Unit - I - Introduction

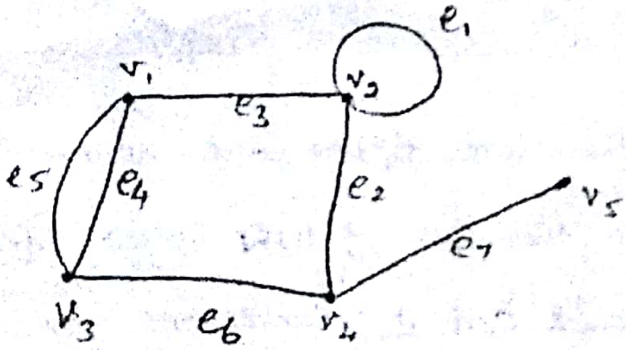
Completed

1.1:

Definition:- Graph

A graph $G = (V, E)$ consists of a set of objects $V = \{v_1, v_2, \dots\}$ called vertices and another set $E = \{e_1, e_2, \dots\}$ whose elements are called edges, such that each edge e_k is identified with an unordered pair (v_i, v_j) of vertices. The vertices v_i, v_j associated with edge e_k are called the end vertices of e_k.

Example:-



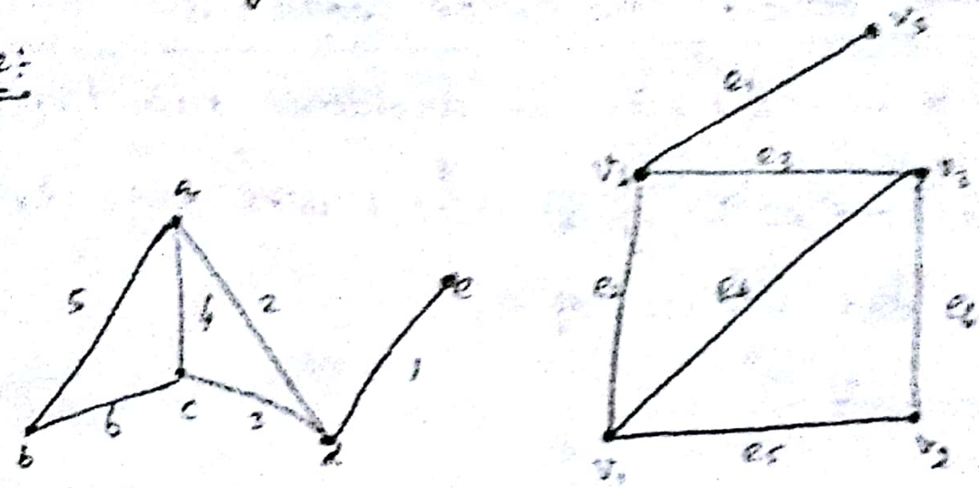
An edge to be associated with a vertex pair (v_i, v_i) such an edge having the same vertex as both its end vertices is called a self-loop or simply a loop.

2.1. Isomorphism: Definition:

Two graphs G and G' are said to be isomorphic if there is a one-to-one correspondence between their vertices and between their edges such that the incidence relationship is preserved.

Otherwise suppose that edge e is incident on vertices v_1, v_2 in G: then the corresponding edge e' in G' must be incident on the vertices v_1' & v_2' that correspond to v_1 & v_2 respectively.

Example:

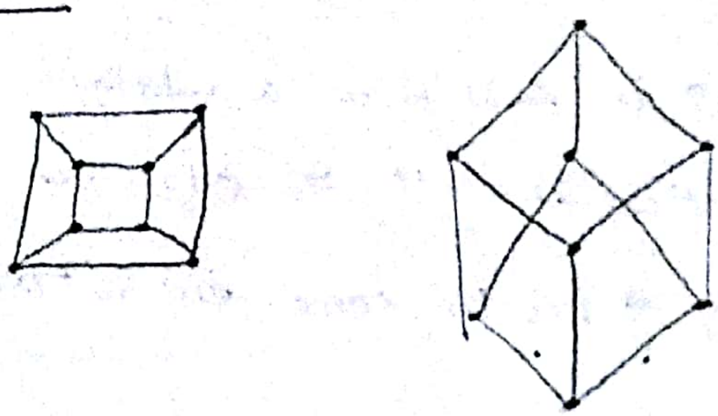


The two graphs are isomorphic graphs

The two graphs in the above figure are isomorphic. The correspondence between the two graphs is as follows: The vertices a, b, c, d and e correspond to v_1, v_2, v_3, v_4 & v_5 respectively. The edges 1, 2, 3, 4, 5 & 6 correspond to e_1, e_2, e_3, e_4, e_5 and e_6 respectively.

Except for the labels of their vertices and edges isomorphic graphs are the same graph, perhaps drawn differently.

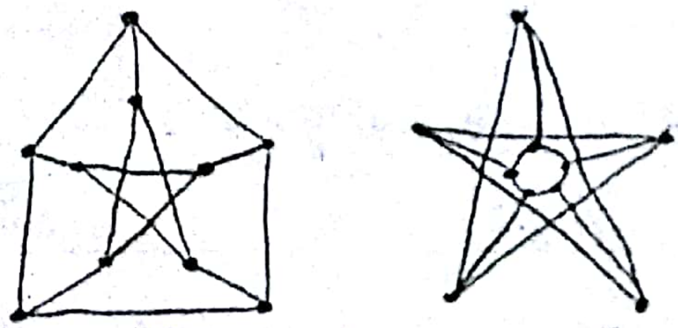
Example:



isomorphic graphs

By the definition of isomorphism that two isomorphic graphs must have

1. The same number of vertices
2. The same number of edges
3. An equal number of vertices with a given degree.



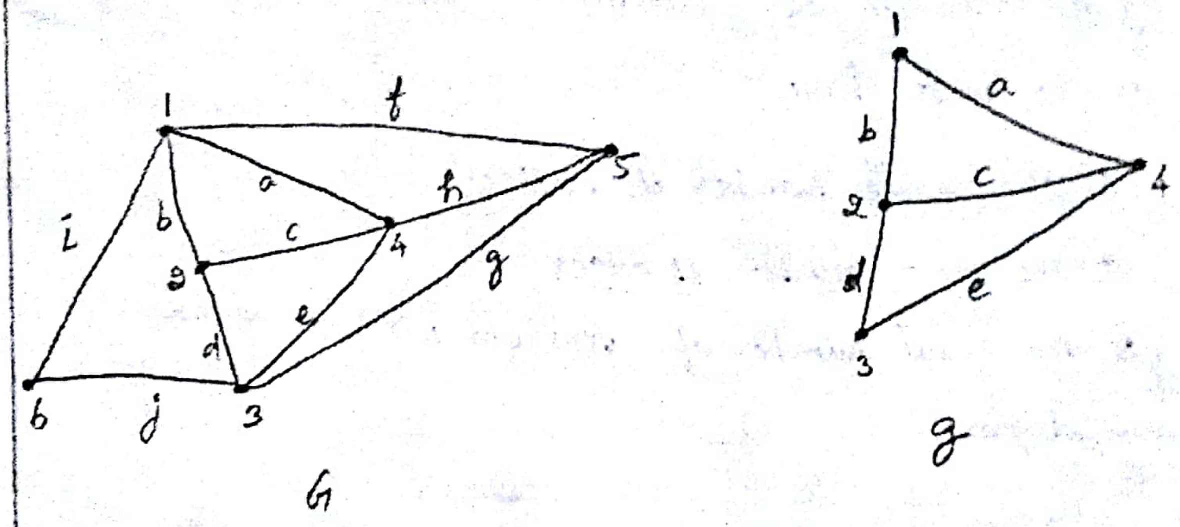
Isomorphic graphs

Two graphs that are not isomorphic

2.2. Subgraphs:

A graph g is said to be a subgraph of a graph G if all the vertices and all the edges of g are in G and each edge of g has the same end vertices in g as in G.

Example:



The symbol from set theory $g \subset G$ is used in stating "g is a subgraph of G".

The following observations can be made immediately:

1. Every graph is its own subgraph
2. A subgraph of a subgraph of G is a subgraph of G.
3. A single vertex in a graph G is a subgraph of G.
4. A single edge in G, together with its end vertices is also a subgraph of G.

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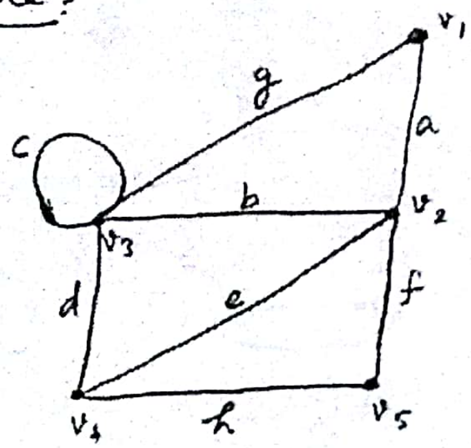
2.4. Walks, paths and circuits:

Definition: walk:

A walk is defined as a finite alternating sequence of vertices and edges beginning and ending with vertices, such that each edge is incident with the vertices preceding and following it. No edge appears more than once in a walk.

Example:

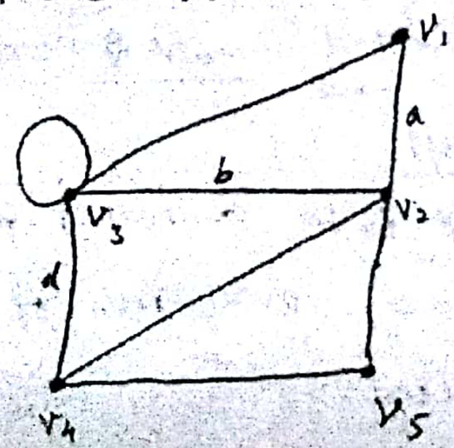


An open walk

Definition: path

An open walk in which no vertex appears more than once is called a path.

Example:



In the above example $v_1 a v_2 b v_3 d v_4$ is a path.

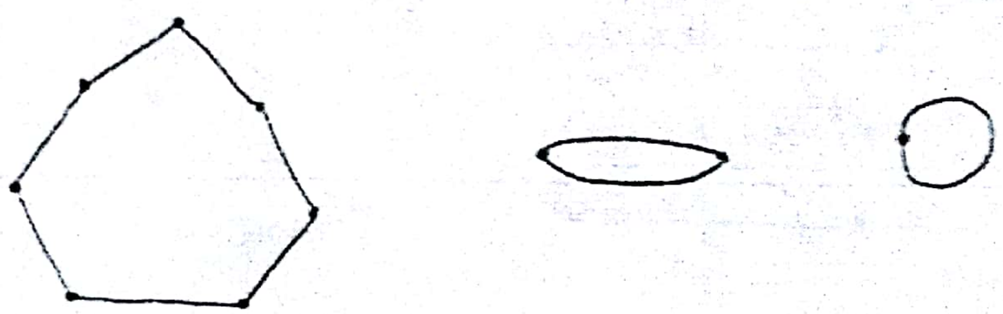
Whereas $v_1 a v_2 b v_3 c v_3 d v_4 e v_2 f v_5$ is not a path.

The number of edges in a path is called the length of a path.

Definition: Circuits:-

A closed walk in which no vertex appears more than once is called a circuit.

Example:-



Three different circuits.

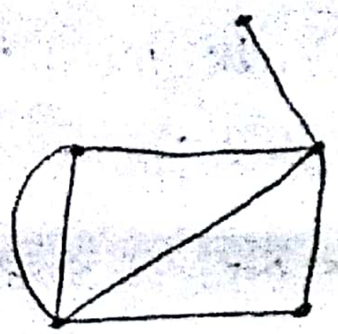
From the example, clearly every vertex in a circuit is of degree two.

Connected Graphs, disconnected Graphs & Components:-

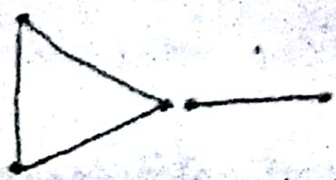
Definition: Connected graph

A graph G is said to be connected if there is at least one path between every pair of vertices in G. Otherwise G is disconnected.

Example:



Connected graph



Disconnected graph

Theorem 2.1:

A graph G is disconnected iff its vertex set V can be partitioned into two nonempty, disjoint subsets V_1 & V_2 such that there exists no edge in G whose one end vertex is in subset V_1 and the other in subset V_2

Proof:

Suppose that such a partitioning exists.

Consider two arbitrary vertices a and b of G, such that $a \in V_1$ and $b \in V_2$.

No path can exist between vertices a and b.

Otherwise there would be at least one edge whose one end vertex would be in V_1 and the other in V_2.

Hence if a partition exists G is not connected.

Conversely, let G be a disconnected graph. Consider a vertex a in G. Let V_1 be the set of all vertices that are joined by paths to a. Since G is disconnected, V_1 does not include all vertices of G. The remaining vertices will form a set V_2.

No vertex in V_1 is joined to any in V_2 by an edge.

Hence the partition.

Theorem 2.2:-

If a graph has exactly two vertices of odd degree, there must be a path joining these two vertices.

Proof:

Let G be a graph with all even vertices except vertices v_1 and v_2 which are odd.

For every graph and therefore for every component of a disconnected graph, no graph can have an odd number of odd vertices.

Therefore in graph G, v_1 and v_2 must belong to the same component and hence must have a path between them.

Theorem 2.3:

A simple graph with n vertices and k components can have at have most $(n-k)(n-k+1)/2$ edges.

Proof:

Let the number of vertices in each of the k components of a graph G be $n_1, n_2, \dots, n_k$.

Thus we have

$$n_1 + n_2 + \dots + n_k = n.$$

$$n_i \geq 1.$$

$$\sum_{i=1}^{k} n_i^2 \leq n^2 - (k-1)(2n-k) \quad \text{——} \quad (1)$$

Now the maximum number of edges in the i^{th} component of G is $\frac{1}{2} n_i (n_i - 1)$.

Therefore, the maximum number of edge in G is

$$\frac{1}{2} \sum_{i=1}^{k} (n_i - 1) n_i = \frac{1}{2} \left(\sum_{i=1}^{k} n_i^2 \right) - \frac{n}{2} \quad \text{——} \quad (2)$$

$$\leq \frac{1}{2} \left[n^2 - (k-1)(2n-k) - \frac{n}{2} \right] \text{ from (1).}$$

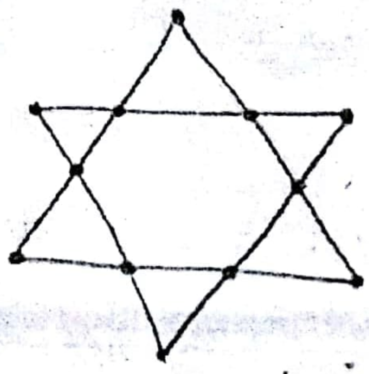
$$= \frac{1}{2} (n-k)(n-k+1) \quad \text{——} \quad (3)$$

2.6 . Euler graphs :-

Definition:-

A closed walk in a graph ~~which~~ contains all the edges of the graph, then the walk is called an Euler line and the graph an Euler graph.

Example :



Euler graph

Theorem 2.1:

A given connected graph G is an Euler graph iff all vertices of G are of even degree.

Proof:-

Suppose that G is an Euler graph.

It therefore contains an Euler line.

In tracing this walk we observe that every time the walk meets a vertex v it goes through two "new" edges incident on v with one

This is true not only of all intermediate vertices of the walk but also of the terminal vertex because we 'exited' and entered the same vertex at the beginning and end of the walk, respectively.

Thus if G is an Euler graph, the degree of every vertex is even.

To prove the sufficiency of the condition, assume that all vertices of G are of even degree.

Now we construct a walk starting at an arbitrary vertex v and going through the edges of G such that no edge is traced more than than once.

We continue tracing as far as possible.

Since every vertex is of even degree, we can exit from every vertex, the tracing cannot stop at any vertex but v.

And since v is also of even degree, we shall eventually reach v when the tracing comes to an end.

If this closed walk h, we just traced includes all the edges of G, G is an Euler graph.

If not, we remove from G all the edges in h and obtain a subgraph h' of G formed by the remaining edges.

Since both G and h have all their vertices of even degree, the degrees of vertices of h' are also even.

Moreover h' must touch h at least at one vertex a, because G is connected.

Starting from a, we can again construct a new walk in graph h'.

Since all the vertices of h' are of even degree, this walk in h' must terminate at vertex a; but this walk in h' can be combined with h to form a new walk, which starts and ends at vertex v and has more edges than h.

This process can be repeated until we obtain a closed walk that traverses all the edges of G.

Thus G is an Euler graph.

Theorem 2.5:-

In a connected graph G with exactly $2k$ odd vertices, there exist k edge-disjoint subgraphs such that they together contain all edges of G and that each is a unicursal graph.

Proof:

Let the odd vertices of the given graph G be named $v_1, v_2, \ldots, v_k$: $w_1, w_2, \ldots, w_k$ in any arbitrary order.

Add k edges to G between the vertex pairs $(v_1, w_1), (v_2, w_2) \ldots (v_k, w_k)$ to form a new graph G'.

Since every vertex of G' is of even degree G' consists of an Euler line P. Now if we remove from P the k edges we just added, P will be split into k walks, each of which is a unicursal line. The first removal will leave a single unicursal line, the second removal will split that into two unicursal lines; and each successive removal will split a unicursal line into two unicursal lines, until there are k of them.

Thus the theorem.

More on Euler Graphs:

Theorem 2.6:

A connected graph G is an Euler graph iff it can be decomposed into circuits.

Proofs:-

Suppose graph G can be decomposed into Circuits. that is G is a union of edge-disjoint Circuits.

Since the degree of every vertex in a Circuit is two the degree of every vertex in G is even.

Hence G is an Euler graph.

Conversely,

Let G be an Euler graph.

Consider a vertex v_1. There are at least two edges incident at v_1.

Let one of these edges be between v_1 & v_2.

Since vertex v_2 is also of even degree. it must have at least another edge, say between v_2 and v_3.

Proceeding in this fashion. we eventually arrive at a vertex that has previously been traversed. thus forming a circuit Γ.

Let us remove Γ from G.

All vertices in the remaining graph must also be of even degree.

From the remaining graph remove another circuit in exactly the same way as we removed Γ from G. Continue this process until no edges are left.

Hence the theorem.

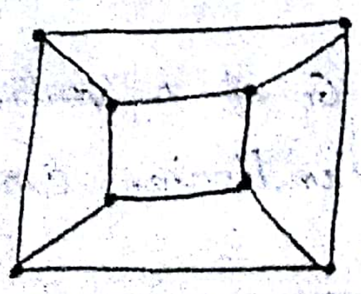
2.9. Hamiltonian Paths and Circuits:-

Definition: Hamiltonian

A circuit in a connected graph G is said to be Hamiltonian if it includes every vertex of G. Hence a Hamiltonian circuit in a graph of n vertices consists of exactly n edges.

Definition: Hamiltonian path

If we remove any one edge from a Hamiltonian circuit, we are left with a path. This path is called a Hamiltonian path.



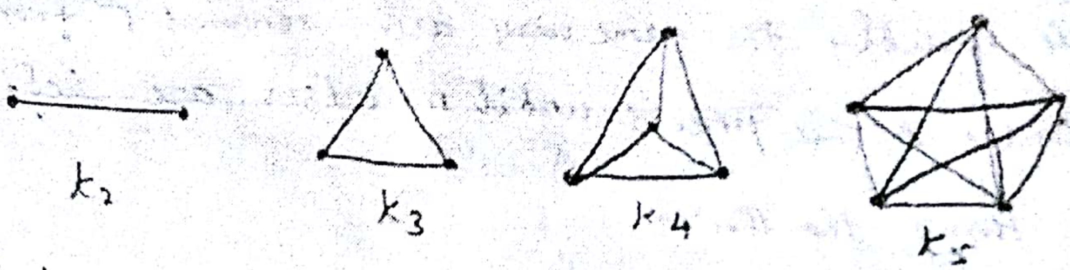
Hamiltonian Circuit

Complete graph:-

Definition:-

A simple graph in which there exists an edge

Example:-



Note:-

1. A complete graph is sometimes also referered to as a universal graph or clique.

2. Since every vertex is joined with every other vertex through one edge, the degree of every vertex is $n-1$ in a complete graph G of n vertices and the total number of edges in G is $n(n-1)/2$.

Theorem 2.8:-

In a complete graph with n vertices there are $(n-1)/2$ edges-disjoint Hamiltonian circuits if n is an odd number ≥ 3.

Proof:-

A complete graph G of n vertices has $n(n-1)/2$ edges, and a Hamiltonian circuit in G consist of n edges.

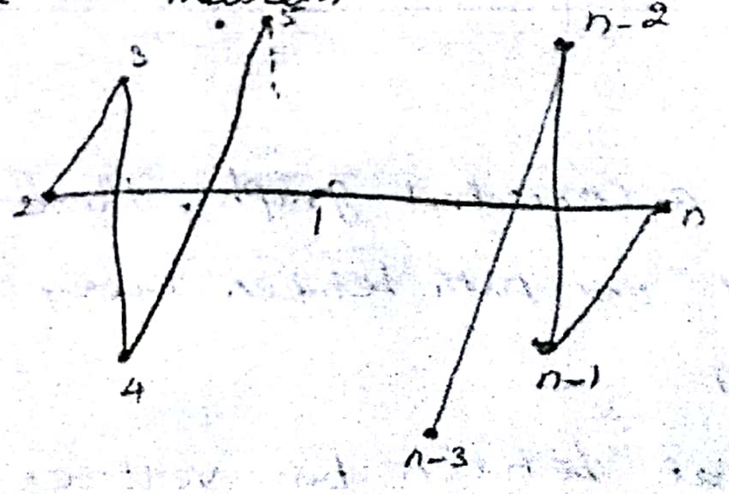
Therefore the number of edge-disjoint Hamiltonian circuits in G cannot exceed $(n-1)/2$.

That there are $(n-1)/2$ edge-disjoint Hamiltonian circuits when n is odd keeping the vertices fixed on a circle rotate the polygonal pattern clockwise by $360/(n-1)$; $2 \cdot 360/(n-1)$, $3 \cdot 360/(n-1)$ ... $(n-3)/2 \cdot 360/(n-1)$ degrees.

Observe that each rotation produces a Hamiltonian circuit that has no edge in common with any of the previous ones.

Thus we have $(n-3)/2$ new Hamiltonian circuits, all edge disjoint from the one in the diagram and also edge disjoint among themselves.

Hence the theorem

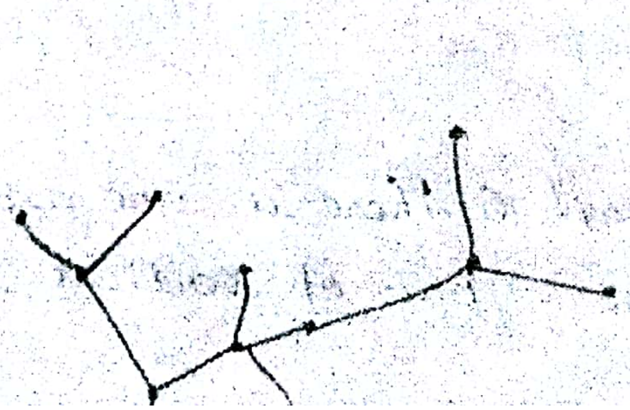


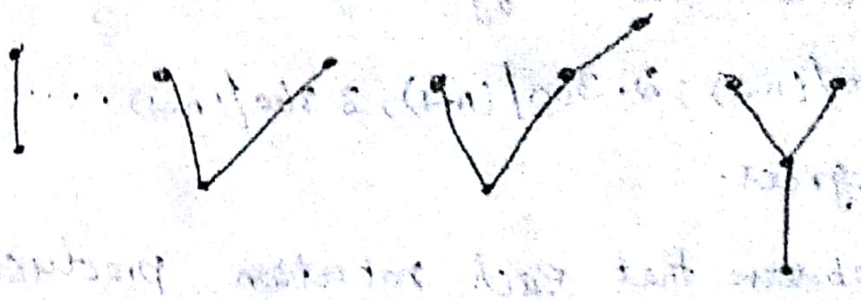
3.1. Trees :-

Definition:

A tree is a connected graph without any circuits.

Example:





Trees with one, two, three, and four vertices

3.2. Some properties of Trees:-

Theorem 3.1:-

There is one and only one path between every pair of vertices in a tree T.

Proof:-

Since T is a connected graph, there must exist at least one path between every pair of vertices in T.

Now suppose that between two vertices a and b of T there are two distinct paths.

The union of these two paths will contain a circuit and T cannot be a tree.

Theorem 3.2:-

If is a graph G there is one and only one path between every pair of vertices, G is a tree.

A circuit in a graph implies that there is at least one pair of vertices a, b such that there are two distinct paths between a and b. Since G has one and only one path between every pair of vertices, G can have no circuit.

Therefore G is a tree.

Theorem 3.3:

A tree with n vertices has n-1 edges.

Proof:

The theorem will be proved by induction on the number of vertices.

Clearly the theorem is true for n = 1, 2 and 3.

Assume that the theorem holds for all trees with fewer than n vertices.

Let us now consider a tree T with n vertices.

In T let e_k be an edge with end vertices v_i & v_j. According to theorem 3.1, there is no other path between v_i & v_j except e_k.

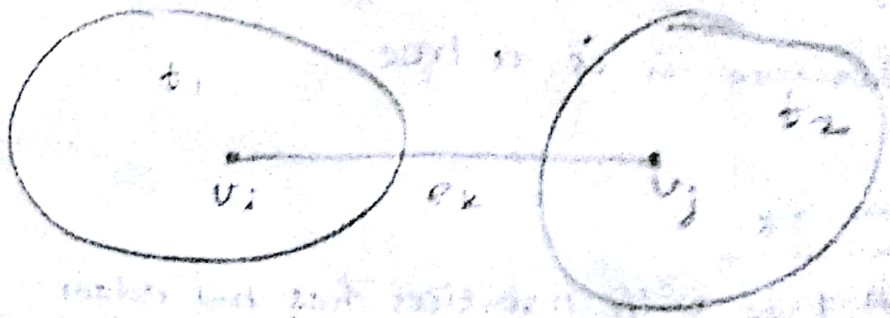
Therefore deletion of e_k from T will disconnect the graph as shown in the diagram.

Moreover T e_k consists of exactly two

By induction hypothesis, each contains one less edge than the number of vertices in it.

∴ Thus $T - e_k$ consists of $n-2$ edges.

Hence T has exactly $n-1$ edges.



Tree T with n vertices

Theorem 3.4:

Any connected graph with n vertices and $n-1$ edges is a tree.

~~Proofn~~

Theorem 3.5:

A graph is a tree iff it is minimally connected.

Theorem 3.6:

A graph G with n vertices, $n-1$ edges and no circuits is connected.

Proof:

Suppose there exists a circuitless graph G

without loss of generality.

let G consists of two Components g_1 & g_2.

Add an edge e between a vertex v_1 in g_1 and v_2 in g_2.

Since there was no path between V_1 & V_2 in G, adding e did not create a circuit.

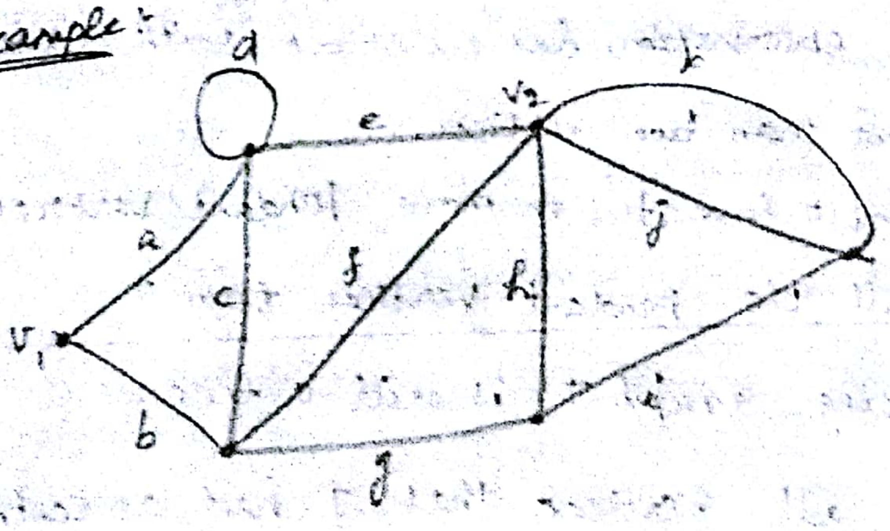
Thus $G \cup e$ is a circuitless connected graph of n vertices and n edges, which is not possible.

3.4. Distance and Centres in a tree:-

Definition: Distance:-

In a connected graph G, the distance $d(v_i, v_j)$ between two of its vertices v_i & v_j is the length of the shortest path.

Example:-



In this diagram, Distance between V_1 & V_2 is two.

... satisfy certain requirements.

1. Non negativity: $f(x,y) \geq 0$ & $f(x,y) = 0$ iff $x = y$

2. Symmetry: $f(x,y) = f(y,x)$

3. Triangle inequality: $f(x,y) \leq f(x,z) + f(z,y)$ for any z

Theorem 2.8:

The distance between vertices of a connected graph is a metric.

Theorem 2.9:

Every tree has either one or two centres.

Proof:

The maximum distance $\max d(v, v_i)$ from a given vertex v to any other vertex v_i occurs only when v_i is a pendant vertex.

With this observation, let us start with a tree T having more than two vertices.

Tree T must have two or more pendant vertices.

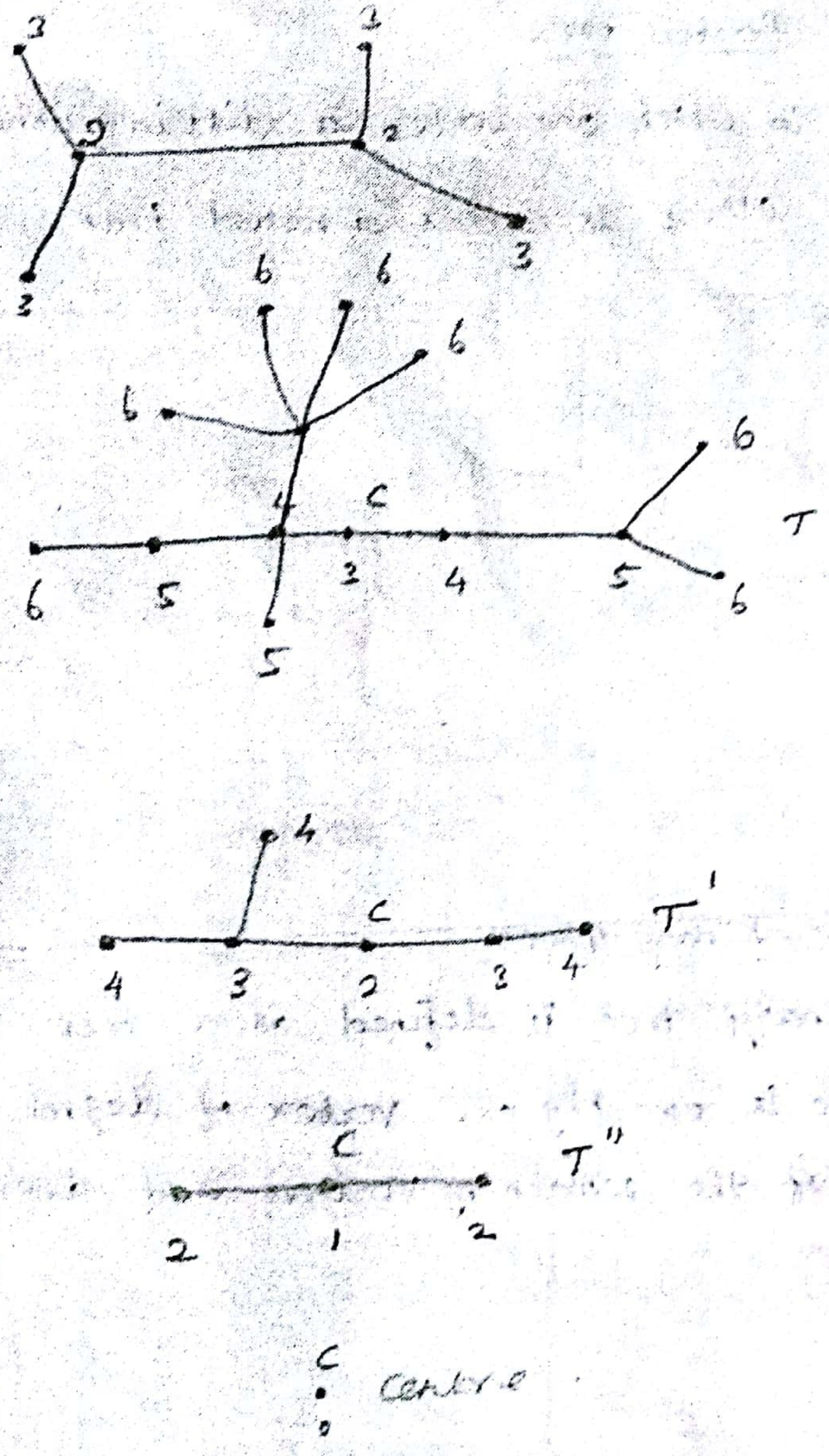
Delete all the pendant vertices from T.

The resulting graph T' is still a tree.

Therefore all vertices that T had as centres will still remain centres in T'.

From T' we can again remove all pendant

Example:



Finding a center of a tree.

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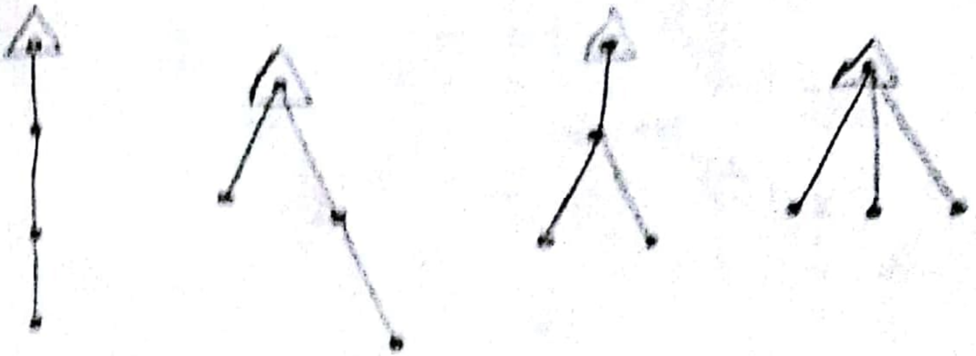
2.5 Rooted and Binary Trees:

Definition: Rooted tree

A tree in which one vertex is distinguished from all the others is called a rooted tree.

Example:

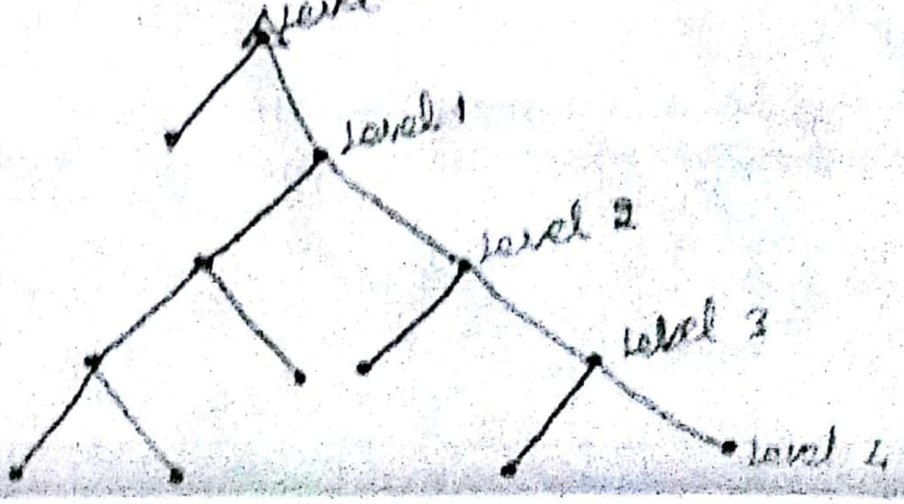


Rooted trees with four vertices

Definition: Binary Trees:

A binary tree is defined as a tree in which there is exactly one vertex of degree two, and each of the remaining vertices is of degree one or three.

Example:



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Note :-

1. Every binary tree is a rooted tree.

2. Two properties of binary trees follow directly from the definition:

   i) The number of vertices n is a binary tree is always odd. This is because there is exactly one vertex of even degree and the remaining $n-1$ vertices are of odd degrees.

   ii) Let p be the number of pendant vertices in a binary tree T. Then $n-p-1$ is the number of vertices of degree three. Therefore the number of edges in T equals

$$\frac{1}{2}\left[p+3(n-p-1)+2\right]=n-1$$

hence $p=\frac{n+1}{2}$.

Example :

For $n=11$, binary trees realizing both these extremes are shown in the below diagram.