⇒ Commutative law holds in G

⇒ G is abelian

eg. ① $Z_7 = \{1,2,3,4,5,6\}, \times_7$

$e = 1$ $3 \times_7 3 \times_7 3 \times_7 3 \times_7 3 \times_7 3 = 1$

$o(3) = 1$

(ii) $Z_3 = \{0,1,2\}, +_3$

$o(0) = 1$ (identity)

$o(1) = o(2) = 3$

* **Coset:** If G be a group and H be a subgroup of G. Let $a \in G$, then the set $Ha = \{ha : h \in H\}$ is called right coset of ~~~~ H by a in G and $aH = \{ah : h \in H\}$ is called left coset of H by a in G.

If binary operation is addition, then right ~~and~~ left coset of H in G is given by $H+a = \{h+a : h \in H\}$ & $a+H = \{a+h : h \in H\}$

eg. $(Z, +)$ is a group

$H = 2Z = \{\ldots -6, -4, -2, 0, 2, 4, 6 \ldots\}$ and

$0, 1, 2, \ldots \in G$

$H+0 = \{\ldots -6, -4, -2, 0, 2, 4, 6 \ldots\}$ Right coset of 0 in G

$H+1 = \{\ldots -5, -3, -1, 1, 3, 5, 7 \ldots\}$ Right coset of 1 in G

$H+2 = \{\ldots -4, -2, 0, 2, 4 \ldots\}$

⇒ ~~G has~~ We have only two distinct cosets $H+0$ and $H+1$.

Result:

1. If e is identity element of G, then $\boxed{He = eH = H}$
2. If $a \in H$, then $\boxed{Ha = aH = H}$
3. Cardinality of H is equal to cardinality of coset i.e. $\boxed{|H| = |aH| = |Ha|}$

* **Theorem:** Any two left (right) cosets of a subgroup H of G is either disjoint or identical.

27 − 3

81, 405, 2430, 729

Proof:

Let H be a subgroup of a group G and $a, b \in G$ and let Ha and Hb are two distinct coset of H in G.

Then we have to show that either $Ha \cap Hb = \phi$ or $Ha = Hb$.

Let $Ha \cap Hb \neq \phi$

$\Rightarrow \exists$ an element c such that $c \in Ha \cap Hb$

$\Rightarrow c \in Ha$ and $c \in Hb$

~~$\Rightarrow c = H_1 a$ and $c = Hb$~~

$\Rightarrow c = h_1 a$ and $c = h_2 b$

$\Rightarrow h_1 a = h_2 b$

$\Rightarrow h_1^{-1}(h_1 a) = h_1^{-1}(h_2 b)$ {as H is subgroup & $h_1 \in H \Rightarrow h_1^{-1} \in H$}

$\Rightarrow e \cdot a = h_1^{-1} h_2 b$

$\Rightarrow a = h_1^{-1} h_2 b$

$\Rightarrow Ha = H(h_1^{-1} h_2) b$ {as $h_1^{-1} h_2 \in H$}

$\Rightarrow Ha = Hb$

Result:
A group G can be expressed as union of all of its cosets.

Lagrange's Theorem:

If H be a subgroup of G, then the order of H is divisor of order of G.

Proof:

Let G be a group of order n, i.e. $G = \{a_1, a_2, \ldots a_n\}$ and H be a subgroup of G of order m, ~~then~~ i.e.

~~$H = \{a_1, a_2, \ldots a_m\}$~~ $H = \{h_1, h_2, \ldots h_m\}$

Now, let $a_i \in G$, then

$Ha_i = \{$ ~~ha_1, ha_2~~ $\{h_1 a_i, h_2 a_i, \ldots h_m a_i\}$ is right coset of H by a_i in G.

Then, Ha_i have m distinct elements.

If possible, let $h_j a_i = h_k a_i$

$\Rightarrow h_j = h_k$

which is a contradiction.

Hence, Ha_i has m distinct elements.

Now, as we know that group G can be written as

$G = Ha_1 \cup Ha_2 \cup Ha_3 \cdots \cup Ha_k$ {where $Ha_1, Ha_2 \ldots Ha_k$ are distinct cosets of H}

Then, $|G| = |Ha_1| + |Ha_2| + \cdots + |Ha_k|$

$\Rightarrow n = m + m + \cdots k$ times

$\Rightarrow n = mk$

$\Rightarrow k = \frac{n}{m} = \frac{O(G)}{O(H)}$

Hence, $O(H)$ is divisor of $O(G)$.

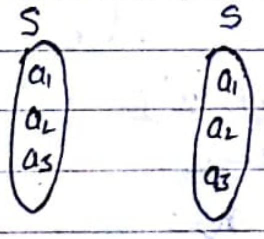
20/9/18
2.8.15 **

Permutation:

$S, |S| = n \quad f: S \longrightarrow S$

If f is one-one & onto, then f is defined as a permutation on set S of degree n.

If $S = \{a_1, a_2, a_3\}$



No. of one-one onto functions defined $= \lfloor n = \lfloor 3 = 6$

If $S = \{a_1, a_2, \ldots, a_n\}$, then

$$f = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ f(a_1) & f(a_2) & \cdots & f(a_n) \end{pmatrix}$$

then f is permutation on S of degree n.

Permutation is a one-one onto map from a set to itself.

No. of maps $= n(n-1)(n-2) \cdots 2 \cdot 1 = \lfloor n$

Identity Permutation:

$S, |S| = n, S = \{a_1, a_2, a_3 \ldots a_n\}$

I is said to be IP if each element is its self-image.

i.e. $I(a) = a \ \forall \ a \in S$

$$f = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ f(a_1) & f(a_2) & & f(a_n) \\ = a_1 & a_2 & & a_n \end{pmatrix}$$

Composition of Permutation:

S, $|S| = n$, $S = \{a_1, a_2, a_3, \ldots, a_n\}$

If f and g are two permutations of S, then ~~~~ composition of permutation . . . .

$$fog = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ (gof)a_1 & (gof)a_2 & \cdots & (gof)a_n \end{pmatrix}$$

eg. $S = \{1, 2, 3, 4\}$

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \quad g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} o \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ (gof)(1) & gof(2) & gof(3) & gof(4) \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

$\{ gof(1) = g(f(1)) = g(2) = 2 \}$

$1 \to 2, 2 \to 2 \Rightarrow 1 \to 2$

$2 \to 1, 1 \to 3 \Rightarrow 2 \to 3$

$3 \to 4, 4 \to 4 \Rightarrow 3 \to 4$

$4 \to 3, 3 \to 1 \Rightarrow 4 \to 1$

$$\begin{pmatrix} 1 & 3 & 4 & 2 & 5 \\ 2 & 5 & 3 & 1 & 4 \end{pmatrix} o \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 1 & 5 & 2 \end{pmatrix}$$

Inverse of Permutation:

If f is a permutation on S such that

$$f = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{pmatrix}$$ having one-one onto mapping

then

$$f^{-1} = \begin{pmatrix} b_1 & b_2 & \cdots & b_n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}$$

such that

$$\boxed{f \circ f^{-1} = I}$$

def (i part) ** Permutation Group:

Let S be a non-empty finite set such that cardinality of S is equal to n, then the set S_n, of all one-one onto mapping from set S to itself form a group which is called symmetric group or permutation group on n symbols.

i.e. if $S_n = \{ f \mid f : S \xrightarrow[\text{onto}]{\text{one-one}} S \}$, then S_n forms a group w.r.t composition of mapping, and if $f \in S$, then f is known as permutation of degree n.

Note:
① If $n \leq 2$ then S_n is abelian group
② If $n \geq 3$ then S_n is non-abelian group. If S has 3 or more elements then $f \circ g \neq g \circ f$.

Cyclic Permutation:

S, $|S| = n$; f is a perm^n of degree n $\quad S = \{a_1, a_2, \ldots a_n\}$

$$f = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_2 & a_3 & a_4 & & a_1 \end{pmatrix}$$ is cyclic permutation and written as

$$f = (a_1\ a_2\ a_3\ a_4 \ldots a_{n-1}\ a_n)$$

No. of symbols = length of permutation, $f = n$

eg $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$ or $(1\ 2\ 3\ 4)$

Disjoint Cycle:

No common elements.

eg. $S = \{1,2,3,4,5,6\}$ $\quad f = (1\ 2\ 5)$ $\quad g = (3\ 6)$

The elements absent in cycle are self-image i.e. $(3,4,6)$

$$\begin{pmatrix} 1 & 2 & 5 & 3 & 4 & 6 \\ 2 & 5 & 1 & 3 & 4 & 6 \end{pmatrix} = (1\ 2\ 5)(3)(4)(6)$$
$$= (1\ 2\ 5)$$

Remark:

Every perm^n can be written as product of disjoint cycles.

eg $S = \{1,2,3,4,5,6\}$ $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 4 & 6 \end{pmatrix}$

$= (1\ 2\ 3) \circ (4\ 5) \circ (6)$ or

$= (1\ 2\ 3) \circ (4\ 5)$

Transposition:

1. Every cycle of length 2 is called transposition.

eg $(a\ b)$, $(1\ 2)$, $(3\ 5)$ etc.

* $(1\ 2\ 3\ 4\ 5) = (1\ 2)(1\ 3)(1\ 4)(1\ 5)$

$(1\ 2\ 3) = (1\ 2) \circ (1\ 3)$ ...

$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} = \begin{matrix} 1 \\ 2 \end{matrix}$

$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} (= 1\ 2\ 3)$

2. Every cycle can be written as comp^n of transposition and every perm^n.

2. Every permutation can be expressed as product of transpositions

eg $A = \{1, 2, 3, 4, 5\}$ then ~~comp comp~~ compute

$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 2 & 4 & 5 \\ 4 & 5 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 3 & 4 \\ 3 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 1 & 5 & 2 \end{pmatrix}$

If a symbol is repeated below side, then it is wrong.

$= (1\ 4\ 5\ 2\ 3)$

Imp (ii) part

$I =$ even perm^n

Even and Odd Permutation:

$(1\ 2\ 3\ 4\ 5) = (1\ 2)(1\ 3)(1\ 4)(1\ 5)$ $n=4$

cycle of even length is an odd permutation, & vice versa. $\Rightarrow$ even permutation

$(1\ 2\ 3\ 4) = (1\ 2)(1\ 3)(1\ 4)$ $n=3 \Rightarrow$ odd permutation

Note: $S = \{1, 2, 3 \dots n\}$ S_n, $|S_n| = \lfloor n \Rightarrow \lfloor n/2$ even perm & $\lfloor n/2$ odd perm

eg. $S = \{1,2,3\}$ $S_3 = \{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$

9.2. **Homomorphism:**

$(G, *)$, (G', o)

$f : (G, *) \longrightarrow (G', o)$ then

f is said to be a homomorphism if

$\boxed{f(a * b) = f(a) \, o \, f(b)}$ $\forall \; a, b \in G$

eg. $(R, +)$, $(R^*, \cdot)$ $f : (R, +) \longrightarrow (R^*, \cdot)$ $R^* = R - \{0\}$

and f is defined as $f(x) = e^x$

Let $x, y \in R$

$f(x+y) = e^{x+y} = e^x \cdot e^y = f(x) \, f(y)$

$\therefore$ f is homomorphism from $(R, +)$ to $(R^*, \cdot)$

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4/10/2018

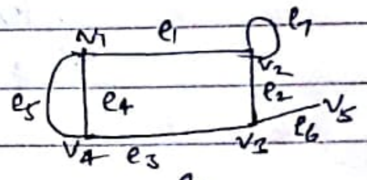
GRAPH THEORY

Graph: A graph $G = (V, E)$ consists of set of objects $V = \{v_1, v_2, v_3 \ldots\}$ whose elements are called vertices and another set $E = \{e_1, e_2, e_3 \ldots\}$ whose elements are called edges such that each edge e_k is identified with ordered pair (v_i, v_j) of vertices. v_i and v_j which are associated with edge e_k are called end vertices of edge e_k.

The common representation of a graph is by the means of a diagram.

$|V|$ = No. of vertices = 5

$|E|$ = No. of edges = 7

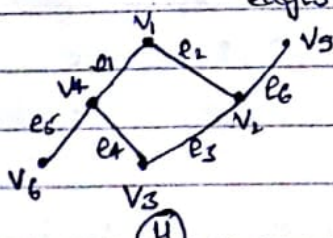


G_1

Self-Loop: An edge which has same vertex as its end vertices is called a self-loop. eg. e_7

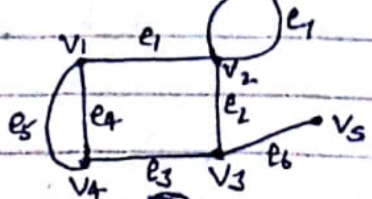
Parallel Edge: Edges which have same end vertices are called parallel edges. eg e_4 and e_5

Simple Graph: A graph which has neither self-loop nor parallel edges is called a simple graph.



(H)

Multi-Graph: A graph which has self-loop as well as parallel edges is called multi-graph.

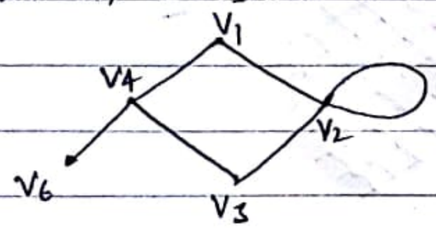


(G)

Incidence and Adjacence:

Let e_k be ~~the~~ an edge joining two vertices v_i and v_j in a graph $G = (V, E)$, then e_k is called incident on vertices v_i and v_j.
The vertices v_i and v_j are called adjacent if there exists an edge joining two vertices v_i and v_j.
eg. In graph H, e_1 is incident on v_1 and v_4. v_1 and v_2 are adjacent but v_1 and v_3 are not.

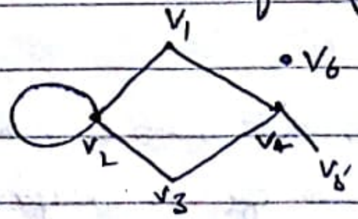
Degree of a Vertex: The degree of a vertex v in a graph is denoted by $d(v)$ and is equal to the no of edges incident on ~~it~~ with self-loop counted twice.
eg. In H, $d(v_5)=1$



$d(v_2)=4$
$d(v_4)=3$
$d(v_3)=2$
$d(v_6)=1$

Isolated vertex: A vertex v of a graph G is said to be isolated if degree of v is zero, i.e. $d(v)=0$.

eg.



$d(V_5=0) \Rightarrow V_5$ is isolated

Pendant Vertex: A vertex v of a graph G is said to be pendant if degree of v is 1, i.e. $d(v)=1$
eg. In above example, v_5 is pendant vertex as $d(v_5)=1$.

Null Graph: A graph $G = (V, E)$ is said to be null graph if set of vertices V is non-empty and set of edges E is empty.

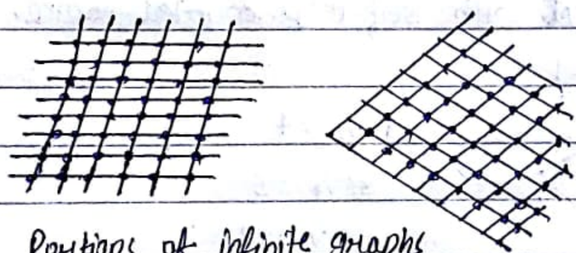
eg.

ie. A graph is said to be null graph if all of its vertices are isolated.

Finite and Infinite Graph: A graph in which no. of vertices as well as no. of edges are finite is called finite graph otherwise, it is called infinite graph.

eg. Finite graph: all graphs discussed till now.

Infinite graph:



Portions of infinite graphs

Theorem: The sum of degree of all vertices in a graph is equal to twice the no. of edges.

Proof:

Let $G = (V, E)$ be a graph, then

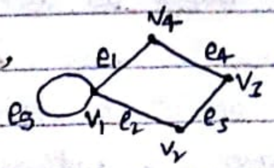
No. of edges in G is $|E|$.

Since, each edge incident on two vertices, therefore it contributes two to the degree of the graph.

Therefore the sum of degree of vertices in a graph is equal to twice the no. of edges in the graph, ie.

$$\boxed{\sum_{v \in V} d(v) = 2|E|}$$

eg.



No. of edges $= |E| = 5$

Sum of degree of vertices $= d(v_1) + d(v_2) + d(v_3) + d(v_4) = 4 + 2 + 2 + 2 = 10$

$= 2|E|$

Even and Odd Vertices: A vertex v is called even or odd according to its degree whether it is even or odd.

Theorem: The no. of odd vertices in a graph is always even.

Proof:

Let $G = (V, E)$ be a graph and V_o and V_e denote the set of odd vertices and even vertices respectively, then

$$V_o \subseteq V \text{ and } V_e \subseteq V$$

such that $V_o \cap V_e = \phi$ and $V_o \cup V_e = V$

thus,

$$\sum_{v \in V} d(v) = \sum_{v \in V_o} d(v) + \sum_{v \in V_e} d(v)$$

$$2|E| = \sum_{v \in V_o} d(v) + \cancel{\sum_{v \in V_e} d(v)}\ 2K \qquad \{2K = \text{even no.}\}$$

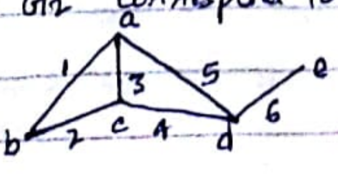
$$\Rightarrow \sum_{v \in V_o} d(v) = 2(|E| - K) = \text{even} \quad — ①$$

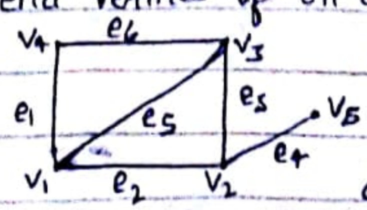
LHS of eqn ① is sum of odd no's but their sum is even only when no. of terms in the LHS of eqn ① is even,

$$\boxed{\text{i.e. } |V_o| = \text{even no.}}$$

Isomorphic Graph: Two graphs G_1 and G_2 are said to be isomorphic if there is one to one correspondance b/w their vertices and in b/w their edges such that ~~incidence relation~~ corresponding edge incidence relation is preserved, i.e., two graphs G_1 ~~and G_2~~ $= (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be isomorphic if there is one-to-one correspondance b/w V_1 and V_2 and in b/w E_1 and E_2 such that corresponding edges of G_1 and G_2 correspond to end vertices of G_1 and G_2.

eg.





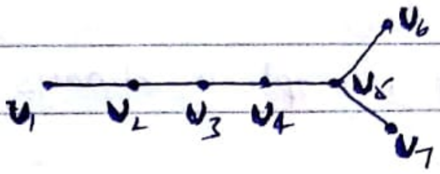
vertex correspondance:

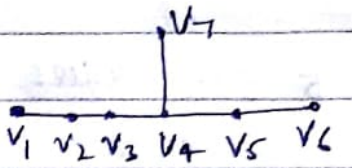
$a \leftrightarrow v_1$, $b \leftrightarrow v_4$, $c \leftrightarrow v_3$

$d \leftrightarrow v_2$, $e \leftrightarrow v_5$

Edge correspondance : $1 \leftrightarrow e_1$, $2 \leftrightarrow e_6$, $3 \leftrightarrow e_5$, $4 \leftrightarrow e_3$

$5 \leftrightarrow \cancel{e_3}\ e_2$, $6 \leftrightarrow e_4$





These graphs are not isomorphic as ~~second graph~~ vertex u_5 has two pendant vertices but in second graph vertex v_4 ~~has~~ (which is corresponding to u_5) has only one pendant vertex.

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Pigeonhole Principle:

If m pigeons are assigned to n pigeonholes where $m > n$, then at least one pigeonhole contains two or more pigeons.

Proof:

Let $H_1, H_2, H_3, \ldots H_n$ denote pigeonholes and $P_1, P_2, P_3, \ldots P_n$, $P_{n+1}, \ldots P_m$ denote pigeons.

Then we can consider assignment of pigeons as follows

Pigeon P_1 is assigned to pigeonhole H_1.

Pigeon P_2 is assigned to pigeonhole H_2

$\vdots$

Pigeon P_n is assigned to pigeonhole H_n

Since, $m > n$, hence $(m-n)$ pigeons are not assigned.

Thus, at least one pigeonhole must contain second pigeon.

$\Rightarrow$ At least one pigeonhole must contain two or more pigeons.

eg. If 8 people are chosen in any way from some room, then at least two of them will have to be born on the same day of the week.

8 people $= m =$ pigeons

$n = 7 =$ no. of days in a week

$\because$ $8 > 7$ ie. $m > n$

$\therefore$ By pigeonhole principle, at least two people are born on the same day of the week.

9 single shoes are selected from 8 pairs. Show that we have one pair of shoes.

m 8 pairs $= 16 =$ pigeons.

n $= 9$ single shoes $=$ pigeonholes

$\because$ $m > n$

$\therefore$ By PHP, we have a pair of shoes.

General Form of Pigeonhole Principle:

If n pigeonholes are occupied by $kn+1$ or more pigeons $(k \in N)$, then at least one pigeonhole is occupied by $k+1$ or more pigeons.

eg. Find the minimum no of students in a class such that at least 3 of them are born in the same month.

$n = 12$ = no. of months = pigeonholes

and given $k+1 = 3$

$k = 2$

Thus minimum no. of students $= nk+1 = 25$

Q. What is the min no. of students required in a class to be that at least 5 students will recieve the same grade if there are 4 possible grades A, B, C and D?

$k+1 = 5$

$n = 4$

$nk+1 = 17$

Extended Pigeonhole Principle:

If m pigeons are assigned to n pigeonholes, then one of the pigeonholes must contain at least $[m/n+1]$ pigeons or $[(m-1)/n]+1$ pigeons.

Q. Show that among 1 lakh people, there are 2 people who are born on the same time.

$m = 100000$ people = no. pigeons

$n = 24 \times 60 \times 60 = 86400$ s

Hence, by extended pigeonhole principle, at least

$$\left[\frac{m}{n}+1\right] = \left[\frac{100000}{86400}+1\right] = [1.157+1] = [2.157] = 2$$

people born on the same time.

$n=4$, at least 5 students

$$\left[\frac{m-1}{4}\right]+1=5$$

$$\left[\frac{m-1}{4}\right]=4$$

$$m-1=16$$
$$m=17$$

Show that among 1000 people, there are at least 84 people who were born in the same month.

$m=1000$, $n=12$

$$\left[\frac{1000}{12}+1\right]=[83.3+1]=84 \text{ people}$$

Recurrence Relation: and Recursive Formula:

A formula which expresses any term of a sequence in ~~its~~ as a function of its previous term is called recursive formula and the relation is called recurrence relation.

eg. 3, 8, 13, 18, 23 ....

$a_n = a_{n-1}+5$ → Recursive formula for the given sequence.

$a_1=3$, $n=2$ → Recurrence relation.

initial condition or boundary condition.

Remark:

Recurrence relation may be written in following forms:

$f(x+3h)+3f(x+2h)+7f(x+h)+f(x)=0$

↳ $a_{x+3}+3a_{x+2}+7a_{x+1}+a_x=0$

or

$y_{n+3}+3y_{n+2}+7y_{n+1}+y_n=0$

or

$u_{x+3}+3u_{x+2}+7u_{x+1}+u_x=0$

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Order of Recurrence Relation:

The order of recurrence relation is defined as difference of highest and lowest subscript of a_n or y_n or u_n

eg. Order of $a_{n+2} + a_{n-1} = n+2-(n-1) = 3$

Degree of Recurrence Relation:

The degree of recurrence relation is defined as highest power of a_n or y_n or u_n.

eg. Degree of $y_{n+3}^5 + y_{n+1} + y_n = 0$ is 5.

$u_{n+3} + u_{n+2} + u_n = f(n) \Rightarrow$ degree $= 1$

Homogeneous Recurrence Relation:

A recurrence relation is called homogeneous if it contains no term of that depend only on the variable n ($n = n, h, x$). If recurrence relation is not homogeneous ie. it contains a term which depends on the variable n, then it is called non-homogeneous recurrence relation.

eg. $a_{n+3} + 2a_{n+2} + a_n = b^n$ is non-homogeneous RR of degree 1 and order 3.

Linear Recurrence Relation with constant coefficient:

The general form of linear recurrence relation with constant coefficients

$$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} = f(n)$$

which is of the order K where $c_0, c_1, c_2 \dots c_k$ are constants and $f(n)$ is some function which satisfy given equation.

Note:

If $f(n) = 0$, then above equation is called linear homogeneous recurrence relation.

If $f(n) \neq 0$, then above equation is called linear non-homogeneous recurrence relation. eg. $a_{n+5} + 3a_{n+2} + a_n = 2^n$

Solution of Linear Homogeneous Recurrence Relation with const coefficient:

Consider $C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k} = 0$ — ①

Solⁿ of eqⁿ ① is of the form $A\alpha^k$ where α is characteristic root and A is constant which can be determined by initial condition.

Put $a_n = A\alpha^n$, $a_{n-1} = A\alpha^{n-1}$, $a_{n-2} = A\alpha^{n-2}$ ..... $a_{n-k} = A\alpha^{n-k}$

then, from ①

~~Co a~~ $C_0 A\alpha^n + C_1 A\alpha^{n-1} + C_2 A\alpha^{n-2} + \cdots + C_k A\alpha^{n-k} = 0$

$$A\alpha^{n-k}[C_0\alpha^k + C_1\alpha^{k-1} + C_2\alpha^{k-2} + \cdots + C_k] = 0$$

$$\Rightarrow \boxed{C_0\alpha^k + C_1\alpha^{k-1} + C_2\alpha^{k-2} + \cdots + C_k = 0} \quad - ②$$

This eqⁿ is called characteristic equation and roots of this equation are called characteristic roots.

Case I. If all the roots of eqⁿ ② are distinct and real.

If $\alpha_1, \alpha_2, \ldots, \alpha_k$ are distinct, then solⁿ is

$$a_n = A_1\alpha_1^n + A_2\alpha_2^n + \cdots + A_k\alpha_k^n$$

where $A_1, A_2, A_3, \ldots, A_k$ are constants.

Case II. If some roots are equal.

If $\alpha_1 = \alpha_2 = \cdots = \alpha_m = \alpha$ and $\alpha_{m+1}, \alpha_{m+2}, \ldots, \alpha_n$ are distinct, then solution is

$$a_n = (A_1 + A_2 n + A_3 n^2 + \cdots + A_m n^{m-1})\alpha + A_{m+1}\alpha_{m+1}^n + \cdots + A_k\alpha_k^n$$

Case III. If roots are complex.

If $\alpha + i\beta$, $\alpha - i\beta$ be the roots of equation $a_{n+2} + 2c a_{n+1} + d a_n = 0$ then solution is

$$a_n = \rho^n[A_1 \cos n\theta + A_2 \sin n\theta]$$

where $\rho = \sqrt{\alpha^2 + \beta^2}$ and $\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$

Q. Solve $a_{n+2} - 5a_{n+1} + 6a_n = 0$ (i) given that $a_2 = 3$ and $a_3 = 3$.

Put $a_n = A\alpha^n$, $a_{n+1} = A\alpha^{n+1}$, $a_{n+2} = A\alpha^{n+2}$ in eqⁿ (i)

$$A\alpha^n [\alpha^2 - 5\alpha + 6] = 0$$

$$\alpha^2 - 5\alpha + 6 = 0 \quad \text{as} \quad A\alpha^n \neq 0$$

$$\alpha = 2, 3$$

Solution is

$$a_n = A_1 2^n + A_2 3^n \quad \text{— (ii)}$$

Put $n = 2$ and $n = 3$ in eqⁿ (ii)

$$a_2 = 4A_1 + 9A_2 = 3$$

$$a_3 = 8A_1 + 27A_2 = 3$$

$$18A_2 - 27A_2 = 3$$

$$A_2 = \frac{-1}{3}$$

$$4A_1 + 9\left(\frac{-1}{3}\right) = 3$$

$$A_1 = \frac{3}{2}$$

$$a_n = \frac{3}{2} \cdot 2^n - \frac{1}{3} \cdot 3^n$$

$$a_n = 3 \cdot 2^{n-1} - 3^{n-1}$$

Q. Solve $a_n + 6a_{n-1} + 12a_{n-2} + 8a_{n-3} = 0$ — (i)

Put $a_n = A\alpha^n$, $a_{n-1} = A\alpha^{n-1}$, $a_{n-2} = A\alpha^{n-2}$, $a_{n-3} = A\alpha^{n-3}$ in eqⁿ (i)

$$A\alpha^n (\alpha^3 + 6\alpha^2 + 12\alpha + 8) = 0$$

$$\alpha^3 + 6\alpha^2 + 12\alpha + 8 = 0$$

$$\alpha^2(\alpha + 2) + 4\alpha(\alpha + 2) + 4(\alpha + 2) = 0$$

$$(\alpha^2 + 4\alpha + 4)(\alpha + 2) = 0$$

$$(\alpha + 2)^3 = 0$$

$$\alpha = -2, -2, -2$$

Solution is

$$a_n = [A_1 + A_2 n + A_3 n^2](-2)^n$$

Q. $a_n - 4a_{n-1} + 13a_{n-2} = 0$ — ①

Put $a_n = A\alpha^n$, $a_{n-1} = A\alpha^{n-1}$, $a_{n-2} = A\alpha^{n-2}$ in eqⁿ ①, then

$$A\alpha^n(\alpha^2 - 4\alpha + 13) = 0$$

$\alpha^2 - 4\alpha + 13 = 0$ is ch eqⁿ

$$\alpha = \frac{4 \pm \sqrt{16-52}}{2}$$

$$\alpha = 2 \pm 3i$$

Solution is

$$a_n = \rho^n[A_1 \cos n\theta + A_2 \sin n\theta]$$

$$\rho = \sqrt{2^2+3^2} = \sqrt{13}$$

$$\theta = \tan^{-1}\left(\frac{3}{2}\right)$$

$$a_n = (\sqrt{13})^n\left[A_1 \cos n \tan^{-1}\left(\frac{3}{2}\right) + A_2 \sin n \tan^{-1}\left(\frac{3}{2}\right)\right]$$

Solution of Linear Non-Homogeneous Recurrence Relation with constant coefficient:

Consider

$$C_0a_n + C_1a_{n-1} + C_2a_{n-2} + \dots + C_ka_{n-k} = f(n)$$

$$f(n) \neq 0$$

Then its solution contains two parts,
first is homogeneous solution, and
second is particular solution.

Then total solution

$$\boxed{a_n = \text{homog sol}^n + \text{particular sol}^n}$$

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~~Solution of Linear non-homogeneous recurrence relation:~~

$f(n)$	Trial Solution a_n
b^n	$a_n = Ab^n$
A polynomial of degree K in n	$a_n = A_0 + A_1 n + A_2 n + \dots + A_K n^K$
b^n (poly of degree K in n)	$a_n = b^n (A_0 + A_1 n + A_2 n + \dots A_K n^K)$
sin bn or cos bn	$a_n = A_1 \cos bn + A_2 \sin bn$

Type 3 - Q. Find PS of $a_{n+2} + 2a_{n+1} + a_n = 2^n$ — ①

The ps of eqn ① is of form

$$a_n = A2^n$$

$$a_{n+2} = A \cdot 2^{n+2}$$

$$a_{n+1} = A\,2^{n+1}$$

Putting these values in eqn ①

$$A2^{n+2} - 2A2^{n+1} + A2^n = 2^n$$

$$\boxed{A=1}$$

Hence, required PS is $\boxed{a_n = 2^n}$

$$\alpha^2 - 2\alpha + 1 = 0$$

$$\alpha = 1, 1$$

homg Soln $a_n = (A_1 + A_2 n)\, 1^n$

Total soln = hom soln + PS

$$a_n = (A_1 + A_2 n) 1^n + 2^n$$

$$a_0 = 2,\ a_1 = 1$$

Q. Find PS of $a_n - 2a_{n-1} = 7n$ — ①

PS of eqn ① is of the form

$$a_n = A_0 + A_1 n$$

$$a_{n-1} = A_0 + A_1(n-1)$$

Putting these values in eqn ①

$$A_0 + A_1 n - 2(A_0 + A_1(n-1)) = 7n$$

$$A_0 + A_1 n - 2A_0 - 2A_1(n-1) = 7n$$

$(A_0 - 2A_0 + 2A_1) + n(A_1 - 2A_1) = 7n$

Comparing like terms on both sides

$-A_0 + 2A_1 = 0$

$A_1 = -7$

$A_0 = -14$

Required PS $a_n = -14 - 7n$

Type II

Q. Find PS of $a_n + 5a_{n-1} + 6a_{n-2} = 3n^2 - 2n + 1$ — ①

PS is of the form

$a_n = A_1 + A_2 n + A_3 n^2$

$a_{n-1} = A_1 + A_2(n-1) + A_3(n-1)^2$

$a_{n-2} = A_1 + A_2(n-2) + A_3(n-2)^2$

Putting these values in eqn ①

$A_1 + A_2 n + A_3 n^2 + 5A_1 + 5A_2 n - 5A_2 + 5A_3 n^2 + 5A_3 - 10A_3 n + 6A_1 +$
$6A_2 n - 12A_2 + 6A_3 n^2 - 24A_3 n + 24A_3 = 3n^2 - 2n + 1$

$(12A_1 - 17A_2 + 29A_3) + n(12A_2 - 34A_3) + n^2(12A_3) = 3n^2 - 2n + 1$

Comparing like terms on both sides,

$12A_3 = 3 \Rightarrow A_3 = 1/4$

$12A_2 - 34A_3 = -2 \Rightarrow 12A_2 - 34/4\, A_3 = -2 \Rightarrow A_2 = 13/24$

$12A_1 - 17 \times \frac{13}{24} + \frac{29}{4} = 1$

$$A_1 = \frac{1}{12}\left(\frac{24 + 221 - 174}{24}\right) = \frac{71}{288}$$

Required PS is

$$a_n = \frac{71}{288} + \frac{13}{24}n + \frac{1}{4}n^2$$

Type III

Q. Find PS of $a_n + a_{n-1} = 3n \cdot 2^n$ — ①

PS is of the form

$a_n = (A_1 + A_2 n)\, 2^n$

$a_{n-1} = [A_1 + A_2(n-1)]\, 2^{n-1}$

$$A_1 2^n + A_2 n 2^n + A_1 2^{n-1} + A_2 n 2^{n-1} - A_2 2^{n-1} = 3n 2^n$$

$$n 2^n \left(A_2 + \frac{A_2}{2}\right) + A_1 2^n + A_1 2^{n-1} - A_2 2^{n-1} = 3n 2^n$$

$$n 2^n \left(\frac{3}{2} A_2\right) + 2^n \left(\frac{3A_1}{2} - \frac{A_2}{2}\right) = 3n 2^n$$

Comparing like terms on both sides,

$$\frac{3A_2}{2} = 3$$

$$A_2 = 2$$

$$\frac{3A_1}{2} - \frac{A_2}{2} = 0$$

$$A_1 = \frac{2}{3}$$

Hence, required PS is

$$a_n = \left(\frac{2}{3} + 2n\right) 2^n$$

Discrete Numeric Function:

A function whose domain is set of non-negative integers and range is real numbers is called discrete numeric function or simply numeric function i.e.

$$a : I^+ \cup \{0\} \longrightarrow R$$

then a is called discrete numeric function and value of function a at $0, 1, 2, 3 \cdots$ is denoted by $a_0, a_1, a_2, a_3, \ldots$, i.e.

$a(0) = a_0$, $a(1) = a_1$, $a(2) = a_2$, $a(3) = a_3 \ldots$

Generating Function:

If a be the discrete numeric function i.e.

$$a = a_0, a_1, a_2 \ldots$$

then, generating function of discrete numeric function a is denoted by $A(z)$ or $G(x)$ and defined as

$$A(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots$$

$$\boxed{A(z) = \sum_{n=0}^{\infty} a_n z^n}$$

Q. Find generating function of sequence $1, -1, 1, -1, 1, -1 \ldots$

$$A(z) = \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots$$

$$A(z) = 1 - z + z^2 - z^3 + z^4 \ldots$$

$$= (1+z)^{-1}$$

$$= \frac{1}{1+z}$$

$1, 1, 1, 1 \ldots \Rightarrow A(z) = \frac{1}{1-z}$