Set: A set is a well-defined collection of (distinct) objects (the objects are called the elements or members of the set). These elements may be anything: numbers, people, letters of the alphabet, points in geometry, etc.

e.g. (i). The set of the first four natural numbers.

(ii) The set of the rivers in India

(iii). The vowels of alphabets.

(iv) The set of colors of the India flag.

Sets denoted by A, B, C, D, ...

elements → a, b, c, d, ...

The statement "x" is an element of A or equivalently "x belongs to A" is written as

$x \in A$.

The statement "x is not an element of A" is written as $x \notin A$.

well defined → there is no ambiguity

- not well defined → who is the president? is a well-defined collection. Does it mean only president of nations, or does it include president of companies? of universities? of Clubs?

∴ The collection of good students at "MNMVT" is not a set. Good < past / present.

Representation of a Set: mainly there are two ~~types~~ ways of representing a set.

(i). Roster or Tabular form
(ii). Rule method or Set builder form.

e.g. (i) The set of vowels in the English alphabets i.e. $A = \{a, e, i, o, u\}$

e.g. (ii) The set of A consisting of elements a, e, i, o, u can be written as

$A = \{x : x \text{ is a vowel in the English alphabets}\}$

$N = \{x : x \text{ is a natural numbers}\}$

$Z = \{x : x \text{ is an integers}\}$

$R = \{x : x \text{ is a real number}\}$

Finite and Infinite set.

A set with finite numbers of elements in it, is called a finite set.

An infinite set is a set which contains infinite numbers of elements.

e.g. The set of months in a year. ← finite

The set of integers numbers. ← infinite

Null Set: A set which contains no elements at all is called the Null set. (Empty set or Void set)

It is denoted by the symbol ϕ.

e.g. $A = \{x : x^2 + 4 = 0, x \text{ real}\}$

Singleton Set: A set which has only one element is called singleton set

one or more than one

e.g. $A = \{9\}$

Subset. If A and B are sets such that every element of A is also an element of B, then A is said to be a subset of B. (or A is contained in B). and is denoted by $A \subseteq B$. In other words,

$A \subseteq B$, if $x \in A$ and $x \in B$.

If A is not a subset of B, $A \not\subseteq B$.

(i). Every set A is a subset of itself. i.e $A \subseteq A$

(ii). $\phi \subseteq A$

(iii) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Number of subsets of a set.

If a set contains n elements, then the number of subsets $= 2^n$.

e.g. $A = \{2, 3\}$.

No. of subsets $= 2^2 = 4$, $\phi, \{1\}, \{2\}, \{1, 2\}$

Super Set: $B \supset A$ B is super set of A.

Proper Set: If A is a subset of B, then

Any subset A is said to be proper subset of another set B if A is a subset of B, but there is at least one element of B which doesn't belong to A i.e. if $A \subseteq B$ but $A \neq B$ it is written as $A \subset B$

Equal Set: Two sets A and B are said to equal iff element of A is an element of B and consequently every element of B is an element of A; that is $A \subseteq B$ and $B \subseteq A$ and it is written as $A = B$.

Symbolically, $A = B$ if $x \in A$ & $x \in B$.

Universal Set: A universal set is a set which contains all objects, including itself. Universal set or Universe of Discourse set and is denoted by U. (upper case italic letter U).

e.g.
The set of letters in alphabets is the universal set from which the letters of any word may be chosen to form a set.

(i) In a study of human population, all people in the world may be assumbed to form the universal set. The people of any continent, country, religion is a subset of this universal set.

operations on sets.

Union. The union of two sets A and B, denoted by $A \cup B$, pronounced as 'A union B' is the set of all elements which belong to A or B or to both, i.e.,

$$A \cup B = \{ x : x \in A \text{ or } x \in B \}$$

Intersection. The intersection of two sets A and B, denoted by $A \cap B$, pronounced as 'A intersection B', is the set of elements which belong to both A and B, that is,

$$A \cap B = \{ x : x \in A \text{ and } x \in B \}.$$

If $A \cap B = \phi$, i.e., if A and B do not have

If any elements in common, then A and B are said to be disjoint or non interesting.

Complements. Let U be the universal set and A be any subset of U. The complement of A, denoted by A' or A^c is the set of elements which belong to U but which do not belong to A, i.e.,

$$A' = \{x : x \in U \text{ and } x \notin A\}$$

If A and B are two sets, the relative complement of B w.r.t. A or, simply, the difference of A and B, denoted by A - B, is the set of elements which belongs to A but which do not belong to B, that is,

$$A - B = \{x : x \in A \text{ and } x \notin B\}.$$

e.g. Let $N = \{1, 2, 3, 4, \ldots\}$ be the universal set and $A = \{1, 3, 5, \ldots\}$, then

$$A' = N - A = \{2, 4, 6, \ldots\}.$$

... or more than one

Symmetric difference:

$A \triangle B$ or $A \oplus B = (A - B) \cup (B - A)$

Boolean sum $= \{x : x$ belongs to exactly one of A and B$\}$

Algebra of Sets:

① Idempotent Laws:

(a) $A \cup A = A$ (b) $A \cap A = A$

② Associative Laws.

(a) $(A \cup B) \cup C = A \cup (B \cup C)$ (b) $(A \cap B) \cap C = A \cap (B \cap C)$

③ Commutative Laws

3(a) $A \cup B = B \cup A$ 3(b) $A \cap B = B \cap A$.

④ Distributive Laws.

(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Identity Laws

(a). $A \cup \phi = A$ (b) $A \cap U = A$

6(a) $A \cup U = U$ 6(b) $A \cap \phi = \phi$

(7). Involution Laws

$(A')' = A$

(8). Complement Laws

(a) $A \cup A' = U$ (b) $A \cap A' = \phi$

9(a) $U' = \phi$ 9(b) $\phi' = U$

(10). De Morgan's Laws

(a). $(A \cup B)' = A' \cap B'$ (b). $(A \cap B)' = A' \cup B'$

10(a)

Proof: Let $x \in (A \cup B)'$. Then

$x \in (A \cup B)' \Rightarrow x \notin (A \cup B)$

$\Rightarrow x \notin A$ and $x \notin B$

$\Rightarrow x \in A'$ and $x \in B'$

$x \in (A \cup B)$
$\Rightarrow x \in A$ or $x \in B$

$x \in (A \cap B)$
$x \in A$ and $x \in B$

$x \notin (A \cup B)$
$x \notin A$ and $x \notin B$

$x \notin (A \cap B)$
$x \notin A$ or $x \notin B$

$\Rightarrow x \in (A' \cap B')$

or $(A \cup B)' \subseteq A' \cap B'$ ——— (i)

Conversely, let $x \in A' \cap B'$. Then

$x \in (A' \cap B') \Rightarrow x \in A'$ and $x \in B'$

$\Rightarrow x \notin A$ and $y \notin B$

$\Rightarrow x \notin (A \cup B)$

$\Rightarrow x \in (A \cup B)'$.

Thus, $A' \cap B' \subseteq (A \cup B)'$ ——— (ii).

So, from eqn. (i) and (ii), we obtain

$$\boxed{(A \cap B)' = A' \cup B'}$$

(b). let $x \in (A \cap B)'$. Then

$x \in (A \cap B)' \Rightarrow x \notin (A \cap B)$

$\Rightarrow x \notin A$ ~~or~~ $x \notin B$

$\Rightarrow x \in A'$ or $x \in B'$

$\Rightarrow x \in (A' \cup B')$

$(A \cap B)' \subseteq (A' \cup B')$ —— (i)

Conversely, let $x \in (A' \cup B')$, Then

$x \in (A' \cup B') \Rightarrow x \in A'$ or $x \in B'$

$\Rightarrow x \notin A$ or $x \notin B$

$\Rightarrow x \notin (A \cap B)$

$\Rightarrow x \in (A \cap B)'$

Thus, $(A' \cup B') \subseteq (A \cap B)'$ —— (ii).

From (i) and (ii), we have

$$A' \cup B' = (A \cap B)'$$

Extra: Prove that $A - B = A \cap B'$

Solution: let $x \in (A - B)$. Then

$x \in (A - B) \Rightarrow x \in A$ and $x \notin B$

$\Rightarrow x \in A$ and $x \in B'$

$\Rightarrow x \in A \cap B'$

$A - B \subseteq A \cap B'$ —— (1).

Conversely, let $x \in (A \cap B')$. Then

$x \in (A \cap B') \Rightarrow x \in A$ and $x \in B'$

$\Rightarrow x \in A$ and $x \notin B$

$\Rightarrow x \in (A - B)$

$A \cap B' \subseteq (A - B)$ —— (11)

Hence from (1) and (11), we obtain

Extra Prove that $A - (B \cap C) = (A - B) \cup (A - C)$

Proof: Let $x \in A - (B \cap C)$. Then

$x \in A - (B \cap C) \Rightarrow x \in A$ and $x \notin (B \cap C)$

$\Rightarrow x \in A$ and $(x \notin B$ or $x \notin C)$

$\Rightarrow$ ~~$x \in A$ and $x \in B$~~

$\Rightarrow (x \in A$ and $x \notin B)$ or $(x \in A$ and $x \notin$

$\Rightarrow x \in (A-B)$ or $x \in (A-C)$

$\Rightarrow x \in (A-B) \cup (A-C)$

So, $A-(B \cap C) \subseteq (A-B) \cup (A-C)$ — (1)

Conversely, let $x \in (A-B) \cup (A-C)$. Then

$x \in (A-B) \cup (A-C) \Rightarrow x \in (A-B)$ or $x \in (A-C)$

$\Rightarrow (x \in A$ and $x \notin B)$ or $(x \in A$ and $x \notin C)$

$\Rightarrow x \in A$ and $(x \notin B$ or $x \notin C)$

$\Rightarrow x \in A$ and $x \notin (B \cap C)$

$\Rightarrow x \in A - (B \cap C)$.

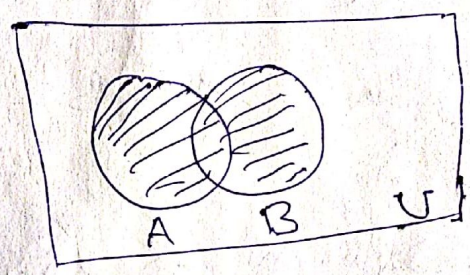
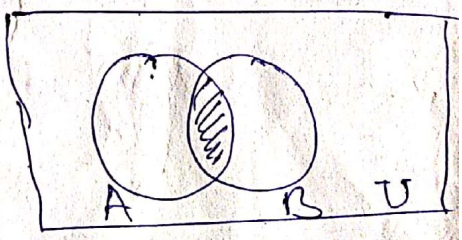
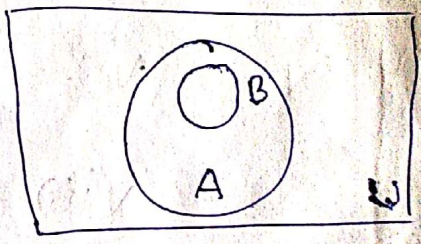
$(A-B) \cup (A-C) \subseteq A-(B \cap C)$ — (2)

Proved.

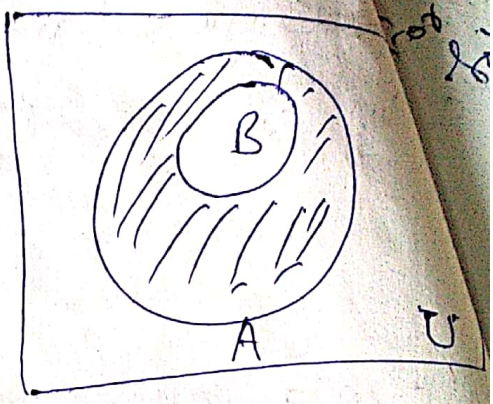
Venn Diagram: A Venn diagram is a pictorial representation of sets which are used to show relationship between sets and also the operations on them. The universal set is representated by the interior of a

rectangle and its subsets are represented by circular areas drawn within the rectangle.

The Venn diagrams of set operations are shown in the following.

Set operations	Symbole	Venn Diagram
(i) The union of sets A and B	$A \cup B$	
(ii) The intersection of sets A and B	$A \cap B$	
(iii) Set B is a proper subset of A	$B \subset A$	
(iv) The Complement of Set A	A' or A^c	

(v) The difference of set A $A-B$ and B



Example. Let $A=\{1,2,3,4,5,6\}$, $B=\{3,6,8,12,17,$ and $C=\{2,3,5,6,12,17\}$. Then find

(i) $A \cup B$ (ii) $A \cap B$ (iii) $A \cap C$ (iv) $B \cup C$

(v) $A-B$ (vi) $B-A$ (vii) $C-B$

Solution (i) $A \cup B=\{1,2,3,4,5,6,8,12,17,18\}$

(ii) $A \cap B=\{3,6\}$

(iii) $A \cap C=\{2,3,5,6\}$

(iv) $B \cup C=\{2,3,5,6,8,12,17,18\}$

(v) $A-B=\{1,2,4,5\}$

(vi) $B-A=\{8,12,17,18\}$

(vii) $C-B=\{2,12,17\}$

For any two sets A and B, we have the following formula

(i). $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(ii) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B)$
$- n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$.

(iii) $n(A - B) + n(A \cap B) = n(A)$
or

(iv) $n(B - A) + n(A \cap B) = n(B)$

(v) $n(A' \cap C) = n(C) - n(A \cap C)$
or
$n(A \cap B') = n(A) - n(A \cap B)$

(vi) $n(A' \cap B \cap C') = n(B) - n(A \cap B') + n(B \cap C)$
$- n(A \cap B' \cap C)$.

Candidates → A → Mathematics
B → Physics
C → aggregate

(a) The no. of students who failed in math. but not in Physics is $n(A \cap B')$

(b) The no. of students who failed in the aggregate

but in maths. is $n(A' \cap C)$

(c). the no. ... failed in Physics. but not in aggregate & in maths is $n(A' \cap B \cap C')$.

Example. Out of 80 students in a class, 60 play football, 53 play hockey, and 35 both the games. How many students.

(a). donot play their games

(b) play only hockey but not football.

Solution. Let A and B be sets of students who play football and hockey respectively. Then, Given

$n(A \cup B) = 80$, $n(A) = 60$, $n(B) = 53$.

$n(A \cap B) = 35$. Let x students donot play their games. Hence.

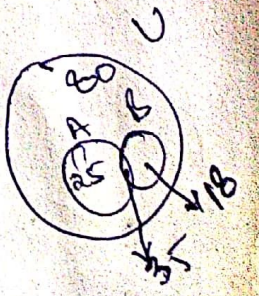
$\therefore$

(a) $n(A \cup B) - x = n(A) + n(B) - n(A \cap B)$

$80 - x = 60 + 53 - 35$

$= 113 - 35 = 78$

$x = 80 - 78 = 2$.



i.e. no. of students who play hockey but not football

$$n(B-A) + n(A \cap B) = n(B)$$

$$n(B-A) = 53 - 35 = 18 \quad \text{Ans.}$$

Ex: In a class of 25 students, 12 have taken mathematics 8 have taken mathematics but not biology. Find the no. of students who have taken mathematics and Biology and that who have taken Biology but not mathematics.

Sol: Let A and B be sets of students who have taken mathematics and Biology respectively. Then $(A-B)$ is the set of students who have taken maths but not biology. So

$$n(A) = 12, \quad n(A \cup B) = 25, \quad n(A-B) = 8$$

we have.

$$n(A-B) + n(A \cap B) = n(A)$$

$$8 + n(A \cap B) = 12 \quad \text{or} \quad n(A \cap B) = 4$$

Thus. 4 students have taken both math and Bio.

Again $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$$25 = 12 + n(B) - 4$$

$$\Rightarrow n(B) = 17$$

i.e. more than and

Now $n(B-A) + n(A \cap B) = n(B)$

$\Rightarrow n(B-A) = 17 - 4 = 13.$

Ordered pair: An ordered pair is a pair of objects whose components occur in a special order. In the ordered pair (a,b), a is called the first component and b, the second component.

Cartesian product: Let A and B be sets. Cartesian product of A and B, denoted by $A \times B$, is defined as

$$A \times B = \{(a,b) : a \in A \text{ and } b \in B\}$$

that the $A \times B$ is the set of all possible ordered pairs whose first component comes from A and whose second component comes from B.

For example: If $A = \{a, b\}$ and $B = \{1\}$,

then $A \times B = \{(a,1), (b,1)\}$.

$B \times A = \{(1,a), (1,b)\}$, $A \times A = \{(a,a), (a,b), (b,a), (b,b)\}$

i.e. $A \times B \neq B \times A$

or more than one

Review of Relations on Sets

Let A and B be two sets as follows

A = {Lucknow Uttar Pradesh, West Bengal, Madhya Pradesh, Bihar} and B = {Yogi Aditiyanath, Mamta Banerji, Kamalnath, Nitish Kumar}.

There is a relation 'is a CM of' between the elements of the sets A and B. If R is used for the relation 'is a Capital of' then

UP R Yogi Aditiyanath, West Bengal R ~~Kamalnath~~ Mamta Banerji

MP R Kamalnath, Bihar R Nitish Kumar.

(Omitting the letter R between pairs of names)

$R = \{(UP, YA), (WB, MB), (MP, K), (B, NK)\}$

$= \{(x, y) : x \in A, y \in B, xRy\}$.

Thus $(x, y) \in R$ iff xRy.

Definition: Let A and B be two sets. A relation from A to B is a subset of the Cartesian product $A \times B$.

Suppose R is a relation from A to B. Then R is a set of ordered pairs (a, b), where $a \in A$ and $b \in B$.

Note. aRb read as 'a is related to b by R'
$(a, b) \notin R$ the a is not related to b by R.

Example Let $A = \{2, 3, 4\}$ and $B = \{3, 4, 5\}$. List the elements of each relation R defined below and the domain and range.

(a). $a \in A$ is related to $b \in B$, that is, aRb if and only if $a < b$

(b). $a \in A$ is related to $b \in B$, i.e., aRb iff a and b are both odd numbers.

Sol (a). $2 \in A$ is less than $3 \in B$, then $2R3$.
Similarly, $2R4$, $2R5$, $3R4$, $3R5$, $4R5$. Therefore,

$$R = \{(2,3), (2,4), (2,5), (3,4), (3,5), (4,5)\}$$

$$Dom(R) = \{2, 3, 4\} \text{ and } Rng(R) = \{3, 4, 5\}.$$

(b). Since $3 \in A$ and $3 \in B$ are both odd then $3R3$. Similarly, $3R5$. Therefore,

$$R = \{(3,3), (3,5)\}$$

or more than one

Dom (R) = {3} and Ran (R) = {3, 5}.

* $n(A) = n$, $n(B) = m$, Total no. of distinct relations from A to B $= 2^{mn}$.

Types of Relations on a set:

(1) **Inverse Relation**: Let R be any relation from a set A to a set B. The inverse of R, denoted by R^{-1} is the relation from B to A, i.e.,

$R^{-1} = \{(b, a) : (a, b) \in R\}$. Consequently

$x R y \Leftrightarrow y R^{-1} x$.

e.g. $R = \{(a, b), (c, a), (e, f)\}$.

$R^{-1} = \{(b, a), (d, c), (f, e)\}$.

$(R^{-1})^{-1} = R$.

Identity Relation: A relation R in a set A is said to be identity relation, generally denoted by I_A, if

$I_A = \{(x, x) : x \in A\}$

Properties of Relations:

① **Reflexive Relation:** A relation R on a set A is reflexive if aRa for every $a \in A$, i.e., $(a,a) \in R \ \forall \ a \in A$.

For Example:

(a) If $R_1 = \{(1,3), (1,2), (1,1), (2,4), (2,3), (3,3)\}$ be a relation on $A = \{1,2,3\}$. Then R_1 is reflexive relation since $\forall \ a \in A$, $(a,a) \in R_1$.

(b) $R_2 = \{(1,1), (1,2), (2,3), (3,3)\}$ be relation on $A = \{1,2,3\}$. Then R_2 is not a reflexive relation since for $2 \in A$, $(2,2) \notin R_2$.

(c) $R_3 = \{(x,y) \in R^2 : x \leq y\}$ is a reflexive relation since $x \leq x$ for any $x \in R$ (a set of real numbers)

Irreflexive Relation: A relation R on a set A is irreflexive if, for every $a \in A$, $(a,a) \notin R$. In other words, there is no $a \in A$ s.t. aRa.

Note: Reflexive means that aRa is true for all a, and irreflexive means that aRa is true for no a. For example,

(4). The relation $R_1 = \{(1,2), (1,3), (2,1), (2,3)\}$ on $A = \{1,2,3\}$ is irreflexive relation since $(x,x) \notin R_1$ $\forall$ $x \in R_1$ (a set of real numbers).

(5). The relation $R_2 = \{(x,y) \in K^2 : x < y\}$ is an irreflexive relation since $x < x$ for no $x \in K$ (K set of real numbers)

Symmetric Relation:

A relation R on a set A is symmetric if whenever $(a,b) \in R$ then $(b,a) \in R$, i.e., if $aRb \Rightarrow bRa$. for example,

(3). $R_1 = \{(1,1), (1,2), (1,3), (2,2), (2,1), (3,1)\}$ on $A = \{1,2,3\}$ is a symmetric relation.

(6). $R_2 = \{(x,y) \in K^2 : x^2+y^2 = 1\}$ is a symmetric relation on K since if $x^2+y^2=1$ then $y^2+x^2=1$ too i.e. if $(x,y) \in R_2$ then $(y,x) \in K_2$.

---

is denoted by I_A

Asymmetric Relation:

Antisymmetric Relation:

aRb and $bRa \Rightarrow a = b$

$$R = \{(x,y) \in R^2 : x \leq y\}$$

Since $x \leq y$ and $y \leq x$

$\Rightarrow x = y$

(6) Transitive Relation.

aRb and $bRC \Rightarrow aRC$

$$R = \{(x,y) \in R^2 : x < y\}$$

$a < b$ and $b < c$, then

$a < c$

also parallel

Extra

① Reflexive Relation.

$R_1 = \{(x,y) \in \mathbb{R}^2 : x \leq y\}$, $x \leq x$ reflexive $\forall x \in \mathbb{R}$.

② Irreflexive Relation

$R_2 = \{(x,y) \in \mathbb{R}^2 : x < y\}$. Since $x < x$ for no $x \in \mathbb{R}$.

③ Symmetric Relation:

If $(a,b) \in R \Rightarrow (b,a) \in R$, i.e.,

$aRb \Rightarrow bRa$.

$R = \{(x,y) : x \| y\}$.

④ A symmetric Relation:

$(a,b) \in R$ then $(b,a) \notin R$ for $a \neq b$

$aRb \Rightarrow b \not R a$

$R = \{(x,y) : x < y\}$ $x < y$ but $y \not< x$

xRy but $y \not R x$.

Summarizes the above properties:

Property	Meaning
1. Reflexivity	$(a,a) \in R$, i.e., aRa for all $a \in A$
2. Irreflexivity	$(a,a) \notin R$, i.e., $a \not R a$ for all $a \in A$
3. Symmetry	$(a,b) \in R \Rightarrow (b,a) \in R$, i.e., $aRb \Rightarrow bRa$ for all $a,b \in A$
4. Asymmetry	$(a,b) \in R \Rightarrow (b,a) \notin R$, i.e., $aRb \Rightarrow b \not R a$ for all $a,b \in A$
5. Antisymmetry	$(a,b) \in R$ & $(b,a) \in R \Rightarrow a=b$, i.e., aRb and $bRa \Rightarrow a=b$ $\forall a,b \in A$
6. Transitivity	$(a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R$ i.e. aRb and $bRc \Rightarrow aRc$ for all $a,b,c \in A$.

Equivalence Relation:

A Relation on a set A is called an equivalence relation or RST relation if it is reflexive, symmetric and transitive. That is, R is an equivalence relation on A if it has the following three properties:

1. $(a,a) \in R$ for all $a \in A$ (reflexive)

2. $(a,b) \in R$ implies $(b,a) \in R$ (Symmetric)

3. (a,b) and $(b,c) \in R$ imply $(a,c) \in R$ (transitive)

Example. If R be a relation in the set of integers Z defined by

$R = \{(x,y) : x \in Z, y \in Z, (x-y)$ is divisible by $6\}$.

Then prove that R is an equivalence relation.

Solution. Let $x \in Z$. Then $x - x = 0$ and 0 is divisible by 6.

Therefore, xRx for all $x \in Z$.

Hence, R is reflexive.

Again, $xRy \Rightarrow (x-y)$ is divisible by 6

$\Rightarrow -(y-x)$ is " " 6

$\Rightarrow (y-x)$ " " " 6

$\Rightarrow yRx$.

Hence, R is Symmetric.

xRy and $xRz \Rightarrow (x-y)$ is divisible by 6 and $(y-z)$ is divisible by 6

$\Rightarrow [(x-y)+(y-z)]$ is divisible by 6

$\Rightarrow (x-z)$ is divisible by 6

$\Rightarrow xRz$.

Hence, R is transitive. Thus R is an equivalence relation.

Theorem: Let R and S be relation from A to B. Show that

(i) If $R \subseteq S$, then $R^{-1} \subseteq S^{-1}$

(ii) $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$

(iii) $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$

Proof (i) Suppose $R \subseteq S$. If $(a,b) \in R^{-1}$, then $(b,a) \in R$ and also $(b,a) \in S$ (since $R \subseteq S$). Again $(b,a) \in S$ implies $(a,b) \in S^{-1}$. Therefore

$$\boxed{R^{-1} \subseteq S^{-1}}$$

(ii) Let $(a,b) \in (R \cap S)^{-1}$. Then $(b,a) \in (R \cap S)$

So, $(b,a) \in R$ and $(b,a) \in S$.

xRy and $xRz \Rightarrow (x-y)$ is divisible by 6 and $(y-z)$ is divisible by 6

$\Rightarrow [(x-y)+(y-z)]$ is divisible by 6

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(ii) Let $(a,b) \in (R \cap S)^{-1}$. Then $(b,a) \in (R \cap S)$

So, $(b,a) \in R$ and $(b,a) \in S$.

This implies $(a,b) \in R^{-1}$ and $(a,b) \in S^{-1}$. Hence $(a,b) \in R^{-1} \cap S^{-1}$. Therefore

$$\boxed{(R \cap S)^{-1} \subseteq R^{-1} \cap S^{-1}} \quad \text{---(1)}$$

Conversely, let $(a,b) \in R^{-1} \cap S^{-1}$. Then $(a,b) \in R^{-1}$ and $(a,b) \in S^{-1}$. This implies $(b,a) \in R$ and $(b,a) \in S$. So, $(b,a) \in R \cap S$. Hence $(a,b)^{-1} \in (R \cap S)^{-1}$. Therefore,

$$R^{-1} \cap S^{-1} \subseteq (R \cap S)^{-1} \quad \text{---(2)}$$

From eqn. (1) and (2), we obtain

$(R \cup S)^{-1} = R^{-1} \cup S^{-1}$ proved.

Theorem: Suppose R and S are relations on a set A. Prove that

(i). If R and S are reflexive, then $R \cup S$ and $R \cap S$ are reflexive.

(ii). If R and S are symmetric, then $R \cup S$ and $R \cap S$ are symmetric.

6. If R and S are transitive, then R∩S is transitive.

Proof. (i). Suppose R and S are reflexive. Then $(a,a) \in R$ and $(a,a) \in S$ for all $a \in A$. Therefore, $(a,a) \in R \cup S$ and $(a,a) \in (R \cap S)$. Hence, $R \cup S$ and $R \cap S$ are reflexive.

> **Note.** R is symmetric iff $R = R^{-1}$

(ii).

(iii)

Theorem. If R and S are equivalence relations on the set A, prove that

(i). R^{-1} is an equivalence relation

(ii). $R \cap S$ is an equivalence relation.

Proof. (i). Let R be an equivalence relation in a set A. Therefore, R is reflexive, symmetric and transitive.

Let $a, b, c \in A$ be arbitrary.

The relation R^{-1} is

(1) Reflexive: $(a,a) \in R^{-1}$, since $(a,a) \in R$

for all $a \Rightarrow (a,a) \in R^{-1}$

(ii) Symmetric: $(a,b) \in R^{-1}$.

Since $(a,b) \in R^{-1} \Rightarrow (b,a) \in R$

$\Rightarrow (a,b) \in R$ (as R is symmetric)

$\Rightarrow (b,a) \in R^{-1}$

(3). transitive. $(a,b) \in R^{-1}$ and $(b,c) \in R^{-1} \Rightarrow (a,c) \in$

Since $(a,b) \in R^{-1}$, $(b,c) \in R^{-1} \Rightarrow (b,a), (c,b) \in R$

$\Rightarrow (c,b), (b,a) \in R$

$\Rightarrow (c,a) \in R$ (as R is transitive)

$\Rightarrow (a,c) \in R^{-1}$.

Therefore, R^{-1} is reflexive, symmetric and transitive. Hence R^{-1} is an equivalence relation in A.

(B). $(a,b) \in (R \cap S) \Rightarrow (a,b) \in R$ and $(a,b) \in S$

$\Rightarrow (b,a) \in R$ and $(b,a) \in S$, since R and S are symmetric

$\Rightarrow (b,a) \in R \cap S$.

Hence, $R \cap S$ is symmetric.

(A). for all $a \in A$, $(a,a) \in R$ and $(a,a) \in S$,

Since R and S are equivalence relations.

Hence for all $a \in A$, $(a,a) \in R \cap S$.

Hence, $R \cap S$ is reflexive.

(C). $(a,b) \in R \cap S$, $(b,c) \in R \cap S \Rightarrow$

$\Rightarrow (a,b) \in R$, $(b,c) \in R$ and

$(a,b) \in S$, $(b,c) \in S$

$\Rightarrow (a,c) \in R$ and $(a,c) \in S$, since R and S are transitive. Hence

$\Rightarrow (a,c) \in R \cap S$. Hence.

$R \cap S$ is transitive. Thus $R \cap S$ is an equivalence relation.

Composition of Relations:

Let A, B, C be sets. Let R be a relation from A to B and let S be a relation from B to C. That is R is subset of A×B and S is a subset of B×C. The composite of R and S, denoted by RoS, is the relation consisting of ordered pairs (a,c) where $a \in A$ and $c \in C$, and for which there exists an element $b \in B$ such that $(a,b) \in R$ and $(b,c) \in S$. Thus

$$RoS = \{(a,c) \in A \times C : \text{for some } b \in B, (a,b) \in R \text{ and } (b,c) \in S\}$$

That is a (RoS) c if for some $b \in B$ we have aRb & bSc.

Note: (i) RoS is empty if the intersection of the range of R and the domain of S is empty.

(ii) RoS is non-empty, if there is at least one ordered pair $(a,b) \in R$ such that the second member $b \in B$ of the ordered pair is a first member in an ordered pair in S.

For the relation RoS, the domain is a subset of A and the range is a

subset of C.

(iii) If R be a relation from A to B, S is a relation from B to C, and P be a relation from C to D, one can find out R o (S o P) and also (R o S) o P which are relations from A to D.

(iv) $R \circ R = R^2$, $R \circ R \circ R = R^3$

$$R^i = R \circ R^{i-1} \quad \text{for } i > 0$$

Example: Let $A = \{1,2,3\}$, $B = \{p, q, r\}$, $C = \{x, y, z\}$ and let $R = \{(1,p), (1,r), (2,q), (3,q)\}$ and $S = \{(p,y), (q,x), (r,z)\}$. Compute R o S.

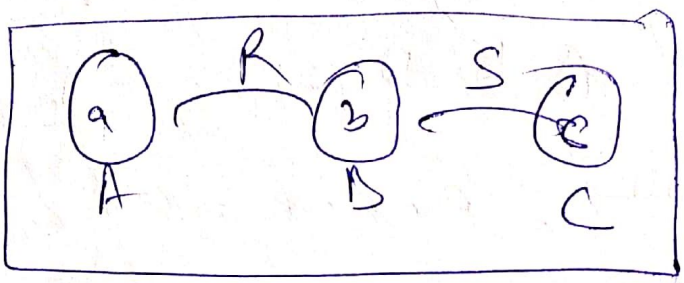
Solution: The ordered pair $(1,p)$ in R and (p,y) in S produce the ordered pair $(1,y)$ in R o S. For other pairs of R o S,

- $(1,r) \in R$ and $(r,z) \in S$, then $(1,z) \in R \circ S$
- $(2,q) \in R$ and $(q,x) \in S$, then $(2,x) \in R \circ S$

$(1,p) \in R \& (p,y) \in S \Rightarrow (1,y) \in R \circ S$

• $(3,2) \in R$ and $(2,x) \in S$, then $(3,x) \in ROS$. So, we get

$ROS = \{(1,7), (1,8), (4,4), (3,x)\}$.



Thm: Let A, B, C and D be four non-empty sets. Let R_1 be a relation from A to B R_2 is a relation from B to C and R_3 is a relation from C to D. Then

(i). $(R_1 \circ R_2)^{-1} = R_2^{-1} \circ R_1^{-1}$

(ii). $(R_1 \circ R_2) \circ R_3 = R_1 \circ (R_2 \circ R_3)$

Proof: (i). Since R_1 is a relation from A to B and R_2 is a relation from B to C, therefore $R_1 \circ R_2$ is a relation from A to C. $\therefore (R_1 \circ R_2)^{-1}$ is a relation from C to A. Again. R_1^{-1} is a relation from B to A and R_2^{-1} is a relation

from C to B. $\therefore R_2^{-1} \circ R_1^{-1}$ is a relation from C to A.

$\therefore Dom((R_1 \circ R_2)^{-1}) = Dom(R_2^{-1} \circ R_1^{-1})$ and $Ran((R_1 \circ R_2)^{-1}) = Ran(R_2^{-1} \circ R_1^{-1})$.

For $z \in C$ and $x \in A$,

$z (R_1 \circ R_2)^{-1} x \Leftrightarrow x (R_1 \circ R_2) z$

$\Leftrightarrow \exists\, y \in B$ s.t. $(x R_1 y$ and $y R_2 z)$

$\Leftrightarrow \exists\, y \in B$ s.t. $(y R_1^{-1} x$ and $z R_2^{-1} y)$

$\Leftrightarrow \exists\, y \in B$ s.t. $(z R_2^{-1} y$ and $y R_1^{-1} x)$

(∵ $\wedge$ is commutative)

$\Leftrightarrow z (R_2^{-1} \circ R_1^{-1}) x$.

$\therefore (R_1 \circ R_2)^{-1} = R_2^{-1} \circ R_1^{-1}$. ☐

(iii). Since $R_1 \circ R_2$ is a relation from A to C and R_3 is a relation from C to D, therefore $(R_1 \circ R_2) \circ R_3$ is a relation from A to D.

Again, R_1 is a relation from A to B and

$R_2 \circ R_3$ is a relation from B to D, so $R_1 \circ (R_2 \circ R_3)$ is a relation from A to D. Now

$$\text{Dom}\left((R_1 \circ R_2) \circ R_3\right) = \text{Dom}\left(R_1 \circ (R_2 \circ R_3)\right) \text{ and}$$

$$\text{Ran}\left((R_1 \circ R_2) \circ R_3\right) = \text{Ran}\left(R_1 \circ (R_2 \circ R_3)\right).$$

Now for $x \in A$ and $t \in D$,

$$x\left((R_1 \circ R_2) \circ R_3\right) t \iff \exists\, z \in C \text{ s.t. } x(R_1 \circ R_2) z \wedge z R_3 t$$

$$\iff \exists\, y \in B \text{ and } \exists\, z \in C \text{ s.t. } (x R_1 y \wedge y R_2 z) \wedge (z R_3 t)$$

$$\iff \exists\, y \in B \text{ and } z \in C \text{ s.t. } (x R_1 y) \wedge (y R_2 z \wedge z R_3 t)$$

($\because \wedge$ is associative)

$$\iff \exists\, y \in B \text{ s.t. } (x R_1 y) \wedge y (R_2 \circ R_3) t$$

$$\iff x\left(R_1 \circ (R_2 \circ R_3)\right) t$$

$$\therefore (R_1 \circ R_2) \circ R_3 = R_1 \circ (R_2 \circ R_3). \quad \square$$

NO FUNCTION AVAILABLE

CONTACT TO SIR FOR MURE INFO

Binary operations:

If $f: G \times G \to G$, then f is said to be binary operation on G. Thus a binary operation on G is a function that assign each ordered pairs of elements of G is unique element of G.

The symbols $+, \cdot, \circ, *$ etc are used to denote binary operations on a set. Thus $+$ will be a binary operation on G iff

$a+b \in G \quad \forall a, b \in G$ and $a+b$ is unique.

Similary, $*$ will be a binary operation on G iff

$a * b \in G \quad \forall a, b \in G$ and $a * b$ is unique.

This is said to be the closure property of the binary operation and the set G is said to be closed w.r.t. the binary operation. e.g. $+, \times$ are B.O. on $\mathbb{N}$. Thus

$\mathbb{N}$ is closed w.r.t. $+, \times$

$-$ is not binary operation on $\mathbb{N}$, because $3-4 = -1 \notin \mathbb{N}$. But $-$ is B.O. on $\mathbb{Z}$.

Algebraic Structure.

A non-empty set together with one or more than one

binary. e.g. $(\mathbb{N}, +)$, $(\mathbb{Z}, +)$, $(\mathbb{R}, +, \cdot)$ are all algebraic structures. Therefore, $(\mathbb{R}, +, \cdot)$ is an algebraic structure equipped with two operations.

Laws of Binary operations:

Associative law. A binary o.* on a set S is said to be associative iff $\forall\, a, b, c \in S$.

$$a * (b * c) = (a * b) * c$$

Commutative Law. A b.o. * on the elements of the set is commutative iff. for any two elements $a, b \in S$

$$a * b = b * a$$

e.g. $(\mathbb{Z}, +)$, $(\mathbb{Z}, \cdot)$ commutative, associative w.r.t. + and multiplication.

e.g. $(\mathbb{Z}, -)$, the B.O. of subtraction on $\mathbb{Z}$ is neither associative nor commutative since

$$4 - (5 - 6) = 5 \neq (4 - 5) - 6$$

and also $3 - 4 \neq 4 - 3$

Identity element: An element e in a set S is called an identity element w.r.t. the binary operation * if for any element a in S.

$$a * e = e * a = a$$

If $a * e = a$, then e is called the right identity element for the operation * and if $e * a = a$.

e is called the left identity element for the operation *.

e.g. $\mathbb{Q}$ → the set of rational number,

Obviously, 0 is the identity element, w.r.t. addition since $0 + x = x + 0 = x$ for every $x \in \mathbb{Q}$.

and 1 is the ... w.r.t. "." since $1 \cdot x = x \cdot 1 = x$

e.g. $\mathbb{N}$ → No identity w.r.t. "+"

" " " "."

1 is ...

Thm The identity element (if it exists) of any algebraic structure is unique.

Inverse element Consider a set S having the identity element e w.r.t. the binary operation *. If corresponding to each element $a \in S$ there exists an element $b \in S$ s.t. $a * b = b * a = e$. Then b is said to be the inverse of a. and is usual denoted by $\bar{a}^1$. e.g.

R: the set of real numbers, R has "0" as the identity element w.r.t. +. Then for any $a \in R$, we see that

$(-a) + (a) = a + (-a) = 0$. Thus $(-a)$ is its inverse. This is called the additive inverse

Similarly, for the set Q of rational numbers 1 is the identity element for the binary operation multiplication. Then, for any $a \in Q$ we see that

$$a \cdot (1/a) = (1/a) \cdot a = 1$$

Thus for any "a" $\in Q$, its reciprocal is its inverse. This is called the multiplicative inverse.

Note: The inverse of the identity element is identity element itself.

Group:

Let $(G, *)$ be an algebraic structure, where $*$ is a binary operation, then $(G, *)$ is called a group under this operation if the following conditions are satisfied.

1. Closure Law: The binary $*$ is a closed operation i.e., $a * b \in G$ for all $a, b \in G$.

2. Associative Law. The binary operation $*$ is an associative operation i.e. $a * (b * c) = (a * b) * c$ for all $a, b, c \in G$.

3. Identity element. There exists an identity element, i.e., for some $e \in G$, $e * a = a * e = a$, $a \in G$.

4. Inverse element: For each 'a' in G, there exists an element a' (the inverse of a)

in G such that $a * a' = a' * a = e$.

Note: A group G is said to be Abelian if the commutative law holds, i.e.

$$a * b = b * a \quad \forall a, b \in G.$$

Additive group → $(+)$
Multiplication group → $(\times)$

Example:

(a) The set R of real numbers, for the binary operation of addition, is a group, with 0 as identity element and $(-a)$ as the inverse of a. The same is true for $\mathbb{Z}, \mathbb{Q}$.

(b) R^* → the set of non-zero real numbers
Group w.r.t multiplication
1 is identity and $1/a$ as the inverse of a. The same is true for Q^*, C^*.

(c) The set Z^+ of positive integers with operation $+$ is not a group. There is no identity element for $+$ in Z^+. The set Z^+ with operation multiplication is not a group. There is an identity element 1, but no inverse of 3

Example: Show that the set $\{1, -1, i, -i\}$ is an abelian group multiplicative group

Thm. (Cancellation Law). If $(G, *)$ is a group and a, b are in G, then

(i). $a*b = a*c \Rightarrow b=c$ (left cancellation law)

(ii). $b*a = c*a \Rightarrow b=c$ (right cancellation law)

Proof (i). Since $a^{-1} \in G$, So from $a*b = a*c$, we can write

$$a^{-1} * (a*b) = a^{-1} * (a*c)$$

$$\Rightarrow (a^{-1}*a)*b = (a^{-1}*a)*c$$

$$\Rightarrow e*b = e*c$$

$$\Rightarrow b=c.$$

(ii). We have

$$b*a = c*a$$

operating on the right by a^{-1}, we obtain

$$(b*a)*a^{-1} = (c*a)*a^{-1}$$

$$\Rightarrow b*e = c*e$$

$$\Rightarrow b=c. \text{ Hence}$$

$b*a = c*a \Rightarrow b=c.$

Thm The left identity is also right identity, i.e.

$$e*a = a = a*e \quad \text{for all } a \in G.$$

Pf. If a^{-1} be the left inverse of a, then

$$a^{-1}*(a*e) = (a^{-1}*a)*e$$

or $$a^{-1}*(a*e) = e*e$$

$$= e$$

$$= a^{-1}*a.$$

Thus $$a^{-1}*(a*e) = a^{-1}*a.$$

$$a*e = a. \quad \text{- by above thm.}$$

Hence e is also the right identity element of a group.

Thm The left inverse of an element is also its right inverse, i.e.,

$$a^{-1}*a = e = a*a^{-1}.$$

Thm. In a group $(G, *)$

(i) $(a^{-1})^{-1} = a$

(ii) $(ab)^{-1} = b^{-1}a^{-1}$

Example Show that if a, b are arbitrary elements of a group G, ~~Suppose $(ab)^2 = a^2b^2$~~ then $(ab)^2 = a^2b^2$ iff G is abelian

Solution. Let a and b be arbitrary elements of a group G. Suppose $(ab)^2 = a^2b^2$ — (1)

To prove G is abelian, we have to show that

$ab = ba$.

$(ab)^2 = a^2b \Rightarrow (ab)(ab) = (aa)(bb)$

$\Rightarrow a(ba)b = a(ab)b$ by associative law

$\Rightarrow (ba)(b) = (ab)b$ by left Cancellation law

$\Rightarrow ba = ab$ by right Cancellation law.

Again, Suppose G is abelian so that

$(ab) = ba \quad \forall a, b \in G$ — (2)

To prove that $(ab)^2 = a^2b^2$

$(ab)^2 = (ab)(ab) = a(ba)b$

$= a(ab)b = (aa)(bb) = a^2b^2$

Hence proved.

Example. G is group and there exists two relatively prime positive integers m and n such that $a^m b^m = b^m a^m$ and $a^n b^n = b^n a^n$ for all $a, b \in G$. Prove that

G is abelian.

3. Proof. Straightforward.

Order of an element:

The order of an element g in a Group G is the smallest positive integer n s.t. $g^n = e$. If no such integer exists, we say g has infinite order.

The order of an element g is denoted by $o(g)$.

e.g.

Note. Example: Find the order of every element in the multiplicative group $G = \{a, a^2, a^3, a^4, a^5, a^6 = e\}$.

Soln: The identity element of G is $a^6 = e$.

$a^6 = e \Rightarrow o(a) = 6$

$(a^2)^3 = a^6 = e \Rightarrow o(a^2) = 3$

$(a^3)^2 = a^6 = e \Rightarrow o(a^3) = 2$.

$(a^4)^3 = a^{12} = (a^6)^2 \Rightarrow e^2 = e \Rightarrow o(a^4) = 3$

$(a^5)^6 = (a^6)^5 = e^5 = e \Rightarrow o(a^5) = 6$

$a^6 = e \Rightarrow o(a^6) = 1$

Groupoid.

Let $(S, *)$ be an algebraic structure with $*$ is a binary operation. Then S is closed w.r.t. the operation $*$. Then such structure is called a groupoid.

Semigroup

An algebraic structure $(S, *)$ is called a semigroup if the following conditions are satisfied.

(i). $a * b \in S \quad \forall a, b \in S$ (closure law)

(ii). $a * (b * c) = (a * b) * c \quad \forall a, b, c \in S$ (ass. law)

Monoid.

An algebraic structure $(S, *)$ is called a monoid if the following conditions are satisfied.

(i). $a * b \in S \quad \forall a, b \in S$.

(ii). $a * (b * c) = (a * b) * c \quad \forall a, b, c \in S$.

(iii). for some $e \in S$, $e * a = a * e = a \quad \forall a \in S$.

e.g. (i) $N :=$ the set of natural numbers, $(N, +)$ is groupoid because N is closed under addition. But the set of odd integers is not a groupoid.

under addition operation since $3+3=6$ do not belong to the set of odd integers and hence is not closed.

(ii). Z := the set of integers, then $(Z, +)$ and $(Z, \cdot)$ are semi-group as these two operations are closed and associative in Z.

Subgroup. Let $(G, *)$ be a group and H is a subset of G. $(H, *)$ is said to be subgroup of G if $(H, *)$ is also group by itself.

Note. Since every set is a subset of itself. Therefore if G is a group, then G itself is a subgroup of G.

Also if e is the identity element of G. Then the subset of G containing only identity element is also a subgroup of G. These two subgroups $(G, *)$ and $(\{e\}, *)$ of the group $(G, *)$ are called improper or trivial subgroups, others are called proper or non-trivial subgroups. e.g. (i). The multiplicative group $\{i, -i\}$ is a subgroup of the multiplicative group $\{1, -1, i, -i\}$.

(iii). The additive group of even integers is a subgroup of the additive group of all integers.

Thm. The identity element of a subgroup is the same as that of the group.

Thm: A non-empty subset H of a group G is a subgroup of G iff

(i). $a \in H; b \in H \Rightarrow a * b \in H$

(ii) $a \in H \Rightarrow a^{-1} \in H$.

Proof: **Necessary Condition.** Suppose H is a subgroup of G. Then H must be closed with respect to operation $*$, i.e., $a \in H, b \in H \Rightarrow$

$\Rightarrow a * b \in H$.

Let $a \in H$ and let a^{-1} be the inverse of a in G. Then the inverse of a in H is also a^{-1}. Since H itself is a group, therefore each element of H must possess inverse. therefore $a \in H \Rightarrow a^{-1} \in H$, ~~itself is a group~~.

Sufficient Condition. We observe that the binary operation $*$ in G is also a binary operation in H. Hence H is closed under the operation.

As the elements of H is also the elements of G and the elements of G satisfy the associative

law for the binary operation, therefore, the elements of H will also satisfy the associative law.

Now $a \in H \Rightarrow a^{-1} \in H$.

From the condition (ii), we obtain

$a \in H, a \in H \Rightarrow a a^{-1} \in H \Rightarrow e \in H$

which shows the existence of identity element in H. Thus

H is a subgroup of G.

Thm 12.14: The necessary and sufficient condition for a non-empty subset H of a group $(G, *)$ to be group is

$$a \in H, b \in H \Rightarrow a * b^{-1} \in H$$

where b^{-1} is the inverse of b in G.

Thm The intersection of any two sub-groups of a group $(G, *)$ is again a sub-group of $(G, *)$.

Pf Let H_1 and H_2 from any two sub-groups of $(G, *)$

We have $H_1 \cap H_2 \neq \phi$. Since at least the identity element is common to both H_1 and H_2

Let $a \in H_1 \cap H_2$ and $b_2 \in H_1 \cap H_2$.

Now, $a \in H_1 \cap H_2 \Rightarrow a \in H_1$ and $a \in H_2$

$b \in H_1 \cap H_2 \Rightarrow b \in H_1$ and $b \in H_2$

Since H_1 and H_2 form sub-groups under the group $(G, *)$, we have

$a \in H_1,\ b \in H_1 \Rightarrow a * b^{-1} \in H_1$

$a \in H_2,\ b \in H_2 \Rightarrow a * b^{-1} \in H_2$.

So, from above, we obtain

$a * b^{-1} \in H_1 \cap H_2$. Thus we see,

$a \in H_1 \cap H_2,\ b \in H_1 \cap H_2 \Rightarrow a * b^{-1} \in H_1 \cap H_2$.

Therefore, $H_1 \cap H_2$ forms a sub-group under $(G, *)$. ☒

Note Let G be the additive group of integers. Then

$H_1 = \{\ldots -6, -4, -2, 0, 2, 4, 6 \ldots\}$ and

$H_2 = \{\ldots -12, -9, -6, -3, 0, 3, 6, 9, 12 \ldots\}$.

are both subgroups of G. Now

$H_1 \cup H_2 = \{\ldots, -4, -3, -2, 0, 2, 3, 4, 6, \ldots\}$. Obviously $H_1 \cup H_2$ is not closed with respect to addition as $2 \in H_1 \cup H_2,\ 3 \in H_1 \cup H_2$ but $2 + 3 = 5 \notin H_1 \cup H_2$

Therefore, $H_1 \cup H_2$ is not a subgroup of G

Cosets: Let H be a subgroup of a group G and let $a \in G$. Then the set $\{a*h : h \in H\}$ is called the left coset generated by a and H and is denoted by aH.

Similarly the set $Ha = \{h*a : h \in H\}$ is called the right coset and is denoted by Ha

It is evident that both aH and Ha are subsets of G.

If e be the identity element of G, then $e \in H$ and $He = H = eH$. Therefore, H itself is a right as well as a left coset.

In general $aH = Ha$, but in the abelian group each left coset coincides with the corresponding right coset.

If the group operation be addition, then the right coset of H in G generated by a is defined as $H+a = \{h+a : h \in H\}$. Similarly, the left coset $a+H = \{a+h : h \in H\}$.

Index of a subgroup in a group: If H is a subgroup of a group G, the number of distinct

right (left) cosets of H in G is called the index of H in G and is denoted by $[G:H]$ or by by $i_G(H)$. e.g. Let G be the additive group of integers, i.e;

$$G = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

Let H be the subgroup of G obtained on multiplying each element of G by 3. Then

$$H = \{\ldots -9, -6, -3, 0, 3, 6, 9 \ldots\}.$$

Since the group G is abelian any right coset will be equal to the corresponding left coset. Let us from the right cosets in G.

we have $0 \in G$ and

$$H = H+0 = \{\ldots, -9, -6, -3, 0, 3, 6, 9 \ldots\}.$$

Again $1 \in H$ and $H+1 = \{\ldots, -8, -5, -2, 1, 4, 7, 10, \ldots\}$

Then $2 \in H$ and $H+2 = \{\ldots, -7, -4, -1, 2, 5, 8, 11, \ldots\}$.

we see that the right cosets H, H+1 and H+2 are all distinct and moreover these are disjoint, i.e; have no element common.

Now, $3 \in G$ and $H+3 = \{\ldots -6, -3, 0, 3, 6, 9, 12, \ldots\}$.

We see that $H+3 = H$. Also we observe that $3 \in H$. Again $4 \in G$ and $H+4 = \{\ldots, -5, -2, 1, 4, 7, 10, 13, \ldots\} = H+1$. The union of all right cosets of H in G will be equal to G, i.e.,

$$G = H \cup (H+1) \cup (H+2).$$

The index of H in $G = 3$.