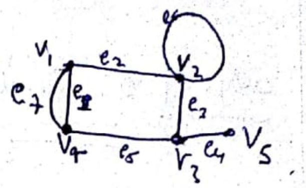
4/10/18

UNIT-III

Graph Theory



Graph: A graph is a collection of edges & vertices.

$G=(V,E)$ is consist of set of objects

$V=\{v_1, v_2, v_3 ---\}$ whose elements are called vertex & another set $E=\{e_1, e_2, e_3 ....\}$ whose elements are called edges such that each edge e_K is identified with ordered pair (V_i, V_j) of vertices V_i, V_j which is associated with edge e_K are called end vertices of edge e_K.

The common representation of a graph is by the means of a diagram.

$|V|$ = no. of vertex = 5

$|e|$ = no. of edges = 6.

Self loop:- An edge which has same vertex as its end vertices is called a self-loop.

ex- e_6 is a self-loop.

Parallel edges:- Edges having same end vertices are K/as || edges.

ex- e_7 & e_1 are || edges.

Simple Graph:- A graph without self-loop & || edges.

Multi Graph:- A Graph which has self-loop & || edges.

Incidence & Adjacency:- Let e_K be ~~the~~ an edge joining two vertices V_i & V_j in a graph $G=(V,E)$, then e_K is called incident vertices V_i & V_j.

The vertices V_i & V_j are called adjacent if there exist an edge joining two vertices V_i & V_j.

→ **Degree of a vertex**:- The degree of a vertex 'v' in a graph is denoted by $d(v)$ & defined as the no. of edges incident on $v(G)$. with self-loop counted twice.

08/10/18: Indian Airforce Day

→ **Isolated vertex**:- A vertex 'v' of graph 'G' is said to be isolated if degree of 'v' is 0.

i.e. $d(v) = 0$

→ **Pendant vertex**:- A vertex 'v' of graph 'G' is said to be pendant if degree of 'v' is 1.

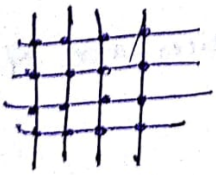
i.e. $d(v) = 1$.

→ **Null Graph**:- A graph $G = (V, E)$ is said to be Null graph if set of vertices V is non-empty & set of edges 'E' is empty, i.e. a graph is said to be Null graph if all of its vertices are isolated.

→ **Finite & Infinite Graph**:-

A graph in which no. of vertices as well as no. of edges are finite, is called finite graph, otherwise infinite graph.

ex -



Portion of an infinite graph.

⇒ **Theorem** :- The sum of degree of all vertices in a graph is equal to twice to the number of edges.

Proof:- each edge in a graph contribute two degrees to the total no. of degree, hence no. total degree $= 2 \times$ edge. in any graph,

here, 'e' is no. of edges.

$$\sum_{v \in V} d(v) = 2|E|$$

Scanned by CamScanner

⇒ Even & odd vertices:-
A vertex 'v', having even degree is called even vertex and with odd degree is called odd vertex.

⇒ Theorem :- The number of odd vertices in a graph is always even.

Proof: Let $G = (V, E)$ be a graph & V_o & V_e be set of odd vertices & even vertices, then

$$V_o \subseteq V \quad \& \quad V_e \subseteq V$$

$$V_o \cap V_e = \phi \quad \& \quad V_o \cup V_e = V.$$

Thus,
$$\sum_{v \in V} d(v) = \sum_{v \in V_o} d(v) + \sum_{v \in V_e} d(v)$$

$$2|E| = \sum_{v \in V_o} d(v) + \sum_{v \in V_e} d(v)$$

$$2|E| = \sum_{v \in V_o} d(v) + 2k \qquad \{2k : \text{even}\}$$

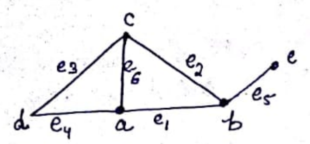
$$\Rightarrow \sum_{v \in V_o} d(v) = 2(|E| - k) = \text{even} \quad (1)$$

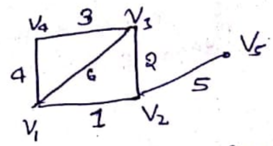
L.H.S of eqn is sum of odd no. can be even, iff no. of terms are even, hence no. of odd degree vertex is even. # Proved.

⇒ Isomorphic Graphs:- Two Graphs G_1 & G_2 are said to be isomorphic if there is one to one correspondence b/w their vertices & in between their edges s.t. ~~relation~~ corresponding edge incidence relation is preserved ~~incidence~~ i.e.
Two graphs $G_1 = (V_1, E_1)$ & $G_2 = (V_2, E_2)$ are said to b isomorphic if there is one-to-one correspondence b/w V_1 & V_2 & in between E_1 & E_2 such that, corresponding edges of G_1 & G_2 corresponds to end vertices of G_1 & G_2.

Scanned by CamScanner

Ex





Vertex correspondence :-

$a \longleftrightarrow V_1$

$b \longleftrightarrow V_2$

$c \longleftrightarrow V_3$

$d \longleftrightarrow V_4$

$e \longleftrightarrow V_5$

Edge correspondence

$e_1 \longleftrightarrow 1$

$e_2 \longleftrightarrow 2$

$e_3 \longleftrightarrow 3$

$e_4 \longleftrightarrow 4$

$e_5 \longleftrightarrow 5$

$e_6 \longleftrightarrow 6$.

UNIT - IV

⇒ **Pigeonhole Principle :-** If 'm' pigeons are assigned to 'n' pigeonhole where $m > n$, then atleast one pigeonhole contains two or more than two pigeons.

Proof :- let $H_1, H_2, \ldots H_n$ denots pigeonholes.

$P_1, P_2 \ldots P_n \ldots P_m$, denotes pigeons.

then, we can consider assignment of pigeons as follows.

Pigeon P_1 is assigned to pigeonhole H_1

" P_2 " " " " H_2

⋮

" P_n " " " " H_n

since, $m > n$, then $(m-n)$ pigeons are not assigned

thus, atleast one pigeonhole must contain second pigeon.

⇒ atleast one pigeonhole must contain two or more than two pigeons.

Q. If 8 people are choosen in any way from some rule then atleast 2 of them will have to born on same day of the week.

Soln

8 people $= m$ pigeons

$n = 7$ = no. of days in a week

since $8 > 7$ i.e. $m > n$, then by PH Principle, atleast 2 people must born on same day of week.

Ques:- 8 Pair of shoes
9 single shoes selected.

Sol:- $m = 8$ pairs $= 16$ pigeons
$n = 9$ shoes i.e. 9 pigeon holes.

$m > n$, $16 > 9$ ~~hence atleast~~
therefore by PH principle we must have atleast one pair of shoes.

⇒ General form of pigeonhole principle :-
If 'n' pigeonholes are occupied by $kn+1$ or more pigeons & 'k' is natural number then atleast one pigeonhole is occupied by '$k+1$' or more pigeons.

Ques:- Find the minm no. of students in a class s.t atleast 3 of them born in the same month.

Sol:-
$n = 12$ = no. of months = pigeonholes.
25 · & $k+1 = 3 \Rightarrow k = 2$.
25 Thus, minm no. of students $= nk+1$

$12 \times 2 + 1 = 25$.

Ques:- What is the minm no. of students required in a class to be s that atleast 5 students will receive the same grade if there are 4 possible grades. A B C D.

$k+1 = 5 \Rightarrow k = 4$.

$4 \times 4 + 1 = 17$.

$n = 4$,
$\left[\frac{m-1}{4}\right]+1 = 5$
$m-1 = 16$
$\boxed{m = 17}$

⇒ Extended Pigeonhole Principle :- If 'm' pigeons are assigned to 'n' pigeonhole, then one of the pigeonhole must contain atleast $\left[\frac{m}{n}+1\right]$ pigeons. $\left[\frac{m-1}{n}\right]+1$ pigeons.

Ques. Show that among 1 lakh people, there are 2 people who born on the same time.

Sol:- $\left[\frac{100000}{24 \times 60 \times 60} + 1\right] = \left[\frac{100000}{86400} + 1\right]$

$= [1.157 + 1] = [2.157]$

$n = (24 \times 60 \times 60)$ sec.

= 2 people are born on same time.

Ques. Show that among one thousand people there are atleast 84 people who were born in same month.

Sol:- $m = 1000$, $n = 12$

$$\left[\frac{1000}{12} + 1\right] = [83.33 + 1] = [84.33]$$

= 84 people.

⇒ Recurrence Relation :- & Recursive Formula :-

A formula which express any term of the sequence as a function of its previous term is called recursive formula & relation is called recurrence relation.

Ex :-

3, 8, 13, 18, 23 . . .

$\boxed{a_n = a_{n-1} + 5}$; $a_1 = 3 \longrightarrow$ initial cond. or boundary cond.

$n \geq 2$.

recursive formula.

recurrence relation or difference reln.

Remark :- Recurrence reln may be written in foll. form as

$f(x+3h) + 3f(x+2h) + 7f(x+h) + f(x) = 0$.

$a_{r+3} + 3a_{r+2} + 7a_{r+1} + a_r = 0$.

or

$y_{n+3} + 3y_{n+2} + 7y_{n+1} + y_n = 0$.

or

$u_{x+3} + 3u_{x+2} + 7u_{x+1} + u_x = 0$.

⇒ Order of Recurrence relation :- The order of recurrence relⁿ is defined as diff. of highest & lowest subscript of a_r or y_h or u_x.

→ Degree of Recurrence Relⁿ :- It is defined as highest power of a_r or y_h or u_x.

$$y_{h+3}^5 + y_{h+1} + y_h = 0 \quad \rightarrow \text{degree } 5.$$

$$u_{n+3} + u_{n+2} + u_n = f(n) \quad \rightarrow \text{degree } 1$$

→ Homogeneous Recurrence Relⁿ :- A recurrence relation is called homogeneous, if it contains no term that depend only on the variable 'n' (r, h, x). If recurrence relⁿ is not homogeneous i.e. it contains a term which depend on the variable n (r, h, x), then it is called non homogeneous recurrence relⁿ.

$$a_{r+3} + 2a_{r+2} + a_r = b^r$$

is non-homogeneous recurrence relⁿ of degree 1 & order 3.

A recurrence relⁿ of degree one is called Linear Recurr. Relⁿ.
" " " " two " " Quadratic " ".

→ Linear recurrence relⁿ with const. coeff. :-
Linear recurrence relⁿ with const. coeff. is : The genⁿ form of

$$C_0 a_r + C_1 a_{r-1} + C_2 \cdot a_{r-2} + \dots + C_k a_{r-k} = f(r).$$

of order 'k'.
where $C_0, C_1, \dots, C_k$ are constant & $f(r)$ is some funⁿ which satisfy given eqⁿ.

Note: (i) If $f(r) = 0$, then above eqⁿ is called linear Homogeneous recurrence relⁿ.
(ii) If $f(r) \neq 0$, then above eqⁿ is called linear non-homogeneous recurrence relⁿ.

⇒ Solution of Linear Homogeneous recurrence relation with const. coeff.

Consider,

$$C_0 a_r + C_1 \cdot a_{r-1} + C_2 \cdot a_{r-2} + \dots\dots + C_k \cdot a_{r-k} = 0 \quad \text{——— (1)}$$

Sol^n of Eq^n (1) is of the form $A\alpha^k$ where α is characteristic root & 'A' is constant which can be determined by initial condition.

put $a_r = A\alpha^r$, $a_{r-1} = A\alpha^{r-1}$, $a_{r-2} = A\alpha^{r-2}$ ...., $a_{r-k} = A\alpha^{r-k}$.

Eqn will be,

$$C_0 \cdot A\alpha^r + C_1 \cdot A\alpha^{r-1} + C_2 \cdot A\alpha^{r-2} + \dots + C_k \cdot A\alpha^{r-k} = 0$$

$$A\alpha^{r-k}\left[C_0\,\alpha^k + C_1\alpha^{k-1} + C_2 \cdot \alpha^{k-2} + \dots + C_k\right] = 0$$

$$\Rightarrow \quad C_0 \cdot \alpha^k + C_1 \cdot \alpha^{k+1} + \dots + C_k = 0$$

This Eqn is called characteristic Eqn and roots of this Eqn is called characteristic root.

Case I: If all the roots of Eq^n (2) are distinct & real.

If $\alpha_1, \alpha_2 \dots, \alpha_k$ are distinct, then sol^n is

$$a_r = A_1\alpha_1^r + A_2\alpha_2^r + \dots + A_k\alpha_k^r$$

where, $A_1, A_2 \dots, A_k$ are constant.

case II: If some roots are equal.

$\alpha_1 = \alpha_2 = \alpha_3 \dots = \alpha_m = \alpha$ &

$\alpha_{m+1}, \alpha_{m+2} \dots \alpha_k$ are distinct, then Sol^n is

$$a_r = (A_1 + A_2 r + A_3 r^2 + \dots + A_m r^{m-1})\alpha^r + A_{m+1}\alpha_{m+1}^r + \dots + A_k\alpha_k^r$$

Case III: If roots are complex.

Let ~~roots~~ $\alpha + i\beta$ & $\alpha - i\beta$, be the root of eqn

$$a_{r+2} + 2c\,a_{r+1} + d\,a_r = 0.$$

then Sol^n is

$$a_r = \rho^r\left[A_1 \cos r\theta + A_2 \sin r\theta\right]$$

when $\rho = \sqrt{\alpha^2 + \beta^2}$ & $\theta = \tan^{-1}(\beta/\alpha)$

Ques:- Solve $a_{r+2} - 5a_{r+1} + 6a_r = 0$., $a_2 = 3$, $a_3 = 3$

Sol. put $a_r = A\alpha^r$, $a_{r+1} = A\alpha^{r+1}$, $a_{r+2} = A\alpha^{r+2}$

$$A\alpha^{r+2} - 5A\alpha^{r+1} + 6A\alpha^r = 0 .$$

$$A\alpha^r\left[\alpha^2 - 5\alpha + 6\right] = 0$$

$\Rightarrow \alpha^2 - 5\alpha + 6 = 0$ since $A\alpha^r \neq 0$.

$$(\alpha-3)(\alpha-2) = 0$$

$$\alpha = 3, 2 .$$

Sol^n is

$$\boxed{a_r = A_1 \cdot 2^r + A_2 3^r}$$

put $r = 2$

$$a_2 = 3 = A_1 2^2 + A_2 3^2$$

$$4A_1 + 9A_2 = 3 \quad \text{—(i)}$$

put $r=3$

$$a_3 = 3 = A_1 2^3 + A_2 \cdot 3^3$$

$$8A_1 + 27A_2 = 3 \quad \text{(ii)}.$$

$$\underline{-8A_1 \mp 18A_2 = -6}$$

$$9A_2 = -3$$

$$A_2 = -\frac{1}{3} \quad \text{ \& } A_1 = \frac{3+3}{4} = \frac{3}{2}.$$

$$\boxed{a_r = \frac{3}{2} \cdot 2^r - \frac{1}{3} \cdot 3^r} = 3 \cdot 2^{r-1} - 3^{r-1}$$

$$\boxed{a_r = 3 \cdot 2^{r-1} - 3^{r-1}}.$$

Scanned by CamScanner

Ques :- Solve :-

$$a_r + 6a_{r-1} + 12a_{r-2} + 8a_{r-3} = 0$$

$$A\alpha^r + 6A\alpha^{r-1} + 12A\alpha^{r-2} + 8A\alpha^{r-3} = 0$$

$$A\alpha^{r-3}(\alpha^3 + 6\alpha^2 + 12\alpha + 8) = 0.$$

$$\alpha^3 + 6\alpha^2 + 12\alpha + 8 = 0.$$

$$(\alpha+2)^3 = 0$$

$$\Rightarrow \alpha = -2 \text{ (3 times)}.$$

hence, soln is

$$a_r = [A_1 + A_2 r + A_3 r^2](-2)^r$$

$\cancel{-1+6-12+8}$

$\cancel{-3+4}$

$-8+24-24+8$

$\alpha^2 + 4\alpha + 4$

$\alpha+2 \,)\, \alpha^3 + 6\alpha^2 + 12\alpha + 8 \,($

$\alpha^3 + 2\alpha^2$

$4\alpha^2 + 12\alpha + 8$

$4\alpha^2 + 8\alpha$

$4\alpha + 8$

$\alpha^2 + 4\alpha + 4$

$(\alpha+2)^2$

Ques :-
$$a_r - 4a_{r-1} + 13a_{r-2} = 0.$$

$$A\alpha^r - 4A\alpha^{r-1} + 13A\alpha^{r-2} = 0$$

$$A\alpha^{r-2}(\alpha^2 - 4\alpha + 13) = 0$$

$$\Rightarrow \alpha^2 - 4\alpha + 13 = 0.$$

$$\alpha = \frac{+4 \pm \sqrt{16-52}}{2} = \frac{+4 \pm 6i}{2}$$

$$\alpha = +2 \pm 3i$$

$$\alpha = +2+3i, \; +2-3i$$

$$\alpha = 2+3i, \; 2-3i.$$

$$\rho = \sqrt{4+9} = \sqrt{13}.$$

$$\theta = \tan^{-1}\left(\frac{3}{2}\right)$$

$$a_r = (\sqrt{13})^r [A_1 \cos r\theta + A_2 \sin r\theta]$$

Required Solution :-

$$a_r = (\sqrt{13})^r \left[A_1 \cos\left[r \tan^{-1}\left(\frac{3}{2}\right)\right] + A_2 \sin\left(r\left[\tan^{-1}\frac{3}{2}\right]\right)\right]$$

Scanned by CamScanner

⇒ Solution of Linear Non-homogeneous Recurrence Relation with const. coeff :-

Consider,

$$C_0 a_r + C_1 \cdot a_{r-1} + C_2 \cdot a_{r-2} + \cdots + C_k \cdot a_{r-k} = f(r)$$

$$f(r) \neq 0$$

Then its solution contains two parts,

① homogeneous solution

② Particular Solution.

Thus, total solⁿ $\boxed{a_r = \text{hom. sol}^n + \text{Particular sol}^n}$

To find Particular solⁿ :-

Terms in $f(r)$	Trial Solⁿ a_r.
b^r e.g. $2^r, 3^r \ldots$	$A b^r$
Polynomial of degree k	~~$A_1 + A_2 r + A_3 \ldots A_{k-1} \cdot r^{k-1}$~~ $A_0 + A_1 r + A_2 r^2 + \cdots + A_k r^k$
$b^r \times$ [Polynomial of deg. k]	$[A_0 + A_1 r + A_2 r^2 + \cdots + A_k r^k] \cdot b^r$
$\sin br$ or $\cos br$	$A \sin br + B \cos br$.
$b^r \sin br$ or $b^r \cos br$	$b^r [A \sin br + B \cos br]$

Soln of linear non-homogeneous recurrence rel.

Q:- Find P.S of

$$a_{r+2} - 2a_{r+1} + a_r = 2^r \quad \text{--- (1)}$$

The P.S. of eqn (1) is of the form

$$a_r = A \cdot 2^r$$

$$a_{r+2} = A \cdot 2^{r+2}$$

$$a_{r+1} = A \cdot 2^{r+1}$$

Putting in eqn (1)

$$A \cdot 2^{r+2} - 2 \cdot A \cdot 2^{r+1} + A \cdot 2^r = 2^r$$

$$4A - 4A + A = \cancel{2^r}\ 1$$

$$\boxed{A = 1}$$

hence, required P.S is

$$\boxed{a_r = 2^r}$$ Ans.

Q:- Find P.S of $a_r - 2a_{r-1} = 7r$. --- (1)

The P.S of eqn (1) is of form

$$a_r = A_0 + A_1 r$$

$$a_{r-1} = A_0 + A_1(r-1)$$

$$A_0 + A_1 r - 2(A_0 + A_1 r - A_1) = 7r$$

$$A_0 + A_1 r - 2A_0 - 2A_1 r + 2A_1 = 7r$$

$$-A_0 - A_1 r + 2A_1 = 7r$$

$$2A_1 - A_0 = r(7 + A_1)$$

$$r(7 + A_1) + (A_0 - 2A_1) = 0$$

Equating coeff.

$7 + A_1 = 0 \Rightarrow A_1 = -7$ & $A_0 = 2A_1 = -14$.

Scanned by CamScanner

$$\boxed{a_r = -14 - 7r}$$

Que Find P.S. of $a_r + 5a_{r-1} + 6a_{r-2} = 3r^2 - 2r + 1$.

P.S is of form :-

$$a_r = A_0 + A_1 r + A_2 r^2$$

$$a_{r-1} = A_0 + A_1(r-1) + A_2(r-1)^2.$$

$$a_{r-2} = A_0 + A_1(r-2) + A_2(r-2)^2.$$

Putting in eq^n.

$$A_0 + A_1 r + A_2 r^2 + 5A_0 + 5A_1(r-1) + 5A_2(r-1)^2 + 6A_0 + 6A_1(r-2) + 6A_2(r-2)^2 = 3r^2 - 2r + 1$$

$$12A_0 + A_1 r + A_2 r^2 + 5A_1 r - 5A_1 + 5A_2 r^2 + 5A_2 - 10A_2 r + 6A_1 r - 12A_1 + 6A_2 r^2 + 24A_2 - 24A_2 r = 3r^2 - 2r + 1$$

$\cancel{r^2(6A_2}$

$$12A_2 r^2 + (12A_1 - 34A_2)r + (12A_0 - 17A_1 + 29A_2) = 3r^2 - 2r + 1$$

$12A_2 = 3 \Rightarrow \boxed{A_2 = \frac{1}{4}}$

$12A_1 - 34A_2 = -2$

$12A_1 = -2 + \frac{17}{2} = \frac{13}{2} \Rightarrow \boxed{A_1 = \frac{13}{24}.}$

$$12A_0 = 1 + 17 \times \frac{13}{24} - 29 \times \frac{1}{4}$$

$$= \frac{24 + 17 \times 13 - 6 \times 29}{24} =$$

$$A_0 = \frac{245 - 174}{12 \times 24} = \frac{71}{288}$$

Required P.S.

$$\boxed{a_r = \frac{71}{288} + \frac{13}{24}r + \frac{1}{4}r^2}$$

Ques:- Find P.S of

$a_r + a_{r-1} = 3r \cdot 2^r$.

Sol:-

It is of form $b^r \cdot$ Pol. of deg 1.

P.S-

$a^r = (A_0 + A_1 r) 2^r$

$a_{r-1} = (A_0 + A_1(r-1)) 2^{r-1}$.

Putting the values,

$(A_0 + A_1 r) 2^r + (A_0 + A_1 r - A_1) 2^{r-1} = 3r \cdot 2^r$.

$\cancel{A_0 + A_1 r + A}$

$(A_0 + A_1 \cancel{r} + A_1) 2^r + (A_0 - A_1) \cancel{2}^{r-1} = 3r \cdot \cancel{2}^r$

$A_0 - A_1 = 0$

$\Rightarrow \boxed{A_0 = A_1}$

$(A_1 + A_0) + A_1 \cancel{r} = 3r$

$A_1 = 3$

$r 2^r \left(A_1 + \frac{A_1}{2} \right) + A_0 2^r + A_0 \cdot 2^{r-1} - A_1 \cdot 2^{r-1} = 3r \cdot 2^r$

$r 2^r \left(\frac{3A_1}{2} \right) + 2^r \left(A_0 + \frac{A_0}{2} - \frac{A_1}{2} \right) = 3r \cdot 2^r$

$\frac{3A_1}{2} = 3 \Rightarrow \boxed{A_1 = 2}$

$\frac{3A_0}{2} - \frac{A_1}{2} = 0 \Rightarrow \boxed{A_0 = \frac{2}{3}}$

Scanned by CamScanner

$$\boxed{a_r = \left(\frac{2}{3} + 2r\right) 2^r}$$

⇒ Discrete numeric function :-

A function whose domain is set of non-negative integers & range is real number is called discrete numeric function or simply numeric function i.e.

$$a : I^+ \cup \{0\} \longrightarrow R$$

then 'a' is k/as Discrete Numeric Function.

and value of function 'a' at 0, 1, 2, 3. -- denoted by $a_0, a_1, a_2, a_3, \ldots$

i.e.

$a(0) = a_0$; $a(1) = a_1$; $a(2) = a_2$ ------

⇒ Generating Function :- If 'a' be the discrete numeric function i.e. $a = (a_0, a_1, a_2 \ldots)$ then generating function of discrete numeric function a is denoted by ~~a(z)~~ or $A(z)$ or $G(x)$ and defined as $A(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 \ldots$

$$\boxed{A(z) = \sum_{r=0}^{\infty} a_r \cdot z^r}$$

Example:- Find Generating function of sequence 1, -1, 1, -1, ----

Soln:-

$$A(z) = \sum_{r=0}^{\infty} a_r z^r = a_0 + a_1 z + a_2 z^2 + \ldots$$

$$= 1 - z + z^2 - z^3 + \ldots$$

$$= (1+z)^{-1}$$

$$\boxed{A(z) = \frac{1}{(1+z)}}$$