14

CHAPTER

Graph Theory - I

14.1. Introduction

Many situations that occur in computer Science, Physical Science, Communication Science, onomics and many other areas can be analysed by using techniques found in a relatively new area mathematics called graph theory. The graphs can be used to represent almost any problem involving crete arrangements of objects, where concern is not with the internal properties of these objects with relationship among them. In this chapter, we begin with some basic graph terminology and en discuss some important concepts in graph theory with many applications of graphs.

14.2. Basic Terminology

A graph G consists of two sets :

(*i*) a non-empty set V whose elements are called **vertices, nodes or points** of G.

The set $V(G)$ is called the **vertex set** of G.

(*ii*) a set E of edges such that each edge $e \in E$ associated with ordered or unordered pairs of elements of V. The set $E(G)$ is called the **edge set** of G.

$$E(G) = \{(u, v) : u, v \in V(G)\}, E(G) \subseteq V(G) \times V(G)$$

More formally, a graph G is an algebraic structure (V, E, ψ) in which the set V is called the set vertices, the set E is the set of edges and ψ is a mapping from the set E to the set of unordered or dered pairs of the elements of V.

Mostly, the graph G with vertices V and edges E is written as $G = (V, E)$ or $G(V, E)$, Committing e function ψ.

If an edge $e \in E$ is associated with an ordered pair (u,v) or an unordered pair (u,v), where u, $\in V$, then e is said to **connect** u and v and u and v are called end points of e. An edge is said to incident with the vertices it joins. Thus, the edge e that joins the vertices u and v is said to be cident on each of its end points u and v. Any pair of vertices that is connected by an edge in a aph is called **adjacent vertices.**

In a graph a vertex that is not adjacent to another vertex is called an **isolated vertex.**

A graph $G(V, E)$ is said to be **finite** if it has a finite number of vertices and finite number of ges; otherwise, it is a **infinite** graph. If G is a finite, $|V(G)|$ denotes the number of vertices in G d is called the **order** of G and $|E(G)|$ denotes the number of edges in G and is called the **size** of . We shall often refer to a graph of order n and size m an (n, m) graph.

Although graphs are frequently stored in a computer as list of vertices and edges, they are ctured as diagrams in the plane in a natural way. Vertex set of graph is represented as a set of points a plane and edge is represented by a line segment or an arc (not necessarily straight). The objects own in Fig. 14.1 are graphs.

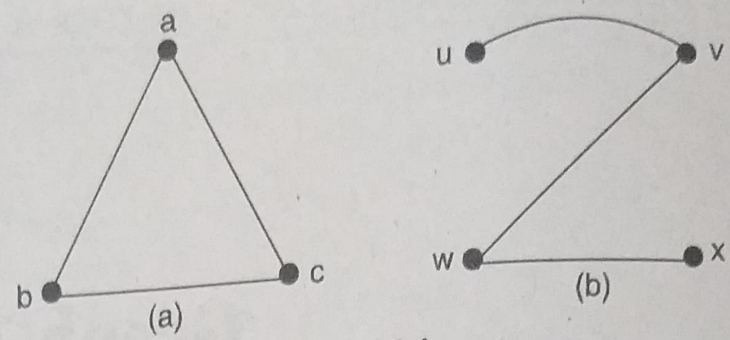


Fig. 14.1

It helps when discussing a graph to label each vertex, often with lower case letters as shown above. In Fig. 14.1 (a), $V = \{a, b, c,\}$ and $E = \{(a, b), (a, c), (b, c)\}$ the member of vertices and edges are $|v(G)| = 3$ and $|E(G)| = 3$ in this graph, the vertices a and b, a and c and b and c are adjacent vertices.

In Fig. 14.1 (b), $V = \{u, v, w, x\}$ and $E = \{(u, v), (v, w), (w, x)\}$

Here vertices u and v, v and w, w and x are adjacent, whereas u and w, u and x and v and x are non adjacent. The number of vertices and edges are $|v(G)| = 4$ and $|E(G)| = 3$.

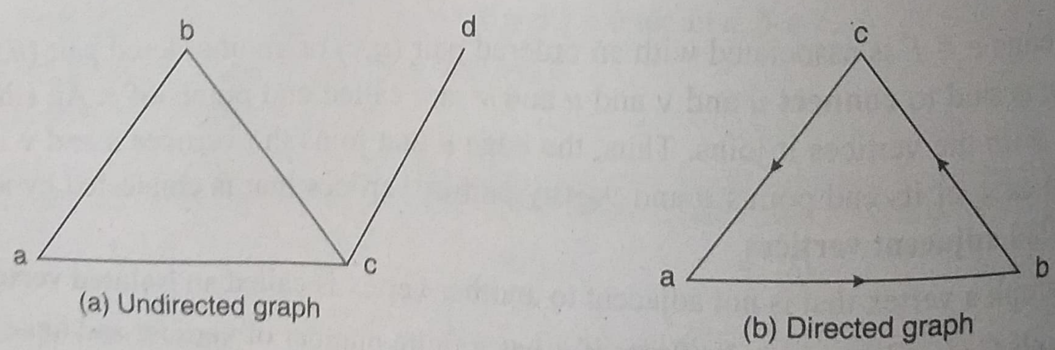
The definition of a graph contains no reference to the length or the shape and the positioning of the edge or arc joining any pair of nodes, nor does it prescribe any ordering of positions of the nodes. Therefore, for a given graph there is no unique diagram that represents the graph, and it can happen that two diagrams that look entirely different from one another may represent the same graph. It is to be noted that, in drawing a graph, it is immaterial whether the lines are drawn straight or curved, long or short, what is important is the incidence between edges and vertices are the same in both cases.

Undirected and Directed Graph

An **undirected** graph G consists of set V of vertices and a set E of edges such that each edge $e \in E$ is associated with an unordered pair of vertices.

Fig. 14.2 (a) is an example of an undirected graph we can refer to an edge joining the vertex pair i and j as either (i, j) or (j, i).

A **directed** graph (or **digraph**) G consists of a set V of vertices and a set E of edges such that $e \in E$ is associated with an ordered pair of vertices. In other words, if each edge of the graph G has a direction then the graph is called directed graph. In the diagram of directed graph, each edge $e = (u, v)$ is represented by an arrow or directed curve from initial point u of e to the terminal point v. Fig. 14.2 (b) is an example of a directed graph.



(a) Undirected graph

(b) Directed graph

Fig. 14.2

Suppose $e = (u,v)$ is a directed edge in a digraph, then

(i) u is called the **initial vertex** of e and v is the **terminal vertex** of e

(ii) e is said to be **incident from** u and to be **incident** to v.

(iii) u is adjacent to v, and v is **adjacent from** u

In specifying any edge of a digraph by its end-points, the edge is understood to be directed from the first vertex towards the second

Graphs, both directed and undirected, occur widely in all sorts of problems and before introducing more terminology we give examples of how graph arise in some familiar contexts.

Utilities Problem

An old problem concerns with three houses H_1, H_2, H_3, each to be connected to each of the three utilities – Water (W), Gas (G) and Electricity (E). Is it possible to connect each utility with each of the three houses without any two connections crossing each other ?

We can represent the connection of the three houses to the utilities by the graph of Fig. 14.3. Here we have diagram of six vertices, three of which represent the houses, denoted by H_1, H_2, H_3 the other three represent the utilities, denoted by G, W, E. An edge joins two vertices if and only if one vertex denotes a house and other vertex a utility. We shall see that it is not possible to draw this graph without edges crossing over. Thus the answer to this problem is no.

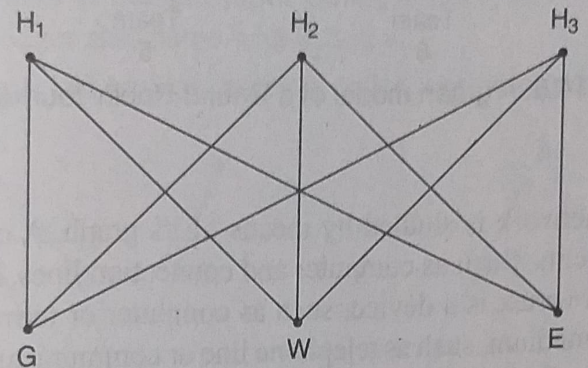


Fig. 14.3

Travelling Salesman Problem

Suppose a salesman territory includes several cities with highways connecting certain pair of these cities. He is required to visit each city personally exactly once. Graph theory can be used to solve this transportation system. The system can be represented by a graph G whose vertices correspond to the cities and such that two vertices are joined by an edge if and only if a highway connects the corresponding cities.

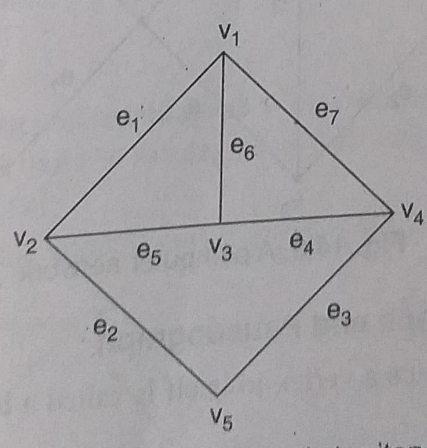


Fig. 14.4 Travelling salesman's territory graph

Starting at vertex v_1, salesman can visit each vertex by taking the edges e_1, e_2 e_3, e_4 and e_6 and back to v_1.

Round Robin Tournaments

A tournament where each team plays each other team exactly once is called a round robin tournament. Such tournaments can be modelled using directed graphs where each team is represented by a vertex. Note that (a, b) is an edge if team a beats team b. Such a directed graph model is presented in Fig. 14.5. Note that Team 2 is undefeated in this tournament, and Team 4, is winless.

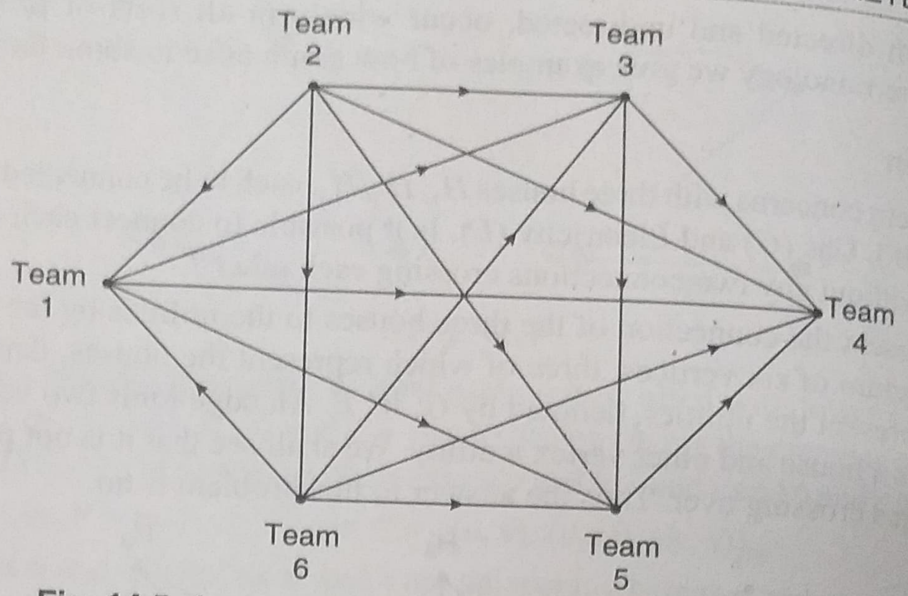


Fig. 14.5. A graph model of a Round Robin Tournament

Networking Problem

The topology of a network is studied by means of its graph. A computer network typically consists of variety of elements. Such as computer and connection lines. In a graph representation of a computer network, each vertex is a device, such as computer or terminal, and each edge or like denotes a communication medium, such as telephone line or communication cable. We are interested to identify those connection lines that must stay in service to avoid disconnecting the network given in Fig. 14.6. There is no single line whose disruption will disconnect the network, but the network will become disconnected if the two lines represented by the edges e_4 and e_5 are disconnected.

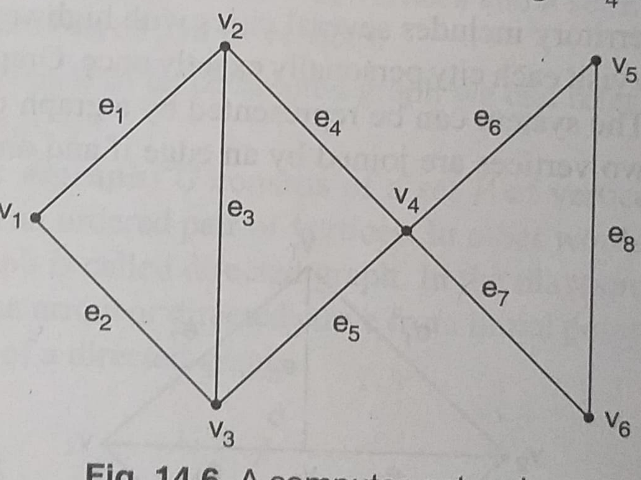


Fig. 14.6. A computer network

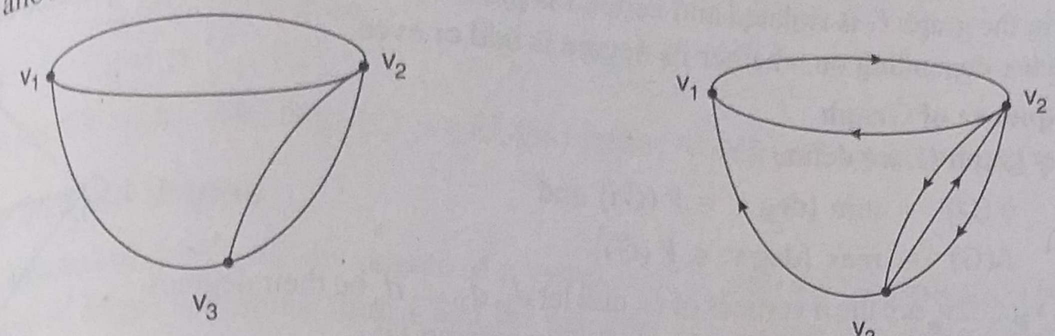
14.3. Simple Graph, Multigraph and Psuedograph

An edge of a graph that joins a vertex to itself is called a **loop** or **self loop** *i.e.*, a loop is an edge (v_i, v_j) where $v_i = v_j$.

In some directed as well as undirected graphs, we may have contain pair of vertices joined by more than one edges, such edges are called **multiple** or **parallel** edges. Two edges (v_i, v_j) and (v_f, v_r) are parallel edges if $v_i = v_f$ and $v_j = v_r$. Note that in case of directed edges, the two possible edges between a pair of vertices which are opposite in direction are considered distinct. So more than one directed edge in a particular direction in the case of a directed graph is considered parallel.

A graph which has neither loops nor multiple edges *i.e.*, where each edge connects two distinct vertices and no two edges connect the same pair of vertices is called a **simple graph**. Fig. 14.2 (*a*) and (*b*) represents simple undirected and directed graph because the graphs do not contain loops and the edges are all distinct.

Any graph which contains some multiple edges is called a **multigraph.** In a multigraph, no loops are allowed.

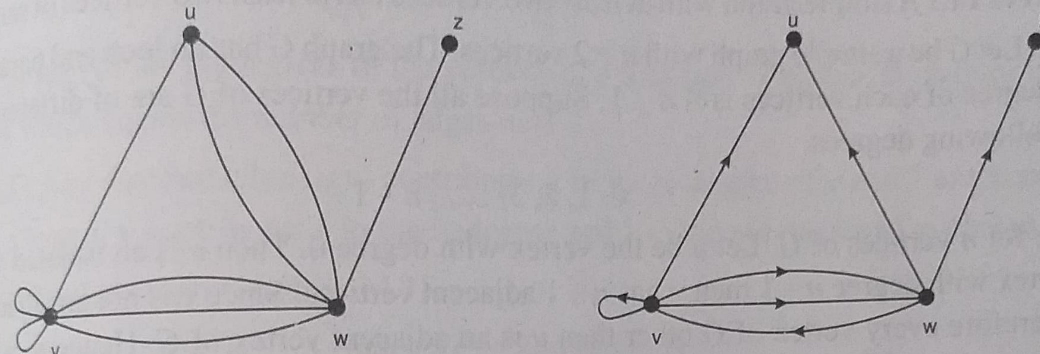


(a) Undirected multigraph (b) Directed multigraph

Fig. 14.7

In Fig. 14.7 (*a*) there are two parallel edges joining nodes v_1 and v_2 and v_2 and v_3. In Fig. 14.7. (*b*), there are two parallel edges associated with v_2 and v_3.

A graph with self-loops and multiple edges is called a **pseudograph**.



(a) Undirected pseudograph (b) Directed pseudograph

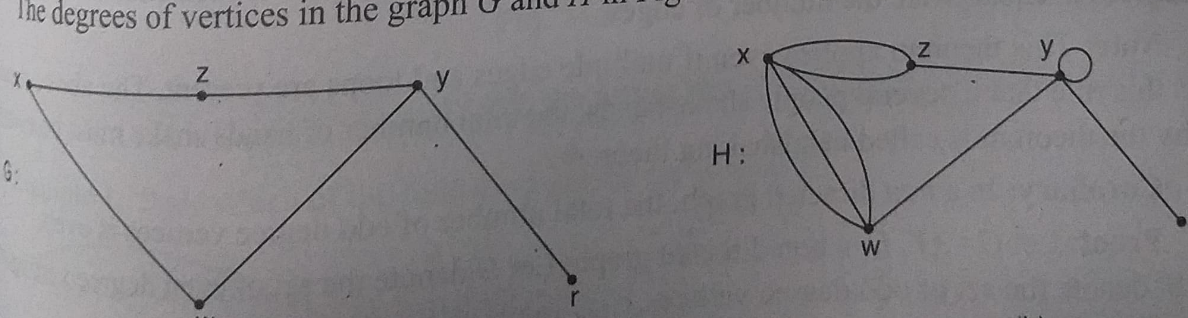
Fig. 14.8

It may be noted that there is some lack of standardisation of terminology in graph theory. Many words have almost obvious meaning, which are the same from book to book, but other terms are used differently by different authors.

14.4. Degree of a Vertex

The degree of a vertex of an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex. The degree of the vertex v in a graph G may be denoted by $deg_G(v)$.

The degrees of vertices in the graph *G* and *H* in Fig. 14.9 are given below.



(a) (b)

Fig. 14.9

In *G* as shown in Fig. 14.9 (*a*)

$deg_G(x) = 2 = deg_G(z) = deg_G(w), deg_G(y) = 3$, $deg_G(r) = 1$ and $deg_G(s) = 0$ and in H as shown in Fig. 14.9 (*b*),

$deg_H(x) = 5$, $deg_H(z) = 3$, $deg_H(y) = 5$, $deg_H(w) = 4$ and $deg_H(r) = 1$.

A vertex of degree 0 is called **isolated vertex**. A vertex is **pendant** if and only if it has a degree 1. Vertex s in the graph G is isolated and vertex r is pendant. A vertex of a graph is called **odd vertex** or **even vertex** depending on whether its degree is odd or even.

Degree Sequence of Graph

In any graph G, we define

$$\delta(G) = \min\{\deg v : \in V(G)\} \text{ and}$$

$$\Delta(G) = \max\{\deg v : \in V(G)\}$$

If $v_1, v_2, \ldots\ldots v_n$ are the n vertices of G, and let $d_1, d_2, \ldots, d_n$ be their degrees. If the sequence $(d_1, d_2, \ldots, d_n)$ is monotonically increasing *i.e.*

$$\delta(G) = d_1 \le d_2 \le \ldots\ldots \le d_n = \Delta(G).$$

then it is called the degree sequence of graph G.

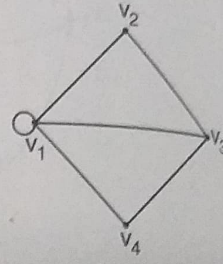


Fig. 14.10

For example, the degree sequence of the graph shown in Fig. 14.10. is (2,2,3,5) as $\deg(v_2) = \deg(v_4) = 2$, $\deg(v_3) = 3$ and $\deg(v_1) = 5$.

Theorem 14.1 A simple graph with at least two vertices has at least two vertices of same degree.

Proof. Let G be a simple graph with $n \ge 2$ vertices. The graph G has no loop and parallel edges. Hence the degree of each vertices is $\le n - 1$. Suppose all the vertices of G are of different degrees. Hence the following degrees

$$0, 1, 2, 3, \ldots\ldots, n-1$$

are possible for n vertices of G. Let u be the vertex with degree 0. Then u is an isolated vertex. Let v be the vertex with degree $n - 1$ then v has $n - 1$ adjacent vertices. Since v is not an adjacent vertex of itself, therefore every vertex of G other than u is an adjacent vertex of G. Hence u cannot be an isolated vertex, this contradiction proves that a simple graph contains two vertices of same degree.

The converse of the above theorem is not true.

Theorem (the **Handshaking theorem**) **14.2.** If $G = (V, E)$ be an undirected graph with e edges. Then

$$\sum_{\vartheta \in V} \deg_G(v) = 2e$$

i.e., the sum of degrees of the vertices in an undirected graph is even.

Proof: Since the degree of a vertex is the number of edges incident with that vertex, the sum of the degree counts the total number of times an edge is incident with a vertex. Since every edge is incident with exactly two vertices, each edge gets counted twice, once at each end. Thus the sum of the degrees is equal twice the number of edges.

Note: This theorem applies even if multiple edges and loops are present. The above theorem holds this rule that if several people shake hands, the total number of hands shake must be even that is why the theorem is called handshaking theorem.

Corollary: In a non directed graph, the total number of odd degree vertices is even.

Proof: Let $G = (V, E)$ a non directed graph. Let U denote the set of even degree vertices in G and W denote the set of odd degree vertices.

Then
$$\sum_{v_i \in V} \deg_G(v_i) = \sum_{v_i \in U} \deg_G(v_i) + \sum_{v_i \in W} \deg_G(v_i)$$

$$\Rightarrow \quad 2e - \sum_{v_i \in U} \deg_G(v_i) = \sum_{v_i \in W} \deg_G(v_i) \qquad \ldots(1)$$

Now $\sum_{v_i \in U} \deg_G(v_i)$ is even as the sum of degrees of even degree vertices is always even.

Therefore, from (1)

$$\sum_{v_i \in W} \deg_G(v_i) \text{ is even}$$

$\therefore$ Since for each $v_i \in W$, $\deg_G(v_i)$ is odd, the number of odd vertices in G must be even.

degree and out degree

In a directed graph G, the out degree of a vertex v of G, denoted by $\text{outdeg}_G(v)$ or $\deg_G^{+}(v)$, he number of edges beginning from v and the indegree of v, denoted by $\text{indeg}_G(v)$ or $\deg_G^{-}(v)$, he number of edges ending at v. The sum of the in degree and out degree of a vertex is called the al **degree** of the vertex. A vertex with zero in degree is called a **source** and a vertex with zero out ree is called a **sink**.

Theorem 14.3. If $G = (V, E)$ be a directed graph with e edges, then

$$\sum_{v \in V} \deg_G^{+}(v) = \sum_{v \in V} \deg_G^{-}(v) = e$$

i.e., the sum of the outdegrees of the vertices of a diagraph G equals the sum of in degrees of vertices which equals the number of edges in G.

Proof : Any directed edge (u,v) contributes 1 to the in degree of v and 1 to the out degree of urther, a loop at v contributes 1 to the in degree and 1 to the out degree of v. Hence the proof.

In the directed graph G in Fig. 14.11.

$\text{Indeg}_G(a) = 2$, $\text{Indeg}_G(b) = 1$, $\text{Indeg}_G(c) = 2$, $\text{Indeg}_G(d) = 3$.

$\text{Outdeg}_G(a) = 1$, $\text{Outdeg}_G(b) = 5$, $\text{Outdeg}_G(c) = 1$, $\text{Outdeg}_G(d) = 1$.

Note that, the sum of the in degrees and the sum of the out degrees each equal to 8, the number dges.

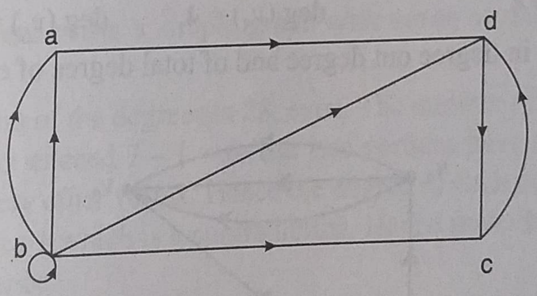
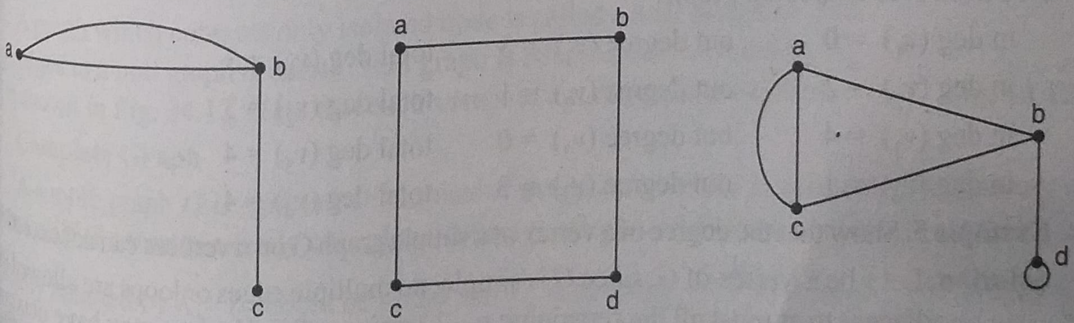


Fig. 14.11

SOLVED EXAMPLES

Example 1. State which of the following graphs are simple ?



(a) (b) (c)

Therefore,

$$\sum_{v_i \in W} \deg_G(v_i) \text{ is even}$$

$\therefore$ Since for each $v_i \in W$, $\deg_G(v_i)$ is odd, the number of odd vertices in G must be even.

In degree and out degree

In a directed graph G, the out degree of a vertex v of G, denoted by $\text{outdeg}_G(\text{v})$ or $\deg_G^{+}(\text{v})$, is the number of edges beginning from v and the indegree of v, denoted by $\text{indeg}_G(\text{v})$ or $\deg_G^{-}(\text{v})$, is the number of edges ending at v. The sum of the in degree and out degree of a vertex is called the total degree of the vertex. A vertex with zero in degree is called a **source** and a vertex with zero out degree is called a **sink**.

Theorem 14.3. If $G = (V, E)$ be a directed graph with e edges, then

$$\sum_{\text{v} \in \text{V}} \deg_G^{+}(\text{v}) = \sum_{\text{v} \in \text{V}} \deg_G^{-}(\text{v}) = e$$

i.e., the sum of the outdegrees of the vertices of a diagraph G equals the sum of in degrees of the vertices which equals the number of edges in G.

Proof : Any directed edge (u,v) contributes 1 to the in degree of v and 1 to the out degree of u. Further, a loop at v contributes 1 to the in degree and 1 to the out degree of v. Hence the proof.

In the directed graph G in Fig. 14.11.

$\text{Indeg}_G(a) = 2$, $\text{Indeg}_G(b) = 1$, $\text{Indeg}_G(c) = 2$, $\text{Indeg}_G(d) = 3$.

$\text{Outdeg}_G(a) = 1$, $\text{Outdeg}_G(b) = 5$, $\text{Outdeg}_G(c) = 1$, $\text{Outdeg}_G(d) = 1$.

Note that, the sum of the in degrees and the sum of the out degrees each equal to 8, the number of edges.

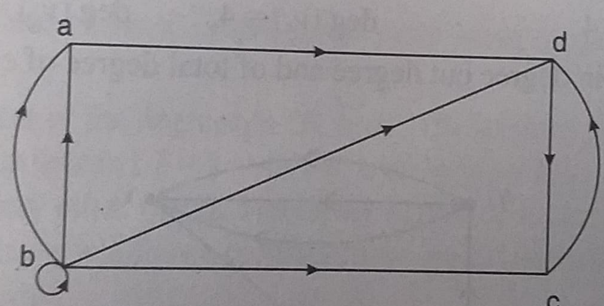


Fig. 14.11

SOLVED EXAMPLES

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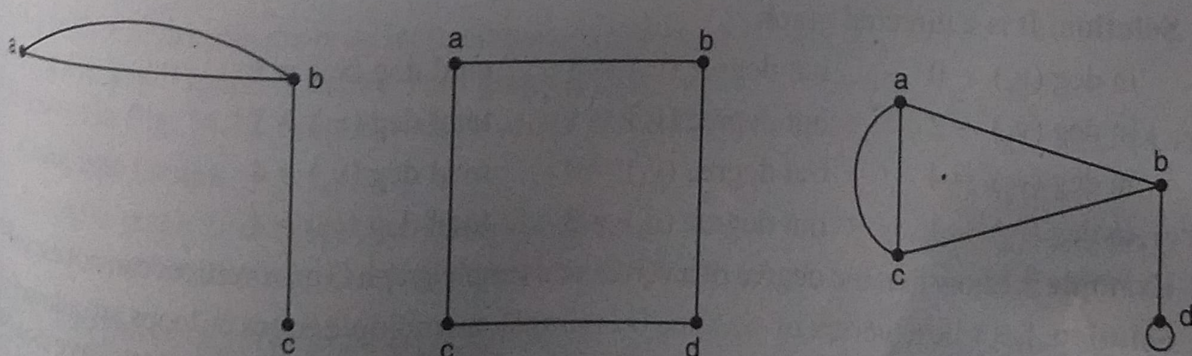


(a) (b) (c)

Solution. (*a*) The graph is not a simple graph, since it contains parallel edge between two vertices *a* and *b*.

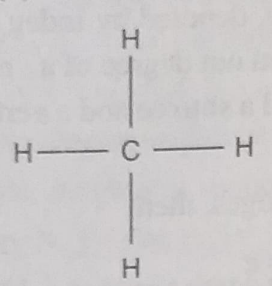
(*b*) The graph is a simple graph, it does not contain loop and parallel edge.

(*c*) The graph is not a simple graph since it contains parallel edge and a loop.

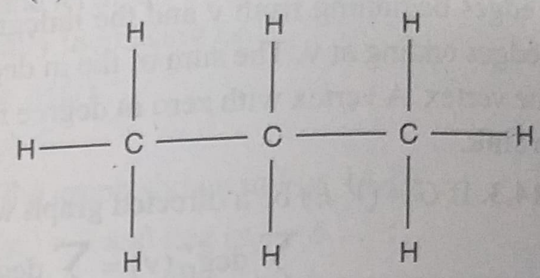
Example 2. Draw the graphs of the chemical molecules of

(*a*) methane (CH_4) (*b*) propane (C_3H_8)

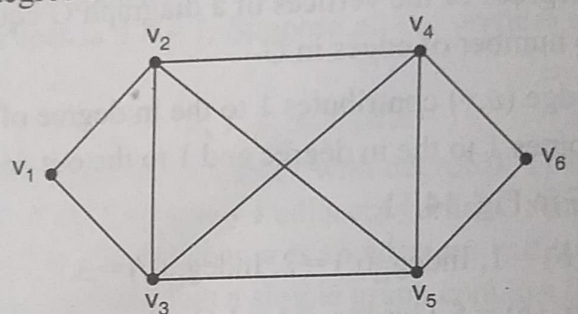
Solution. (*a*)



(*b*)



Example 3. Find the degree of each vertex of the following graph.

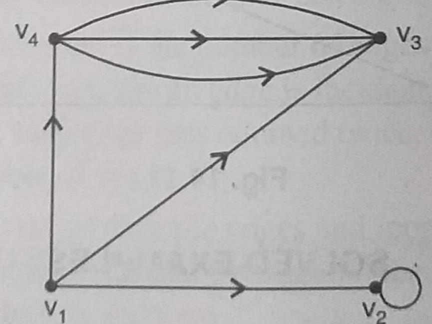


Solution. It is an undirected graph. Then

deg (v_1) = 2, deg (v_2) = 4, deg (v_3) = 4

deg (v_4) = 4 deg (v_5) = 4, deg (v_6) = 2

Example 4. Find the in degree out degree and of total degree of each vertex of the following graph.



Solution. It is a directed graph

in deg (v_1) = 0	out degree (v_1) = 3	total deg (v_1) = 3
in deg (v_2) = 2	out degree (v_2) = 1	total deg (v_2) = 3
in deg (v_3) = 4	out degree (v_3) = 0	total deg (v_3) = 4
in deg (v_4) = 1	out degree (v_4) = 3	total deg (v_4) = 4

Example 5. Show that the degree of a vertex of a simple graph G on *n* vertices can not exceed *n*–1.

Solution. Let *v* be a vertex of *G*, since *G* is simple, no multiple edges or loops are allowed in *G*. Thus *v* can be adjacent to at most all the remaining $n-1$ vertices of *G*. Hence *v* may have maximum degree $n-1$ in *G*. then $0 \le \deg_G(v) \le n-1$ for all $v \in V(G)$.

Example 6. Show that the maximum number of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$

Solution. By the handshaking theorem

$$\sum_{i=1}^{n} d(v_i) = 2e,$$

where e is the number of edges with n vertices in the graph G.

$$\Rightarrow \quad d(v_1) + d(v_2) + \ldots d(v_n) = 2e \qquad \ldots(1)$$

Since the maximum degree of each vertex in the graph G can be $(n-1)$. Therefore, equation (1) reduces to

$$(n-1) + (n-1) + \ldots\ldots \text{ to } n \text{ terms} = 2e$$

$$\Rightarrow \quad n(n-1) = 2e \Rightarrow e = \frac{n(n-1)}{2}$$

Hence the maximum number of edges in any simple graph with n vertices is $\frac{n(n-1)}{2}$.

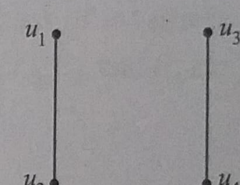
Example 7. Is there a simple graph corresponding to the following degree sequences ?

(*i*) (1, 1, 2, 3) (*ii*) (2, 2, 4, 6) (*iii*) (1, 1, 1, 1)

Solution. (*i*) Since the sum of degrees of vertices is odd, there exist no graph corresponding to this degree sequence.

(*ii*) Number of vertices in the graph sequence is four and the maximum degree of a vertex is 6, which is not possible as in a simple graph the maximum degree cannot exceed one less than the number of vertices.

(*iii*) The sum of the degrees of all vertices is 4, even. The number of odd vertices is 4, even. Hence a simple disconnected graph is possible which has 4 vertices of degree 1 each. The number of edges is 4/2 = 2



Example 8. Does there exists a simple graph with seven vertices having degree sequence (1,3,3,4,5,6,6)?

Solution. Here the sum of the degrees in 28, even. The number of odd vertices is 4, even. The maximum degree 6 does not exceed 7 – 1 = 6. But two vertices have degree 6, each of these two vertices is adjacent with every other vertex. Hence the degree of each vertex is at least 2, so that the graph has no vertex of degree 1 which is a contradiction. Hence there does not exist a simple graph with the given degree sequence.

14.5. Types of Graphs

Some important types of graphs are introduced here. These graphs are often used as examples and arise in many applications.

Null Graph

A graph which contains only isolated node is called a **null graph** *i.e.*, the set of edges in a null graph is empty. Null graph is denoted on n vertices by N_n : N_4 is shown in Fig. 14.12. Note that each vertex of a null graph is isolated.

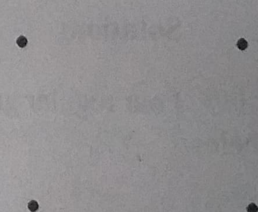


Fig. 14.12

Complete Graph

A simple graph G is said to be complete if every vertex in G is connected with every other vertex *i.e.*, if G contains exactly one edge between each pair of distinct vertices. A complete graph is usually denoted by K_n. It should be noted that K_n has exactly $\frac{n(n-1)}{2}$ edges. The graphs K_n for n=1,2,3,4,5,6 are shown in Fig. 14.13.

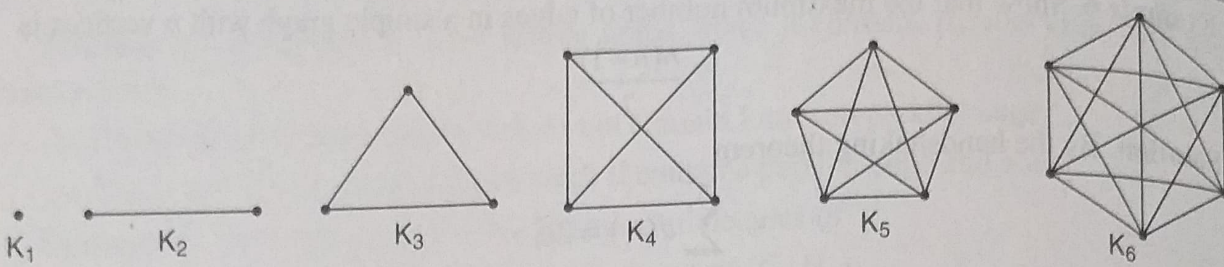
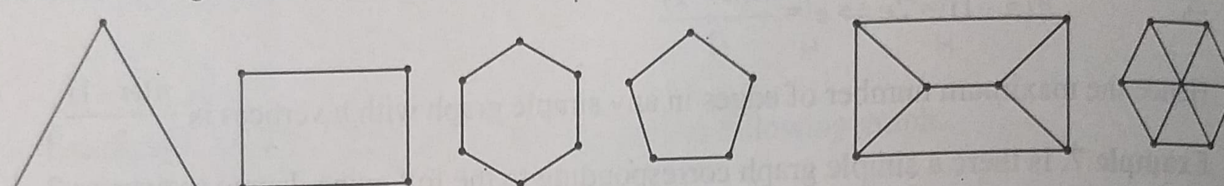


Fig. 14.13

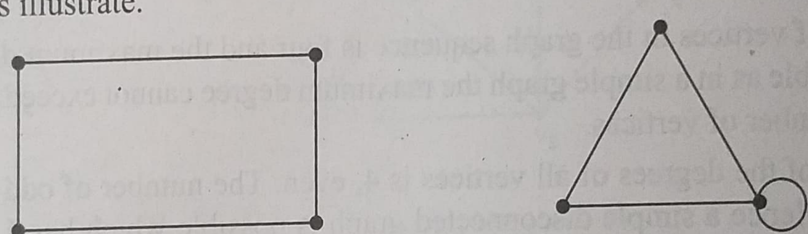
Regular Graph

A graph in which all vertices are of equal degree is called a regular graph. If the degree of each vertex is r, then the graph is called a **regular graph of degree r**. Note that every null graph is regular of degree zero, and that the complete graph K_n is a regular of degree $n-1$. Also, note that if G has n vertices and is regular of degree r, then G has $(1/2)\ r\ n$ edges.



Regular graph of degree 2 Regular graph of degree 3

Note that complete graph need not be regular and a regular graph need not be complete as the following examples illustrate.



2 regular but not complete Complete but not regular

Example 9. What is the size of an r-regular (p, q)-graph ?

Solution: Since G is a r-regular graph, by the definition of regularity of G, we have, $\deg_G(v_i) = r$, for all $v_i \in V(G)$.

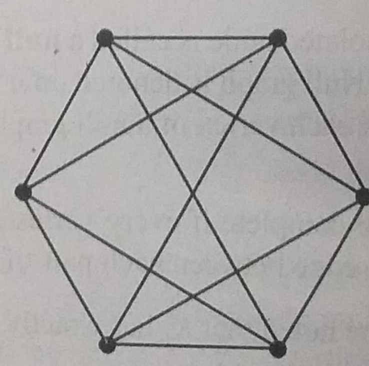
By the hand shaking theorem, $2q = \sum \deg_G(v_i)$

$$2q = \sum r = p \times r \Rightarrow q = \frac{p \times r}{2}$$

Example 10. Does there exists a 4-regular graph on 6 vertices ? If so construct a graph.

Solution: $$q = \frac{p \times r}{2} = \frac{6 \times 4}{2} = 12.$$

Four regular graph on 6 vertices is possible and it contains 12 edges. One of the graph is shown below.



Platonic Graph

Of special interest among the regular graphs are the so-called Platonic graph, the graphs formed by the vertices and edges of the five regular (Platonic) solids –The tetrahedron, octahedron cube, dodecahedron, and icosahedron. The graphs are shown in Fig. 14.14.

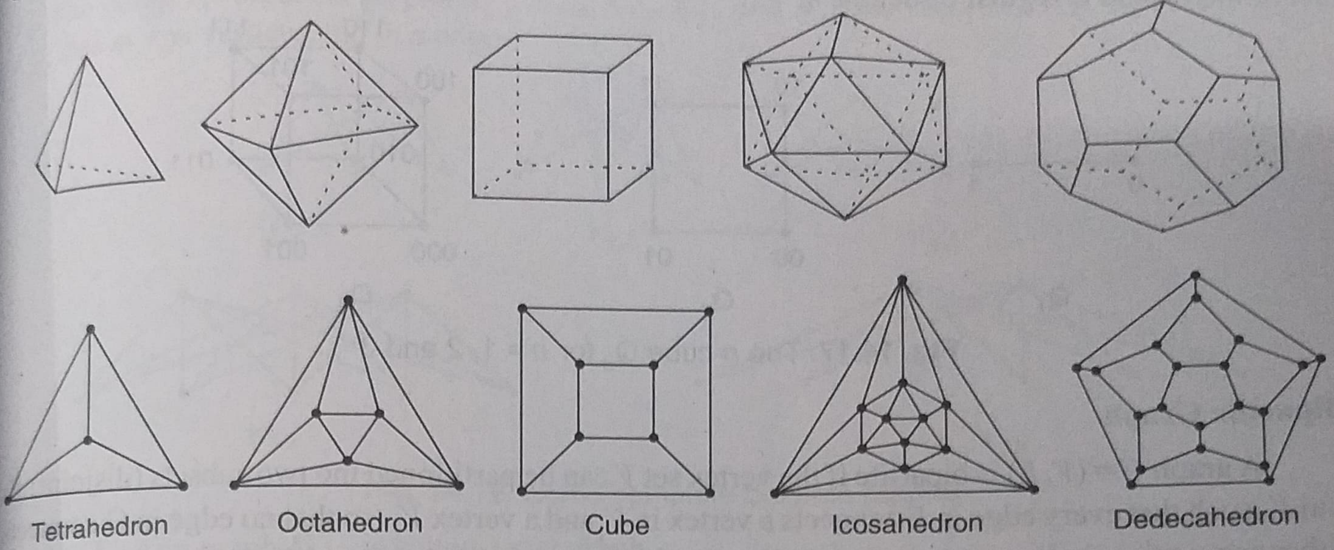


Fig. 14.14.

Cycles

The cycle graph C_n ($n \geq 3$), of length n is a connected graph which consists of n vertices $v_1, v_2 ..... v_n$ and n edges $\{v_1, v_2\}, \{v_2, v_3\}, ... \{v_{n-1}, v_n\}$, and $\{v_n, v_1\}$. The cycles C_3, C_4, C_5, and C_6 are shown in Fig. 14.15.

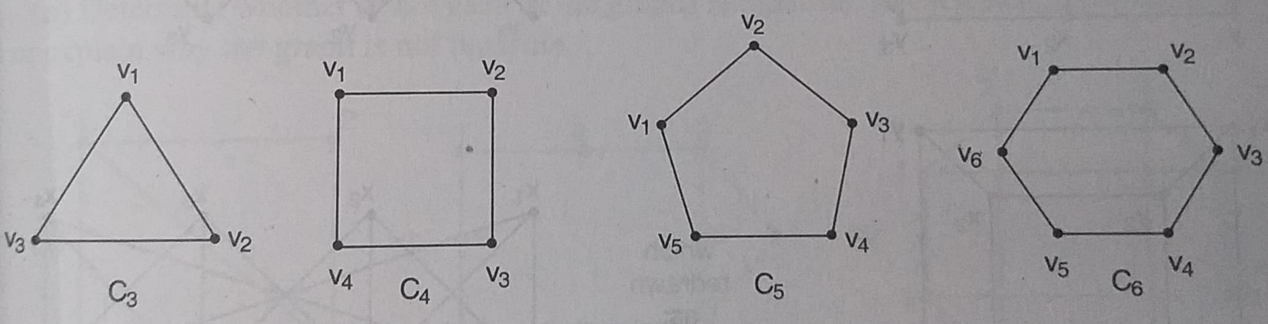


Fig. 14.15. Cycles C_3, C_4, C_5 and C_6

C_n is a regular graph of degree 2.

Wheels

The wheel graph W_n ($n > 3$) is obtained from C_{n-1} by adding a vertex v inside C_{n-1} and connecting it to every vertex in C_{n-1}. The wheels W_4, W_5, W_6 and W_7 are displayed in Fig. 14.16.

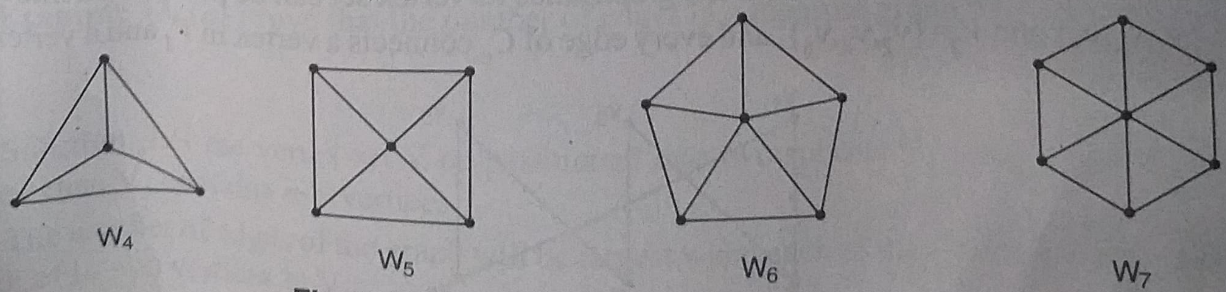


Fig. 14.16. The Wheels W_4, W_5, W_6 and W_7

W_n is a regular graph for $n = 4$. It has n vertices and $2n - 2$ edges.

N-cube

The N-cube denoted by Q_n, is the graph that has vertices representing the 2^n bit strings of length n. Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position. The graphs Q_1, Q_2, Q_3 are displayed in Fig. 14.17. Thus Q_n has 2^n vertices and $n.2^{n-1}$ edges, and is regular of degree n.

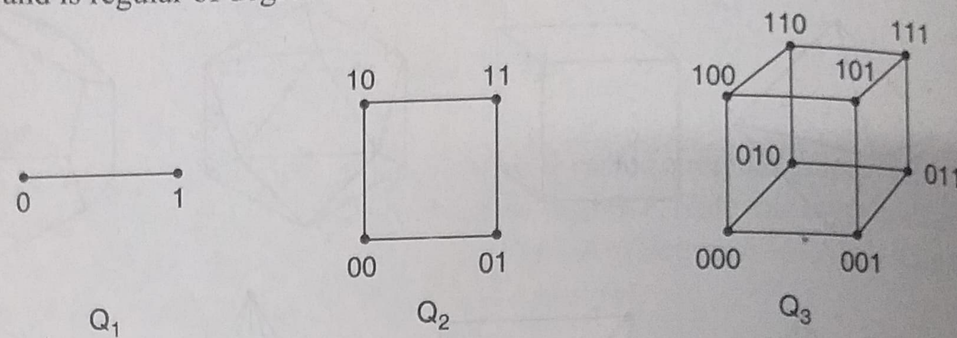


Fig. 14.17. The n-cube Q_n for $n = 1$, 2 and 3

Bipartite Graph

A graph $G = (V, E)$ is bipartite if the vertex set V can be partitioned ino two subsets (disjoint) V_1 and V_2 such that every edge in E connects a vertex in V_1 and a vertex V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2). (V_1, V_2) is called a bipartition of G: Obviously, a bipartite graph can have no loop as loop connects the same vertex which is not permitted in bipartite graph.

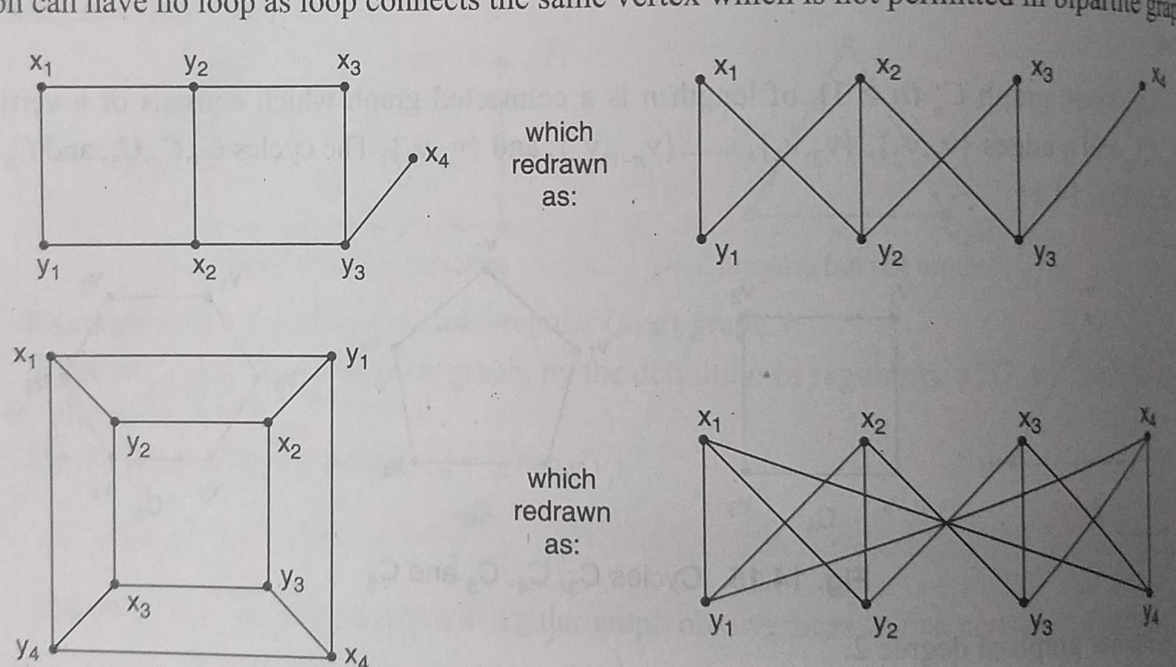
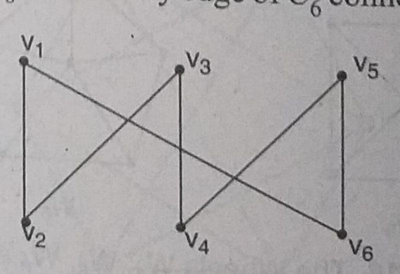


Fig. 14.18. Some bipartite graphs

Example 11. Show that C_6 is a bipartite graph.

Solution. C_6 in Fig. 14.15 is a bipartite graph since its vertex set can be partitioned into the two sets $V_1 = \{v_1, v_3, v_5\}$ and $V_2 = \{v_2, v_4, v_6\}$, and every edge of C_6 connects a vertex in V_1 and a vertex in V_2.



Note : Q_n is a bipartite graph

Complete bipartite graph

A bipartite graph $G = (V, E)$ is said to be complete bipartite if each vertex of V_1 os connected to each vertex of V_2 where V_1 and V_z are the two distinct partitions of the vertex set V. Complete bipartite graph G is denoted by $K_{m,n}$, where m and n are the number of vertices m vertex sets V_1 and V_2. The complete bipartite graphs $K_{2,3}$, $K_{2,4}$, $K_{3,3}$, $K_{3,5}$ and $K_{2,6}$ are shown in Fig. 14.19. Note that $K_{m,n}$ has $m + m$ vertices and m,n edges.

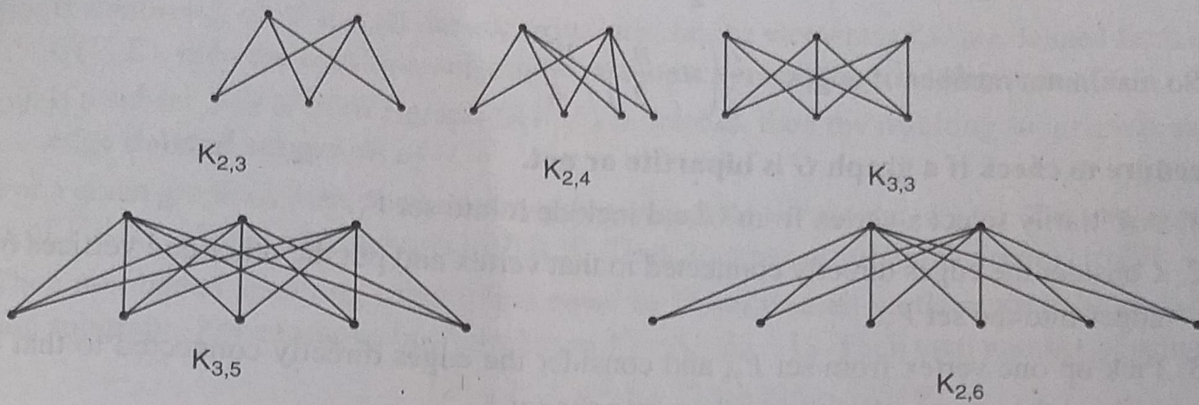


Fig. 14.19. Some complete bipartite graphs

Note : 1. Any graph $K_{1, n}$ is called a star graph.

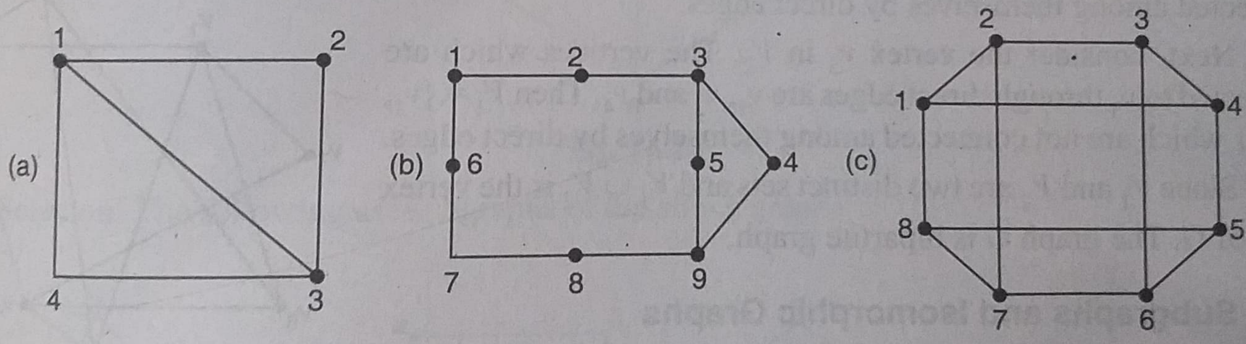
2. A complete bipartite graph $K_{m,n}$ is not a regular if $m \neq n$.

3. K_5 and $K_{3,3}$ are called Kuratowski graphs.

Example 12. (*i*) Prove that a graph which contains a triangle can not be bipartite.

Solution: At least two of the three vertices must lie in one of the bipartite sets since these two are joined by an edge, the graph can not be bipartite.

(*ii*) Determine whether or not each of the graphs is bipartite. In each case, give the bipartition sets or explain why the graph is not bipartite.



Solution (*a*) The graph is not bipartite because it contains triangles (in fact two triangles)

(*b*) This is a bipartite graph and the bipartite sets are $V_1 = \{1,3,7,9\}$ and $V_2 = \{2,4,5,6,8\}$ such that each edge is incident on a vertex in V_1 and a vertex in V_2.

(*c*) This is bipartite and the bipartite sets are $\{1,3,5,7\}$ and $\{2,4,6,8\}$

Example 13. (*a*) Prove that the number of edges in a bipartite graph with n vertices is at most $\frac{n^2}{4}$.

Solution: Let the vertex set V be partitioned into two subsets V_1 and V_2. Let V_1 contains x vertices, then V_2 contains n–x vertices.

The number of edges of the graph will be largest when each of the x vertices in V_1 is connected to each of $(n-x)$ vertices in V_2.

Hence, the largest number of edges

$$f(x) = x(n-x) \qquad \text{which is a function of } x$$

$\therefore \qquad f'(x) = n - 2x \quad , \quad f''(x) = -2$

Now $\qquad f'(x) = 0 \quad \Rightarrow \quad n - 2x = 0$ *i.e.* $x = \dfrac{n}{2}$

and $\qquad f''(x) < 0.$

Hence, $f(x)$ is maximum when $x = \dfrac{n}{2}$.

So maximum number of edges $= \dfrac{n}{2}\left(n - \dfrac{n}{2}\right) = \dfrac{n^2}{4}$

A procedure to check if a graph *G* is bipartite or not.

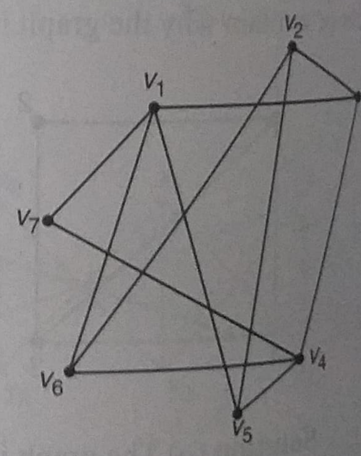
1. Arbitrarily select a vertex from G and include it into set V_1.
2. Consider the edges directly connected to that vertex and put the other end vertices of these edges into the set V_2.
3. Pick up one vertex from set V_2, and consider the edges directly connected to that vertex, and put the other end of these edges into the set V_1.
4. At each step, check if there is any edge among the vertices of set V_1 and set V_2.

   If so, the given graph is not bipartite graph, and then return.

   Elese continue 2 and 3 alternately until all the vertices are included in the union of sets V_1 and V_2.
5. If two computed sets are distinct, then the graph is bipartite.

Example 13. (*b*) Show that the following graph G is bipartite.

Solution: Select a vertex v_1. The vertices joined to v_1 through direct edges are v_3, v_5, v_6 and v_7. So, $V_1 = \{v_1\}$ and $V_2 = \{v_3, v_5, v_6, v_7\}$ and vertices in V_2 are not connected among themselves by direct edges.



Next, consider the vertex v_5 in V_2. The vertices which are connected to v_5 through direct edges are v_1, v_2 and v_4. Then $V_1 = \{v_1, v_2, v_4\}$ which are not connected among themselves by direct edges.

Since V_1 and V_2 are two distinct sets and $V_1 \cup V_2$ is the vertex set V of G. The graph G is bipartite graph.

14.6. Subgraphs and Isomorphic Graphs

Subgraph

Some graph applications are concerned with only a part of entire graph. Such a graph is called a subgraph of the original graph.

Consider a graph $G = (V, E)$. A graph $H = (V', E')$ is called a subgraph of G if the vertices and edges of H are contained in the vertices and edges of G, that is, if $V' \subseteq V$ and $E' \subseteq E$. So, if H is a subgraph of G, then

(*i*) All the vertices of H are in G.

(*ii*) All the edges of H are in G and

(*iii*) Each edge of H has the same end points in H as in G.

Any subgraph of a graph G can be obtained by removing certain vertices and edges from G. It is understood that the removal of an edge leaves its points in place, whereas the removal of a vertex necessitates the removal of any edges with that vertex as an end point.

Different Types of Subgraphs

(i) If $V' \subset V$ and $E' \subset E$, then H is called a **proper subgraph** of G.

(ii) A subgraph H of G, is called a **spanning subgraph** of G if and only if $V(H) = V(G)$. *i.e.*, H contains all vertices G but not necessary all edges of G.

(iii) A subgraph H (V', E') of $G(V, E)$ is called the **induced subgraph** of G if $V' \subseteq V$ and its edge set E' contains all those edges in G that joins the vertices of set V' in G.

(iv) If a subset U of V and all the edges incident on the elements of U are deleted from a graph $G(V, E)$, then the resulting subgraph is called a **vertex deleted subgraph** of $G(V, E)$.

(v) If a subset S of E from a graph $G(V, E)$ is deleted, then the resulting subgraph is called an **edge deleted subgraph** *of* G.

For a given graph G, there can be many subgraphs. Let $|V| = m$ and $|E| = n$. The total non-empty subsets of V is $2^m - 1$ and total subsets of E is 2^n. Thus, number of subgraph is equal to $(2^m - 1) \times 2^n$.

Then number of spanning subgraph is equal to 2^n because all vertices are to be included in a spanning subgraph. For example, in Q_3 we have $|V| = 8, |E| = 12$. Then total number of subgraphs is

$$(2^8 - 1) \times 2^{12} = 127 \times 4096 = 520192$$

and the total number of spanning subgraphs is $2^{12} = 4096$.

Example 14. Consider the graph shown in Fig. 14.20. Find the different subgraphs of this graph.

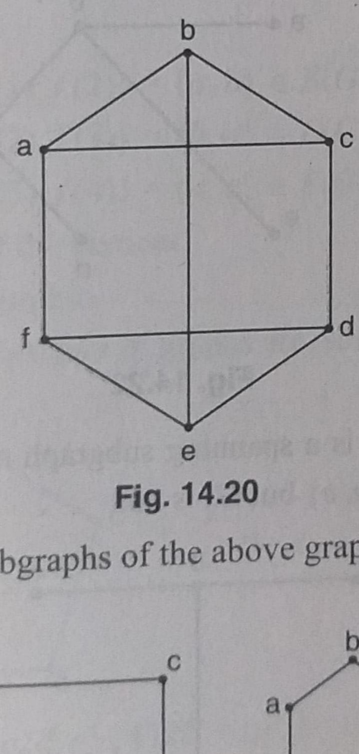
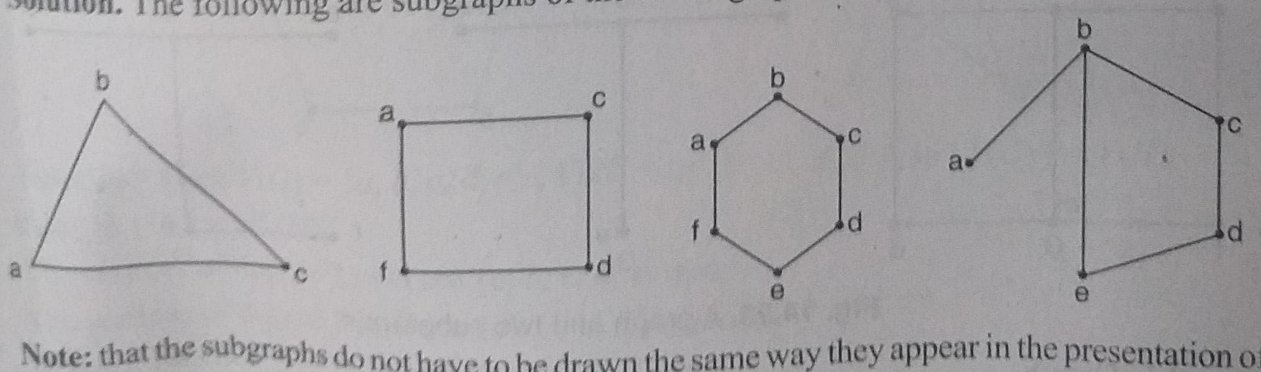


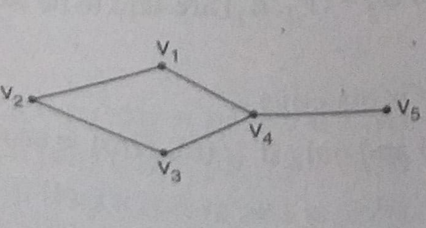
Fig. 14.20

Solution. The following are subgraphs of the above graph.

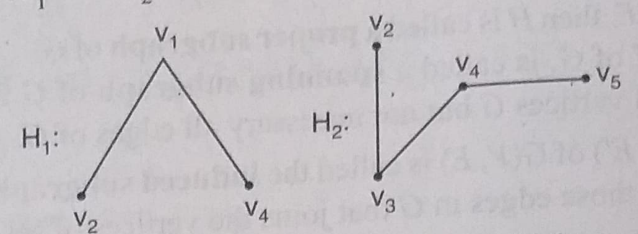


Note: that the subgraphs do not have to be drawn the same way they appear in the presentation of G.

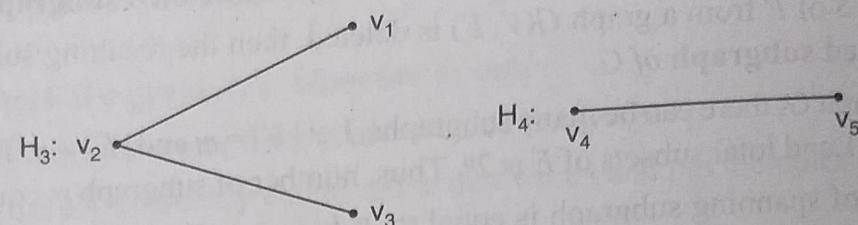
Example 15. Construct two edge deleted subgraphs and two vertex deleted subgraphs of a graph G shown below.



Solution: The graphs H_1 and H_2 are two edge deleted subgraphs of G.



The graphs H_3 and H_4 are vertex deleted subgraphs of G which are also edge deleted subgraphs of G.



Example 16. In Fig. 14.22, G_1 is a subgraph of G but not an induced subgraph where as G_2 is an induced subgraph and induced by {a, b, e, d}.

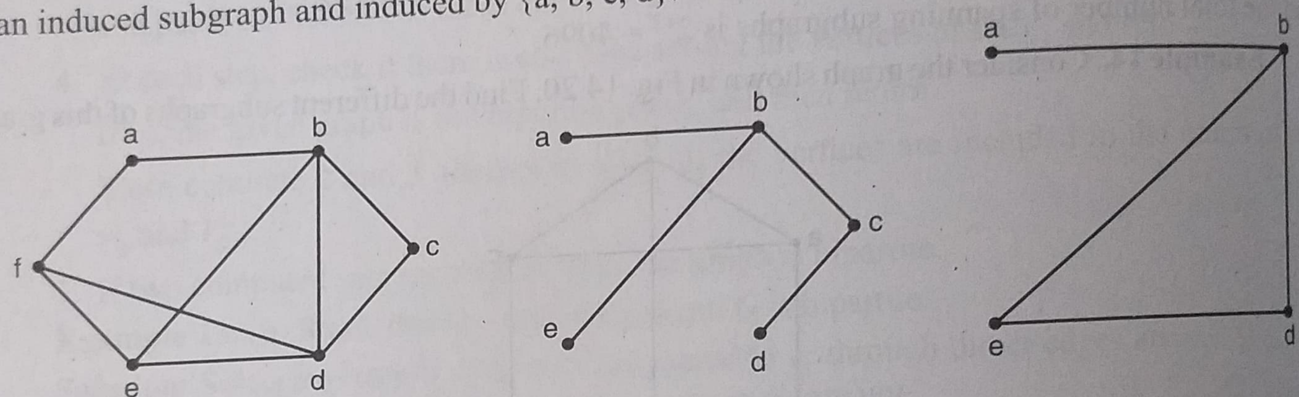


Fig. 14.22

Example 17. In Fig. 14.23 G_2 is a spanning subgraph of G but G_1 is not, whereas G_1 is an induced subgraph induced by $\{a, b, c, e\}$ but G_2 is not.

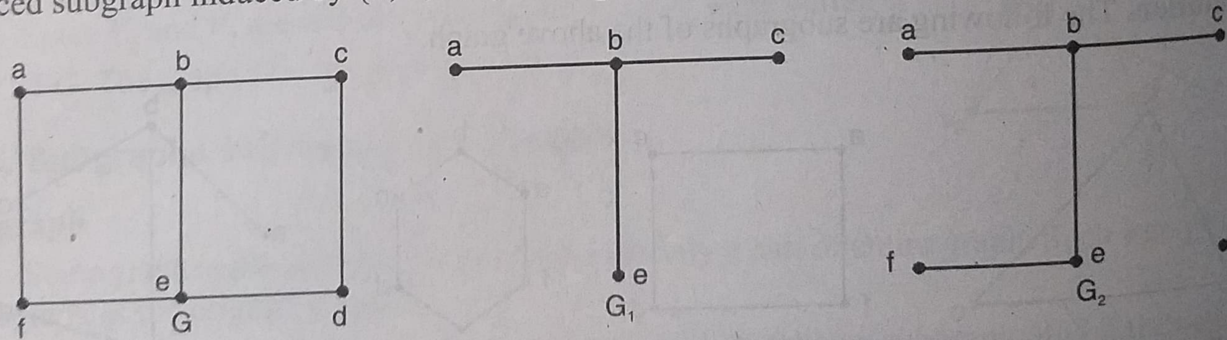


Fig. 14.23. A graph and two subgraphs

Isomorphic Graph

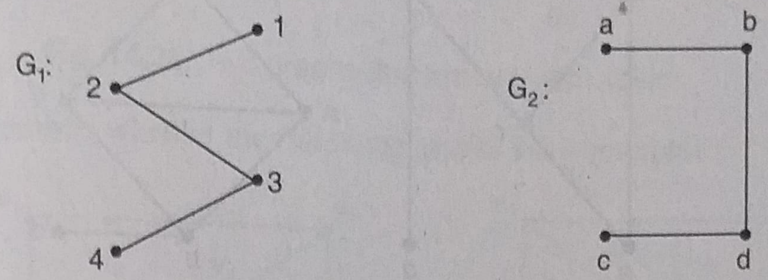
Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be isomorphic if there exists a functions $f: V_1 \to V_2$ such that

(i) f is one-to-one onto *i.e.*, f is bijective.

(ii) $\{a, b\}$ is an edge in E_1, if and only if $\{f(a), f(b)\}$ is an edge in E_2 for any two elements a, $b \in V_1$.

The condition (*ii*) says that if vertices a and b are adjacent in G_1 then $f(a)$ and $f(b)$ are adjacent in G_2. In other words the function f preserves adjacency relationship and consequently the corresponding vertices in G_1 and G_2 will have the same degree. Any function f with the above properties is called an **isomorphism** between G_1 and G_2.

Example 18. Show that the given pair of graphs are isomorphic



Solution. Here, $V(G_1) = \{1,2,3,4\}$, $V(G_2) = \{a,b,c,d\}$, $E(G_1) = \{\{1,2\}, \{2,3\}, \{3,4\}\}$ and $E(G_2) = \{\{a,b\}, \{b,d\}, \{d,c\}\}$. Hence $|V(G_1)| = |V(G_2)|$ and $|E(G_1)| = |E(G_2)|$

The vertices of degree 1 in G_1 are $\{1, 4\}$ and in G_2 are $\{a, c\}$

The vertices of degree 2 in G_1 are $\{2, 3\}$ and in G_2 are $\{b, d\}$

Define a function $f: V(G_1) \to V(G_2)$ as $f(1) = a, f(2) = b, f(3) = d$ and $f(4) = c$.

f is clearly one–one and onto.

Further,

$\{1,2\} \in E(G_1)$ and $\{f(1), f(2)\} = \{a, b\} \in E(G_2)$

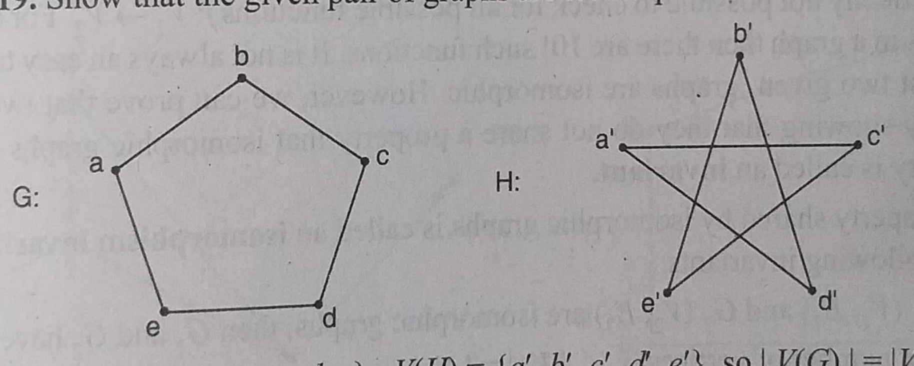
$\{2,3\} \in E(G_1)$ and $\{f(2), f(3)\} = \{b, d\} \in E(G_2)$

$\{3,4\} \in E(G_1)$ and $\{f(3), f(4)\} = \{d, c\} \in E(G_2)$

Hence f preserves adjacency of the vertices.

Therefore, G_1 and G_2 are isomorphic.

Example 19. Show that the given pair of graphs are isomorphic.



Solution: Here $V(G) = \{a, b, c, d, e\}$, $V(H) = \{a', b', c', d', e'\}$, so $|V(G)| = |V(H)|$

Also, $|E(G)| = |E(H)|$

The vertices of degree 2 in G are $\{a, b, c, d, e\}$ and

The vertices of degree 2 in H are $\{a', b', c', d', e'\}$

Define a function $f: V(G) \to V(H)$ as

$f(a) = a', f(b) = c', f(c) = e'\ f(d) = b'$ and $f(e) = d'$

f is clearly one-one and onto.

Further,

$\{a, b\} \in E(G)$ and $\{f(a), f(b)\} = \{a', c'\} \in E(H)$

$\{a, e\} \in E(G)$ and $\{f(a), f(e)\} = \{d', d'\} \in E(H)$

$\{b, c\} \in E(G)$ and $\{f(b), f(c)\} = \{c', e'\} \in E(H)$

$\{c, d\} \in E(G)$ and $\{f(c), f(d)\} = \{e', b'\} \in E(H)$

$\{d, e\} \in E(G)$ and $\{f(d), f(e)\} = \{b', d'\} \in E(H)$

Hence f preserves adjacency of the vertices. Therefore, G and H are isomorphic.

We hereby give some examples of isomorphic graphs.

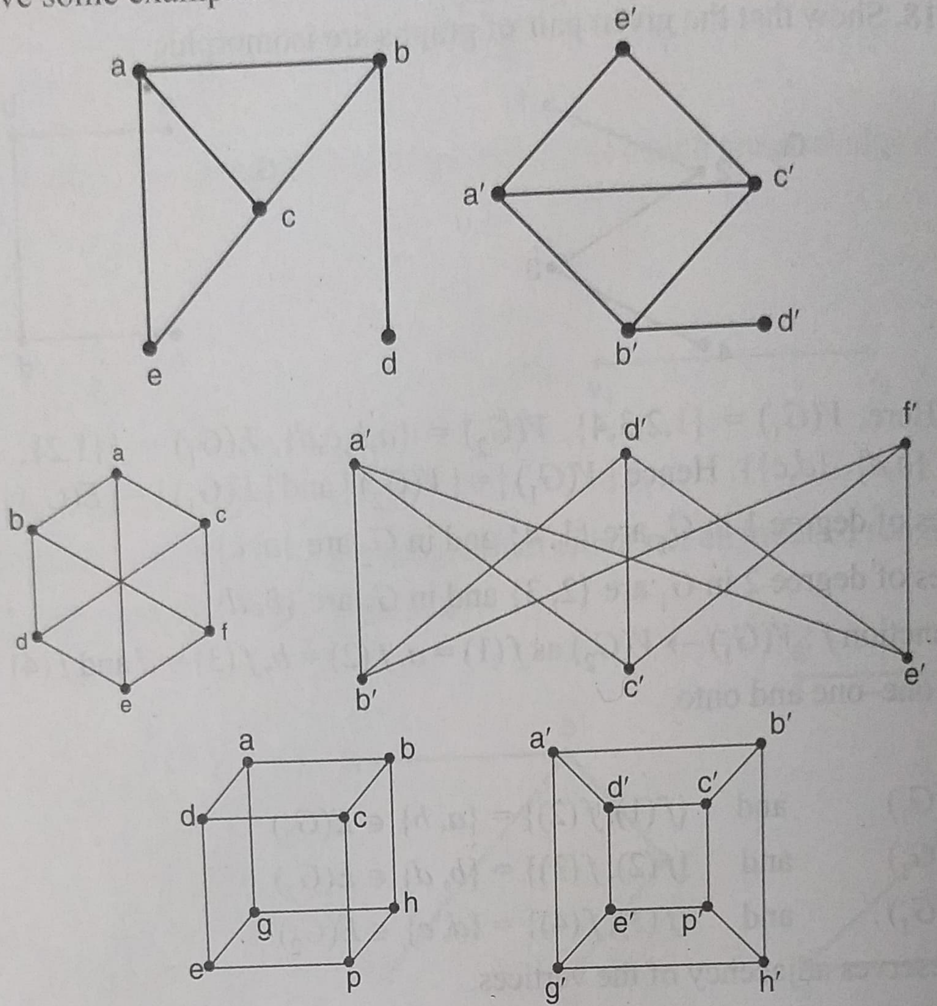


Fig. 14.24. Isomorphic pairs of graphs

It is practically not possible to check for all possible functions $f: V_1 \to V_2$. For example, if there are 10 vertices in a graph then there are 10! such functions. It is not always an easy task to determine whether or not two given graphs are isomorphic. However, we can prove that two graphs are not isomorphic by showing that they do not share a property that isomorphic graphs must both have, such a property is called an **invariant.**

A property shared by isomorphic graphs is called an **isomorphism invariant.** We may list some of the following invariants.

If $G_1 (V_1, E_1)$ and $G_2 (V_2, E_2)$ are isomorphic graphs, then G_1 and G_2 have the

(*i*) same number of vertices, *i. e.*, $|V_1| = |V_2|$

(*ii*) same number of edges, *i.e.*, $|E_1| = |E_2|$

(*iii*) same degree sequences *i.e.*, if the degree of a vertex v_i in G_1 is m, then the degree of the vertex $f(v_i)$ in G_2 must also be m.

(*iv*) If $\{v, v\}$ is a loop in G_1 then $\{f(v), f(v)\}$ is also a loop in G_2.

If any of these quantities differ in two graphs, they cannot be isomorphic. However, these conditions are by no means sufficient. For instance, the two graphs shown in Fig. 14.25 satisfy all three conditions, yet they are not isomorphic. That the graphs in Fig. 14.25 (*a*) and (*b*) are not isomorphic can be shown as follows : if the graph in Fig. 14.25 (*a*) are to be isomorphic to the one in (*b*), vertex x must correspond to y, because there are no other vertices of degree three. Now in (*b*) there is only one pendant vertex w, adjacent to y, while in (*a*) there are two pendant vertices u and v, adjacent to x.

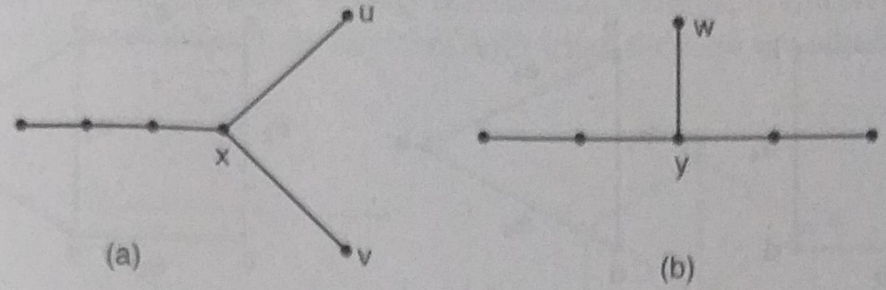
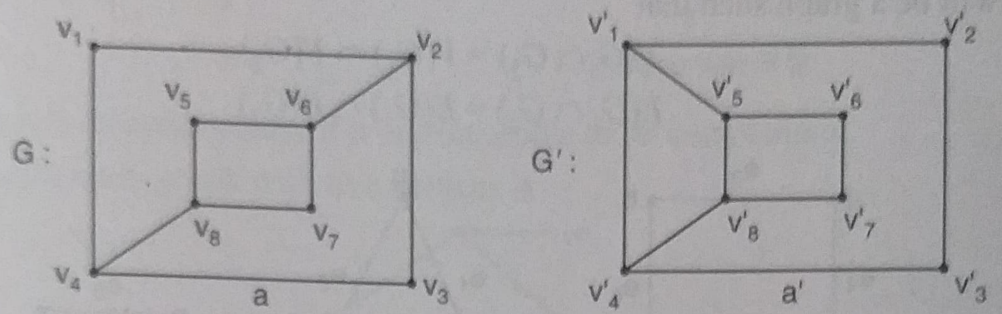


Fig. 14.25. Two graphs that are not isomorphic.

Example 20. Determine whether the following graphs are isomorphic.



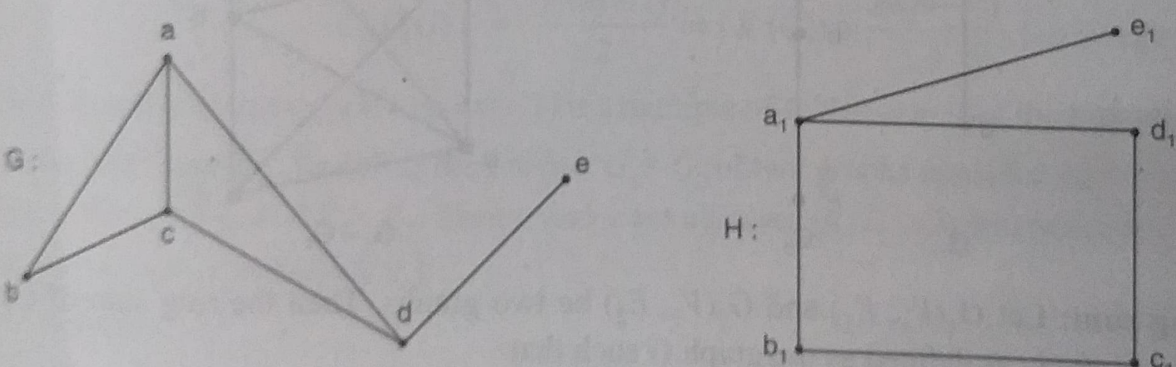
Solution. The graphs G and G' both have 8 vertices and 10 edges. They both have 4 vertices each of degree 3 and 4 vertices each of degree 2.

Now consider degree $(v_1) = 2$ in G. Then v_1 must correspond to either v_2', v_3', v_6', v_7', since these are vertices of degree 2 in G'.

However, each of these vertices in G' is adjacent to another vertex of degree 2 in G' viz. v'_2 is adjacent to v'_3, v'_6 is adjacent to v'_7 but v_1 is adjoint to v_2 and v_4 in G which are of degree 3. Thus the preservation of adjacency of the vertices is not maintained.

$\therefore$ G and G' are not isomorphic graphs.

Example 21. Examine G and H for isomorphism.



Solution. The graph G has 5 vertices and 6 edges. The graph H has 5 vertices and 5 edges. So $|E(G)| \neq |E(H)|$ *i.e.* the number of edges in G is not equal the number of edges in H.

$\therefore$ G and H are not isomorphic.

14.7. Operations of Graphs

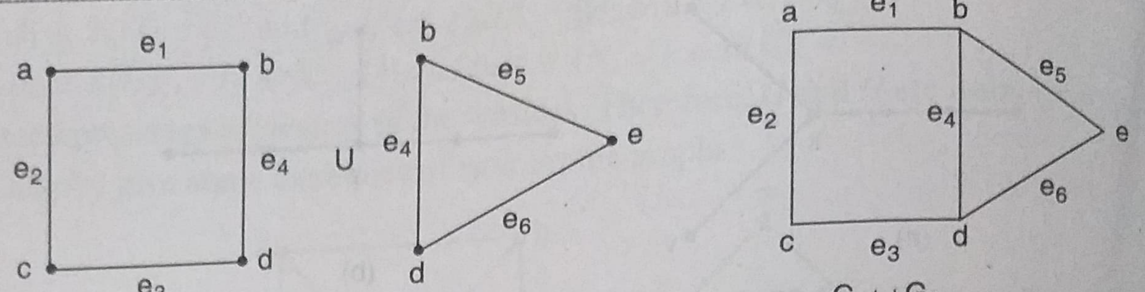
In this section we would learn about some operations on the graph.

Union: Given two graphs G_1 and G_2 their union will be a graph such that

$$V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$$

and

$$E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$$



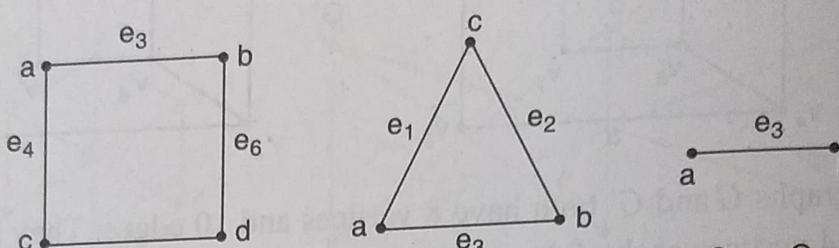
G₁ G₂ $G_1 \cup G_2$

Intersection: Given two graphs G_1 and G_2 with at least one vertex in common then their intersection will be a graph such that

$$V(G_1 \cap G_2) = V(G_1) \cap V(G_2)$$

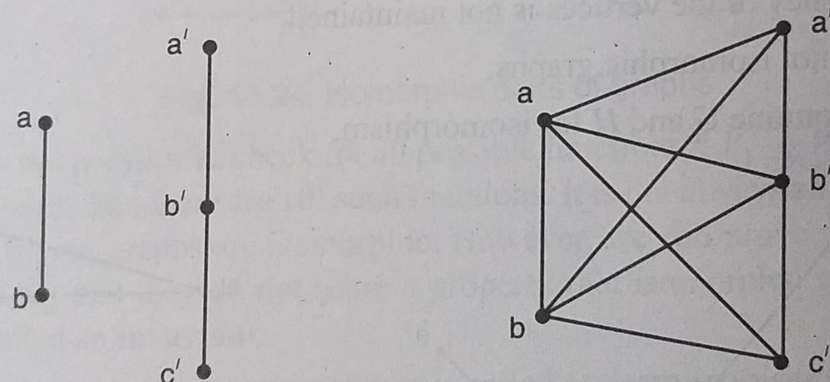
and

$$E(G_1 \cap G_2) = E(G_1) \cap E(G_2)$$



G_1 G_2 $G_1 \cap G_2$

Sum of Two Graphs: If the graphs G_1 and G_2 such that $V_1(G_1) \cap V(G_2) = \phi$; then the sum $G_1 + G_2$ is defined as the graph whose vertex set is $V(G_1) + V(G_2)$ and the edge set is consisting those edges, which are in G_1 and in G_2 and the edges obtained, by joining each vertex of G_1 to each vertex of G_2. For example,

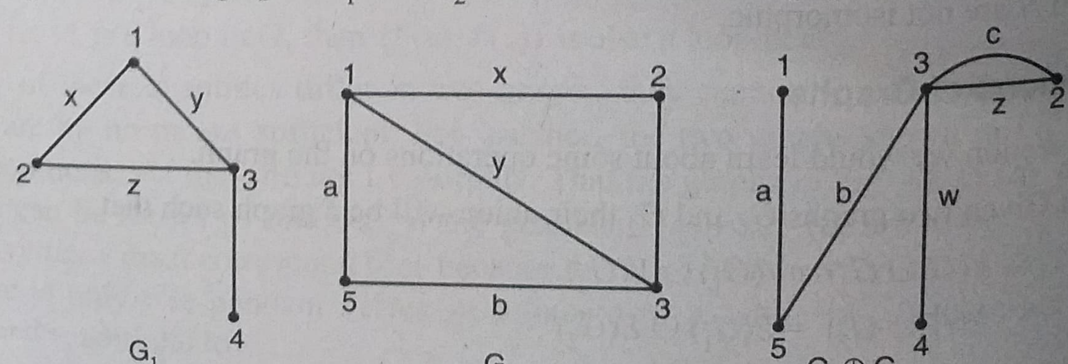


G_1 G_2 $G_1 + G_2$

Ring sum: Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be two graphs. Then the ring sum of G_1 and G_2, denoted by $G_1 \oplus G_2$, is defined as the graph G such that

(*i*) $V(G) = V(G_1) \cup V(G_2)$

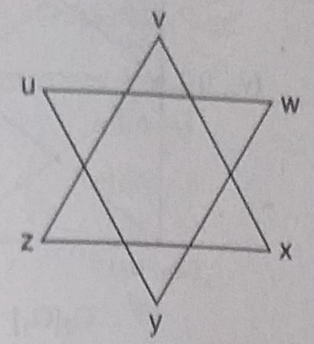
(*ii*) $E(G) = E(G_1) \cup E(G_2) - E(G_1) \cap E(G_2)$ *i.e.*, the edges that either in G_1 or G_2 but not in both. The Ring sum of two graphs G_1 and G_2 is shown on the next page.



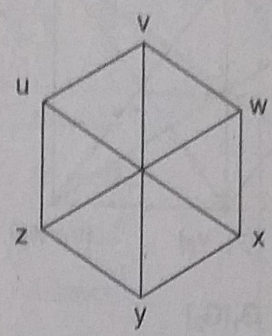
G_1 G_2 $G_1 \oplus G_2$

Complement: The complement G' of G is defined as a simple graph with the same vertex set as G and where two vertices u and v are adjacent only when they are not adjacent in G.

For example,



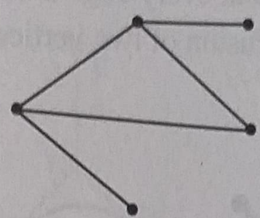
G



Complement of G

A graph G is self-complementary if it is isomorphic to its complement.

A self-complemenrary graph with five vertices is



Example 22. Show that a graph G is self complementary if it has $4n$ or $4n + 1$ vertices (n is a non-negative integer).

Solution: Let $G(V, E)$ be a self complimentary graph with m vertices. Since G is self complementary, G is isomorphic to G' (complement of G)

$$\therefore \quad |E(G)| = |E(G')|.$$

$$\text{Again} \quad |E(G) + E(G)'| = \frac{m(m-1)}{2}$$

$$\therefore \quad 2|E(G)| = \frac{m(m-1)}{2} \Rightarrow |E(G)| = \frac{m(m-1)}{4} \text{ (an integer)}$$

This is possible if and only if m or $(m-1)$ is a multiple of 4. Hence m is of the form $4n$ or $4n+1$.

Product of Graphs : To define the product $G_1 \times G_2$ of two graphs consider any two points $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $V = V_1 \times V_2$. Then u and v are adjacent in $G_1 \times G_2$ whenever $[u_1 = v_1$ and u_2 adj $v_2]$ or $[u_2 = v_1$ and $u_1$adj $v_1]$

For example,

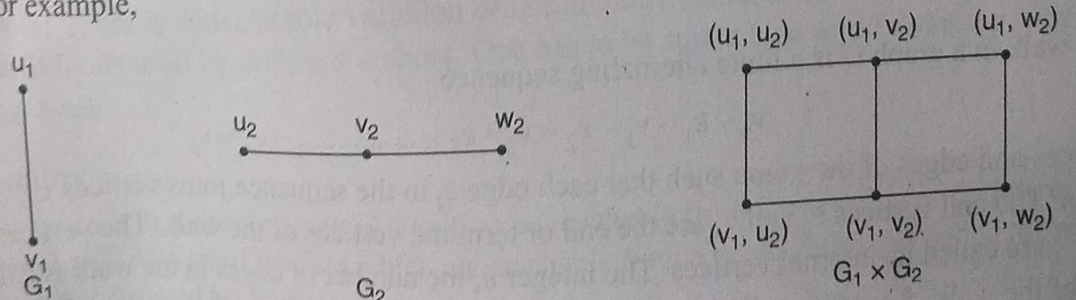


Fig. 14.26. The product of two graphs

Composition: The composition $G = G_1[G_2]$ also has $V = V_1 \times V_2$ as its point set, and $u = (u_1, u_2)$ is adjacent with $v = (v_1, v_2)$ whenever $[u_1$adj $v_1)$ or $[u_1 = v_1$ and u_2 adj $v_1]$. For the graphs G_1 and G_2 of Fig. 14.26 both compositions $G_1[G_2]$ and $G_2[G_1]$ are shown in Fig. 14.27.

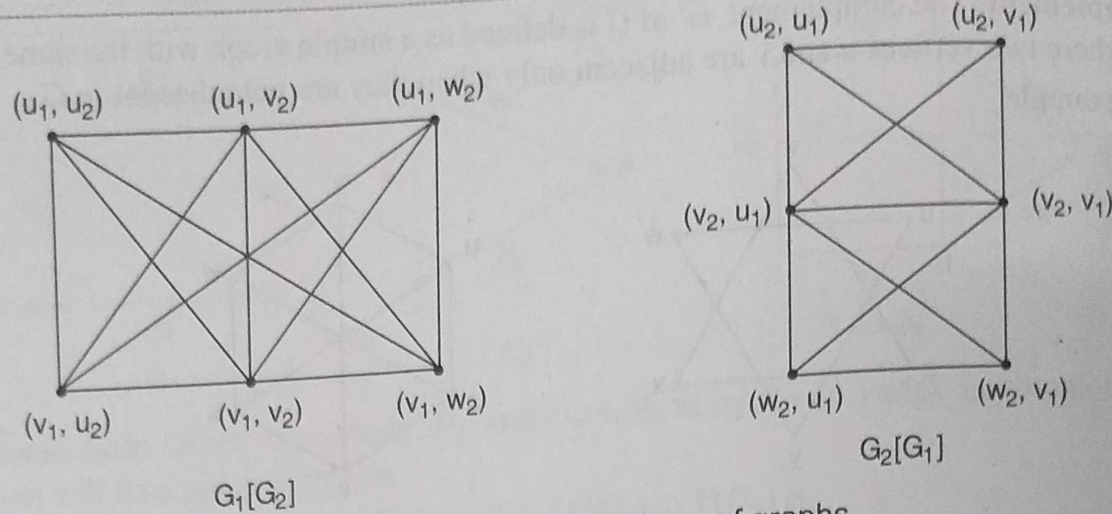
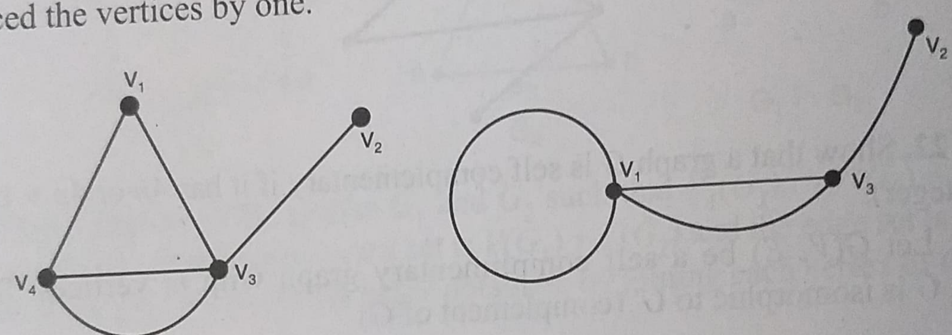


Fig. 14.27. Two compositions of graphs

Fusion: A pair of vertices v_1 and v_2 in graph G is said to be 'fused' if these two vertices are replaced by a single new vertex v such that every edge that was adjacent to either v_1 or v_2 or both is adjacent to v. Thus we observe that the fusion of two vertices does not alter the number of edges of graph but reduced the vertices by one.



v_1 is fused with v_4

Notes

1. Let v be the vertex obtained by the fusion of two vertices a and b in G and G' be the graph obtained after the fusion. Then $\deg_{G'}(v) = \deg_G(a) + \deg_G(b)$ and $\deg_{G'}(x) = \deg_G(x)$ for all vertices of G which are not a or b.

2. Fusion of two adjacent vertices always produce a loop at the point of fusion and the number of loops is equal to the number of edges between the vertices which are fused together.

14.8. Paths, Circuits, Cycles and Connectivity

In this section we introduce some additional terminology associated with a graph.

Walk

A walk in a graph G is a finite alternating sequence.

$$v_0 - e_1 - v_1 - e_2 - v_2 - e_3 \ldots\ldots\ldots\ldots\ldots - e_n - v_n$$

of vertices and edges of the graph such that each edge e_i in the sequence joins vertices v_{i-1} and v_i, $1 \le i \le n$. The end vertices v_0 and v_n are the end or terminal vertices of the walk. The vertices $v_1, v_2, \ldots, v_{n-1}$ are called its internal vertices. The integer n, the number of edges in the walk is called the length of the walk. A walk is called open when the terminal vertices are distinct. For the same end terminal vertices, it is termed as closed. Note that a walk may repeat both vertices and edges.

Special Types of Walk

A walk is called a **trail** if all its edges are distinct. A trail is open or closed depends on whether its end vertices are distinct or not.

A closed trail is called a **circuit.** A walk is called a **path** if all its vertices and edges are distinct. A path in which only repeated vertex is the first vertex is called a **cycle** to describe such a closed path.

	Repeated Edge	*Repeated Vertex*	*Starts and Ends at same points ?*
Walk (open)	allowed	allowed	no
Walk (closed)	allowed	allowed	yes
Trail	no	allowed	no
Circuit	no	allowed	yes
Path	no	no	no
Cycle	no	first and last only	yes

For example, in the graph given in Fig. 14.28.

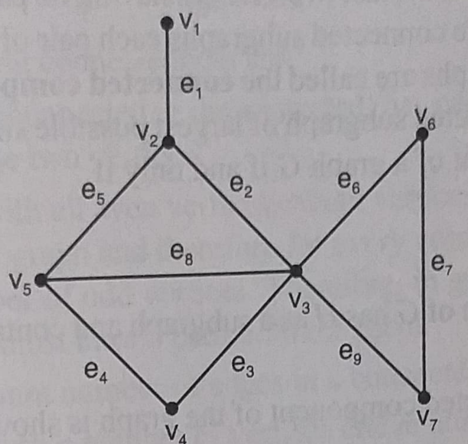


Fig. 14.28

(*i*) The sequence $v_1 - e_1 - v_2 - e_5 - v_5 - e_8 - v_3 - e_3 - v_4 - e_4 - v_5 - e_5 - v_2 - e_2 - v_3 - e_6 - v_6$ walk of length 8. It contains repeated vertices v_2, v_3 and v_5 and repeated edge e_5.

(*ii*) The sequence $v_1 - e_1 - v_2 - e_5 - v_5 - e_3 - v_3 - e_3 - v_4 - e_4 - v_5$ is a trail. It contains repeated vertex v_5 but does not contain repeated edge.

(*iii*) The sequence $v_1 - e_1 - v_2 - e_5 - v_5 - e_8 - v_3 - e_3 - v_4$ is a path. It does not contain repeated vertex and repeated edge.

(*iv*) The sequence $v_2 - e_2 - v_3 - e_3 - v_4 - e_4 - v_5 - e_5 - v_2$ is a cycle. It does not conatain repeated vertex and repeated edge except the first and last vertex.

Note: There is considerable variation of terminology concerning the concepts defined in the above definitions used by different authors. One has to be sure which set of definitions are used in particular book.

Reachability

A vertex v in a simple graph G is said to be reachable from the vertex u of G if there exists a path from u to v. The set of vertices which are reachable from a given vertex v is called the reachable set of v and is denoted by $R(v)$.

For any subset U of the vertex set V, the reachable set of U is the set of all vertices which are reachable from any vertex set of S and this set is denoted by $R(S)$. For example, in the directed graph given here

$$R(v_1) = \{v_2, v_3, v_4\},\ R(v_2) = \{v_1, v_3, v_4\}, \text{ and } R(\{v_1, v_2\}) = \{v_3, v_4\}.$$

Solution. The graph shown in 14.32 (*a*) is a connected graph since for every pair of distinct vertices there is a path between them.

The graph shown in Fig. 14.32 (*b*) is not connected since there is no path in the graph between vertices *b* and *d*.

The graph shown in Fig. 14.32 (*c*) is not connected. In drawing a graph two edges may cross at a point which is not a vertex. The graph can be redrawn as

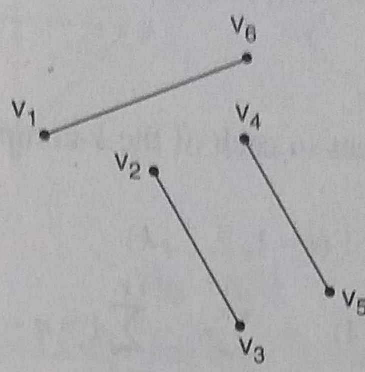


Fig. 14.33

Two useful results involving connectedness are:

Theorem 14.4. If a graph (connected or disconnected) has exactly two vertices of odd degree, there must be a path joining these two vertices.

Proof: Let G be a graph with all even vertices except vertices v_1 and v_2, which are odd. From Theorem, which holds for every graph and therefore for every component of a disconnected graph, no graph can have an odd number of odd vertices. Therefore, in graph G, v_1 and v_2 must belong to the same component and hence must have a path between them.

Theorem 14.5. The minimum number of edges in a connected graph with n vertices is $n - 1$.

Proof: Let G be a graph with n number of vertices and m number of edges. We have to prove that $m \geq n-1$. We prove this by method of induction on m. If $m = 0$ then obviously $n = 1$, otherwise G will be disconnected, clearly then $m \geq n-1$ (as $0 \geq 1-1$)

Let the result holds good for all connected graphs with $\leq k$ number of edges. We shall show tha the result is true for $m = k + 1$. Let G be a graph with $k + 1$ edges. Let e be an edge of G. Then the subgraph $G - e$ has k number of edges and n number of vertices.

If (*i*) $G - e$ is a connected graph then by our hypothesis

$$k \geq n-1 \quad i.e. \quad k+1 \geq n > n-1$$

(*ii*) $G - e$ is a disconnected graph then it would have two connected components. Let the two connected components have k_1 and k_2 number of edges and n_1, n_2 number of vertices respectively. So by our hypothesis

$$k_1 \geq n_1 - 1 \text{ and } k_2 \geq n_2 - 1. \quad \text{So,} \quad k_1 + k_2 \geq n_1 + n_2 - 2 \text{ i.e. } k \geq n_2 - 2$$

$$i.e. \ k + 1 \geq n - 1 \quad (\because k_1 + k_2 = k, n_1 + n_2 = n)$$

Hence, the result is true for $m = k + 1$.

Thus by induction, the result is true for any connected graph with n number of vertices.

Theorem 14.6. The minimum number of edges in a simple graph (not necessarily connected) with n vertices is $n - k$, where k is the number of connected components of the graph.

Example 24. Suppose G is a non-directed graph with 12 edges. If G has 6 vertices each of degree and the rest have degree less than 3, find the minimum number of vertices G can have.

Solution. Let G has n vertices, say, $v_1, v_2, \ldots\ldots, v_n$ among these $v_1, v_2, \ldots, v_6$ have degree 3 each and the rest $v_7, v_8, \ldots, v_n$ have degree less than 3.

$\therefore$ ... $\leq 18 + 3(n - 6)$ ...(1)

We know, $\quad d(v_1) + d(v_2) + \ldots d(v_n) = 2 \times$ no. of edges

$$= 2 \times 12 = 24$$

Therefore from (1),

$$24 < 18 + 3(n-6), \text{ or } 3n < 24, \quad \text{or} \quad n > 8.$$

Therefore, the minimum number of vertices of G is 9.

Theorem 14.7. A simple graph with n vertices and k components cannot have more than $\dfrac{(n-k)(n-k+1)}{2}$ edges.

Proof: Let the number of vertices in each of the k-components of a graph G be $n_1, n_2, \ldots, n_k$, then we get

$n_1 + n_2 + \ldots + n_k = n$ where $n_i \geq 1$ $(i = 1, 2, \ldots, k)$

Now,
$$\sum_{i=1}^{k}(n_i - 1) = \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} 1 = n - k$$

$$\therefore \quad \left(\sum_{i=1}^{k}(n_i - 1)\right)^2 = n^2 + k^2 - 2nk$$

or
$$\sum_{i=1}^{k}(n_i - 1)^2 + 2\sum_{i=1}^{k}\sum_{\substack{j=1\\ i \neq j}}(n_i - 1)(n_j - 1) = n^2 + k^2 - 2nk$$

or
$$\sum_{i=1}^{k}(n_i - 1)^2 + 2(\text{non-negative terms}) = n^2 + k^2 - 2nk \qquad [\because n_i - 1 \geq 0, n_j - 1 \geq 0]$$

or
$$\sum_{i=1}^{k}(n_i - 1)^2 \leq n^2 + k^2 - 2nk$$

or
$$\sum_{i=1}^{k} n_i^2 + \sum_{i=1}^{k} 1 - 2\sum_{i=1}^{k} n_i \leq n^2 + k^2 - 2nk$$

or
$$\sum_{i=1}^{k} n_i^2 + k - 2n \leq n^2 + k^2 - 2nk$$

or
$$\sum_{i=1}^{k} n_i^2 - n \leq n^2 + k^2 - 2nk - k + 2n$$
$$= n(n-k+1) - k(n-k+1)$$
$$= (n-k)(n-k+1) \qquad \ldots(1)$$

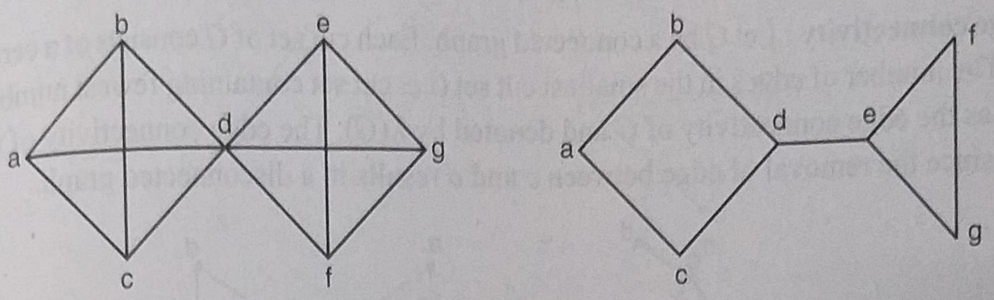
We know that the maximum number of edges in the ith component of $G = {}^{n_i}C_2 = \dfrac{n_i(n_i - 1)}{2}$

Therefore, the maximum number of edges in G is,

$$\frac{1}{2}\sum_{i=1}^{k} n_i(n_i - 1) = \frac{1}{2}\left(\sum_{i=1}^{k} n_i^2 - \sum_{i=1}^{k} n_i\right)$$
$$= \frac{1}{2}\left(\sum_{i=1}^{k} n_i^2 - n\right)$$
$$\leq \frac{1}{2}(n-k)(n-k+1) \text{ by (1)}$$

Distance and Diameter

In a connected graph G, the distance between the vertices u and v, denoted by $d(u,v)$ is the length of the shortest path. In Fig. 14.34 (a), $d(a, f) = 2$ and in Fig. 14.4(b), $d(a,e) = 3$



(a) (b)

Fig. 14.34

The distance function as defined above has the following properties. If u,v and w are any three vertices of a connected graph then

(i) $d(u, v) \geq 0$ and $d(u, v) = 0$ iff $u = v$

(ii) $d(u, v) = d(v, u)$ and

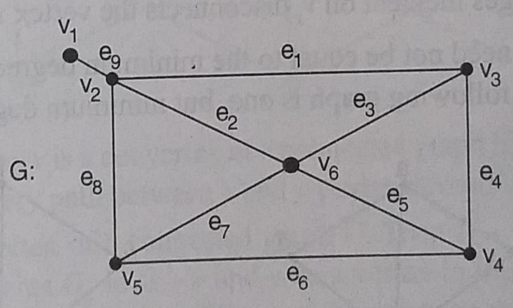
(iii) $d(u, v) \geq d(u, w) + d(w, v)$

This shows that distance in a graph is metric.

The diameter of a connected graph G, written as diam (G) is the maximum distance between any two vertices in G. In Fig. 14.34 (a), diam $(G) = 2$ and in Fig. 14.34 (b), diam $(G) = 4$.

Cut Set and Cut Vertices

Cut Set : A cut set of a connected graph G is a set of edges whose removal (without removing the end vertices) from G leaves the G disconnected provided removal of no proper subsets of these edges disconnects G.



For example, in graph above

(i) The edge $\{e_9\}$ is a cut set

(ii) The set of edges $\{e_1, e_2, e_6, e_7\}$ and $\{e_1, e_2, e_8\}$ and cut sets.

(iii) The set of edges $\{e_1, e_2, e_8, e_7\}$ is not a cut set because one of its proper subset $\{e_1, e_2, e_8\}$ is a cut set.

Thus a cut set S of G satisfy the followings:

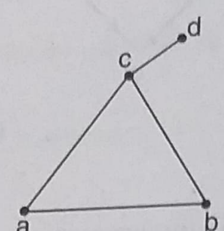
1. S is a subset of the edge set E of G.
2. Removal of edge/edges from a connected graph G disconnects the graph.
3. No proper subset of G satisfy the condition.

Note : Every edge of a tree is a cut set.

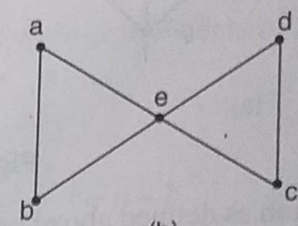
Connectivity

To study the measure of connectedness of a graph G we consider the minimum number of vertices and edges to be removed from the graph in order to disconnect it.

Edge connectivity : Let G be a connected graph. Each cut set of G consists of a certain number of edges. The number of edges in the smallest cut set (*i.e.* cut set containing fewest number of edges) is defined as the edge connectivity of G and denoted by $\lambda(G)$. The edge connectivity of the graph in (a) is one since the removal of edge between c and d results in a disconnected graph.



(a)



(b)

In (b), the edge connectivity is two since the removal of at least two edges viz (a, b) and (a, c) disconnects the graph.

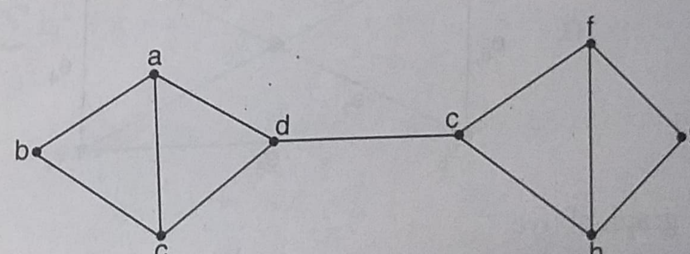
Note :

1. The edge connectivity of a tree is one because removal of an edge from the tree disconnects the graph.
2. The edge connectivity of the complete bipartite graph $K_{m,n}$ is a p where $p = \min(m, n)$
3. The edge connectivity of a complete graph of n vertices is $n - 1$.

Theorem 14.8. The edge connectivity of a connected graph G cannot exceed the degree of the vertex having the smallest degree in G.

Proof: Let G be a connected graph and v_i be the vertex having smallest degree. Let $\deg(v_i) = r$. Then the removal of all r edges incident on v_i disconnects the vertex v_i from G. Hence the theorem.

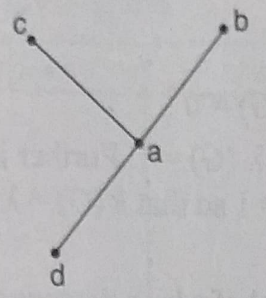
The edge connectivity need not be equal to the minimum degree of a vertex of G, we see that the edge connectivity of the following graph is one, but minimum degree of a vertex of G is two.



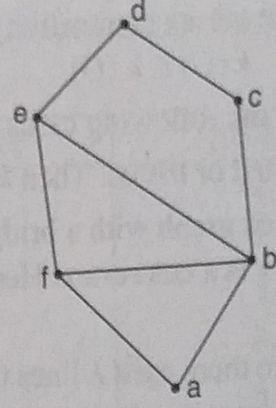
Theorem 14.9. The edge connectivity is less than or equal $[2e/n]$ where $[2e/n]$ represents the integral part of $2e/n$.

Proof: Let G be a connected graph having n number of vertices and e number of edges. Let degree of each of the n vertices be greater than or equal to d, therefore, $nd \le 2e$ or $d \le 2e/n$. Since d must be an integer, then $d \le [2e / n]$ where $[2e / n]$ represents the integral part of $2e / n$. Hence by previous theorem, the edge connectivity is less than or equal to $[2e / n]$.

Vertex Connectivity : The vertex connectivity of a connected graph G denoted by $k(G)$ is defined as the minimum number of vertices whose removal (together with the edges incident on it) from G leaves the remaining graph disconnected. The vertex connectivity of the graph in Fig. (a) and (b) are one and two respectively.



(a)



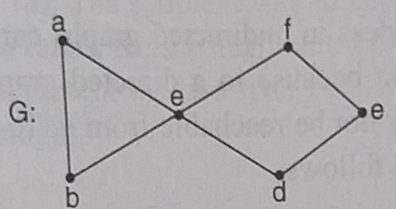
(b)

Note : 1. It G is a disconnected graph $k(G) = 0$

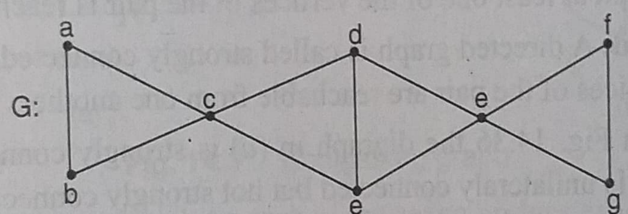
2. The complete graph K_n can not be disconnected by removing any number of vertices, but the removal of $n-1$ vertices results in a trivial graph, hence $k(K_n) = n-1$.

3. The vertex connectivity of a path is one and that of cycle C_n $(n \geq 4)$ is two.

Separable Graph : A connected graph G is said to be separable if its vertex connectivity is one. All other graphs are known as non-separable. The graph shown below is separable.



Cut Vertex : A cut vertex is a vertex in a separable graph whose removal disconnects the graph. The cut vertex is also called an **articulation point.** A given separable graph can have more than one cut vertex. The graph shown below is separable and both c and e are cut vertices.



Theorem 14.10. A vertex v is a cut vertex of a connected graph if there exist two vertices x and y distinct from v such that every path between x and y passes through v.

Proof: Let v is a cut vertex of a connected graph G. Then $G - v$ is disconnected. Let x be a vertex in one of the components G_1 is $G - v$ and y be a vertex in the complement of G_2 in $G - v$. Since x and y are in different components of $G - v$, there exists no path between x and y in $G - v$. But G is connected and it implies there exist paths between x and y in G. Hence every path between x and y passes through v.

Conversely, if every path from x to y contains the vertex v, then removal of v from G disconnects x and y. Hence x and y lies in different components of G, which implies $G - v$ is disconnected graph. Thus v is a cut vertex of G.

Theorem 14.11. The edge connectivity of a graph G cannot exceed the minimum degree of a vertex in G *i.e.* $\lambda(G) \leq \delta(G)$.

Proof: Let G be a connected graph and v be a vertex of minimum degree in G. Then the removal of edges incident with the vertex v disconnects the vertex u from the graph G. Thus the set of all edges incident with the vertex v forms a cut set of G. But edge connectivity is the minimum cardinality of the cut set of G, this implies that the edge connectivity is always less than the number of edges (degree of v) incident with the vertex v.

Theorem 14.12. The vertex connectivity of a graph G is always less than or equal to the edge connectivity of G *i.e.* $k(G) \le \lambda(G)$

Proof: We consider the following cases.

(*i*) If G is disconnected or trivial. Then $k(G) = \lambda(G) = 0$

(*ii*) If G is a connected graph with a bridge x, then $\lambda(G) = 1$. Further in this case $G = k_2$ or one of the point incident with x is a cut vertex. Hence $k(G) = 1$ so that $k(G) = \lambda(G) - 1$. $\therefore k(G) \le \lambda(G)$ when $\lambda(G) = 0$ or 1.

(*iii*) If $\lambda(G) \ge 2$. Then there exist λ lines the removal of which disconnects the graph. Hence the removal of $\lambda - 1$ of these lines results in a graph G with a bridge $x = \{u, v\}$. For each of these λ–1, lines select an incident point different from u or v. The removal of these $\lambda - 1$, points removes all the $\lambda - 1$, lines. If the resulting graph is disconnected, then $k \le \lambda$–1. If not x is a bridge of this subgraph and hence the removal of u or v results in a disonnected or trivial graph.

Hence $k \le \lambda$ in each case and this completes the proof.

Definition. A graph G is said to be n–connected if $k(G) \ge n$ and n-line connected if $\lambda(G) \ge n$. Thus a nontrivial graph is 1–connected if it is connected:

Connectedness in Directed Graphs

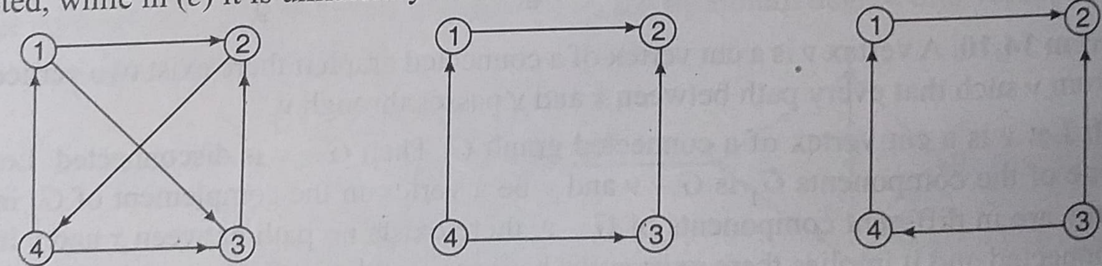
The definition of connectedness in undirected graphs can not be applied to directed graphs without some further modification because in a directed graph if a vertex u is reachable from another vertex v, the vertex v may not be reachable from u. To overcome this difficulty we define the connectedness in a diagraph as follows.

Connected or weakly connected: A directed graph is called connected or weakly connected if it is connected as an undirected graph in which each directed edge is converted to an undirected graph.

Unilaterally connected: A simple directed graph is said to be unilaterally connected if for any pair of vertices of the graph at least one of the vertices of the pair is reachable from other vertex.

Strongly connected: A directed graph is called strongly connected if for any pair of vertices of the graph both the vertices of the pair are reachable from one another.

For the digraphs in Fig. 14.36 the digraph in (*a*) is strongly connected, in (*b*) it is weakly connected, while in (*c*) it is unilateraly connected but not strongly connected.

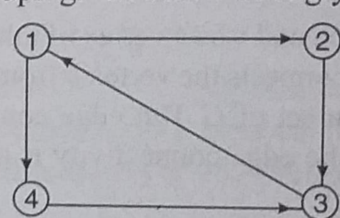


(a) Strongly connected (b) Weakly connected (c) Unilaterally connected

Fig. 14.36. Connectivity in digraphs

Note that a unilaterally connected digraph is weakly connected but a weakly connected digraph is not necessarily unilaterally connected. A strongly connected digraph is both unilaterally and weakly connected.

Example 25. Is the directed graph given below strongly connected ?



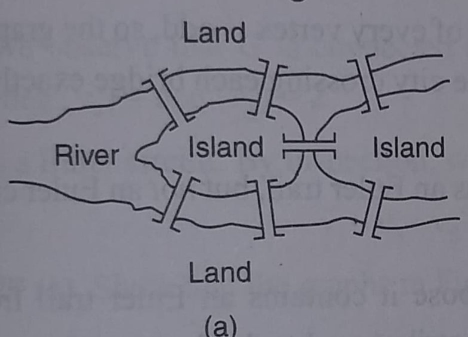
Solution. The possible pairs of vertices and the forward and the backward paths between them are shown below for the given graph

Pairs of vertices	*Forward path*	*Backward path*
(1,2)	1 – 2	2 – 3 – 1
(1,3)	1 – 2 – 3	3 – 1
(1,4)	1 – 4	4 – 3 – 1
(2,3)	2 – 3	3 – 1 – 2
(2,4)	2 – 3 – 1 – 4	4 – 3 – 1 – 2
(3,4)	3 – 1 – 4	4 – 3

Therefore, we see that between every pair of distinct vertices of the given graph there exists a forward as well as backward path, and hence it is strongly connected.

14.9. Eulerian and Hamiltonian Graph

One of the oldest problem involving graphs is the Konigsberg bridge probelm. There were two islands linked to each other and to the banks of the Pregel River (earlier known as Konigsberg) by seven bridges shown in Fig. 14.37 (*a*). The problem was to begin at any of the four land areas to walk across each bridge once and to return the starting point. Euler drew a graph like Fig. 14.37 (*b*) for the problem in which *a* and *c* represent the two river banks; *b* and *d* the two islands. The arcs joining them represent the seven bridges.



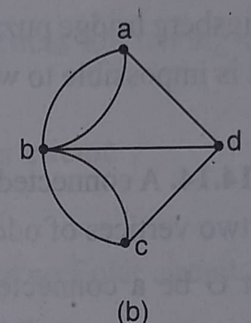


Fig. 14.37. Konigsberg graph

It is clear that the problem of walking each of the seven bridges exactly once and returning to the starting point is equivalent to finding a circuit in graph (*b*) that traverses each of the edge exactly once. Euler discovered a very simple criterion for determining whether such a circuit exists in a graph.

Eulerian Graph

A circuit in a connected graph is an **Euler circuit** if it contains every edge of the graph exactly once. A connected graph with an Euler circuit is called an Euler graph or Eulerian graph.

If there is trail from vertex a to b in G and this trail traverses each edge in G exactly once, then the trail is called an Euler trail.

In Fig. 14.38, $d\ c\ e\ a\ b\ c$ represents an Euler trail since it contains all the edges exactly once and vertex c is repeated but start and end vertex are not the same. It is not an Euler circuit as starting and ending at the same vertex is not possible without repeating an edge cd.

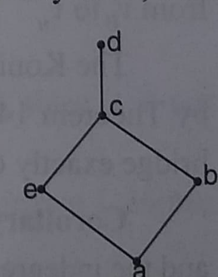


Fig. 14.38

The existence of Euler circuit and trail depends on the degree of vertices.

The next theorem provides necessary and sufficient condition for characterising Euler graph.

Theorem 14.13. A nonempty connected graph G is Eulerian if and only if its vertices are all of even degree.

Proof: Suppose G is Eulerian. Then G contains an Eulerian circuit, say, from v_0 to $v_0 : v_0 - e_1 - v_1 - e_2 - \ldots\ldots v_{n-1} - e_n - v_0$. Both edges e_1 and e_n contribute a 1 to the degree of v_0; so deg (v_0) is at least two. Each time the circuit passes through a vertex (including v_0), the degree of the vertex is increased by 2. Consequently, the degree of every vertex, including v_0, is an even integer.

Conversely, suppose every vertex of G has even degree. Now we construct an Euler circuit starting at an arbitrary vertex v and going through the edges of G exactly once. Since every vertex is of even degree, we can exit from every vertex we enter. Since v is even degree, one reaches v, when tracing comes to an end. If this closed circuit H_1 contains all the edges of G, then G is an Euler graph. If this circuit does not contain all the edges of G then we remove from G all the edges in H_1 and obtain a subgraph $H_2 = G - H_1$ of G formed out of the remaining edges. Since both G and H_1 have all their vertices of even degree, the degrees of vertices of H_2 are also even. Since G is connected, H_2 and H_1 must have atleast one common vertex say v_1. Starting from v_1 one can construct another new circuit in H_2. This new circuit in H_2 can be combined with H_1 to form a new larger circuit. If it is Eulerian, then G is. If it is not, we continue this procedure to form an Euler circuit. This procedure must terminate since the number of edges in G is finite. Thus G contains an Euler circuit and hence is Eulerian.

Note: The theorem does not give any method to find the Euler circuit.

In the Konigsberg bridge puzzle, the degree of every vertex is odd, so the graph is not Eulerian. Consequently, it is impossible to walk through the city crossing each bridge exactly once and return home.

Theorem 14.14. A connected graph contains an Euler trail, but *not* an Euler circuit, if and only if it has exactly two vertices of odd degree.

Proof: Let G be a connected graph. Suppose it contains an Euler trail from v_o to v_n say, $v_0 - e_1 - v_1 - e_2 \ldots\ldots\ldots v_{n-1} - e_n - v_n$. Edges e_1 and e_n contributes a 1 to the degrees of v_0 and v_n, respectively. Every time the path passes through a vertex, it contributes a 2 to its degree. So every time the path passes through v_0, its degree it increased by 2; similarly for v_n. Consequently, the degrees of v_o and v_n are odd. The degree of each internal vertex $v_1, v_2, \ldots v_{n-1}$ remains even. Thus the graph contains exactly two vertices of odd degree.

Conversely, suppose G contains exactly two vertices of odd degree, say v_0 and v_n. Adding a new edge with v_0 and v_n as the end vertices to G results in a graph G_1 with all even degree vertices. Therefore, by Theorem 14.13, G_1 is Eulerian. Removing edge $\{v_0, v_n\}$ from G_1 shows that G contains an Euler trail from v_0 to v_n

The Konigsberg bridge model in Fig. 14.37 contains four vertices, all odd degree. Therefore, by Theorem 14.14 no Euler trail exists. In other words, no walk through the city, traversing each bridge exactly once, is possible.

Corollary 1. A directed multigraph G has an Euler path if and only if it is unilaterally connected and the indegree of each vertex is equal to its outdegree with the possible exception of two vertices, for which it may be that the indegree of one is larger than its outdegree and the indegree of the other is oneless than its outdegree.

Corollary 2. A directed multigraph G has an Euler circuit if and only if G is unilaterally connected and the indegree of every vertex in G is equal to its outdegree.

Thus to determine whether a graph G has an Euler circuit, we note the follow points.

1. List the degree of all vertices in the graph.
2. If any value is zero, the graph is not connected and hence it can not have Euler path or Euler circuit.
3. If all the degrees are even, then G has both Euler trail and Euler circuit.
4. If exactly two vertices are odd degree, then G has Euler trail but no Euler circuit.

Example 27(*a*). Let G be a graph of Fig. 14.39. Verify that G has an Euler circuit,

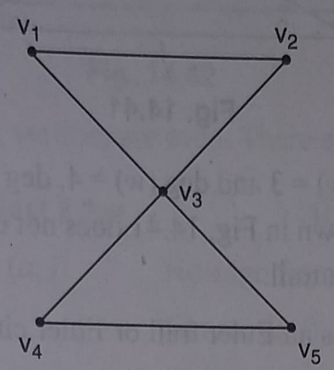


Fig. 14.39

Solution. We observe that G is connected and all the vertices are having even degree deg $(v_1) = \deg(v_2) = \deg(v_4) = \deg(v_5) = 2$.

Thus G has a Euler circuit. By inspection, we find the Euler circuit.

$$v_1 - v_3 - v_5 - v_4 - v_3 - v_2 - v_1$$

Example 27 (*b*). Show that the graphs in Fig. 14.40 contain no Euler circuit.

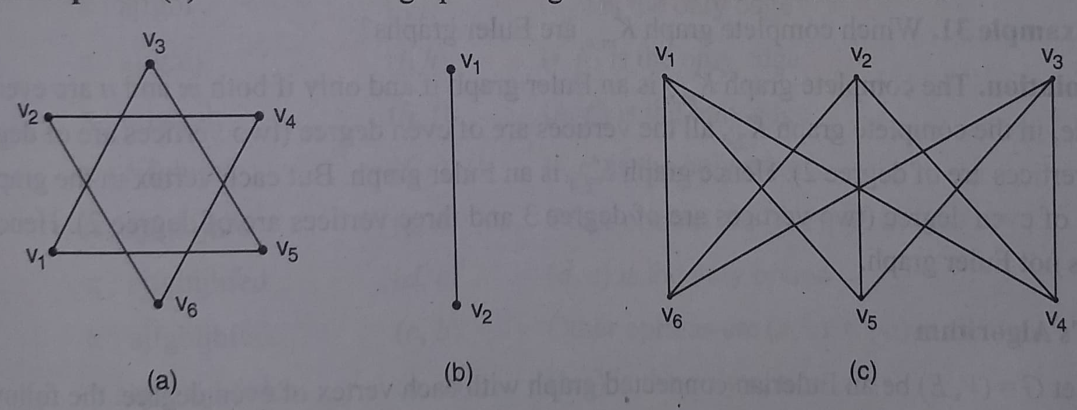


Fig. 14.40

Solution. The graph shown in Fig. 14.40 (*a*) does not contain Euler circuit since it is not connected.

The graph shown in Fig. 14.40 (*b*) is connected but vertices v_1 and v_2 are of degree 1. Hence it does not contain Euler circuit.

All the vertices of the graph shown in Fig. 14.40 (*c*) are of degree 3, hence it does not contain Euler circuit.

Example 29. Show that the graph shown in Fig. 14.41 has no Eulerian circuit but has a Eulerian trail.

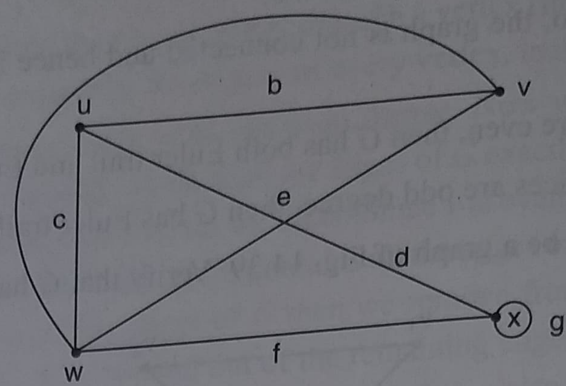


Fig. 14.41

Solution. Here deg (u) = deg $(v) = 3$ and deg $(w) = 4$, deg $(x) = 4$. Since u and v have only two vertices of odd degree, the graph shown in Fig. 14.41 does not contain Eulerian circuit, but the path $b-a-c-d-g-f-e$ is an Eulerian trail.

Once known that a graph G has an Euler trail or Euler circuit, a algorithm due to Fleury can be applied to find them.

Example 30. For what values of n is the graph of K_n Eulerian?

Solution. We know that K_n the complete graph of n vertices is a connected graph in which degree of each vertex is $n-1$. Since a graph is an Eulerian graph if and only if it is connected and degree of each vertex is even, we conclue that K_n is an Eulerian if and only if n is odd.

For example, K_5 the complete graph of 5 vertices is connected graph and degree of each vertex is $5-1 = 4$. Therefore K_5 is Eulerian.

Example 31. Which complete graph $K_{m,n}$ are Euler graphs?

Solution. The complete graph $K_{m,n}$ is an Euler graph if and only if both m and n are even. For example, in the complete graph $K_{2,4}$ all the vertices are of even degree (two vertices are of degree 4 and 4 vertices are of degree 2). Hence graph $K_{2,4}$ is an Euler graph. But each vertex in the graph $K_{2,3}$ is not of even degree (two vertices are of degree 3 and three vertices are of degree 2). Hence the graph is not Euler graph.

Fleury's Algorithm

Let $G = (V, E)$ be an Eulerian connected graph with each vertex of even degree. the following steps construct an Eulerian circuit.

Step 1: Select any vertex u from V as the starting vertex of Euler circuit π. Initialize π to u.

Step 2: Select an edge $e = (u,v)$. If there are many such edges, select one that is not a bridge. Extend the path π to πv and set $E = E-\{e\}$. If e is a bridge (select only if there is no alternative) then set $V = V-\{u\}$. Now from vertex v proceed further.

Step 3. Repeat step 2 until $E = \phi$.

Note. The same algorithm can be used to find an Euler path in a graph with a modification in step 1. The choice of selection of staring vertex is limited to one of the two odd degree vertices. If all the vertices are of even degree no modification is required.

Example 32. Using Fleury's algorithm, find Euler circuit in the graph of Fig. 14.42.

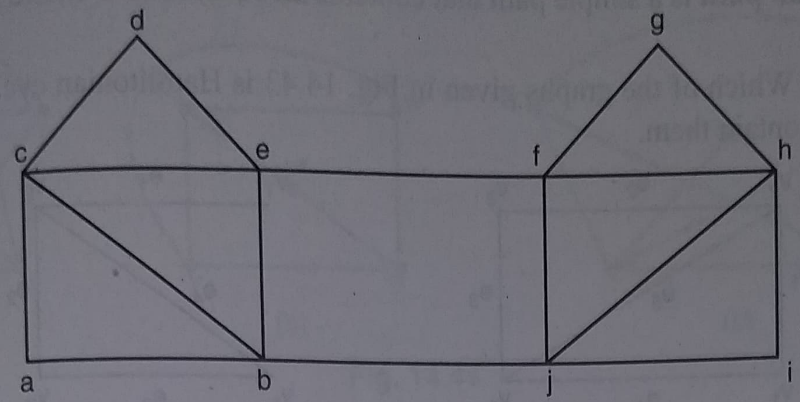


Fig. 14.42

Solution. The degrees of all the vertices are even. There exists an Euler circuit in it.

Current Path	Next Edge	Remark
$\pi : a$	(a, j)	No edge from a is a bridge. Choose (a, j). Add j to π and remove (a, j) from E.
$\pi : aj$	(j, f)	No edge from j is a bridge. Choose (j, f). Add f to π and remove (j, f) from E.
π : ajf	(f, g)	(f, e) is a bridge and (f, g) is not a bridge. Other option (f, h)
π : ajfg	(g, h)	(g, h) is the only edge
π : ajfgh	(h, i)	(h, i) is the other option
π : ajfghi	(i, j)	(i, j) is the only edge
π : ajfghij	(j, h)	(j, h) is the only edge
π : ajfghijh	(h, f)	(h, f) is the only edge
π : ajfghijhf	(f, e)	(f, e) is the only edge
π : ajfghijhfe	(e, d)	Other options are (e, c), (e, a)
π : ajfghijhfed	(d, c)	(d, c) is the only option
π : ajfghijhfedc	(c, b)	Other options are (c, e), (c, a)
π : ajfghijhfedcb	(b, a)	(b,a) is the only option
π : ajfghijhfedcba	(a, c)	Other options are (a, e)
π : ajfghijhfedcbac	(c, e)	(c, e) is the only option
π : ajfghijhfedcbace	(e, a)	(e, a) is the only option
π : ajfghijhfedcbacea	No edge is remaining in E This is the Euler circuit.	

Hamiltonian Graphs

Hamiltonian graphs are named after Sir William Hamilton, an Irish mathematician who introduced the problems of finding a circuit in which all vetices of a graph appear exactly once.

A circuit in a graph G that contains each vertex in G exactly once, except for the starting and ending vertex that appears twice is known as **Hamiltonian cycle.**

A graph G is called a **Hamiltonian cycle** if it contains a Hamiltonian cycle.

A Hamiltonian path is a simple path that contains all vertices of G where the end points may be distinct.

Example 33. Which of the graphs given in Fig. 14.43 is Hamiltonian cycle. Give the cycles on the graphs that contain them.

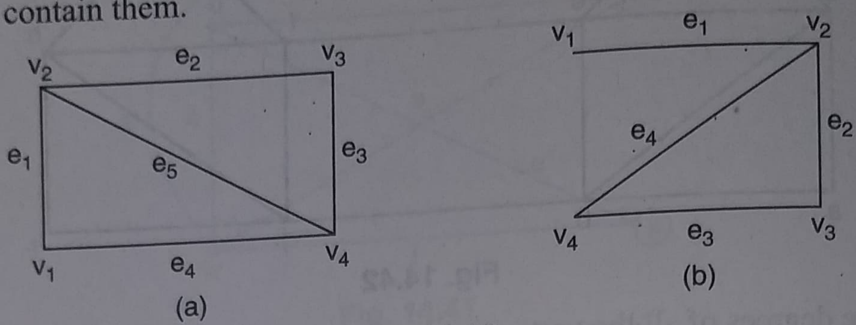


Fig. 14.43

Solution. The graph shown in Fig. 14.43 (*a*) has Hamilitonian cycle given by $v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_1$. Note that all vertices appear in this a cycle but not all edges. The edge e_5 is not used in the cycle.

The graph shown in Fig.14.43 (*b*) does not contain Hamiltonian cycle since every cycle containing every vertex must contain the e_1 twice. But the graph does have a Hamiltonian path $v_1 - v_2 - v_3 - v_4$.

There are no known simple necessary and sufficient criteria for the existence of Hamilton cycle nor is there even an efficient algorithm. For finding such a cycle many theorems exist that establish either necessary or sufficient conditions for a connected graph to have Hamilton cycle or path.

Theorem 14.15 (Dirac's Theorem) A simple connected graph G with $n \geq 3$ vertices is Hamiltonian if deg $(v) \geq n / 2$ for every vertex v in G.

The graph K_5 in Fig. 14.44 (*a*) has Hamiltonian circuit. It has deg $(v) = 4$ for each v and also have $V(G) = 5$, so it satisfies the condition of the theorem.

The graph in Fig. 14.44 (*b*) has $n = |V(G)| = 5$ and has a vertex of degree 2. So it does not satisfy the condition of the theorem but nevertheless Hamiltonian. The next two theorems provides other sufficient conditions for graph to be Hamiltonian.

Theorem 14.16 A simple connected graph with n vertices and m edges is Hamiltonian if $m \geq [(n-1)(n-2)]/2 + 2$.

The Hamiltonian graph of Fig. 14.44 (*c*) has $n = 5$, which gives $[(n-1)(n-2)]/2 + 2 = 8$ and so it satisfies the condition and hence the conclusion of the theorem.

The Hamiltonian graph of Fig. 14.44 (*b*) also has $n = 5$ but it has only 7 edges. Hence it fails to satisfy the condition of the theorem 14.11, as well as theorem 14.10 but it has Hamiltonian circuit. The graph satisfies the condition of the next theorem.

Theorem 14.17 (Ore's Theorem) Let G be a single connected graph with $n \geq 3$ vertices. If

$$\deg(v) + \deg(w) \geq n$$

for each vertices v and w not connected by an edge *i.e.* for every pair of non-adjacent vertices of v and w, then G is Hamiltonian.

For the graph in Fig. 14.44 (*b*), $n = 5$. There are three pairs of distinct vertices that are not connected by an edge. We verify the hypotheses of Theorem 14.12 by examining them:

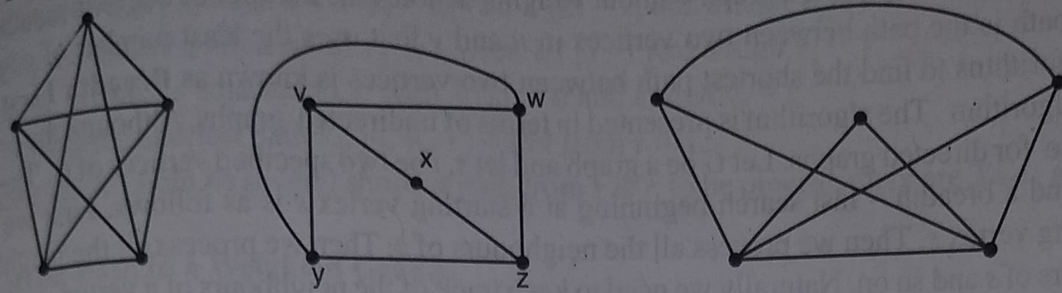
for $\langle v, z\rangle$ $\quad \deg(v) + \deg(z) = 3 + 3 = 6 \geq 5$

for $\langle w, x\rangle$ $\quad \deg(w) + \deg(x) = 3 + 2 = 5 \geq 5$

for $\langle x, y\rangle$ $\quad \deg(x) + \deg(y) = 2 + 3 = 5 \geq 5$

It may be noted that all the theorem stated above provide only sufficient criterion. They are not necessary.

A few helpful hints for trying to find a Hamilton cycle in a graph $G = (V, E)$ is given below :



(a) (b) (c)

Fig. 14.44

1. If G has Hamilton cycle, then for all $u \in V$, deg $(u) \geq 2$.

2. If $a \in V$ and deg $(a) = 2$, then the two edges incident with vertex a must appear in every Hamilton circuit for G.

3. If $a \in V$ and deg $(a) > 2$, then as we try to build a Hamilton cycle, once we pass through vertex a, any unused edges incident with a are deleted from further consideration.

A circular graph has both Hamiltonian path and cycle. The complete graph $K_n (n \geq 3)$ is Hamiltonian. The complete bipartite graph $K_{m,n}$ is Hamiltonion if and only if $m = n$ and $n > 1$. $Q_n (n \geq 2)$ and W_n $(n \geq 3)$ has a Hamiltonian cycle.

Example 34. Give an example of a graph which contains

(*i*) an Eulerian circuit that is also a Hamiltonian cycle.

(*ii*) an Eulerian circuit and a Hamiltonian cycle that are distinct.

(*iii*) an Eulerian circuit, but not a Hamiltonian cycle.

(*iv*) A Hamiltonian cycle but not an Eulerian circuit.

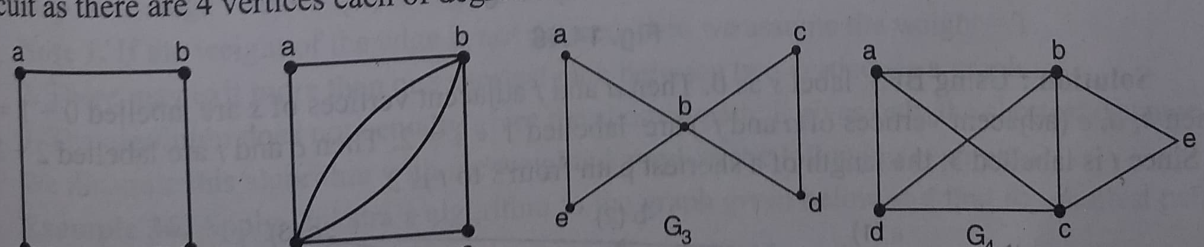
Solution. (*i*) The circuit $a - b - c - d - a$ in G_1 consists all edges, hence it is Eulerian circuit. The cycle $a - b - c - d - a$ also consists all vertices of G_1, hence it is Hamiltonian cycle.

Hence G_1 contains both Eulerian circuit and Hamiltonian cycle.

(*ii*) The circuit $a - b - d - b - c - d - a$ in G_2 consists all edges, hence it is Eulerian circuit. The cycle $a - b - c - d - a$ consists all vertices of G_2, hence it is Hamiltonian cycle. But they are different.

(*iii*) The circuit $a - b - c - d - b - e - a$ is G_3 consists all edges, hence it is an Eulerian circuit. But G_3 does not contain any Hamiltonian cycle.

(*iv*) G_4 contains $d - a - b - e - c - a$ as Hamiltonian cycle. But it does not contain any Eulerian circuit as there are 4 vertices each of degree 3.



(*i*) (*ii*) (*iii*) (*iv*)

Fig. 14.45

14.10. Shortest Path Problems

Two algorithms related to shortest path problems one in a graph without weights and another in a weighted graphs are discussed here.

Shortest Path in a Graph without Weights

The length of a path in a graph without weights denotes the number of edges in a path and the shortest path is the path between two vertices in u and v that uses the least number of edges. One of the algorithms to find the shortest path between two vertices is known as **Breadth First Search** (B F S) algorithm. The algorithm is presented in terms of undirected graphs. Although the procedure works also for directed graphs. Let G be a graph and let s, t be two specified vertices of G. The general idea behind a breadth – fast search beginning at a starting vertex s is as follows. First we process the starting vertex s. Then we process all the neighbours of s. Then we process all the neighbours of neighbours of s and so on. Naturally we need to keep track of the neighbours of a vertex, and we need to guarantee that no vertex is processed twice. The algorithm involves assigning labels to vertices.

Algorithm : The Breadth First Search Algorithm (BFS)

Step 1: Label vertex s with 0, set $i = 0$.

Step 2: Find all unlabeled vertices in G which are adjacent to vertices labelled i. If there are no such vertices then i is not connected to S.

If there are such vertices, label them $i + 1$.

Step 3: If t is labelled,go to step 4. If not, increase i to $i + 1$ and go to step 2.

Step 4: The length of a shortest path from s to t is $i + 1$. Stop.

Once the length of the shortest path is found from the previous algorithm, we use the Backtracking algorithm to find the actual shortest path from s to t. This algorithm uses the label λ (v) which are generated in the BFS algorithm.

Algorithm: The back-tracking Algorithm for a Shortest Path

Step 1: Set $i = \lambda\ (t)$ and assign $v_i = t$

Step 2: Find a vertex u adjacent to v_t and with $\lambda(u) = i - 1$. Assign $v_{i-1} = u$.

Step 3: If $i = 1$ stop.

If not, decrease, i to $i - 1$ and go to step 2.

In general, there may be many shortest paths from s to t and the previous algorithm finds just one of them.

Example 35. Find the shortest path from vertex s to t and its length from the graph given below.

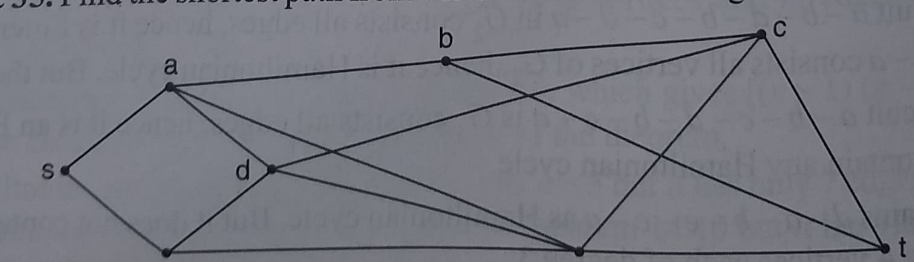
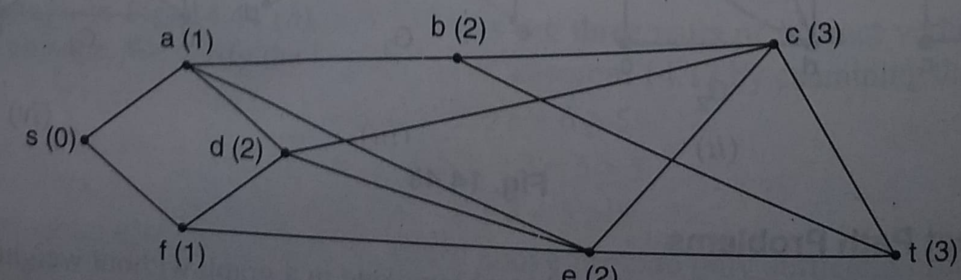


Fig. 14.46

Solution : Using BFS, label s as 0. Then a and f adjacent vertices of s are labelled $0 + 1 = 1$. Then b, d, e (adjacent vertices of a and f) are labelled $1 + 1 = 2$. Then c and t are labelled $2 + 1 = 3$. Since t is labelled 3, the length of a shortest path from s to t is 3.



Now, using second algorithm, since $\lambda(t) = 3$. We start with $i = 3$ and $v_i = t$.

We choose e (or b) adjacent to $v_3 = t$, with $\lambda(e) = 2$, and assign $v_2 = e$.

Next, we choose f adjacent to $v_2 = e$ with $\lambda(f) = 1$ and assign $v_1 = f$

Finally, we take s adjacent to f with $\lambda(s) = 0$ and assign $v_0 = s$.

This gives the shortest path $v_0, v_1, v_2, v_3 = sfet$ from s to t.

Note : There could be several shortest path from s to t, to be precise, there are 3 *i.e.*, *saet*, *sabt* and *sfet*.

Shortest Path in a Weighted Graph

Many problems can be modelled using graphs with weights assigned to their edges, Graphs that have a number assigned to each edge are called **weighted graphs**. The number may represent mileage cost, computer time or some other quantity.The length of a path in a weighted graph is the sum of the weights of the edges of this path and the shortest path between the two vertices is the minimum length of the path. There are several different algorithms that find the shortest path between two vertices in a weighted graph. We discuss here one discovered by E.W. Dijkstra.

The algorithm involves assigning labels to vertices. Let $G = (V, E)$ be a connected weighted graph. Let a and z be any two vertices where a be the starting point and z be the terminal point. Let L (v) denotes the labels at vertex v. At any point some vertices have temporary labels and the rest have permanent labels. It begins with by labelling starting vertex a, say, by 0 and other vertices with ∞. We next label all neighbours v of a by L (v), is the weight of the edge from a to v. Let u be the vertex among those v for which $L(u)$ is minimum. Now find those neighbours of w of u and, for those w not already premanently labelled assign the label $L(w) = L(u) + w(e)$, $w(e)$ being the weight of the edge from u to w, while for those w already labelled $L(w)$ change the label to $L(u) + w(e)$ if this is smaller. Each iteration of the algorithm changes the status of one label from temporary to permanent, thus we may terminate the algorithm when z receives a permanent label.

Dijkstra's Algorithm

Input: A connected weighted graph G. If G has self-loops, delete them. If G has parallel edges two vertices, delete all except that having least weight.

Output: $L(z)$, the length of shortest distance from a to z

Step 1: Initially set the starting vertex a permanently with 0 *i.e.* $L(a) = 0$ and set $L(v) = \infty$ for all vertices $v \neq a$.

T = {vertices having temporary labels of G}

Step 2: Let u be a vertex in T for which $L(u)$ is minimum and hence the permanent label of u.

Step 3: If $u = z$, stop.

Step 4: For every edge $e = (u, v)$, incident with u, if $v \in T$, change $L(v)$ to min $(L(v), L(u) + w(e))$.

Step 5: Change T to $T - \{u\}$ and go to step 2.

Note 1. If the weight of the edge is not defined, then we assume the weight = 1.

2. There may exit more than one shortest path between two vertices in a graph.

3. The algorithm does not actually gives the shortest path, it gives only the shortest distance.

We illustrate this algorithm with the weighted graph in the following example.

Example 36. Apply Dijkstra's algorithm to the graph given below and find the shortest path from a to f

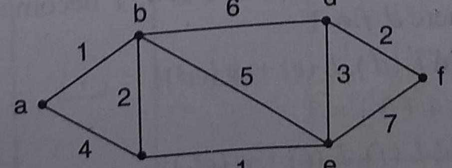


Fig. 14.48

Solution. The intial labelling is given by

Vertex V	a	b	c	d	e	f
$L(v)$	0	∞	∞	∞	∞	∞
T	$\{a,$	$b,$	$c,$	$d,$	$e,$	$f\}$

Iteration 1: $u = a$ has $L(u) = 0$. T becomes $T - \{a\}$. There are two edges incident with a i.e. ab and ac where b and $C \in T$.

$$L(b) = \min \{\text{old } L(b), L(a) + w(ab)\}$$
$$= \min \{\infty, 0 + 1\} = 1$$
$$L(c) = \min \{\text{old } L(c), L(a) + w(ac)\}$$
$$= \min \{\infty, 0 + 4\} = 4$$

Hence minimum label is $L(b) = 1$

Vertex V	a	b	c	d	e	f
$L(v)$	0	1	4	∞	∞	∞
T	$\{$	$b,$	$c,$	$d,$	$e,$	$f\}$

Iteration 2: $u = b$, the permanent label of b is 1. T becomes $T - \{b\}$ there are three edges incident with b i.e. bc, bd and be where $c, d, e \in T$.

$$L(c) = \min \{\text{old } L(c)\ L(b) + w(bc)\}$$
$$= \min \{4, 1 + 2\} = 3$$
$$L(d) = \min \{\text{old } L(d), L(b) + w(bd)\}$$
$$= \min \{\infty, 1 + 6\} = 7$$
$$L(e) = \min \{\text{old } L(e),\ L(b) + w(be)\}$$
$$= \min \{\infty, 1 + 5\} = 6$$

Vertex V	a	b	c	d	e	f
$L(v)$	0	1	3	7	6	∞
T	$\{$		$c,$	$d,$	$e,$	$f\}$

Thus minimum label is $L(c) = 3$

Itreation 3 : $u = c$, the permanent label of c is 3, T becomes $T - \{c\}$. There is one edge incident with c i.e. ce where $e \in T$.

$$L(e) = \min \{\text{old } L(e), L(c) + w(ce)\}$$
$$= \min \{6, 3 + 1\} = 4$$

Thus minimum label is $L(e) = 4$

Vertex V	a	b	c	d	e	f
L(v)	0	1	3	7	4	∞
T	$\{$			$d,$	$e,$	$f\}$

Iteration 4 : $u = e$, the permanent label of e is 4, T becomes $T - \{e\}$. There are two edges incident with e i.e. ed and ef where $d, f \in T$

$$L(d) = \min \{\text{old } L(d), L(e) + w(ed)\}$$
$$= \min \{7, 4 + 3\} = 7$$
$$L(f) = \min \{\text{old } L(f), L(e) + w(ef)\}$$
$$= \min \{\alpha, 4 + 7\} = 11$$

Vertex V	a	b	c	d	e	f
$L(v)$	0	1	3	7	4	11
T	{			d,		f}

Thus minimum label is $L(d) = 7$.

Iteration 5 : $u = d$. The permanent label of d is 7. T becomes $T - \{d\}$. There is one edge incident with d i.e. df where $f \in T$

$$L(f) = \min \{\text{old } L(f), L(d) + w(df)\} = \min \{11, 7+2\} = 9$$

Vertex V	a	b	c	d	e	f
$L(v)$	0	1	3	7	4	9
T	{					f}

The minimum label is $L(f) = 9$.

Since $u = f$, the only choice, iteration stops. Thus the shortest distance between a and f is 9.

The steps in the above algorithm can be presented in a table as given below. The permanent lables are enclosed in a square □ and the most recently assigned permanent label is indicated by a tick □✓. The shortest path càn be obtained by backward technique.

Table – Labelling of vertices for shortest path

a	b	c	d	e	f	Reasons
$\boxed{0}$ ✓	∞	∞	∞	∞	∞	Starting vertex a is permanently labelled 0, others are temporarily labelled ∞.
$\boxed{0}$	min (∞, 0 + 1) = $\boxed{1}$ ✓	min (∞, 0 + 4) = 4	∞	∞	∞	There are two edges incident with $a \to ab$ and ac, b is permanently labelled 1 since it is minimum in this row.
$\boxed{0}$	$\boxed{1}$	min (4, 1 + 2) = $\boxed{3}$	min (∞, 1 + 6) = 7	min (∞, 1 + 5) = 6	∞	There are three edges incident with $b \to bc$, bd and be, c is permanently labelled 3 since it is minimum in this row.
$\boxed{0}$	$\boxed{1}$	$\boxed{3}$	7	min (6, 3 + 1) = $\boxed{4}$ ✓	∞	There is one edge incident with $c \to ce$. e is permanently labelled 4 since it is minimum in this row.
$\boxed{0}$	$\boxed{1}$	$\boxed{3}$	min (7, 4 + 3) = $\boxed{7}$ ✓	$\boxed{4}$	min (∞, 4 + 7) = 11	There are two edges incident with $e \to ed$ and ef. d is permanently labelled 7.
$\boxed{0}$	$\boxed{1}$	$\boxed{3}$	$\boxed{7}$	$\boxed{4}$	min (11, 7 + 2) = $\boxed{9}$ ✓	There is one edge incident with $d \to df$, f is permanently labelled 9. The algorithm stops because f is the final vertex.

Since the permanent label of final vertex (f) is 9, the shortest distance between a and f is 9.

To find the shortest path

We start with permanent label of final vertex f (*i.e.*, $\boxed{9}$) and go backwards along the previous assigned temporary labels of f until there is a change in label. We reach the temporary label in the 5th row. In the row the newly assigned permanent label is $\boxed{7}$ to d. Again we go backward until there is a change in label. We reach the temporary label ∞ in the second row. In that the newly generated permanent label is $\boxed{1}$ to b. We go backward until there is a change in label. We reach temporary label ∞ in the first row and find the newly generated permanent number in that row is to a. Hence the shortest path is

$$a \xrightarrow{1} b \xrightarrow{6} d \xrightarrow{2} f$$

Example 37 : Determine a shortest path between the vertices a to z as shown below.

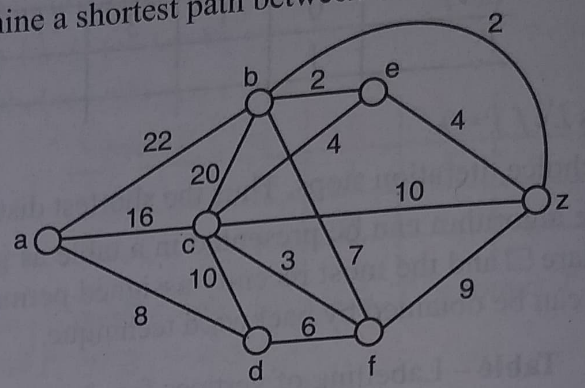


Fig. 14.49

Solution. The initial labelling is given by

Vertex V	a	b	c	d	e	f	z
$L(v)$	0	∞	∞	∞	∞	∞	∞
T	$\{a,$	$b,$	$c,$	$d,$	$e,$	$f,$	$z\}$

Iteration 1: $u = a$ has $L(u) = 0$. T becomes $T - \{a\}$. There are three edges incident with i.e. ab, ac, and ad where $b, c, d, \in T$.

$$L(b) = \min \{\text{old } L(b), L(a) + w(ab)\}$$
$$= \min \{\infty, 0 + 22\} = 22$$
$$L(c) = \min \{\text{old } L(c), L(a) + w(ac)\}$$
$$= \min \{\infty, 0 + 16\} = 16$$
$$L(d) = \min \{\text{old } L(d), L(a) + w(ad)\}$$
$$= \min \{\infty, 0 + 8\} = 8$$

Hence minimum label is $L(d) = 8$

Vertex V	a	b	c	d	e	f	z
$L(v)$	0	22	16	8	∞	∞	∞
T	$\{$	$b,$	$c,$	$d,$	$e,$	$f,$	$z\}$

Iteration 2 : $u = d$, the permenent label of d is 8. T becomes $T - \{d\}$. There are two edges incident with d i.e. dc and df where $c, f \in T$.

$$L(c) = \min \{\text{old } L(c), L(d) + w(dc)\}$$
$$= \min \{16, 8 + 10\} = 16$$
$$L(f) = \min \{\text{old } L(f), L(d) + w(df)\}$$
$$= \min \{\infty, 8 + 6\} = 14$$

Hence minimum label is $L(f) = 14$.

Vertex V	a	b	c	d	e	f	z
$L(v)$	0	22	16	8	∞	14	∞
T	{	b,	c,		e,	f,	z}

Iteration 3 : $u = f$, the permanent label of is 14. T becomes $T - \{f\}$. There are three edges incident with f *i.e.* fc, fb and fz where $c, b, z \in T$.

$$L(c) = \min\{\text{old } L(c), L(f) + w(fc)\}$$
$$= \min\{16, 14 + 3\} = 16$$
$$L(b) = \min\{\text{old } L(b), L(f) + w(fb)\}$$
$$= \min\{22, 14 + 7\} = 21$$
$$L(z) = \min\{\text{old } L(z), L(f) + w(fz)\}$$
$$= \min\{\infty, 14 + 9\} = 23$$

Hence minimum label is $L(c) = 16$.

Vertex V	a	b	c	d	e	f	z
$L(v)$	0	21	16	8	∞	14	23
T	{	b,	c,		e,		z}

Iteration 4 : $u = c$, the permanent label of is 16. T becomes $T - \{c\}$. There are three edges incident with c *i.e.* cb, ce and cz where $b, e, z \in T$.

$$L(b) = \min\{\text{old } L(b), L(c) + w(cb)\}$$
$$= \min\{21, 16 + 20\} = 21$$
$$L(e) = \min\{\text{old } L(e), L(c) + w(ce)\}$$
$$= \min\{\alpha, 16 + 4\} = 20$$
$$L(z) = \min\{\text{old } L(z), L(c) + w(cz)\}$$
$$= \min\{23, 16 + 10\} = 23$$

Hence minimum label is $L(e) = 20$.

Vertex V	a	b	c	d	e	f	z
$L(v)$	0	21	16	8	20	14	23
T		b,			e,		z

Iteration 5: $u = e$, the permanent label of e is 20. T becomes $T - \{e\}$. There are two edges incident with e, *i.e.* eb and ez where $b, z \in T$.

$$L(b) = \min\{\text{old } L(b), L(e) + w(eb)\}$$
$$= \min\{21, 20 + 2\} = 21$$
$$L(z) = \min\{\text{old } L(z), L(e) + w(ez)\}$$
$$= \min\{23, 20 + 4\} = 23$$

Hence minimum label is $L(b) = 21$.

Vertex V	a	b	c	d	e	f	z
$L(v)$	0	21	16	8	20	14	23
T		b,					z

Iteration 6 : $u = b$, the permanent label of b is 21. T becomes $T - \{b\}$. There is one edges incident with b *i.e.* bz where $z \in T$.

$$L(z) = \min \{\text{old } L(z), L(b) + w(bz)\}$$
$$= \min \{23, 21 + 4\} = 23$$

Hence minimum label is $L(z) = 23$.

Vertex V	a	b	c	d	e	f	z
$L(v)$	0	21	16	8	20	14	23
T.	{						z}

Since $u = z$, the only choice, iteration stops. Thus the length of the shortest path is 23. The steps in the above algorithm can be presented is the following table.

Table : Labelling of vertices for shortest path

a	b	c	d	e	f	z	Reasons
$\boxed{0}$ ✓	∞	∞	∞	∞	∞	∞	Starting vertex a is permanently labelled 0, others are temporarily labelled ∞.
$\boxed{0}$	min $(\infty, 0 + 22) = 22$	min $(\infty, 0 + 16) = 16$	min $(\infty, 0 + 8) = \boxed{8}$ ✓	∞	∞	∞	There are three edges incident with $a \to ab, ac, ad$. d is permanently labelled 8. It is minimum among all temporarily label.
$\boxed{0}$	22	min $(16, 8 + 10) = 16$	$\boxed{8}$	∞	min $(\infty, 8 + 6) = \boxed{14}$ ✓	∞	There are two edges in incident with $d \to dc, df$. f is permanently labelled 14.
$\boxed{0}$	min $(22, 14 + 7) = 21$	min $(16, 14 + 3) = \boxed{16}$	$\boxed{8}$	∞	$\boxed{14}$	min $(\infty, 14 + 9) = 23$	There are three edges incident with $f \to fc, fb, fz$. c is permanently labelled 16.
$\boxed{0}$	min $(21, 16 + 20) = 21$	$\boxed{16}$	$\boxed{8}$	min $(\infty, 16 + 4) = \boxed{20}$ ✓	$\boxed{14}$	min $(23, 16 + 10) = 23$	There are three edges incident with $c \to cb, ce, cz$. e is permanently labelled 20.
$\boxed{0}$	min $(21, 20 + 2) = \boxed{21}$ ✓	$\boxed{16}$	$\boxed{8}$	$\boxed{20}$	$\boxed{14}$	min $(23, 20 + 4) = 23$	There are two edges incident with $e \to eb, ez$. b is permanently labelled 21.

$\boxed{0}$	$\boxed{21}$	$\boxed{16}$	$\boxed{8}$	$\boxed{20}$	$\boxed{14}$	min (23, 21 + 4) = $\boxed{23}$ ✓	There is only one edge incident with $b \to bz$. z is permanently labelled 23.

Since the permanent label of final vertex z is 23, the shortest distance between a and z is 23.

To find out shortest path.

We start with permanent label of final vertex z (*i.e.*, [23]) and go backward along the previously assigned temporary labels of z until there is a change in label. We reach the temporary label in the 3rd row.

In that row, the newly assigned permanent label is 14 to f. Following the same procedure, the newly assigned permanent label in the 2nd row is 8 to d and then to the starting vertex a. Hence the shortest path is

$$a \xrightarrow{8} d \xrightarrow{6} f \xrightarrow{9} z$$

14.11. Representation of Graphs

Although a diagrammatic representation of a graph is very convenient for a visual study but this is only possible when the number of vertices and edges is reasonably small. Two types of representation are given below.

Matrix Representation. The matrix are commonly used to represent graphs for computer processing. The advantages of representing the graph in matrix form lies on the fact that many results of matrix algebra can be readily applied to study the structural properties of graphs from an algebraic point of view. There are number of matrices which one can associate with any graph. We shall discuss adjacency matrix and the incidence matrix.

Adjacency Matrix

(*a*) Representation of Undirected Graph

The adjacency matrix of an undirected graph G with n vertices and no parallel edges is an n by n matrix $A = [a_{ij}]$ whose elements are given by

$a_{ij} = 1$, if there is an edge between ith and jth vertices, and

$= 0$, if there is no edge between them.

Observations

(*i*) A is symmetric *i.e.* $a_{ij} = a_{ji}$ for all i and j.

(*ii*) The entries along the principal diagonal of A all 0's if and only if the graph has no self loops. A self loop at the vertex corresponds to $a_{ii} = 1$.

(*iii*) If the graph is simple (no self loop, no parallel edges), the degree of vertex equals the number of 1's in the corresponding row or column of A.

(*iv*) The (i, j) entry of A^m is the number of paths of length (no. of occurence of edges) m from vertex v_i to vertex v_j.

(*v*) If G be a graph with n vertices $v_1, v_2, \ldots, v_n$ and let A denote the adjacency matrix of G with respect to this listing of the vertices. Let B be the matrix.

$$B = A + A^2 + A^3 + \ldots + A^n \ (n > 1)$$

Then G is a connected graph iff B has no zero entries.

This result can be used to check the connectedness of a graph by using its adjacency matrix.

Adjacency can also be used to represent undirected graphs with loops and multiple edges. A loop at the vertex v_i must have the element a_{ii} equal to 1 in the adjacency matrix. When multiple edges are present, the adjacency matrix is no longer a zero - one matrix, since the (i, j) th entry equals the number of edges that are associated between v_i and v_j. All undirected graphs, including multigraphs and pseudographs, have symmetric adjacency matrices.

(*b*) Representation of Directed Graph

The adjacency matrix of a diagraph D, with n vertices is the matrix $A = [a_{ij}]_{n \times n}$ in which

$$a_{ij} = 1 \text{ if arc } (v_i, v_j) \text{ is in } D$$
$$= 0 \text{ otherwise.}$$

Observations

(*i*) A is not necessary symmetric, since there may not be an edge from v_i to v_j when there is an edge from v_j to v_i

(*ii*) The sum of any column of j of A is equal to the number of arcs directed towards v_j.

(*iii*) The sum of entries in row i is equal to the number of arcs directed away from vertex v_i (out degree of vertex v_i)

(*iv*) The (i, j) entry of A^m is equal to the number of path of length m from vertex v_i to vertex v_j.

(*v*) The diagonal elements of $A . A^T$ show that out degree of the vertices. The diagonal entries of $A^T . A$ shows the in degree of the vertices.

The adjacency matrices can also be used to represent directed multigraphs. Again such matrices are not zero – one matrices when there are multiple edges in the same direction connecting two vertices. In the adjacency matrix for a directed multigraph, a_{ij} equals the number of edges that are associated to (v_i, v_j).

Example 38 : Use adjacency matrix to represent the graphs shown in Fig. 14.50.

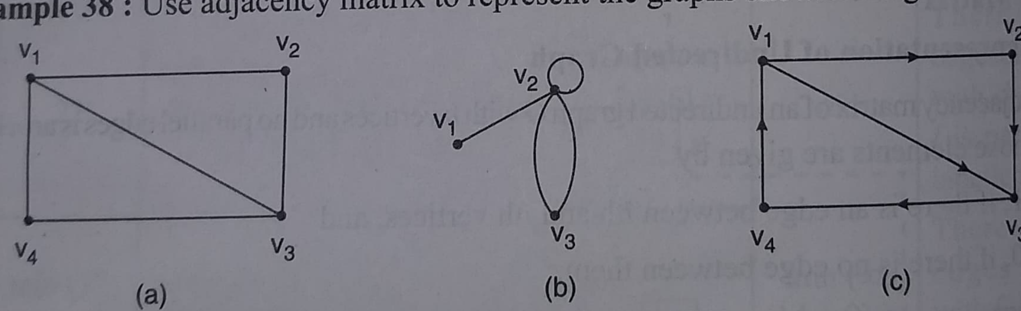


Fig. 14.50

Solution. We order the vertices in Fig 14.51 (a) as v_1, v_2, v_3 and v_4. Since there are four vertices, , the adjacency matrix representing the graph will be a square matrix of order four. The required adjacency matrix A is

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

We order the vertices in Fig. 14.51 (*b*) as v_1, v_2 and v_3. The adjacency matrix representing the graph with loop and multiple edges is given by

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 0 \end{bmatrix} \end{array}$$

Taking the order of the vertices in Fig. 14.51 (*c*) as v_1, v_2, v_3 and v_4. The adjacency matrix representing the digraph is given by

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Example 39 : Draw the undirected graph represented by adjacency matrix A given by

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Solution. Since the given matrix is a square of order 5, the graph G has five vertices v_1, v_2, v_3, v_4 and v_5. Draw an edge from v_i to v_j where $a_{ij} = 1$. The required graph is drawn in Fig. 14.52.

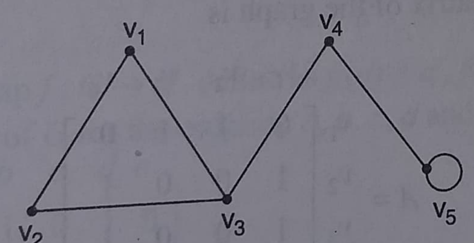


Fig. 14.51

Example 40: Draw the digraph G corresponding to adjacency matrix.

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Solution. Since the given matrix is square matrix of order four, the graph G has 4 vertices v_1, v_2, v_3 and v_4. Draw an edge from v_i to v_j where $a_{ij} = 1$. The required graph is shown in Fig.14.53.

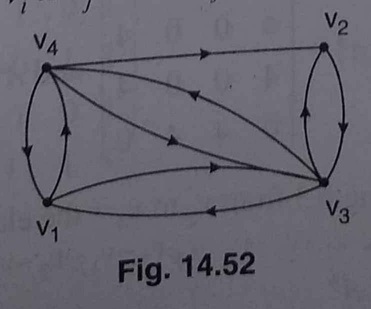


Fig. 14.52

Example 41: Draw the undirected graph G corresponding to adjacency matrix.

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

Solution. Since the given adjacency matrix is square matrix of order 4, G has four vertices v_1, v_2, v_3 and v_4. The matrix is not a zero-one matrix. Draw n edges from v_i to v_j where $a_{ij} = n$. Also draw n loop at v_i where $a_{ij} = n$. The required graph is shown in Fig.14.53.

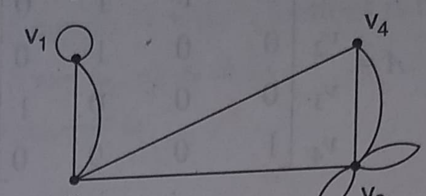


Fig. 14.53

Example 42. Consider the graph shown in Fig. 14.55. Find the number of walks of length three from v_2 to v_4 and also check the connectedness of the graph.

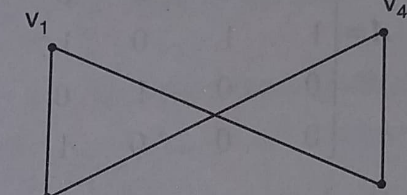


Fig. 14.54

Solution. The adjacency matrix of the graph is

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Now
$$A^2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}$$

Similarly, using matrix multiplication, we get

$$A^3 = \begin{bmatrix} 0 & 4 & 4 & 0 \\ 4 & 0 & 0 & 4 \\ 4 & 0 & 0 & 4 \\ 0 & 4 & 4 & 0 \end{bmatrix} \text{ and } A^4 = \begin{bmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix}$$

Now the number of walks of length 3 from v_2 to v_4 = the element in the (2, 4) th entry in $A^3 = 4$. The four different edge sequence are $v_2 - v_1 - v_3 - v_4, v_2 - v_1 - v_2 - v_4, v_2 - v_4 - v_2 - v_4$, and $v_2 - v_4 - v_3 - v_4$

Again $\quad B = A + A^2 + A^3 + A^4$

$$\therefore \quad B = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 4 & 4 & 0 \\ 4 & 0 & 0 & 4 \\ 4 & 0 & 0 & 4 \\ 0 & 4 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix}$$

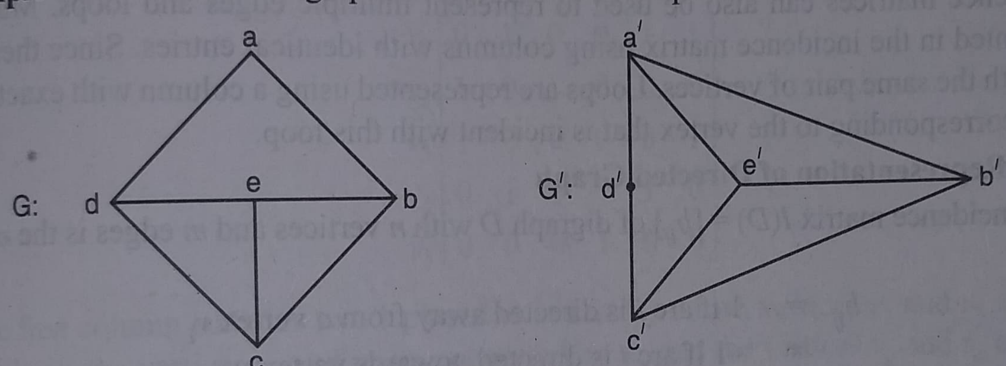
or $$B = \begin{bmatrix} 10 & 5 & 5 & 10 \\ 5 & 10 & 10 & 5 \\ 5 & 10 & 10 & 5 \\ 10 & 5 & 5 & 10 \end{bmatrix}$$ Which contains no zero element.

Hence the graph is a connected graph.

Note that one can use the adjacency matrix to check whether or not the given graphs G and G' are isomorphic. Two graphs are isomorphic if and only if their vertices can be labeled in such a way that the corresponding adjacency matrices are equal.

Theorem 14.18. Two graphs G_1 and G_2 are isomorphic if and only if the adjacency matrix of one is obtained from that of the other by interchanging rows and also columns in the same way.

Example 43. Show that the graphs G and G' are isomorphic.



Solution. Consider the map $f : G \to d'$, define as $f(a) = d', f(b) = a', f(c) = a', f(d) = c'$, and $f(e) = e'$,. The adjacency matrix of G for the ordering a, b, c, d and e is

$$A(G) = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

The adjacency matrix of G′ for the ordering d', a', b', c' and e' is

$$A(G') = \begin{array}{c} \\ d' \\ a' \\ b' \\ c' \\ e' \end{array} \begin{array}{c} \begin{array}{ccccc} d' & a' & b' & c' & e' \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

i.e. $A(G) = A(G')$

Incidence Matrix

(*a*) **Representation of Undirected Graph**

Consider a undirected graph $G = (V, E)$ which has n vertices and m edges all labelled. The incidence matrix $I(G) = [b_{ij}]$, is then $n \times m$ matrix, where

$$b_{ij} = 1 \quad \text{when edge } e_j \text{ is incident with } v_i$$
$$= 0 \quad \text{otherwise}$$

Observations

(*i*) Each column of B comprises exactly two unit entries.

(*ii*) A row with all 0 entries corresponds to an isolated vertex.

(*iii*) A row with a single unit entry corresponds to a pendant vertex.

(*iv*) The number of unit entries in row i of B is equal to the degree of the corresponding vertex v_i.

(*v*) The permutation of any two rows (any two columns) of $I(G)$ corresponds to a relabelling of the vertices (edges) of G.

(*vi*) Two graphs are isomorphic if and only if their corresponding incidence matrices differ only by a permutation of rows and columns.

(*vii*) If G is connected with n vertices then the rank of $I(G)$ is $n - 1$.

Incidence matrices can also be used to represent multiple edges and loops. Multiple edges are represented in the incidence matrix using columns with identical entries. Since these edges are incident with the same pair of vertices. Loops are represented using a column with exactly one entry equal to l, corresponding to the vertex that is incident with this loop.

(*b*) **Representation of Directed Graph**

The incidence matrix $I(D) = [b_{ij}]$ of digraph D with n vertices and m edges is the $n \times m$ matrix in which.

$$b_{ij} = 1 \text{ if arc } j \text{ is directed away from } a \text{ vertex } v_i$$
$$= -1 \text{ if arc } j \text{ is directed towards vertex } v_i$$
$$= 0 \text{ otherwise.}$$

Example 44 : Find the incidence matrix to represent the graph shown in Fig.14.55.

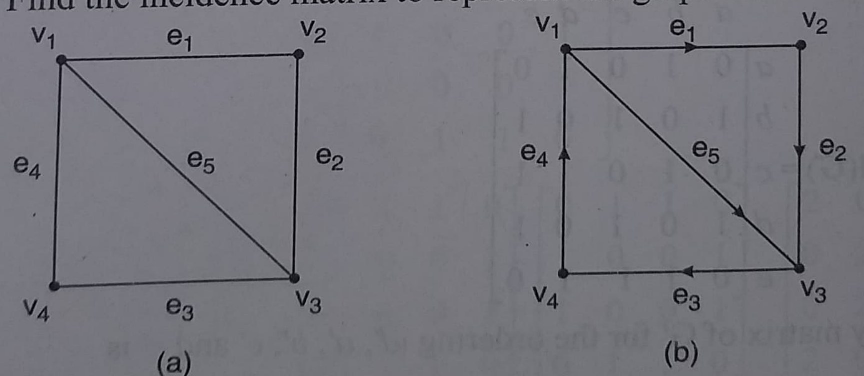


Fig. 14.55

Solution. The incidence matrix of Fig.(*a*) is obtained by entering for row v and column e is 1 if e is incident on v and 0 otherwise. The incidence matrix is.

$$I(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

The incidence matrix of the digraph of Fig.(*b*) is

$$I(D) = \begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix}$$

Example 45. Draw the graph whose incidence matrix is

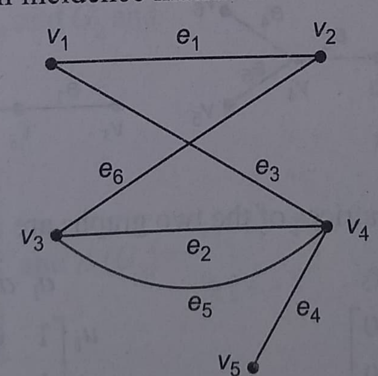
$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Solution. Since the given matrix has 5 rows and 6 columns, its corresponding graph has 5 vertices and 6 edges. We re-write the incidence matrix as follows:

$$\begin{array}{c c} & \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Since first column contains two 1 placed at 2nd and 3rd row, vertices v_2 and v_3 are connected by e_1. Similarly, the vertices v_1 and v_4 are connected by e_2, and the vertices v_2 and v_4 are connected by e_3 and so on.

Therefore, the graph of the given incidence matrix is



Example 46. Draw the graph whose incidence matrix is given below:

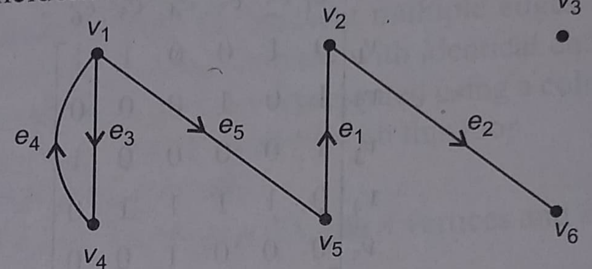
$$\begin{bmatrix} 0 & 0 & 1 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix}$$

Solution. Since the given matrix has 6 rows and 5 columns, its corresponding graph has 6 vertices and 5 edges. Since the matrix has the element – 1, it is a digraph. We rewrite the incidence matrix as follows:

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix} \end{array}$$

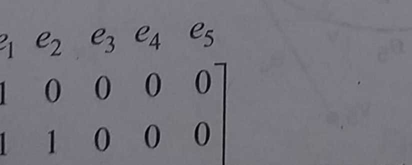
Since first column contains -1 placed at 2nd row, and 1 placed at 4th row, the edge e_1 is directed from v_4 to v_2. The edge e_2 has no direction and it is connected by v_2 and v_5, the edge e_3 has direction from v_1 to v_6, the edge e_4 from v_6 to v_1, the edge e_5 from v_1 to v_4 but the vertex v_3 is not connected to any vertex. Hence v_3, is isolated vertex and the graph is not connected.

Hence, the graph of the incidence matrix is

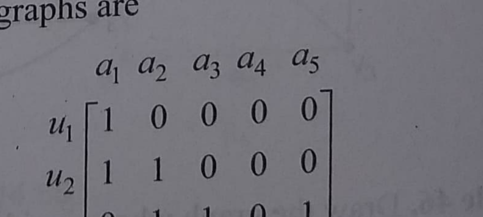


Theorem 14.19. Two graphs G_1 and G_2 are isomorphic if and only if the incidence matrix of one is obtained from that of the other by permutation of rows/columns of the matrix.

Example 47. Using incidence matrix find whether the two given graphs G_1 and G_2 are isomorphic.



G_1



G_2

Solution. The incidence matrices of the two graphs are

$$I(G_1) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \text{ and } I(G_2) = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{ccccc} a_1 & a_2 & a_3 & a_4 & a_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Now, if $I(G_2)$ can be obtained from $I(G_1)$ by interchanging rows and columns, then G_1 and G_2 are isomorphic.

We see 1st three columns of $I(G_1)$ and $I(G_2)$ are same. To make the fourth column identical we interchange 5th row and 6th row of $I(G_1)$.

and get a matrix $B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

We see first four columns of $I(G_2)$ and B are same. The only fifth column of B is not same as that of $I(G_2)$.

So, we conclude $I(G_2)$ can not be obtained from $I(G_1)$ by interchanging rows and columns. Hence the two graphs G_1 and G_2 are not isomorphic.

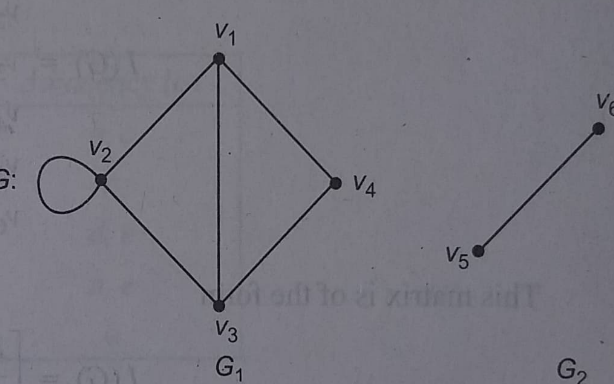
Matrix Representation of disconnected Graph and Digraphs.

1. Adjaceny Matrix: Let G be a disconnected graph (digraph) having two components G_1 and G_2. Then the adjacency matrix $A(G)$ of G can be partitioned as

$$A(G) = \left[\begin{array}{c|c} A(G_1) & O \\ \hline O & A(G_2) \end{array}\right]$$

where $A(G_1)$ and $A(G_2)$ are the adjacency matrices of G_1 and G_2 respectively and O is the null matrix.

Example 48. Find the adjacency of graph G



Solution. The adjacency matrix of a disconnected graph is given by

$A(G) = \left[\begin{array}{c|c} A(G_1) & O \\ \hline O & A(G_2) \end{array}\right]$ where $A(G_1)$ and $A(G_2)$ are the adjacency matrices of G_1 and G_2 and O is the null matrix.

$$\text{Here, } A(G_1) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array} \text{ and } A(G_2) = \begin{array}{c} \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cc} v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}.$$

Hence the required adjacency matrix is

$$A(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

2. Incidence Matrix Let G be a disconnected graph (digraph) having two components G_1 and G_2. Then the incidence matrix $I(G)$ of G can be partitioned as

$$I(G) = \left[\begin{array}{c|c} I(G_1) & O \\ \hline O & I(G_2) \end{array}\right]$$

where $I(G_1)$ and $I(G_2)$ are the incidence matrices of G_1 and G_2 respectively and O is the null matrix.

Example 49. Write down the graph corresponding to the following incidence matrix

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

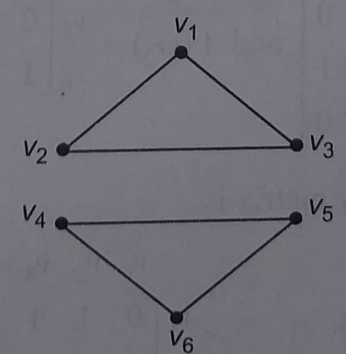
Solution. Since the given matrix has 6 rows and 6 columns, its corresponding graph has 6 vertices and 6 edges. We write the incidence matrix as follows:

$$I(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

This matrix is of the form

$$I(G) = \left[\begin{array}{c|c} I(G_1) & O \\ \hline O & I(G_2) \end{array}\right]$$

Hence the graph is disconnected. Further submatrix $I(G_1)$ and $I(G_2)$ gives components of G. They represent the following graphs.



Linked Representation

In this representation, a list of vertices adjacent to each vertex is maintained. This representation is also called adjacency structure representation. In case of a directed graph, *a* care has to be taken, according to the direction of an edge, while placing a vertex in the adjacent structure representation of another vertex. The linked representation is described by means of two examples.

Example 50. Write adjacency structure for the graphs shown in Fig.14.56.

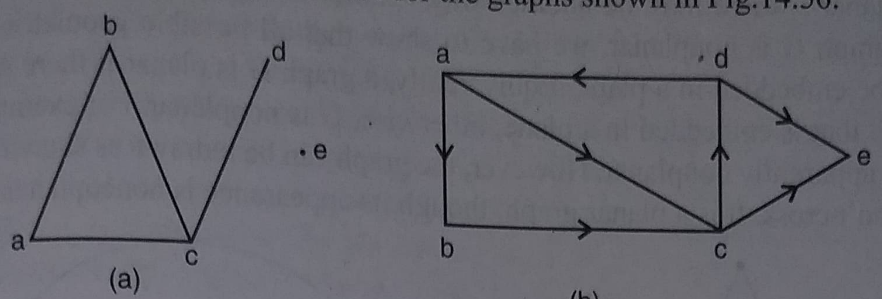


Fig. 14.56

Solution. The adjacency structure representation is given in the table for Fig 14.56(a). Here the symbol ϕ is used to denote the empty list.

Vertex	*Adjacency list*
a	b, c
b	a, c
c	a, b, d
d	e
e	ϕ

The adjacency structure representation is given in the table for the directed graph shown in Fig. 14.57 (b)

Vertex	*Adjacency list*
a	b, c
b	c
c	d, e
d	a, e
e	ϕ

14.12. Planar Graphs

A graph G is said to be planar if there exists some geometric representation of G which can be drawn on a plane such that no two of its edges intersect except only at the common vertex. The points of intersection are called **Crossovers**. A graph that cannot be drawn on a plane without a crossover between its edges is called a nonplanar graph. The graphs shown in Fig. 14.57 (a) and (b) are planar graphs.

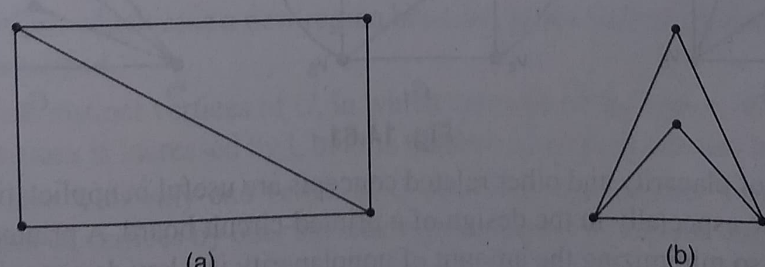
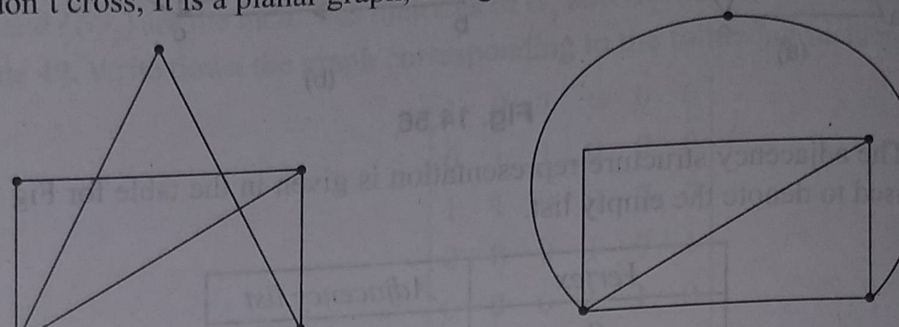


Fig. 14.57

A drawing of a geometric representation of a graph on any surface such that no edges intersect is called **embedding**. An embedding of a planar graph G on a plane is called a plane or planar representation of G. Note that if a graph G has been drawn with crossing edges. this does not mean that G is nonplanar – there may be another way to draw the graph without crossovers. Thus to declare that a graph G is nonplanar, we have to show that all possible geometric representations of G none can be embedded in a plane. Equivalently, a graph G is planar if there exists a graph G' isomorphic to G that is embedded in a plane, otherwise, G is nonplanar. For example, the graph in Fig, 14.58(a) is apparently nonplanar. However, the graph can be redrawn as shown in Fig. 14.58(b) so that edges don't cross, It is a planar graph, though its appearance is noncoplanar.



(a) (b)

Fig. 14.58

As another example the two isomorphic diagrams in Figs. 14.59 and 14.60 are different representation of the same graph. One of the diagram is a plane representation : the other one is not. The graph is planar.

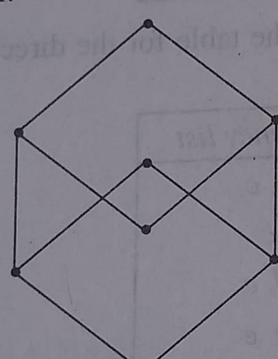


Fig. 14.59. A graph G.

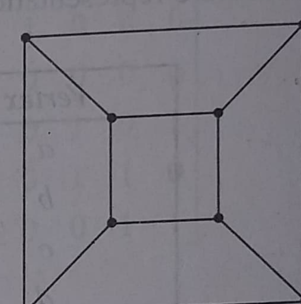


Fig. 14.60. A Planar representation of G

A graph may have more than one plane representation. For example, the graph G has two plane representations G_1 and G_2.

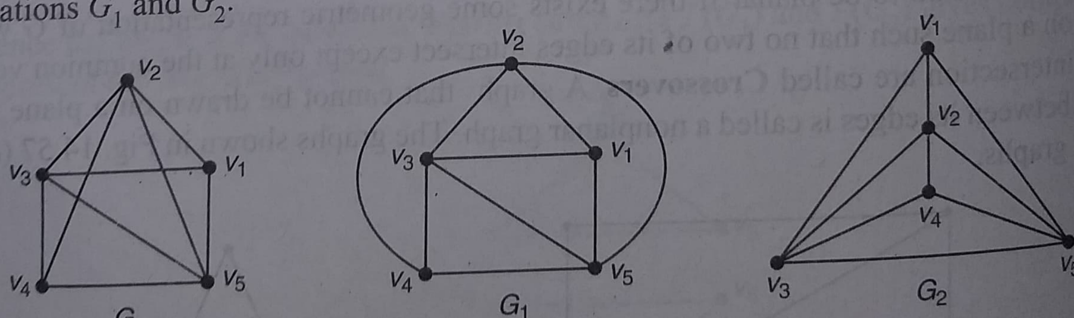


Fig. 14.61

The question of planarity and other related concepts are useful in applictation of graph theory to Computer Science especially in the design of a printed-circuit board. A printed circuit board is a planar network and so minimizing the amount of nonplanarity is a key design criterion.

Region of a Graph

A region of a planar graph is defined to be an area of the plane that is bounded by edges and is not further divided into subareas. The set of edges which bound a region of a planar graph is called its boundaries. If the area of the region is finite then the region is called a finite region. If the region

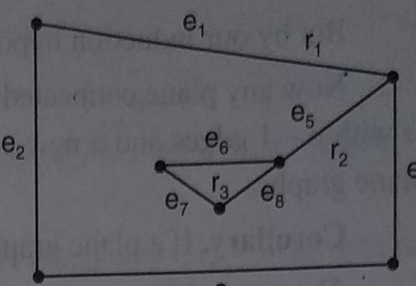


Fig. 14.62

is infinite it is called infinite, outer or unbounded region. The degree of a region of a planar graph is the number of encounters with edges in a walk round the boundary of the region. If an edge encounters twice along a boundary or a region, the edge counted twice in calculation of degree of that region. For example, the Fig. 14.62 has three regions, two are finite and one is infinite. The infinite region r_1 is characterised by the set $\{e_1, e_2, e_3, e_4\}$ and hence has degree 4, the region r_3 characterised by $\{e_6, e_7, e_8\}$ has degree 3 and region r_2 has degree 9 (note that one edge is encountered twice, once on each side).

Note: A planar graph has only one infinite region.

Euler's Formula

The basic result about the planar gaph is known as Euler's formula; Euler initially studied it in the context of polyhedra (a polyhedron is a solid bounded by a finite number of regions, each of which is a polygonal). This result is often called **Euler's polyhedron formula**, since it relates the number of vertices, edges and regions of a convex polyhedron.

Theorem 14.20. If a connected planar graph G has n vertices, e edges and r region, then $n-e+r=2$.

Proof. We prove the theorem by induction on e, number of edges of G.

Basis of induction. If $e=0$, then G must have just one vertex *i.e*, $n=1$ and one infinite region *i.e.*, $r=1$. Then $n-e+r=1-0+1=2$.

If $e=1$ (though it is not necessary), then the number of vertices of G is either 1 or 2, the first possibility of occurring when the edge is a loop. These two possibilities give rise to two regions and one region respectively, as shown in Fig. 14.63.

Fig. 14.63. Connected plane graphs with one edge.

In the case of loop, $n-e+r=1-1+2=2$ and in case of non-loop, $n-e+r=2-1+1=2$. Hence the result is true.

Induction hypothesis: Now, we suppose that the result is true for any connected plane graph G with $e-1$ edges.

Induction step: We add one new edge k to G to form a connected supergraph of G which is denoted by $G+k$. There are following three possibilities.

(*i*) k is a loop, in which case a new region bounded by the loop is created but the number of vertices remains unchanged.

(*ii*) k joins two distinct vertices of G, in which case one of the region of G is split into two, so that number of regions is increased by l, but the number of vertices remains unchanged.

(*iii*) k is incident with only one vertex of G on which case another vertex must be added, increasing the number of vertices by one, but leaving the number of regions unchanged.

If let n', e' and r' denote the number of vertices, edges and regions in G and n, e and r denote the same in $G+k$. Then

In case (*i*) $n-e+r=n'-(e'+1)+(r'+1)=n'-e'+r'$.

In case (*ii*) $n-e+r=n'-(e'+1)+(r'+1)=n'-e'+r'$.

In case (*iii*) $n-e+r=(n'+1)-(e'+1)+r'=n'-e'+r'$.

But by our induction hypothesis, $n' - e' + r' = 2$. Thus in each case $n - e + r = 2$.

Now any plane connected graph with e edges is of the form $G + k$, for some connected graph G with $e - 1$ edges and a new edge k. Hence by mathematical induction the formula is true for all plane graphs.

Corollary. If a plane graph has k components, then $n - e + r = k + 1$.

The result follows on applying Euler's formula to each component separately, remembering not to count the infinte region more than once.

Corollary. If G is connected simple planar graph with n (≥ 3) vertices and e edges, then $e \leq 3n - 6$.

Proof. Each region is bounded by at least three edges (since the graphs discussed here are simple graphs, no multiple edges that could produce regions of degree 2, or loops that could produce regions of degree l, are permitted) and edges belong to exactly two regions.

$$2e \geq 3r \text{ i.e., } 3r \leq 2e$$

If we combine this with Euler's formula, $n - e + r = 2$, we get

$3r = 6 - 3n + 3e \leq 2e$ which is equivalent to $e \leq 3n - 6$.

Corollary. If G is connected simple planar graph with n (≥ 3) vertices and e edges and no circuits of length 3, then $e \leq 2n - 4$.

Proof. If the graph is planar, then the degree of each region is at least 4. Hence the total number of edges around all the regions is at least $4r$. Since every edge borders two regions, the total number of edges around all the regions is $2e$, so we established that $2e \geq 4r$. Which is equivalent to $2r \leq e$. If we combine this with Euler's formula $n - e + r = 2$, we get

$$2r = 4 - 2n + 2e \leq e$$

which is equivalent to $e \leq 2n - 4$.

Example 51. Show that the graph K_5 **(Kuratowski's first graph)** is non-planar.

Solution. Since K_5 is a simple graph, the smallest possible length for any cycle of K_5 is three. We shall suppose that the graph is planar. The graph has 5 vertices and 10 edges so that $n = 5, e = 10$. Now $3n - 6 = 3.5 - 6 = 9 < e$. Thus the graph violates the inequility $e \leq 3n - 6$ and hence it is non-planar. It can be seen that K_n is planar for $n = 1, 2, 3$ and 4.

It may be noted that the inequality $e \leq 3n - 6$ is only a necessary condition but not a sufficient condition for the planarity of a graph. For example, graph $K_{3,3}$ satisfies the inequality because $e = 9 \leq 3.6 - 6 = 12$, yet the graph is non-planar.

Example 52. Show that the graph $K_{3,3}$ **(Kuratowski's second graph)** is non-planar.

Solution. Since $K_{3,3}$ has no circuits of length 3 (it is bipartite) and has 6 vertices and 9 edges *i,e.*, $n = 6$ and $e = 9$ so that $2n - 4 = 2.6 - 4 = 8$. Hence the inequality $e \leq 2n - 4$ does not satisfy and the graph is non-planar.

Example 53: A connected plane graph has 10 vertices each of degree 3. Into how many regions, does a representation of this planar graph split the plane ?

Solution. Here $n = 10$ and degree of each vertex is 3.

$$\Sigma \deg(v) = 3 \times 10 = 30$$

But $\Sigma \deg(v) = 2e \Rightarrow 30 = 2e \Rightarrow e \Rightarrow 15$

By Euler's formula, we have $n - e + r = 2$

$$10 - 15 + r = 2 \Rightarrow r = 7.$$

The planarity of a graph is not effected if

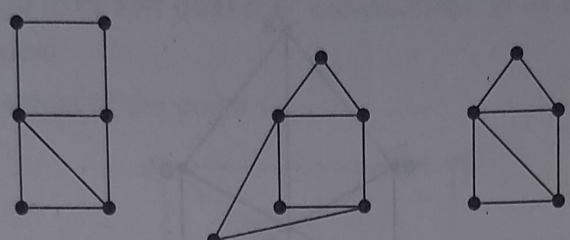
(*i*) An edge in divided into two edges by inserting a now vertex of degree 2.

(*ii*) A path of length 2 is converted into a single edge by dropping the intermediate vertex.

Such an operation is called an **elementary subdivision.**

Definition : Two graphs G_1 and G_2 are said to be **homographic** if and only if G_1 can be obtained from G_2 by insertion or deletion of one or more vertices of degree 2.

The graph G_1 and G_2 in Fig. 14.64 shown below are homeomorphic since both are obtainable from the graph G in that figure by adding a vertex to one of its edges.



G_1 G_2 G_3

Fig. 14.64

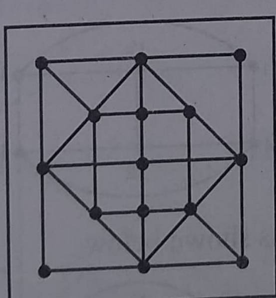
Kuratowski's Theorem

The concept of subdivision gives an important characterisation of planarity due to Kuratowski.

Theorem 14.21. A graph is a planar if and only if it does not contains any subgraph homeomorphic to K_5 or $K_{3,3}$.

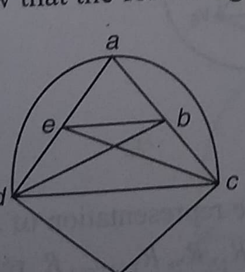
The proof of this topological result is beyond the scope of this book. In practice, theorem is not of much use, because it can be extremely difficult to identify these subgraphs. However, there are several algorithms available for testing for planarity, which can be found in more advanced textbooks.

Example 55. Determine the number of vertices, the number of edges, and the number of region in the graphs shown below. Then show that your answer satisfy Euler's throrem for connected planar graphs.

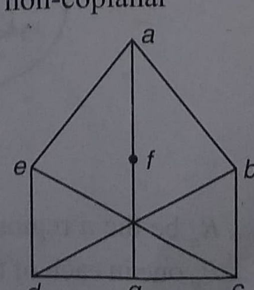


Solution. There are 17 vertices, 34 edges and 19 regions. So $n - e + r = 17 - 34 + 19 = 2$ which verifies Euler's theorem.

Example 56. Show that the following graphs are non-coplanar

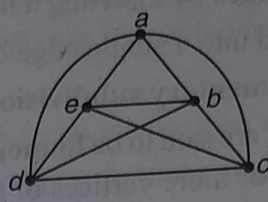


(*i*)



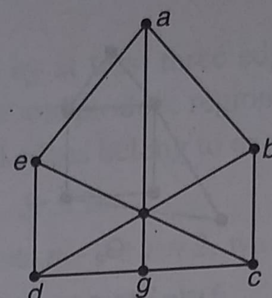
(*ii*)

Solution. (*i*) Deleting the vertex f and the edges incident with from the given graph, we get the subgraph.



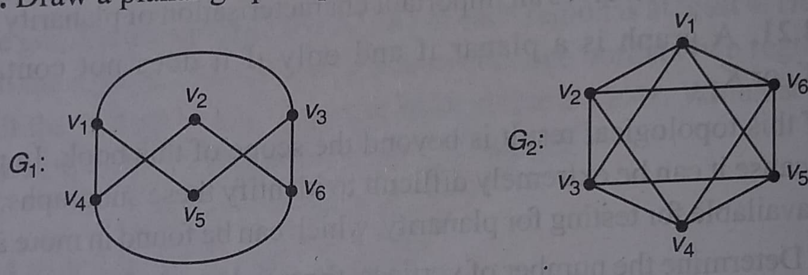
This subgraph is isomorphic to K_5. Using Kuratowski's theorem the given graph is non-planer.

(*ii*) Deleting the vertex f in the path $a-f-g$, we get the sub graph

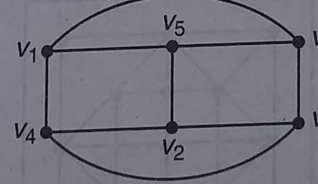


This subgraph is nothing but $K_{3,3}$. Using Kuratowski's theorem the given graph is non-planar.

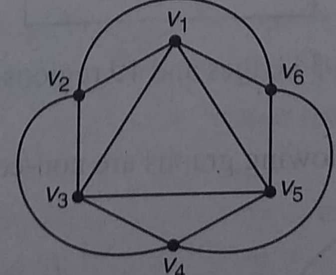
Example 57. Draw a planar graph representation of each graph if possible.



Solution. Redrawing the positions of the vertices v_2 and v_5 we get a planar representation of the graph G_1



The planar representation of G_2 is shown below:



Dual of a Graph

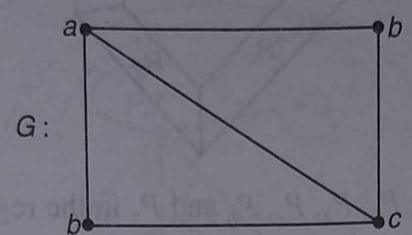
Let $R_1, R_2, R_3, \ldots, R_n$ be the n regions in a plane representation of a planar graph G. Let us place n points $P_1, P_2, \ldots, P_n$ one in each of the regions $R_1, R_2, R_3, \ldots, R_n$ respectively. We join these points according to the following rules.

- If two regions R_i and R_j are adjacent (*i.e.* they have a common edge) draw a line joining points P_i and P_j that intersects the common edge between R_i and R_j exactly once.
- If there is more than one edge common between R_i and R_j draw one line between points P_i and P_j for each of the common edges.
- For an edge e lying entirely in one region, say, R_k, draw a self loop at point P_k intersecting e exactly once.

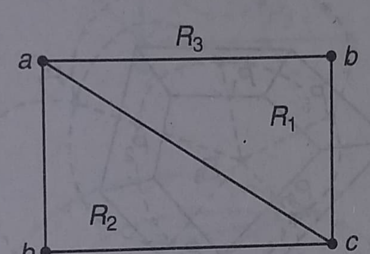
The new graph G thus obtained consisting of n vertices $P_1, P_2, ..., P_n$ and of edges joining these vertices are called a dual or a geometric **dual of the graph G.**

A planar graph G is said to be **self dual** if G is isomorphic to its dual G. The complete graph K_4 with four vertices is self dual.

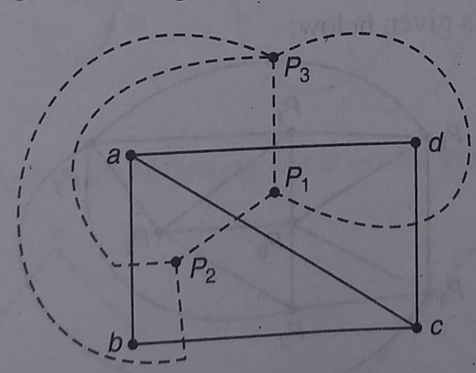
Example 58. Draw the dual of the graph G



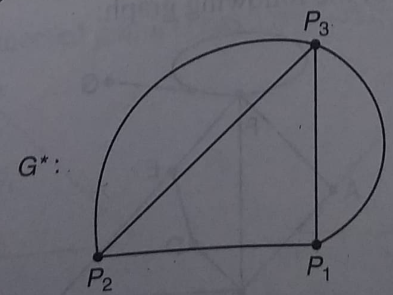
Solution. The planar graph has 3 regions R_1, R_2 and R_3 as shown below :



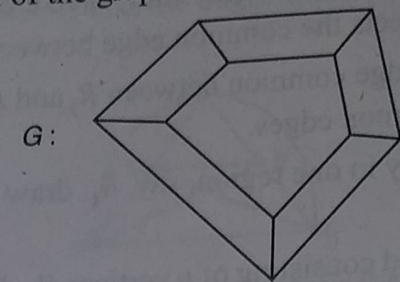
We place 3 points P_1, P_2 and P_3 in R_1, R_2 and R_3. We join these points according to the above procedure and get a new graph G^* consisting of three vertices and edges shown in dotted lines.



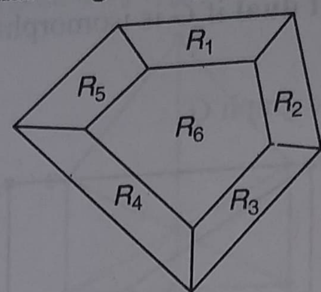
The dual graph G^* of G is given below,



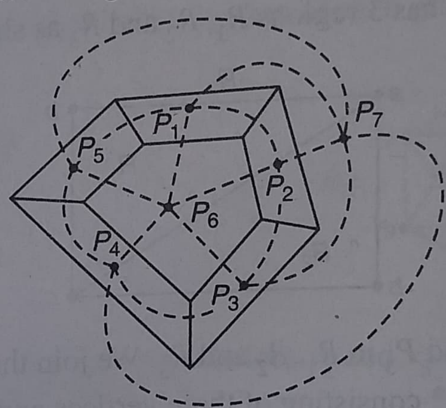
Example 59. Draw the dual of the graph.



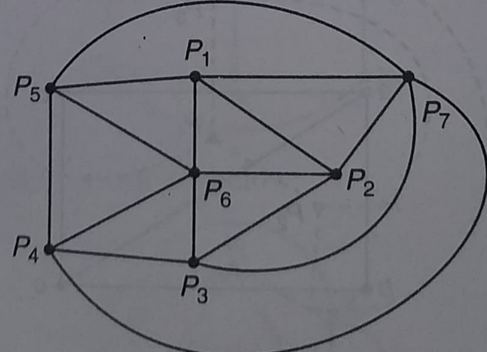
Solution. The planar graph G has 7 regions $R_1, R_2, R_3, R_4, R_5, R_6$ and R_7 which are shown below:



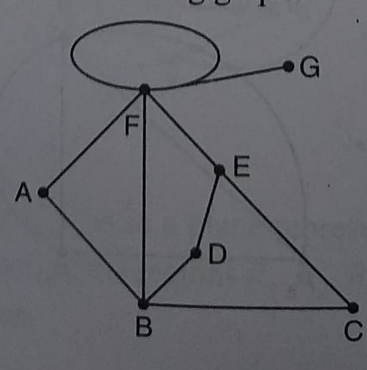
We take the 7 points $P_1, P_2, P_3, P_4, P_5, P_6$ and P_7 in the regions $R_1, R_2, R_3, R_4, R_5, R_6$ and R_7 respectively. We join the points according to rule as shown by dotted lines.



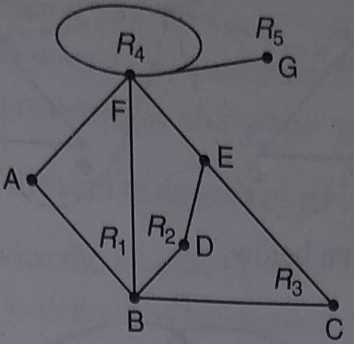
The dual graph G^* of G is given below:



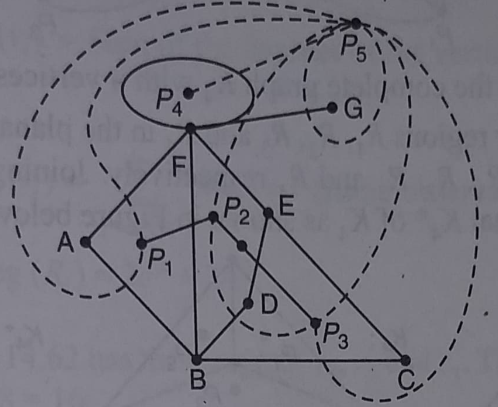
Example 60. Draw the dual of the following graph:



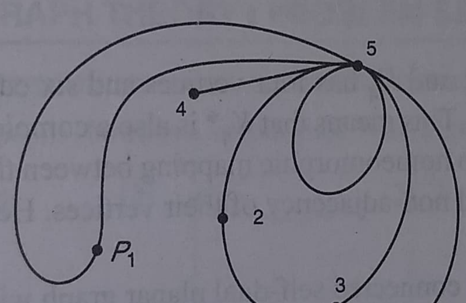
Solution. The planar graph has 5 regions R_1, R_2, R_3, R_4 and R_5 as shown below



We take 5 points P_1, P_2, P_3, P_4 and P_5 in the regions R_1, R_2, R_3, R_4 and R_5 respectively and joined by dotted lines as described in rule to get the dual G^* of G.

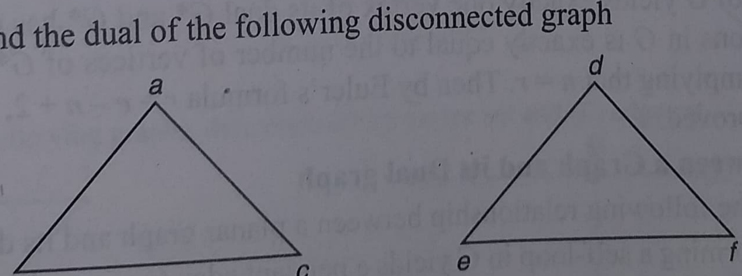


The dual graph G^* of G is

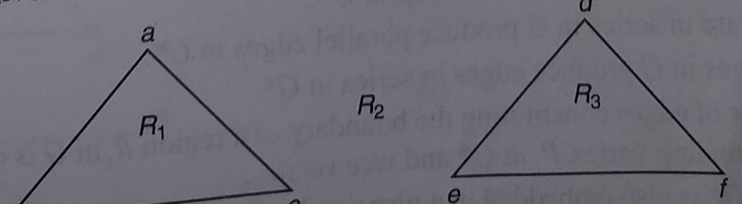


Example 61. Find the dual of the following disconnected graph

G:



Solution. The given disconnected graph G has 3 regions R_1, R_3 and R_3 which are shown below:



We take 3 points P_1, P_2 and P_3 in the regions R_1, R_2 and R_3 respectively and join the points according to rule by dotted lines as shown below: