1/10/2018

Divided Difference:

Let corresponding to the arguments $x_0, x_1, x_2, \dots x_n$, the entries of $f(x_0), f(x_1), f(x_2) \dots f(x_n)$, then

First Divided Difference is defined as

$$\underset{x_1}{\Delta} f(x_0) = f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

~~Second Divided Difference~~:

$$\underset{x_2}{\Delta} f(x_1) = f(x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Similarly,

$$\underset{x_n}{\Delta} f(x_{n-1}) = f(x_{n-1}, x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

Second Divided Difference:

$$\underset{x_1 x_2}{\Delta} f(x_0) = f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$

$$\underset{x_2 x_3}{\Delta} f(x_1) = f(x_1, x_2, x_3) = \frac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1}$$

$$\vdots$$

$$\underset{x_{n-1} x_n}{\Delta} f(x_{n-2}) = f(x_{n-2}, x_{n-1}, x_n) = \frac{f(x_{n-1}, x_n) - f(x_{n-2}, x_{n-1})}{x_n - x_{n-2}}$$

Third Divided Difference:

$$\underset{x_1 x_2 x_3}{\Delta} f(x_0) = f(x_0, x_1, x_2, x_3) = \frac{f(x_1, x_2, x_3) - f(x_0, x_1, x_2)}{x_3 - x_0}$$

$$\underset{x_2 x_3 x_4}{\Delta} f(x_1) = f(x_1, x_2, x_3, x_4) = \frac{f(x_2, x_3, x_4) - f(x_1, x_2, x_3)}{x_4 - x_1}$$

$$\vdots$$

$$\underset{x_{n-2} x_{n-1} x_n}{\Delta} f(x_{n-3}) = f(x_{n-3}, x_{n-2}, x_{n-1}, x_n) = \frac{f(x_{n-2}, x_{n-1}, x_n) - f(x_{n-3}, x_{n-2}, x_{n-1})}{x_n - x_{n-3}}$$

n th Divided Difference:

$$\boxed{\underset{x_1 x_2 \dots x_n}{\Delta} f(x_0) = f(x_0, x_1, x_2 \dots x_n) = \frac{f(x_1, x_2, \dots x_n) - f(x_0, x_1, x_2, \dots x_{n-1})}{x_n - x_0}}$$

Divided Difference Table:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
x_0	$f(x_0)$	$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$	$f(x_0, x_1, x_2)$	$f(x_0, x_1, x_2, x_3) = \frac{f(x_1, x_2, x_3) - f(x_0, x_1, x_2)}{x_3 - x_0}$
x_1	$f(x_1)$	$f(x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$	$f(x_1, x_2, x_3)$	
x_2	$f(x_2)$	$f(x_2, x_3) = \frac{f(x_3) - f(x_2)}{x_3 - x_2}$		
x_3	$f(x_3)$			

Q. Find the third divided difference with arguments 2, 4, 9, 10 of the function $f(x) = x^3 - 2x$.

$f(2) = 4,\ f(4) = 56,\ f(9) = 711,\ f(10) = 980$

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
2	4			
		26		
4	56		15	
		131		1
9	711		23	
		269		
10	980			

3/10/2018

Newton's Divided Difference Interpolation Formula:

$$f(x) = f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0, x_1, x_2) + (x-x_0)(x-x_1)(x-x_2) f(x_0, x_1, x_2, x_3) + \dots + (x-x_0)(x-x_1)\dots(x-x_{n-1}) f(x_0, x_1, x_2 \dots x_n)$$

Q. Use Newton's Divided difference formula to calculate $f(x)$ from the following table. Hence find the value of $f(3)$.

x	0	1	2	4	5	6
$f(x)$	1	14	15	5	6	19

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
0	1					
		13				
1	14		$-13/2$			
		1		$47/24$		
2	15		$4/3$		$-21/40$	
		5		$-2/3$		$41/240$
4	5		$-4/3$		$1/2$	
		1		$11/6$		
5	6		6			
		13				
6	19					

$$f(x) = 1 + 13x + x(x-1)(-13/2) + x(x-1)(x-2)\, 47/24 + x(x-1)(x-2)(x-4)(-21/40) + x(x-1)(x-2)(x-4)(x-5)\, 41/240$$

$$= x^3 - 9x^2 + 21x + 1$$

$f(3) = 10$

Derivative:

$$f(x) = f(x_0) + (x-x_0) f(x_0, x_1) + \{x^2 - (x_0+x_1)x + x_0x_1\} f(x_0, x_1, x_2) + \{x^3 - (x_0+x_1+x_2)x^2 + ((x_0+x_1)x_2 + x_0x_1)x - x_0x_1x_2\} f(x_0, x_1, x_2, x_3) + \dots$$

$$f'(x) = f(x_0, x_1) + \{2x - (x_0+x_1)\} f(x_0, x_1, x_2) + \{3x^2 - 2x(x_0+x_1+x_2) + (x_0x_1 - x_0x_2 - x_1x_2)\} f(x_0, x_1, x_2, x_3) + \dots$$

$$f''(x) = 2 f(x_0, x_1, x_2) + \{6x - 2(x_0+x_1+x_2)\} f(x_0, x_1, x_2, x_3) + \dots$$

Q. From the following table find the first derivative at $x=4$ using Newton's divided difference formula.

x	1	2	4	8	10
$f(x)$	0	1	5	21	27

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1	0				
		1			
2	1		1/3		
		2		0	
4	5		1/3		-1/144
		4		-1/16	
8	21		-1/6		
		3			
10	27				

$f'(x) = 1 + (8-3)\frac{1}{3} + (48 - 8(7) + (-10)4 - 8)0 + \{256 - 48(6) + 8\cdot 8 - 48\cdot 3 + 3\cdot 6\cdot 8$
$-18 + 16 - 6\cdot 2\}(-1/144)$
$= 2.66$

Q. Find out first and second derivative from the following table using Newton's divided difference interpolation formula.

x	1	3	-4
$f(x)$	3	-5	4

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$
1	3		
		-4	
3	-5		-19/35
		-9/7	
-4	4		

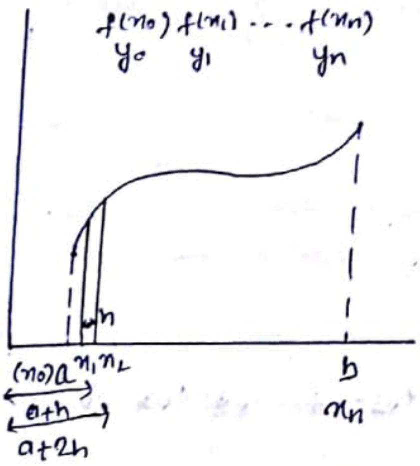
$f'(x) = -4 + (2x-4)\left(-\frac{19}{35}\right) = -4 - \frac{38x}{35} + \frac{76}{35} = \frac{-140+76-38x}{35} = \frac{-64-38x}{35}$

$= -137.82 - 1.08x$

$f''(x) = 2\left(-\frac{19}{35}\right) = -1.085$

5/10/2018

NUMERICAL INTEGRATION



$$I = \int_a^b f(x)\,dx$$

$$x = x_0 + ph$$

$$\frac{x - x_0}{h} = p$$

$$\frac{dx}{h} = dp$$

$$dx = h\,dp$$

$$\Rightarrow I = \int_a^b f(x)\,dx = \int_{x_0}^{x_n} y\,dx.$$

$$I \doteq \int_{x_0}^{x_n} \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{\lfloor 2}\Delta^2 y_0 \ldots\right] h\,dp$$

$$I = h\int_0^n \left[y_0 + p\Delta y_0 + \left(\frac{p^2-p}{2}\right)\Delta^2 y_0 + \left(\frac{p^3-3p^2+2p}{6}\right)\Delta^3 y_0 + \ldots\right] dp$$

$$I = h\left[p y_0 + \frac{p^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{p^3}{3} - \frac{p^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{p^4}{4} - p^3 + \frac{2p^2}{2}\right)\Delta^3 y_0 + \ldots\right]_0^n$$

$$\boxed{I = h\left[n y_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{n^4}{4} - n^3 + n^2\right)\Delta^3 y_0 + \ldots\right]}$$

This formula is known as Newton-Cotes general quadrature formula.

Trapezoidal Rule:

$n=1$

$$\int_{x_0}^{x_0+h} f(x)\,dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right] = h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right] = \frac{h}{2}(y_0 + y_1)$$

$$\int_{x_0+h}^{x_0+2h} f(x)\,dx = \frac{h}{2}(y_1 + y_2)$$

$\vdots$

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx = \frac{h}{2}(y_{n-1} + y_n)$$

Adding above equations,

$$I = \int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[(y_0+y_1) + (y_1+y_2) + \ldots + (y_{n-1}+y_n)\right]$$

$$\boxed{\int_a^b f(x)\,dx = \frac{h}{2}(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})}$$

Simpson's 1/3 Rule:

$n=2$

$$\int_{x_0}^{x_0+2h} f(x)\,dx = h\left[2y_0 + \frac{2^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{2^3}{3} - \frac{2^2}{2}\right)\Delta^2 y_0\right]$$

$$= h\left[2y_0 + 2\overset{(y_1-y_0)}{\cancel{\Delta y_0}} + \frac{1}{2}\left(\frac{8}{3} - 2\right)\overset{(y_2-2y_1+y_0)}{\cancel{\Delta^2 y_0}}\right]$$

$$= \cancel{h\left[2y_0 + 2\Delta y_0\right.} = h\left[\frac{y_0}{3} + \frac{4}{3}y_1 + \frac{y_2}{3}\right] = \frac{h}{3}(y_0 + 4y_1 + y_2)$$

$$\int_{x_0+2h}^{x_0+4h} f(x)dx = \frac{h}{3}[y_2+4y_3+y_4]$$

$$\int_{x_0+4h}^{x_0+6h} f(x)dx = \frac{h}{3}[y_4+4y_5+y_6]$$

$\vdots$

$$\int_{x_0+(n-2)h}^{x_0+nh} f(x)dx = \frac{h}{3}[y_{n-2}+4y_{n-1}+y_n]$$

Adding above equations,

$$I = \int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{3}\Big[(y_0+4y_1+y_2)+(y_2+4y_3+y_4)+(y_4+4y_5+y_6)+ \dots + (y_{n-2}+4y_{n-1}+y_n)\Big]$$

$$\int_a^b f(x)dx = \frac{h}{3}\Big[(y_0+y_n)+4(y_1+y_3+y_5+\dots)+2(y_2+y_4+y_6+\dots)\Big]$$

$$\boxed{I = \int_a^b f(x)dx = \frac{h}{3}\Big[(y_0+y_n)+4\Sigma\text{odd terms}+2\Sigma\text{even terms}\Big]}$$

Simpson's 3/8 Rule:

$n=3$

$$\int_{x_0}^{x_0+3h} f(x)dx = h\Big[3y_0+\frac{3^2}{2}\Delta y_0+\frac{1}{2}\Big(\frac{3^3}{3}-\frac{3^2}{2}\Big)\Delta^2 y_0+\frac{1}{6}\Big(\frac{3^4}{4}-3^3+3^2\Big)\Delta^3 y_0\Big]$$

$$= h\Big[3y_0+\frac{9}{2}(y_1-y_0)+\frac{9}{4}(y_2-2y_1+y_0)+\frac{3}{8}(y_3-3y_2+3y_1-y_0)\Big]$$

$$= h\Big[\frac{3}{8}y_0+\frac{9}{8}y_1+\frac{9}{8}y_2+\frac{3}{8}y_3\Big]$$

$$= \frac{3}{8}h(y_0+3y_1+3y_2+y_3)$$

$$\int_{x_0+3h}^{x_0+6h} f(x)dx = \frac{3}{8}h(y_3+3y_4+3y_5+y_6)$$

$$\int_{x_0+(n-3)h}^{x_0+nh} f(x)dx = \frac{3}{8}h(y_{n-3}+3y_{n-2}+3y_{n-1}+y_n)$$

Adding above equations,

$$\boxed{\int_{x_0}^{x_0+nh} f(x)dx = \frac{3h}{8}\Big[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\dots)+2(y_3+y_6+y_9+\dots)\Big]}$$

Q. Calculate $\int_{-3}^{3} x^4 dx$ by taking 7 equidistant intervals by (i) Trapezoidal rule (ii) Simpson's 1/3 rule (iii) Simpson's 3/8 rule.

x	-3	-2	-1	0	1	2	3
$f(x)$	81	16	1	0	1	16	81
x^4	y_0	y_1	y_2	y_3	y_4	y_5	y_6

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(i) $\int_{-3}^{3} x^4 dx = \frac{h}{2}\left[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)\right]$

$= \frac{1}{2}\left[(81+81)+2(16+1+0+1+16)\right]$

$= \frac{1}{2}\left[162+68\right]$

$= \frac{230}{2}$

$= 115$

(ii) $\int_{-3}^{3} x^4 dx = \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$

$= \frac{1}{2}\left[(81+81)+4(16+0+16)+2(1+1)\right]$

$= \frac{1}{2}(162+128+4)$

$= \frac{196}{2}$

$= 98$

(iii) $\int_{-3}^{3} x^4 dx = \frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2y_3\right]$

$= \frac{3}{8}\left[(81+81)+3(16+1+1+16)+2\times 0\right]$

$= \frac{3}{8}(162+102)$

$= \frac{3}{8}\times 264$

$= 99$

$\int_{-3}^{3} x^4 dx = \left[\frac{x^5}{5}\right]_{-3}^{3} = \frac{243-(-243)}{5} = \frac{486}{5} = 97.2$

Q. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by dividing range into 6 equal parts by the 3 rules and also find the value of π.

x	0	1/5	2/5	3/5	4/5	1
$f(x)$	1	25/26	25/29	$\frac{25}{34}$	$\frac{41}{25}$	$\frac{1}{2}$
	y_0	y_1	y_2	y_3	y_4	y_5

i) $\int_0^1 \frac{dx}{1+x^2} = \frac{1}{10}\left[\left(1+\frac{1}{2}\right)+2\left(\frac{25}{26}+\frac{25}{29}+\frac{25}{34}+\frac{25}{41}\right)\right] = \frac{1}{10}\left[\frac{3}{2}+2(.96+.861+.73+.60)\right]$

$= .78$

x	0	1/6	2/6	3/6	4/6	5/6	1
$f(x)$	1	$\frac{36}{37}$	$\frac{36}{40}$	$\frac{36}{45}$	$\frac{36}{52}$	$\frac{36}{61}$	$\frac{1}{2}$
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

i) $\int_0^1 \frac{dx}{1+x^2} = \frac{1}{12}\left[1.5+2(.973+.9+.8+.692+.59)\right] = 0.7841$

10/10/2018

Solution of Ordinary Differential Equation of I order (with initial condition):

Taylor Series Method:

$$y = y_0 + (x-x_0)\frac{dy}{dx} + \frac{(x-x_0)^2}{2!}\frac{d^2y}{dx^2} + \frac{(x-x_0)^3}{3!}\frac{d^3y}{dx^3} + \cdots$$

Q. Apply Taylor series method to obtain approx value of y at $x=0.2$ for the differential equation $\frac{dy}{dx} = 2y + 3e^x$ with the initial condition $y(0)=0$

$$\cancel{\frac{dy}{dx} = 2\times0 + 3e^0 = 3}$$

$$\frac{dy}{dx} = 2y + 3e^x \qquad \frac{dy}{dx} = 3$$

$$\frac{d^2y}{dx^2} = 2\frac{dy}{dx} + 3e^x = 2\times3 + 3e^0 = 9$$

$$\frac{d^3y}{dx^3} = 2\frac{d^2y}{dx^2} + 3e^x = 2\times9 + 3e^0 = 21$$

$$y = 0 + (x-0)(3) + \frac{(x-0)^2}{2!}9 + \frac{(x-0)^3}{3!}21 + \cdots$$

$$= 3x + \frac{9}{2}x^2 + \frac{7}{2}x^3 + \cdots$$

$$y(0.2) = 3\times0.2 + \frac{9}{2}\times0.04 + \frac{7}{2}\times0.008 + \cdots$$

$$= 0.6 + 0.18 + 0.028$$

$$= 0.808$$

$\frac{4.5}{100}\times2$

$\frac{3.82}{240}$

$$\frac{dy}{dx} - 2y = 3e^x$$

$$IF = e^{-2\int dx} = e^{-2x}$$

$$ye^{-2x} = 3\int e^{-x}dx$$

$$ye^{-2x} = -3e^{-x} + c$$

$\cancel{y = 3}$ $\quad c = -1$

$\cancel{c = 3}$ $\quad ye^{2x} = e^{3x} - 1$

$\cancel{y = -3e^{x} + 3e^{2x}}$ $\quad y = e^x - e^{-2x}$

$$y(0.2) = 1.221 - 0.670 = 0.551$$

Error $= 0.808 - 0.551 = 0.257$

Q. Use Taylor series method to solve the following ordinary DE $\frac{dy}{dx} = 3x + y^2$ with initial condition $y(0) = 1$, and hence find the value of y for $x = 0.1$.

$$\frac{dy}{dx} = 3x + y^2 = 1$$

$$\frac{d^2y}{dx^2} = 3 + 2y\frac{dy}{dx} = 3 + 2\cdot1 = 5$$

$$\frac{d^3y}{dx^3} = 2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 10 + 2 = 12$$

$\frac{\cancel{54}\ \cancel{27}\ 9}{4\times3\times2}$

$$\frac{d^4y}{dx^4} = 2y\frac{d^3y}{dx^3} + 2\frac{dy}{dx}\frac{d^2y}{dx^2} + 4\frac{dy}{dx}\frac{d^2y}{dx^2}$$

2.25

$$= 24 + 10 + 20 = 54$$

$$y = 1 + (x-0)1 + \frac{x^2}{2!}5 + \frac{x^3}{3!}12 + \frac{x^4}{4!}\cdot 54 + \dots$$

$$y = 1 + x + \frac{5}{2}x^2 + 2x^3 + \frac{9}{4}x^4 + \dots$$

$$y(0.1) = 1.1 + 0.025 + 0.002 + 0.000225$$
$$= 1.127225$$

Picards Method:

$$\frac{dy}{dx} = f(x,y),\ y(x_0) = y_0$$

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x,y)\,dx$$

$$y - y_0 = \int_{x_0}^{x} f(x,y)\,dx$$

$$\boxed{y = y_0 + \int_{x_0}^{x} f(x,y)\,dx}$$

First approximation, $y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx$$
$$\vdots$$
$$y_{n+1} = y_0 + \int_{x_0}^{x} f(x, y_n)\,dx$$

Q. Using Picards method find solution of $\frac{dy}{dx} = 1 + xy$ upto III approximation with IC $y(0) = 0$.

$$y_1 = 0 + \int_0^x (1 + xy_0)\,dx$$
$$= \int_0^x (1 + x\times 0)\,dx$$
$$= \int_0^x dx$$
$$y_1 = x$$

$$y_2 = 0 + \int_0^x (1 + x^2)\,dx = \left(x + \frac{x^3}{3}\right)_0^x = x + \frac{x^3}{3}$$

$$y_3 = 0 + \int_0^x \left[1 + x\left(x + \frac{x^3}{3}\right)\right]dx = x + \frac{x^3}{3} + \frac{x^5}{15}$$

22/10/2018

Euler's Method:

$$\frac{dy}{dx} = f(x,y) \quad , \quad y(x_0) = y_0$$

Let $y = \phi(x)$ be a solution

$x_0, x_1, x_2 \ldots\ldots x_n, x_{n+1}$

$y_0, y_1, y_2 \ldots\ldots y_n, y_{n+1}$

$x_1 = x_0 + h$

$x_{n+1} = x_n + h$

$$y_{n+1} = \phi(x_{n+1}) = \phi(x_n + h) = \phi(x_n) + h\phi'(x_n) + \frac{h^2}{2!}\phi'' x_n + \ldots$$

$$= y_n + h f(x_n, y_n)$$

$$\left\{ \because \phi'(x_n) = \frac{dy}{dx} = f(x_n, y_n) \right\}$$

$$\boxed{y_{n+1} = y_n + h f(x_n, y_n)}$$

Q. Using Euler's method, find an approximate value of y corresponding to $x = 2$ given that $\frac{dy}{dx} = x + 2y$ with the initial condition $y_1 = 1$, $h = 0.1$.

$$y_1 = y_0 + h f(x_0, y_0) = y_0 + h(x_0 + 2y_0)$$
$$= 1 + 0.1(1 + 2.1)$$
$$= 1.3$$

$y_2 = 1.3 + 0.1(1.1 + 2\times1.3) = 1.3 + 0.1(\cancel{2}\cdot37) = 1.67$

$y_3 = 1.67 + 0.1(1.2 + 2\times1.67) = 1.67 + .454 = 2.124$

$y_4 = 2.124 + 0.1(1.3 + 2\times2.124) = 2.6788$

$y_5 = 2.6788 + 0.1(1.4 + 2\times2.6788) = 3.3545$

$y_6 = 3.3545 + 0.1(1.5 + 2\times3.3545) = 4.1754$

$y_7 = 4.1754 + 0.1(1.6 + 2\times4.1754) = 5.17048$

$y_8 = 5.17048 + 0.1(1.7 + 2\times5.17048) = 6.3745$

$y_9 = 6.3745 + 0.1(1.8 + 2\times6.3745) = 7.8294$

$y_{10} = 7.8294 + 0.1(1.9 + 2\times7.8294) = 9.58528$

$h = 0.5$

$y_1 = 1 + 0.5(1 + 2\times1) = 1 + 1.5 = 2.5$

$y_2 = 2.5 + 0.5(1.5 + 2\times2.5) = 2.5 + 0.5(1.5+5) = 5.75$

Q. Apply Euler's method to solve $\frac{dy}{dx} = x + 3y$ where $y(0) = 1$ and hence find an approximate value of y at $x = 1$, by taking $h = 0.1$.

$y_1 = 1+0.1(0+3\times1) = 1.3$

$y_2 = 1.3+0.1(0.1+3\times1.3) = 1.7$

$y_3 = 1.7+0.1(0.2+3\times1.7) = 2.23$

$y_4 = 2.23+0.1(0.3+3\times2.23) = 2.929$

$y_5 = 2.929+0.1(0.4+3\times2.929) = 3.8477$

$y_6 = 3.8477+0.1(0.5+3\times3.8477) = 5.05201$

$y_7 = 5.05201+0.1(0.6+3\times5.05201) = 6.6276$

$y_8 = 6.6276+0.1(0.7+3\times6.6276) = 8.68588$

$y_9 = 6.68588+0.1(0.8+3\times8.68588) = 9.3716$

$y_{10} = 9.3716+0.1(0.9+3\times9.3716) = 12.27308$

~~$y_{11} = 12.27308+0.1(1+3\times12.27308) =$~~

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Euler's Modified Method:

$y_{n+1} = y_n + h\phi'(x) + \frac{h^2}{2!}\phi''(x) + \cdots$

$\left(\frac{dy}{dx}\right)_{n+1} = \left(\frac{dy}{dx}\right)_n + h'^2\phi'' f(x) + \frac{h^2}{2!}\phi''' f(x) + \cdots$

$\left(\frac{dy}{dx}\right)_n = f(x_n, y_n)$

$\left(\frac{dy}{dx}\right)_{n+1} = f(x_{n+1}, y_{n+1})$

$f(x_{n+1}, y_{n+1}) = f(x_n, y_n) + h\phi'' f(x) + \frac{h^2}{2!}\phi''(x) + \cdots$

$① - \frac{h}{2}$ ⑪

$y_{n+1} - \frac{h}{2}f(x_{n+1}, y_{n+1}) = y_n + hf(x_n, y_n) - \frac{h}{2}f(x_n, y_n) + \frac{h^3}{6}\phi'''(x) - \frac{h^3}{4}\phi'''(x)$

$= y_n + \frac{h}{2}f(x_n, y_n) + \left(\frac{2h^3 - 3h^3}{12}\right)\phi'''(x)$

Neglecting h^3 and higher powers

$y_{n+1} - \frac{h}{2}f(x_{n+1}, y_{n+1}) = y_n + \frac{h}{2}f(x_n, y_n)$

$$\boxed{y_{n+1} = y_n + h\left[\frac{f(x_n, y_n) + f(x_{n+1}, y_{n+1})}{2}\right]}$$

Q. Use Euler's modified formula to calculate value of y at $x = 2$ ~~correct to~~ ~~decimal places~~. $\frac{dy}{dx} = x+2y$, $y(1) = 1$.

$y_1 = y_0 + hf(x_0, y_0) = 1+0.1(1+2\times1) = 1.3$

$y(1.1) = y_n + h\left[\frac{f(x_n, y_n) + f(x_{n+1}, y_{n+1})}{2}\right]$

$= 1+0.1\left[\frac{1+2+1.1+2\times1.3}{2}\right] = 1.335$

$y_2 = y_1 + hf(x_1, y_1)$

$= 1.335 + 0.1\,(1.1 + 2\times 1.338) = 1.712$

$\cancel{y_1 = y_0 + h\left[\frac{f(x_0, y_0) + f(x_1, y_1)}{2}\right]}$

$= 1 + 0.1\,[\,3 +$

$y(0.2) = 1.3 + 0.1\left(\frac{1.1 + 2.6 + 1.2 + 3.424}{2}\right) = \cancel{1.3+0.4}\ 1.7162$

$y_3 = 1.7162 + 0.1\,(1.2 + 3.424) = 2.1786$

$y(1.3) = 1.712 + 0.1\left(\frac{1.2 + 3.424 + 1.3 + 4.3572}{2}\right) = 2.226$

$y_4 = y_3 + hf(x_3, y_3) = 2.1786 + 0.1\,(1.3 + 2.1786) = 2.5264$

$y(1.4) = y_3 + h\left[\frac{f(x_3, y_3) + f(x_4, y_4)}{2}\right] = 2.1786 + 0.1\left(\frac{5.6572 + 6.4528}{2}\right)$

$= 2.7841$

$y_5 = y_4 + h(x_4, y_4) = 2.5264 + 0.1\,(1.4 + 5.0528) = 3.1716$

$y(1.5) = y_4 + h\left[\frac{f(x_4, y_4) + f(x_5, y_5)}{2}\right] =$

RK Method of Fourth Order : (Runge Kutta Method)

$$\boxed{y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)}$$

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

Q. Apply RK method of IV order to find the value of y when $x = 0.2$ given that $\frac{dy}{dx} = x + y$, $y(0) = 1$, $h = 0.1$

$$k_1 = hf(x_0, y_0) = 0.1(0+1) = 0.1$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1\left[0 + \frac{0.1}{2} + 1 + \frac{0.1}{2}\right] = 0.11$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1\left(0 + \frac{0.1}{2} + 1 + \frac{0.11}{2}\right) = 0.1105$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1(0 + 0.1 + 1 + 0.1105) = 0.12105$$

$$y_{0.1} = 1 + \frac{1}{6}(0.1 + 0.22 + 0.221 + 0.12105) = 1.1103$$

$$x_0 = 0.1, y_0 = 1.1103$$

$$k_1 = 0.1(0.1 + 1.1103) = 0.12103$$

$$k_2 = 0.1\left(0.1 + \frac{0.1}{2} + 1.1103 + \frac{0.12103}{2}\right) = 0.1381$$

$$k_3 = 0.1\left(0.1 + \frac{0.1}{2} + 1.1103 + \frac{0.1381}{2}\right) = 0.19509$$

$$k_4 = 0.1(0.1 + 0.1 + 1.1103 + 0.19509) = 0.3261$$

$$y(0.2) = 1.1103 + \frac{1}{6}(0.12103 + 0.2762 + 0.39018 + 0.3261) = 1.2958$$

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STATISTICAL METHODS

Moment:

Moment about Mean:

r^{th} moment about mean is defined as

$$\boxed{\mu_r = \frac{1}{N}\sum_{i=1}^{n}(x_i-\bar{x})^r f_i}$$

f_i = frequency $\qquad N = \sum f_i$

x_i = random variable

$$\mu_0 = \frac{1}{N}\sum f_i(x_i-\bar{x})^0 = \sum \frac{f_i}{N} = 1$$

$$\mu_1 = \frac{1}{N}\sum f_i(x_i-\bar{x}) = \frac{1}{N}\sum f_i x_i - \sum \frac{f_i \bar{x}}{N} = \bar{x}-\bar{x} = 0$$

$$\mu_2 = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i-\bar{x})^2$$

$$\mu_3 = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i-\bar{x})^3$$

$$\mu_4 = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i-\bar{x})^4$$

Moment about an arbitrary point a:

It is defined as

$$\boxed{\mu_r' = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i-a)^r}$$

$$\mu_0' = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i-a)^0 = 1$$

$$\mu_1' = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i-a) = \sum_{i=1}^{n}\frac{f_i x_i}{N} - a\frac{\sum f_i}{N} = \bar{x}-a$$

$$\mu_2' = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i-a)^2$$

$$\mu_3' = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i-a)^3$$

$$\mu_4' = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i-a)^4$$

Moment about origin:

$$\boxed{\nu_r = \frac{1}{N}\sum f_i x_i^r}$$

$$\nu_0 = 1$$

$$\nu_1 = \frac{\sum f_i x_i}{N} = \bar{x}$$

$$\nu_2 = \frac{\sum f_i x_i^2}{N}$$

$$\nu_3 = \frac{\sum f_i x_i^3}{N}$$

$$\nu_4 = \frac{\sum f_i x_i^4}{N}$$

Relation b/w Moment about mean and moment about any arbitrary point:

$$\mu_r = \frac{1}{N}\sum f_i (x_i - \bar{x})^r$$

$$= \frac{1}{N}\sum f_i \{(x_i - a) - (\bar{x} - a)\}^r$$

$$= \frac{1}{N}\sum f_i ((x_i - a) - \mu_1')^r \qquad \{\because \mu_1' = \bar{x} - a\}$$

$$= \frac{1}{N}\sum f_i \{(x_i-a)^r - {}^rC_1 (x_i-a)^{r-1}\mu_1' + {}^rC_2 (x_i-a)^{r-2}\mu_1'^2 - {}^rC_3 (x_i-a)^{r-3}\mu_1'^3 + {}^rC_4 (x_i-a)^{r-4}\mu_1'^4 - \cdots\}$$

$$= \frac{1}{N}\sum f_i (x_i-a)^r - \frac{{}^rC_1}{N}\sum f_i (x_i-a)^{r-1}\mu_1' + \frac{{}^rC_2}{N}\sum f_i (x_i-a)^{r-2}\mu_1'^2 - \frac{{}^rC_3}{N}\sum f_i (x_i-a)^{r-3}\mu_1'^3 + \frac{{}^rC_4}{N}\sum f_i (x_i-a)^{r-4}\mu_1'^4 - \cdots$$

$$\boxed{\mu_r = \mu_r' - {}^rC_1 \mu_{r-1}' \mu_1' + {}^rC_2 \mu_{r-2}' \mu_1'^2 - {}^rC_3 \mu_{r-3}' \mu_1'^3 + {}^rC_4 \mu_{r-4}' \mu_1'^4 - \cdots}$$

$\mu_1 = \mu_1' - 1\cdot\mu_0'\cdot\mu_1' = \mu_1' - \mu_1' = 0 \Rightarrow \boxed{\mu_1 = 0}$

$\mu_2 = \mu_2' - {}^2C_1 \mu_1'\mu_1' + {}^2C_2 \mu_0' \mu_1'^2 = \mu_2' - \mu_1'^2 \Rightarrow \boxed{\mu_2 = \mu_2' - \mu_1'^2}$

$$\boxed{\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3}$$

$$\boxed{\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4}$$

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Relation b/w moment about any arbitrary point and moment about mean:

$$\boxed{\mu_1' = \mu_1 - a}$$

$$\boxed{\mu_2' = \mu_2 + \mu_1'^2}$$

$$\boxed{\mu_3' = \mu_3 + 3\mu_2\mu_1' + \mu_1'^3}$$

$$\boxed{\mu_4' = \mu_4 + 4\mu_3\mu_1' + 6\mu_2\mu_1'^2 + \mu_1'^4}$$

Relation b/w moment about origin and moment about mean:

$$\boxed{\nu_1 = \bar{x}}$$

$$\boxed{\nu_2 = \mu_2 + \bar{x}^2}$$

$$\boxed{\nu_3 = \mu_3 + 3\mu_2\bar{x} + \bar{x}^3}$$

$$\boxed{\nu_4 = \mu_4 + 4\mu_3\bar{x} + 6\mu_2\bar{x}^2 + \bar{x}^4}$$

Q. The first four moments of a distribution about the value 4 of the variant ~~value 4~~ are $-1.5, 17, -30, 108$ calculate the first four moments about the mean.

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$\mu_1 = 0$

$\mu_2 = \mu_2' - \mu_1'^2 = 17 - (1.5)^2 = 17 - 2.25 = 14.75$

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 = -30 - 3(17)(-1.5) + 2(-1.5)^3$

$= -30 + 76.5 - 3.375 \times 2 = 39.75$

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4 = 108 - 4(-30)(-1.5) + 6(17)(-1.5)^2 - 3(-1.5)^4$

$= 142.31$

Skewness (β_1)

Lack of symmetry is known as skewness. If graph is skewed on the left, it is called negatively skewed while a graph skewed on the right is called positively skewed. $\boxed{\beta_1 = \frac{\mu_3^2}{\mu_2^3}}$

Kurtosis: (β_2)

Measurement of peak of a graph is known as kurtosis.

i) Platykurtic $\beta_2 < 3$, Peak resembles plate
ii) Mesokurtic $\beta_2 = 3$
iii) Leptokurtic $\beta_2 > 3$, Peak is sharp

$$\boxed{\beta_2 = \frac{\mu_4}{\mu_2^2}}$$

Q. The first four moments of a distribution about the value 4 of the variant are −1.5, 17, −30 and 108. Find the moment about mean, skewness and kurtosis. Comment on skewness and kurtosis.
Also find moment about the origin and about the point $x = 2$.

$\mu_1 = 0,\ \mu_2 = 14.75,\ \mu_3 = 39.75,\ \mu_4 = 142.31$

$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(39.75)^2}{(14.75)^3} = 0.4922 \Rightarrow$ Positively skewed

$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{142.31}{(14.75)^2} = 0.6541 \Rightarrow$ Platykurtic

$\cancel{\bar{x} = \frac{\sum f_i x_i}{N} = \frac{-1.5 + 17 - 30 + 108}{4} = 23.375}$

$\bar{x} - A = \mu_1'$
$\bar{x} = -1.5 + 4$
$= 2.5$

$\cancel{\nu_1 = \bar{x} = 23.375}$

$\cancel{\nu_2 = \mu_2 + \bar{x}^2 = 14.75 + (23.375)^2}$

$\cancel{\nu_3 = \mu_3 + 3\mu_2\bar{x} + \bar{x}^3 = -30 + 3(17)(23.375) + (23.375)^3}$

$\nu_1 = \bar{x} = 2.5$

$\nu_2 = \mu_2 + \bar{x}^2 = 14.75 + (2.5)^2 = 21$

$\nu_3 = -30 + 3 \times 14.75 \times 2.5 + (2.5)^3 = 166$

$\nu_4 = 108 + 4(39.75)\,2.5 + 6(14.75)(2.5)^2 + (2.5)^4 = 1132$

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$u_1' = \bar{x} - a = 2.5 - 2 = 0.5$

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Corelation:

Covariance:

$(x_1, y_1), (x_2, y_2), \ldots \ldots (x_n, y_n)$

The covariance of x and Y is defined as

$$\mathrm{Cov}(X,Y) = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{n} = \frac{(x_1-\bar{x})(y_1-\bar{y}) + (x_2-\bar{x})(y_2-\bar{y}) + \ldots + (x_n-\bar{x})(y_n-\bar{y})}{n}$$

Working rule to find covariance:

1. Find $\sum x_i$, $\sum y_i$, $\sum x_i y_i$
2. Divide these by n
3. $\mathrm{cov}(X,Y) = \frac{\sum x_i y_i}{n} - \frac{\sum x_i}{n} \cdot \frac{\sum y_i}{n}$

eg. Find the covariance of following pairs of observations:

$(1,2), (2,3), (4,5), (6,7)$

$\sum x_i = 13$, $\sum y_i = 17$, $\sum x_i y_i = 70$

$\mathrm{cov} = \frac{70}{4} - \frac{13}{4} \cdot \frac{17}{4} = \frac{280 - 221}{16}$

$= 3.68$

Corelation: Whenever two variables x and y are so related that an increase in one results increase or decrease in the other, then the variables are said to be corelated.

eg. Yield of crop depends upon the amount of rainfall.

Positive Corelation:

If an increase or decrease in one variable results increase or decrease in the second variable, then there is a positive corelation.

Negative Corelation:

If an increase or decrease in one variable results in decrease or increase in the other variable, then they are said to be negatively corelated.

Linear Corelation:

If all the plotted points lie on approximately on a straight line, then the two quantities are said to be linearly corelated.

Methods to Find Coefficient of Corelation:

Karl Pearson Coefficient of Corelation:

$$r = \frac{\mathrm{covariance}(X,Y)}{\sqrt{\mathrm{var}X}\sqrt{\mathrm{var}Y}} = \frac{\frac{\sum XY}{N}}{\sqrt{\frac{X^2}{N}}\sqrt{\frac{Y^2}{N}}} = \frac{\sum XY}{\sqrt{\sum X^2}\sqrt{\sum Y^2}}$$

$$\boxed{r = \frac{\Sigma XY}{\sqrt{\Sigma X^2}\sqrt{\Sigma Y^2}}}$$

$X = x - \bar{x}$
$Y = y - \bar{y}$

Q. Calculate the correlation coeff. b/w x and y for the following data;

x	21	23	30	54	57	58	72	78	87	90
y	60	71	72	83	110	84	100	92	113	135

$\bar{x} = \frac{570}{10} = 57$

$\bar{y} = \frac{920}{10} = 92$

x	y	$x-\bar{x}$	$y-\bar{y}$	X^2	Y^2	XY
21	60	-36	-32	1296	1024	1152
23	71	-34	-21	1156	441	714
30	72	-27	-20	729	400	540
54	83	-3	-9	9	81	27
57	110	0	18	0	324	0
58	84	1	-8	1	64	-8
72	100	15	8	225	64	120
78	92	21	0	441	0	0
87	113	30	21	900	441	630
90	135	33	43	1089	1849	1419
570	920			5846	4688	4594

$$r = \frac{4594}{\sqrt{5846}\sqrt{4688}} = \frac{4594}{76.48 \times 68.46} = 0.87$$

Spearman Rank Corelation Coefficient:

$$\boxed{r = 1 - \frac{6\Sigma d^2}{n(n^2-1)}}$$

$d = R_1 - R_2$

R_1 = Rank of ~~first data~~ highest data

R_2 = Rank of second data

Q. Calculate the spearman rank corelation coefficient for the ~~previous~~ given data.

x	y
25	8
36	10
32	15
35	17
37	20
40	23
42	24
45	25

x	y	R_1	R_2	$d = R_1 - R_2$	d^2
68	62	4	4		
64	58	5	6		
73	68	3	3		
50	45	7	7		
65	81	6	1		
80	60	1	5		
75	69	2	2		

If two data have similar rank, then Spearman rank corr coeff is:

$$\boxed{r = 1 - \frac{6\left[\Sigma d^2 + \frac{m_1^3 - m_1}{12} + \frac{m_2^3 - m_2}{12} + \dots\right]}{n(n^2-1)}}$$

Q.

x	y	R_1	R_1
68	62	4	4
64	58	2.5	5.5
75	68	7	2.5
50	45	5.5	7
64	81		5.5
80	60		1
75	68		2.5

$75 \to m_1 = 2$

$64 \to m_2 = 2$

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Regression:

$y = a + bx$

$e_1 = y_1 - (a + bx_1)$

$e_2 = y_2 - (a + bx_2)$

$\vdots$

$e_n = y_n - (a + bx_n)$

$S = e_1^2 + e_2^2 + \dots + e_n^2$

$= \sum_{i=1}^{n} \{y_i - (a + bx_i)\}^2$

$\frac{\partial S}{\partial a} = 0, \frac{\partial S}{\partial b} = 0$

$-2\sum_{i=1}^{n} \{y_i - (a + bx_i)\} = 0$

$\sum_{i=1}^{n} \{y_i - (a + bx_i)\} = 0$

$\Sigma y_i = na + \sum_{i=1}^{n} bx_i$

$\boxed{\Sigma y = na + b\Sigma x}$ — (i)

$-2x_i \sum_{i=1}^{n} \{y_i - (a + bx_i)\} = 0$

$-2\sum_{i=1}^{n} x_i \{y_i - (a + bx_i)\} = 0$

$\boxed{\Sigma xy = a\Sigma x + b\Sigma x^2}$ — (ii)

$\frac{\Sigma y}{n} = a + b\frac{\Sigma x}{n}$

$\bar{y} = a + b\bar{x}$ ⇒ Line passes through $(\bar{x}, \bar{y})$

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$$\Sigma(x-\bar{x})(y-\bar{y}) = a\Sigma(x-\bar{x}) + b\,\Sigma(x-\bar{x})^2$$ [Origin is shifted for (ii)]

$\because x-\bar{x}=0$

$$b = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(x-\bar{x})^2} = \frac{\Sigma XY}{\Sigma X^2} \quad \text{— (iv)}$$

$$r = \frac{\Sigma XY}{\sqrt{\Sigma X^2}\sqrt{\Sigma Y^2}} = \frac{\Sigma XY}{n \times \sqrt{\frac{\Sigma X^2}{n}}\sqrt{\frac{\Sigma Y^2}{n}}} = \frac{\Sigma XY}{n\sigma_x\sigma_y}$$ {σ = std. deviation}

$$\Rightarrow \Sigma XY = rn\sigma_x\sigma_y$$

Now (iv) becomes

$$b = \frac{nr\sigma_x\sigma_y}{\Sigma X^2}$$

$$b = \frac{r\sigma_x\sigma_y}{\frac{\Sigma X^2}{n}}$$

slope is $b = \dfrac{r\sigma_x\sigma_y}{\sigma_x^2}$

The equation of line of regression is of x on y

$$\boxed{(y-\bar{y}) = r\frac{\sigma_y}{\sigma_x}(x-\bar{x})}$$

Similarly we can find line of regression of y on x.

$$\boxed{(x-\bar{x}) = r\frac{\sigma_x}{\sigma_y}(y-\bar{y})}$$

Angle b/w the two lines,

$$\tan\theta = \left|\frac{m_2-m_1}{1+m_1m_2}\right|$$

$$= \frac{r\frac{\sigma_x}{\sigma_y} - r\frac{\sigma_y}{\sigma_x}}{1 + r\frac{\sigma_x}{\sigma_y}\cdot r\frac{\sigma_y}{\sigma_x}} = \frac{\frac{r\sigma_x^2 - r\sigma_y^2}{\sigma_x\sigma_y}}{1+r^2}$$

Q. Two lines of regression are given by $5y-8x+17=0$ and $2y-5x+14=0$. If $\sigma_y^2=16$, then find (i) mean of x and y (ii) σ_x^2, and (iii) coeff of correlation b/w x and y.

(i) Replace x, y with $\bar{x}, \bar{y}$ and solve.

(ii) Multiply slopes

(iii) Divide slopes.

Binomial Distribution:

Happening of an event r times in n trials is known as binomial distribution and its probability distribution function is given by

$$\boxed{P(r) = {}^nC_r\, p^r q^{n-r}}$$

Q. Find the probability of getting 4 heads in 6 ~~trials~~ tosses of a fair coin.

$$P(4) = {}^6C_4 p^4 q^2 = {}^6C_4 \left(\tfrac{1}{2}\right)^4 \left(\tfrac{1}{2}\right)^2 = \frac{6\times5}{2\times2^6} = \frac{15}{64}$$

*

0	q^n
1	${}^nC_1 pq^{n-1}$
2	${}^nC_2 p^2q^{n-2}$
⋮	
n	p^n

Remark:

$$\boxed{\begin{array}{l}\text{Mean of the BD} = np \\ \text{Standard Deviation} = \sqrt{npq}\end{array}}$$

Q. If mean and variance of a BD are 4 and 2 respectively then find the probability of (i) exactly 2 success (ii) less than 2 success (iii) at least 2 success.

$$4 = np$$
$$npq = 2$$
$$4q = 2$$
$$q = \frac{1}{2}$$
$$p = \frac{1}{2}$$
$$n = \frac{4}{1/2} = 8$$