Unit - III

Matrices, colouring and Directed graph

3.1. Chromatic number - Chromatic partitioning - Chromatic polynomial :-

Definition :- chromatic number.

The minimum number of colours required to colour G is called the chromatic number of G. It is denoted by $\chi(G)$.

Note :-

1. If $\chi(G)=k$, then G is said to be k-chromatic.
2. A graph consisting of only isolated vertices is 1-Chromatic.
3. The chromatic number of the complete graph K_n is n for all $n \geq 1$.
4. Every tree with 2 or more vertices is 2-Chromatic.

Definition :- K-chromatic graph.

A graph G that requires k different colours for it's proper colouring and no less, is called a k-chromatic graph, and the number k is called the chromatic number of G.

Theorem 3.1.1 :-

Every tree with two or more vertices is 2-Chromatic.

Proof :-

Let T be a tree with two or more vertices.

Select any vertex v in T and paint colour
Paint all vertices adjacent to v with colo
Paint all the vertices adjacent to those
vertices which have been used Colour 2 w
colour 1.

Continue this process, until every vert
of T has been painted.

Now, in T we find that all vertices at
odd distances from v have colour 2, w
v and vertices at even distances from
have colour 1.

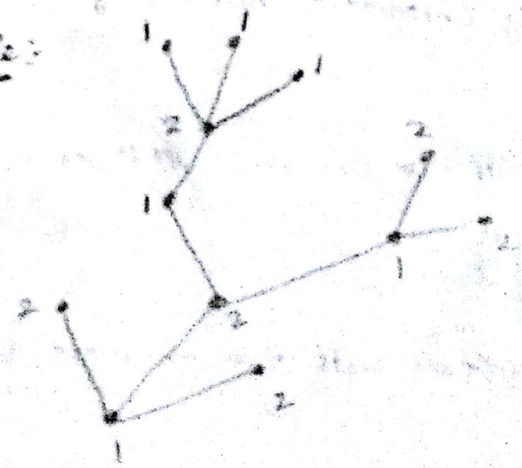
Since, there is one and only one path
between any two vertices in a tree,
two adjacent vertices will have the
same colour.

Thus T has been properly coloured with tw
Colours.

One colour would not have been enough.

Thus T is 2-chromatic.

Example:



Theorem 9.1.8 - Konig's theorem

A graph with atleast one edge is 2-chromatic iff it has no circuits of odd length.

Proof:

Let G be a connected graph with circuits of even length only.

Let T be a spanning tree from G.

We know that,

every tree with two or more vertices is 2-chromatic

∴ T is 2-chromatic

Now add the edges to T one by one.

By adding the edges circuit of even length will be created.

Since G has no odd circuit, the end vertices of every edge will be coloured with different colours.

Thus G can be properly coloured with 2 colours only.

So G is 2-chromatic

Conversely,

Let G be a 2-chromatic graph.

If G has a circuit of odd length then it will require 3 colours.

But G is 2-chromatic. So, it cannot have a circuit of odd length.

Hence the theorem

Theorem 2.1.2:-

If d_{max} is the maximum degree of the vertices in a graph G, chromatic number of G $\leq 1 + d_{max}$.

(i.e) $\chi(G) \leq 1+\Delta(G)$.

(or)

Prove that the chromatic number of a will not exceed by more than one the max degree of the vertices in a graph.

Proof:-

Suppose $\chi > \Delta+1$

$\therefore \chi \geq \Delta+2$

$\chi-1 \geq \Delta+1$.

we know that,

there are atleast χ vertices each of whose degree is atleast $\chi-1$.

Therefore, atleast there are χ vertices eac degree atleast $\Delta+1$.

This is a $\Rightarrow\Leftarrow$ for no vertex can be degree greater than Δ.

So our assumption $\chi > \Delta+1$ is wrong.

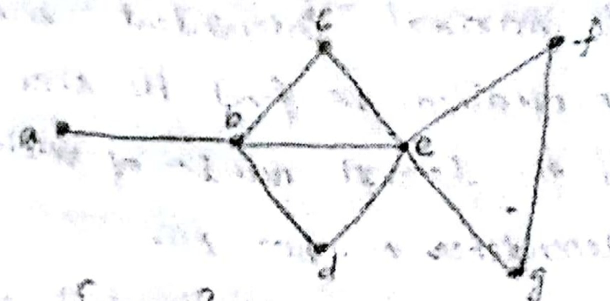
$\therefore \chi \leq \Delta+1$.

Hence the theorem.

Definition: Independent set.

A set of vertices in a graph is said to be an independent set if no two vertices in the set are adjacent.

Example:



$\{a, c, d\}$ is an independent set

Definition: Maximal Independent set

A maximal independent set is an independent set in which no other vertex can be added without destroying its independence property.

Example:-

In the above figure $\{a, c, d, f\}$, $\{b, f\}$ are the maximal independent.

Definition: Independence number $\beta(G)$:

The number of vertices in the largest independent set of a graph G is called the independent number.

Note:

Finding all maximal independent set

Let each vertex in the graph be treated as a Boolean variable. Let the logical sum $a+b$ denote the operation of including vertex a or b or both; let the logical multipl[illegible] ab denote the operation of including both vertices a and b, and the Boolean Complem[illegible] a' denote that vertex a is not included

Finding independence and chromatic numbers

Once all the maximal independent [illegible] of G have been obtained, we find the si[illegible] of the one with the largest number of ve[illegible] to get the independence number $\beta(G)$.

Chromatic Partitioning:

Given a simple connected graph G, part[illegible] all vertices of G into the smallest pos[illegible] number of disjoint, independent sets. This problem known as the chromatic partiti[illegible] of graphs, is perhaps the most importan[illegible] problem in partitioning of graphs.

Uniquely colourable graphs:

A graph that has only one chroma[illegible] partition is called a uniquely colourab[illegible] graphs.

Dominating sets:

A dominating set in a graph G is a [illegible] of vertices that dominates every vert[illegible] v in G in the following sense: Either [illegible] is included in the dominating set is adjacent to one or more vertices included in the dominating set.

Minimal dominating set:-

A minimal dominating set is a dominating set from which no vertex can be removed without destroying its dominance property.

Definition:- Chromatic polynomial

A polynomial which gives the number of different ways the graph G can be properly coloured using the minimum number of colours from λ is called chromatic polynomal of graph G and is denoted by $P_n(\lambda)$.

Theorem 3.1.5:-

A graph of n vertices is a complete graph iff its chromatic polynomial is

$$P_n(\lambda) = \lambda(\lambda-1)(\lambda-2)\ldots(\lambda-n+1).$$

Proof:-

Let G be a complete graph with n vertices.

Let λ be the number of colours.

1^{st} vertex of G can be coloured in λ ways

2^{nd} vertex of G can be coloured in $(\lambda-1)$ ways

3^{rd} " " " " " $(\lambda-2)$ ways

⋮

n^{th} " " " " " $\lambda-(n-1)$ ways

A complete graph G can be coloured in

$\lambda(\lambda-1)(\lambda-2)\ldots(\lambda-n+1)$ ways.

Let $P_n(\lambda)$ be the Chromatic polynomial, H

$$P_n(\lambda) = \lambda(\lambda-1)(\lambda-2)\ldots(\lambda-n+1)$$

Hence the theorem.

Theorem 3.1.6 :-

An n-vertex graph is a tree iff its Chromatic polynomial

$$P_n(\lambda) = \lambda(\lambda-1)^{n-1}.$$

Proof :-

Given n-vertex graph is a tree.

$\therefore$ 1^{st} vertex can be coloured by λ way

2^{nd} vertex can be coloured by $\lambda-1$ wa

⋮

n^{th} vertex " " " by $\lambda-1$ wa

Hence $P_n(\lambda) = \lambda(\lambda-1)^{n-1}$, $n \geq 2$.

Theorem 3.1.7 :-

Let a and b be two non adjacent vertices in a graph G. Let G' be a gr obtained by adding an edge between a Let G'' be a simple graph obtained f G by fusing the vertices a and b toget and replacing sets of parallel edges with single edges. Then

$$P_n(\lambda) \text{ of } G = P_n(\lambda) \text{ of } G' + P_{n-1}(\lambda) \text{ of}$$

Proof:-

The number of ways of properly colouring G can be grouped into two cases.

1. vertices a and b are of the same colour.
2. vertices a and b are of the different colour.

The number of ways of properly colouring G such that a and b

i) have different colours = number of ways of properly colouring G'.

ii) have the same colours = number of ways of properly colouring G''.

$\therefore P_n(\lambda)$ of $G = P_n(\lambda)$ of $G' + P_{n-1}(\lambda)$ of G''.

Problem :-

A graph G is bipartite iff any circuit in G has even length.

Proof:-

Assume that G is bipartite.

To prove: A circuit in G also has even length.

Since G is bipartite all of its edges must connect a left vertex with a right vertex.

This means that any ~~vertex~~ circuit found within G will alternate back and forth from left to right vertices.

Therefore, any circuit will contain a

even number of vertices.

Since within a circuit the number of

is equal to the number of vertices.

the number of edges must also be e

Therefore since the number of edges

even.

By definition G has even length.

Conversely,

Assume that every circuit in G h

even length.

To prove: G is bipartite.

Take any vertex, lets starts with

example circuit.

Put all vertices of odd length

from A on the right side of our graph

Next. Put all the vertices of even len

away from A on the left side of

new graph near A.

Lastly use our circuit to connect

the vertices in our new graph.

As we can see from the example

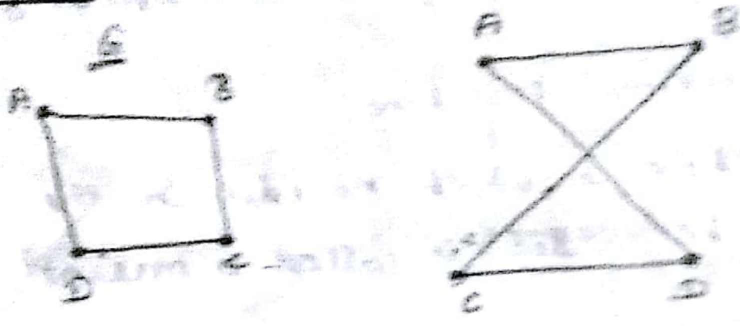
a bipartite graph G has been

constructed where no two vertices

on the left or right are adjacent.

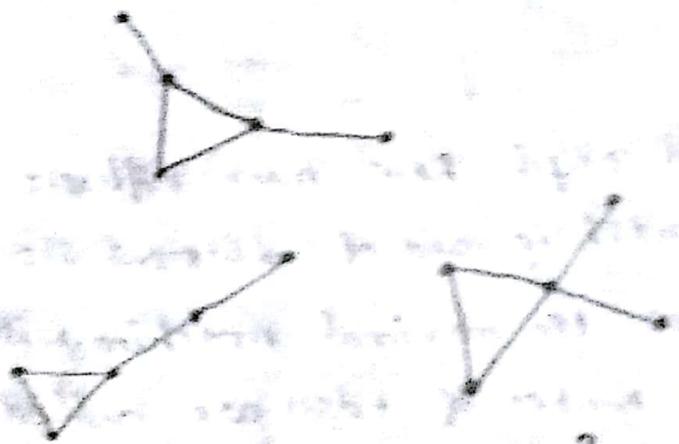
If two vertices on the same side were by chance connected then our circuit would have had an odd length

Example:



Sketch two different (ie non-isomorphic) graphs that have the same chromatic polynomial.

sol:



Chromatic polynomial = $\lambda(\lambda-1)^3(\lambda-2)$

3.2 Matching - Covering - Four Colour problem

3.2.1 Matching:-

Definition:-

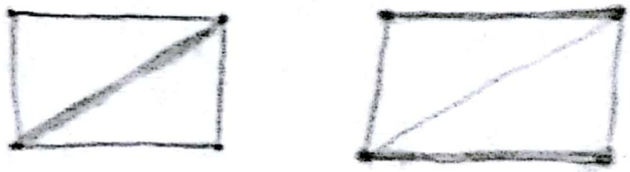
Matching in a graph G is a subset o edges in which no two edges are adjacent. A single edge in a graph G is a matchi

Definition: Maximal Matching

A matching in which no edge in the graph can be added is called a maxima matching.

Example:-

In a complete graph of three vertic any single edge is a maximal matching



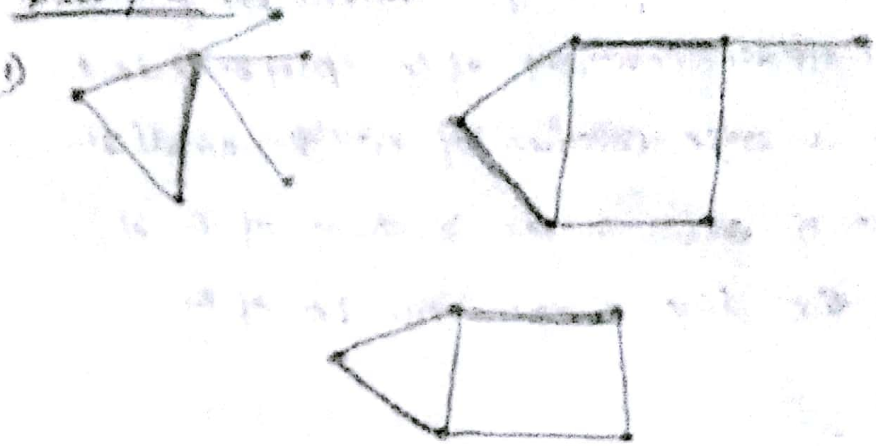
Clearly a graph have many different maximal matching and of different siz Among these, the maximal matching w the largest number of edges are called the largest maximal matching.

In above example, a largest maxim matching is shown in heavy lines.

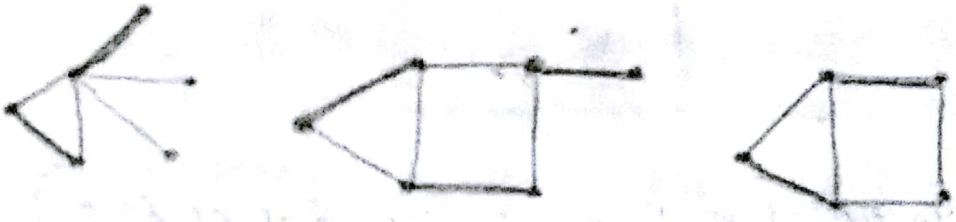
The number of edges in a largest maxi matching is called the matching number of the graph.

Example :- maximal matching

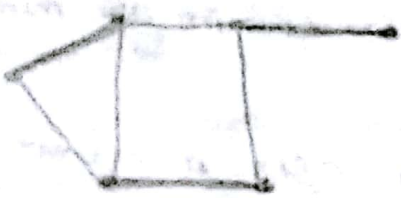
1)



2) Maximum matching



3) Perfect matching



Here $v = 6$

$\frac{v}{2} = 3$ edges

The three edges are shown in heavy lines.

A matching of G is perfect if it has $\frac{|V|}{2}$ edges.

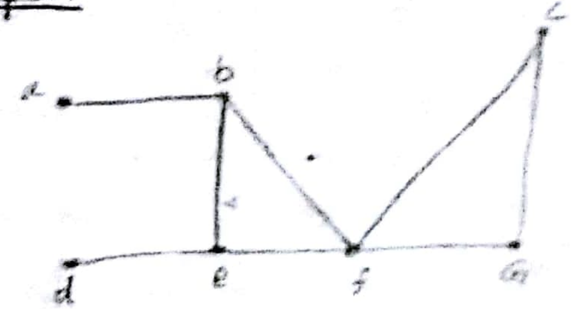
3.2.2. Covering:

Definition:-

In a graph G, a set g of edges is said to cover G if every vertex in G is incident on atleast one edge in G. A set of edges that covers a graph G is said to be an edge covering or covering subgraph or covering of G.

An edge covering g of G is called a minimal edge covering if no proper sub of g is an edge covering of G. The smalle number of edges in any covering of G is called the line covering number of G.

Example:



In the above graph, the sets {ab, de, fg {ab, de, bf, gc}, {ab, de, fc, fg} are edge covering of graph. These covering are mi covering of graph.

The smallest number of edges in a cove of graph is 4.

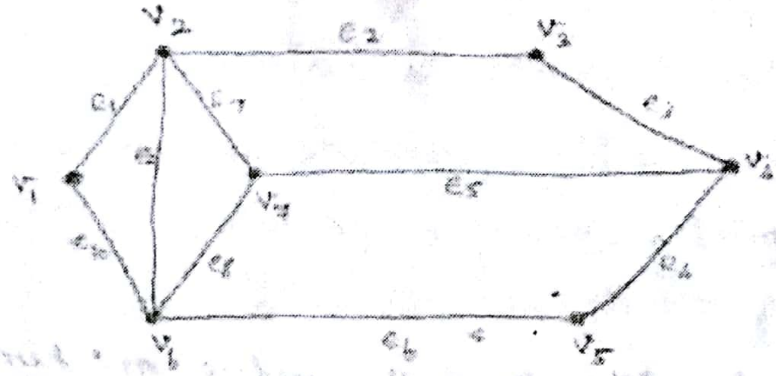
Definition: vertex covering

A subset W of V is called a vertex covering or a vertex cover of G if every in G is incident on atleast one vertex in W.

A vertex cover of a graph is a subg of the graph. V itself is a vertex cov of G. This is known as trivial vertex co

A vertex covering W of G is called a minimal vertex covering if no proper subset of W is a vertex covering of G.

Example:



In the graph, the set $W = \{v_2, v_4, v_6\}$ is a vertex covering. And $\{v_1, v_2\}$, $\{v_1, v_3\}$, $\{v_2, v_3\}$, $\{v_1\}$, $\{v_2\}$, $\{v_3\}$ are not vertex covering of the graph.

Thus no proper subset of W is a vertex covering.

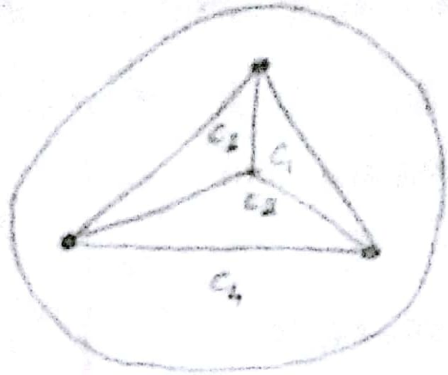
Hence, W is a minimal vertex covering and the set $\{e_1, e_3, e_6, e_8\}$ is an edge covering.

3.2.3. Four Colour Problem:-

The four colour problem states that every planar graph can be coloured with four colours in such a way that no two adjacent regions have the same colour. The regions are said to be adjacent if there is a common edge between them.

Example:

The graph is an example that needs



Complete matching:-

Theorem 3.8.3:

In a bipartite graph a complete ma[tching] of V_1 into V_2 exists if there is a positi[ve] integer m for which the following conditi[on is] satisfied degree of every vertex in $V_1 \geq m \geq$ degree of every vertex in V_2.

Proof:-

let us assume a subset of r vertices in

These r vertices have atleast $m.r.$ edges in[cident] on them.

Each $m.r.$ edge is incident to some vertex i[n]

Since degree of every vertex in V_2 is gr[eater] than m_1, then $m.r.$ edges are incident on at [least] $(m.r)/m = m$ vertices in V_2.

Therefore any subset of r vertices in V_1 a[re] collectively adjacent to r or more vertices i[n]

Hence by theorem there exists a complete matching of V_1 into V_2.

Theorem 3.2.5 :

A covering g of a graph is minimal iff g contains no paths of length three or more.

Proof :

Suppose that a covering H contains a path of length 3 and let it be $v_1 e_1 v_2 e_2 v_3 e_3 v_4$.

Edge e_2 may be removed without leaving vertices v_2 and v_3 uncovered.

Therefore, H is not a minimal covering.

Conversely,

if the covering does not contain a path length 3 or more, Then all its components will be a star graphs.

And from star graph no edge can be removed with out uncovering a vertex.

Therefore H is a minimal covering.

Example :

Theorem 3.2.6 :

The vertices of every planar graph can be properly coloured with five colours.

Proof :

We shall prove the result by induction.

If the number of vertices in a graph is less than or equal to 5.

Then we can properly colour these vertic[es]
with five colours
Hence, the result is obvious
we assume that the result holds for all
planar graphs having less than n vertices
Consider a graph with n vertices
Then it must have a vertex v with deg[ree 5]
or less.
(i) v be the vertex of degree ≤ 5 and con[sider]
the induced subgraph G' of G on $G - v$
By hypothesis 'G'' graph requires not more [than]
five colours.
(ii) the induced subgraph G' requires ≤ 5 [colours]
Paint the vertices of G' using five colours
and now add it to the vertex v along wi[th]
all edges incident on v.
If the degree of v is less than 5 we can [assign]
a proper colour to v and obtain a proper
colouring of G.

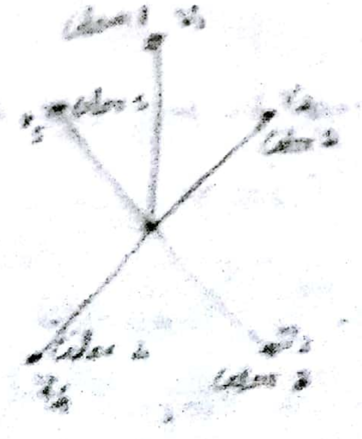


Figure (a)

Figure (b)

The only case, which is left is when $d(v)=5$ and all the colours have been used in colouring the vertex v_1, v_2, v_3, v_4 and v_5 adjacent to v as in figure (a).

Let H be a subgraph of G on the vertices which have been assigned the colour 1 and colour 3.

Hence v_1, v_3 are in H.

If v_1 and v_3 belong to different components of H, then we can inter change the colours 1 and 3 in the component which contains vertex v_1 without destroying the proper colouring of $G-v$

This ~~paint~~ interchange will paint vertices v_1 and v_3 with colour 3.

Now colour 1 can be assigned to vertex v and thus G is 5-chromatic.

On the other hand, if v_1 and v_3 are in same component of H then there is a path P from v_1 to v_3 whose vertices are painted alternately with colours 1 and 3.

The path P together with the edges (v_1, v) and (v_3, v) form a circuit C which encloses v_2. (or v_4).

Now, consider the subgraph K of G on the vertices painted with colours 2 or 4.

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Since C encloses v_2 or v_4 but not b[...]

vertices v_2 and v_4 belong to different

components of K.

Therefore we can interchange the colour[...]

and 4 in the component containing v_2

without destroying the proper colourin[...]

G-v.

This interchanging will paint vertices

and v_4 with colour 4.

Now colour 2 can be assigned to vert[...]

Hence the theorem.

3.3. Directed graphs:-

Definition: Digraph

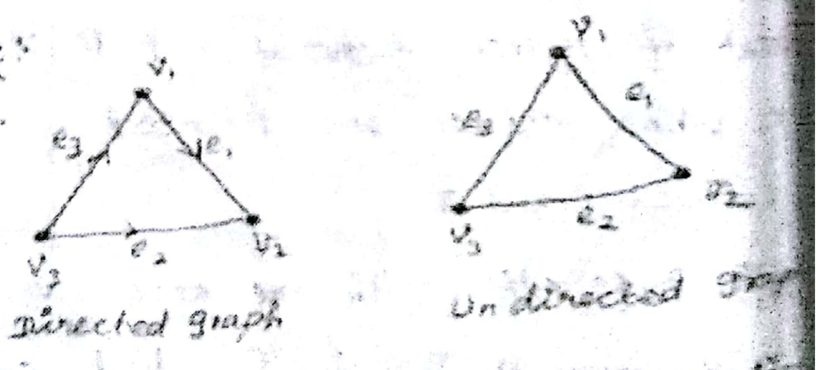
A graph in which every edge is di[...]

is called a directed graph or simply a di[...]

A graph in which every edge is undirec[...]

is called an undirected.

Example:



The edge e_1 is incident with the verti[...]

v_1 and v_2 also the vertex v_1 is incide[...]

with e_1 and e_2.

Problem:

In every digraph D, the sum of the out-degrees of all vertices is equal to the sum of the in degrees of all vertices, each sum being equal to the number of edges in the digraph D.

Sol:

Suppose that D has n vertices $v_1, v_2, \dots, v_n$ and m edges.

Let r_1 be the number of edges going out of v_1, r_2 be the number of edges going out of v_2 and so on.

Then $d^+(v_1) = r_1$; $d^+(v_2) = r_2, \dots, d^+(v_n) = r_n$ ——— ①

Since every edge terminates at some vertex and since there are m edges, we should have

$$r_1 + r_2 + \dots + r_n = m \quad \text{——— ②}$$

We get, $d^+(v_1) + d^+(v_2) + \dots + d^+(v_n) = r_1 + r_2 + \dots + r_n$
$= m$

(by ① & ②)

Similarly if s_1 is the number of edges coming into v_1, s_2 is the number of edges coming into v_2, and so on, we get

$$d^-(v_1) + d^-(v_2) + \dots + d^-(v_n) = s_1 + s_2 + \dots + s_n$$
$$= m$$

Thus $\sum_{i=1}^{n} d^+(v_i) = \sum_{i=1}^{n} d^-(v_i) = m$

Hence the proof.

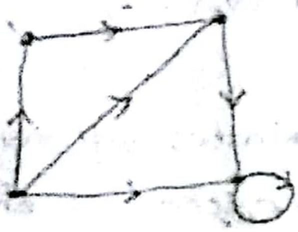
of directed graphs:-

1. Simple graph:-

A digraph without self loops and par[allel]
arcs is called a simple digraph.

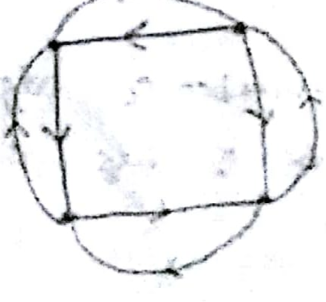
2. Antisymmetric graph:-

A digraph having atmost one arc be[tween]
a pair of vertices and with or without s[elf]
loops is called an antisymmetric digraph.



3. Symmetric digraph:-

A digraph in which for each arc (u,v) there [is]
also an arc (v,u) is called a symmetric graph.

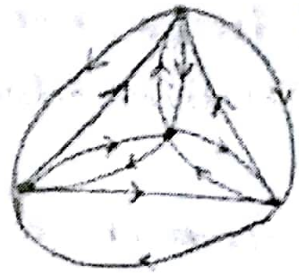


4. A complete symmetric digraph:-

A complete symmetric digraph is [a]
simple digraph in which there is ex[actly]

one arc directed from every vertex to every other vertex.

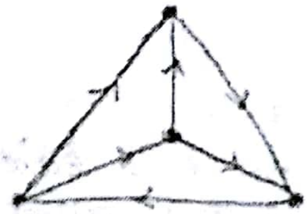
If D is a complete symmetric digraph on n vertices, the number of arcs is $n(n-1)$



5. A complete asymmetric digraph:-

An asymmetric digraph in which there is exactly one arc between every pair of vertices.

If n is the number of vertices of a complete asymmetric digraph the number of arc is $\frac{n(n-1)}{2}$.



6. Pseudo symmetric digraph or Isograph or balanced

A digraph is said to balanced if for every vertex v, the indegree and out degree are equal. that is $d^-(v) = d^+(v) \ \forall \ v \in D$.

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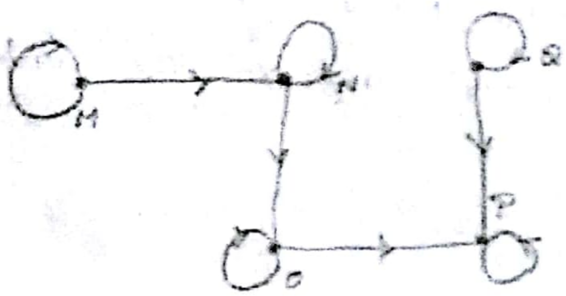
Digraphs and Binary relations :-

a) Reflexive relation :-

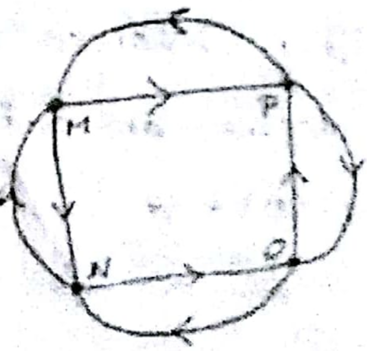
W.K.T, a relation R on a set A is [said] to be reflexive if $aRa\ \forall a \in A$. Thus a di[-] graph having a self-loop at each of its vertices defines a reflexive relation [and] vice versa.

Such a digraph represents a refle[xive] binary relation on its vertex set m[ay] be called a reflexive digraph. A di[graph] in which no vertex has a self loop is called an irreflexive digraph.



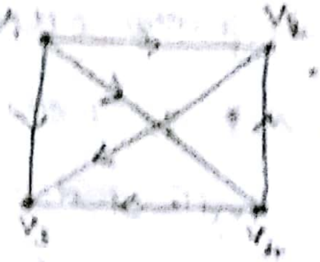
b) Symmetric relation :-

A relation R on set A is said to [be] Symmetric if $aRb \Rightarrow bRa$.



c) Transitive relation:

A relation R on A is said to be transitive if aRb and bRc implies aRc. A digraph for transitive relation should contain a directed path between from a to c whenever there is a directed path from ~~a to~~ a to b and b to c.



d) Equivalence relation:

A relation R on a non-empty set A is called an equivalence relation if it is reflexive, symmetric and transitive.

Directed Paths and Connectedness:

Definition: Directed walk

A directed walk from a vertex v_i to v_j is an alternating sequence of vertices and edges, beginning with v_i and ending with v_j such that each edge is oriented from the vertex preceding it to the vertex following it.

Definition: Semi walk.

A semi walk in a directed graph is a walk in the corresponding undirected graph, but is not a directed walk.

Connected Digraphs:-

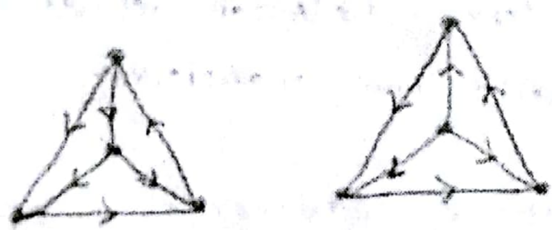
A graph was defined as Connected there was atleast one path between pair of vertices.

There are two different types of paths

1. A digraph G is said to be Strongly Connected if there at least one directe from every vertex to every other ve

2. A digraph G is said to be weakly C if its Corresponding undirected grap Connected but G is not Strongly Conne

Example:



Euler digraphs:-

Definition: Eulerian trail

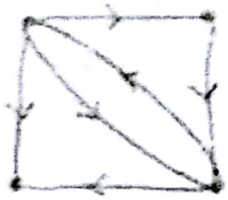
A closed directed trail containin all the edges in a digraph D is called Eulerian trail.

Definition: Eulerian digraph

A digraph containing an Eulerian trail is called an Eulerian digraph.

Example -

D_1



D_2

D_3

D_1, D_2 & D_3 are Eulerian digraphs