11 CHAPTER

Recurrence Relation and Generating Function

11.1. Introduction

An important use of recurrence relation is the analysis of the complexity of algorithms. A recurrence relation that defines a sequence can be directly converted to an algorithm to compute the sequence.

Generating functions are important tools in discrete mathematics and their use is by no means confined to solve linear recurrence relation. The function can be used to solve many types of counting problems.

11.2. Recurrence Relation

A sequence can be defined by giving a general formula for its nth term or by writing few of its terms. An alternative approach is to write the sequence by finding a relationship among its terms. Such a relationship is called a **recurrence relation** (also called **difference equation**).

For example, let us consider a sequence $S = \{3^1, 3^2, 3^3, \ldots\ldots, 3^n, \ldots\ldots\}$

This sequence can be defined by giving an explicit formula for its nth term *i.e.*, $S = \{S_n\}$ where $S_n = 3^n$. Since the value of S_n is three times the value of S_{n-1} for all n, once the value of S_{n-1} is known, the value of S_n can be computed. The same sequence S can be described by the relation

$$S_n = 3\,S_{n-1},\ n \geq 2 \text{ with } S_1 = 3.$$

The information $S_1 = 3$ is called initial condition.

Definition : A recurrence relation for the sequence $\{S_n\}$ is an equation that relates S_n in terms of one or more of the previous terms of the sequence, $S_0, S_1, \ldots\ldots S_{n-1}$ for all integer $n \geq n_0$, where n_0 is a non negative integer. The specification of the values of S_n, $n < n_0$ is called the initial conditions of a recurrence relation. For example,

(*i*) *The recurrence relation of the sequence*

$$S = \{5, 8, 11, 14, 17 \ldots\ldots\}$$

is

$$S_n = S_{n-1} + 3,\ n \geq 2$$

with initial condition $S_1 = 5$.

(*ii*) *The recurrence relation of the Fibonnaci Sequence of numbers.*

$$S = \{1, 1, 2, 3, 5, 8, 13 \ldots\ldots\}$$

is

$$S_n = S_{n-1} + S_{n-2},\ n \geq 3 \text{ with initial conditions } S_1 = S_2 = 1$$

Note that the recurrence relation in example (*i*) requires only one initial condition S_1 to start up. The recurrence relation in example (*ii*) expresses S_n in terms of two previous values and requires two initial conditions S_1 and S_2 before all values of the recurrence are uniquely determined.

Example 1. *Find the first four terms of each of the following recurrence relation*

(*a*) $a_k = 2a_{k-1} + k$, *for all integers* $k \geq 2$, $a_1 = 1$

(*b*) $a_k = a_{k-1} + 3a_{k-2}$, *for all integers* $k \geq 2$, $a_0 = 1$, $a_1 = 2$.

(*c*) $a_k = k\,(a_{k-1})^2$, *for all integers* $k \geq 1$, $a_0 = 1$.

Solution. (*a*)

$$
\begin{aligned}
a_1 &= 1 \\
a_2 &= 2a_1 + 2 = 2.1 + 2 = 4 \\
a_3 &= 2a_2 + 3 = 2.4 + 3 = 11 \\
a_4 &= 2a_3 + 4 = 2.11 + 4 = 26
\end{aligned}
$$

(*b*)

$$
\begin{aligned}
a_0 &= 1 \\
a_1 &= 2 \\
a_2 &= a_1 + 3a_0 = 2 + 3.1 = 5 \\
a_3 &= a_2 + 3a_1 = 5 + 3.2 = 11
\end{aligned}
$$

(*c*)

$$
\begin{aligned}
a_0 &= 1 \\
a_1 &= 1.\,(a_0)^2 = 1.1 = 1 \\
a_2 &= 2.\,(a_1)^2 = 2.1 = 2 \\
a_3 &= 3.\,(a_2)^2 = 3.4 = 12
\end{aligned}
$$

Example 2. *Show that the sequence*

$\{2, 3, 4, 5, \ldots\ldots\ldots, 2 + n, \ldots\ldots\ldots\}$

for $n \geq 0$ *satisfies the recurrence relation* $a_k = 2\,a_{k-1} - a_{k-2}$, $k \geq 2$.

Solution. Let a_n (*n*th term of the sequence) $= 2 + n$

$$
\begin{aligned}
a_k &= 2 + k \\
a_{k-1} &= 2 + (k - 1) = 1 + k \\
a_{k-2} = 2 + (k - 2) &= k
\end{aligned}
$$

Now $\quad 2a_{k-1} - a_{k-2} = 2\,(1 + k) - k = 2 + k = a_k$

$\therefore \quad a_k = 2a_{k-1} - a_{k-2}$

A recurrence relation for a particular sequence can be written more than one way. For example

$$a_n = 3\,a_{n-1}, \quad n \geq 2 \text{ with initial condition } a_1 = 3$$

can also be written as

$$a_{n+1} = 3a_n, \quad n \geq 1 \text{ with initial condition } a_1 = 3$$

A sequence defined recursively need not start with a subscript of zero.

A given recurrence relation may be satisfied by many different sequences, the actual terms of the sequence are determined by initial conditions. For example,

$$a_n = 3a_{n-1} \text{ with initial condition } a_1 = 1$$

and

$$b_n = 3b_{n-1} \text{ with initial condition } b_1 = 2.$$

define the sequence,

$a_2 = 3a_1 = 3.1 = 3$	and	$b_2 = 3b_1 = 3.2 = 6$
$a_3 = 3a_2 = 3.3 = 9$		$b_3 = 3b_2 = 3.6 = 18$
$a_4 = 3a_3 = 3.9 = 27$		$b_4 = 3b_3 = 3.18 = 54$
....		....
....		....

Thus the two different sequences are

$3, 9, 27, \ldots\ldots\ldots$

and $\quad 6, 18, 54, \ldots\ldots\ldots$

3 Recurrence Relation Models

We can use recurrence relation to model a wide variety of problems both inside and outside puter science. Modelling of problems with recurrence relation are illustrated below.

Example 3. (*Compound Interest*) *A person invests ₹ 10,000/- @ 12% interest compounded ally. How much will be there at the end of 15 years.*

Solution. Let A_n represents the amount at the end of n years. So at the end of $n-1$ years, the unt is A_{n-1}. Since the amount after n years equals the amount after $n-1$ years plus interest for th year. Thus the sequence $\{A_n\}$ satisfies the recurrence relation.

$$\begin{aligned} A_n &= A_{n-1} + (0.12)\, A_{n-1}, \\ &= (1.12)\, A_{n-1},\ n \geq 1. \end{aligned}$$

h initial condition $A_0 = 10{,}000$

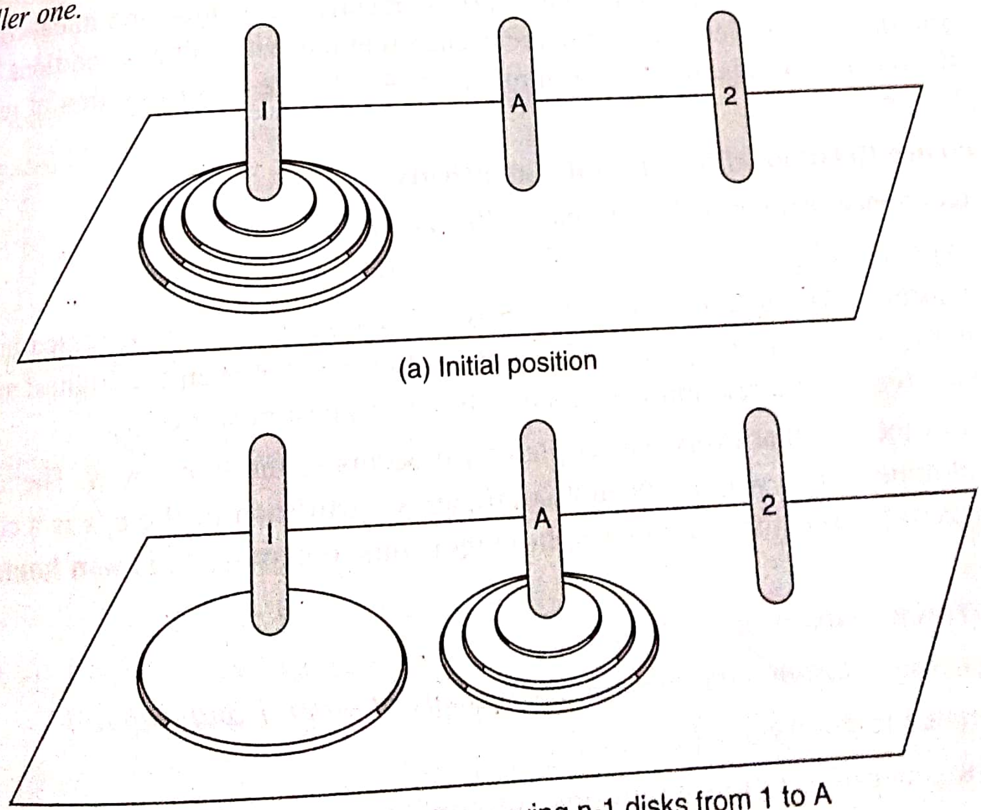
The recurrence relation with the initial condition allow us to compute the value of A_n for any For example,

$$\begin{aligned} A_1 &= (1.12)\, Ao \\ A_2 &= (1.12)\, A_1 = (1.12)^2\, Ao \\ A_3 &= (1.12)\, A_2 = (1.12)^3\, Ao \\ A_n &= (1.12)^n\, Ao. \end{aligned}$$

ich is an explicit formula and the required amount can be derived from the formula by putting $= 15$.

So,
$$A_{15} = (1.12)^{15}\,(10000).$$

Example 4. (*Tower of Hanoi*) : *The Tower of Hanoi is a puzzle consisting of three pegs nted on a board with disks of different diameters. Initially, these disks are placed on the first g in order of size, with the largest on the bottom (as shown in Fig. 11.1). These disks are to be nsfered on the second peg with their relative positions unchanged. The rules of the puzzle allow k to be moved one at a time from one peg to another as long as a disk is never placed on the p of smaller one.*



(a) Initial position

(b) Position after moving n-1 disks from 1 to A

Fig. 11.1

Solution. Suppose there are n disks on peg 1. Let C_n denotes the number of moves needed to move them from peg 1 to 2 with the given restrictions. If there is only one disk, we simply move it to the second peg. If we have $n > 1$, we can transfer $n - 1$ disks to peg A using C_{n-1} moves. During the moves, the largest disk at the bottom of peg 1 stays fixed. There we use one move to transfer the largest disk to the second peg. We can now move again $n - 1$ disks on peg A to peg 2 using C_{n-1} moves. Therefore,

$$C_n = 2\,C_{n-1} + 1, \; n > 1.$$

with initial condition $C_1 = 1$.

Example 5. *Find a recurrence relation and give initial conditions for the number of bit strings of length n that do not contain the pattern 11.*

Solution. Let, C_n denote the number of bit strings of length n that do not contain the pattern 11, and can be counted as

(*a*) that begin with 0,

(*b*) that begin with 10.

Since, the sets of strings of types (*a*) and (*b*) are disjoint, by the sum rule, C_n will be equal to the sum of the numbers of strings of type (*a*) and (*b*). Suppose that an n bit string begins with 0 and does not contain the pattern 11. Since $(n - 1)$ bit string not containing 11 can follow the initial 0, there are C_{n-1} such bit string of type (*a*). In an n bit string begin with 10 and does not contain the pattern 11, the $(n - 2)$ bit string following the initial 10 can not contain the pattern 11 : therefore there are C_{n-2} bit strings of type (*b*).

Thus,

$$C_n = C_{n-1} + C_{n-2}, \; n \geq 3.$$

The initial conditions are $C_1 = 2$, since both bit strings of length 1 and 0 do not contain the pattern 11 and $C_2 = 3$, since the valid bit string of length two is 00, 01 and 10.

11.4 Solution of Recurrence Relation

Suppose we have a sequence that satisfies a certain recurrence relation and initial conditions. An explicit formula for a_n which satisfy the recurrence relation with initial conditions is called a solution to the recurrence relation. For example, $a_n = A \cdot 2^n, n \geq 0$ is a solution of recurrence relation $a_n - 2a_{n-1} = 0$.

Linear Recurrence Relation with Constant Coefficients.

A linear recurrence relation with constant coefficients is of the form

$$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \ldots\ldots\ldots\ldots + c_k a_{n-k} = f(n) \quad \ldots (1)$$

where $c_i s$ are constants. The **order** of the relation is k, because a_n can be expressed in terms of previous k terms of the sequence *i.e.*, the order is the difference between the highest and lowest subscript of a. The **degree** of a relation is defined to be the highest power of a_n.

Linear refer to the fact that every subscripted term occurs to the first power. The terms such as $a_n \cdot a_{n-1}$ do not appear. The words constant coefficients mean each of the c_i's is a constant. If $f(n)$ is identically zero the relation is known as **homogeneous**, otherwise, it is **non homogeneous**. For example,

(*a*) *The recurrence relation* $a_n = 2a_{n-1}$
is a linear homogeneous relation with constant coefficients of order 1 and degree 1.

(*b*) The recurrence relation $a_n = 2\,a_{n-1}\,a_{n-2}$
is not a linear homogeneous relation with constant coefficients as term such as $a_{n-1} a_{n-2}$ is not

(c) The recurrence relation $a_n - a_{n-1} = 3$

is not a linear homogeneous relation with constant coefficients because the expression of the right hand side is not zero because of the constant term 3.

(d) The recurrence relation $a_n = a_{n-1} + a_{n-2}$

is a linear homogeneous relation with constant coefficients of order 2 and degree 1.

Note that the recurrence relation $a_n = 2a_{n/2} + n$ does not have a constant order. This type of relation is generally harder to solve.

Methods of Solving Linear Recurrence Relation

The three methods of solving linear recurrence relations with constant co-efficient are discussed here.

1. Iteration
2. Characteristic roots and
3. Generating functions.

Iteration Method: In this method the recurrence relation for a_n is used repeatedly to look for a pattern to find a general expression for a_n in terms of n. The mechanics of this method are described in terms of examples.

Example 6. Solve the recurrence relation

$$a_n = a_{n-1} + 2, \qquad n \geq 2$$

subject to initial condition $a_1 = 3$

Solution. We backtrack the value of a_n by substituting the expression of a_{n-1}, a_{n-2} and so on, until a pattern is clear.

Given $\quad a_n = a_{n-1} + 2 \quad \ldots (1)$

Replacing n by $n - 1$ in (1), we obtain

$$a_{n-1} = a_{n-2} + 2$$

From (1) $\quad a_n = a_{n-1} + 2 = (a_{n-2} + 2) + 2 = a_{n-2} + 2.2 \quad \ldots (2)$

Replacing n by $n - 2$ in (1), we obtain

$$a_{n-2} = a_{n-3} + 2$$

So, from (2) $\quad a_n = (a_{n-3} + 2) + 2.2 = a_{n-3} + 3.2$

In general $\quad a_n = a_{n-k} + k \,.\, 2$

For $\quad k = n - 1$

$$a_n = a_{n-(n-1)} + (n-1).2 = a_1 + (n-1).\,2 = 3 + (n-1).\,2$$

which is an explicit formula.

More generally, if c is a constant then one can solve

$$a_n = c\,a_{n-1} + f(n), \text{ for } n > 1 \text{ in a similar way} \qquad (3)$$

Given $\quad a_n = ca_{n-1} + f(n)$

So, $\quad a_{n-1} = ca_{n-2} + f(n-1)$

From (3) $\quad a_n = c(ca_{n-2} + f(n-1)) + f(n)$

$$= c^2 a_{n-2} + cf(n-1) + f(n) \qquad \ldots (4)$$

Replaing n by $n - 2$ in (3), we obtain

$$a_{n-2} = ca_{n-3} + f(n-2)$$

From (4) $\quad a_n = c^2\,[(ca_{n-3}) + f(n-2)] + cf(n-1) + f(n)$

$$= c^3\,a_{n-3} + c^2 f(n-2) + cf(n-1) + f(n)$$

$$a_n = c^k a_{n-k} + c^{k-1} f(n-(k-1)) + \dots\dots\dots + cf(n-1) + f(n)$$

In general,

For $k = n$

$$a_n = c^n a_0 + c^{n-1} f(1) + c^{n-2} f(2) + \dots\dots\dots + cf(n-1) + f(n)$$

$$= c^n a_0 + \sum_{k=1}^{n} c^{n-k} f(k)$$

In Tower of Hanoi problem, we deduced the recurrence relation as

$$a_n = 2a_{n-1} + 1 \qquad n > 1$$

with the initial condition

$$a_1 = 1$$

Using general formula, we get

$$a_n = 2^n a_0 + \sum_{k=1}^{n} 2^{n-k} .1, \text{ taking } a_0 = 0$$

$$= 2^{n-1} + 2^{n-2} + \dots\dots + 2^0 = (2^n - 1)/(2-1) = 2^n - 1$$

The Method of Characteristic Roots

In this method the solution is obtained as the sum of two parts, the homogeneous solution which satisfy the recurrence relation when the right hand of the relation is set to 0 *i.e.*, $f(n) = 0$, and the particular solution, which satisfies the relation with $f(n)$ on the right hand side.

***(a)* Homogeneous Recurrence Relation**

The basic approach for solving homogeneous relation [$f(n) = 0$] is to look for solutions of the form $a_n = r^n$. Now $a_n = r^n$ is a solutions of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots\dots\dots\dots + c_k a_{n-k}$$

if and only if

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \dots\dots\dots + c_k r^{n-k}$$

Dividing both sides by r^{n-k}

$$r^k = c_1 r^{k-1} + c_2 r^{k-2} + \dots\dots\dots + c_k$$

or

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \dots\dots\dots - c_k = 0$$

which is called the **characteristic or auxiliary equation** of the recurrence relation. The solutions of this equation are called the **characteristic roots** of the recurrence relation. A characteristic equation of the kth degree has k characteristic roots.

Distinct roots: If the characteristics equation has distinct roots $r_1, r_2, \dots\dots\dots r_k$, then the general form of the solutions for homogeneous equation is

$$a_n = b_1 r_1^{\,n} + b_2 r_2^{\,n} + b_3 r_3^{\,n} + \dots + b_k r_k^{\,n}$$

where $b_1, b_2, b_3, \dots\dots\dots\dots b_k$ are constants which may be chosen to satisfy any initial conditions.

Example 7. Solve $a_n = a_{n-1} + 2a_{n-2}$, $n \geq 2$ with the initial conditions $a_0 = 0$, $a_1 = 1$.

Solution. The given recurrence relation

$$a_n = a_{n-1} + 2a_{n-2}$$

i.e.

$$a_n - a_{n-1} - 2\,a_{n-2} = 0 \qquad \dots(1)$$

is a second order linear homogeneous recurrence relation with constant coefficients

Let $a_n = r^n$ is a solution of (1). The characteristic equation is

$$r^2 - r - 2 = 0$$

$$(r-2)(r+1) = 0$$

or, $r = 2, -1$ distinct real roots. So the general solution is

$$a_n = b_1 (2)^n + b_2 (-1)^n \qquad \dots(2)$$

Again $a_0 = 0$ implies $b_1 + b_2 = 0$

And $a_1 = 1$ implies $2b_1 - b_2 = 1$.

The solution of these two equations are $b_1 = 1/3$ and $b_2 = -1/3$. Hence the explicit solution is given by

$$a_n = (1/3)2^n - (1/3)(-1)^n$$

Example 8. Solve the recurrence relation of the Fibonaci sequence of numbers $f_n = f_{n-1} + f_{n-2}$, $n \geq 2$ with the initial condition $f_0 = 0, f_1 = 1$.

Solution. Let, $f_n = r^n$ is a solution of the given equation, the characteristic equation is $r^2 - r - 1 = 0$

$r = (1 \pm \sqrt{5})/2$ *i.e.* $r_1 = (1+\sqrt{5})/2$, $r_2 = (1-\sqrt{5})/2$ are two distinct roots. So the general solution is

$$f_n = b_1\left(\frac{1+\sqrt{5}}{2}\right)^n + b_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Again $f_0 = 0$ implies $b_1 + b_2 = 0$

$f_1 = 1$ implies $b_1(1+\sqrt{5})/2 + b_2(1-\sqrt{5})/2 = 1$

Solving these two equations we get

$$b_1 = \frac{1}{\sqrt{5}}, \quad b_2 = -\frac{1}{\sqrt{5}}$$

Thus the nth Fibonacci number is given explicitly by

$$f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

Multiple Roots: If the characteristic equation of a homogeneous recurrence relation is $(r-2)^3 = 0$, then $r = 2$ is a required root of multiplicity 3. Then the general solution is

$$a_n = (b_1 + n\,b_2 + n^2\,b_3)2^n$$

In general, if r is a root of the characteristic equation of mth order of a given recurrence relation with multiplicity m, then the general form of the solution is

$$a_n = (b_1 + n\,b_2 + n^2\,b_3 + \ldots\ldots + n^{m-1}b_m)r^n$$

where $b_1, b_2, \ldots\ldots, b_m$ are constants which may be chosen to satisfy any initial conditions.

Mixed Roots: A combination of distinct and multiple roots is also possible *i.e.*, some roots of a characteristic equation are distinct and some root are equal. If the characteristic equation of a homogeneous recurrence relation of 5th order is $(r-2)(r-4)(r-3)^3 = 0$, then $r = 2, 4, 3, 3, 3$. Then the general solution is

$$a_n = b_1\,2^n + b_2 4^n + (b_3 + nb_4 + n^2 b_5)3^n$$

Example 9. Solve the recurrence relation

$$a_n = 4(a_{n-1} - a_{n-2}) \text{ with initial conditions } a_0 = a_1 = 1 \quad \ldots (1)$$

Solution. The given relation is $a_n - 4a_{n-1} + 4a_{n-2} = 0$

Let $a_n = r^n$ is a solution of (1).

Then the characteristic equation is $r^2 - 4r + 4 = 0$ which gives $r = 2, 2$.

Thus the general solution is $a_n = (b_1 + nb_2)2^n$

So, $a_0 = b_1$ and $a_1 = 2(b_1 + b_2)$

Now $a_0 = 1$ gives $b_1 = 1$
and $a_1 = 1$ gives $2(b_1 + b_2) = 1 \Rightarrow b_2 = -1/2$
So, the required solution is

$$a_n = (1 - \tfrac{1}{2} n) 2^n$$

Example 10. Solve the recurrence relation

$$a_n - 8a_{n-1} + 21a_{n-2} - 18a_{n-3} = 0$$

Solution. Let $a_n = r^n$ be a solution of the given equation.
The characteristic equation is $r^3 - 8r^2 + 21r - 18 = 0$

i.e. $r^3 - 2r^2 - 6r^2 + 12r + 9r - 18 = 0$

or $r^2(r-2) - 6r(r-2) + 9(r-2) = 0$

or $(r-2)(r^2 - 6r + 9) = 0$

or $(r-2)(r-3)^2 = 0$ which gives $r = 2, 3, 3$

So the general solution is $a_n = (b_1 + b_2 n) 3^n + b_3 2^n$.

Note : We restrict our examples to linear recurrence relation whose characteristic equations have only real roots.

(b) Non Homogeneous Recurrence Relation

A second order non homogeneous linear recurrence relation with constant coefficients is of the form

$$a_{n-2} + 5a_{n-1} + 6a_n = f(n)$$

Its solution a_n consists of two parts

1. Homogeneous solution $a_n^{(h)}$ of the given recurrence relation by keeping $f(n) = 0$.
2. Particular solution $a_n^{(p)}$ of the given recurrence relation with $f(n)$ on the right hand side.

The required general solution is

$$a_n = a_n^{(h)} + a_n^{(p)}$$

Particular Solution

There is no general method for finding the particular solution of a recurrence relation for every function $f(n)$. We shall discuss method of undetermined co-efficients for finding particular solution. This method is useful when the function $f(n)$ consists of special forms. Depending on certain forms of $f(n)$, we consider a trial solution containing a number of unknown constant coefficients which are to be determined by substitution in the recurrence relation. The trial solutions to be used in each case are shown in the following table where A, A_0, A_1, ... represent the unknown constant coefficients to be determined.

Form of $f(n)$	Trial Function
b^n (if b is not a root of characteristic equation)	Ab^n
Polynomial P (n) of degree m	$A_0 + A_1 n + A_2 n^2 + \dots\dots\dots + A_m n^m$
c^n P (n) (if c is not a root of characteristic equation)	$c^n (A_0 + A_1 n + A_2 n^2 + \dots\dots\dots + A_m n^m)$
b^n (if b is a root of characteristic equation of multiplicity s)	$A_0 n^S b^n$
c^n P (n) (if c is a root of characteristic equation of multiplicity t)	$n^t (A_0 + A_1 n + A_2 n^2 + \dots\dots\dots + A_m n^m) c^n$
$\sin bn$ or $\cos bn$	$A_0 \sin bn + A_1 \cos bn$
$b^n \sin bn$ or $b^n \cos bn$	$b^n (A_0 \sin bn + A_1 \cos bn)$

Note: (*i*) If $f(n)$ is constant i.e. a polynomial of degree zero, the trial solution is taken as A.
(*ii*) If $f(n)$ is a linear combination of the above forms, the trial solution is taken as the sum of corresponding trial functions with different unknown constant coefficients to be determined.

Solved Examples

(1) When $f(n) = \alpha$, a constant

Example 11. Solve. $a_{n+2} - 5a_{n+1} + 6a_n = 2$ with initial condition $a_0 = 1$ and $a_1 = -1$.

Solution. The associated homogeneous recurrence relation is

$$a_{n+2} - 5a_{n+1} + 6a_n = 0 \quad \ldots (1)$$

Let $a_n = r^n$ be a solution of (1).

The characteristic equation is $r^2 - 5r + 6 = 0 \Rightarrow r = 3, 2$.

So, the solution of (1) is $a_n^{(h)} = C_1 3^n + C_2 2^n$.

To find the particular solution of the given equation, let $a_n = A$. Substituting in the given equation.

$$A - 5A + 6A = 2 \Rightarrow A = 1$$

$\therefore$ Particular solution $\quad a_n^{(p)} = 1$

Hence the general solution is $\quad a_n = a_n^{(h)} + a_n^{(p)}$

$$= C_1 3^n + C_2 2^n + 1 \quad \ldots (2)$$

To find C_1 and C_2, put $n = 0$ and $n = 1$ in (2)

$$a_0 = C_1 + C_2 + 1$$

or, $\quad 1 = C_1 + C_2 + 1 \quad \ldots (3)$

or, $\quad C_1 + C_2 = 0$

Again, $\quad a_1 = 3C_1 + 2C_2 + 1$

or, $\quad -1 = 3C_1 + 2C_2 + 1 \quad \ldots (4)$

or, $\quad 3C_1 + 2C_2 = -2$

Solving (3) and (4), we get $\quad C_1 = -2$ and $C_2 = 2$.

Putting the values of C_1 and C_2 in (2), the required solution is

$$a_n = -2.3^n + 2.2^n + 1.$$

(2) When $f(n)$ X is a polynominal

Example 12. *Solve the following*

$$y_{n+2} - y_{n+1} - 2y_n = n^2$$

Solution. Substituting $y_n = r^n$ in the associated homogeneous recurrence relation, the characteristic equation is $r^2 - r - 2 = 0$

or, $\quad (r + 1)(r - 2) = 0$

This gives $\quad r = -1, r = 2$

The solution of the associated homogeneous recurrence relation is

$$y_n^{(h)} = C_1 (-1)^n + C_2.2^n$$

Let the particular solution of the given equation be

$$y_n = A_0 + A_1 n + A_2 n^2 \text{ (Since } f(n) \text{ is a polynomial of degree 2)}$$

Substituting in the given equation, we have

$A_0 + A_1 (n+2) + A_2 (n+2)^2 - [A_0 + A_1 (n+1) + A_2 (n+1)^2] - 2(A_0 + A_1 n + A_2 n^2) = n^2$

or, $(-2A_0 + A_1 + A_2) + (-2A_1 + 2A_2)n - 2A_2 n^2 = n^2$

On comparing the coefficients of like powers of n, we have

$$-2A_0 + A_1 + 3A_2 = 0 \quad \ldots (1)$$

$$-2A_1 + 2A_2 = 0 \quad \ldots (2)$$

$$-2A_2 = 1$$

From (3) $\quad A_2 = -1/2$

We have from (2) $\quad A_1 = A_2 = -1/2$

From (1) $\quad -2A_0 - 1/2 - 3/2 = 0$

$$A_0 = -1$$

Therefore, particular solution of given recurrence relation is

$$y_n^{(p)} = -1 - (1/2)\,n - (1/2)\,n^2$$

Hence the general solution of the given recurrence relation is

$$y_n = y_n^{(h)} + y_n^{(p)}$$

$$y_n = C_1\,(-1)^n + C_2\,.\,2^n - 1 - n/2 - (1/2)\,n^2$$

(3) $f(n) = a^n$, were a is a root of characteristic equation

Example 13. *Solve* $a_{n+2} - 4a_{n+1} + 4a_n = 2^n$

Solution. Let $a_n = r^n$ be a solution of the associated homogeneous recurrence relation

$$a_{n+2} - 4a_{n+1} + 4a_n = 0$$

The characteristic equation is

$$r^2 - 4r + 4 = 0$$

or $\quad (r-2)^2 = 0$

has a single root 2 of multiplicity two. So the solution of associated homogeneous relation is $a_n^{(h)} = (C_1 + C_2 n)2^n$.

To find the particular solution of the given relation, we note $b = 2$ is a root of characteristic equation with multiplicity $s = 2$. So, the particular solution has the form $a_n = A_0 n^2\,.\,2^n$. Substituting in the given relation, we get

$$A_0\,(n+2)^2\,2^{n+2} - 4\,A_0\,(n+1)^2\,2^{n+1} + 4\,A_0\,n^2\,.\,2^n = 2^n$$

$$\Rightarrow 4A_0\,(n+2)^2 - 8\,A_0\,(n+1)^2 + 4\,A_0\,n^2 = 1$$

$$\Rightarrow \quad A_0 = 1/8 \quad \therefore \quad \text{Particular solution } a_n^{(p)} = \frac{1}{8}n^2 2^n$$

Hence the general solution is

$$a_n = a_n^{(h)} + a_n^{(p)}$$

$$= (C_1 + C_2 n)2^n + (1/8)n^2\,.\,2^n.$$

Example 14(*a*). *What form does a particular solution of the linear homogeneous recurrence relation* $a_{n+2} - 6a_{n+1} + 9a_n = f(n)$ *have when* $f(n) = 3^n$, $f(n) = n\,.\,3^n$ *and* $f(n) = (n^2+1)\,3^n$?

Solution. Let, $a_n = r^n$ be a solution of the associated homogeneous recurrence relation

$$a_{n+2} - 6a_{n+1} + 9a_n = 0$$

The characteristic equation $\quad r^2 - 6r + 9 = 0$

or $\quad (r-3)^2 = 0$

has a single root 3 of multiplicity two. Thus if $f(n) = 3^n$, the particular solution has the form $A_0\,n^2 3^n$. If $f(n) = n3^n$, the particular solution has the form $n^2\,(A_0 + A_1 n)\,3^n$. If $f(n) = (n^2+1)\,3^n$, the particular solution has the form $n^2\,(A_0 + A_1 n + A_2 n^2)3^n$.

Example 14(*b*). *Solve*

(*i*) $a_n - 7a_{n-1} + 12a_{n-2} = n\,.\,4^n$

(*ii*) $a_n - 4a_{n-1} + 4a_{n-2} = n + 4^n$

Solution. (*i*) Let $a_n = r^n$ be a solution of the associated homogeneous recurrence relation

$$a_n - 7a_{n-1} + 12a_{n-2} = 0$$

The characteristic equation is

$$r^2 - 7r + 12 = 0$$
$$(r-4)(r-3) = 0$$
$$r = 4,\ r = 3.$$

So, the solution of associated homogeneous relation is

$$a_n^{(h)} = C_1(4)^n + C_2(3)^n.$$

To find the particular solution of the given relation, we note 4 is a root of the characteristic equation with multiplicity $s = 1$.

So the particular solution is of the form

$$a_n = n(A_0 + A_1 n)(4)^n.$$

Substituting in the given relation,

$$n(A_0 + A_1 n)(4)^n - 7\{(n-1)(A_0 + A_1(n-1)).(4)^{n-1}\}$$
$$+ 12\{(n-2) + A_0 + A_1(n-2).(4)^{n-2}\} = n(4)^n$$

or
$$(nA_0 + A_1 n^2) - \frac{7}{4}\{n(A_0 - 2A_1) + (A_1 - A_0) + n^2 A_1\}$$
$$+ \frac{12}{16}\{n(A_0 - 4A_1) + (4A_1 - 2A_0) + n^2 A_1\} = n$$

or
$$\frac{n}{2}A_1 + n^2.(0) + \frac{A_0}{4} + \frac{5A_1}{4} = n$$

$\Rightarrow$ $\frac{A_1}{2} = 1$ or $A_1 = 2$ and $\frac{A_0}{4} + \frac{5A_1}{4} = 0 \Rightarrow A_0 = -10$

Hence
$$a_n^{(p)} = n(-10 + 2n).(4)^n$$

So, the general solution is
$$a_n = a_n^{(h)} + a_n^{(p)}$$
$$= 4(4)^n + C_2(3)^n + n(2^n - 10).(4)^n.$$

(ii) Let $a_n = r^n$ be a solution of the associated homogeneous recurrence relation

$$a_n - 4a_{n-1} + 4a_{n-2} = 0$$

The characteristic equation is $r^2 - 4r + 4 = 0 \Rightarrow r = 2, 2.$

So,
$$a_n^{(h)} = (C_1 + C_2 n)\, 2^n.$$

To find the particular solution of the given relation, we note n is a polynomial of degree one and 4^n is an exponential function and 4 is not a root of the characteristic equation so the particular solution is of the form.

So,
$$a_n = (A_0 + A_1 n) + A_2(4)^n.$$

Substituting in the given recurrence relation, we get

$$\{(A_0 + A_1 n) + A_2(4)^n\} - 4\{A_0 + A_1(n-1) + A_2(4)^{n-1}\}$$
$$+ 4\{A_0 + A_1(n-2) + A_2(4)^{n-2}\} = n + (4)^n$$

or
$$nA_1 + (A_0 - 4A_1) + (4)^n \frac{A_2}{4} = n + (4)^n$$

$$\Rightarrow \quad A_1 = 1,\ A_2 = 4,\ A_0 = 4$$

$$\therefore \quad a_n^{(p)} = (4 + n) + 4.4^n = 4 + n + 4^{n+1}$$

So, the general solution is

$$a_n = a_n^{(h)} + a_n^{(p)}$$
$$= (C_1 + C_2 n)2^n + (4 + n) + 4^{n+1}.$$

11.5. Solution of Non-linear Recurrence Relations

Sometimes non-linear recurrence relations can be solved by converting them into linear recurrence by suitable substitution.

Example 15. *Solve the recurrence relations*

$$a_n^2 - 2a_{n-1}^2 = 4 \quad \text{for } n \geq 1 \text{ and } a_0 = 3.$$

Solution. Let $b_n = a_n^2$. Then the given relation reduces to $b_n - 2b_{n-1} = 4$ and $b_0 = 9$. The characteristic equation is $r - 2 = 0 \Rightarrow r = 2$. Hence $b_n^{(h)} = C.2^n$.

The general form of a particular solution corresponding to the constant 4 is A. Substituting in the equations A in place of b_n, we get

$$A - 2A = 4 \Rightarrow A = -4$$

$\therefore$ $$b_n^{(p)} = -4$$

Hence the general solution is

$$b_n = b_n^{(h)} + b_n^{(p)}$$
$$= C.2^n - 4$$

$\therefore$ $$b_0 = C.2^0 - 4 \Rightarrow 9 = C - 4 \quad i.e. \ C = 13$$

Hence $$b_n = 13.2^n - 4$$

$\therefore$ $$a_n = \sqrt{13.2^n - 4}$$ which is the required solution.

Example 16. *Solve the recurrence relation*

$$a_n = 7a_{n/3} + 5 \text{ where } n = 3^k \text{ and } a_1 = 1$$

Solution. Let $b_k = a_n = a_{3^k}$. Then the given relation reduces to

$$b_k = 7b_{k-1} + 5 \quad i.e. \quad b_k - 7b_{k-1} = 5 \quad \text{and} \quad b_0 = a_1 = 1$$

The linear relation has the characteristic equation $r - 7 = 0 \Rightarrow r = 7$. Hence $b_k^{(h)} = C7^k$

The general form of a particular solution corresponding to the constant 5 is A. Substituting in the equation A in place of b_k, we get

$$A - 7A = 5 \Rightarrow A = -\frac{5}{6} \quad \therefore \ b_k^{(p)} = -\frac{5}{6}$$

Hence the general solution is

$$b_k = b_k^{(h)} + b_k^{(p)}$$
$$= C7^k - \frac{5}{6}$$

$\therefore$ $$b_0 = C - \frac{5}{6} \Rightarrow 1 = C - \frac{5}{6} \quad \therefore \ C = \frac{11}{6}$$

Hence $$b_k = \frac{11}{6}7k - \frac{5}{6}$$

$\therefore$ $$a_n = \frac{11}{6}7^{\log_3 n} - \frac{5}{6} \qquad [\because n = 3^k \Rightarrow k = \log_3 n]$$

$$= \frac{11}{6}n^{\log_3 7} - \frac{5}{6} \qquad [\because 7^{\log_3 n} = n^{\log_3 7}]$$

Generating Function

The generating function for the sequence $a_0, a_1, \ldots, a_k, \ldots$ of real numbers is infinite series,

$$G(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_k x^k + \ldots = \sum_{k=0}^{\infty} a_k x^k \qquad \ldots (1)$$

For example, the generating functions for the sequences $\{a_k\}$ where $a_k = 2$, $a_k = 3^k$ and $a_k = (k+1)$ are $\sum_{k=0}^{\infty} 2x^k$, $\sum_{k=0}^{\infty} 3^k x^k$ and $\sum_{k=0}^{\infty} (k+1)x^k$ respectively.

It is often possible to find a **closed form expression** for $G(x)$ which can be manipulated algebraically to provide useful combinatorial information.

Some Special Generating Functions

1. The generating function of the sequence 1, 1, 1, ... is

$$G(x) = 1 + x + x^2 + \ldots = \sum_{k=0}^{\infty} x^k$$

which can be written in closed form as

$$G(x) = (1-x)^{-1} = \frac{1}{1-x}$$

2. The generating function of the sequence 1, 2, 3, 4, ... is

$$G(x) = 1 + 2x + 3x^2 + 4x^3 + \ldots = \sum_{k=0}^{\infty} (k+1)x^k$$

$$= (1-x)^{-2} = \frac{1}{(1-x)^2} \text{ in closed form } |x| < 1$$

3. The generating function of the sequence 0, 1, 2, 3, ... is

$$G(x) = 0 + 1x + 2x^2 + 3x^3 + \ldots = \sum_{k=0}^{\infty} k\,x^k$$

$$= x(1 + 2x + 3x^2 + \ldots)$$

$$= x(1-x)^{-2} = \frac{x}{(1-x)^2} \text{ in closed form.}$$

4. The generating function of the sequence $1, a, a^2, a^3, \ldots$ is

$$G(x) = 1 + ax + a^2x^2 + a^3x^3 + \ldots = \sum_{k=0}^{\infty} a^k x^k$$

$$= (1-ax)^{-1} = \frac{1}{1-ax} \text{ in closed form } |ax| < 1$$

We can derive a closed form expression for $G(x)$ by involving the formula for the sum of a geometric series. We can also apply indirect method as follows.

Example 17. Find the generating function for the sequence $1, a, a^2, \ldots\ldots$ where a is a fixed constant.

Solution. Let $\quad G(x) = 1 + ax + a^2x^2 + a^2x^3 + \ldots$

So, $\quad G(x) - 1 = ax + a^2x^2 + a^3x^3 + \ldots$

or $\quad \dfrac{G(x)-1}{ax} = 1 + ax + a^2x^2 + \ldots$

or $$\frac{G(x)-1}{ax} = G(x) \Rightarrow G(x) = \frac{1}{1-ax}$$

The required generating function is $\frac{1}{1-ax}$.

The summary of various types of sequence and the corresponding generating functions are given in the following table.

Sl. No.	General term of Sequence a_k	Generating Function G (x)
1.	1	$\frac{1}{1-x}$
2.	$(-1)^k$	$\frac{1}{1+x}$
3.	$k+1$	$\frac{1}{(1-x)^2}$
4.	k	$\frac{x}{(1-x)^2}$
5.	$k(k+1)$	$\frac{2x}{(1-x)^3}$
6.	$(k+1)(k+2)$	$\frac{2}{(1-x)^3}$
7.	a^k	$\frac{1}{1-ax}$
8.	$(-a)^k$	$\frac{1}{1+ax}$
9.	$\frac{1}{k!}$	e^x

Addition and Multiplication of two Generating Functions

Arithmetic operations allow us to create new generating functions from old ones. Suppose that two generating functions are given :

$$F(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots\ldots\ldots = \sum_{k=0}^{\infty} a_k x^k,$$

$$G(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + \ldots\ldots\ldots\ldots = \sum_{k=0}^{\infty} b_k x^k.$$

$$F(x) + G(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \ldots\ldots\ldots$$

$$= \sum_{k=0}^{\infty} (a_k + b_k)x^k \qquad \ldots (1)$$

Then F(x) + G(x) is the generating function of $a_k + b_k$

Multiplication is modelled on the familiar rules for multiplying two polynomials.

g generating functions are

Function G (x)

Suppose that

... (1)

$$F(x) \cdot G(x) = (a_0 + a_1x + a_2x^2\ a_3x^3 + \ldots\ldots) \cdot (b_0 + b_1x + b_2x^2 + b_3x^3 + \ldots\ldots),$$
$$= a_0b_0 + (a_0b_1 + a_1b_0)\,x + (a_0b_2 + a_1b_1 + a_2b_0)\,x^2$$
$$+ (a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0)\,x^3 + \ldots + \left(\sum_{i=0}^{k} a_ib_{k-i}\right)x^k + \ldots$$
$$= \sum_{k=0}^{\infty}\left(\sum_{i=0}^{k} a_ib_{k-i}\right)x^k. \qquad \ldots(2)$$

Formula (2) is the definition of the product of two generating functions, and it has become customary to distinguish this special kind of multiplication by calling it **convolution.** Then F(x) . G(x) is the generating function $a_k \times b_k$ (convolution of a_k and b_k)

Shifting Properties of Generating Function

1. If $G(x) = \sum_{n=0} a_nx^n$ generates the sequence $(a_0, a_1, a_2 \ldots\ldots\ldots)$, then x G (x) generates the sequence $(0, a_0, a_1, a_2, \ldots\ldots\ldots)$; x^2 G (x) generates $(0, 0, a_0, a_1, a_2, \ldots\ldots\ldots)$, and, in general x^k G (x) generates $(0, 0, \ldots\ldots\ldots, 0, a_0, a_1, a_2, \ldots\ldots\ldots)$ where there are k zeros before a_0.

For instance, we know that $1/(1-x) = \sum_{n=0}^{\infty} x^n$ generates the sequence (1, 1, 1, ...), that is the sequence $\{a_n\}$ where $a_n = 1$ for each $n \geq 0$.

Thus, $\dfrac{x}{1-x} = \sum_{n=0}^{\infty} x^{n+1} = \sum_{r=1}^{\infty} x^r$ generates (0, 1, 1, 1, ...),

and $\dfrac{x^2}{1-x} = \sum_{n=0}^{\infty} x^{n+2} = \sum_{r=2}^{\infty} x^r$ generates (0, 0, 1, 1, 1, ...).

2. If $G(x) = \sum_{n=0}^{\infty} a_nx^n$ generates $(a_0, a_1, a_2, \ldots\ldots\ldots)$, then $G(x) - a_0 = \sum_{n=1}^{\infty} a_nx^n$ generates $(0, a_1, a_2, \ldots\ldots\ldots)$, $G(x) - a_0 - a_1x = \sum_{n=2}^{\infty} a_nx^n$ generates $(0, 0, a_2, a_3, \ldots\ldots)$ and in general $G(x) - a_0 - a_1x \ldots\ldots - a_{k-1}\,x^{k-1}$ generates $(0, 0, \ldots, 0, a_k, a_{k+1} \ldots\ldots\ldots)$, where there are k zeros before a_k.

3. Dividing by powers of x shifts the sequence to the left. For instance, $(G(x) - a_0)/x = \sum_{n=1}^{\infty} a_nx^{n-1} = \sum_{n=0}^{\infty} a_{n+1}x^n$ generates the sequence $(a_1, a_2, a_3, \ldots\ldots\ldots)$; $(G(x) - a_0 - a_1x)/x^2 = \sum_{n=2}^{\infty} a_nx^{n-2} = \sum_{n=0}^{\infty} a_{n+2}x^n$ generates $(a_2, a_3, a_4, \ldots\ldots\ldots)$; and in general, for $k \geq 1$, $(G(x) - a_0 - a_1x - \ldots - \ldots - a_{k-1}\,x^k)/x^k$ generates $(a_k, a_{k+1}, a_{k+2}, \ldots)$.

Example 18. Find a closed form for the generating function for each of the following sequence.

(*a*) 0, 0, 1, 1, 1, ...

(*b*) 1, 1, 0, 1, 1, 1, 1, 1, ...

(*c*) 1, 0, –1, 0, 1, 0, –1, 0, 1, ...

(*d*) $C(8, 0)$, $C(8, 1)$, $C(8, 2)$, ..., $C(8, 8)$, 0, 0, ...

(*e*) 3, –3, 3, –3, 3, –3, ...

Solution. (*a*) We know

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \text{.........} \infty = \sum_{n=0}^{\infty} x^n$$

So, the generating function of 1, 1, 1, ... is $\frac{1}{1-x}$

Now $$\frac{x^2}{1-x} = \sum_{n=0}^{\infty} x^{n+2}$$

Hence $\frac{x^2}{1-x}$ is the generating function of 0, 0, 1, 1, 1, ...

(*b*) Here $\frac{1}{1-x} - x^2 = 1 + x + x^3 + \text{.........} \infty = \sum_{\substack{n=0 \\ n\neq 2}}^{\infty} x^n$

So, the generating function of 1, 1, 0, 1, 1, 1, ... is $\frac{1}{1-x} - x^2$.

(*c*) We know $$\frac{1}{1+x^2} = (1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + x^8 \text{.........} \infty.$$

$$= 1 + 0\,.\,x + (-1)\,x^2 + 0\,.\,x^3 + 1\,.\,x^4 + 0.x^5 + (-1)\,x^6 + \text{.........}$$

So, the generating function of 1, 0, – 1, 0, 1, 0, – 1, ... is $\frac{1}{1+x^2}$

(*d*) We know $$(1+x)^8 = C(8, 0)\,x^0 + C(8, 1)\,x + \text{.........} + C(8, 8)\,x^8 + 0 + 0 + \text{.........}$$

$$= \sum_{n=0}^{\alpha} C\,(8, n) x^n.$$

So, the generating function of C (8, 0), C (8, 1), ..., C (8, 8), 0, 0 .... is $(1+x)^8$.

(*e*) We have $$\frac{3}{1+x} = 3\,(1+x)^{-1} = 3\,(1 - x + x^2 - x^3 + \text{.........})$$

$$= 3 + (-3)\,x + 3x^2 + (-3)\,x^3 + \text{.........}$$

$$= \sum_{n=0}^{\infty} (-3)^n x^n.$$

Hence, the required generating function is $\frac{3}{1+x}$.

Example 19. (***a***) Find the generating function of a sequence $\{a_k\}$ if $a_k = 2 + 3k$.

Solution. The generating function of a sequence whose general term is 2 is

$$F(x) = \frac{2}{1-x}$$

The generating function of a sequence whose general term is $3k$ is

$$G(x) = \frac{3x}{(1-x)^2}.$$

Hence the required generating function is

$$F(x) + G(x) = \frac{2}{1-x} + \frac{3x}{(1-x)^2}.$$

(a) Determine the generating function of the following sequences

(i) $a_r = \begin{cases} 2^r & \text{if} \quad r \text{ is even} \\ -2^r & \text{if} \quad r \text{ is odd} \end{cases}$

(ii) $a_r = (r+1)\,3^r$

(iii) $a_r = 5^r + (-1)^r\,3^r + 8^r + {}^3C_r$

(iv) $f_r \dfrac{r(r+1)}{2}, (r \geq 0)$

Solution. (i) The required generating function is given by

$$\begin{aligned} G(x) &= \sum_{r=0}^{\infty} a_r x^r = \sum_{r=0}^{\infty} (-1)^r 2^r x^r \\ &= 2^0 2^0 - 2x + 2^2 x^2 - 2^3 x^3 + \ldots\ldots \\ &= 1 - 2x + (2x)^2 - (2x)^3 + \ldots\ldots \\ &= (1+2x)^{-1} = \frac{1}{1+2x} \end{aligned}$$

(ii) The required generating function is given by

$$\begin{aligned} G(x) &= \sum_{r=0}^{\infty} a_r x^r = \sum_{r=0}^{\infty} (r+1) 3^r x^r \\ &= \sum_{r=0}^{\infty} r 3^r x^r + \sum_{r=0}^{\infty} 3^r x^r \\ &= \{0 + 1\,(3x)^1 + 2(3x)^2 + 3(3x)^3 + \ldots\ldots\} + \{1 + (3x)^1 + (3x)^2 + (3x)^3 + \ldots\ldots\ldots\} \\ &= (3x)\,\{1 + 2(3x) + 3(3x)^2 + \ldots\ldots\ldots\ldots\} + \{1 + (3x) + (3x)^2 + (3x)^3 + \ldots\ldots\ldots\ldots\} \\ &= 3x(1-3x)^{-2} + (1-3x)^{-1} \\ &= \frac{3x}{(1-3x)^2} + \frac{1}{(1-3x)} = \frac{1}{(1-3x)^2} \end{aligned}$$

(iii) The required generating function is given by

$$\begin{aligned} G(x) &= \sum_{r=0}^{\infty} a_r x^r = \sum_{r=0}^{\infty} \left\{5^r + (-1)^r 3^r + 8^r + {}^3C_r\right\} x^r \\ &= \sum_{r=0}^{\infty} 5^r x^r + \sum_{r=0}^{\infty} (-1)^r 3^r x^r + \sum_{r=0}^{\infty} 8^r x^r + \sum_{r=0}^{\infty} {}^3C_r x^r \\ &= (1 + 5x + (5x)^2 + (5x)^3 + \ldots\ldots\ldots\} + \{1 + 3x + (3x)^2 + (3x)^3 + \ldots\ldots\ldots\} \\ &\quad + \{1 + 8x + (8x)^2 + (8x)^3 + \ldots\ldots\} + \{{}^3C_0 + {}^3C_1 + {}^3C_2 x^2 + 3C_3 x^3 + 0 + 0 + \ldots\} \\ &= (1-5x)^{-1} + (1+3x)^{-1} + (1-8x)^{-1} + (1+x)^3 \\ &= (1-5x)^{-1} + (1+3x)^{-1} + (1-8x)^{-1} + (1+x)^3 \\ &= \frac{1}{1-5x} + \frac{1}{1+3x} + \frac{1}{1-8x} + (1+x)^3 \\ &= \frac{4 - 27x - 25x^2 + 94x^3 + 367x^4 + 361x^5 + 120x^6}{(1-5x)(1+3x)(1-8x)(1+x)^3} \end{aligned}$$

(*iv*) The required generating function is given by

$$G(x) = \sum_{r=1}^{\infty} f_r x^r \ (r \geq 1)$$

$$= \sum_{r=1}^{\infty} \frac{r(r+1)}{2} x^r$$

$$= x + 3x^2 + 6x^3 + 10x^4 + \ldots\ldots$$
$$= x(1 + 3x + 6x^2 + 10x^3 + \ldots\ldots)$$
$$= x(1-x)^{-3} = \frac{x}{(1-x)^3}$$

Example 20. Find the sequences corresponding to the ordinary generating functions (*a*) $(3 + x)^3$, and (*b*) $3x^3 + e^{2x}$.

Solution. (*a*) $(3 + x)^3 = 27 + 27x + 9x^2 + x^3$; the sequence is $(27, 27, 9, 1, 0, 0, 0, \ldots)$.

(*b*) $3x^3 + e^{2x} = 1 + 2x + \frac{2^2}{2!}x^2 + \left(3 + \frac{2^3}{3!}\right)x^3 + \frac{2^4}{4!}x^4 + \frac{2^5}{5!}x^5 + \ldots$

The sequence is $(1, 2, 2^2/2!, 2^3/3! + 3, 2^4/4!, \ldots)$.

11.7 Solutions of Linear Recurrence Relations using Generating Functions

We can find the solution to a recurrence relation with initial conditions by finding an explicit formula for the associated generating function. This is illustrated by the following examples.

Example 21. Use generating functions to solve the recurrence relation.

(*i*) $a_n = 3a_{n-1} + 2$ $\qquad a_0 = 1$

(*ii*) $a_n - 9a_{n-1} + 20a_{n-2} = 0$ $\qquad a_0 = -3, a_1 = -10$

(*iii*) $a_{n+2} - 2a_{n+1} + a_n = 2^n$ $\qquad a_0 = 2, a_1 = 1.$

Solution. (*i*) Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$ where $G(x)$ is the generating function for the sequence $\{a_n\}$.

Multiplying each term in the given recurrence relation by x^n and summing from 1 to ∞, we get

$$\sum_{n=1}^{\infty} a_n x^n = 3\sum_{n=1}^{\infty} a_{n-1} x^n + 2\sum_{n=1}^{\infty} x^n$$

$$G(x) - a_0 = 3x\,G(x) + 2\left[\frac{1}{1-x} - 1\right]$$

$$\left(\text{Since } \quad xG(x) = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1} x^n\right)$$

$$\therefore \quad G(x) - 3x\,G(x) = 1 + \frac{2x}{1-x} \quad (a_0 = 1)$$

or $$G(x) = \frac{1+x}{(1-x)(1-3x)} = \frac{2}{1-3x} - \frac{1}{1-x} \quad \text{(by partial fraction)}$$

$$\therefore \quad \sum_{n=0}^{\infty} a_n x^n = 2\sum_{n=0}^{\infty} 3^n x^n - \sum_{n=0}^{\infty} x^n$$

Hence $a_n = 2 \cdot 3^n - 1$ which is the required solution.

(*ii*) Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$ where $G(x)$ is the generating function for the sequence $\{a_n\}$.

Multiplying each term in the given recurrence relation by x^n and summing from 2 to ∞, we get

$$\sum_{n=2}^{\infty} a_n x^n - 9\sum_{n=2}^{\infty} a_{n-1} x^n + 20\sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

or $$[G(x) - a_0 - a_1 x] - 9x\,[G(x) - a_0] + 20x^2\, G(x) = 0$$

or $$G(x)\,[1 - 9x + 20x^2] = a_0 + a_1 x - 9a_0 x$$

or $$G(x) = \frac{a_0 + a_1 x - 9a_0 x}{1-9x+20x^2} = \frac{-3-10x+27x}{1-9x+20x^2}$$

$(\because a_0 = -3 \text{ and } a_1 = -10)$

$$= \frac{-3+17x}{(1-5x)(1-4x)}$$

$$G(x) = \frac{2}{1-5x} - \frac{5}{1-4x} \qquad \text{(by partial fraction)}$$

or $$\sum_{n=0}^{\infty} a_n x^n = 2\sum 5^n x^n - 5\sum 4^n x^n$$

$\therefore$ $$a_n = 2.5^n - 5.4^n \text{ which is the required solution.}$$

Hence

(iii) Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$ where $G(x)$ is the generating function for the sequence $\{a_n\}$.

Multiplying each term in the given recurrence relation by x^n and summing from 0 to ∞, we get

$$\sum_{n=0}^{\infty} a_{n+2} x^n - 2\sum_{n=0}^{\infty} a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} 2^n x^n$$

or $$\frac{G(x) - a_0 - a_1 x}{x^2} - 2\left(\frac{G(x) - a_0}{x}\right) + G(x) = \frac{1}{1-2x}$$

or $$\frac{G(x) - 2 - x}{x^2} - 2\left(\frac{G(x) - 2}{x}\right) + G(x) = \frac{1}{1-2x}$$

or $$(x^2 - 2x + 1)\, G(x) = 2 + 3x + \frac{x^2}{1-2x}$$

or $$G(x) = \frac{2}{(1-x)^2} + \frac{3x}{(1-x)^2} + \frac{x^2}{(1-2x)(1-x)^2}$$

By partial fraction $$\frac{x^2}{(1-2x)(1-x)^2} = \frac{1}{1-2x} - \frac{1}{(1-x)^2}$$

$\therefore$ $$G(x) = \frac{1}{(1-x)^2} + \frac{3x}{(1-x)^2} + \frac{1}{1-2x}$$

$\therefore$ $$\sum a_n x^n = \sum (n+1)x^n + 3\sum n x^n + \sum 2^n x^n$$

Hence $$a_n = (n+1) + 3n + 2^n = 1 + 4n + 2^n.$$

11.8 Counting Problems and Generating Functions

As mentioned earlier, the method of generating function can be applied to many areas of discrete mathematics besides the solution of recurrence relation. One such area of application is to a problem of combination such as the number of ways to select or distribute the objects of different kinds, subject to a variety of constraints.

Example 22. *Using generating function show that combination of n objects taking r at a time with repetition is C (n + r – 1, r).*

Solution. Since each object can be selected as many times as we wish, the generating function is

$$G(x) = (1 + x + x^2 + \ldots \infty)^n$$

where each factor $1 + x + x^2 + \ldots$ corresponds to the number of times one of the objects is selected.

To find the number of ways of selecting r objects amongst n objects with unlimited repetitions we find the co-efficient of x^r in $G(x)$.

Now $\quad G(x) = (1 + x + x^2 + \ldots \infty)^n = (1 - x)^{-n}$

Hence the co-efficient of $\quad x^r = (-1)^r \dfrac{(-n)(-n-1)\ldots(n-r+1)}{r!}$

$$= \frac{(n+r-1)!}{r!\,(n-r)!} = C(n+r-1, r).$$

Example 23. Give a combinatorial interpretation of the coefficient of x^6 in the expansion $(1 + x + x^2 + x^3 + \ldots\ldots)^3$ and hence find the coefficient.

Solution. Here x^6 in the expansion $(1 + x + x^2 + \ldots\ldots)^3 = (1 + x + x^2 + \ldots\ldots)(1 + x + x^2 + \ldots\ldots)(1 + x + x^2 + \ldots\ldots)$ correspond to the ways that the three terms can be selected, with one from each factor that have exponenents adding up to 6 which is same as to find the number of 6-combination from a set with 3 elements when repetition of elements is allowed.

Now $(1 + x + x^2 + \ldots\ldots)^3 = [(1 - x)^{-1}]^3 = (1 - x)^{-3}$

Hence the required number is $C(n + r - 1, r) = C(3 + 6 - 1, 6) = C(8, 6) = 28$.

Example 24. Use generating functions to find the number of ways to select r objects of n different kinds if at least one object of each kind is selected.

Solution. Let us consider the sequence $\{a_n\}$ where a_n denotes the number of ways to select r objects of n different kinds which includes at least one object of each kind.

Let $G(x) = \Sigma a_r x^r$ be the generating function. Each of the n kind objects constitutes the factor $(x + x^2 + \ldots\ldots)$.

$$\begin{aligned}
\text{Now,}\quad (x + x^2 + x^3 + \ldots\ldots)^n &= x^n(1 + x + x^2 + \ldots\ldots)^n \\
&= x^n(1 - x)^{-n} = x^n \Sigma C(n + r - 1, r)x^r \\
&= \Sigma C(n + r - 1, r)x^{n+r} \\
&= \Sigma C(z - 1, z - n)\, x^Z = \Sigma C(r - 1, r - n)\, x^r
\end{aligned}$$

Hence the number of ways to select r objects of n different kinds which includes at least one object of each kind is $C(r - 1, r - n)$.

Note: The above result can be used to find the number of ways of distributing r similar balls into n numbered boxes where each box is non-empty.

Example 25. *Find the coefficient of*

(*a*) x^{10} in $(1 + x^5 + x^{10} + \ldots\ldots)^3$

(*b*) x^{12} in $(x^3 + x^4 + x^5 + \ldots\ldots)^3$

(*c*) x^{18} in $(x + x^2 + x^3 + x^4 + x^5)(x^2 + x^3 + x^4 + \ldots\ldots)^5$

Solution. (*a*) We know

$$\begin{aligned}
(1 + x^5 + x^{10} + x^{15} + \ldots\ldots)^3 &= [(1 - x^5)^{-1}]^3 \\
&= (1 - x^5)^{-3} = \Sigma C(3 + r - 1, r)\, x^{5r}
\end{aligned}$$

Since we have to find the coefficients of x^{10}, $5r = 10 => r = 2$

The required coefficient is $C(3 + 2 - 1, 2) = C(4, 2) = 6$

(*b*) We have

$$\begin{aligned}
(x^3 + x^4 + x^5 + \ldots\ldots)^3 &= x^9(1 + x + x^2 + \ldots\ldots)^3 \\
&= x^9[(1 - x)^{-1}]^3 = x^9(1 - x)^{-3} \\
&= x^9 \Sigma C(3 + r - 1, r)\, x^r = \Sigma C(3 + r - 1, r)\, x^{9+r}
\end{aligned}$$

Since we have to find the coefficient of 12, $r + 9 = 12 \Rightarrow r = 3$

The required coefficient is $C(3 + 3 - 1, 3) = C(5, 2) = 10$

(*c*) We know

$$
\begin{aligned}
(x + x^2 + x^3 + x^4 + x^5)\,(x^2 + x^3 + x^4 + \ldots\ldots\ldots)^5 &= x\,(1 + x + x^2 + x^3 + x^4)\,x^{10}\,(1 + x + x^2 + \ldots\ldots\ldots\ldots)^5 \\
&= x^{11}\,(1 + x + x^2 + x^3 + x^4)\,[(1 - x)^{-1}]^5 \\
&= x^{11}\,(1 - x^5)\,(1 - x)^{-1}\,(1 - x)^{-5} \\
&= (x^{11} - x^{16})\,(1 - x)^{-6} \\
&= (x^{11} - x^{16})\,\Sigma\, C\,(6 + r - 1, r)\,x^r
\end{aligned}
$$

Hence the coefficient of x^{18} is $C(6 + 7 - 1, 7) - C(6 + 2 - 1, 2) = C(12, 7) - C(7, 2)$.

Generating function can be used in counting the number of integral solution of equations of the form

$$x_1 + x_2 + x_3 + \ldots + x_n = r \quad \ldots (1)$$

where r is a constant and x_i is a non-negative that may be subjected to a specified constraints.

In this case we have only to find the coefficient of x^r in the product of generating function $A_1(x), A_2(x), \ldots\ldots, A_n(x)$, where the exponents of $A_i(x)$ reflect the constraints on x_i. The coefficient of x^r gives the number of integral solutions of equation (1).

Example 26. Find the generating function which will give the number of integral solutions of $x + y + z = 5$ *if*

(*a*) *each variable is non-negative.*

(*b*) *each variable is non-negative and at most 3.*

(*c*) *each variable is at least 2 and at most 4.*

(*d*) $0 \le x \le 5$; $2 \le y \le 6$, $5 \le z \le 8$, *x is even and y is odd.*

Solution. (*a*) For each variable there is a factor equal $(1 + x + x^2 + x^3 + x^4 + x^5)$. Hence, the number of solutions is the coefficient of x^5 in the generating function.

$$(1 + x + x^2 + x^3 + x^4 + x^5)\,(1 + x + x^2 + x^3 + x^4 + x^5)\cdot(1 + x + x^2 + x^3 + x^4 + x^5)$$

$$= (1 + x + x^2 + x^3 + x^4 + x^5)^3. = \left(\frac{1-x^6}{1-x}\right)^3 = (1-x^6)^3(1-x)^{-3} = (1-x^6)^3\,\Sigma^{3+r-1}\,C_r x^r$$

In $(1 - x^6)^3$, each term is of degree greater than 6 except the first which is 1. Hence, for x^5, $r = 5$ and the co-efficient of x^5 is ${}^{3+5-1}C_3 = 21$. Hence there are 21 solutions.

(*b*) For each variable there is a factor equal $(1 + x + x^2 + x^3)$. Hence, the number of solution is the coefficient of x^5 in the generating function $(1 + x + x^2 + x^3)^3$.

(*c*) For each variable there is a factor equal $(x^2 + x^3 + x^4)$. Hence, number of solution is the coefficient of x^5 in the generating function $(x^2 + x^3 + x^4)^3$.

(*d*) For variable x; there is a factor equal $(1 + x^2 + x^4)$, for variable y there is a factor equal $(x^3 + x^5)$ and for variable z there is a factor equal $(x^5 + x^6 + x^7 + x^8)$. Hence, the number of solution is the coefficient of x^5 in the generating function $(1 + x^2 + x^4)\,(x^3 + x^5)\,(x^5 + x^6 + x^7 + x^8)$.

Example 27. Find the coefficient of x^{10} in $(x + x^2 + x^4)(x^2 + x^3 + x^4)(x^2 + x^4)$.

Solution. We have,

$$(x + x^2 + x^4)\,(x^2 + x^3 + x^4)(x^2 + x^4) = x^5(1 + x + x^3)(1 + x + x^2)(1 + x^2)$$

The problem is then to find the coefficient of x^5 from the product of last three terms. The term x^5 can be found by choosing a term x^a from the first factor, a term x^b from the second factor and a term x^c from the third factor and finding the product $x^a.x^b.x^c = x^{a+b+c}$ such that $a + b + c = 5$ subje[...] to $a = 0, 1, 3$, $b = 0, 1, 2$ and $c = 0, 2$. This reduces to find the number of ways of choosing thr[...] integers a, b, c so that the above conditions are satisfied.

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When $c = 0$, we get $a + b = 5$, then $a = 3$ and $b = 2$

When $c = 2$, we get $a + b = 3$, then $a = 1$, $b = 2$ and $a = 3$, $b = 0$

So, there are three ways of choosing a, b and c and hence the coefficient of x^{10} is 3.

Example 28. In how many different ways can eight identical cookies be distributed among three distinct children if each child receives at least two cookies and no more than four cookies?

Solution. Since each child receives at least two but no more than four cookies, for each child there is a factor equal to $(x^2 + x^3 + x^4)$ in the generation function. Since there are three children, the generating function is $(x^2 + x^3 + x^4)(x^2 + x^3 + x^4)(x^2 + x^3 + x^4)$. There are eight identical cookies, we then need the coefficient of x^8 in this product. The terms x^8 in the expansion correspond to the ways that three terms can be selected, with one from each factor, that have exponents adding up to 8.

Now, $(x^2 + x^3 + x^4)(x^2 + x^3 + x^4)(x^2 + x^3 + x^4) = x^6(1 + x + x^2)(1 + x + x^2)(1 + x + x^2)$

The problem is then to find the coefficient of x^2 from the product of last three terms. This can easily be shown equal 6 which is then the coefficient of x^8. Hence the required number is 6.

Example 29. Determine the sequence corresponding to each of the following generating functions.

(*i*) $\dfrac{1}{5 - 6x + x^2}$ (*ii*) $\dfrac{(1+x)^2}{(1-x)^4}$ (*iii*) $\dfrac{x^5}{5 - 6x + x^2}$

Solution. (*i*) We have

$$G(x) = \frac{1}{5 - 6x + x^2} = \frac{1}{(5-x)(1-x)}$$

$$= \frac{1}{4}\left[\frac{1}{1-x} - \frac{1}{5-x}\right] = \frac{1}{4}\cdot\frac{1}{1-x} - \frac{1}{20}\,\frac{1}{\left(1-\dfrac{x}{5}\right)}$$

Therefore, the required sequence corresponding to G (x) is $\{a_n\}$ where

$$a_n = \frac{1}{4}\cdot 1 - \frac{1}{20}\left(\frac{1}{5}\right)^n, \quad n \ge 0$$

$$= \frac{1}{4}\left[1 - \left(\frac{1}{5}\right)^{n+1}\right], \quad n \ge 0$$

Second Method

$$G(x) = \frac{1}{4}\,\frac{1}{1-x} - \frac{1}{20\left(1-\dfrac{x}{5}\right)} = \frac{1}{4}(1-x)^{-1} - \frac{1}{20}\left(1-\frac{x}{5}\right)^{-1}$$

$$= \frac{1}{4}(1 + x + x^2 + \ldots + x^n + \ldots) - \frac{1}{20}\left(1 + \frac{x}{5} + \left(\frac{x}{5}\right)^2 + \ldots + \left(\frac{x}{5}\right)^n + \ldots\right)$$

$$= \left[\left(\frac{1}{4} - \frac{1}{20}\right) + \left(\frac{1}{4} - \frac{1}{20\times 5}\right)x + \left(\frac{1}{4} - \frac{1}{20\times 5^2}\right)x^2 + \ldots + \left(\frac{1}{4} - \frac{1}{20\times 5^n}\right)x^n + \ldots\right]$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{4} - \frac{1}{4\times 5^{n+1}}\right)x^n = \sum_{n=0}^{\infty}\frac{1}{4}\left[1 - \left(\frac{1}{5}\right)^{n+1}\right]x^n$$

...nt of x^{10} is 3.
...be distributed among ...than four cookies? ...okies, for each child ...e three children, then ...ht identical cookies, ...n correspond to the ...ents adding up to 8. ...$)(1+x+x^2)$ ...ee terms. This can ...umber is 6. ...owing generating

Therefore, the required sequence corresponding to G (x) is $\{a_n\}$ where $a_n = \frac{1}{4}\left[1-\left(\frac{1}{5}\right)^{n+1}\right]$

(ii) We have

$$G(x) = \frac{(1+x)^2}{(1-x)^4} = (1+x)^2(1-x)^{-4}$$

$$= (1+2x+x^2)\left(1+4x+\frac{4.5}{2!}x^2+\frac{4.5.6}{3!}x^3+\ldots+\frac{4.5.6\ldots(k+3)}{k!}x^k+\ldots\right)$$

$$= 1+2x+\left(\frac{4.5}{2!}+2.4+1\right)x^2+\left(\frac{4.5.6}{3!}+\frac{2.4.5}{2!}+4\right)x^3+\ldots$$

$$+\left(\frac{4.5.6\ldots(k+3)}{k!}+2.\frac{4.5.6\ldots(k+2)}{(k-1)!}+\frac{4.5.6\ldots(k+1)}{(k-2)!}\right)x^k+\ldots$$

Therefore, the co-efficient of x^k

$$a_k = \frac{4.5.6\ldots(k+3)}{k!}+2.\frac{4.5.6\ldots(k+z)}{(k-1)!}+\frac{4.5.6\ldots(k+1)}{(k-2)!}$$

$$= \frac{4.5.6\ldots(k+1)}{(k-2)!}\left[\frac{(k+2)(k+3)}{k(k-1)}+\frac{2(k+2)}{k-1}+1\right]$$

and hence the corresponding sequence to G (x) is $\{a_k\}$.

(iii) We have

$$G(x) = \frac{x^5}{5-6x+x^2} = x^5\left[\frac{1}{(5-x)(1-x)}\right]$$

$$= x^5\left[\frac{1}{4(1-x)}-\frac{1}{4(5-x)}\right] = x^5\left[\frac{1}{4(1-x)}-\frac{1}{20\left(1-\frac{x}{5}\right)}\right]$$

$$= \frac{1}{4}x^5(1-x)^{-1}-\frac{1}{20}x^5\left(1-\frac{x}{5}\right)^{-1}$$

$$= \frac{1}{4}x^5(1+x+x^2+\ldots)-\frac{1}{20}x^5\left(1+\frac{x}{5}+\left(\frac{x}{5}\right)^2+\ldots\right)$$

$$= \frac{1}{4}(x^5+x^6+x^7+\ldots+x^k+\ldots)-\frac{1}{20}\left(x^5+\frac{x^6}{5}+\ldots\frac{x^k}{5^{k-5}}+\ldots\right)$$

Therefore, the required sequence corresponding to the generating function is $\{a_k\}$ where

$$a_k = \begin{cases} 0 & \text{if } 0 \le k \le 4 \\ \frac{1}{4}-\frac{1}{20}\left(\frac{1}{5^{k-5}}\right) & \text{if } k \ge 5 \end{cases}$$

$$= \begin{cases} 0 & \text{if } 0 \le k \le 4 \\ \frac{1}{4}\left(1-\frac{1}{5^{k-4}}\right) & \text{if } k \ge 5 \end{cases}$$

...$^4 + \ldots]$

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Example 30. *Find the closed form generating function of*

(i) 1, (1 + 2), (1 + 2 + 3), ...

(ii) 1^2, $(1^2 + 2^2)$, $(1^2 + 2^2 + 3^2)$, ...

Solution. (*i*) We know

$G_1(x) = \dfrac{1}{1-x}$ is the generating function of the sequence 1, 1, 1, ... and

$G_2(x) = \dfrac{1}{(1-x)^2}$ is the generating function of the sequence 1, 2, 3, ...

Then
$$G(x)\,G_2(x) = \frac{1}{1-x}\cdot\frac{1}{(1-x)^2}$$
$$= (1 + x + x^2 + ...)(1 + 2x + 3x^2 + ...)$$
$$= 1 + (1 + 2)\,x + (1 + 2 + 3)\,x^2 + ...$$

$\therefore$
$$G(x) = G_1(x)\,G_2(x) = \frac{1}{(1-x)^3},\ |x| < 1$$

is the generating function of the sequence 1, (1 + 2), (1 + 2 + 3), ...

(*ii*) We know
$$\frac{1}{(1-x)^2} = (1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + ...,\ |x| < 1$$

Multiplying both sides by x, we get
$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \;.....\; + kx^k + ...$$

Differentiating both sides with respect to x
$$\frac{1+x}{(1-x)^3} = 1^2 + 2^2x + 3^2x^2 + 4^2x^3 + \;.....\; + k^2x^{k-1} + ...$$

Again
$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ...$$

$\therefore$
$$\frac{1}{1-x}\cdot\frac{1+x}{(1-x)^3} = (1 + x + x^2 + x^3 + ...)(1^2 + 2^2x + 3^2x^2 + 4^2x^3 + ...)$$
$$= 1^2 + (1^2 + 2^2)\,x + (1^2 + 2^2 + 3^2)\,x^2 + (1^2 + 2^2 + 3^2 + 4^2)\,x^3 + ...$$

$\therefore$ $G(x) = \dfrac{1+x}{(1-x)^4}$, $|x| < 1$ is the generating function of the sequence 1^2, $(1^2 + 2^2)$, $(1^2 + 2^2 + 3^2)$, $(1^2 + 2^2 + 3^2 + 4^2)$, ...

Example 31. *Solve the recurrence relation*
$$a_n + a_{n-1} = 3n\,2^n$$

Solution. The associated homogeneous recurrence relation is
$$a_n + a_{n-1} = 0 \qquad (1)$$

Let $a_n = r^n$ be a solution of (1)

The characteristic equation is $r + 1 = 0 \Rightarrow r = -1$

So, the solution of (1) is $a_n^{(h)} = C_1(-1)^n$.

As 2 is not a characteristic root, let $a_n^{(p)} = (A_0 + A_1 n)\,2^n$

Substituting in the given recurrence relation, we get

$(A_0 + A_1 n)\, 2^n + [A_0 + A_1 (n-1)]\, 2^{n-1} = 3n2^n$

or $(A_0 + A_1 n)\, 2 + A_0 + A_1 (n-1) = 3n \times 2$

or $3A_0 - A_1 + 3A_1 n = 6n$

Equating corresponding coefficients

$$3A_1 = 6 \Rightarrow A_1 = 2$$

and

$$3A_0 - A_1 = 0 \Rightarrow 3A_0 = A_1 = 2 \Rightarrow A_0 = \frac{2}{3}$$

$\therefore$

$$a_n^{(p)} = \left(\frac{2}{3} + 2n\right) 2^n.$$

Hence the general solution of the given recurrence relation is

$$a_n = C_1 (-1)^n + \left(\frac{2}{3} + 2n\right) 2^n.$$

Example 32. *Find a recurrence relation for a_n the number of ways to group $2n$ people into pairs.*

Solution. We first select a person and a partner for that person. Since the partner can be taken to be any of the other $2n - 1$ persons in the original group, there are $2n - 1$ ways to form this first pair.

We are now left of grouping $2n - 2$ persons into persons, and the number of doing this is p_{n-1}. Thus, by the product rule, we have

$$p_n = (2n-1)\, p_{n-1} \text{ for } n \geq 1$$

Since two people can be paired in one way, $p_1 = 1$.

Example 33. *There are n guests in a party. Each person shakes hand with everyone else exactly once. Form a recurrence relation representing the number of hand shakes occurred in the party. Also, solve the recurrence relation.*

Solution. Set H_n denotes the number of handshakes occurred in this party. So,

H_n = Number of hand shakes occurred among the gathering of $(n-1)$ guests + Number of handshakes made by the nth guests with $(n-1)$ number of guests

$\therefore$ $H_n = H_{n-1} + (n-1)$ is the required recurrence relation.

Since hand shakes is occurred only between two persons

$$H_1 = 0 \qquad (1)$$

$$H_n = H_{n-1} + (n-1)$$

Now

$$H_{n-1} = H_{n-2} + (n-2) \qquad (2)$$

$\therefore$

$$H_n = H_{n-2} + (n-2) + (n-1)$$

From (1)

$$H_{n-2} = H_{n-3} + (n-3)$$

Again

From (2)

$$H_n = H_{n-3} + (n-3) + (n-2) + (n-1)$$

$$\text{-------------------------}$$

$$= H_{n-k} + (n-k) + (n-k+1) + \ldots + (n-1)$$

$$\text{-------------------------}$$

$$= H_2 + 2 + \ldots + (n-3) + (n-2) + (n-1)$$

$$= H_1 + 1 + 2 + \ldots + (n-3) + (n-2) + (n-1)$$

$$= 0 + \frac{(n-1)(n-1+1)}{2} = \frac{n(n-1)}{2}.$$

Example 34. *Solve, by iterative method, the recurrence relation*

$$a_n = 2a_{n-1} + 3^n,\ n \geq 1 \text{ and } a_0 = 1.$$

Solution. Given $\quad a_n = 2a_{n-1} + 3^n \qquad (1)$

$\therefore \quad a_{n-1} = 2a_{n-2} + 3^{n-1}$

From (1) $\quad a_n = 2\,(2a_{n-2} + 3^{n-1}) + 3^n$

$a_n = 2^2\,a_{n-2} + 2\,.\,3^{n-1} + 3^n \qquad (2)$

Again $\quad a_{n-2} = 2a_{n-3} + 3^{n-2}$

From (2) $\quad a_n = 2^2\,(2a_{n-3} + 3^{n-2}) + 2\,.\,3^{n-1} + 3^n$

$$= 2^3 a_{n-3} + 2^2\,.\,3^{n-2} + 2\,.\,3^{n-1} + 3^n$$

$$a_n = 2^k a_{n-k} + 2^{k-1}\,.\,3^{n-k+1} + 2^{k-2}\,.\,3^{n-k+2} + \ldots + 2\,.\,3^{n-1} + 3^n$$

$$= 2^{n-1} a_1 + 2^{n-2}\,.\,3^2 + 2^{n-3}\,.\,3^3 + \ldots + 2\,.\,3^{n-1} + 3^n \quad (k = n-1)$$

$$= 2^{n-1}(2a_0 + 3) + 2^{n-2}\,.\,3^2 + 2^{n-3}\,.\,3^3 + \ldots + 2\,.\,3^{n-1} + 3^n$$

$$= 2^n + 2^{n-1}\,.\,3 + 2^{n-2}\,.\,3^2 + 2^{n-3}\,.\,3^3 + \ldots + 2\,.\,3^{n-1} + 3^n \quad (a_0 = 1)$$

$$= \frac{2^n\left\{\left(\frac{3}{2}\right)^{n+1} - 1\right\}}{\frac{3}{2} - 1} \qquad \left[\begin{array}{l}\text{G.P series of } (n+1) \text{ terms} \\ \text{with common ratio } \frac{3}{2} > 1\end{array}\right]$$

$$= 2\,.\,2^n\left\{\left(\frac{3}{2}\right)^{n+1} - 1\right\}$$

$$= (3^{n+1} - 2^{n+1})$$

Example 35. *Evaluate $1^2 + 2^2 + 3^2 + \ldots + r^2$ using generating function.*

Solution. We know that

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots + x^r + \ldots$$

Differentiating both sides with respect to x, we get

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \ldots + r\,x^{r-1} + \ldots$$

Multiplying both sides by x we get

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + \ldots + r\,x^r + \ldots$$

Differentiating both sides with respect to x we get

$$\frac{1+x}{(1-x)^3} = 1^2 + 2^2 x + 3^2 x^2 + \ldots + r^2\,x^{r-1} + \ldots$$

Multiplying both sides by x we get

$$\frac{x\,(1+x)}{(1-x)^3} = 0^2 + 1^2 x + 2^2 x^2 + 3^2 x^3 + \ldots + r^2 x^r + \ldots$$

This shows that $x(1+x)/(1-x)^3$ is the generating function of the numeric function $0^2, 1^2, \ldots, r^2$.

Now $\frac{1}{1-x} \times \frac{x(1+x)}{(1-x)^3} = (1 + x + x^2 + \ldots + x^r + \ldots) \times (0^2 + 1^2 x + 2^2 x^2 + 3^2 x^3 + \ldots + r^2 x^2 + \ldots)$

$$= 0^2 + (0^2 + 1^2)x + (0^2 + 1^2 + 2^2)x^2 + \ldots + (0^2 + 1^2 + 2^2 + \ldots + r^2)x^r + \ldots$$

Thus $0^2 + 1^2 + 2^2 + \ldots + r^2$ = coefficient of x^r in the expansion of $x(1+x)/(1-x)^4$.

But from Binomial theorem we know that the coefficient of x^r in the expansion of $1/(1-x)^4$ is

$$\frac{(-4)(-4-1)\ldots(-4-r+1)}{r!}(-1)^r = \frac{4 \cdot 5 \cdot 6 \cdot \ldots (r+3)}{r!}$$

$$= \frac{(r+1)(r+2)(r+3)}{1 \cdot 2 \cdot 3}$$

So, the coefficient of x^r in the expansion of $x(1+x)/(1-x)^4 = \frac{x}{(1-x)^4} + \frac{x^2}{(1-x)^4}$ is

$$\frac{r(r+1)(r+2)}{1 \cdot 2 \cdot 3} + \frac{(r-1)r(r+1)}{1 \cdot 2 \cdot 3} = \frac{r(r+1)(2r+1)}{6}.$$

Hence,

$$1^2 + 2^2 + 3^2 + \ldots + r^2 = \frac{r(r+1)(2r+1)}{6}.$$

Example 36. *Find a generating function for a_r = the number of non-negative integral solution of $e_1 + e_2 + e_3 + e_4 + e_5 = r$ where $0 \le e_1 \le 3$, $0 \le e_2 \le 3$, $2 \le e_3 \le 6$, $2 \le e_4 \le 6$, e_5 is odd, and $1 \le e_5 \le 9$.*

Solution. The number of solution with the indicates constraints is the co-efficient of x^r in the expansion of

$$(x^0 + x^1 + x^2 + x^3)^2 (x^2 + x^3 + x^4 + x^5 + x^6)^2 (x^1 + x^3 + x^5 + x^7 + x^9)$$

Hence the corresponding generating function is

$$G(x) = (1 + x + x^2 + x^3)^2 (x^2 + x^3 + x^4 + x^5 + x^6)^2 (x + x^3 + x^5 + x^7 + x^9)$$

RECURRENCE RELATION AND GENERATING FUNCTION

Problem Set 11.1

1. Identify each of the given recurrence as linear homogeneous of constant co-efficients. If the relation is a linear homogenous relation, give its degree.

(a) $a_n = 3a_{n-1}$ (b) $a_n = 2^n a_{n-1}$ (c) $a_n = 6a_{n-1} + 7$

(d) $a_n = 3a_{n-1} - 2a_{n-2}$ (e) $a_n = a_{n-4}\, a_{n-2}$ (f) $a_n = 3n\, a_{n-1}$

(g) $a_n = a^2_{n-1} + a_{n-2}$

2. Find the first four terms of the sequence defined by each of the recurrence relations and initial conditions.

(a) $a_n = 2a_{n-1} + n$; $n \ge 2, a_1 = 1$

(b) $a_n = a^2_{n-1}$; $n \ge 2, a_1 = 12$

(c) $a_n = n(a_{n-1})^2$; $n \ge 1, a_0 = 1$

(d) $b_k = b_{k-1} + 2b_{k-2}$; $k \ge 2, b_0 = 1, b_1 = 1$

(e) $b_k = k\, b_{k-1} + k^2 b_{k-2}$, $k \ge 2, b_0 = 1, b_1 = 1$

3. ... $-2\, 4^n + 3n\, 4^n$ are all solutions of the same recurrence relation