same colour as every two vertices of K_n are adjacent. Thus at least n colours are required for proper colouring of K_n. Hence the chromatic number of K_n is n.

Note: (*i*) The chromatic number of the graph $K_n - v_i$ obtained by deletion of a vertex v_i from K_n is $n - 1$.

If a vertex is deleted from a complete graph then the graph remains to be complete. So, $K_n - v_i$ is a complete graph having $n - 1$ number of vertices.

$$\therefore \quad \chi(K_n - v_i) = n - 1$$

(*ii*) If K_n is a subgraph of a graph G. Then $x(G) \geq n$.

Chromatic Number of cycle (C_n)

Theorem 15.4. The chromatic number of a cycle with n vertices (C_n) is

(*i*) 2 if n is even (*ii*) 3 if n is odd

Proof. Let the cycle C_n has the vertices $v_1, v_2, \ldots, v_n$ appearing in order of the cycle. If we assign colour-1 to v_1, v_2 must be coloured with different colour say with colour-2 because v_2 is adjacent to v_1. Now v_3 is adjacent to v_2 but not to v_1, So we can assign colour 1 to the vertex v_3. In this way we can alternately assign colour-2, colour-1, colour-2 .... to the vertices $v_4, v_5, \ldots$ respectively.

(*i*) If n is even the last vertex v_n which is again adjacent to v_1 will be assigned colour-2. Thus if n is even C_n can be coloured by 2 colours only but not less than 2. Hence $\chi(C_n) = 2$ if n is even.

(*ii*) If n is odd the last vertex cannot be assigned any of colour-1 and colour-2 because this vertex is adjacent to v_{n-1} (which is assigned colour-2) and v_1 (which is assigned colour-1). So the last vertex v_n should be assigned a new colour, say colour-3. Thus if n is odd then C_n can be coloured by 3 colours but not less than 3. Hence $\chi(C_n) = 3$ if n is odd.

The graphs of C_3, C_4 and C_5 are shown below.

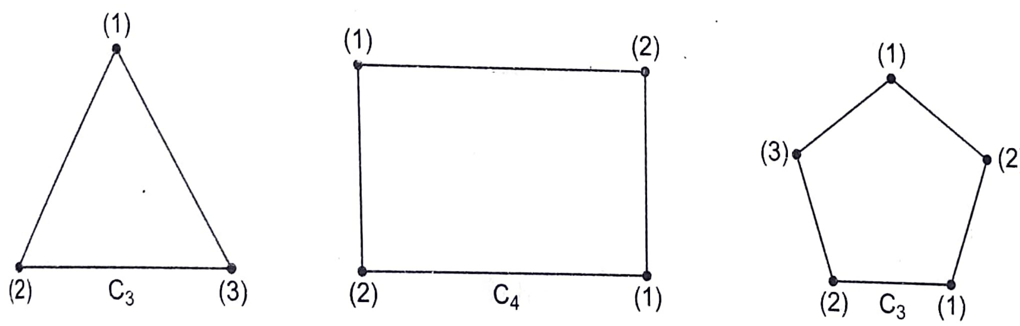
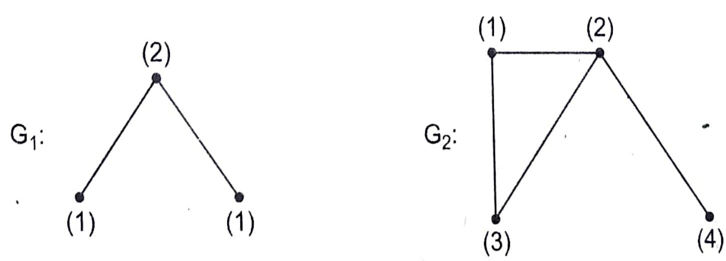


Fig. 15.2.

Note (*i*) If C_n is a subgraph of a graph G then

(*a*) $x(G) \geq 2$ if n even (*b*) $x(G) \geq 3$ if n is odd.

(*ii*) The converse of the theorem is not true as shown in the following example:



None of G_1 and G_2 is cycle but $\chi(G_1) = 2$ and $\chi(G_2) = 3$.

Theorem 15.5. A graph G with a least one edge is 2-chormatic if and only if one of the following conditions is satisfied.

(*a*) G is a tree.

(*b*) Every cycle of G has even length

(*c*) G is a bipartite graph.

Vertex Colouring Algorithm

There are a number of colouring algorithms which gives approximation to the number of colours for proper colouring of graph G and here we present the following algorithm.

Welch-Powell algorithm

Input is a graph G.

1. Order the vertices of G in decreasing degree (such an ordering may not be unique since some vertices may have the same degree).
2. Assign one colour to paint the first vertex and to paint, in sequential order, each vertex on the list that is not adjacent to a vertex previously painted with this colour.
3. Repeat 2 the process of painting previously non-coloured vertices using a second colour.
4. Repeat 3 with a third colour, then a fourth colour, and so on until all vertices are coloured.
5. Exit.

The algorithm only gives an upper bound for $\chi(G)$, *i.e.*, the algorithm does not always give the minimum number of colours which is needed to paint G.

Example 1. Consider the graph G in Fig. 15.3, use the Wetch – powell algorithm to colour G and find $\chi(G)$.

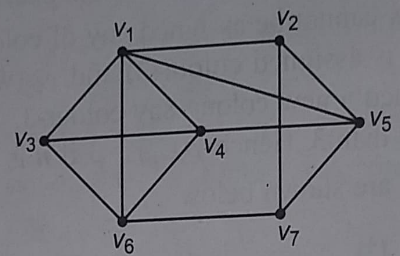


Fig. 15.3.

Solution. We list the vertices of G in decreasing degrees as shown below. We refer to colours as a, b, c, etc. Assign colour a to the first vertex v_1 in the list. The next and only vertex in the list not adjacent to v_1 is v_7, assign a colour a to v_7. Now, we move to next uncoloured vertex v_4 in the list and assign a colour b to v_4. The uncoloured vertex in the list not adjacent to v_4 is v_2, assign a colour b to v_2. Since all the remaining uncoloured vertices are adjacent to v_4 or v_2, move to colour c. Now assign a colour c to v_5. The uncoloured vertex in the list not adjacent to v_5 is v_6, assign a colour c to v_6. Hence v_3 is also not adjacent with v_5 but it is adjacent with v_6. Hence v_3 cannot be coloured with c. Assign colour d to only left out vertex v_3.

Thus, all the vertices are painted. The completed assignment is listed below.

Vertex	v_1	v_4	v_5	v_6	v_2	v_3	v_7
Degree	5	4	4	4	3	3	3
Colour	a	b	c	c	b	d	a

Upper Bound of Chromatic Number

There is no efficient and convenient procedure to find out the chromatic number of any arbitrary graph G. However there are various results which give upper bounds for the chromatic number of an arbitrary graph if the degree of their vertices are known.

Theorem 15.6. For any graph G, $\chi(G) \leq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of a vertex in G.

Proof: We prove this theorem by induction on n, where n is the number of verticles of G.

Basic step: Let $n = 1$. Then G is a graph with only one vertex and G has no edge. Hence $\chi(G) = 1$ and $\Delta(G) = 0$. This implies that $\chi(G) \leq \Delta(G) + 1$ for $n = 1$.

Inductive hypothesis: Suppose that $k > 1$ is an integer such that for any simple graph G, with k vertices, $\chi(G) \le \Delta(G) + 1$.

Inductive step: Let G be a simple graph with $k + 1$ vertices. Consider a vertex v of G and construct the graph $G_1 = G - \{v\}$. The graph G_1 is obtained from G by deleting the vertex v and also deleting all the edges incident with v. Clearly, $\Delta(G_1) \le \Delta(G)$. The graph G_1 is a simple graph with k vertices. Thus by the inductive hypothesis $\chi(G_1) \le \Delta(G_1) + 1$. Then $\chi(G_1) \le \Delta(G) + 1$. This implies that G_1 can be properly coloured by at most $\Delta(G) + 1$ colours. Now v has at most $\Delta(G)$ adjacent vertices. Because $\Delta(G) \le \Delta(G) + 1$, it follows that not all the $\Delta(G) + 1$ colours are needed to colour these $\Delta(G)$ adjacent vertices. Thus from these $\Delta(G) + 1$ colours one unused colour is definitely available to colour vertex v. Hence $\chi(G) \le \Delta(G) + 1$.

Note. The inequality in the above theorem becomes an equality when G is a

(*i*) complete graph K_n (*ii*) cycle C_n, n is odd

In K_n, the degree of each vertex is $n - 1$ and $\chi(K_n) = n$

$$\therefore \quad \chi(K_n) = n = n - 1 + 1 = \Delta(n) + 1$$

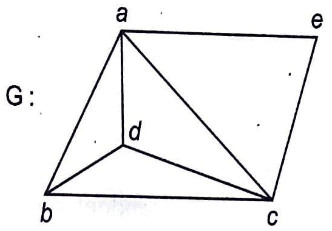
In cycle C_n, the degree of each vertex is 2 and $\chi(C_n) = 3$ if n is odd.

$$\therefore \quad \chi(C_n) = 3 = 2 + 1 = \Delta(C_n) + 1.$$

Theorem 15.7. (Brook's Theorem) If G is a connected graph other than a complete graph with $\Delta(G) \ge 3$, then $\chi(G) \le \Delta(G)$

Note: The theorem reduces the upper of $\chi(G)$ by 1 for a particular class of graphs.

Example 2. Find the chromatic number of the graph G.



Solution. The graph induced by four vertices a, b, c and d is a complete subgraph of G.

$$\chi(G) \ge 4. \quad \text{...(1)}$$

Since $\deg(a) = \deg(b) = 4$; $\deg(c) = \deg(d) = 3$; $\deg(e) = 2$

$$\Delta(G) = \text{maximum of the degrees of all vertices} = 4$$

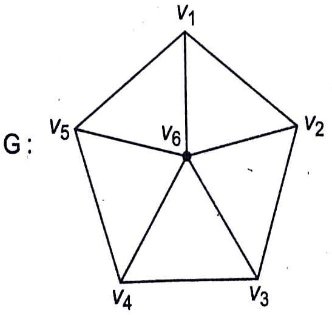
Again, G is a connected graph other than a complete graph with $\Delta(G) = 4 > 3$, by Brook's theorem

$$\chi(G) \le \Delta(G) \text{ i.e. } \chi(G) \le 4 \quad (2)$$

From (1) and (2) $\chi(G) = 4$.

Example 3. Find the chromatic number of wheel graph W_6.

Solution.



Since there is a triangle subgraph of G

$$\chi(G) \ge 3 \quad \text{...(1)}$$

Again $\deg(v_6) = 5$ and all other vertices have degree 3.

$$\Delta(G) = \text{maximum of the degrees of all vertices} = 5$$

$\therefore$

$$\chi(G) \leq 1 + \Delta(G) = 1 + 5 = 6 \qquad \ldots(2)$$

From (1) and (2), we get

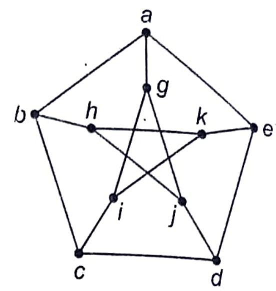
$$3 \leq \chi(G) \leq 6.$$

If $\chi(G) \geq 5$, there would be at least 5 vertices of degree $\geq 5 - 1 = 4$. But this is not true for W_6. $\chi(G) \leq 4$

Hence $3 \leq \chi(G) \leq 4$.

Example 4. Define Peterson graph and find its chromatic number.

Solution. The graph with 10 vertices and 15 edges shown in adjacent figure is called the Peterson graph.



As C_5 induced by vertices a, b, c, d and e is a subgraph of G (the Peterson graph)

Here

$$\chi(G) \geq 3. \qquad \ldots(1)$$

$$\Delta(G) = \underset{v \in V(G)}{\text{Max}} \deg(v) = 3.$$

Since G is a connected other than a complete graph with $\Delta(G) = 3$, by Brook's theorem we get

$$\chi(G) \leq \Delta(G)$$

i.e.

$$\chi(G) \leq 3 \qquad \ldots(2)$$

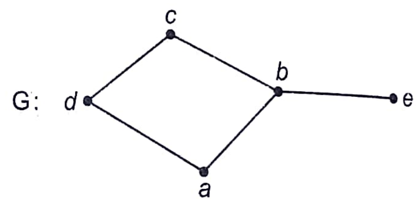
From (1) and (2), we have

$$\chi(G) = 3.$$

Lower Bounds of Chromatic Number

Independent Set

A set of vertices in a graph is said to be independent set of vertices or simply independent set if not two vertices in the set are adjacent. An independent set is also called stable set.



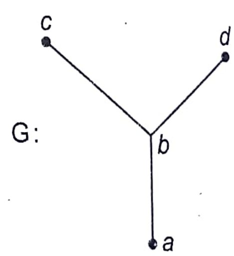
In the above graph G, $\{a, c, e\}$ is an independent set.

Note: 1. A single vertex in any graph is always an independent set.

2. Every subset of an independent set is an independent set.

3. Vertices in an independent set can have the same colour.

Maximal Independent Set: An independent set of G which is not a proper subset of any independent set of G is called maximal independent set of G *i.e.* an independent set of a graph G to which no other vertex can be added without destroying its independence property. The set $\{a, c, d\}$ is a maximal independent set of the graph. The set $\{a, c\}$ is not a maximal independent set as it is a proper subset of the independent set $\{a, c, d\}$.



Independence Number: A graph may have many different maximal independent set and they are of different sizes. The number of vertices in the largest maximal independent set of G is called the independence number and denoted by $\beta(G)$. The independence number of the above graph is 3.

Hence, the independence number is the largest number of vertices that can be painted with the same colour in a vertex colouring of G. For a null graph N_n, a complete graph K_n and a complete bipartite graph $K_{m,n}$, we get

$$\beta(N_n) = n; \quad \beta(K_n) = 1 \quad \text{and} \quad \beta(K_{m,n}) = \max\{m, n\}$$

Chromatic number: The minimum number of all maximal independent sets whose union is the vertex set V of G is the chromatic number of G. In this example, chromatic number is two since $\{b, d\} \cup \{a, c, e\} = V$.

Method of Finding all Maximal Independent Sets in a Graph

One of the methods to find all maximal independent sets of a graph G using Boolean algebra is discussed here. We treat all the vertices of the graph as Boolean variables. We define

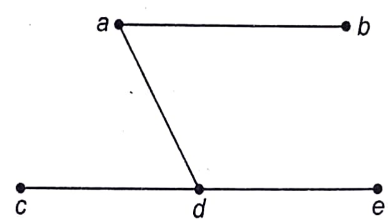
1. $a + b$ denotes the operations of including vertex a or b or both.
2. $a.b$ denotes the operations of including both the vertices a and b.
3. a' denotes the operation of excluding the vertex a.
4. Express an edge (x, y) as a Boolean product xy.
5. $\varphi = \Sigma xy$, for all (x, y) in G to get a Boolean expression.
6. $\phi' = \Pi\,(x' + y') = f_1 + f_2 + \ldots\ldots + f_k$ where ϕ' is the Boolean complement of ϕ and each f_i is the maximal independent set of G in its complement form.

For a vertex set is maximal independent set if and only if $\phi = 0$

But $\phi = 0$ if and only if $\phi' = 1$

If and only if $f_i = 1$ for some i.

Example 5. Find all possible maximal independent sets of the following graph using Boolean algebra.



Solution. The graph has 4 edges.

$\therefore\ \varphi = \Sigma xy = ab + ad + cd + de$ by expressing each edge (x, y) as product xy.

By De-Morgan's rule, we get

$$\begin{aligned}
\varphi' &= (a' + b')\,(a' + d')\,(c' + d')\,(d' + e') \\
&= \{a'(a' + d') + b'\,(a' + d')\}\ \{c'(d' + e') + d'\,(d' + e')\} \\
&= \{a' + b'a' + b'd'\}\ \{c'd' + c'e' + d')\} \qquad \text{as } a(a + b) = a \\
&= \{a'(1 + b') + b'd')\}\ \{d'(c' + 1) + c'e'\} \\
&= \{a' + b'd'\}\ \{d' + c'e'\} \qquad \text{as } 1 + a' = 1 \\
&= \{a'd' + a'c'e' + b'd' + b'c'd'e'\} \\
&= a'd' + a'c'e' + b'd'\,(1 + c'e') \\
&= a'd' + a'c'e' + b'd'
\end{aligned}$$

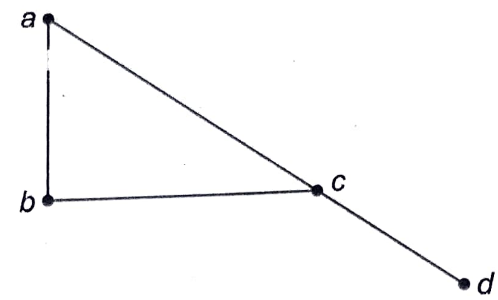
Thus $\quad f_1 = a'd',\ \ f_2 = a'c'e' \ \text{ and } \ f_3 = b'd'$

Excluding from the vertex set V of G the vertices appearing in any of the three terms, we get the maximal independent sets and they are

$$V - \{a, d\} = \{b, c, e\},\ V - \{a, c, e\} = \{b, d\}$$

and $\quad V - \{b, d\} = \{a, c, e\}.$

Example 6. Using Boolean algebra find all maximal independent set. Hence find the independence and chromatic number.



Solution. The graph has four edges.

Here $$\varphi = \Sigma xy = ab + ac + bc + cd$$

Using De-Morgan's rule, we get

$$\begin{aligned}
\varphi' &= (a' + b')(a' + c')(b' + c')(c' + d') \\
&= \{a' + b'c'\}\{c' + b'd'\} \qquad \text{as } (a + b)(a + c) = a + bc \\
&= a'c' + a'b'd + b'c' + b'c'd' \\
&= a'c' + a'b'd' + b'c' + (1 + d') \\
&= a'c + a'b'd' + b'c' \qquad \text{as } 1 + a = 1
\end{aligned}$$

Thus $f_1 = a'c', \quad f_2 = a'b'd' \quad f_3 = b'c'$

The maximal independent sets are

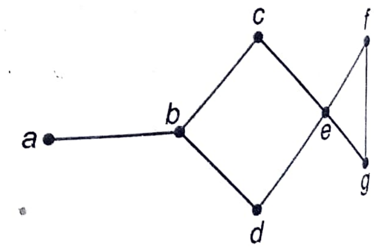
$V - \{a, c\} = \{b, d\}$, $V - \{a, b, d\} = \{c\}$ and $V - \{b, c\} = \{a, d\}$

Since maximum number or vertices among all maximal independent set is 2. Hence the independence number is 2.

Since $V = \{b, d\} \cup \{c\} \cup \{a, d\}$, the chromatic number is 3.

Chromatic Partitioning

Let G be a simple connected graph. Let S be the set of all disjoint independent sets of G whose union is V. Then S is the partition of the vertex set V of G. The smallest possible number of partitioning of vertices of G is called chromatic partitioning. Some of the disjoint indenpendent set that are partitions of G are



(i) $\{a, c, d, f\}$ $\{b, g\}$, $\{e\}$

(ii) $\{a, c, d, g\}$ $\{b, f\}$, $\{e\}$

(iii) $\{a, c, d, f\}$ $\{b, e\}$, $\{g\}$

(iv) $\{a, c\}$ $\{b, e\}$, $\{d, g\}$ $\{f\}$

(v) $\{a, d\}$ $\{b, e\}$, $\{c, g\}$ $\{f\}$

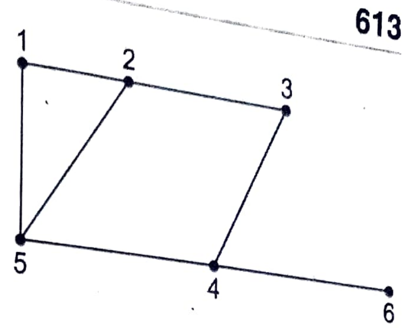
Out of the sets, the set (*i*), (*ii*) and (*iii*) contains smallest number of independent sets. Hence they are chromatic partitions of G.

The graph that has only one chromatic partition of vertices is called uniquely colourable.

Clique

A clique in an undirected graph $G = (V, E)$ is a subset of the vertex set $C \subseteq V$, such that for ...ing that

In the above graph, the vertex set $\{1, 2, 5\}$ is a clique.



A ***maximal clique*** is a clique that cannot be extended by including *one more* adjacent vertex. In other words a maximal clique is a clique which does not exist exclusively within the vertex set of a larger clique.

A ***maximum clique*** is a clique of the largest possible size in a given graph.

The ***clique number*** of a graph G is the number of vertices of the maximum clique in G and is denoted by $\omega(G)$. In the adjacent graph $\omega(G) = 3$.

For a complete graph and complete bipartite graph, we have $w(K_n) = n$ and $w(K_{m,n}) = 2$ for $m, n \geq 2$.

Note: (*i*) Clique of a null graph does not exist.

(*ii*) $\omega(G) \geq 2$ for any graph G.

(*iii*) The complete subgraph of a graph G is a clique of G.

(*iv*) The chromatic number and clique number are related by the inequality.

$$\chi(G) \geq \omega(G)$$

For example, every bipartite graph G with at least one edge satisfies $\chi(G) = \omega(G) = 2$. But in a cycle graph $c_n (n > 3)$ satisfies $\chi(c_n) = 3 > 2 = \omega(c_n)$.

Relation Between Clique and Independent Set

Let $\bar{G} = (V_G, \bar{E}_G)$ be the *complementary graph* of G. Since $\bar{G}$ has the same set of vertices as G, and two vertices are adjacent in $\bar{G}$ if an only if they are not adjacent in G. It follows from definitions that, for a subset U of V_G, U is an independent set of G if and only if U is a clique of $\bar{G}$, and U is a clique of G if and only if U is an independent set of $\bar{G}$. In particular, we have

$$\alpha(G) = w(\bar{G}) \text{ and } w(G) = \alpha(\bar{G})$$

The following two theorems give the lower bound of chromatic number of a graph.

Theorem 15.8 For a graph G with n vertices and $\alpha(G)$ independence number, $\chi(G) \geq \dfrac{n}{\alpha(G)}$ where $\chi(G)$ is the chromatic number of G.

Theorem 15.9 For a graph with n vertices and $w(G)$ clique number, $\chi(G) \geq w(G)$

Example: Every bipartite graph G with at least one edge satisfies $\chi(G) = w(G) = 2$. But in a cycle graph C_n of odd order $n > 3$ with n edges arranged in a cycle satisfies $\chi(C_n) = 3 > 2 = w(C_n)$.

A *clique-partition* of a graph G is a partition of its vertices into cliques. The smallest number of cliques in a clique-partition of G is the *clique-partition number* of G, denoted by $\theta(G)$. We have

$$\chi(G) = \theta(\bar{G}), \theta(G) = \chi(\bar{G})$$

Perfect Graph

A graph G is called a perfect graph if and only if $\chi(H) = \omega(H)$, for every induced subgraph H of G.

Examples:

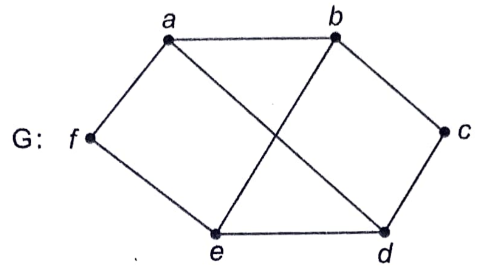
(*i*) Edgeless graphs *i.e.* the graphs $V = \{1, 2, ..., n\}$ and $E = \varphi$ are perfect. These graphs and all of their subgraphs have both chromatic and clique number 1.

(*ii*) The complete graph K_n are all perfect. Any induced subgraph H of K_n on κ vertices is itself a complete graph on κ vertices therefore, $\kappa = \chi(H) = \omega(H)$ for any such H.

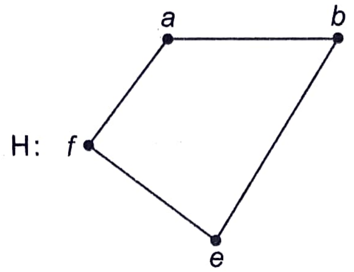
(*iii*) Bipartite graphs are perfect. This is because

(*a*) if G is not edgeless, $\chi(G) = 2 = \omega(G)$ as bipartite graphs do not contain any *odd cycles. Therefore,* ...graph of a bipartite G, it is either edgeless or bipartite ... bipartite graph.

Example 7. Show that the following graph is a perfect graph



Solution. Let the vertex a be assigned a colour-1. Since b, d and f are adjacent to a, they are assigned colour-2. Since c and e are not adjacent to a they can be assigned the same colour of a i.e. colour-1. Hence two colours are required for proper colouring of G. i.e. $\chi(G) = 2$.



Let H be a induced subgraph of G induced by set $S = \{a, b, e, f\}$. It is clear that $\omega(\chi) = 2$ and this is true for every induced subgraph of G. Hence $\chi(H) = \omega(H)$ i.e. the graph G is a perfect graph.

Theorem 15.10. (five-colour theorem) A planar graph G is 5-colourable.

Proof. The proof is by induction on the number n of vertices of G. If $n \leq 5$, then the theorem obviously holds, since the graph is 5-colourable. We assume that the theorem holds for every planar graph with $n - 1$ vertices, we shall prove that it is true for a planar graph with n vertices. Since G is a planar and is connected it must have a vertex v such that degree $(v) \leq 5$. Let G' be the graph obtained by deleting v from G. By the induction hypothesis, G' requires not more than 5 colours. When v has degree 1, 2, 3, 4, there is no difficulty, since we can give to v one more colour. hence, it suffices to consider the case when G' has 5 colours with which its vertices are properly coloured as shown in Fig. 15.5 (a).

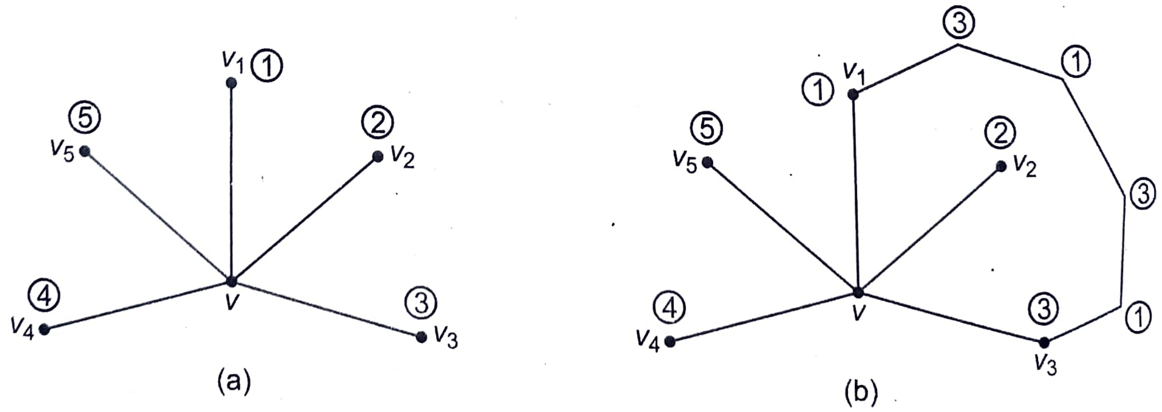


Fig. 15.5

Suppose there is a path between the vertices v_1 and v_3 coloured alternately with colours 1 and 3 (shown in Fig. 15.5 (b)) then, a similar path between v_2 and v_4 cannot exist, since this path, if it exists, will intersect the path between v_1 and v_3, contradicting that G is planar. Hence, we can interchange the colours of all vertices connected to v_2, giving colour 4 to v_2, while v_4 has still colour 4. Thus, we can colour v with colour 2, and hence the graph $G + v = G$ is 5-colourable. Similarly, if we have assumed that no path exists between v_1 and v_3, we could colour v_3 with colour 1 and v_1 with colour 3. Thus, a planar graph with 5 vertices is 5-colourable.

Hence, by induction, it follows that a planar graph with n vertices is 5-colourable.

This result can be strengthened further, and this is what was formerly one of the most famous unsolved problem in mathematics – the four problem.

Theorem 15.11. (four-colour problem) Any planar graph is 4-colourable.

The problem was originally posed as a conjecture in the 1850s. It was finally proved by the American mathematician Kenneth Appel and Wolfgang Haken in 1976. Their proof, which took them several years and a substantial amount of computer time ultimately depends on complicated extension of the ideas is the proof of the five-colour theorem.

Chromatic Polynomial

The evaluation or computation of the total number of ways of proper colouring of a graph G of n vertices using λ or fewer colours is expressed by means of a polynomial known as chromatic polynomial. It is denoted by $P(G, \lambda)$, $P_n(G, \lambda)$ or $P_n(\lambda)$

The smallest +ve integer λ for which $P_n(\lambda) \neq 0$ is nothing but $\chi(G)$, the chromatic number of G.

15.3. Determination of $P_n(\lambda)$,

Example 8. Write down the chromatic polynomial of a given graph on n vertices.

Solution. Let G be a graph on n vertices. Let c_i denote the different ways of properly colouring G using exactly i distinct colours. These i colours can be chosen out of λ colours in ${}^{\lambda}C_i$ distinct ways. Thus total number of distinct ways a proper colouring to a graph with i colours out of λ colours is possible in ${}^{\lambda}C_i$ ways. Hence the chromatic polynomial is given by

$$\begin{aligned} P_n(G, x) &= \sum_{i=1}^{n} c_i \, {}^{\lambda}C_i \\ &= c_1 {}^{\lambda}C_1 + c_2 {}^{\lambda}C_2 + \dots\dots\dots + c_n {}^{\lambda}C_n \\ &= c_1\lambda + c_2 \frac{\lambda(\lambda-1)}{2!} + \dots\dots\dots + c_n \frac{\lambda(\lambda-1)(\lambda-2)\dots\dots(\lambda-n+1)}{n!} \end{aligned}$$

Each c_i can be evaluated individually.

Note: (*i*) Any non-null graph requires at least two colour for proper colouring, and therefore $C_1 = 0$

(*ii*) For a graph with one edge and two vertices $C_1 = 0$, $C_2 = 2!$ as the graph can be coloured with two colours which can be arranged in $2!$ ways $P_2(\lambda) = 0.\lambda + 2!\dfrac{\lambda(\lambda-1)}{2!} = \lambda^2 - \lambda$.

Theorem 15.12 If G is disconnected graph with components $G_1, G_2, \dots\dots, G_n$, then

$$P(G, \lambda) = P(G_1, \lambda)\, P(G_2, \lambda) \dots\dots\dots P(G_n, \lambda).$$

Chromatic Polynomial of Some Standard Graph

Theorem 15.13 Chromatic polynomial of a complete graph with n vertices (K_n) is

$$P_n(\lambda) = \lambda(\lambda-1)(\lambda-2) \dots\dots\dots (\lambda - n + 1).$$

Proof. Let G be a graph of n vertices $v_1, v_2, \dots, v_n$. Given λ colours, the vertex v_1 can be coloured in λ different ways. Now for each colouring v_1 there remain $\lambda - 1$ colours for v_2 as v_1 is adjacent ot v_2 being a complete graph. Therefore v_2 can be coloured in $(\lambda - 1)$ ways. For each colouring of v_1 and v_2, v_3 can be coloured in $(\lambda - 2)$ ways and so on. Thus by the product rule of counting

$$P_n(\lambda) = \lambda(\lambda-1)(\lambda-2) \dots\dots\dots (\lambda - n + 1)$$

For example, if $G = K_4$, then

$P_G(\lambda) = \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)$ and the chromatic number is 4 since $P_G(\lambda) > 0$ first time when $\lambda = 4$.

Theorem 15.14. The chromatic polynomial of a tree with n vertices is

$$P_n(\lambda) = \lambda(\lambda - 1)^{n-1}$$

Proof. Let G be a tree with n vertices. We prove the theorem by induction. If $n = 1$, then G has one vertex which can be coloured in λ distinct ways.

Then $$P_1(\lambda) = \lambda.$$

If $n = 2$, then G has one edge, so two colours are required for the proper colouring of the graph. Here $c_1 = 0$, $c_2 = 2$. Then $P_2(\lambda) = 0 + [\lambda(\lambda - 1)]2!]2 = \lambda(\lambda - 1)$.

We assume the result is true for all trees having lesser than n vertices. Now consider a tree with n vertices. Since G is a tree it must have at least one pendant vertex. Let v be a pendant vertex of G. Let $G' = G - v$. Then G' is a tree with $n - 1$ vertices. The by inductive hypothesis, $P_{n-1}(\lambda) = \lambda(\lambda - 1)^{n-2}$.

Now for each proper colouring of G' the given graph can be properly coloured by painting the vertex v with the colour other than the vertex adjacent to the vertex v (one and only vertex is adjacent to v as v is a pendant vertex).

Thus v can be coloured with $(\lambda - 1)$ colours for each proper colouring of G'. Thus the number of ways of colouring G is

$$P_n(\lambda) = \lambda(\lambda - 1)^{n-2}(\lambda - 1) = \lambda(\lambda - 1)^{n-1}$$

Example 9. For each graph, the constant term in its chromatic polynomial is zero.

Solution. Let the chromatic polynomial of a graph G of n be vertices be

$$P_n(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \ldots.. + a_{n-1}\lambda + a_n$$

Since $\chi(G) \geq 1$ for any graph G.

$$P_n(\lambda) = 0 \text{ for } \lambda = 0$$

$$\therefore \quad a_0 \times 0 + a_1 \times 0 + \ldots.. + a_{n-1} \times 0 + a_n = 0$$

$$\Rightarrow \quad a_n = 0.$$

Example 10. If a graph has at least one edge then the sum of the coeffecients in its chromatic polynomial is 0.

Solution. Let $G = (V, E)$, $|E| \geq 1$ be a graph and its chromatic polynomial be

$$P_n(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \ldots.. + a_{n-1}\lambda + a_n$$

Since G has at least one edge then at least two vertices of G are adjacent. So we cannot properly colour G with only one colour. Consequently, $\chi(G) \geq 2 \quad \therefore \quad P_n(1) = 0$

or $$a_0 \cdot 1^n + a_1 1^{n-1} + \ldots.. + a_{n-1} \cdot 1 + a_n = 0$$

or $$a_0 + a_1 + \ldots.. + a_{n-1} + a_n = 0.$$

Example 11. Explain why each of the following polynomials cannot be a chromatic polynomial,

(*a*) $\lambda^3 + 5\lambda^2 - 3\lambda + 5$

(*b*) $\lambda^4 + 3\lambda^3 - 3\lambda^2$

Solution. (*a*) The polynomial cannot be a chromatic polynomial since the constant term is 5, not 0.

(*b*) The polynomial cannot be a chromatic polynomial since the sum of the coefficients is 1, not 0.

Example 12. Determine the chromatic polynomial of a connected graph with three vertices.

Solution. Since the given graph, say G has three vertices, so its chromatic polynomial is

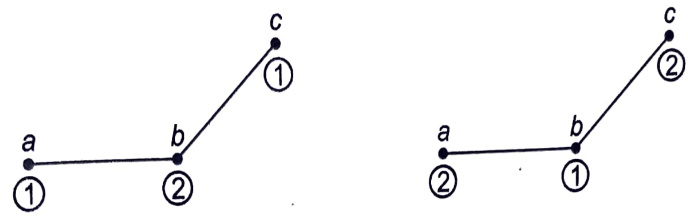
$$P_3(\lambda) = c_1{}^{\lambda}C_1 + c_2{}^{\lambda}C_2 + c_3{}^{\lambda}C_3.$$

G must have at least one edge, since it is connected. So, only one colour cannot properly colour G *i.e.* $c_1 = 0$. There are two possibilities.

Case 1: If the three vertices of G form a triangle, then $c_2 = 0$ and $c_3 = 3! = 6$. So, in this case the chromatic polynomial becomes

$$P_3(\lambda) = 3!\,{}^{\lambda}C_3 = 3!\frac{\lambda(\lambda-1)(\lambda-2)}{3!} = \lambda(\lambda-1)(\lambda-2).$$

Case 2: If the three vertices does not make a triangle then it makes a path (*i.e.*, a tree).

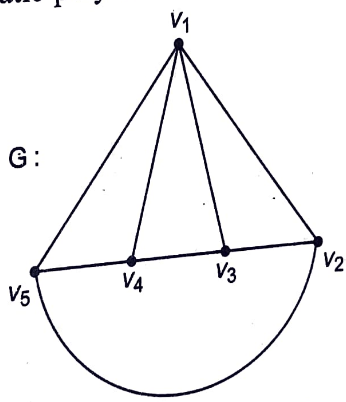


So, $c_2 = 2$ and $c_3 = 3!$

Therefore, in this case the chromatic polynomial is

$$P_3(\lambda) = 2\,{}^{\lambda}C_2 + 3!\,{}^{\lambda}C_3 = 2.\frac{\lambda(\lambda-1)}{2!} + 3!\frac{\lambda(\lambda-1)(\lambda-2)}{3!} = \lambda(\lambda-1)^2.$$

Example 13. Find the chromatic polynomial of the following graph



Solution. Since the graph consists of 5 vertices, the chromatic polynomial is denoted by

$$P_5(\lambda) = c_1\lambda + c_2\frac{\lambda(\lambda-1)}{2!} + c_3\frac{\lambda(\lambda-1)(\lambda-2)}{3!} + c_4\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!} + c_5\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{5!}$$

The graph G has triangles as subgraph, so $\chi(G) \geq 3$ and hence $c_1 = c_2 = 0$.

When we have three colours, the clique vertices $\{v_1, v_2, v_3\}$ can be coloured with 3 colours in $3! = 6$ different ways. Hence, $c_3 = 6$

Similarly, when we have 4 colours, then the clique vertices $\{v_1, v_2, v_3\}$ can be coloured in $3! \times C(4, 3) = 24$ ways and the fourth colour can be assigned to v_4 or v_5 which give two choices.

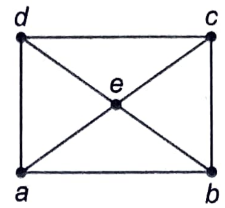
Hence $c_4 = 24 \times 2 = 48$ and $c_5 = 5! = 120$

$$P_5(\lambda) = \frac{6\lambda(\lambda-1)(\lambda-2)}{3!} + 48\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!} + 120\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{5!}$$

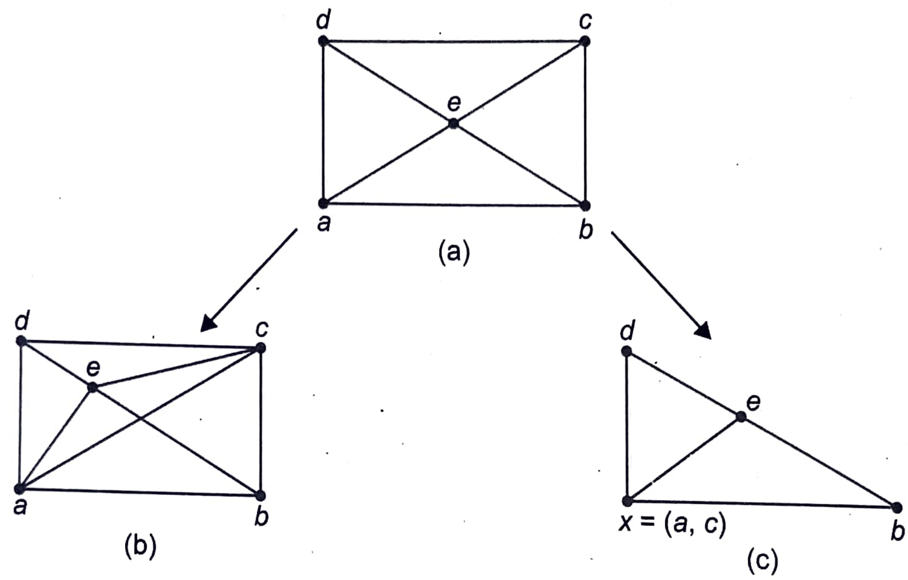
$$= \lambda(\lambda-1)(\lambda-2)[1 + 2(\lambda-3) + (\lambda-3)(\lambda-4)]$$

$$= \lambda(\lambda-1)(\lambda-2)(\lambda^2 - 5\lambda + 7).$$

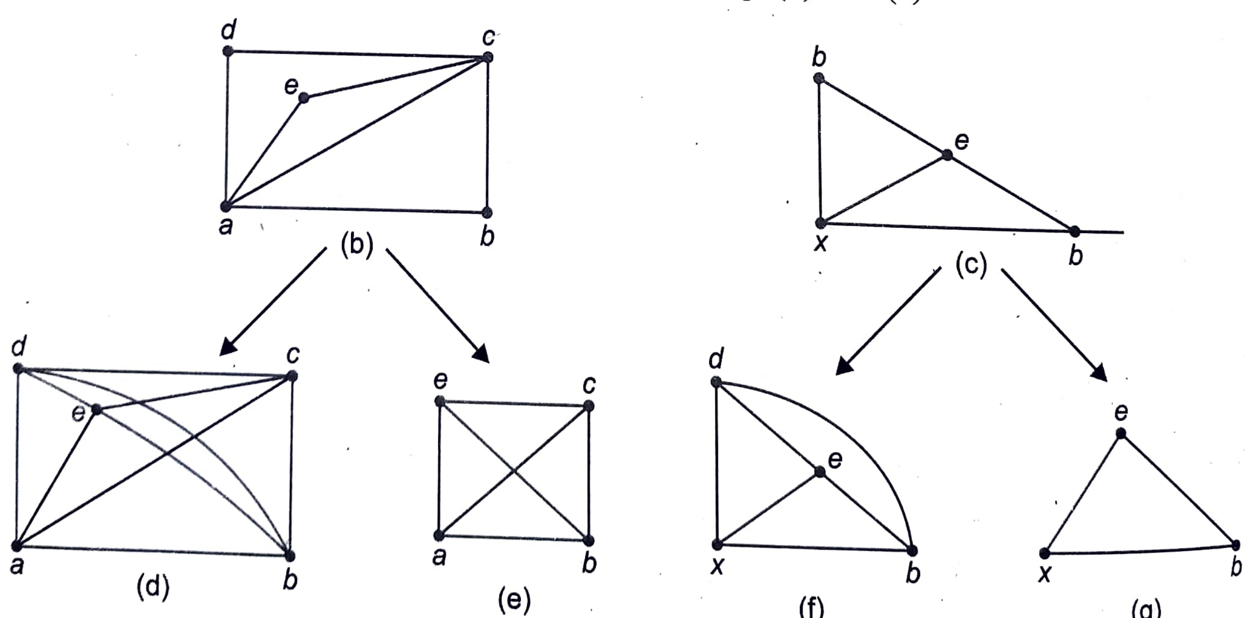
Example 17. Using decomposition theorem find the chromatic polynomial of the graph given below:



Solution: We choose two non-adjacent vertices a and c and draw an edge between a and c to obtain G'. We also fuse a and c to obtain G'' as shown in the following figure.



Again taking b and d as non-adjacent vertices in Fig. (b) and (c)



Now all the fours graphs (d), (e) , (f) and (g) are complete graphs K_5, K_4, K_4 and K_3 respectively. Hence the chromatic polynomial of the given graph G is

$$\begin{aligned} P_5(\lambda) &= [\lambda(\lambda-1)(\lambda-2)(\lambda-3) + (\lambda-4)] + [\lambda(\lambda-1) + (\lambda-2)(\lambda-3)] \\ &\quad + [\lambda(\lambda-1) + (\lambda-2) + (\lambda-3)] + [\lambda(\lambda-1)(\lambda-2)] \\ &= \lambda(\lambda-1)(\lambda-2)[(\lambda-3)(\lambda-4) + 2(\lambda-3) + 1] \\ &= \lambda(\lambda-1)(\lambda-2)(\lambda^2 - 5\lambda + 7). \end{aligned}$$

Example 18. Using decomposition theorem find the chromatic polynomial and hence the chromatic number for the graph given in Fig. 15.5 (a)

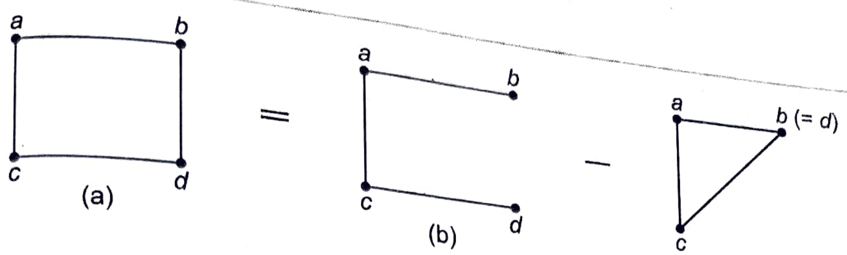


Fig. 15.5

Solution. Deleting the edge e from G, we get $G' = G - e$ as shown in Fig. 15.5 (b). Now G' is a tree with 4 vertices. Then the chromatic polynomial of G' is

$$P(G', \lambda) = \lambda(\lambda - 1)(\lambda - 2).$$

By merging the endpoints of e *i.e.*, b and d, we get G'' as shown in Fig. 15.5 (c). G'' is a complete graph K_3 and hence chromatic polynomial of G'' is

$$P(G'', \lambda) = \lambda(\lambda - 1)^3.$$

Now, by decomposition theorem, the chromatic polynomial of G is

$$P(G, \lambda) = \lambda(\lambda - 1)^3 - \lambda(\lambda - 1)(\lambda - 2)$$

$$= \lambda(\lambda - 1)[(\lambda - 1)^2 - (\lambda - 2).]$$

i.e.
$$P_4(\lambda) = \lambda(\lambda - 1)(\lambda^2 - 3\lambda + 3)$$

$$= \lambda^4 - 4\lambda^3 + 6\lambda^2 - 3\lambda.$$

$$\chi(G) = 2 \text{ since } P_4(1) = 0 \text{ and } P_4(2) \neq 0.$$

Theorem 15.16 The chromatic polynomial of any cycle C_n of length n is

$$P_n(\lambda) = (\lambda - 1)^n + (-1)^n(\lambda - 1).$$

Proof. The cycle C_n of length n has n vertices where $n \geq 3$. We prove the theorem by induction.

For $n = 3$, C_3 becomes K_3. Then, chromatic polynomial of $C_3 = \lambda(\lambda - 1)(\lambda - 2)$.

Also from the given statement

$$P_3(\lambda) = (\lambda - 1)^3 (-1)^3 (\lambda - 1)$$

$$= (\lambda - 1)(\lambda^2 - 2\lambda + 1 - 1)$$

$$= \lambda(\lambda - 1)(\lambda - 2).$$

$\therefore$ The given statement so true for $n = 3$.

Let us assume that the given statement is true for some positive integer $m \geq 3$.

Then for any cycle of length m, ...(1)

$$P_m(\lambda) = (\lambda - 1)^m + (-1)^m(\lambda - 1)$$

Let us consider the cycle C_{m+1} of length $(m + 1)$.

We delete an edge from the cycle C_{m+1}. Since deletion of an edge from a cycle does not alter the number of vertices in the cycle, the graph is a path, which is a tree with $(m + 1)$ vertices.

Therefore, its chromatic polynomial is $\lambda(\lambda - 1)^m$

Again contracting an edge from C_{m+1} yields a cycle of length m, which by inductive hypothesis has a chromatic polynomial $(\lambda - 1)^m + (-1)^m(\lambda - 1)$.

$\therefore$ By deletion-contraction property of chromatic polynomial, the chromatic polynomial for

C_{m+1}

$$= (\lambda - 1)^m - \{(\lambda - 1)^m + (-1)^m(\lambda - 1)\}$$

$$= (\lambda - 1)^m(\lambda - 1) - (-1)^m(\lambda - 1)$$

$$= (\lambda - 1)^{m+1} + (-1)^{m+1}(\lambda - 1)$$

Thus the given statement is true for $n = m + 1$.

Hence by principle of mathematical induction, the chromatic polynomial of any C_n of length n is

$$P_n(\lambda) = (\lambda - 1)^n + (-1)^n (\lambda - 1).$$

Chromatic Polynomial for certain graphs are:

Triangle K_3	$\lambda(\lambda - 1)(\lambda - 2)$
Path graph P_n	$\lambda(\lambda - 1)^{n-1}$
Complete graph K_n	$\lambda(\lambda - 1)(\lambda - 2) \ldots\ldots (\lambda - (n - 1))$
Tree with n vertices	$\lambda(\lambda - 1)^{n-1}$
Cycle C_n	$(\lambda - 1)^n + (-1)^n (\lambda - 1)$

Example 19. Find the chromatic polynomials of each of the six connected simple graphs on four vertices.

Solution. G $P_G(\lambda)$

 $= L_4$ $\lambda(\lambda - 1)^3 = \lambda^4 - 3\lambda^3 + 3\lambda^2 - \lambda$

 $= K_{1,3}$ $\lambda(\lambda - 1)^3 = \lambda^4 - 3\lambda^3 + 3\lambda^2 - \lambda$

 $= C_4$ $(\lambda - 1)^4 + (-1)^4 (\lambda - 1) = \lambda^4 - 4\lambda^3 + 6\lambda^2 - 3\lambda$

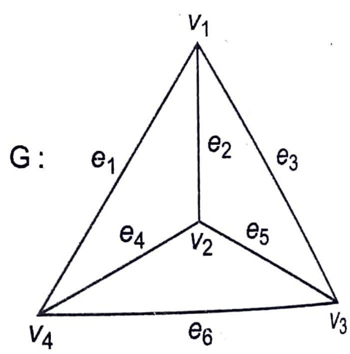
 $\lambda(\lambda - 1)^2 (\lambda - 2) = \lambda^4 - 4\lambda^3 + 5\lambda^2 - 2\lambda$

 $\lambda(\lambda - 1)(\lambda - 2)^2 = \lambda^4 - 5\lambda + 8\lambda^2 - 4\lambda$

 $= K_4$ $\lambda(\lambda - 1)(\lambda - 2)(\lambda - 3) = \lambda^4 - 6\lambda^3 + 12\lambda^2 - 6\lambda$

15.4. Edge Colouring

An edge colouring of a graph G is an assignment of colours to the edges of G so that no two edges with a common vertex receive the same colour. The minimum number of colours required in an edge colouring of G is called the **chromatic index** of G and is denoted by $\chi'(G)$. Fig. 15.23 shows a graph G for which $\chi'(G) = 3$.



edge	e_1	e_2	e_3	e_4	e_5	e_6
co-lour	1	2	3	3	1	2

If G has K-edge colouring, then G is said to be K edge colourable.

Lower Bound of $\chi'(G)$

To obtain a lower bound for $\chi'(G)$, we look for the largest vertex-degree $\Delta(G)$ in given G, which gives

$$\chi'(G) \geq \Delta(G)$$

Upper Bound of $\chi'(G)$

To obtain an upper bound for $\chi'(G)$, we note that if G has m edges, then

$$\chi'(G) \leq m$$

However, this upper bound has been improved by V. G. Vizing and C.E. Shannon.

Theorem 15.17 (Vizing Theorem): If G be a simple graph with maximum vertex-degree $\Delta(G)$, then

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1.$$

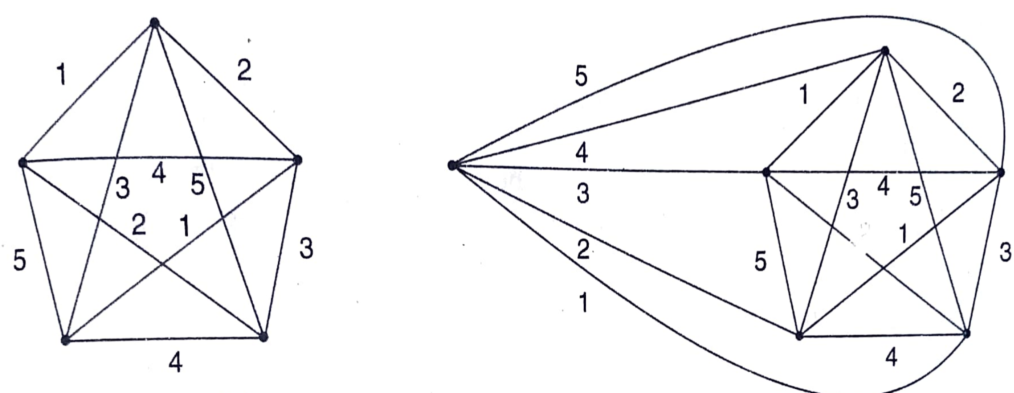
Theorem (Konig's theorem)

$$\chi'(G) \leq \Delta(G) \text{ for all bipartite graph.}$$

Theorem 15.18 $\chi'(K_n) = n$ if n is odd $(n \neq 1)$, and $\chi'(K_n) = n - 1$ if n is even.

Proof: The result is trivial if $n = 2$. We, therefore assume $n \geq 3$.

If n is odd, any matching in K_n can have at most 1/2 $(n - 1)$ edges. So at most 1/2 $(n - 1)$ edges can be given any colour. But there are 1/2 $n(n - 1)$ edge in K_n, so at least n colours are needed. We can colour the edges of K_n in the following way. Represent K_n as a regular n-gon, with all diagonals drawn. Colour the edges around the boundary with a different colour for each edge, and then colouring each remaining edge parallel to it. This gives an edge colouring using n colours. The case $n = 5$ is illustrated in Fig. 15.6 (a).



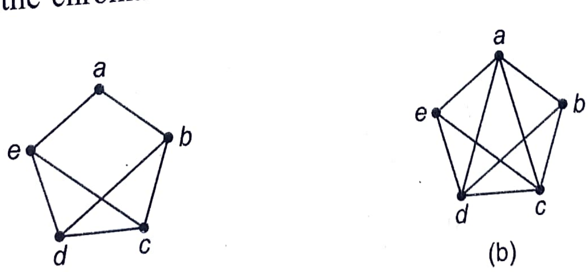
(a) (b)

Fig. 15.6.

If n is even, we can colour K_{n-1}, using $n - 1$ colours as described above. Now we first obtain K_n by joining the complete graph K_{n-1} to a single vertex, then there is one colour missing at each vertex, and these missing colours are all different. So we can use these n–1 colours to colour the added edges. This gives an edge colouring of K_n using n–1 colours. This has been illustrated in Fig. 15.6 (b).

Note: $\chi'(C_n) = 2$ or 3 depending on whether n is even or odd.

Example 20. Find the chromatic index (or edge-chromatic number) of the graph G, where G is:



(a) (b)

Solution.

(a) Here G has maximum degree 3 (all vertices having degree 3 except a, which has degree 2). So obviously at least 3 colours are needed to edge colour G properly ($e.g.$, the three edges incident to b require three different colours). By Vizing's theorem, $\chi'(G)$ is either 3 or 4.

Try to colour G properly with just 3 colour: R, W and B, say.

Let we colour, $(a,b) = R$, $(b, c) =$ and $(d, e) = W$.

... lour is needed to colour (a, e) properly. Hence $\chi'(G) = 4$.

(b) Vizing's theorem here show $\chi'(G)$ is either 4 (the maximum vertex degree) or 5. Try using just 4 colours: R, B, W and Y, say.

Let $(a, b) = R$, $(a, c) = W$, $(a, d) = B$ and $(a, e) = Y$. Then (c, d) cannot be coloured with W or B, so is either R or Y— by symmetry we can assume $(c, d) = R$.

Hence $(d, e) = W$, and so $(b, d) = Y$ and then $(b, c) = B$, and now a fifth colour is needed for (c, e). Hence $\chi'(G) = 5$.

15.5. Applications of Colouring

There are many applications of colouring of graphs. We discuss here two applications

1. A Storage Problem

A company manufactures n chemicals $C_1, C_2, \ldots, C_n$. Certain pairs of these chemicals are incompatible and would cause explosions if brought into contact with each other. As a precautionary measure the company wishes to partition its warehouse into compartments, and store incompatible chemicals in different compartments. What is the least number of compartments into which the warehouse should be partitioned?

We obtain a graph G on the vertex set $(v_1, v_2, \ldots, v_n)$ by joining two vertices v_i and v_j if and only if the chemicals C_i and C_j are incompatible. It can easily be seen that the least number of compartments into which the warehouse should be partitioned is equal to the chromatic number of G since the chromatic number of a graph is the least number of independent sets into which its vertex set can be partitioned.

2. Examination Schedule Problem

How can the examination at a university be scheduled so that no student has two examinations at the samė time?

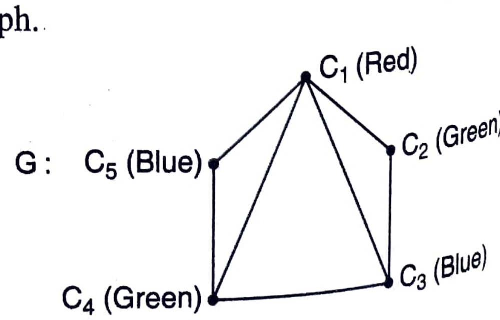
Here vertices corresponds to courses and there is an edge between two vertices if there is a common student in the courses. Each time slot is repeated by different colours. Thus a scheduling of the examinations corresponds to a colouring of the associated graph.

Example 21. A company has to store five chemicals C_1, C_2, C_3, C_4, C_5 in a warehouse in compartments. Some of these chemicals are incompatible with some other chemicals and they cannot be kept together. In the table, an asterisk indicates the chemicals can't be kept together. What is the minimum number of compartments need to avoid any explosion?

	C_1	C_2	C_3	C_4	C_5
C_1	—	*	*	*	*
C_2	*	—	*	—	—
C_3	*	*	—	*	—
C_4	*	—	*	—	*
C_5	*	—	—	*	—

Solution. We construct a graph G with varieties of chemicals as vertices. There is an edge between two vertices if the chemicals cannot be kept together. Then the minimum number of compartment required is the chromatic number of the graph.

Since the graph contains triangles as subgraph of G, $\chi(G) \geq 3$. The maximum degree, $\Delta(G) = 4$. Since the graph G is connected and not complete graph by Book's therorem $\chi(G) \leq 4$. The graph is 3 colourable as shown and so its chromatic number is 3. Hence the minimum number of compartment required is 3.



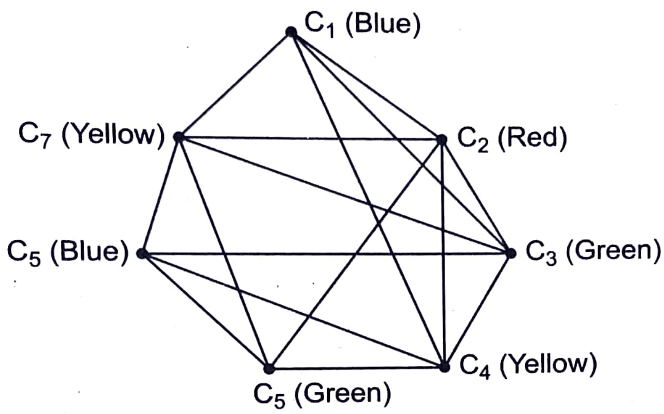
Example 22. Seven courses $C_1, C_2, ..., C_7$ are to be scheduled at a university examinations, where the following pains of courses have common students

(C_1, C_2); (C_1, C_3), (C_1, C_4), (C_1, C_7), (C_2, C_3), (C_2, C_4), (C_2, C_5), (C_2, C_7), (C_3, C_4), (C_3, C_6), (C_3, C_7), (C_4, C_5), (C_4, C_6), (C_5, C_6), (C_5, C_7) and (C_6, C_7)

How can the examination be scheduled so that no students has two examination at the same time?

Solution. We draw the graph where vertices represent the courses. The edges between the vertices are drawn if there is a common student in the course represented by the vertices.

Each time slot of the examination is represented by different colours. A scheduling of the examination corresponds to the colouring of the associated graph G.



The graph has K_4 as a subgraph induced by C_1, C_2, C_3 and C_4. Hence the chromatic number $\chi(G) \geq 4$. We see that the graph is 4 colourable. Hence $\chi(G) = 4$ so the seven courses can be scheduled in minimum four time slots as given below:

Time slot	I	II	III	IV
Courses	C_1, C_6	C_2	C_3, C_5	C_4, C_7

15.6. Matching

Given a graph $G = (V, E)$, a subset M of the edges is said to be a matching if no two of edges in M are incident on the same vertex *i.e.* no two edges are adjacent.

A single edge in a graph is a matching.

A vertex is said to be matched or **saturated** if it is an end point of one of the edges in the matching; otherwise the vertex is said to be unmatched.

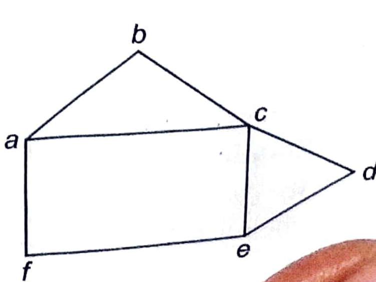
Maximal Matching: A maximal matching is a matching to which no edge can be added without violating the matching property. In other words, M is a maximal matching if it is not a proper subset of any other matching in the graph G.

Note (*i*) A graph may have many different maximal matchings with different sizes.

(*ii*) In K_3, every single edge forms a maximal matching.

Maximum Matching: A maximum matching is a maximal matching that contains the largest possible number of edges. There may exist more than one maximum matching in a graph. The size of a maximum matching M is denoted by $\nu(G)$ and is called the **matching number** of G.

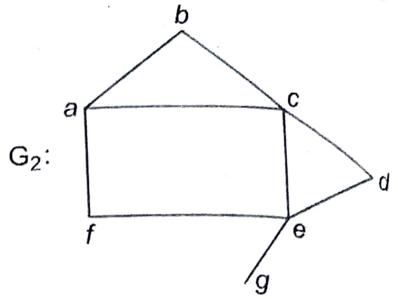
G_1:



In G_1

$M_1 = \{(a, b), (c, e)\}$ and $M_2 = \{(a, b), (c, d), (e, f)\}$ are both matching and they are maximal matching but M_2 is a maximum matching and $\gamma(G_1) = 3$. In M_1 the vertices a, b, c and e are saturated while f and d are unsaturated.

G_2:

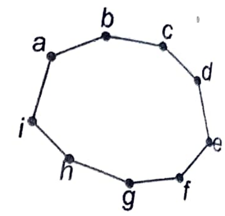


In G_2,

$$M_1 = \{(b, c), (a, f), (e, g)\}$$
$$M_2 = \{(a, b), (e, g), (c, d)\}$$
$$M_3 = \{(b, c), (a, f), (e, d)\}$$

are all maximal matchings and they are also maximum matchings. So, $\gamma(G_2) = 3$.

Example 23. C_9 is a cycle (i.e. a circular chain) with nine vertices $a, b, c, d, e, f, g, h, i$. How many distinct maximum matching of size five in C_9 contain the edge ab ?



Solution. There are only four maximal matchings of size four in C_9 each containing the edge $(a, \underline{b})$ and these are

$M_1 = \{(a, b), (c, d), (e, f), (g, h)\}$ $M_2 = \{(a, b), (c, d), (e, f), (h, i)\}$

$M_3 = \{(a, b), (d, e), (f, g)(h, i)\}$

$M_4 = \{(a, b), (c, d), (f, g), (h, i)\}$

Perfect Matching: A matching M is a perfect matching if and only if, every vertex of G is saturated *i.e.*, if and only if every vertex of G is incident to exactly one edge of the matching.

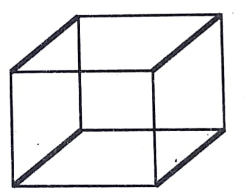
In G_1, M_2 is perfect matching since every vertex of G_1 is incident in source edge in M_2

Note: (*i*) Not all graphs have perfect matching.

(*ii*) Every perfect matching is a maximum matching but the converse is not tree. In G_2, the matchings M_1, M_2 and M_3 are maximum matchings but none of these are perfect matching.

(*iii*) In n-cube Q_n $(n \geq 1)$ each maximal matching has 2^{n-1} edges and it is a perfect matching.

Q_3:



Theorem 15.19 The number of perfect matchings in K_n is o if n is odd and $(n-1)(n-3)$ ..... 5 . 3 . 1 if n is even.

Proof: If $n = 1$, there is no perfect matching and if $n = 2$, there is only 1 perfect matching.

Assume the theorem for all complete graph with k vertices where $k < n$. K_n has no perfect matching if n is odd, we assume n even.

Let us consider vertex v_1 which can be matched to any of the other $n-1$ vertices. Suppose v_1 is matched to v_2. Then remove v_1, v_2 and all edges incident to v_1 and v_2 to obtain the graph K_{n-2} with vertices $\{v_3, v_4, \ldots\ldots v_n\}$. By inductive hypothesis, since $n-2$ is even, the member of perfect matches in k_{n-2} is $(n-3)(n-5) \ldots 5 . 3 . 1$. Since there are $n-1$ ways to match v_1, and for each of these there are $(n-3)(n-5) \ldots. 5 . 3 . 1$ ways to match the remaining $n-2$ vertices. The total number of perfect matches is

$$(n-1)(n-3)(n-5) \ldots\ldots\ldots 5 . 3 . 1.$$

Note (*i*) In K_6, the number of matches is

$$5 \times 3 \times 1 = 15.$$

Example 24. Let a_n denotes the number of perfect matching in the complete graph K_{2n}. Find a recurrence relation for a_n and solve.

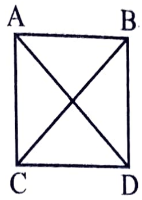
Solution. Let $V(K_{2n}) = (v_1, v_2, \ldots., v_{2n}\}$.

We can pair the vertex v_1 with any of the other $2n-1$ vertices, we are then confronted in the case when $n \geq 2$, with finding a perfect matching for the graph K_{2n-2}. Hence

$$a_n = (2n-1)a_{n-1}, \qquad a_1 = 1$$

We find

$$\begin{aligned} a_n &= (2n-1)a_{n-1} = (2n-1)(2n-3)a_{n-2} \\ &= (2n-1)(2n-3)(2n-5)a_{n-3} \\ &= (2n-1)(2n-3)(2n-5)\ldots(5)(3)(1) \\ &= \frac{(2n)(2n-1)(2n-2)(2n-3)\ldots(4)(3)(2)(1)}{(2n)(2n-2)\ldots(4)(2)} = \frac{(2n)!}{2^n.n!}. \end{aligned}$$

For example, the graph K_4: 

has $\dfrac{4!}{2^2 . 2!} = 3$ perfect matchings and they are $M_1 = \{AC, BD\}$, $M_2 = \{AB, DC\}$ and $M_3 = \{BC, AD\}$.

Example 25(a). Determine necessary and sufficient conditions for the complete bipartite graph $k_{m,n}$ to have a perfect matching.

Solution. First we assume $m = n$. Let the vertex sets be $V_1 = \{u_1, u_2, \ldots, u_m\}$ and $V_2 = \{v_1, v_2, \ldots, v_m\}$. Then $\{u_1v_1, u_2v_2, \ldots, u_mv_m\}$ is a perfect matching. Conversely, let we have a perfect matching and $m \le n$. Since, each edge in a matching must join a vertex V_1 to vertex a V_2, there can be at most m edges.

In $m < n$, some vertex in V_2 would not be a part of any edge in the matching which is a contradiction. Thus $m = n$.

Example 25(b). Find the number of perfect matchings in the complete bipartite graph $k_{n,n}$.

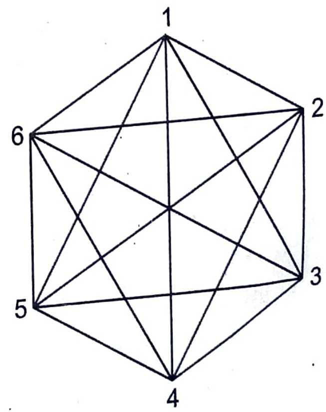
Solution. Let $V_1 = \{u_1, u_2, \ldots u_n\}$ and $V_2 = \{v_1, v_2, \ldots v_n\}$ be a bipartition of $k_{n,n}$.

We see that any matching of $k_{n,n}$ that saturates vertex of V_1 is a perfect matching. The vertex u_1 can be saturated in n ways by choosing any of the edges $u_1v_1, u_1v_2, \ldots u_1v_n$. Having saturated u_1, the vertex u_2 can be saturated in $n - 1$ ways. In general having saturated $u_1, u_1, \ldots, u_i$, the next vertex u_{i+1} can be saturated in $n - i$ ways. Thus the number of perfect matchings in $k_{n,n}$ is $n(n-1)(n-2)\ldots 3.2.1 = n!$

Example 26. Write down the perfect matchings of K_6.

Solution. The number of perfect matchings are 15 and they are

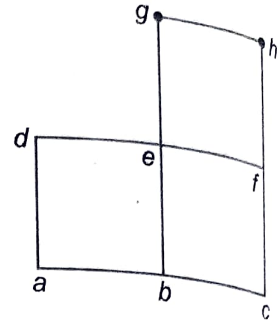
$M_1 = \{(1, 2), (3, 4), (5, 6)\}$
$M_2 = \{(1, 2), (3, 5), (4, 6)\}$
$M_3 = \{(1, 2), (3, 6), (4, 5)\}$
$M_4 = \{(1, 3), (2, 6), (4, 5)\}$
$M_5 = \{(1, 3), (2, 5), (4, 6)\}$
$M_6 = \{(1, 3), (2, 4), (3, 6)\}$
$M_7 = \{(1, 4), (2, 3), (5, 6)\}$
$M_8 = \{(1, 4), (2, 5), (3, 6)\}$
$M_9 = \{(1, 4), (2, 6), (3, 5)\}$
$M_{10} = \{(1, 4), (2, 3), (4, 6)\}$
$M_{11} = \{(1, 5), (2, 4), (3, 6)\}$

K_6: 

$M_{13} = \{(1, 6), (2, 3), (4, 5)\}$
$M_{14} = \{(1, 6), (2, 4), (3, 5)\}$

$M_{12} = \{(1, 5), (2, 6), (4, 3)\}$ $M_{15} = \{(1, 6), (2, 5), (4, 3)\}$

Definition: For matching M in a graph G, an **M-alternating path** is a simple path whose edges are alternately matched and unmatched *i.e.* in which the edges belong alternatively between M and not in M.



An M-augmenting path is an M-alternating path that starts and ends at unsaturated vertices.

Maximum matchings have been characterized by the following theorem.

Theorem 15.20 The matching M of a graph G is maximum if, and only if, G has no M-augmenting path.

The matching $M_1 = \{ad, be, fh\}$ is a maximal matching in the adjacent graph but not maximum matching. The alternating path $cb - be - eg$ is an augmenting path since two end points c and g are unsaturated.

The maximum matching in $M_2 = \{cb, ad, ef, hg\}$. The alternating path is $cb - ba - ad - de - ef - fh - gh$.

Application of Matching

- pairing of compatible partners
  - perfect matching: nobody "left out"
- jobs and qualified workers
  - perfect matching: full employment, and all jobs filled
- clients and servers
  - perfect matching: all clients served, and no server idle

Matchings in bipartite graphs

Matching problems are often concerned with bipartite graphs. The following problems motivate the study of matchings in bipartite graphs.

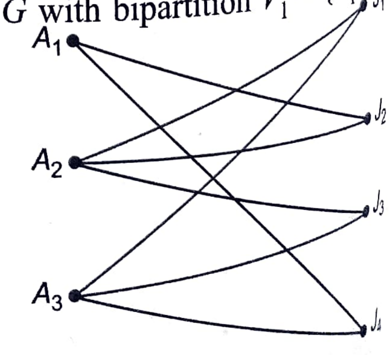
(*i*) Assignment Problem

Suppose three persons A_1, A_2 and A_3 apply for four jobs J_1, J_2, J_3 and J_4 in a company. Each person is qualified for one or more jobs. The following information is obtained from the applications of the candidates.

Applicants	Jobs
A_1	J_2, J_4
A_2	J_1, J_2, J_3
A_3	J_1, J_3, J_4

Is it possible to assign one job for each applicant for which he is qualified?

The given information can be represented a bipartite graph G with bipartition $V_1 = \{A_1, A_2, A_3\}$ and $V_2 = \{J_1, J_2, J_3, J_4\}$, where V_1 represents applicants and V_2 the jobs. An edge joins A_i to J_f if and only if A_i is qualified for the job J_f.



Then the assignment problem translates into the graph problem: Is it possible to find a matching in G that saturates all the vertices in V_1?

A particular solution to the above problem is for the match $M = \{A_1J_2, A_2J_1, A_3J_3\}$ A_1 is assigned job J_2, A_2 is assigned job

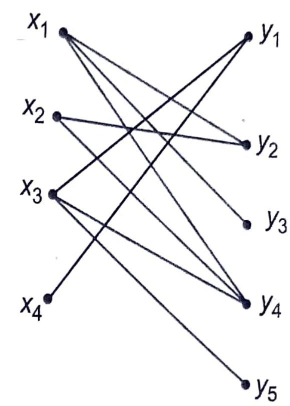
J_1 and A_3 is assigned job J_3. Thus each applicant can be assigned one of the jobs for which he or she is qualified.

If there are more applicants than the jobs *i.e.* $|V_1| > |V_2|$, then it is not possible to provide job to every applicant. Thus the condition $|V_1| \leq |V_2|$ is the necessary condition for the assignment of job to every applicant, that is the set of vertices v_1 will be saturated.

(ii) The Marriage Problem

Let we have n boys and each boy has one or more girl friends. Under what conditions can we arrange marriage in such a way that each boy marries one of his girl friends? This problem is known as the marriage problem. (We assume no boy can marry more that one girl).

We construct a bipartite graph G with bipartition $V(G) = A \cup B$ where $A = \{x_1, x_2, ..., x_n\}$ represents the set of n boys and $B = \{y_1, y_2, ..., y_m\}$ represents their girl friends, m in all. An edge joins vertex x_i to vertex y_j if and only if y_j is a girl friend of x_i. For example, if there are four boys, x_1, x_2, x_3, x_4 and five girls y_1, y_2, y_3, y_4, y_5 and the relationships are given by



Boy	Girl friend
x_1	y_2, y_3, y_4
x_2	y_2, y_4
x_3	y_1, y_4, y_5
x_4	y_1

The graph drawn from the given correspondence is a bipartite graph. A particular solution to the above problem is the matching $M = \{x_1y_3, x_2y_2, x_3y_4, x_4y_1\}$, x_1 to marry y_3, x_2 to marry y_2, x_3 to marry y_4 and x_4 to marry y_1. Note that though the vertex y_5 is not an end vertex of any edge of M, every vertex of A is end vertex of some edge of M.

We observe that for the solution of the marriage problems, every set of k boys must have at least k girlfriends for $1 \leq k \leq n$. This condition is known as marriage condition. Thus marriage condition is also turns out to be sufficient.

Theorem 15.21 (Hall's marriage theorem) Let $G = (V, E)$, $V = V_1 \cup V_2$ be a bipartite graph suck that $|V_1| \leq |V_2|$. Then G has a complete (perfect) matching that saturates all the vertices of V_1 if and only if,

$$|N(S)| \geq |S|$$

for every subset S of V_1, where $N(S)$ is the neighbourhood subset of V_2 that are adjacent to some vertex in S.

The theorem is known as Hall's marriage theorem because originally elements in X were thought of as women and elements is Y were thought of as men. A matching between the two set corresponded to marrying each woman to a man.

The following is an important consequence of Hall's marriage theorem.

Theorem 15.22 Let G be a k-regular bipartite graph with $k > 0$. Then G has a perfect matching.

Proof. Let G have bipartition $V = X \cup Y$. There are $|X|$ vertices in X and each of these vertices has k edges (all going to vertices in Y) incident with it. Thus there are $k \times |X|$ edges going from X to Y. Similarly since each of the $|Y|$ vertices in Y has k edges incident with it there are $k \times |Y|$ edges going from Y to X. By the bipartite nature of G each edge goes from X to Y (the same as from Y to X) and so G has $k|X| = k|Y|$ edges. Since $k > 0$ and $k|X| = k|Y|$, we get

$$|X| = |Y|$$

... the set of edges incident with vertices in S. Since

G is k-regular, we have

$$|E_1| = k|S| \quad ...(1)$$

Let E_2 denote the set of edges incident with vertices in $N(S)$. Then, since $N(S)$ is the set of vertices which are joined by edges to S, we have $E_1 \subset E_2$. Thus

$$|E_1| \leq |E_2|. \quad ...(2)$$

Again by the k-regularity of G we have

$$|E_2| = k|N(S)|. \quad ...(3)$$

From (1), (2) and (3), we get $k|N(S)| = |E_2| \geq |E_1| = k|S|$ and so

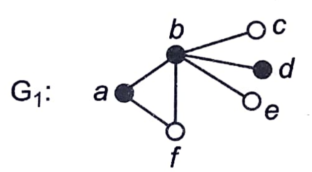
$$k|N(S)| \geq k|S|$$

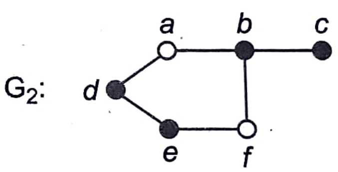
Since $k > 0$ $|N(S)| \geq |S|$. Since S was an arbitrary subset of X, it follows from Hall's marriage theorem that G contains a matching M which saturates every vertex in X. Since $|X| = |Y|$ the edges in the matching M also saturate every vertex in Y. Thus M is a perfect matching in G.

15.7. Graph Covering

An edge (u, v) of a graph G covers the vertices u and v, and a vertex of G is said to cover the edges with which it is incident. Similarly, a vertex G is said to cover the edges with which it is incident. Covering with vertices and edges are closely related with independent set and matching.

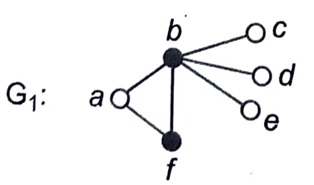
Vertex Covering: A vertex covering of a graph $G(V, E)$ is set of vertices $C \subseteq V$ such that each edge of G is incident to at least one vertex in C. The set C is said to cover the edges of G. The following figure shows examples of vertex covering in two graphs (the set C is marked dark)

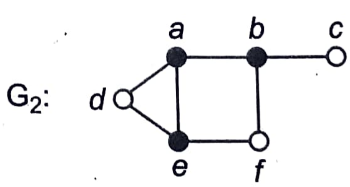




Here $C_1 = \{a, b, d\}$ in G_1 and $C_2 = \{b, c, d, e\}$ A vertex covering $c \subseteq v$ of a graph G is said to be minimal covering if no vertex can be removed without destroying its ability to cover G.

A graph may have many minimal coverings of different sizes. A **minimum vertex covering** is a minimal vertex covering of smallest possible size. The **vertex covering number** T is the size of a minimum vertex covering. The following figure shows examples of minimum vertex covering in the previous graphs





Note: (*i*) The set of all vertices for any graph G is trivially a vertex covering of G.

(*ii*) The end points of any maximal matching form a vertex covering.

(*iii*) The complete bipartite graph $K_{m,n}$ has a minimum covering of size $T(K_{m,n}) = \min(m, n)$

(*vi*) For any graph $G(V, E)$, $T(G)$ + maximum independent set = number of vertices in V.

Edge Covering: An edge covering of a graph $G(V, E)$ is a set of edges $L \subseteq E$ such that each vertex in G is incident with at least one edge in L. The set L is said to *cover* the vertices of G. The following figure shows examples of edge coverings in two graphs.

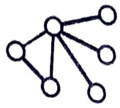
Theorem 15.23. If G is a bipartite graph, then the maximum size of a matching in G is equal to the minimum size of a vertex covering of G.

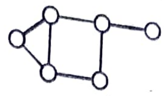
G_1: 

G_2: 

An edge covering L in G is said to be minimal covering if no edge of L can be removed without destroying its ability to cover the graph.

An **minimum edge covering** is a minimal edge covering of smallest possible size. The **edge covering number** $\rho(G)$ is the size of a minimum edge covering. The following figure shows examples of minimum edge coverings. Here $\rho(G_1) = 4$ and $\rho(G_2) = 3$.

G_1: 

G_2: 

Note that the figure on the right is not only an edge cover but also a matching. In particular, it is a perfect matching: a matching M in which each vertex is incident with exactly one edge in M. A perfect matching is always a minimum edge covering.

Note: (*i*) The set of all edges is an edge cover, assuming that there are no degree-0 vertices.

(*ii*) The complete bipartite graph $K_{m,n}$ has edge covering number $L = \max(m, n)$.

(*iii*) Every edge covering of G includes all the pendant edges of the graph.

Theorem 15.23 Let a graph have a matching M and covering C. Then $|M| \le |C|$. Moreover, if $|M| = |C|$, then M is a maximum matching and C is a minimum covering.

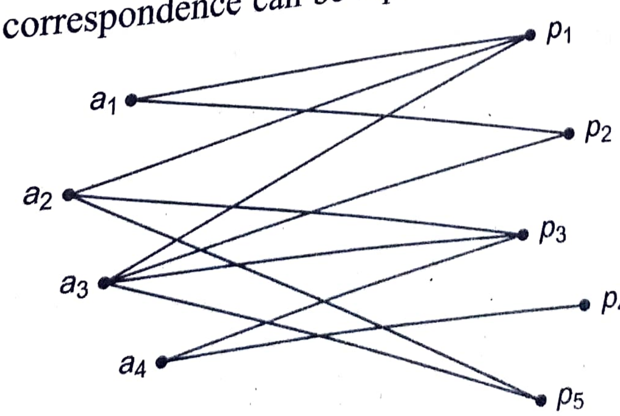
Proof. From the definition of a covering, every edge of the graph, and in particular every edge in M, is incident with some vertex in C. If the edge e is in M, let $v(e)$ be a vertex in C incident with e. Now if e_1 and e_2 are distinct edges in M, then $v(e_1)$ and $v(e_2)$ are also distinct, since, by definition, two edges in a matching cannot share a vertex. Thus there are at least as many vertices in C as edges in M, and so $|M| \le |C|$.

Now suppose $|M| = |C|$. If M were not a maximum matching, there would be a matching M' with $|M'| > |M| = |C|$, contradicting the first part of the theorem. Similarly, if C were not a minimum covering, these would be a covering with fewer than $|M|$ vertices which leads to the same contradiction.

Example 27. Applicants a_1, a_2, a_3 and a_4 apply for five posts p_1, p_2, p_3, p_4 and p_5. The application are done as follows: $a_1 \to \{p_1, p_2\}$, $a_2 \to \{p_1, p_3, p_5\}$, $a_3 \to \{p_1, p_2, p_3, p_5\}$ and $a_4 \to \{p_3, p_4\}$. Using graph theory find (*i*) whether there is any perfect matching of the set of applicants into the set of posts.

Find matching (*ii*) whether every applicant can be offered a single post.

Solution. The given correspondence can be represented in the following graphical form:



This is a bipartite graph with the two disjoint vertex set·

$$V_1 = \{a_1, a_2, a_3, a_4\}$$

and

$$V_2 = \{p_1, p_2, p_3, p_4, p_5\}.$$

(*i*) Here we see the edge set $\{a_1p_1, a_2p_3, a_3p_5, a_4p_4\}$ is maximum matching which is not perfect since every vertex of V_1 is not incident in some edge in V_2.

This graph has no perfect matching

(*ii*) This can be treated as a Marriage Problem. We construct the following table:

Subset of vertices in V_1		Sub-set of adjacent vertices in V_2	
S	$\lvert S \rvert$	$N(S)$	$\lvert N(S) \rvert$
$\{a_1\}$	1	$\{p_1, p_2\}$	2
$\{a_2\}$	1	$\{p_1, p_3, p_5\}$	3
$\{a_3\}$	1	$\{p_1, p_2, p_3, p_5\}$	4
$\{a_4\}$	1	$\{p_3, p_4\}$	2
$\{a_1a_2\}$	2	$\{p_1, p_2, p_3, p_5\}$	4
$\{a_1a_3\}$	2	$\{p_1, p_2, p_3, p_5\}$	4
$\{a_1a_4\}$	2	$\{p_1, p_2, p_3, p_4\}$	4
$\{a_2a_3\}$	2	$\{p_1, p_3, p_5, p_2\}$	4
$\{a_2a_4\}$	2	$\{p_1, p_3, p_5, p_4\}$	4
$\{a_3a_4\}$	2	$\{p_1, p_2, p_3, p_5, p_4\}$	5
$\{a_1a_2a_3\}$	3	$\{p_1, p_2, p_3, p_5\}$	4
$\{a_1a_2a_4\}$	3	$\{p_1, p_2, p_3, p_5, p_4\}$	5
$\{a_2a_3a_4\}$	3	$\{p_1, p_3, p_5, p_2, p_4\}$	5
$\{a_1a_3a_4\}$	3	$\{p_1, p_2, p_3, p_5, p_4\}$	5
$\{a_1a_2a_3a_4\}$	4	$\{p_1, p_2, p_3, p_5, p_4\}$	5

In the above table we see, in every row cardinal number of every set in 4th column is ≥ cardinal number of the set in 2nd column.

Thus by Hall's theorem a solution of this Marriage problem exist which shows that matching is perfect that is every applicant can be offered a single post.

15.8 Network Flows

In this section we consider the arcs of a digraph as pipes through which some commodity (such as number of cars, gallons of oils, bits of information, etc.) is transported from one place to another. The weight on an arc represents the capacity of the pipe, the maximum amount of some commodity that can flow through it in a given unit of time. The general problem in such a situation is to find the maximum flow in all these types of transmission network.

For example,

(*i*) If each arc represents a one-way street and the number of each arc is the maximum flow of traffic along that street, in vehicles per hour, then one may be interested to find the greatest possible number of vehicles that can travel from u to v in one hour.

(*ii*) The network can represent links in a computer network with data transmission capacities. Given two locations s, t in the network, one may be interested to find maximum flow of data (per unit time) from s to t.

Definitions

A **network or transport network** is a simple, connected, weighted directed graph $N(V, E)$ satisfying the following conditions :

(*i*) There is a unique vertex $s \in V$ if it has in-degree 0. This vertex is called the **source.**

(*ii*) There exists a unique vertex $t \in V$ if it has out-degree 0. This vertex is called the **sink.**

(*iii*) Every directed edge $e = (v, w) \in E$ has been assigned a non-negative number called the **capacity** of e, denoted by $c(e) = c(v, w)$. We can think of $c(e)$ as representing the maximum rate at which a commodity can be transported along the edge e.

We assume that the network has exactly one source s and one sink t. Any other vertex of N is called an **intermediate vertex.**

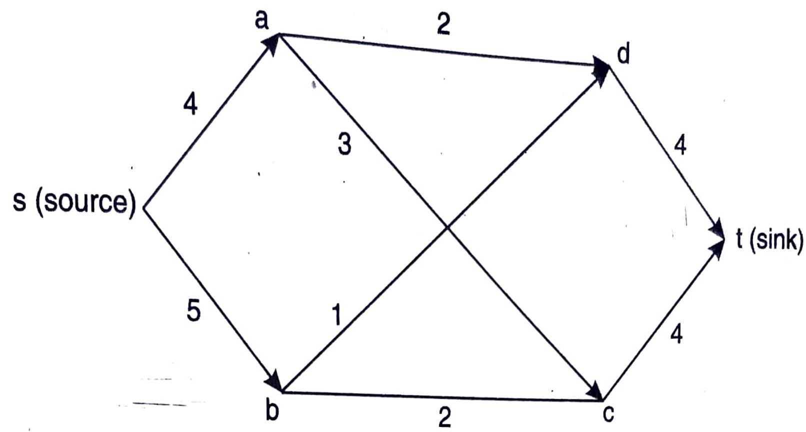


Fig. 15.7. A network showing the capacity on each arc.

A **flow** in a network is a function f that assigns to each arc $e = (v, w) \in E$ a non negative real number such that

(*i*) (**Capacity constraint**) $f(e) \le c(e)$ for each $e \in E$

i.e., $f(v, w) \le c(v, w)$ for each edge (v, w)

(*ii*) (**Flow conservation**) For any intermediate vertex x, total flow into x equals to the total flow out of x. Symbolically

$$\sum_{w \in V} f(w, x) = \sum_{v \in V} f(x, v)$$

(*iii*) $f(e) = 0$ for any edge e incident to the source s or incident from the sink t.

Conditions (*i*) ensures that the amount of material transported along a given edge cannot exceed the given capacity of that edge. (*ii*) enforces that the sum of all incoming flows at vertex x is equal to the sum of all outgoing flows at x and (*iii*) ensures that the flow moves from source to sink and not in the opposite direction.

The flow along an edge $e\,(v, w)$ is said to be **saturated** if $f(e) = c(e)$ *i.e.*, $f(v, w) = c(v, w)$ and if $f(e) < c(e)$, then the flow is said to be **unsaturated.** If an edge is unsaturated, then the **slack or** residual capacity of e in a flow f is defined to be $s(e) = c(e) - f(e)$.

The **value of a flow** is the net amount of flow per unit time leaving the source or equivalently, the net amount of flow per unit time entering the sink. Symbolically

$$val(f) = \sum_{v \in V} f(s, v) \quad \text{where } s \text{ is the source.}$$

$$= \sum_{v \in V} f(v, t) \quad \text{where } t \text{ is the sink}$$

i.e., the total outgoing flow at the source is equal to the total incoming flow at the sink.

Our problem is to calculate the maximum value of a flow for a given network without exceeding the capacity of each edge.

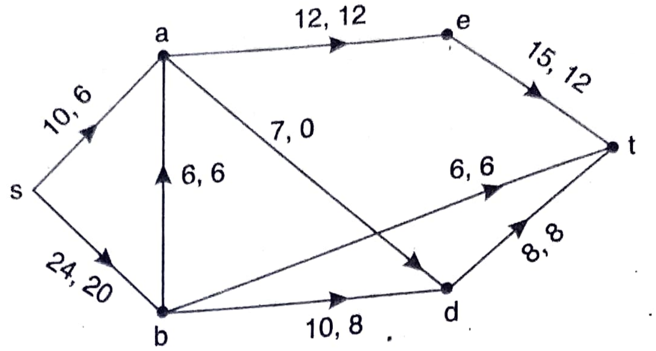


Fig. 15.8

Fig. 15.8 shows a directed network with a source s and sink t. The first number of a label on each edge represents its capacity and the second number represents the flow through it. Thus the label 10, 6 on edge (s, a) means that this arc has capacity 10 and currently a flow of 6. Note that the law of conservation of flow is satisfied at each vertex. for example, consider the vertex a.

The flow into a is $\sum_{v \in V} f(v, a) = f(s, a) + f(b, a) = 6 + 6 = 12$

The flow away from a is $\sum_{v \in V} f(a, v) = f(a, e) + f(a, d) = 12 + 0 = 12$

The value of a flow f, val $(f) = \sum_{v \in V} f(s, v) = f(s, a) + f(s, b) = 6 + 20 = 26$

or $\quad$ val $(f) = \sum_{v \in V} f(v, t) = f(e, t) + f(b, t) + f(d, t) = 12 + 6 + 8 = 26$

The following edges are saturated

(b, a), (a, e) (b, t) and (d, t) as the capacity and the flow are same.

Since $10 < 6$ *i.e.*, $f(s, a) < c(s, a)$, the edge (s, a) is unsaturated. Again $24 < 20$ *i.e.*, $f(s, b) < c(s, b)$, the edge (s, b) is unsaturated.

Similarly the edges (a, d), (b, d) and (e, t) are unsaturated.

The slack of the edges (s, a), (s, b), (a, d), (b, d) and (e, f) are $10 - 6 = 4$, $24 - 20 = 4$, $7 - 0 = 7$, $10 - 8 = 2$ and $15 - 12 = 3$ respectively.

A **maximum flow** or **maximal flow** in a net work is a flow that achieves the largest possible value.

i.e., a flow in N is a maximum flow if there is no flow f in N such that val $(f') >$ val (f).

It is to be noted that in a network there may be more than one maximal flow.

Cut and its Capacity

If $N(V, E)$ is a network and C is a cut-set for the undirected graph associated with N, then C is called **a cut or s-t cut** if the removal of the edges in C from the network separates the source s from the sink t. The notation $(P, \overline{P})$ is used to denote a cut that partitions the vertices into two subsets P and $\overline{P}$ where P contains s and $\overline{P}$ contains t.

If there are 4 vertices other than source and sink, then there are $2^4 = 16$ $s - t$ cuts $(P, \overline{P})$ since any subset of (a, b, c, d) where a, b, c, d are vertices together with s form a possible choice of P. More generally, if there are n intermediate vertices in the network N, then N will have 2^n cuts.

The *capacity* of a cut denoted by $C(P, \overline{P})$ is defined to be the sum of the capacities of those edges directed from the vertices in set P to the vertices in $\overline{P}$ *i.e.*

$$C(P, \overline{P}) = \sum_{\substack{v \in P \\ w \in \overline{P}}} c(v, w)$$

...g. 15.9 and list all s-t cuts.

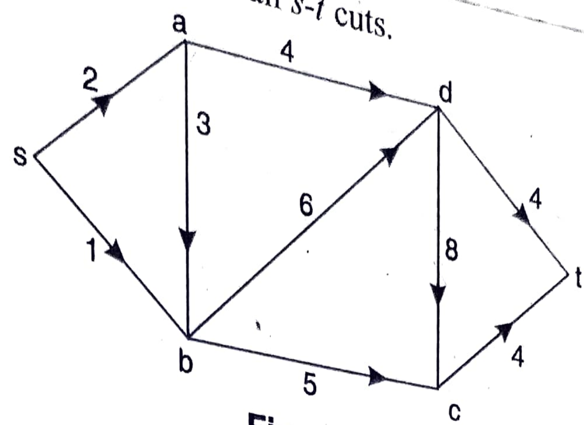


Fig. 15.9

Solution. There are four vertices other than source and sinck in the given network. So, there exists $2^4 = 16$ $s - t$ cut sets. The following table gives all s-t possibilities.

Table Possible s-t cuts.

P	$\overline{P}$	Cut set $(P, \overline{P})$	Capacity $(P, \overline{P})$
$\{s\}$	$\{a, b, c, d, t\}$	$\{sa, sb\}$	$2 + 1 = 3$
$\{s, a\}$	$\{b, c, d, t\}$	$\{sb, ab, ad\}$	$1 + 3 + 4 = 8$
$\{s, b\}$	$\{a, c, d, t\}$	$\{sa, bc, bd\}$	$2 + 5 + 6 = 13$
$\{s, c\}$	$\{a, b, d, t\}$	$\{sa, sb, ct\}$	$2 + 1 + 4 = 7$
$\{s, d\}$	$\{a, b, c, t\}$	$\{sa, sb, dc, dt\}$	$2 + 1 + 8 + 4 = 15$
$\{s, a, b\}$	$\{c, d, t\}$	$\{ad, bc, bd\}$	$4 + 5 + 6 = 15$
$\{s, a, c\}$	$\{b, d, t\}$	$\{sb, ab, ad, ct\}$	$1 + 3 + 4 + 4 = 12$
$\{s, a, d\}$	$\{b, c, t\}$	$\{sb, ab, dc, dt\}$	$1 + 3 + 8 + 4 = 16$
$\{s, b, c\}$	$\{a, d, t\}$	$\{sa, bd, ct\}$	$2 + 6 + 4 = 12$
$\{s, b, d\}$	$\{a, c, t\}$	$\{sa, bc, dc, dt\}$	$2 + 5 + 8 + 4 = 19$
$\{s, c, d\}$	$\{a, b, t\}$	$\{sa, sb, ct, dt\}$	$2 + 1 + 4 + 4 = 11$
$\{s, a, b, c\}$	$\{d, t\}$	$\{ad, bd, ct\}$	$4 + 6 + 4 = 14$
$\{s, a, b, d\}$	$\{c, t\}$	$\{bc, dc, dt\}$	$5 + 8 + 4 = 17$
$\{s, a, c, d\}$	$\{b, t\}$	$\{sb, ab, ct, dt\}$	$1 + 3 + 4 + 4 = 12$
$\{s, b, c, d\}$	$\{a, t\}$	$\{sa, ct, dt\}$	$2 + 4 + 4 = 10$
$\{s, a, b, c, d\}$	$\{t\}$	$\{ct, dt\}$	$4 + 4 = 8$

Note that the capacity of the edge (c, b) does not contribute any value to the capacity of $(P, \overline{P})$ because the edge (c, b) is an edge from $\overline{P}$ to P and not from P to $\overline{P}$.

A cut is called a minimum cut if its capacity does not exceed the capacity of any other cut of the network *i.e.*, we call a cut $(X, \overline{X})$ as a **minimum cut** if there is no $(Y, \overline{Y})$ cut such that the sum of all capacities of edges from Y to $\overline{Y}$ is less than the sum of capabilities of all edges from X to $\overline{X}$ *i.e.*, $C(Y, \overline{Y}) < C(X, \overline{X})$.

The following theorem gives an upper bound on the values of flows in a network.

Theorem 15.24. The value of any flow in given network N from source s to sink t is less than or equal to the capacity of any cut in the network.

Proof. Let f be a flow and $(P, \overline{P})$ be a cut in a network. For the source s

$$\sum_{\text{all } i} f(s,i) - \sum_{\text{all } j} f(j,s) = \sum_{\text{all } i} f(s,i) = \text{val}(f) \quad ...(1)$$

since $f(j, s) = 0$ for any j. For vertex $u \neq s$ in P,

$$\sum_{\text{all } i} f(u,i) - \sum_{\text{all } j} f(j,u) = 0 \quad ...(2)$$

From (1) and (2), we have

$$\text{val}(f) = \sum_{u \in P} [\sum_{\text{all } i} f(u,i) - \sum_{\text{all } j} f(j,u)]$$

$$= \sum_{u \in P;\ \text{all } i} f(u,i) - \sum_{u \in P;\ \text{all } j} f(j,u)$$

$$= \sum_{u \in P;\ i \in P} f(u,i) + \sum_{u \in P;\ i \in \bar{P}} f(u,i) - [\sum_{u \in P;\ j \in P} f(j,u) + \sum_{u \in P;\ j \in \bar{P}} f(j,u)] \quad ...(3)$$

But $\displaystyle\sum_{u \in P;\ i \in P} f(u,i) = \sum_{u \in P;\ j \in P} f(j,u)$

Since both sums run through all the vertices in P. Thus (3) becomes

$$\text{val}(f) = \sum_{u \in P;\ i \in \bar{P}} f(u,i) - \sum_{u \in P;\ j \in \bar{P}} f(j,u) \quad ...(4)$$

since $\Sigma f(j, u)$ is always a nonnegative quantity, we have

$$\text{val}(f) \leq \sum_{u \in P;\ i \in \bar{P}} f(u,i) \leq \sum_{u \in P;\ i \in \bar{P}} c(u,i) = C(P . \bar{P})$$

Note that if f_0 is a flow with maximum possible, and $(P_0, \bar{P}_0)$ is a cut with minimum possible capacity. The above theorem tells us that val $(f_0) \leq C(P_0, \bar{P}_0)$ or more expressively.

Max-flow ≤ Min-cut

Equation (4) can be stated as : For any cut $(P, \bar{P})$, the value of a flow in a network equal the sum of flows in the edges from the vertices in P to the vertices in $\bar{P}$ minus the sum of flows in the edges from the vertices in $\bar{P}$ to the vertices in P.

In view of the above theorem, whenever we can construct a flow of which is equal to the capacity of some cut $(P, \bar{P})$ we can be certain that f is maximum flow. Because if there were a larger flow, its value would exceed the capacity of the cut $(P, \bar{P})$.

Flow Augmenting Path

Let s and t be the source and sink of a network $N(V, E)$. A non-directed path P from s to t is a sequence of edges $e_1, e_2, \ldots e_{k-1}$ and sequence of vertices $x_1, x_2, \ldots, x_{k-1}, x_k$. The edges e_i is called a **forward edge** of P if it is directed from x_i to x_{i+1} and a **backward edge** if it directed from x_{i+1} to x_i.

If a flow f is given, a path P

$$s = x_1, x_2, \ldots, x_{k-1}, x_k = t$$

is called an **f-augmenting path** (or **flow-augmenting path)** if

(*i*) every forward edge of the path has excess capacity (unsaturated) and
(*ii*) every bactward edge on the path has non zero flow.

Symbolically

$f(e) < c(e)$ for each forward edge $e \in E$

i.e., $f(x_i, x_{i+1}) < c(x_i, x_{i+1})$, $(x_i, x_{i+1}) \in E$

and

for each backward edge $e \in E$

i.e., $f(x_{i+1}, x_i) < 0$, $(x_i, x_{i+1}) \in E$

for $1 \le i \le k - 1$. Given such a path, we can increase flow on the forward arcs and decrease the flow on the backward arcs by the same amount, without violating the conservation rule. The amount can be determined as follows:

Calculate the unused capacity $c(e) - f(e)$ of each forward arc and the flow $f(e)$ in each backward arc. The amount to be increased/decreased is the minimum of all these numbers. The existence of an f-augmenting path from s to t enables us to find a new flow f^* such that val $(f^*) >$ val (f).

Theorem 15.25 (Max-flow Min-cut Theorem) In a network, the value of a maximum flow is equal to the capacity of a minimum cut.

Ford-Fulkerson Algorithm for Maximum Flow

This algorithm is iterative. One starts with $f(u, v) = 0$ for all $u, v \in V$, giving an initial flow of value 0. At each iteration, the value of flow is increased by finding an augmenting path. The process is repeated until no augmenting path can be found. The max-flow min-cut theorem will show that on termination, this process yields a maximum flow.

A flow f can be constructed the value of which is equal to the capacity of a cut using the procedure known as the **labeling procedure.** The necessary steps of the procedure is given below:

1. (Initialize flow) Given a network N, define an initial flow f by $f(e) = 0$ for every e in E.

2. (Label source) Label the source s as $(-, \infty)$ (It means the source can supply an infinite amount of material to the other vertices)

3. For each vertex x that is adjacent from s is labeled with $(s^+, \Delta x)$

   (*i*) if $c(s, x) > f(s, x)$ where $\Delta x = c(s, x) - f(s, x)$

   (*ii*) if $c(s, x) = f(s, x)$ then x is not labeled.

   {The $(s^+, \Delta x)$ means that the flow from s into x can be increased by an amount equal to Δx}.

4. As long as there exists $x (\neq s) \in V$ such that x is labeled, and there is an edge (x, y) where y is not labeled, label vertex y with $(x^+, \Delta y)$

   (*i*) if $c(x, y) > f(x, y)$ where $\Delta y = \min[\Delta x, c(x, y) - f(x, y)]$

   (*ii*) if $c(x, y) = f(x, y)$ the y is not labeled.

5. Similarly, As long as there exists $x (\neq s) \in V$ such that x is labeled, and there is an edge (y, x) where y is not labeled, label vertex y with $(x^+, \Delta y)$

   (*i*) if $f(y, x) > 0$ where $\Delta y = \min[\Delta x, f(y, x)]$

   (*ii*) if $f(y, x) = 0$ then y is not labeled

{For a vertex y that is adjacent to a labeled vertex x, the label $(x^+, \Delta y)$ means that, by decreasing the flow from y to x, the total outgoing flow from y to the labeled vertices can be decreased by Δy}

Note that when a vertex can be labeled in more than one way, an arbitary choice can be made.

If we repeat the procedure of labeling the vertices that are adjacent to or from the labeled vertices, one of the following two cases shall arise:

Case 1. If sink t is labeled as $(x^+, \Delta t)$, then the flow in the edge (x, t) can be increased from $f(x, t)$ to $f(x, t) + \Delta t$. Vertex x must be labeled either $(v^+, \Delta(x))$ or $(v^-, \Delta(x))$ with $\Delta x \ge \Delta t$ for some vertex v. If x is labeled as $(v^+, \Delta(x))$, we may regard vertex v as the source for increasing the flow in the edge (v, x) from $f(v, x)$ to $f(v, x) + \Delta(t)$. If x labeled as $(v^-, \Delta(x))$ the flow in the edge (v, x) changes from $f(v, x)$ to $f(v, x) - \Delta(t)$ so that increment in $\Delta(t)$ from x to t is compensated.

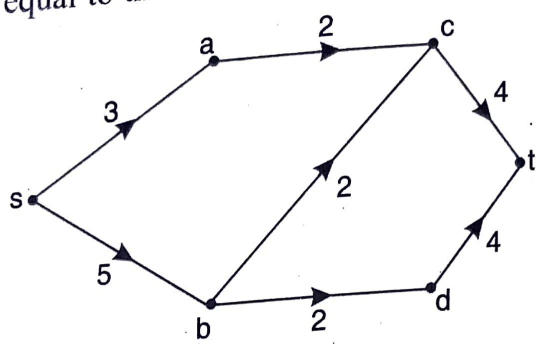
The process is continued back to the source s, each directed edge along a path from s to ... its flow increased or decreased by $\Delta(t)$. The labeling procedure can now be started all over ... to further increase the value of the flow in the network.

The flow, therefore, is a maximum flow.

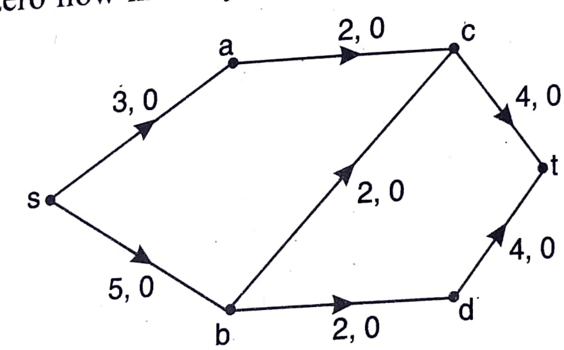
Case 2. If sink t is not labeled then the maximum flow is attained. Let P be labeled ver... and $\overline{P}$ be the all unlabeled vertices. Since the vertices in $\overline{P}$ are not labeled, the flows in the ... (x, y) where $x \in P$ and $y \in \overline{P}$ satisfy $f(x, y) = c(x, y)$ and that the flow in each of the ... incident from the vertices in $\overline{P}$ to the vertices in P is equal to zero. Thus, there is a flow in ... network, when the value of the flow is equal to the capacity to the cut $(P, \overline{P})$.

Example 29. Use Ford–Fulkerson algorithm to find the maximum flow for the follo... network.

Find the cut with capacity equal to this maximum flow.

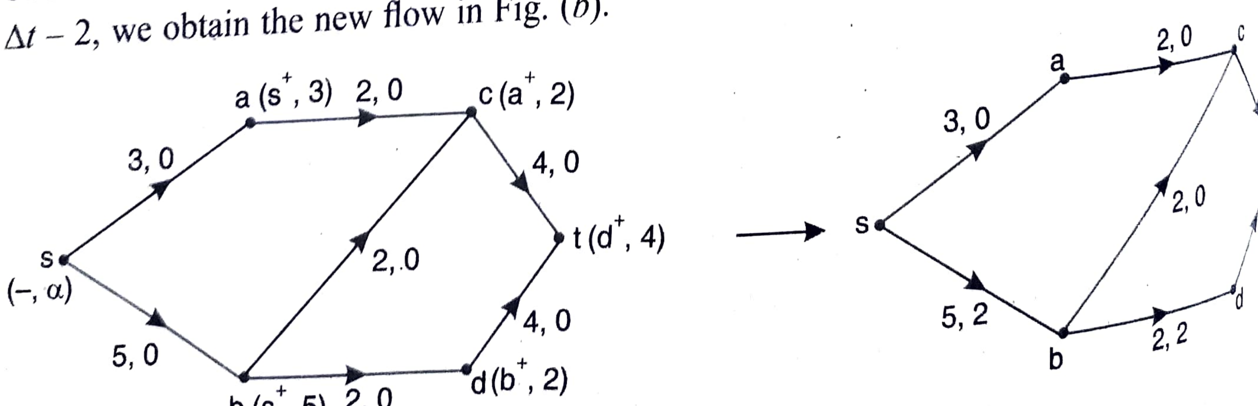


Solution. We start with zero flow in every edge as shown in Fig. (a)



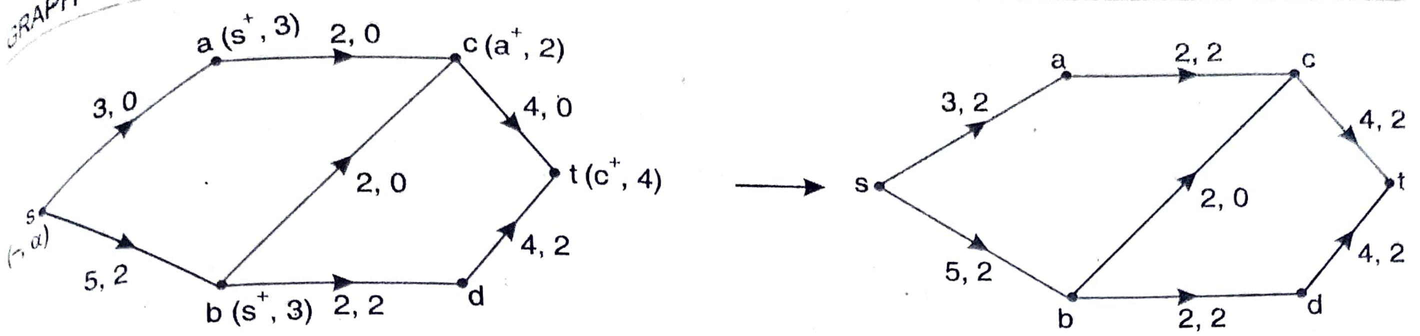
(a)

Label the source s as $(-, \propto)$ Sink t can be labeled either with $(d^+, 4)$ or with $(c^+, 4)$. Le... be labeled with $(d^+, 2)$. Backtracking from t to d to b to s and increasing the flow in each edge ... $\Delta t - 2$, we obtain the new flow in Fig. (b).



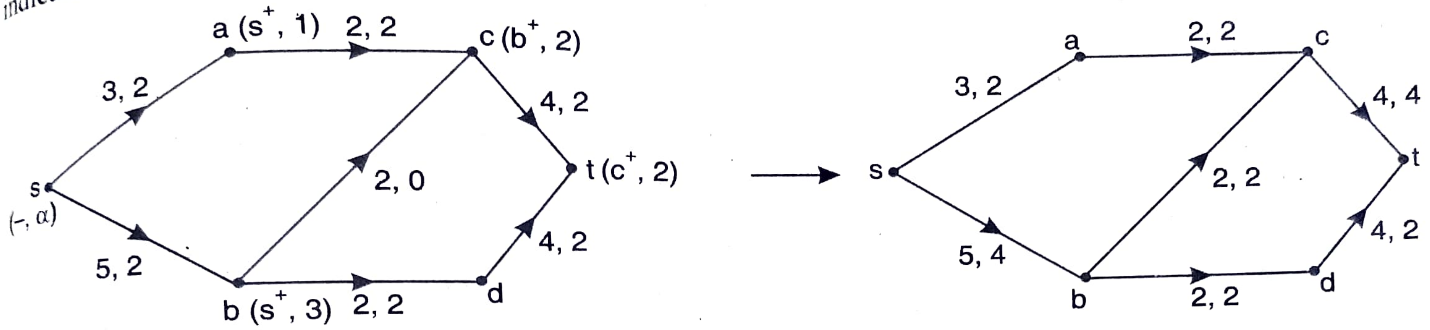
(b)

...hows a second application of the labeling procedure.



(c)

Fig. (d) shows a third applications of the labeling procedure which is the last pass which indicates that we cannot increase the flow any further.



(d)

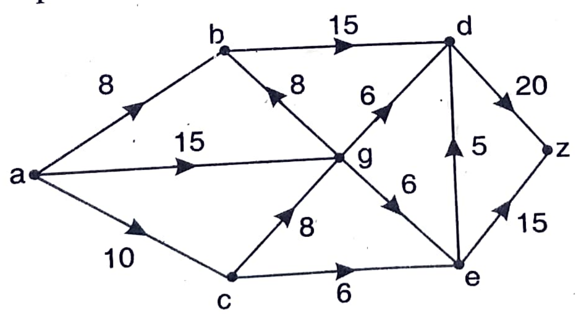
Here sink t is unlabeled and hence the maximum value of flow from source to sink is 6.

Letting $P = \{s, a, b\}$ $\overline{P} = \{c, d, t\}$

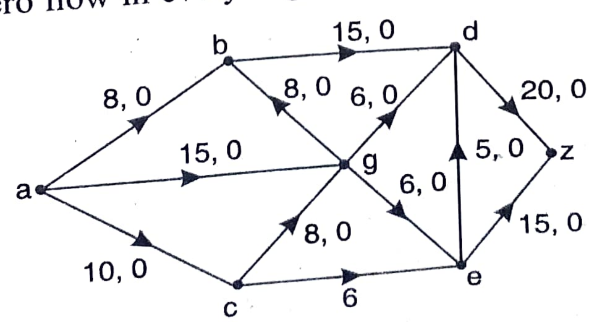
Then $c(a, c) + c(b, c) + c(b, d) = 2 + 2 + 2 = 6$

Example 67. Use Ford-Fulkerson algorithm to find the maximum flow for the following network.

Find a cut with capacity equal to this maximum flow

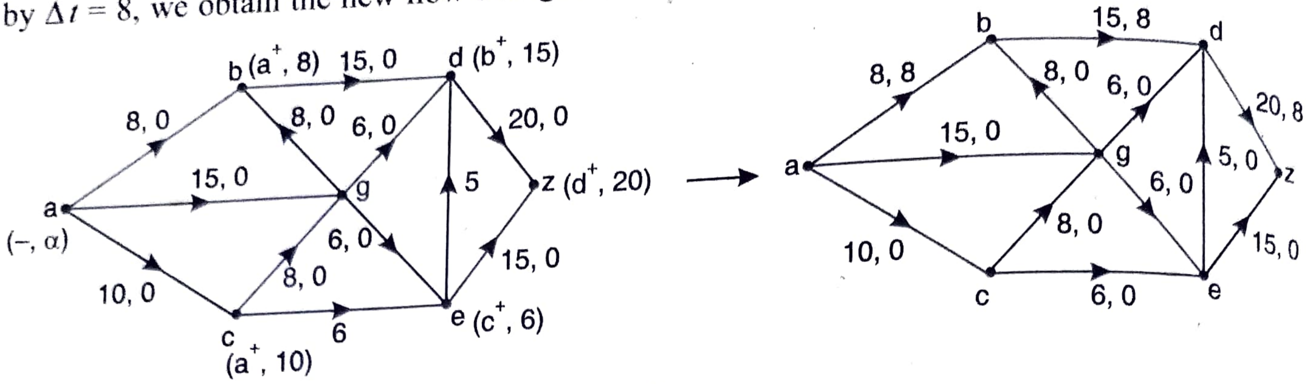


Solution. We start with zero flow in every edge as shown in Fig. (a).



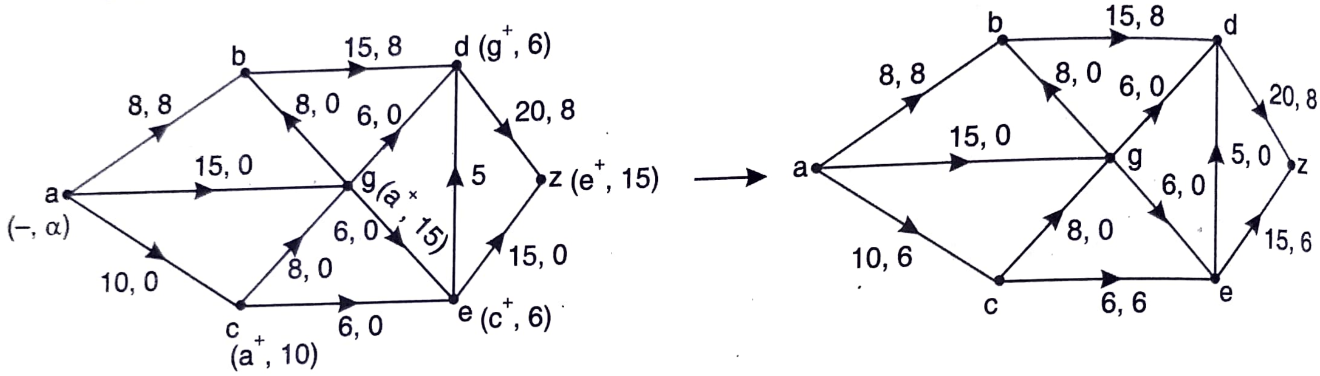
(a)

Label the source a as $(-, \infty)$. Sink z can be labeled either with $(d^+, 20)$ or with $(e^+, 15)$. Let it be labeled with $(d^+, 20)$. Backtracking from z to d to b to a and increasing the flow in each edge by $\Delta t = 8$, we obtain the new flow in Fig. (*b*).



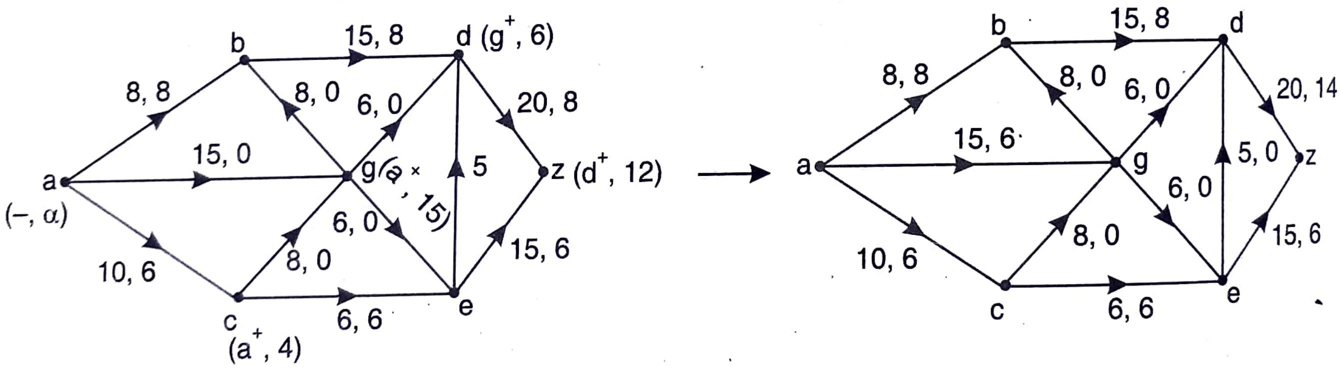
(*b*)

Fig. (*c*) shows the second application of the labeling procedure.



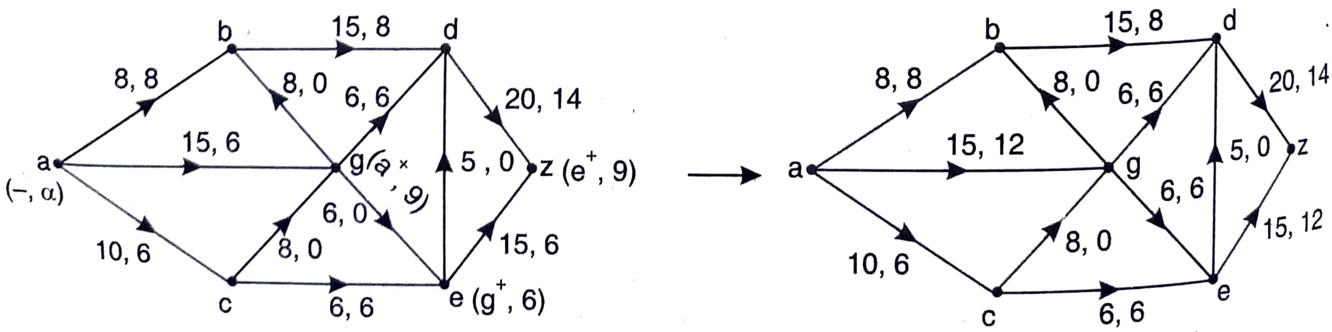
(*c*)

Fig. (*d*) shows the third application of the labeling procedure.



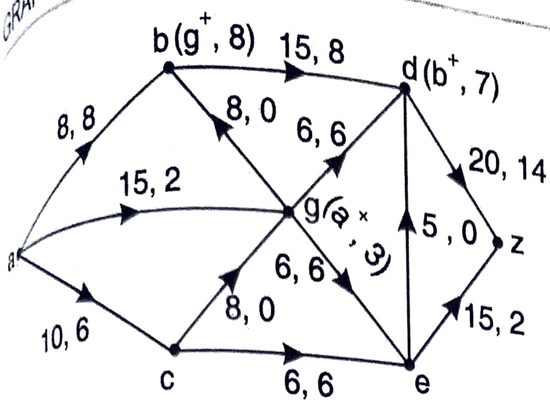
(*d*)

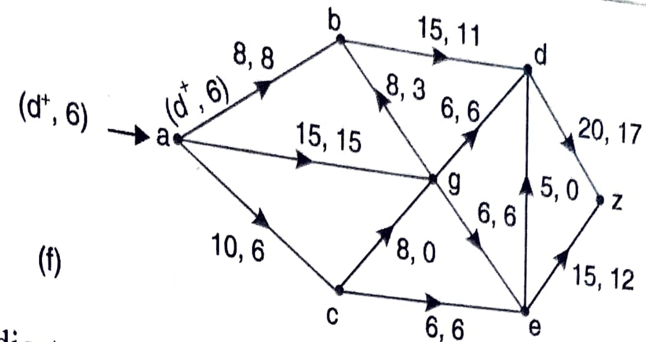
Fig. (*e*) shows the fourth application of the labeling procedure.



(*e*)

Fig. (*f*) shows the fifth application of the labeling procedure.

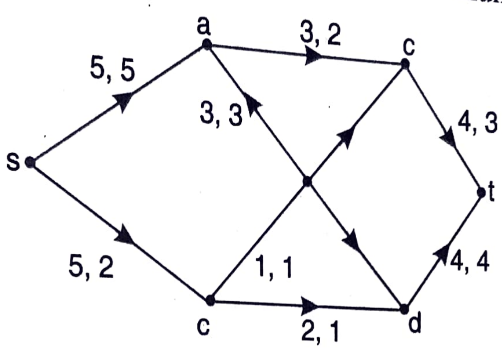




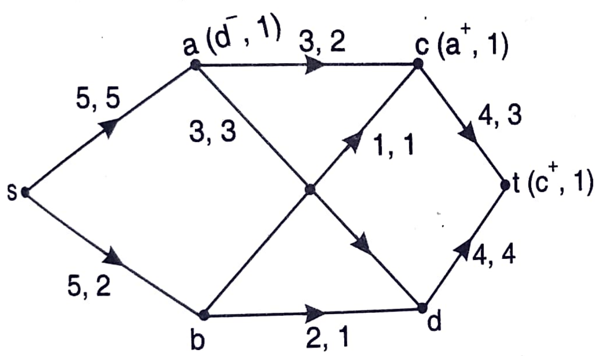
(f)

Fig. (*f*) shows the last pass which indicates that we cannot increase the flow any further. Hence, the maximum flow is 29.

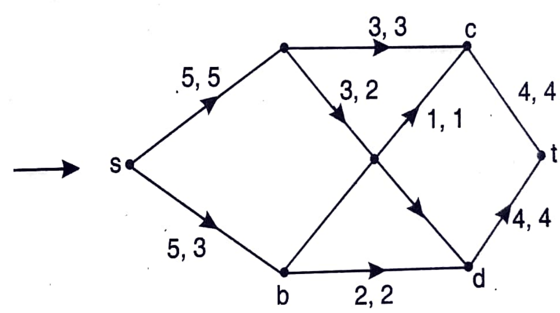
Example 30. Consider the given network to obtain a maximum flow.



Solution. It seems that we cannot increase the flow any further. The labeling procedure can be applied as follows.



(*a*)



(*b*)

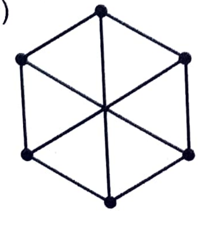
Fig. (*b*) shows the maximum flow network. Note that the vertex a has a negative label. The maximal flow is 8.

GRAPH THEORY II PROBLEM SET

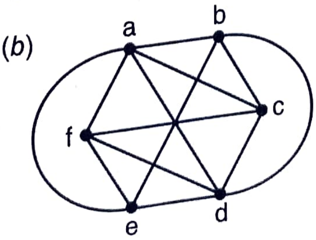
Problem Set 15.1

1. Show that each graph is not planar by finding a subgraph homeomorphic to either k_5 or $k_{3,3}$.

(a)



(*b*)



2. Explain the vertex colouring problem.
3. Distinguish between *k*-colouring of a graph and chromatic number of a graph.