# Divide & Difference

Let corresponding to the arguments $x_0, x_1, x_2 \ldots x_n$, the entries are $f(x_0), f(x_1), \ldots f(x_n)$ then

1st divide & difference is defined as

$$\underset{x_1}{\Delta} f(x_0) = f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$\underset{x_2}{\Delta} f(x_1) = f(x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\underset{x_n}{\Delta} f(x_{n-1}) = f(x_{n-1}, x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

2nd divide & difference :-

$$\underset{x_1 x_2}{\Delta} f(x_0) = f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$

$$\underset{x_2 x_3}{\Delta} f(x_1) = f(x_1, x_2, x_3) = \frac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\underset{x_{n-1} x_n}{\Delta} f(x_{n-2}) = f(x_{n-2}, x_{n-1}, x_n) = \frac{f(x_{n-1}, x_n) - f(x_{n-2}, x_{n-1})}{x_n - x_{n-2}}$$

3rd divide & difference :-

$$\underset{x_1 x_2 x_3}{\Delta} f(x_0) = f(x_0, x_1, x_2, x_3) = \frac{f(x_1, x_2, x_3) - f(x_0, x_1, x_2)}{x_3 - x_0}$$

$$\underset{x_2, x_3, x_4}{\Delta} f(x_1) = f(x_1, x_2, x_3, x_4) = \frac{f(x_2, x_3, x_4) - f(x_1, x_2, x_3)}{x_4 - x_1}$$

$$\underset{x_{n-2} x_{n-1} x_n}{\Delta} f(x_{n-3}) = f(x_{n-3}, x_{n-2}, x_{n-1}, x_n) = \frac{f(x_{n-2}, x_{n-1}, x_n) - f(x_{n-3}, x_{n-2}, x_{n-1})}{x_n - x_{n-3}}$$

## $n^{th}$ divide & diff.

$\Delta_{x_1, x_2, x_3 \dots x_n} x_0 \quad f(x_0, x_1, x_2 \dots x_n) = \dfrac{f(x_1 \dots x_n) - f(x_0 \dots x_{n-1})}{x_n - x_0}$

$\Rightarrow$ DIVIDE & DIFFERENCE TABLE :-

| $x$ | $f(x)$ | $\Delta f(x)$ | $\Delta^2 f(x)$ | $\Delta^3 f(x)$ |
|---|---|---|---|---|
| $x_0$ | $f(x_0)$ | $f(x_0, x_1) = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$ | $f(x_0, x_1, x_2) = \dfrac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$ | |
| $x_1$ | $f(x_1)$ | $f(x_1, x_2) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$ | | $f(x_0, x_1, x_2, x_3) = \dfrac{f(x_1, x_2, x_3) - f(x_0, x_1, x_2)}{x_3 - x_0}$ |
| $x_2$ | $f(x_2)$ | | $f(x_1, x_2, x_3) = \dfrac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1}$ | |
| $x_3$ | $f(x_3)$ | $f(x_2, x_3) = \dfrac{f(x_3) - f(x_2)}{x_3 - x_2}$. | | |

**Q.** Find the 3rd divide & diff with arguments 2, 4, 9, 10 of the function $f(x) = x^3 - 2x$.

**Sol$^n$**

| $x$ | $f(x)$ | $\Delta f(x)$ | $\Delta^2 f(x)$ | $\Delta^3 f(x)$ |
|---|---|---|---|---|
| 2 | 4 | $f(2,4) = \dfrac{56-4}{4-2} = 26$ | $f(2,4,9) = \dfrac{131-26}{7} = 15$ | $f(2,4,9,10) = \dfrac{23-15}{8} = \dfrac{8}{8} = 1$ |
| 4 | 56 | $f(4,9) = \dfrac{711-56}{9-4} = 131$ | $f(4,9,10) = \dfrac{269-131}{6} = 23$ | |
| 9 | 711 | $f(9,10) = \dfrac{940-711}{1} = 269$ | | |
| 10 | 940 | | | |

**Ans :** 1.

2/10/2018

⇒ Newton's Divided Diff. Interpolation formula :-

$$f(x) = f(x_0) + (x-x_0)f(x_0,x_1) + (x-x_0)(x-x_1)f(x_0,x_1,x_2) + (x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3) + \cdots + (x-x_0)(x-x_1)(x-x_2)\cdots(x-x_{n-1})f(x_0,x_1,x_2,\ldots,x_n).$$

Ques. Use Newton's dividend diff. formula to calculate $f(x)$ for the fol. ~~Interval~~ table hence find the value of $f(3)$

| $x$ | $f(x)$ | $\Delta f(x)$ | $\Delta^2 f(x)$ | $\Delta^3 f(x)$ | $\Delta^4 f(x)$ |
|---|---|---|---|---|---|
| 0 | 1 | $f(0,1) = \frac{13}{1} = 13$ | | | |
| | | | $f(0,1,2) = \frac{1-13}{2} = -6$ | | |
| 1 | 14 | $f(1,2) = \frac{1}{1} = 1$ | | $\frac{4}{4} = 1$ | |
| | | | $f(1,2,4)$ $\frac{-5-1}{3} = -2$ | | 0 |
| 2 | 15 | $f(2,4) = \frac{-10}{2} = -5$ | | $\frac{4}{5-1} = 1$ | |
| | | | $f(2,4,5)$ $\frac{6}{3} = 2$ | | 0 |
| 4 | 5 | $f(4,5) = 1$ | | $\frac{4}{6-2} = 1$ | |
| | | | $f(4,5,6)$ $\frac{12}{2} = 6$ | | |
| 5 | 6 | $f(5,6) = 13$ | | | |
| 6 | 19 | | | | |

$$f(x) = 0 + (x-0)(13) + (x-0)(x-1)(-6) + (x-0)(x-1)(x-2)(1)$$

$$= 13x - 6(x^2-x) + (x^2-x)(x-2)$$

$$= 13x - 6x^2 + 6x + (x^2-x)(x-2) = x^3 - 9x^2 + 21x + 1$$

$$f(x) = -5x^2 + 19x - 2.$$

$$f(3) = -45 + 57 - 2 = 55 - 45 = \underline{10}$$

$\cancel{x^3 - 2x^2 - x^2 + 2x + 6x + 13x - 6x^2}$

$x^3 - 9x^2 + 21x$ $\quad$ $27 - 81 + 63 + 1$

$90 - 81 + 1 =$

$f(x) = f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0, x_1, x_2) +$
$(x-x_0)(x-x_1)(x-x_2) f(x_0, x_1, x_2, x_3) + \ldots$

$f(x) = f(x_0) + (x-x_0) f(x_0, x_1) + (x^2 - x x_1 - x_0 x + x_0 x_1) f(x_0, x_1, x_2) +$
$(x^3 - x^2 x_1 - x_0 x^2 + x_0 x_1 x - x^2 x_2 + x x_1 x_2 + x_0 x x_2 - x_0 x_1 x_2)$
$\left\{ \Rightarrow x^3 - x^2(x_1 + x_0 + x_2) + x(x_0 x_1 + x_1 x_2 + x_0 x_2) - x_0 x_1 x_2 \right\} +$
$+$

$f'(x) = f(x_0, x_1) + \{2x - (x_0 + x_1)\} f(x_0, x_1, x_2) +$
$f(x_0, x_1, x_2, x_3)(3x^2 - 2x(x_0 + x_1 + x_2) + (x_0 x_1 + x_1 x_2 + x_0 x_2))$
$+ \ldots$

$f''(x) = 2 f(x_0, x_1, x_2) + \{6x - 2(x_0 + x_1 + x_2)\} f(x_0, x_1, x_2, x_3) \ldots$

Q. From the fol. table find the first derivative at $x = 4$ using newton's divided diff. formula.

Sol:

| $x$ | $f(x)$ | $\Delta f(x)$ | $\Delta^2 f(x)$ | $\Delta^3 f(x)$ | $\Delta^4 f(x)$ |
|---|---|---|---|---|---|
| 1 | 0 | | | | |
| | | $f(1,2) = \frac{1}{1} = 1$ | | | |
| 2 | 1 | | $f(1,2,4) = \frac{1}{3}$ | | |
| | | $f(2,4) = \frac{4}{2} = 2$ | | $f(1,2,4,8) = 0$ | |
| 4 | 5 | | $f(2,4,8) = \frac{2}{6} = \frac{1}{3}$ | | $\frac{-\frac{1}{16}}{9} = \frac{-1}{144}$ |
| | | $f(4,8) = \frac{16}{4} = 4$ | | $f(2,4,8,10) = \frac{-\frac{1}{6} - \frac{1}{3}}{8} = \frac{-9}{18 \times 8} = -\frac{1}{16}$ | |
| 8 | 21 | | $f(4,8,10) = -\frac{1}{6}$ | | |
| | | $f(8,10) = \frac{6}{2} = 3$ | | | |
| 10 | 27 | | | | |

$x^2 - 2x - x + 2$

$(x^2 - 3x + 2)(x-4)$

$x^3 - 3x^2 + 2x - 4x^2 + 12x - 8$

$\{x^3 - 7x^2 + 14x - 8\}(x-8) = x^4 - 7x^3 + 14x^2 - 8x - 8x^3 + 56x^2 - 112x + 64$

$$f(x) = f(1) + (x-1) + (x-1)(x-2)\frac{1}{3} + (x-1)(x-2)(x-4)(0)$$

$$+ (x-1)(x-2)(x-4)(x-8)\left(\frac{-1}{144}\right)$$

$$= (x-1) + \frac{1}{3}(x^2 - 2x - x + 2) + (x^2 - 3x + 2)(x-4)(x-8)\left(\frac{-1}{144}\right)$$

$$= (x-1) + \frac{1}{3}(x^2 - 3x + 2) + (x^3 - 7x^2 + 14x - 8)(x-8)\left(\frac{-1}{144}\right)$$

$$f(x) = (x-1) + \frac{1}{3}(x^2 - 3x + 2) - \frac{1}{144}(x^4 - 15x^3 + 70x^2 - 120x + 64)$$

$$f'(x) = 1 + \frac{(2x-3)}{3} - \frac{1}{144}(4x^3 - 45x^2 + 140x - 120)$$

$$f'(4) = 1\left(\frac{8-3}{3}\right) - \frac{1}{144}(256 - 16\times 45 + 140\times 4 - 120)$$

16 $$= 1 + \frac{5}{3} - \frac{1}{36}(64 - 4\times 45 + 140 - 30)$$

$\frac{164}{256}$ $$= 1 + \frac{5}{3} - \frac{1}{18}(32 - 2\times 45 + 70 - 15)$$

$$= 1 + \frac{5}{3} - \frac{1}{9}(16 - 45 + 5)$$

$\frac{40}{24}$ $$= 1\ \frac{5}{3} + \frac{248}{43}$$

$$\boxed{f'(4) = \frac{13}{3}}$$

$\frac{5}{3} + \frac{1}{6} =$

$\frac{5\times 2}{6}$

$\frac{1}{6}$

1.83

+1

$$\boxed{f'(4) = 2.83}$$

Ques. Find out first & second derivative from fol. table using Newton's divided diff. Interpolation formula.

| | | | |
|---|---|---|---|
| $x$ | 1 | 3 | -4 |
| $f(x)$ | 3 | -5 | 4 |

| $x$ | $f(x)$ | $\Delta f(x)$ | $\Delta^2 f(x)$ |
|---|---|---|---|
| 1 | 3 | | |
| | | $-\frac{8}{2} = -4$ | |
| 3 | -5 | | $\frac{4 - \frac{9}{7}}{-5} = -\frac{19}{35}$ |
| | | $-\frac{9}{7}$ | |
| -4 | 4 | | |

$$f(x) = f(1) + (x-1)(-4) + (x-1)(x-3)\left(-\frac{19}{35}\right)$$

$$= 3 + 4 - 4x + 4 + (x^2 - 4x + 3)\left(\frac{-19}{35}\right)$$

$$= \frac{35(1 - \cancel{4} - x) + 19\,(x^2 - 4x + 3)}{35}$$

$$= \frac{35 - 140x - 19x^2 + 76x - 57}{35}$$

$$f(x) = \frac{-19x^2 - 74x - 22}{35}$$

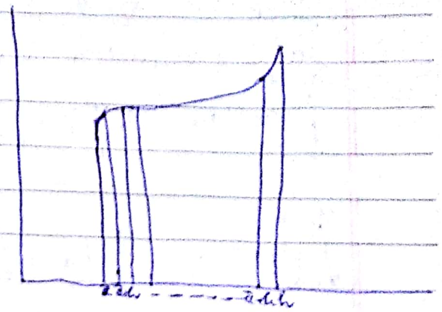
$(p^2 - p)(p-1)$

$p^3 = p^2 - p^2 + p$

5/10/2019

# NUMERICAL INTEGRATION

$I = \int_a^b f(x)\,dx$




$f(x_0), f(x_1), \ldots f(x_n)$

$y_0, y_1, \ldots y_n$

arbitrary '$x$' $\rightarrow \boxed{x = x_0 + ph}$

or

$p = \frac{x - x_0}{h}$

$\frac{dx}{h} = dp$

$\Rightarrow dx = h\,dp$

$I = \int_a^b f(x)\,dx = \int_{x_0}^{x_n} y\,dx$

$f(x) = y = f(x_0) + p\,\Delta f(x_0) + \frac{p(p-1)}{2}\Delta^2 f(x_0) + \ldots$

$I = \int_{x_0}^{x_n} \left(y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \ldots\right) h\,dp$

$I = h\int_0^n \left(y_0 + p\,\Delta y_0 + \frac{p^2-p}{2}\Delta^2 y_0 + \frac{p^3-3p^2+2p}{6}\Delta^3 y_0 + \ldots\right) dp$

$= h\left[p y_0 + \frac{p^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{p^3}{3} - \frac{p^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{p^4}{4} - \frac{3p^3}{3} + \frac{2p^2}{2}\right)\Delta^3 y_0\right]_0^n$

$\int_a^b f(x)\,dx = I = h\left(n y_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{n^4}{4} - n^3 + n^2\right)\Delta^3 y_0 + \ldots\right)$

This is Newton-Cotes Gen$^l$ quadrature Formula :-

① Trapezoidal Rule $\boxed{n=1}$

② Simpson's $\frac{1}{3}$ Rule $\boxed{n=2}$

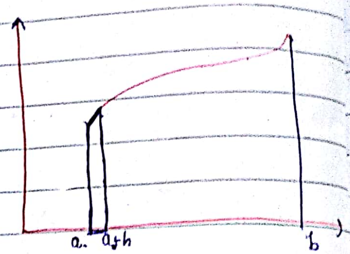
③ Simpson's $\frac{3}{8}$ Rule

① Take one part at a time :-

$$\int_{x_0}^{x_0+h} f(x)\,dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right] = h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right] = \frac{h}{2}[y_0 + y_1]$$

$$\int_{x_0+h}^{x_0+2h} f(x)\,dx = \frac{h}{2}[y_1 + y_2]$$

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx = \frac{h}{2}[y_{n-1} + y_n].$$




Adding above eq^ns.

$$\int_{x_0}^{x_0+h} f(x)\,dx + \int_{x_0+h}^{x_0+2h} f(x)\,dx + \cdots + \int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[y_0 + y_1 + y_1 + y_2 + y_2 + y_3 + \cdots y_{n-1} + y_n\right]$$

$$\boxed{\int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})\right]}$$

Trapezoidal Rule.

② Take 2 parts at a time

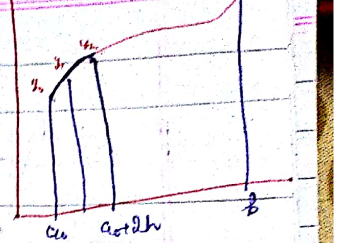
$$\int_{x_0}^{x_0+2h} f(x)\,dx = h\left[2y_0 + \frac{2^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{2^3}{3} - \frac{2^2}{2}\right)\Delta^2 y_0\right]$$

$$= h\left[2y_0 + 2(y_1 - y_0) + \left(\frac{2^2}{3} - 1\right)(y_2 - y_1 - y_1 + y_0)\right]$$

$$= h\left[\cancel{2y_0} + 2y_1 - \cancel{2y_0} + \frac{1}{3}(y_2 - 2y_1 + y_0)\right]$$

$$= h\left[\frac{1}{3}y_1 + \frac{4}{3}y_2 + \frac{1}{3}y_0\right] = \frac{h}{3}\left[y_0 + 4y_1 + y_2\right]$$

3.




$$\int_{x_0+(n-2)h}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left[y_{n-2} + 4y_{n-1} + y_n\right].$$

Adding all above, we get-,

$$\cancel{\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0+4y_1+y_2)+(y_1+4y_2+y_3)+\cdots+(y_{n-2}+4y_{n-1}+y_n)\right]}$$

$$\cancel{\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0+y_n)+5(y_1+y_2+\cdots y_{n-1})\right]}$$

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0+y_n) + 4y_1 + y_2 + y_2 + 4y_3 + y_4 + \cdots y_{n-2} + 4y_{n-1}\right]$$

$$\boxed{\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0+y_n) + 4(y_1+y_3+y_5+\cdots) + 2(y_2+y_4+y_6+\cdots)\right]}$$

$$\boxed{I = \int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0+y_n) + 4(\Sigma \text{ odd terms}) + 2(\Sigma \text{ even terms})\right]}$$

Simpson's $\frac{1}{3}$ Rule.

3. Take 3 parts at a time

$$\int_{x_0}^{x_0+3h} f(x)\,dx = h\left[3y_0 + \frac{3^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{3^3}{3} - \frac{3^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{3^4}{4} - 3^3 + 3^2\right)\Delta^3 y_0 + \cdots\right]$$

$$= h\left[3y_0 + \frac{9}{2}(y_1-y_0) + \frac{9}{4}(y_2-2y_1+y_0) + \frac{1}{6}\left[\frac{9}{4}\right]\left[(y_3-y_2) - 2(y_2-y_1) + y_1-y_0\right]\right]$$

Ques:-

$= h\left[3y_0 + \frac{9}{2}(y_1-y_0) + \frac{9}{4}(y_2-2y_1+y_0) + \frac{3}{8}\left[y_3-3y_2+3y_1-y_0\right]\right]$

$= h\left[3y_0 + \frac{9}{2}\left[y_1-y_0+y_2-2y_1+y_0\right] + \frac{3}{8}\left[y_3-3y_2+3y_1-y_0\right]\right]$

$= h\left[3y_0 + \frac{9}{2}\left[y_2-y_1\right] + \frac{3}{8}\left[y_3-3y_2+3y_1-y_0\right]\right]$

$= h\left[\left(3-\frac{3}{8}\right)y_0 \right.$

Soln.

(I)

$= h\left[3y_0 + \left(-\frac{9}{2}+\frac{9}{4}-\frac{3}{8}\right)y_0 + \left(\frac{9}{2}-\frac{9}{2}+\frac{9}{8}\right)y_1 + \left(+\frac{9}{4}-\frac{9}{2}\right)y_2 + \frac{3}{8}y_3\right]$

$= h\left[\frac{3}{8}y_0 + \frac{9}{8}y_1 + \frac{9}{8}y_2 + \frac{3}{8}y_3\right]$

$\int_{x_0}^{x_0+3h} f(x)dx = \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + y_3\right]$

$\int_{x_0+3h}^{x_0+6h} f(x)dx = \frac{3h}{8}\left[y_3 + 3y_4 + 3y_5 + y_6\right]$ (II)

$\vdots$

$\int_{x_0+(n-3)h}^{x_0+nh} f(x)dx = \frac{3h}{8}\left[y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\right]$

Adding above eqns.

$\int_{x_0}^{x_0+3h} f(x)dx + \int_{x_0+3h}^{x_0+6h} f(x)dx + \dots + \int_{x_0+(n-3)h}^{x_0+nh} f(x)dx = \frac{3h}{8}\left[(y_0+y_n) + 3(y_1+y_2+y_4+y_5 + \dots) + 2(y_3+y_6)\right]$ (III)

**Que:** Calculate Integral $\int_{-3}^{3} x^4\,dx$ by taking 7 equidistant intervals by

i) Trapezoidal

ii) Simpson's $\frac{1}{3}$

iii) Simpson's $\frac{3}{8}$

(i)

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x$ : | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
| $f(x)$ : | 81 | 16 | 1 | 0 | 1 | 16 | 81 |
| | $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ |
| | $y_0$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | $y_6$ |

$$\int f(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$\int_{-3}^{3} x^4\,dx = \frac{1}{2}\left[(81+81)+2(16+1+0+1+16)\right]$$

$$= \frac{1}{2}\left[162+2(34)\right] = 81+34 = \underline{115}$$

(ii)

$$\int_{-3}^{3} f(x)\,dx = \frac{h}{3}\left[(y_0+y_n)+4(\Sigma \text{odd term})+2(\Sigma \text{even ter})\right]$$

$$= \frac{1}{3}\left[(81+81)+4(16+0+16)+2(1+1)\right]$$

$$= \frac{1}{3}\left[162+128+4\right]$$

$$\int_{-3}^{3} x^4\,dx = \frac{294}{3} = 98$$

(iii)

$$\int_{-3}^{3} x^4\,dx = \frac{3}{8}\left[162+3(16+1+1+16)+2(0)\right]$$

$$= \frac{3}{8}\left[162+102\right]$$

$$= \frac{3}{8}\,264 = 3\times33 = 99$$

⇒ Exact Method :-

$$\int_{-3}^{3} x^4\,dx = \left[\frac{x^5}{5}\right]_{-3}^{3} = \frac{\cancel{81+81}\ 243+243}{5}$$

$$= \frac{486}{5}$$

$$= 97.2$$

(iii)

**Ques.** Evaluate $\int_0^1 \frac{1}{1+x^2}\,dx$ by dividing ray into 6 equal parts.

$x : 0 \quad \frac{1}{5} \quad \frac{2}{5} \quad \frac{3}{5} \quad \frac{4}{5} \quad \cancel{\frac{5}{4}} \quad 1$ $\qquad \frac{7}{6}$

$f(x) : 1 \quad \frac{25}{26} \quad \frac{25}{29} \quad \frac{25}{34} \quad \frac{25}{41} \quad \frac{1}{2}$

$y_0 \quad y_1 \quad y_2 \quad y_3 \quad y_4 \quad y_5$

(i) $$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{1}{5\times 2}\left[\left(1+\frac{1}{2}\right) + 2\left[\frac{25}{29} + \frac{25}{39} + \frac{25}{41} + \frac{25}{26}\right]\right]$$

$$= \frac{1}{10}\left[\frac{3}{2} + 2\left[\frac{25}{26} + \frac{25}{29} + \frac{25}{34} + \frac{25}{41}\right]\right]$$

$$= \frac{1}{10}\left[\frac{3}{2} + 2\left[0.862 + 0.735 + 0.609\right]\right]$$

$$= \frac{1}{10}\left[5.912\right] = \cancel{0.5912} \; 0.7835$$

(ii) $$= \frac{1}{5\times 3}\left[\left(1+\frac{1}{2}\right) + 4\left(\frac{25}{26} + \frac{25}{34}\right) + 2\left[\frac{25}{29} + \frac{25}{41}\right]\right]$$

$$= \frac{1}{15}\left[1.5 + 4(0.9615 + 0.735) + 2(0.862 + 0.609)\right]$$

$$= \frac{1}{15}\left[1.5 + 6.786 + 2.936\right]$$

$$= 0.748133$$

(iii) $$\int_0^1 \frac{1}{x^2+1}\,dx = \frac{3}{5\times 8}\left[\left(1+\frac{1}{2}\right) + 3\left(\frac{25}{26} + \frac{25}{29} + \frac{25}{41}\right) + 2\times\frac{25}{34}\right]$$

$$= \frac{3}{5\times 8}\left[1.5 + 3(0.9615 + 0.862 + 0.609) + 2\times 0.735\right]$$

$$= \frac{3}{40}\left[1.5 + 7.2975 + 1.47\right]$$

$$= 0.7700$$

Exact method:-

$$\int_0^1 \frac{1}{x^2+1}\,dx = \tan^{-1}(1) - \tan^{-1}(0)$$

$$= \frac{\pi}{4} - 0$$

$$= \frac{\pi}{4} = 0.7853$$

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Topic: Numerical :-

$\Rightarrow$ Solution of ordinary diff. eqn :- (1st order & 1st degree)

1. Taylor series method:-

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$$

$$y = y_0 + (x-x_0)\frac{dy}{dx} + \frac{(x-x_0)^2}{2!}\frac{d^2y}{dx^2} + \frac{(x-x_0)^3}{3!}\frac{d^3y}{dx^3} + \cdots$$

Ques: Apply Taylor series method to obtain approx. value of $y$ at $x = 0.2$ for the diff. eqn

$$\frac{dy}{dx} = 2y + 3e^x, \quad y(0) = 0$$

Soln: $$y = y_0 + (x-x_0)\frac{dy}{dx} + \frac{(x-x_0)^2}{2!}\frac{d^2y}{dx^2} + \frac{(x-x_0)^3}{3!}\frac{d^3y}{dx^3} + \cdots$$

at (0,0)

$$\frac{dy}{dx} = 2(0) + 3e^0 \Rightarrow \frac{dy}{dx} = 3$$

Now, $$\frac{d^2y}{dx^2} = 2\frac{dy}{dx} + 3e^x$$

$$\cancel{y = y_0 + (x-0)\frac{dy}{dx} + \frac{(x-0)^2}{2!}\frac{d^2y}{dx^2}}$$

$$\left(\frac{d^2y}{dx^2}\right)_{(0,0)} = 2\times 3 + 3e^0 = 9$$

$$\Rightarrow \frac{d^2y}{dx^2} = 9$$

$$\frac{d^3y}{dx^3} = 2\frac{d^2y}{dx^2} + 3e^x = 18 + 3 = 21$$

$$\Rightarrow \frac{d^3y}{dx^3} = 21.$$

$$y = 0 + (x-0)3 + \frac{(x-0)^2}{2!}\times 9 + \frac{(x-0)^3}{3!}\times 21 + \frac{45}{8}x^4 + \frac{81}{40}x^5$$

$$y = 3x + \frac{9}{2}x^2 + \frac{7}{2}x^3 + \frac{45}{8}(0.0016) + \frac{81}{40}(0.0008$$

$$(y)_{0.2} = 0.6 + \frac{9}{2}(0.04) + \frac{7}{2}(0.008) + 0.003 +$$

$(y)_{0.2} = 0.6 + 0.18 + 0.028$

$= 0.78 + 0.028$

$= 0.808$

$\frac{dy}{dx} = 2y + 3e^{x}$.

$I.F = e^{\int +2dx} = e^{+2x}$

$\cancel{\frac{dy}{dx} - Py = Q}$

$\cancel{I.F = e^{\int P dx}}$

$\cancel{y = \int I.F \cdot Q + y}$

$e^{2x} \cdot y = \int e^{2x} \cdot 3e^{x} \, dx + c$

$e^{2x} \cdot y = 3\int e^{3x} dx + c$

$e^{-2x} \cdot y = -3 \, e^{-x} + C$

$\cancel{e^{2x} y = e^{3x} + C}$

$I.F = 9e^{3x} + c$

$y \cdot e^{2x} = \int e^{2x} \cdot 3e^{x} + c$

$\cancel{3e^{3x} \cdot y =}$

$y \cdot e^{2x} = e^{3x} + c$

$y = e^{x} - e^{-2x}$.

$y = e^{0.2} - e^{-0.4}$

$\boxed{e^{-2x} \cdot y = -3e^{-x} + C}$

$y = 1.2214 - 0.67032$

$\boxed{y = 0.5510}$

$y = -3e^{x} + C \cdot e^{2x}$

$(y)_0 = 0 = -3 + C \quad C = 3$.

$y = -3e^{x} + 3e^{2x} = 3(e^{2x} + e^{x})$

$(y)_{0.2} = 3(e^{0.4} +$

Q. Use Taylor Series Method to solve the ordinary diff. Equation & hence find value of for $y(0.1)$.

$$\frac{dy}{dx} = 3x + y^2, \quad y(0) = 1.$$

Sol.

$$\left(\frac{dy}{dx}\right)_{0,1} = 0 \times 3 + 1 = 1$$

$$\left(\frac{d^2y}{dx^2}\right)_{0,1} = 3 + 2y\frac{dy}{dx} = 3 + 2 \times 3 = 9.$$

$$\left(\frac{d^3y}{dx^3}\right) = 2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2$$

$$= 2 \times 1 \times 9 + 2 \times 3^2 = 18 + 18 = 36.$$

$$\frac{d^4y}{dx^4} = 2y\frac{d^3y}{dx^3} + 2\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) + 4\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}$$

$$= 2 \times 1 \times 36 + 2 \times 3 \times 9 + 4 \times 3 \times 9$$

$$\frac{d^4y}{dx^4} = 108 + 72 + 54 = 126 + 108$$

$$= 234$$

$$y = y_0 + (x - x_0)\frac{dy}{dx} + \frac{(x-x_0)^2}{2!}\frac{d^2y}{dx^2} + \frac{(x-x_0)^3}{3!}\frac{d^3y}{dx^3} + \frac{(x-x_0)^4}{4!}\frac{d^4y}{dx^4} + \cdots$$

$$y = 1 + x \times 3 + \frac{x^2}{2}\cdot 9 + \frac{x^3}{3 \times 2} \times 36 + \frac{x^4}{4 \times 3 \times 2} \times 234$$

$$y = 1 + 3x + \frac{9x^2}{2} + 6x^3 + \frac{39}{4}x^4$$

$$y(0.1) = 1 + 0.3 + \frac{9}{2}(0.01) + 6(0.001) + \frac{39}{4}(0.0001)$$

$$= 1.3 + 0.045 + 0.006 + 0.000975$$

$$= 1.345 + 0.006 + 0.000975$$

$$= \boxed{1.351975}$$

$$\frac{dy}{dx} = 3x + y^2$$

$$e^{\int -1\,dx} = e^{-x}$$

$$\frac{dy}{dx} - y^2 = 3x$$

$y$.

⇒ **Picards Method :-**

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x,y)\, dx$$

or

$$y - y_0 = \int_{x_0}^{x} f(x,y)\, dx$$

$$y = y_0 + \int_{x_0}^{x} f(x,y)\, dx.$$

Now,

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx$$

$$\vdots \qquad \vdots$$

$$y_{n+1} = y_0 + \int_{x_0}^{x} f(x, y_n)\, dx.$$

**Ques:** Using Picards' Method find a sol of $\frac{dy}{dx} = 1 + xy$ up to 3rd Approximation with cond$^n$ $y(0) = 0$.

Sol

$$\frac{dy}{dx} = 1 + xy$$

$$y_1 = 0 + \int_0^x f(x, 0)\, dx = \int_0^x (1 + x \cdot 0)\, dx$$

$$y_1 = \int_0^x dx = [x]_0^x = x.$$

$$\Rightarrow \boxed{y_1 = x}$$

$$y_2 = 0 + \int_0^x (1 + x \cdot y_1)\,dx = \int_0^x (1+x^2)\,dx$$

$$y_2 = \left[x + \frac{x^3}{3}\right]_0^x = x + \frac{x^3}{3}$$

$$y_3 = 0 + \int_0^x (1 + x \cdot y_2)\,dx = \int_0^x \left\{1 + \left(x^2 + \frac{x^4}{3}\right)\right\}dx$$

$$y_3 = x + \frac{x^3}{3} + \frac{x^5}{15}$$

⇒ <u>Euler's Method</u> :-

$\frac{dy}{dx} = f(x,y)$ , $y(x_0) = y_0$ ; $y = \phi(x)$ (let)

$x_1 = x_0 + h$ or in general $\boxed{x_{n+1} = x_n + h}$

$x_0 \quad x_1 \quad x_2 \; - - - \; x_n \quad x_{n+1}$

$y_0 \quad y_1 \quad y_2 \; - - - \; y_n \quad y_{n+1}$

$$y_{n+1} = \phi(x_{n+1}) = \phi(x_n + h) = \phi(x_n) + h \cdot \phi'(x_n) + \frac{h^2}{2!}\phi''(x_n) - - -$$

$$= y_n + h \cdot f(x_n, y_n) + - - -$$

$$\Rightarrow \boxed{y_{n+1} = y_n + h \cdot f(x_n, y_n)}$$

Que. Using Euler's method find an approx. value of $y$ corresponding to $x = 2$, given that $\frac{dy}{dx} = x + 2y$, $y(1) = 1$; take $h = 0.1$.

Sol. $\frac{dy}{dx} = x + 2y = f(x, y)$ ; $y(2) = ?$

$y_{n+1} = y_n + h \cdot f(x_n, y_n)$

put $n = 0$

$y_1 = y_0 + 0.1\ f(x_0, y_0)$

$y_1 = 1 + 0.1 \times (1 + 2)$

$y_1 = 1.3$

put $n = 1$ ; $x_1 = x_0 + h = 1 + 0.1 = 1.1$

$y_2 = y_1 + 0.1 \times f(x_1, y_1)$

$y_2 = 1.3 + 0.1 \times (x_1 + 2y_1)$

$= 1.3 + 0.1\ (1.1 + 2 \times 1.3)$

$1.1 + 2.6$

$y_2 = 1.3 + 0.37$

$y_2 = 1.67$

$x_2 = 1.1 + 0.1 = 1.2$

Put $n = 2$

$y_3 = y_2 + h \cdot f(x_2, y_2)$

$= 1.67 + 0.1\ (1.2 + 3.34)$

$= 1.67 + 0.1(4.54)$

$y_3 = 1.67 + 0.454 = 2.124$

$x_3 = 1.2 + 0.1 = 1.3$

$y_4 = y_3 + h f(x_3, y_3) = 2.124 + 0.1(1.3 + 2 \times 2.124)$

$= 2.124 + (0.1)(1.3 + 4.248)$

$= 2.124 + 0.1 \times 5.548$

$y_4 = 2.124 + 0.548 = 2.672$

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$x_4 = 1.4$

$y_5 = y_4 + h f(x_4, y_4)$

$= 2.672 + 0.1 \times (1.4 + 2 \times 2.672)$

$= 2.672 + 0.1 (1.4 + 5.344)$

$= 2.672 + 0.6744$

$y_5 = 3.3464$

$x_5 = 1.5$

$y_6 = y_5 + h (f(x_5, y_5))$

$= 3.3464 + 0.1 \times 1.5 + (2 \times 3.3464)$

$= 3.3464 + (1.5 + 6.6928) 0.1$

$= 3.3464 + 0.81928$

$y_6 = 4.16568$

$x_6 = 1.6$

$y_7 = y_6 + h (f(x_6, y_6))$

$= 4.16568 + 0.1 (1.6 + 8.33136)$

$= 4.16568 + 0.993136$

$y_7 = 5.158816 = 5.1588$

$x_7 = 1.7$

$y_8 = y_7 + h(f(x_7, y_7))$

$= 5.1588 + 0.1 (1.7 + 10.3176)$

$5.1588 + 0.1 \times 12.0176$

$5.1588 + 1.20176$

$\cancel{.16056}$

$y_8 = 6.36056$

$y_9 = 6.36056 + 0.1 (1.8 + 12.72112)$

$6.36056 + 1.452112$

$= 7.812672$

$y_{10} = 7.812672 + 0.1 (1.9 + 7.812672 \times 2)$

$\cancel{y_{10}} = 9.4220$

$x = 1.5$

$y_2 = 1 + 0.5(1.5 + 2)$

$2.5 \times 0.5$

$= 1 + 1.25$

$y_2 = 2.25$

$y_3 = 2.25 + 0.5(2 + 2 \times 2.25)$

$2.25 + 0.5(2 + 4.50)$

$2.25 + 0.5(6.50)$

$= 2.25 + 3.25$

$y_3 = 5.50 \quad 3.5 + 2.25$

$y = 5.75$

Q. Apply Euler

$\frac{dy}{dx} = x + 3y, \quad y(0) = 1$ & hence find an approximate value of $y$ when $x = 1$. taking $h = 0.1$

Sol: $x = 0, y = 1$

$y_1 = 1 + 0.1(0 + 3 \times 1) = 1.3$

$x_1 = 0.1$, $y$

$y_2 = 1.3 + 0.1(0.1 + 3 \times 1.3)$

$1.3 + 0.1(0.1 + 3.9)$

$y_2 = 1.3 + 0.1(4) = 1.7$

$x_2 = 0.2$

$y_3 = 1.7 + 0.1(0.2 + 3 \times 1.7)$

$= 1.7 + 0.1(0.2 + 5.1)$

$= 1.7 + 0.1(5.3)$

$y_3 = 1.7 + 0.53 = 2.23$

$x_3 = 0.3$

$y_4 = 2.23 + 0.1(0.3 + 3 \times 2.23)$

$2.23 + 0.1(0.3 + 6.69)$

$= 2.23 + (6.99)0.1$

$y_4 = 2.23 + 0.699 = 2.929$

$y_4 = 2.92$ ; $x_4 = 0.4$

$y_5 = 2.92 + 0.1(0.4 + 3 \times 2.92)$
$= 2.92 + 0.1(0.4 + 8.76)$
$= 2.92 + 0.1 \times 9.16$
$= 2.92 + 0.916$
$y_5 = 3.836$

$x_5 = 0.5$

$y_6 = 3.836 + 0.1(0.5 + 3 \times 3.836)$
$3.836 + 0.1(0.5 + 11.508)$
$3.836 + 0.1(12.508)$
$3.836 + 1.2508$
$y_6 = 5.0868$
$x_6 = 0.6$

$y_7 = 5.0868 + 0.1(0.6 + 3 \times 5.0868)$
$y_7 = 5.0868 + 1.58604$
$y_7 = 6.67284 = 6.673$
$x_7 = 0.7$

$y_8 = 6.673 + 0.1(0.7 + 3 \times 6.673)$
$y_8 = 8.7449$
$x_8 = 0.8$

$y_9 = 8.745 + 0.1(0.8 + 3 \times 8.745)$
$y_9 = 11.03409$

$y_{10} = 11.03409 + 0.1(0.9 + 11.03409)$
$y_{10} = 14.4343$

22/10/18

Euler's Modified Method :-

$$y_{n+1} = y_n + h\,\phi'(x) + \frac{h^2}{2!}\phi''(x) + \frac{h^3}{3!}\phi'''(x) \quad \text{--- (i)}$$

$$\left(\frac{dy}{dx}\right)_{n+1} = \left(\frac{dy}{dx}\right)_n + h\,\phi''(x) + \frac{h^2}{2!}\phi'''(x) + \dots$$

$$\left(\frac{dy}{dx}\right)_n = f(x_n, y_n)$$

$$\left(\frac{dy}{dx}\right)_{n+1} = f(x_{n+1}, y_{n+1}).$$

$$f(x_{n+1}, y_{n+1}) = f(x_n, y_n) + h\,\phi''(x) + \frac{h^2}{2!}\phi'''(x) + \dots \quad \text{--- (ii)}$$

Multiply by $h/2$ to (ii) & subtract from (i).

$$y_{n+1} - \frac{h}{2} f(x_{n+1}, y_{n+1}) = y_n + h\,f(x_n, y_n)$$
$$- \frac{h}{2} f(x_n, y_n) + \frac{h^2}{2!}\phi''(x) - \frac{h^2}{2\cdot 2}\phi''(x)$$
$$+ \frac{h^3}{3!}\phi'''(x) - \frac{h^3}{2\cdot 2}\phi'''(x) + \dots$$

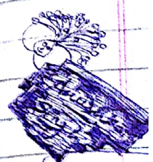
$$= y_n + \frac{h}{2} f(x_n, y_n) - \frac{h^3}{12}\phi'''(x) + \dots$$

neglecting $h^3$ & higher order

$$y_{n+1} - \frac{h}{2} f(x_{n+1}, y_{n+1}) = y_n + \frac{h}{2} f(x_n, y_n).$$

$$\boxed{y_{n+1} = y_n + \frac{h}{2}\left( f(x_{n+1}, y_{n+1}) + f(x_n, y_n)\right)}$$

Ques:



4.45
2.5
6.45

[redacted] sheet

Ques: Use Euler's Modified Formula to calculate $y$ at $x = 2$.

$\frac{dy}{dx} = x + 2y$. given $y(1) = 1$. $h = 0.1$.

$y_1 = 1.3$

Using Modified formula,

$y_1 = 1 + \left\{ \frac{(1+2) + (1.1 + 2 \times 1.3)}{2} \right\} 0.1$

$y_1 = 1 + \left\{ \frac{3 + 1.1 + 2.6}{2} \right\} = 1 + \frac{(0.1)(6.7)}{2}$

$y_1 = \cancel{1.335}\ \frac{3.67}{2} = 1.335$

$y_2 = y_1 + h(x_1 + 2y_1) = 1.712$

$x_2 = 1.2$

using MEF, $y_2 = 1.335 + \frac{0.1}{2}\left[(1.2 + 2 \times 1.67) + (1.1 + 2 \times 1.335)\right]$

$= 1.335 + \frac{0.1}{2}\left[1.2 + 3.34 + 1.1 + 2.67\right]$

$= 1.335 + 0.1 \times \frac{8.31}{2} = 1.335 + 0.4155$

$y_2 = \cancel{1.7505}$ $y_2 = \cancel{1.76}$ 1.7547 $x_2 = 1.2$

$y_3 = 2.124$

$x_3 = 1.3$

$\cancel{y_3 = 1.76}$ +

$y_3 = 1.754 + (0.1)(1.2 + 2 \times 1.754)$

$= 1.754 + 0.1($

$y_3 = 2.2248$; $x_3 = 1.2$.

using MEF

$y_3 = 1.7505 + \frac{0.1}{2}\left[1.2 + 2 \times 1.7505 + 1.3 + 2 \times 2.124\right]$

$= 1.7505 + \frac{0.1}{2}\left[6.001 + 4.248\right]$

$= 1.7505 + 0.51245$

$= 2.26295$

$y_3 = 1.7547 + \frac{0.1}{2}\left[\right.$

$\left[1.3 + 2 \times 2.225 + 1.2 + 2 \times 1.754\right]$

$y_3\ \left[1.3 + 2.45 + 1.2 + 3.508\right] \frac{0.1}{2}$

4.45
2.5
6.95
3.508

~~10.458~~ 5.229

0.17547

$y_7 = 4.758 + 0.1\,(1.6 + 2 \times 4.758)$

$y_7 = 5.86966$

$x_7 = 1.7$

$y_7' = 4.758 + \frac{0.1}{2}\left[1.7 + 2 \times 5.88 + \underbrace{1.6 + 2 \times 4.758}_{9.516}\right]$

$= 5.907$

$y_8 = 5.907 + 0.1\,(1.7 + 2 \times 5.907)$

$= 7.2584$

$y_8' = 5.907 + \frac{0.1}{2}\left[1.8 + \underbrace{2 \times 7.258}_{14.516} + 1.7 + 2 \times 5.907\right]$

$= \cancel{7.3935}\ 8.885$

$y_9 = \cancel{7.3935}^{8.885} + 0.1\,(1.8 + 2 \times \cancel{7.3935}^{8.885})$

$= \cancel{9.0517}\ 10.837$

$\cancel{y_9' = 7.3935 + \frac{0.1}{2}\left[1.9 + 2 \times 9.0517 + 1.\right.}$

21.674

$y_9' = 8.885 + \frac{0.1}{2}\left[1.9 + 2 \times 10.837 + 1.8 + 2 \times 8.885\right]$

$= 10.9522$

Qu

So

⇒ Runge's ~~Mets~~ Kutta Method of Fourth order:—

$$y_1 = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$K_1 = h\, f(x_0, y_0)$$

$$K_2 = h\, f\left(x_0 + \frac{h}{2},\ y_0 + \frac{K_1}{2}\right)$$

$$K_3 = h \cdot f\left(x_0 + \frac{h}{2},\ y_0 + \frac{K_2}{2}\right)$$

$$\cancel{K_4 = h\, f\left(x_0 + \frac{h}{2},\ y_0 + \frac{K_3}{2}\right)}$$

$$K_4 = h\, f(x_0 + h,\ y_0 + K_3)$$

Que Apply RK Method of 4th order to find the value of 'y' when $x = 0.4$, $\frac{dy}{dx} = x + y$, $y(0) = 1$, taking $h = 0.1$.

Sol:

$$K_1 = h\, f(1,0) = 0.1 \times (0+1) = 0.1$$

$$K_2 = 0.1 f(0.05,\ 1 + 0.05) = 0.1 f(0.05, 1.05)$$

$$= 0.1(1.55) = \cancel{0.155}\ 0.11$$

$$K_3 = 0.1 \times f(0.05,\ 1 + 0.055)$$

$$= 0.1(0.05 + 1.55)$$

$$= 0.1105$$

$$K_4 = 0.1(0.1,\ 1.1105) = 0.12105$$

$$y(0.1) = 1 + \frac{1}{6}(0.1 + 0.11 \times 2 + 2 \times 0.1105 + 0.12105)$$

$$= 1 + \frac{1}{6}[0.1 + 0.22 + 0.2210 + 0.12105]$$

$$= 1.11034$$

29/10/18

$K_1 = h \cdot f(x_1, y_1)$

$= 0.1 \times f(0.1, 1.11034)$

$= 0.1 \times 1.21034 = 0.121034$

$K_2 = 0.1 \times f\left(0.1 + \frac{0.1}{2}, 1.11034 + \frac{\overset{0.060517}{\cancel{0.121034}}}{2}\right)$

$= 0.1 \times f(0.155, 1.170857)$

0.055

$= 0.1 \times 1.325857 = 0.13258$

$K_3 = 0.1 \times f\left(0.1 + \frac{0.1}{2}, 1.11034 + \frac{\overset{0.06625}{\cancel{0.1325}}}{2}\right)$

$= 0.1 \times f(0.155, 1.17659)$

$= 0.1 \times 1.33159 = 0.133159$

$K_4 = 0.1 \times f(0.1 + 0.1, 1.11034 + 0.1331)$

$= 0.1 \times (0.2, 1.24344)$

$= 0.1 \times 1.44344$

$= 0.144344$

$y = 1.11034 + \frac{1}{6}[0.121034 + 0.26416 + 0.266318 + 0.144344]$

$y = 1.24284$

1/10/18

# STATISTICAL METHODS

Moment :-

Moment about mean :-

$r^{th}$ moment about mean is defined as

$$\boxed{\mu_r = \frac{1}{N} \sum_{i=1}^{n} (x_i - \bar{x})^r f_i}$$

$N = \sum f_i$

$f_i$ = frequency

$x_i$ = random variable.

$$\mu_0 = \frac{1}{N} \sum f_i (x_i - \bar{x})^0 = \frac{\sum f_i}{N} = 1.$$

$$\mu_1 = \frac{1}{N} \sum f_i (x_i - \bar{x}) = \frac{1}{N} \sum f_i x_i - \frac{\sum f_i \bar{x}}{N} = \bar{x} - \bar{x} = 0$$

$$\mu_2 = \frac{1}{N} \sum_{i=1}^{n} f_i (x_i - \bar{x})^2$$

$$\mu_3 = \frac{1}{N} \sum_{i=1}^{n} f_i (x_i - \bar{x})^3$$

$$\mu_4 = \frac{1}{N} \sum_{i=1}^{n} f_i (x_i - \bar{x})^4$$

Moment about an arbitrary point a:

It is defined as:

$$\boxed{\mu_r' = \frac{1}{N} \sum_{i=1}^{n} f_i (x_i - a)^r}$$

$$\mu_0' = \frac{1}{N} \sum_{i=1}^{n} f_i (x_i - a)^0 = 1$$

$$\mu_1' = \frac{1}{N} \sum_{i=1}^{n} f_i (x_i - a) = \sum_{i=1}^{n} \frac{f_i x_i}{N} - a \sum_{i=1}^{n} \frac{f_i}{N} = \bar{x} - a$$

$\mu_2' = \frac{1}{N} \sum_{i=1}^{n} f_i (x_i - a)^2$

$\mu_3' = \frac{1}{N} \sum_{i=1}^{n} f_i (x_i - a)^3$

Moment about Origin :-

$$\boxed{\nu_r = \frac{1}{N} \sum f_i x_i^r}$$

$\nu_0 = 1$

$\nu_1 = \frac{\sum f_i x_i}{N} = \bar{x}$

$\nu_2 = \frac{1}{N} \sum f_i x_i^2$

$\nu_3 = \frac{1}{N} \sum f_i x_i^3$

⟹ Relation b/w moment about mean & moment about any arbitrary point :-

$\mu_r = \frac{1}{N} \sum f_i (x_i - \bar{x})^r$

$= \frac{1}{N} \sum f_i \{ (x_i - a) - (\bar{x} - a) \}^r$

$= \frac{1}{N} \sum f_i ((x_i - a) - \mu_1')^r \quad \{ \because \mu_1' = \bar{x} - a \}$

$= \frac{1}{N} \sum f_i \{ (x_i - a)^r - {}^rC_1 (x_i - a)^{r-1} \cdot \mu_1' + {}^rC_2 (x_i - a)^{r-2} \mu_1'^2 + {}^rC_3 (x_i - a)^{r-3} \mu_1'^3 + \ldots \}$

$= \frac{1}{N} \sum f_i (x_i - a)^r - {}^rC_1 \frac{1}{N} \sum f_i (x_i - a)^{r-1} \cdot \mu_1' +$

$$\frac{{}^{r}C_2}{N} \sum f_i (x_i - a)^{r-2} \mu_1'^{2} - \frac{{}^{r}C_3}{N} \sum f_i (x_i - a)^{r-3} \mu_1'^{3} + \cdots$$

$$\mu_r = \mu_r' - {}^{r}C_1 \mu_{r-1}' \mu_1' + {}^{r}C_2 \mu_{r-2}' \mu_1'^{2} - {}^{r}C_3 \mu_{r-3}' \mu_1'^{3} + \cdots (-1)^r (\mu_1')^r$$

$$\mu_1 = \mu_1' - \mu_0' \mu_1' = \mu_1' - \mu_1' = 0 \Rightarrow \boxed{\mu_1 = 0}$$

$$\mu_2 = \mu_2' - 2\mu_1' \cdot \mu_1' + \mu_0' \mu_1'^{2} = \mu_2' - \mu_1'^{2} \Rightarrow \boxed{\mu_2 = \mu_2' - \mu_1'^{2}}$$

$$\boxed{\mu_3 = \mu_3' - 3\mu_2' \mu_1' + 2\mu_1'^{3}}$$

$$\boxed{\mu_4 = \mu_4' - 4\mu_3' \mu_1' + 6\mu_2' \mu_1'^{2} - 3\mu_1'^{4}}$$

⇒ Relation b/w moment about any arbitrary pt. & moment about mean.

$$\mu_1' = \mu_1 - a$$

$$\mu_2' = \mu_2 + \mu_1'^{2}$$

$$\mu_3' = \mu_3 + 3\mu_2 \mu_1' + \mu_1'^{3}$$

$$\mu_4' = \mu_4 + 4\mu_3 \mu_1' + 6\mu_2 \mu_1'^{2} + \mu_1'^{4}$$

⇒ Relation b/w moment about origin & moment about mean :-

$$\nu_1 = \bar{x}$$

$$\nu_2 = \mu_2 + \bar{x}^2$$

$$\nu_3 = \mu_3 + 3\mu_2 \bar{x} + \bar{x}^3$$

$$\nu_4 = \mu_4 + 4\mu_3 \bar{x} + 6\mu_2 \bar{x}^2 + \bar{x}^4.$$

Ques:- The first four moment of a distribution about a value 4, of the variate -1.5, 17, -30 & 108 calculate the first four moment about the mean.

$\mu_1' = -1.5$

$\mu_2' = 17$

$\mu_3' = -30$

$\mu_4' = 108$

$\mu_1 = 0$

$\mu_2 = \mu_2' - \mu_1'^2$

$= 17 - (-1.5)^2$

$= 17 - 2.25$

$= 14.75$

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$

$= -30 - 3 \times 17 \times (-1.5) + 2(-1.5)^3$

$= -30 + 7.65 - 2 \times 1.5 \times 2.25$

$= \cancel{-37} \; 39.75$

$\cancel{-106.5 - 3 \times 2.25}$

$= \cancel{-106.5 - 6.75}$

$= \cancel{-113.25} \quad 39.75$

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

$= 108 - 4 \times 30 \times (1.5) + 6 \times 17 \times (1.5)^2 - 3 \times (1.5)^4$

$= \cancel{108 + 180 + 229.5 - 3\times}$

$= 108 - 180 - 15.1875$

$= 142.31$

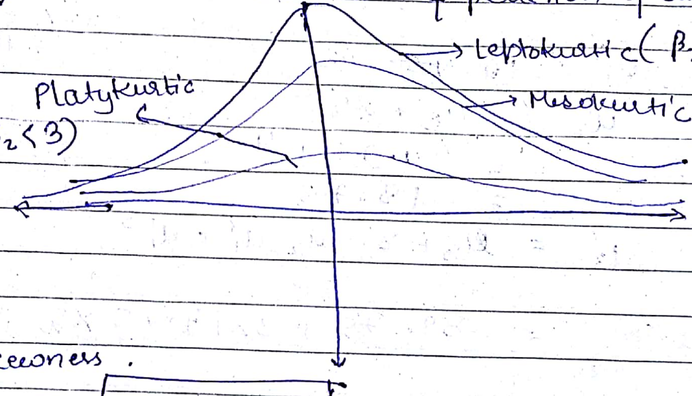
Sol→

$(\beta_1)$

Skewness : Lack of symm. is k/as skewness.

Negatively skewed → enlarged left tail
Positively skewed → enlarged right tail

→ $(\beta_2)$ Kurtosis : Measurement of peakness of curve.




Skewness.

$$\boxed{\beta_1 = \frac{\mu_3^2}{\mu_2^3}}$$

Kurtosis

$$\boxed{\beta_2 = \frac{\mu_4}{\mu_2^2}}$$

Q1. The first four moment of a distribution about the value 4 of the variant are. −1.5, 17, −30, & 108. Find moment about mean, skewness & kurtosis. Also find moment about origin & about pt. $\alpha = 2$

Sol → $\cancel{\beta_1} = \cancel{\frac{-30 \times -30}{17 \times 17 \times 17}}$ $\beta_1 = \frac{39.75 \times 39.75}{14.75 \times 14.75 \times 14.75}$

$= 0.4940.$

$\cancel{\beta_2} = \cancel{108}$

$\beta_2 = \frac{142.31}{14.75 \times 14.75}$ $= 0.6541$

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Moment about origin.

$\nu_0 = 0.$

$\bar{x} - 4 = -1.5 \quad \bar{x} = 2.5$

$\nu_1 = \bar{x} = 2.5$

$\nu_2 = \mu_2 + \bar{x}^2 = 14.75 + 6.25 = 21$

$\nu_3 = \mu_3 + 3\mu_2\bar{x} + \bar{x}^3 = 166 \; ; \; \nu_4 = 142.31 + 4 \times 39.75 \times 1.5 + 6 \times (1.5)^2 \times 14.75 + (1.5)^4 = 1132.$

about $x = 2$.

$\mu_1' = \cancel{12.75} \; 2$

$\mu_2' = \mu_2 + \mu_1'^2$

$= 14.75 + 4.$

$= 18.75$

$\mu_3' = \mu_3 + 3\mu_2\mu_1' + \mu_1'^3$

$= 39.75 + 3 \times 14.75 \times 2 + 8$

$= 136.25.$

$\mu_4' = \mu_4 + 4\mu_3\mu_1' + 6\mu_2\mu_1'^2 + \mu_1'^4$

$= 142.31 + 4 \times 39.75 \times 2 + 6 \times 14.75 \times 4 + 16$

$= 142.31 + 318 + 354 + 16$

$= 830.31.$

about $x = 2$,

$\mu_1' = \bar{x} - a = 0.5$

$\mu_2' =$

## Correlation

## Covariance :

$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$

$$Cov(X,Y) = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{n}$$

$$= \frac{(x_1-\bar{x})(y_1-\bar{y}) + \cdots + (x_n-\bar{x})(y_n-\bar{y})}{n}$$

Shortcut to find covariance :-

Find $\sum x_i$, $\sum y_i$, $\sum x_i y_i$

$$Cov(X,Y) = \frac{\sum x_i y_i}{n} - \frac{\sum x_i}{n} \cdot \frac{\sum y_i}{n}$$

Q. Find the covariance of the pairs of obs.
$(1,2), (2,3), (4,5), (6,7)$

Sol. $\sum x_i = 13$ ; $\sum y_i = 17$
$\sum x_i y_i = 2 + 6 + 20 + 42 = 70$

$$Cov(X,Y) = \frac{70}{4} - \frac{13}{4} \times \frac{17}{4}$$

$$= \frac{70}{4} - \frac{221}{16}$$

$$= \frac{280 - 221}{16} = \frac{59}{16}$$

$$= 3.6875$$

Whenever two variables $x$ & $y$ are so related that an ~~entrtts~~ increase in 1 results increase or decrease in the other, then variables are said to be corelated.

For ex - yield of crop depends upon the amount of rainfall.

Types of Corelation :-

1. Positive corelation : if an increase or decrease in one variable results increase or decrease in the second variable then there is a +ve corelation.

2. -ve corelation : if an increase or decrease in one variable results decrease or increase in the other variable then they are said to be negatively corelated.

3. Linear Corelation :- If all the plotted pts lie on approximately on a st. line then the two Quantities are said to be linearly corelated.

To find coeff. of corelation

1. Karl Pearson coeff. of Corelation :-

$$r = \frac{\text{Covariance}(X,Y)}{\sqrt{(\text{Variance}X)} \times \sqrt{(\text{Variance}Y)}}$$

$$= \frac{\frac{\Sigma XY}{N}}{\sqrt{\frac{X^2}{N}} \times \sqrt{\frac{Y^2}{N}}} = \frac{\Sigma XY}{\sqrt{\Sigma X^2}\sqrt{\Sigma Y^2}}$$

Q.

Sol

=>

$X = x - \bar{x}$ ; $Y = y - \bar{y}$

Q. Calculate the corelation Coff. b/w $x$ & $y$ for the following data.

| $x$ | $y$ | $X$ = $x - \bar{x}$ | $Y$ = $y - \bar{y}$ | $X^2$ | $Y^2$ | $XY$ |
|---|---|---|---|---|---|---|
| 21 | 60 | -36 | -32 | 1296 | 1024 | 1152 |
| 23 | 71 | -34 | -21 | 1156 | 441 | 714 |
| 30 | 72 | 27 | -20 | 729 | 400 | -540 |
| 54 | 83 | -3 | -9 | 9 | 81 | 27 |
| 57 | 110 | 0 | +18 | 0 | 324 | 0 |
| 58 | 84 | 1 | -8 | 1 | 64 | -8 |
| 72 | 100 | 15 | +8 | 225 | 64 | 120 |
| 78 | 92 | 21 | 0 | 441 | 0 | 0 |
| 87 | 113 | 30 | +21 | 900 | 441 | 630 |
| 90 | 135 | 33 | +43 | 1089 | 1849 | 1419 |
| | | | | | 4688 | 4594 |

$\sqrt{\Sigma X^2}$

$5846 = 76.45$

Sol/n $\bar{x} = \frac{\Sigma x_i}{N} = \frac{570}{10} = 57$

$\bar{y} = \frac{\Sigma y_i}{N} = \frac{920}{10} = 92$

$r = \frac{4594}{\sqrt{5846} \times \sqrt{4688}}$

$r = 0.877$

⇒ Spearman Rank Corelation Coefficient.

$$r = 1 - \frac{6 \Sigma d^2}{n(n^2 - 1)}$$

$d = R_1 - R_2$,

$R_1 \to$ Ranking of 1st variable

$R_2 \to$ Ranking of 2nd variable

12/11/18

Q. Calculate the Spearman Rank correlation coeff.

| ~~x~~ | ~~y~~ | X | Y | $R_1$ | $R_2$ | $d = R_1 - R_2$ | $d^2$ |
|---|---|---|---|---|---|---|---|
| ~~25~~ | ~~8~~ | 68 | 62 | 4 | 4 | 0 | 0 |
| ~~36~~ | ~~10~~ | 64 | 58 | 5 | 6 | -1 | 1 |
| ~~22~~ | ~~15~~ | 73 | 68 | 3 | 3 | 0 | 0 |
| ~~35~~ | ~~17~~ | 80 | 45 | 7 | 7 | 0 | 0 |
| ~~37~~ | ~~20~~ | 64 | 81 | 6 | 1 | 5 | 25 |
| ~~40~~ | ~~23~~ | 80 | 60 | 1 | 5 | -4 | 16 |
| | | 75 | 68 | 2 | 2 | 0 | 0 |

$$r = 1 - \frac{6 \times 42}{7(48)} \qquad 1 - \frac{252}{336} = \frac{84}{336}$$

In case of repition:-

$$r = 1 - \frac{6\left[\Sigma d^2 + \frac{m_1^3 - m_1}{12} + \frac{m_2^3 - m_2}{12} + \cdots\right]}{n(n^2+1)}$$

75, $m_1 = 2$

64, $m_2 = 2$.

⇒ Reg

12/11/18

⇒ Regression :-

$$y = a + bx$$

$$e_1 = y_1 - (a + bx_1)$$
$$e_2 = y_2 - (a + bx_2)$$
$$\vdots$$
$$e_n = y_n - (a + bx_n)$$

$$S = e_1^2 + e_2^2 + \cdots + e_n^2$$
$$= \sum_{i=1}^{n} \{ y_i - (a + bx_i) \}^2$$

$$\frac{\partial S}{\partial a} = 0 \; ; \; \frac{\partial S}{\partial b} = 0$$

$$-2 \sum_{i=1}^{n} \{ y_i - (a + bx_i) \} = 0$$

$$\Rightarrow \sum_{i=1}^{n} \{ y_i - (a + bx_i) \} = 0$$

$$\Rightarrow \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} a + b x_i$$

$$\Rightarrow \boxed{\sum y_i = na + b \sum x_i} \quad \text{— (I)}$$

Now, $\frac{\partial S}{\partial b} = 0$

$$\Rightarrow -2 \sum_{i=1}^{n} x_i \{ y_i - (a + bx_i) \}$$

$$\Rightarrow \boxed{\sum_{i=1}^{n} xy = a \sum x + b \sum x^2} \quad \text{— (II)}$$

From (I).

$$\sum y = na + b\sum x$$

$$\frac{\sum y}{n} = a + b\frac{\sum x}{n}$$

$$\bar{y} = a + b\bar{x}$$

$\Rightarrow$ Line pass through $(\bar{x}, \bar{y})$

From (II)

$$\sum xy = a\sum x + b\sum x^2$$

Replacing shifting origin to $(\bar{x}, \bar{y})$

$$\sum (x-\bar{x})(y-\bar{y}) = a \underbrace{\cancel{\sum (x-\bar{x})}}_{\text{moment of mean}}^{0} + b\sum (x-\bar{x})^2$$

$$b = \frac{\sum (x-\bar{x})(y-\bar{y})}{\sum (x-\bar{x})^2} = \frac{\sum XY}{\sum X^2}$$

$$r = \frac{\sum XY}{\sqrt{\sum X^2}\sqrt{\sum Y^2}} = \frac{\sum XY}{n\sqrt{\frac{\sum X^2}{n}}\sqrt{\frac{\sum Y^2}{n}}} = \frac{\sum XY}{n\sigma_x \sigma_y}$$

$$\therefore \sum XY = nr\sigma_x \sigma_y$$

$$\Rightarrow b = \frac{nr\sigma_x \sigma_y}{\sum X^2} = \frac{r\sigma_x \sigma_y}{\frac{\sum X^2}{n}} = \frac{r\sigma_x \sigma_y}{\sigma_x^2}$$

Slope -

$$\boxed{b = r\frac{\sigma_y}{\sigma_x}}$$

Line of regression of X on Y.

$$y - \bar{y} = r \frac{\sigma_y}{\sigma_x}(x - \bar{x})$$

Similarly, we can find line of regression of Y on X.

$$(x - \bar{x}) = r \frac{\sigma_x}{\sigma_y}(y - \bar{y})$$

angle b/w two lines,

$$\tan\theta = \left| \frac{m_2 - m_1}{1 + m_1 m_2} \right|$$

here, $m_1 = r\frac{\sigma_y}{\sigma_x}$ ; $m_2 = \frac{1}{r} \cdot \frac{\sigma_y}{\sigma_x}$.

**Ques:-** Two line of regression are given by $5y - 8x + 17 = 0$ & $2y - 5x + 14 = 0$, if $\sigma_y^2 = 16$ then, find

(I) mean of $x$ & $y$

(II) $\sigma_x^2$ & coefficient of correlation b/w $x$ & $y$.

Sol:- (I) Solve both eqn.

(II) $y = \frac{8x}{5} - \frac{17}{5}$ ; $x = \frac{2y}{5} + \frac{14}{5}$

$r\frac{\sigma_y}{\sigma_x} = \frac{8}{5}$ — (i) ; $r\frac{\sigma_x}{\sigma_y} = \frac{2}{5}$ (ii).

⇒ **Binomial Distribution :-** Happening of an event 'r' times in 'n' trials is known as B.D. and its probability distribution ~~is~~ is given by,

$$P(r) = {}^nC_r \cdot p^r q^{n-r}$$

p : probability of happening an event.
q : " " not " " "

**Ex.** Find the probability of getting four heads in 6 trials.

$p = \frac{1}{2}$ ; $q = \frac{1}{2}$

$n = 6$ ; $r = 4$.

$$P(r) = {}^6C_4 \cdot \left(\frac{1}{2}\right)^4 \cdot \left(\frac{1}{2}\right)^2$$

$$\cancel{\frac{6\times5\times4\times3}{4\times3\times2\times1}} \left(\frac{1}{2}\right)^6$$

$$P(r) = \frac{15}{64}$$

**Note ⇒** Mean of the Binomial distribution is $np$ & Standard Deviation is $\sqrt{npq}$.
Variance = $npq$.

**Ques :-** If mean & variance of a ~~B~~ B. distribution are 4 & 2, then find the prob. of -
(i) exactly 2 success (ii) less than 2 success
(iii) atleast 2 success.

**Sol :-** $np = 4$
$npq = 2$ ⊙ $q = \frac{1}{2}$ ⊙ $p = \frac{1}{2}$ ⊙ $n = 8$.

(i) $P_2$
(ii) $P_0 + P_1$
(iii) $P_2 + P_3 + P_4 + P_5 + P_6 + P_7 + P_8$
$= 1 - (P_0 + P_1)$