Unit - II

Trees, Connectivity & Planarity:

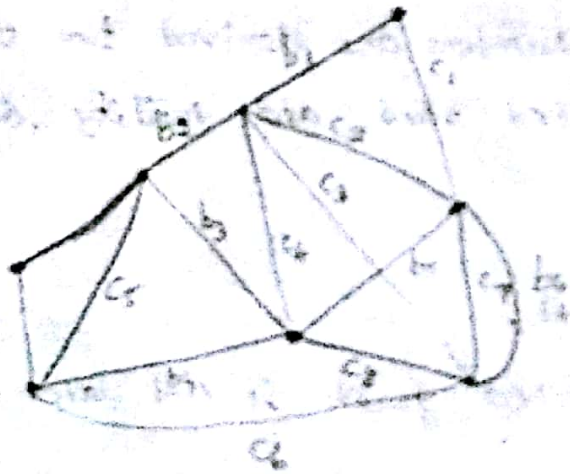
Completed

3.7. Spanning Trees:-

Definition:-

A tree T is said to be a spanning tree of a connected graph G if T is a subgraph of G and T contains all vertices of G.

Example:-



Spanning tree

Theorem 3.11:-

Every connected graph has at least one spanning tree

proof:-

An edge in a spanning tree T is called a branch of T.

An edge of G that is not in a given spanning tree T is called a Chord

Consider a connected graph G has a union of two subgraphs. T and $\bar{T}$ here ...

where T is a spanning tree. $\bar{T}$ is the complement of T in G. since the subgraph $\bar{T}$ is the collection of chords.

Theorem 3.12 :-

With respect to any of its spanning trees, a connected graph of n vertices and e edges has $n-1$ tree branches and $e-n+1$ chords.

Rank and Nullity :-

From these three numbers are derived two other important numbers called rank and nullity, defined as

rank $r = n-k$

nullity $\mu = e-n+k$.

The rank of a connected graph is $n-1$ and the nullity $e-n+1$.

rank of G = number of branches in any spanning tree of G.

nullity of G = number of chords in G.

rank + nullity = number of edges in G.

The nullity of a graph is also referred to as its cyclomatic number.

3.8. Fundamental circuits :-

Theorem 3.13 :-

A connected graph G is a tree iff adding an edge

Definition:- **Fundamental circuit**

Consider a spanning tree T in a connected graph G. Adding any one chord to T will create exactly one circuit. Such a circuit, formed by adding a chord to a spanning tree, is called a fundamental circuit.

3.10. Spanning Trees in a Weighted Graph:-

Definition:-

A spanning tree with the smallest weight in a weighted graph is called a shortest spanning tree or shortest distance spanning tree or minimal spanning tree.

Theorem 3.16:-

A spanning tree T is a shortest spanning tree iff there exists no other spanning tree at a distance of one from T whose weight is smaller than that of T.

Proof:-

Let T_1 be a spanning tree in G satisfying the hypothesis of the theorem.

The proof will be completed by showing that if T_2 is shortest spanning tree in G, the weight of T_1 will also be equal to that of T_2.

Let T_2 be a shortest spanning tree in G.

Consider an edge e in T_2 which is not in T_1.

Adding e to T_1 forms a fundamental circuit with branches in T_1.

Some, but not all of the branches in T_1 that from the fundamental circuit with e may also be in T_2; each of these branches in T_1 has a weight smaller than or equal to that of e, because of the assumption on T_1.

Amongest all those in this circuit which are not in T_2 at least one, say b_j, must from some fundamental circuit containing e.

Because of the minimality assumption on T_2 weight as e.

Hence the spanning tree $T_1' = (T_1 \cup e - b_j)$ obtained from T_1 through one cycle exchange, has the same weight as T_1.

But T_1 has one edge more in common with T_2, and it satisfies the condition of this theorem.

This argument can be repeated, producing a series of trees of equal weight $T_1, T_1, \ldots, T_1, \ldots$ each a unit distance closed to T_2, until we get T_2 itself.

This proves that if none of the spanning trees at a unit distance from T is shorter than T, no ... graph.

9.4. cut sets:

Definition: cut set

Suppose G be a connected graph and let T be spanning tree of G. A cut set containing exactly one edge of the spanning tree T is called a fundamental cut set with respect to T.

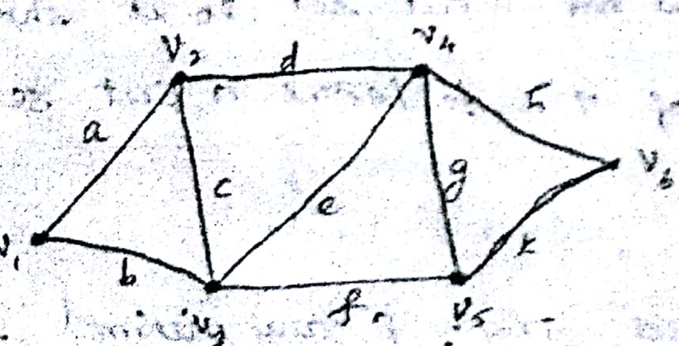
Properties of a cut set:-

i) Every cut set in a connected graph G must contain atleast one branch of every spanning tree of G.

ii) In a connected graph G, any minimal set of edges containing atleast one branch of every spanning tree of G is a cutset.

iii) Every circuit has an even number of edges in common with any cutset.

Example:



Theorem 9.4.1:

Every cut set in a connected graph G must ... branch of every spanning

Proof:

Let S be a cut set of G.

Let T be the spanning tree of G.

Suppose S does not contain any branch of T.

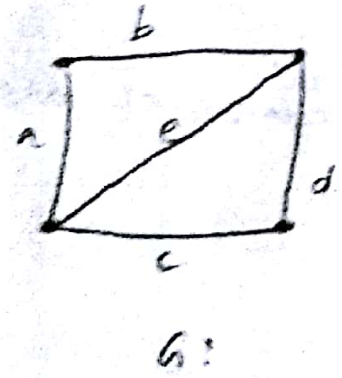
Then all the edges of T present in G-S.

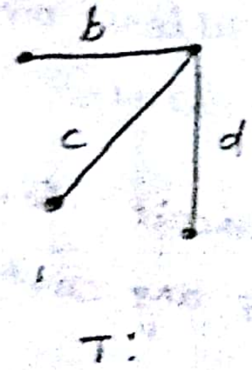
G-S is a connected graph.

⇒ S is not a cut set.

Hence a cut set must contain atleast one branch of every spanning tree of G.

Example:





Let S = {a, e} is not a cut set, so it should contain one branch of T to become a cut set.

Theorem: 2.4.2:

In a connected graph G, any minimal set of edges containing atleast one branch of every spanning tree of G is a cut set.

S contains atleast one branch of every spanning tree, having minimum number of edges.

If G-S is connected it means G-S has spanning tree and no edge of spanning tree present in S.

∴ So, it must contain, an edge of spanning tree, however it any contain more than one branch of spanning tree:

⇒ S is a minimal cut set containing branch of the tree.

Theorem 2.4.2:-

Every circuit has an even number of edges in common with any cut set.

Proof:-

Suppose S be the cut set of G.

⇒ G-S is a disconnected graph because removal of S from G partitions the vertices of G into two sets V_1 and V_2

Consider a circuit Γ in G.

If all the vertices in Γ are entirely

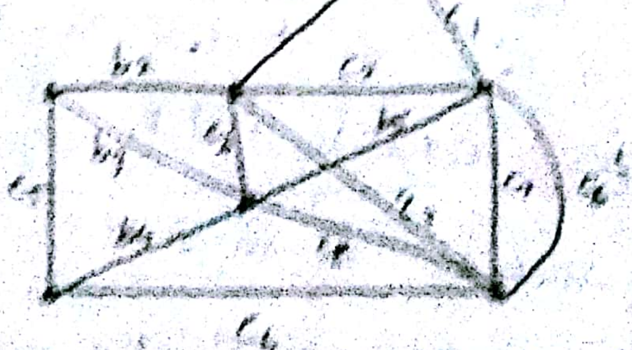
All cut sets in a graph :-

If we want to find the fundamental circuits, a spanning tree is essential because a spanning tree can give the set of fundamental circuits. Similarly we can get a set of fundamental cut sets if a spanning tree is given. So we can say that that to get the fundamental cut set a spanning tree is essential.

Fundamental circuits and cut sets

Let a spanning tree T in a connected graph G. Addition of a chord to T will create exactly one circuit.

Formation of such a circuit due to addition of chord to a spanning tree is called a Fundamental circuit.

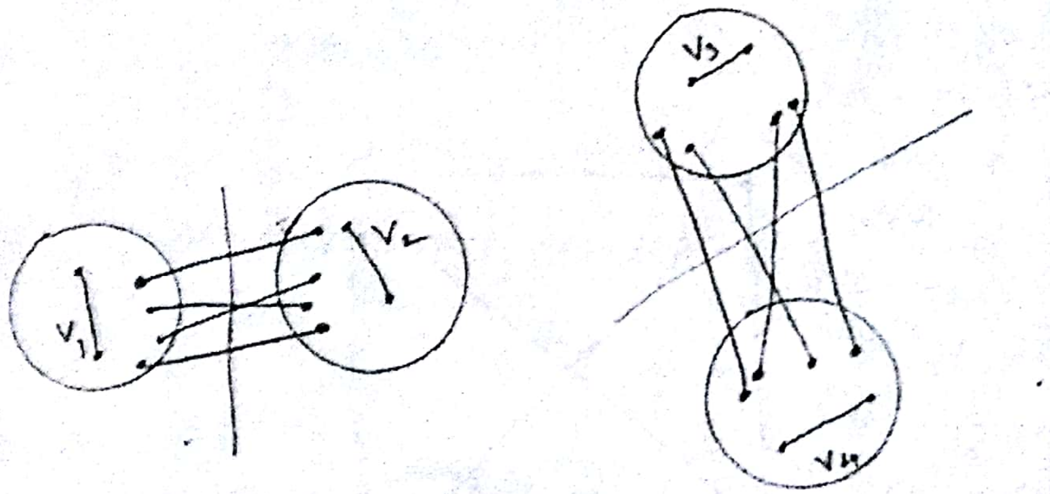


Theorem 2.4.4:-

The ring sum of any two cut set in a graph is either a third cut set or an edge disjoint union of cut sets.

Proof:-

Let S_1 and S_2 be two two sets in a given connected graph G.



Let V_1 and V_2 be the unique and disjoint partitioning of the vertex set V of G corresponding to S_1.

Let V_3 and V_4 be the partitioning corresponding to S_2.

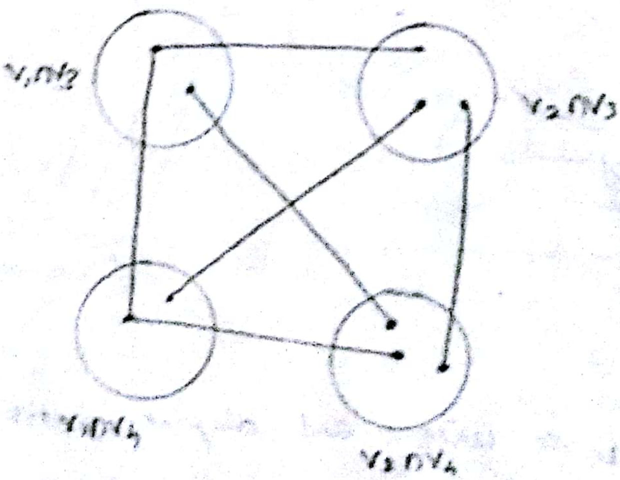
Clearly from the above figure

$V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \phi$

$V_3 \cup V_4 = V$ and $V_3 \cap V_4 = \phi$

Now let the subset $(V_1 \cap V_4) \cup (V_2 \cap V_3)$ be called V_5 and this by definition is the same as the ring sum $V_1 \oplus V_3$.

Similarly let the subset $(V_1 \cap V_3) \cup (V_2 \cap V_4)$ be called V_6 which is the same as $V_2 \oplus V_3$.



The ring sum of the two cut sets $S_1 \oplus S_2$ can be seen to consist only of edges that join vertices in V_5 to those in V_6.

Also there are no edges outside $S_1 \oplus S_2$ that join vertices in V_5 to those in V_6.

Thus the set of edges $S_1 \oplus S_2$ produces a partitiong of V into V_5 and V_6 such that

$$V_5 \cup V_6 = V \text{ \& } V_5 \cap V_6 = \phi.$$

Hence $S_1 \oplus S_2$ is a cut set if the subgraph Containg V_5 and V_6 each remain connected after $S_1 \oplus S_2$ is removed from G.

Otherwise $S_1 \oplus S_2$ is an edge disjoint union of cut sets.

Theorem 2.4.6:-

With respect to given spanning tree T, a branch b_i that determines a fundamental cut-set S is contained in every fundamental circuit associated with the Chords in S and in no others.

Proof:-

Let the fundamental cut set S determined by a branch b_i be $S = \{b_i, c_1, c_2, \ldots c_p\}$ and let Γ_1 be the fundamental circuit determined by chord c_1.

Scanned by CamScanner

$\Gamma_1 = \{c_1, b_1, b_2, \ldots b_q\}$

Since the number of edges common to S and Γ_1

must be even, b_i must be in Γ_1.

The same is true for the fundamental circuit

made by chords $c_2, c_3, \ldots, c_p$.

On the other hand, suppose that b_i occurs in

a fundamental circuit Γ_{p+1} made by a chord

other than $c_1, c_2, \ldots c_p$.

Since none of the chords $c_1, c_2, \ldots, c_p$ is in Γ_{p+1}

there is only one edge b_i common to a circuit

Γ_{p+1} and the cut-set S, which is not possible.

Hence the theorem.

* Connectivity and separability :-

1) Edge Connectivity :-

In a connected graph the number of edges in the smallest cut set is defined as the edge connectivity of G.

2. vertex connectivity:-

The vertex connectivity of a connected graph is defined as the minimal number of vertices whose removal from G leaves the remaining subgraph disconnected.

This is denoted by $k(V)$.

3) separability:-

A connectivity graph is said to be separable if its vertex connectivity is one. A connected graph which is not separable is termed as non-separable graph.

Theorem 2.4.7:-

A vertex v is a connected graph G is a cut-vertex iff there exist two vertices x & y in G such that every path between x & y passes through v.

proof:-

If v is a cut-vertex of G, then $G-V$ is a disconnected graph.

let x & y in two different components $G-V$ —— ①

Then there exists no path from x to y in $G-V$

since G is connected there exists a path

P from x and y in G.

If the path does not contain the vertex V,

then removal of v from G will not disconnects

the vertices x and y which is a contradiction -

to ①.

Hence every path between x and y passes

through v.

<u>Converse Part:-</u>

If every path from x to y contain the

vertex v then removal of V from G disconnects

x and y.

Hence x and y lies in different component of

G.

$\Rightarrow$ $G-V$ is disconnected graph

Then v is a cut-vertex of G.

Theorem 2.4.8:-

The edge connectivity of a graph G can not exceed the degree of the vertex with the smallest degree in G.

Proof:

Let vertex v_i be the vertex with the smallest degree in G.

Let $d(v_i)$ be the degree of v_i vertex can be separted from G by removing $d(v_i)$ edges incident on vertex v_i.

Hence the theorem.

Theorem 2.4.9:-

The vertex connectivity any graph G can never exceed the edge connectivity of G.

Proof:-

Let k denote the edge connectivity of G.

There exists a cut set S in G with k edges.

Let S Partition the vertices of G into Subsets V_1 & V_2

By removing at most k vertices from V_1 or V_2 on which the edges in S are incident, we can effect the removal of S together with all other edges incident on those vertices from G.

Hence the theorem.

<u>Theorem 2.4.10:-</u>

The maximum vertex connectivity one can achieve with a graph G of n vertices and e edges $(e \geq n-1)$ is the integral part of the number $\frac{2e}{n}$; that is $\left[\frac{2e}{n}\right]$.

<u>Proof:-</u>

We know that every edge in G contributes two degrees.

The total $(2e)$ degrees is divided among n vertices.

This implies there must exist atleast one vertex in G whose degree $\leq \frac{2e}{V}$

The vertex connectivity of G cannot exceed this number

To show that one can get maximum connectivity $= \left[\frac{2e}{n}\right]$ first construct an n-vertex regular graph of degree $\left[\frac{2e}{n}\right]$ and then add the remaining $e - \frac{n}{2}$. $\left[\frac{2e}{n}\right]$ edges arbitrarily.

Theorem 2.4.11:-

A connected graph G is k-connected iff every pair of vertices in G is joined by k or more paths that do not intersect. (paths with no common vertices, except the two terminal vertices, are called non intersecting paths or vertex-disjoint paths) and atleast one pair of vertices is joined by exactly k non intersecting paths.

Theorem 2.4.12:-

The edge connectivity of a graph G is k iff every pair of vertices in G is joined by k or more edge-disjoint paths and atleast one pair of vertices is joined by exactly k edge-disjoint paths

2.5. Network Flow:-

A network consists of a set of vertices links by edges and each edge is associated with flow of some type

A flow of some type is associated with network like telephone lines, rail roads, oil or water pipelines etc

Theorem 2.5.1:-

The maximum flow possible between two vertices a and b in a network is equal to the minimum of the minimum of the capacities of all cut sets with respect to a and b.

Proof:

We choose any cut set S w.r.to the vertices a and b in G.

In the subgraph G–S there is no path between a and b.

Therefore every path in G between a and b must contain atleast one edge of S.

Thus every flow from a and b must pass through one or more edges of S.

Hence the total flow rate between these two vertices cannot exceed the capacity of S.

Since, this holds for all cut-sets with respect to a and b, the flow rate cannot exceed the minimum of their capacities.

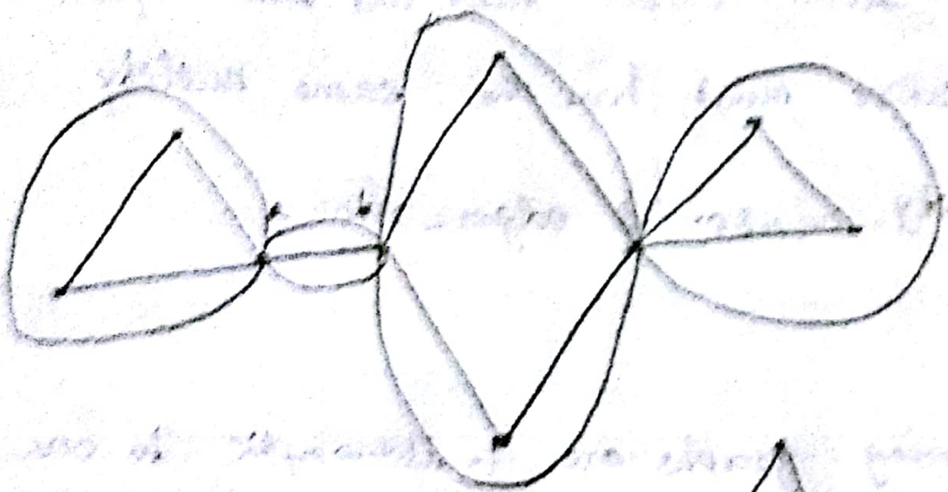
1-Isomorphism:-

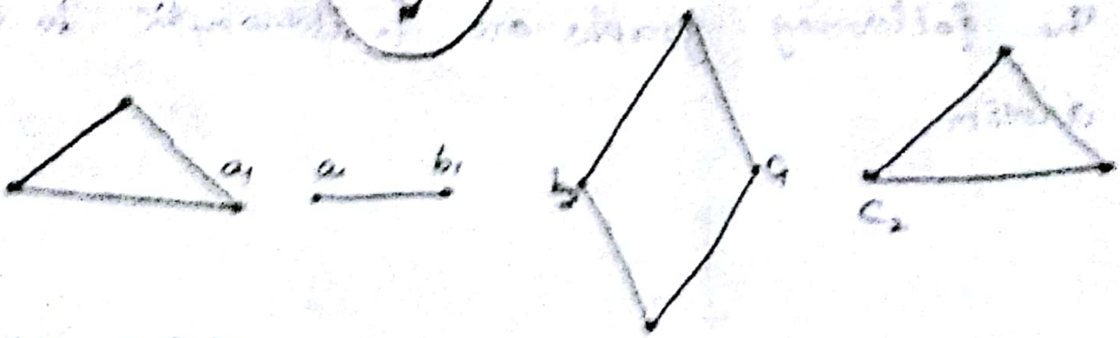
Definition:-

Two graphs G_1 and G_2 are said to be 1-isomorphic if they become isomorphic to each

1-isomorphic if they become isomorphic to each other when we repeatedly "decompose" a cut-vertex into two vertices to produce two disjoint subgraphs.

Example:





Theorem 8.5.8:-

If G_1 & G_2 are two 1-isomorphic graphs, the rank of G_1 equals the rank of G_2 and the nullity of G_1 equals the nullity of G_2.

Proof:

Whenever a cut-vertex in a graph G is split into two vertices the number of components in G increases by 1.

Therefore the rank of G = number of vertices in G − no of components in G.

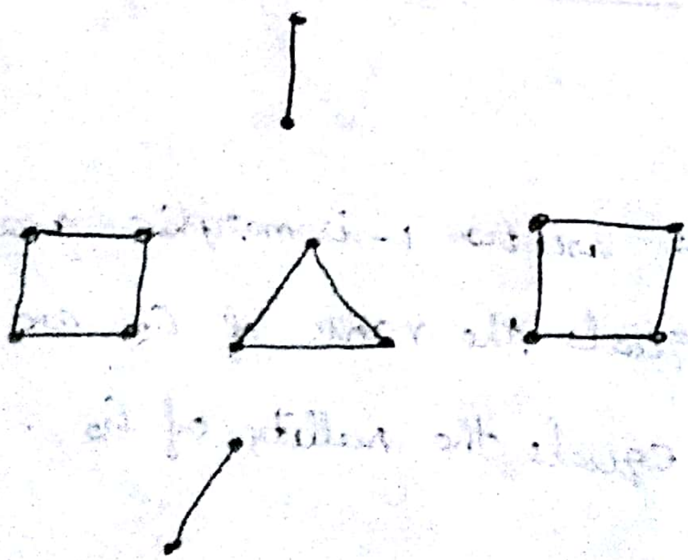
Two 1-isomorphic graphs have the same number of edges since no edges are destroyed or new edges created by operation 1.

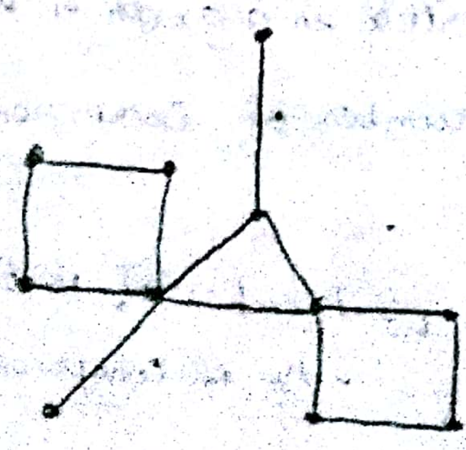
Two graphs awith equal rank and with equal number of edges must have the same nullity.

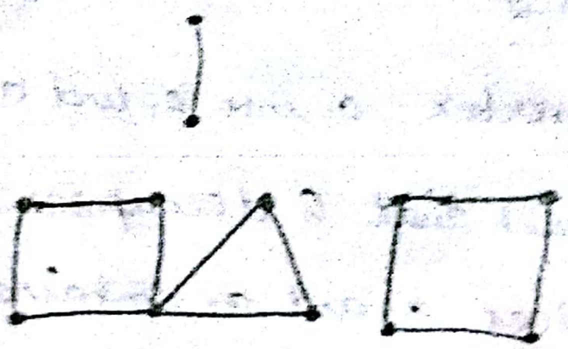
Nullity = Number of edges − rank.

Example:

The following graphs are 1-isomorphic to one another.







Definition: 2-isomorphism:

Two graphs G_1 & G_2 are said to be 2-isomorphic if they become isomorphic after undergoing operation 1 or operation 2 or both any number of times.

Definition: Operation 1:-

Decompose a cut-vertex into two vertices to produce two disjoint subgraphs.

Consider a 2-connected graph G and two vertices a and b. Let the deletion of two vertices a and b from G leave the remaining graph disconnected. The same can also be said in other words, G consists of a subgraph S and its complement $\bar{S}$, and have two vertices a and b in common. The operation 2 on G is .

Definition: Operation 2

Decompose the vertex a into a_1 and a_2 and vertex b into b_1 and b_2 such that G decomposes into S and $\bar{S}$. Suppose the vertices a_1 and a_2 contained in S and b_1 and b_2 contained in $\bar{S}$. Now reconnect the graphs S and $\bar{S}$ by meging a_1 with b_2 and a_2 with b_1.

Theorem 2.5.3:

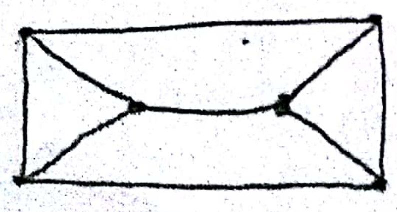
Two graphs are 2-isomorphic iff they have circuit correspondence.

Different representation of a planer graph:

Definition: Planar graph

A graph G is said to be planar if there exists some geometric representation of G which can be drawn on a plane such that no two of its edges intersect.

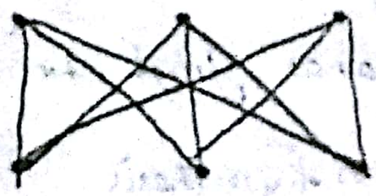
Example:



Definition:- Non-Planar graph

A graph that cannot be drawn on a plane without cross over between its edges is called non-planar.

Example:-



Theorem 2.6.1:-

Any simple planar graph can be embedded in a plane such that every edge is drawn as a straight line segment.

Theorem 2.6.2:-

A graph can be embedded in the surface of a sphere if and if it can be embeded in a plane.

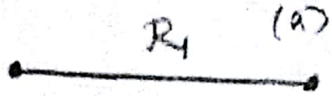
Theorem 2.6.3:-

A planar graph can be embeded in a plane such that any specified region can be made the infinte region

Theorem 2.6.4:-

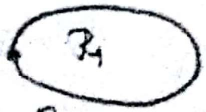
A connected planar graph with n vertices and e edges has $e-n+2$ regions.

(a)



$e=1 ; n=2$

(b)



$e=1 ; n=1$

Let G be a connected planar graph. We will prove that theorem by induction hypothesis.

If $e=1$ then $n=1$ or 2.

When $e=1$ and $n=2$, we have $e-n+2=1$.

Clearly graph in figure (a) has one region.

When $e=1$ and $n=1$ then we have $e-n+2=2$

Also graph in figure (b) above has two regions.

Thus the result is true for $e=1$.

Assume that the result is true for all graphs with atmost $e-1$ edges.

Let G be a connected graph with e edges and R regions.

If G is a tree then $e=n-1$ and number of faces is 1.

Also number of faces $= e - n + 2 = 1$.

Hence the theorem is true.

If G is not a tree, then it has some circuits.

Let a be an edge in some circuit.

Removal of the edge a from the plane representation of G will merge the two regions into one new region.

Thus $G-a$ is a connected graph with n vertices $e-1$ edges and $R-1$ region.

By induction hypothesis, we have

$$R-1 = e-1-n+2$$

$$\Rightarrow R = e-n+2 .$$

This completes the proof.

Corollary:

In any simple, connected planar graph with R regions, n vertices and e edges $(e>2)$ the following inequalities must hold.

$$e \geq \frac{3}{2} R \quad \text{——} \ (1)$$

$$e \leq 3n-6 \quad \text{——} \ (2)$$

Proof:-

Since each region is bounded by atleast three edges and each edge belongs to exactly two regions.

$$2e \geq 3R$$

$$\Rightarrow e \geq 3/2 \, R$$

Substituting for R from Euler' formula

$$e \geq \frac{3}{2}(e-n+2)$$

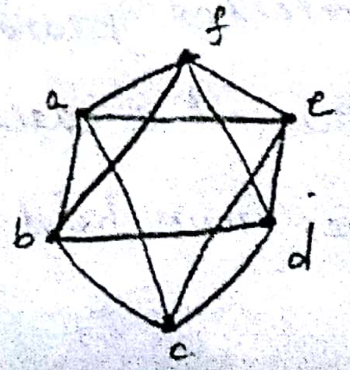
$$\Rightarrow e \leq 3n-6.$$

Theorem 2.6.5:-

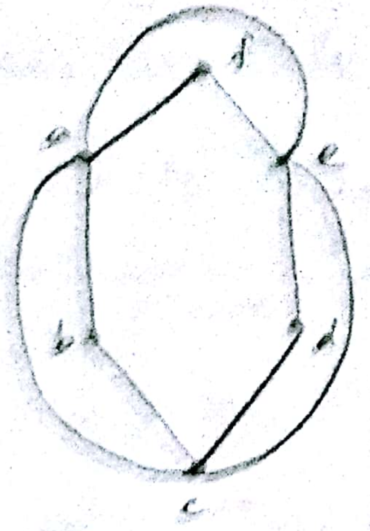
The sphrerical embeding of every planar 3-connected graph is unique.

Example:

Draw the planar representation of the graph



Sol



Example

If every region of a simple planar graph embedded in a plane is bounded by k edges, show that $e = \frac{k(n-2)}{k-2}$

Sol:

we have $kr \le 2e$

(or) $r \le \frac{2e}{k}$

$$e\left(1-\frac{2}{k}\right) \le (n-2)$$

we have $r = e-n+2$ (or) $e = r+n-2$

(i.e.) $e \le \left(\frac{2e}{k}\right) + n - 2$

$$ke \le 2e + k(n-2)$$

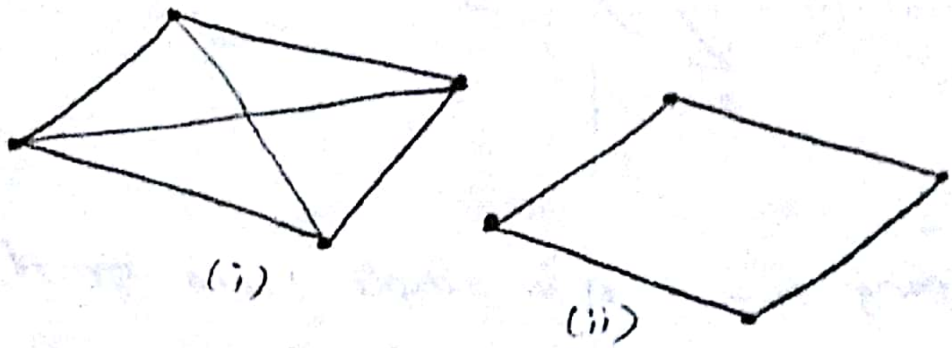
$$(k-2)e \le k(n-2)$$

$$e \le \frac{k(n-2)}{k-2}$$

Example :-

Give an example of plane, connected graph such that i) $e = 3v - 6$; ii) $e < 3v - 6$

Sol :-



(i)

(ii)

Example :

Suppose G is a graph with 1000 vertices and 3000 edges. Is G is planar?

Sol :-

If the graph is planar the

$e \leq 3n - 6$ (or) $e \leq 3v - 6$

Here $e = 3000$; $v = 1000$

$\therefore 3000 \nleq 3(1000) - 6$

$\therefore e \nleq 3v - 6$

Hence the given graph is not planar.

Scanned by CamScanner

Example:-

Draw all planar graphs with 5 vertices which are not isomorphic to each other.

Sol:-