CHAPTER 10

Combinatorics

10.1. Introduction

In many discrete problems, we are confronted with problem of counting. Combinatorics is the branch of discrete mathematics concerned with counting problems. Techniques for counting are important in Mathematics and Computer Science especially in probability theory and in the analysis of algorithms. In this chapter, we will discuss basics of counting and their applications.

10.2. The Fundamental Principles

Sum Rule

If an event can occur in m ways and another event can occur in n ways, and if these two events cannot occur simultaneously, then one of the two events can occur in $m + n$ ways. In general, if E_i ($i = 1, 2, 3 \ldots k$) are k events such that no two of them can occur at the same time, and if E_i can occur in n_i ways, then one of the k events can occur in $n_1 + n_2 + n_3 + \ldots\ldots + n_k$ ways.

Example 1. *If there are 14 boys and 12 girls in a class, find the number of ways of selecting one student as class representative.*

Solution. Using sum rule, there are $14 + 12 = 26$ ways of selecting one student (either a boy or a girl) as class representative.

Example 2. *If a student is getting admission in 4 different Engineering Colleges and 5 Medical Colleges, find the number of ways of choosing one of the above colleges.*

Solution. Using sum rule, there are $4 + 5 = 9$ ways of choosing one of the colleges.

Product Rule

If an event can occur in m ways and a second event in n ways, and if the number of ways the second event occurs, does not depend upon the occurrence of the first event, then the two events can occur simultaneously in mn ways. In general if E_i ($i = 1, 2, \ldots k$) are k events and if E_1 can occur in n_1 ways, E_2 can occur n_2 ways (no matter how E_1 occurs), E_3 can occur in n_3 ways (no matter how E_1 and E_2 occurs),......,E_k occurs in n_k ways (no matter how k-1 events occur), then the k events can occur simultaneously in $n_1 \times n_2 \times n_3 \times \ldots\ldots\ldots \times n_k$ ways.

Example 3. *Three persons enter into car, where there are 5 seats. In how many ways can they take up their seats?*

Solution. The first person has a choice of 5 seats and can sit in any one of those 5 seats. So there are 5 ways of occupying the first seat. The second person has a choice of 4 seats. Similarly, the third person has a choice of 3 seats. Hence, the required number of ways in which all the three persons can seat is $5 \times 4 \times 3 = 60$.

Example 4. *There are four roads from city X to Y and five roads from city Y to Z, find*

(i) how many ways is it possible to travel from city X to city Z via city Y.

(ii) how different round trip routes are there from city X to Y to Z to Y and back to X.

Solution. (*i*) In going from city X to Y, any of the 4 roads may be taken. In going from city Y to Z, any of the 5 roads may be taken. So by the product rule, there are 5.4 = 20 ways to travel from city X to Z via city Y.

(*ii*) A round trip journey can be performed in the following four ways:

1. From city X to Y
2. From city Y to Z
3. From city Z to Y
4. From city Y to X

1. Can be performed in 4 ways, 5 ways to perform 2, 5 ways to perform 3 and 4 ways to perform 4. By product rule, there are 4.5.5.4 = 400 round trip routes.

Example 5. *For a set of six true or false questions, find the number of ways of answering all questions.*

Solution. The number of ways of answering the first question is 2. The second question can also be answered in 2 ways and similarly for other 4 questions. Hence the total number of ways of answering all the questions is $2^6 = 64$.

Many counting problems can be solved using just the sum rule or just the product rule. However, many counting problems can be solved using both of these rules.

Example 6. *In how many ways can one select two books from different subjects from among six distinct computer science books, three distinct mathematics books, and two distinct chemistry books?*

Solution. Using product rule one can select two books from different subjects as follows:

(*i*) one from computer science and one from mathematics in 6.3 = 18 ways.

(*ii*) one from computer science and one from chemistry in 6.2 = 12 ways.

(*iii*) one from mathematics and one from chemistry in 3.2 = 6 ways.

Since these sets of selections are pairwise disjoint, one can use the sum rule to get the required number of ways which is 18 + 12 + 6 = 36.

10.3. Factorial Notation

The product of n consecutive positive integer beginning with 1 is denoted by n ! or $\underline{|n}$ and is read as factorial n. Thus

$$n\,! = 1.\ 2.\ 3.\ \ldots\ldots\ (n-1)n = n(n-1)(n-2)\ \ldots\ldots\ 3.\ 2.\ 1.$$

For example,
$$5! = 1.\ 2.\ 3.\ 4.\ 5 = 5.\ 4.\ 3.\ 2.\ 1 = 120$$
$$4! = 1.\ 2.\ 3.\ 4 = 4.\ 3.\ 2.\ 1 = 24$$

Clearly,
$$\begin{aligned} n! &= 1.\ 2.\ 3.\ \ldots\ldots\ (n-2)(n-1)n \\ &= \{1.\ 2.\ 3.\ \ldots\ldots\ (n-2)(n-1)\}n \\ &= (n-1)!\ n = n.\ (n-1)!. \end{aligned}$$

Thus,
$$\begin{aligned} n! &= n(n-1)! = n.(n-1)(n-2)!. \\ &= n(n-1)(n-2)(n-3)! \text{ etc.} \end{aligned}$$

It r and n are positive integer and $r < n$, then

$$\frac{n!}{r!} = n(n-1)(n-2)\ \ldots\ldots\ (r+1) \text{ and } \frac{n!}{(n-r)!} = n(n-1)(n-2)\ \ldots\ldots\ (n-r+1)$$

For example, $\dfrac{5!}{2!} = 5.\ 4.\ 3 = 60,\ \dfrac{3!}{6!} = \dfrac{1}{6.5.4} = \dfrac{1}{120}$

Example 7. *Prove that* $(2n)! = 2^n\, n!\, \{1.\, 3.\, 5.\, \ldots\ldots\, (2n-1)\}$

Solution. $(2n)! = 2n\,(2n-1)(2n-2)\, \ldots\ldots\, 5.\,4.\,3.\,2.\,1$

$$= [2n\,(2n-2)\, \ldots\ldots\, 4.\,2]\,[(2n-1)(2n-3)\, \ldots\ldots\, 5.\,3.\,1]$$

$$= 2^n\,[n(n-1)\, \ldots\ldots\, 2.\,1]\,\{1.3.5.\, \ldots\ldots\, (2n-3)(2n-1)\}$$

$$= 2^n\, n!\, \{1.\,3.\,5.\, \ldots\ldots\, (2n-3)(2n-1)\}$$

Hence proved.

4. Permutation

The different arrangements which can be made out of a given set of things, by taking some or all of them at a time are called their permutations.

The number of permutations of n different things taken $r (\le n)$ at a time is denoted by $P(n, r)$ or ${}^nP_r$.

The different arrangement of three letters a, b, c taking two at a time are ab, bc, ca, ba, cb, ac and thus the number of arrangements (or permutations) of 3 things taken 2 at a time is $P(3, 2) = 6$.

The value of $P(n, r)$

To find the number of permutations of n different things taken $r (\le n)$ at a time.

The number of permutations of n things taken r at a time is the same as the number of ways in which r blank places can be filled up with n given things.

The first place can be filled up in n ways since any one of the n different things can be put in it.

After having filled up the first place by any one of the n things, there are $(n-1)$ things left. Hence the second place can be filled up in $(n-1)$ ways. Now, corresponding to each way of filling up the first place, there are $(n-1)$ ways of filling up the second place, so the first two places can be filled up in $n(n-1)$ ways.

After having filled up the first two places in any one of the above ways, there are $(n-2)$ things left and so the third place can be filled up in $(n-2)$ ways. Now corresponding to each ways of filling up the first two places, there are $(n-2)$ ways of filling up the third place and so the first three places can be filled up in $n(n-1)(n-2)$ ways.

It may be observed that

(*i*) at every stage the number of factors is equal to the number of places filled up.

(*ii*) each factor is less than the proceeding factor by 1.

Place	1st	2nd	...	$(r-1)$th	rth
Number of ways	n	$n-1$	...	$n-(r-2)$	$n-(r-1)$

Hence the number of ways of filling up all the r places *i.e.*, the number of permutation of n different things taken r at a time.

$$= n(n-1)(n-2)\, \ldots\ldots \text{ to } r \text{ factors}$$

$$= n(n-1)(n-2)\, \ldots\ldots\, [n-(r-1)]$$

Hence $$\boxed{P(n, r) = n(n-1)\,(n-2)\, \ldots\ldots\, (n-r+1)}$$

Now, $$P(n, r) = \frac{n\,(n-1)\,(n-2)\ldots\ldots(n-r+1)\,(n-r)!}{(n-r)!} \quad \ldots (1)$$

$$P(n, r) = \frac{n!}{(n-r)!}$$

Note 1. Number of permutation of n different things taken all n at a time is

$$P(n, n) = n(n-1)(n-2) \ldots\ldots 3.\ 2.\ 1 = n\,!$$

Note 2. Meaning of o!

Putting $r = n$ in (1)

$$P(n, n) = \frac{n!}{(n-n)!}$$

$$n\,! = \frac{n!}{0!} \quad i.e.,\ 0! = \frac{n!}{n!} = 1$$

Note 3. $P(n, r) = n \times P(n-1, r-1)$.

$$n \times P(n-1, r-1) = n \times \frac{(n-1)!}{(n-1-r+1)!} = n \times \frac{(n-1)!}{(n-r)!} = \frac{n!}{(n-r)!} = P(n, r)$$

Example 8. *If ${}^nP_2 = 72$, find the value of n.*

Solution: Here ${}^nP_2 = 72 \quad \Rightarrow \frac{n!}{(n-2)!} = 72$

$$\therefore \quad \frac{n \times (n-1) \times (n-2)!}{(n-2)!} = 72 \quad \Rightarrow n(n-1) = 72$$

$$\text{or} \quad n^2 - n - 72 = 0 \quad \Rightarrow n^2 - 9n + 8n - 72 = 0$$

$$\text{or} \quad (n-9)(n+8) = 0 \quad \therefore\ n = 9, -8$$

But $n \neq -8$ (n cannot be negative). So $n = 9$.

Example 9. *If ${}^{2n+1}P_{n-1} : {}^{2n-1}P_n = 3 : 5$, find the value of n.*

Solution: Here ${}^{2n+1}P_{n-1} = \frac{(2n+1)!}{(n+2)!}$ and ${}^{2n-1}P_n = \frac{(2n-1)!}{(n-2)!}$

$$\therefore \quad {}^{2n+1}P_{n-1} : {}^{2n-1}P_n = \frac{(2n+1)!}{(n+2)!} \times \frac{(n-1)!}{(2n-1)!} = \frac{3}{5}$$

$$\text{or} \quad \frac{(2n+1) \times 2n \times (2n-1)!}{(n+2) \times (n+1) \times n \times (2n-1)!} = \frac{3}{5} \quad \Rightarrow \frac{(2n+1) \times 2n}{(n+2) \times (n+1) \times n} = \frac{3}{5}$$

$$\text{or} \quad \frac{4n+2}{n^2+3n+2} = \frac{3}{5} \quad \Rightarrow 20n + 10 = 3n^2 + 9n + 6$$

$$\text{or} \quad 3n^2 - 11n - 4 = 0$$

$$\text{or} \quad (n-4)(3n+1) = 0 \quad \therefore\ n = 4, -\frac{1}{3}$$

$$\text{But,} \quad n \neq -\frac{1}{3} \quad \therefore\ n = 4$$

Example 10. *How many different numbers lying between 100 and 1000 can be formed with the digits 1, 2, 3, 4 and 5, no digit being repeated.*

Solution: The numbers lying between 100 and 1000 are of 3 digits. They will be formed by taking 3 digits from the given 5 digits. So the required number is equal to the number of permutations of 5 different things taken 3 at a time i.e. $P(5, 3) = \frac{5!}{2!} = 5.4.3 = 60$

Example 11. *Three prizes are to be awarded among 10 candidates. In how many ways can the prizes be given, so that no candidate may get more than one prize?*

Solution: Since no candidate get more than one prize, the required number is equal to the number of permutations of 10 different things taken 3 at a time *i.e.*,

$$P(10,3) = \frac{10!}{7!} = 10 \times 9 \times 8 = 720.$$

Restricted Permutations

1. The numbers of Premutations of n different objects taken r at a time in which k particular objects do not occur is $P(n-k, r)$.
2. The number of permutations of n different object taken r at a time in which k particular object are always present is $P(n-k, r-k) \times P(r, k)$.

Example 12. *In how many of the permutations of 10 things taken 4 at a time will (a) two things always occur (b) never occur ?*

Solution. (*a*) Keeping aside the two particular things which will always occur, the number of permutations of $10 - 2 = 8$ things taken 2 at a time ${}^8P_2$. Now 2 particular things can take up any one of the four places and so can be arranged in ${}^4P_2$ ways.

Hence the total numbers of permutation is ${}^8P_2 \times {}^4P_2 = 8 \times 7 \times 4 \times 3 = 672$

(*b*) Leaving aside the two particular things which will never occur, the number of permutations of 8 things taken 4 at a time $= {}^8P_4 = 8 \times 7 \times 6 \times 5 = 1680$.

3. When certain things not occurring together

Example 13. *Prove that the number of ways in which n books can be arranged on a shelf so that two particular books are never together is* $(n-2) \times (n-1)!$.

Solution. Treating the two particular books as one book, there are $(n-1)$ books which can be arranged in ${}^{n-1}P_{n-1} = (n-1)!$ ways. Now, these two books can be arranged amongst themselves in 2! ways. Hence the total number of permutations in which there two books are placed together is $2!\,(n-1)!$.

The number of permutations of n books without any restriction is $n!$.

Therefore, the number of arrangements in which these two books never occur together $= n! - 2!.\,(n-1)! = n.\,(n-1)! - 2.\,(n-1)! = (n-2)\,.\,(n-1)!$

Example 14. *In how many ways can 7 boys and 5 girls be seated in a row so that no two girls may sit together.*

Solution. Since there is no restriction on boys, first of all we fix the positions of 7 boys. Their positions are indicated as

$$\times\ B_1 \times B_2 \times B_3 \times B_4 \times B_5 \times B_6 \times B_7 \times$$

Now 7 boys can be arranged in 7! ways. Now if 5 girls sit at places (including the two ends) indicated by ×, then no two of the 5 girls will sit together. Clearly, 5 girls can be seated in 8 places in ${}^8P_5$ ways.

Hence the required number of ways of seating 7 boys and 5 girls under the given condition $= {}^8P_5 \times 7!$.

Example 15. *In how many ways 4 boys and 4 girls can be seated in a row so that boys and girls are alternate?*

Solution.

Case I. When a boy sits at the first place:

Possible arrangements will be of the form

B G B G B G B G

Now there are four places for four boys, therefore four boys can be seated in 4! ways. Again, there are four places for four girls, therefore four girls can be seated in 4! ways. Hence the number of ways in this case is 4! × 4!

Case II. When a girl sits at the first place:

Possible arrangements will be of the form

G B G B G B G B

Therefore, the number of arrangements in their case is 4! × 4!

Hence the required number of ways = 4! × 4! + 4! × 4! = 1152.

Example 16. *Given 10 people $P_1, P_2, P_3 \ldots\ldots, P_{10}$*

(i) In how many ways can the people be lined up in a row?

(ii) How many lineups are there if P_2, P_6, P_9 want to stand together?

(iii) How many lineups are there if P_2, P_6, P_9 do not want to stand together?

Solution. (*i*) Without any restriction 10 people can be lined up in a row in 10! ways.

(*ii*) Treating P_2, P_6, P_9 as one, 8 people can be arranged in a row in 8! ways. Again three persons P_2, P_6, P_9 can be arranged among themselves in 3!. So, the number of lined up when particular 3 persons are together is 8! × 3!.

(*iii*) The total number of ways in which P_2, P_6, P_9 do not want to stand together = Total number of arrangements without any restriction – Total number of arrangements in which P_2, P_6, P_9 are always together = 10 ! – 8 ! × 3 !.

4. Formation of numbers with digits

Example 17. *How many number of four digits can be formed with the digits 1, 2, 3, 4 and 5 if repetition of digits is not allowed.*

Solution. The required number of 4–digit numbers = the number of arrangement of 5 distinct digits taken 4 at a time = $^5P_4 = 5 \times 4 \times 3 \times 2 = 120$.

Example 18. *How many numbers lying between 100 and 1000 can be formed with the digits 1, 2, 3, 4, 5 if the repetition of digits is not allowed.*

Solution. Every number lying between 100 and 1000 is a 3–digit number. Therefore, we have to find the number of permutations of 5 digits 1, 2, 3, 4, 5 taken three at a time.

Hence, the required number of 3–digit numbers = $^5P_3 = \dfrac{5\,!}{(5-3)\,!} = 60.$

Example 19. *How many numbers between 400 and 1000 can be formed with the digits 2, 3, 4, 5, 6 and 0, if repetition of digits is not allowed.*

Solution. Any number lying between 400 and 1000 must be of 3–digit only. Since the number should be greater than 400, the hundred's place can be filled up by any one of the three digits 4, 5 and 6 in 3 ways. The remaining two places (ten's and unit's) can be filled up by remaining five digits in $^5P_2$ ways.

$\therefore$ The required number = $3 \times {}^5P_2 = 3 \times \dfrac{5\,!}{3\,!} = 60$

Example 20. *How many four digit numbers are there with distinct digits?*

Solution. The total number of arrangements of ten digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 taking 4 at a time is $^{10}P_4$. But these arrangements also include those numbers which have 0 at thousand's place. Such numbers are not 4–digits numbers.

To find the total number of numbers in which 0 is at thousand's place, we fix 0 at thousand's place and arrange remaining 9 digits by taking 3 at a time. The number of such arrangements is $^9P_3$. Hence, the total number of 4–digit numbers = $^{10}P_4 - {}^9P_3 = 5040 - 504 = 4536$.

5. Word Building

Example 21. *Find the number of permutations that can be had from the letters of the DAUGHTER.*

(i) taking all the letters together.
(ii) beginning with D
(iii) beginning with D and ending with R
(iv) vowels being always together
(v) not all vowels together
(vi) not even two vowels together
(vii) vowels occupying even places.

Solution. The number of letters in the word DAUGHTER is 8 and all are different.

(i) Number of words taking all the letters together is equal to the numbers of arrangements of all 8 letters taken all at a time = 8 ! = 40320.

(ii) Keeping D fixed in the first place, the remaining seven places can be filled up by the remaining 7 letters in 7 ! = 5040 ways.

Hence, the required number of words = 5040.

(iii) Keeping D fixed in the first place and R in the last place, the remaining 6 places can be filled up by the remaining 6 letters in 6 ! = 720 ways.

Hence, the required number of words = 720.

(iv) Treating the three vowels A, U and E as one, we have 8 – 3 + 1 = 6 letter to arrange. This can be done in 6! = 720 ways. Again the three vowels A, U, E can be arranged among themselves in 3! = 6 ways.

Hence, the required number of words = 720.6 = 4320.

(v) The required number of words = Total number of all possible words – Number of words with all vowels together.

$$= 40320 - 4320 = 36000.$$

(vi) Since there is no restriction on consonants D, G, H, T, R. We first fix these five consonants. This can be done 5! = 120 ways.

$$\times \text{C} \times \text{C} \times \text{C} \times \text{C} \times \text{C} \times$$

The three vowels can be arranged in any of the three of six places marked ×, so that no two vowels are together. This can be done in ${}^6P_3 = 6 \times 5 \times 4 = 120$. Hence the required number of permutation = 120 × 120 = 14400.

(viii) Out of 8 places, 2nd, 4th, 6th, 8th places are even places and the three vowels A, E, U can be arranged in 4 places in ${}^4P_3 = 24$ ways.

The four places (1st, 3rd, 5th, 7th) and one remaining even place are to be filled by 5 consonants. This can be done in 5! = 120 ways.

Hence, the required number of ways = 24 × 120 = 2880.

Permutations of things not all different

To find the number of ways in which n things may be arranged among themselves, taking all of them at a time, when p of things are alike of one kind, q of them alike of a second kind, r of them of a third kind, and the rest all different.

Let x be the required number of permutations.

Now, replace p alike things by p distinct things which are also different from others. These p different things may be permuted among themselves in p ! ways without changing the positions of other things.

Thus if p alike things of one kind are replaced by p different things, the number of permutations $= x \times p!$.

Now if in any one of the $x \times p!$ permutations, we replace q alike things of other kind by q other different things, then the number of permutations of these q new things among themselves without disturbing the arrangements of rest will be $q!$.

Thus if both the replacements are done, the number of permutation $= x \times p! \times q!$

Similarly when r alike things are also replaced by r different things, then the number of permutations $= x \times p! \times q! \times r!$.

Now each of these $x \times p! \times q! \times r!$ permutations, is a permutation of n different things, taken all at a time.

$\therefore\ x \times p! \times q! \times r! = n!$

Hence $x = \dfrac{n!}{p!q!r!}$

Example 22. *How many permutations of the letters of the word BANANA?*

Solution. There are 6 letters in the word BANANA of which three are alike of one kind (3A's), two are alike of second kind (2N's) and the rest one letter is different.

Hence the required number of permutations $= \dfrac{6!}{3!2!} = 60$

Example 23. *Find the number of possible ways in which the letters of the word COTTON can be arranged so that the two Ts don't come together.*

Solution. There are two T's, two O's and the rest two letters are different in the word COTTON.

Hence the number of arrangements of the letters without any restriction $= \dfrac{6!}{2!2!} = 180$

Considering the two T's as one letter, the number of letters to be arranged is 5.

So, the number of arrangements in which both T come together $= \dfrac{5!}{2!} = 60$

Hence, the number of ways in which two T's do not come together $= 180 - 60 = 120$.

Example 24. *A coin is tossed 6 times. In how many ways can we obtain 4 heads and 2 tails?*

Solution. Whether we toss a coin 6 times or toss 6 coins at a time, the number of arrangements will be the same.

$\therefore$ The number of arrangements of 4 heads and 2 tails out of 6 is $\dfrac{6!}{4!2!} = 15$.

Example 25. *There are 3 copies each of 4 different books. Find the number of ways of arranging them on a shelf.*

Solution. Total number of books is $3 \times 4 = 12$

Each of the 4 different titles has 3 copies each.

$\therefore$ The required number of ways of arranging them on a shelf is

$$\frac{12!}{3!3!3!3!} = \frac{12!}{(3!)^4} = 369600.$$

Permutation with repetition allowed

The number of permutation of n different things taken r at a time, when things may be repeated any number of times is n^r.

Suppose r places are to be filled with n things. The first place can be filled up in n ways; and when it has been filled up in any one of these ways, the second place can also be filled up in n ways for the thing occupying the first place may occupy the second place also. Thus the first two places can be filled in $n \times n = n^2$ ways. When the two places are filled in any of the n^2 ways, the third place can also be filled up in n ways, and so the three places together can be filled up in $n^2 \times n = n^3$ ways.

Proceeding in this way, we conclude that the r places can be filled in $n \times n \times n \times \ldots\ldots$ r times i.e. n^r ways.

Example 26. *How many numbers of 3 digits can be formed with the digits 2, 4, 6 and 8 when a digit may be repeated any number of times?*

Solution. Since repetition is allowed, each of the 3 places in a 3–digit numbers can be filled up in 4 ways by the given 4 digits.

hundreds place	*tens place*	*units place*
4 ways	4 ways	4 ways

$\therefore$ The required number of 3–digit numbers $= 4^3 = 64$.

Example 27. *In how many ways 3 prizes can be distributed among 4 boys when a boy can get any number of prizes.*

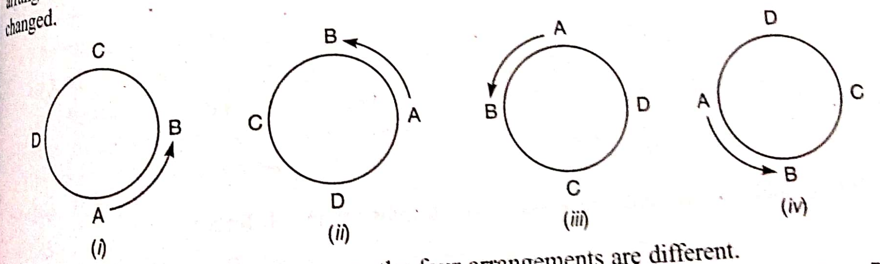
Solution. Each of the three prizes can be given in 4 ways since a boy can receive any number of prizes.

$\therefore$ The number of ways in which all the prizes can be given $= 4 \times 4 \times 4 = 4^3 = 64$.

Circular Permutation

The permutation that has been described above are more properly called linear permutations as the objects are being arranged in a line (or row). If we arrange them in a circle, then the number of permutation change. A circular permutation is an arrangements of objects around a circle.

Consider four persons A, B, C and D who are to be arranged around a circle. The following arrangements are one and the same as relative position of none of the four positions A, B, C, D is changed.



But in case of linear arrangements the four arrangements are different.

A B C D   D A B C   C D A B   B C D A

Thus the circular permutations are different only when the relative order of the objects is changed otherwise they are the same.

As the number of circular permutations depend on the relative positions of objects, the number of permutation of n objects in a circle can be found by keeping one element fixed and arranging remaining $(n - 1)$ elements which can be done in $(n - 1)!$ ways.

$\boxed{\text{Thus number of circular arrangements of } n \text{ different things} = (n-1)!}$

Example 28. *In how many ways 5 boys and 4 girls can be seated at a round table if*

(*i*) *there is no restriction*

(*ii*) *all the four girls sit together*

(*iii*) *all the four girls do not sit together*

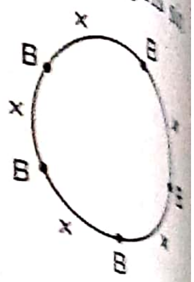
(*iv*) *no two girls sit together.*

Solution: (*i*) Total number of persons = 5 + 4 = 9. These persons can be seated at the round table in (9 – 1)! = 8! ways.

(*ii*) Taking 4 girls as one person, we have 5 + 1 = 6 persons. These 6 persons can be seated at a round table in (6 – 1)! = 5! ways. But 4 girls can be arranged among themselves in 4! ways. So the required number of ways = 5! × 4!.

(*iii*) The number of arrangements when all the 4 girls do not sit together = total number of arrangements without restriction – number of arrangements when all the four girls sit together = 8 ! – 5 ! × 4 !.

(*iv*) Since there is no restriction on boys, first of all we arrange 5 boys to seat at round table and this can be done (5 – 1)! = 4! ways. Now there are 5 places for 4 girls as indicated by ×. Therefore, 4 girls can be seated in ${}^5P_4$ = 5! ways.



So, the required number of ways = 4! × 5!

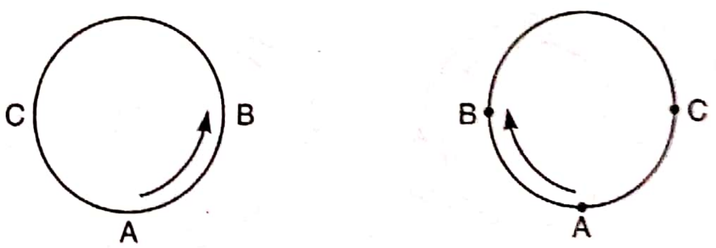
Example 29. *In how many ways can a party of 4 boys and 4 girls be seated at a circular table so that no 2 boys are adjacent.*

Solution: 4 girls can be seated at a circular table in 3! ways. When they have been seated, there remain 4 places for the boys each between two girls. Therefore, the boys can sit in 4! ways. Thus, there are 3! × 4! *i.e.* 144 ways of seating in the party.

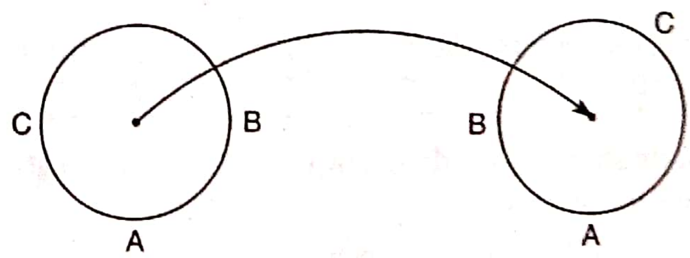
Clockwise and Counter-clockwise Permutations

There are two types of circular permutations

(*i*) Those permutations in which clockwise and anti-clockwise arrangements are distinguishable. Thus while seating 3 persons A, B, C around a table, the following permutations are different.



(*ii*) When clockwise and anti-clockwise permutations are not different.



When clockwise and anti-clockwise arrangements are not different e.g., arrangement of beads

Example 30. *In how many ways 8 different beads can be arranged to form a necklace?*

Solution. Eight different beads can be arranged in circular form in $(8-1)! = 7!$ ways. Since there is no distinction between the clockwise and anti-clockwise arrangements, the required number of arrangements $= \frac{1}{2}(7!)$.

Example 31. *In how many ways can seven persons sit around a table so that all shall not have same neighbours in any two arrangements?*

Solution. Seven persons can sit at a round table in $(7-1)! = 6!$ ways. But, in clockwise and anti-clockwise arrangements, each person will have the same neighbours.

So, the required number of ways $= \frac{1}{2}(6!) = 360$.

10.5. Combination

The different groups or selections that can be made out of a given set of things by taking some or all of them at a time irrespective of the order are called their combinations.

The number of combinations of n different things taken $r (\le n)$ at a time is denoted by $C(n, r)$ or ${}^nC_r$.

The selection of two letters from three letters a, b, c are ab bc ca and thus, the number of combinations of 3 letters taken 2 at a time is $C(3, 2) = 3$

The number of combinations of 4 letters a, b, c, d taken two at a time is $C(4, 2) = 6$ and they are

$$ab,\ ac,\ ad,\ bc,\ bd,\ cd.$$

Difference between a Permutation and Combination

1. In a combination only selection is made where in a permutation not only a selection is made but also an arrangement in a definite order is considered.
2. In a combination, the ordering of the selected objects is immaterial. In a permutation, this ordering is essential. For example, a, b and b, a are same in combination but they are different in permutation.

Generally we use the word arrangements for permutations and the word selections for combinations.

The value of $C(n, r)$

To find the number of combination of n distinct objects taken $r (\le n)$ of a time.

Let x be the required number of combinations *i.e.*, $x = C(n, r)$. Now each combination contains r different things which can be arranged among themselves in $r!$ ways. Hence x such combinations give rise to $x \times r!$ arrangements giving the number of permutations of n different things taken r at a time *i.e.*, $P(n, r)$

$$x \times r! = P(n, r) = \frac{n!}{(n-r)!}$$

$$\therefore \quad x = \frac{P(n, r)}{r!} = \frac{n!}{r!(n-r)!}$$

$$\therefore \quad C(n, r) = \frac{n!}{r!(n-r)!} \quad \ldots (1)$$

or

Cor I. $C(n, n) = C(n, 0) = 1$

Putting $r = n$ in (1) $C(n, n) = \dfrac{n!}{n!(n-n)!} = \dfrac{1}{0!} = 1$

Putting $r = 0$ in (1) $C(n, 0) = \dfrac{n!}{0!(n-0)!} = \dfrac{n!}{0!\,n!} = 1$

Properties of ^nC_r or $C(n, r)$

Prop 1. $\quad {}^nC_r = {}^nC_{n-r} \quad (0 \le r \le n).$

We have
$$^nC_{n-r} = \frac{n!}{(n-r)!(n-n+r)!} = \frac{n!}{(n-r)!\,r!} = {}^nC_r$$

Prop 2. $\quad {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r \quad (0 \le r \le n).$

We have
$$\begin{aligned}
^nC_r + {}^nC_{r-1} &= \frac{n!}{(n-r)!\,r!} + \frac{n!}{[n-(r-1)]!\,(r-1)!} \\
&= n!\left[\frac{1}{(n-r)!\,r!} + \frac{1}{(n-r+1)!\,(r-1)!}\right] \\
&= n!\left[\frac{1}{(n-r)!\,r(r-1)!} + \frac{1}{(n-r+1)\,(n-r)!\,(r-1)!}\right] \\
&= \frac{n!}{(n-r)!\,(r-1)!}\left[\frac{1}{r} + \frac{1}{n-r+1}\right] \\
&= \frac{n!}{(n-r)!\,(r-1)!}\left[\frac{n+1}{r\,(n-r+1)}\right] \\
&= \frac{(n+1)\,n!}{r\,(r-1)!\,(n-r+1)\,(n-r)!} \\
&= \frac{(n+1)!}{r!\,(n-r+1)!} \\
&= {}^{n+1}C_r.
\end{aligned}$$

$\therefore \quad {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r.$

Prop 3. $\quad {}^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n.$

We have $\quad {}^nC_x = {}^nC_y$

$\Rightarrow \quad {}^nC_x = {}^nC_{n-y} \qquad [\because {}^nC_r = {}^nC_{n-r}]$

$\Rightarrow \quad x = y$ or $x = n - y$

$\Rightarrow \quad x = y$ or $x + y = n.$

Relation between ^nC_r and $^{n+1}C_r$.

$$\frac{^{n+1}C_r}{^nC_r} = \frac{(n+1)!/r!\,(n-r+1)!}{n!/r!\,(n-r)!} = \frac{(n+1)!}{n!} \times \frac{(n-r)!}{(n-r+1)!}$$
$$= \frac{n+1}{n-r+1}$$

Example 32. *Evaluate the following*

(i) $^6C_3$ $\qquad$ *(ii)* $^{10}C_8$

Solution. *(i)*
$$^6C_3 = \frac{6 \times 5 \times 4 \times 3!}{3!\,(6-3)!} = \frac{120}{3!} = 20$$

(ii) $\quad {}^{10}C_8 = {}^{10}C_2 = \dfrac{10!}{(10-2)!\,2!} = \dfrac{10\times9\times8!}{8!\times2!} = 45. \qquad [\because {}^nC_r = {}^nC_{n-r}]$

Example 33. *If ${}^{18}C_r = {}^{18}C_{r+2}$, find the value of ${}^rC_5$.*

Solution. As $r \neq r+2$ so $r + (r+2) = 18 \Rightarrow r = 8$

$\therefore \quad {}^rC_5 = {}^8C_5 = {}^8C_3 = \dfrac{8\times7\times6}{3!} = 56.$

Example 34. *If ${}^nC_x = 56$ and ${}^nP_x = 336$, find n and x*

Solution. Here $\quad {}^nP_x = 336$ i.e., $\dfrac{n!}{(n-x)!} = 336$

and $\quad {}^nC_x = 56$ i.e., $\dfrac{n!}{x!\,(n-x)!} = 56$

$\Rightarrow \quad \dfrac{n!}{x!\,(n-x)!} \times \dfrac{(n-x)!}{n!} = \dfrac{56}{336}$

$\Rightarrow \quad \dfrac{1}{x!} = \dfrac{1}{6} = \dfrac{1}{3!}$

$\therefore \quad x = 3.$

Again $\quad {}^nP_3 = 336$ i.e., $\dfrac{n!}{(n-3)!} = 336$

or $\quad \dfrac{n(n-1)(n-2)(n-3)!}{(n-3)!} = 336$

or $\quad n(n-1)(n-2) = 336 = 8\times7\times6$

or $\quad n(n-1)(n-2) = 8\times(8-1)\times(8-2)$

$\therefore \quad n = 8$

Example 35. *If ${}^{1000}C_{98} = {}^{999}C_{97} + {}^xC_{901}$; find x.*

Solution. Here $\quad {}^{1000}C_{98} = {}^{999}C_{97} + {}^xC_{901}$

or $\quad {}^{1000}C_{902} = {}^{999}C_{902} + {}^xC_{901} \qquad (\because {}^nC_r = {}^nC_{n-r})$

or $\quad {}^{999+1}C_{902} = {}^{999}C_{902} + {}^xC_{902-1}$

As $\quad {}^{n+1}C_r = {}^nC_r + {}^nC_{r-1}$

we get $\quad x = 999.$

Example 36. *In how many ways can 4 questions be selected from 7 questions?*

Solution. The required number of ways $= {}^7C_4 = {}^7C_3 = \dfrac{7\times6\times5}{3.2.1} = 35.$

Example 37. *If there are 12 persons in a party, and if each two of them shake hands with each other, how many handshakes happen in the party?*

Solution. When two persons shake hands, it is counted as one handshake, not two. Therefore, the total number of handshakes is the same as number of ways of selecting 2 persons from among 12 persons. This can be done

$${}^{12}C_2 = \frac{12\times11}{2\times1} = 66 \text{ ways.}$$

Example 38. *In how many ways can a committee of 5 teachers and 4 students be chosen from 9 teachers and 15 students?*

Solution. The four students can be chosen out of 15 students in ${}^{15}C_4$ and the nine teachers can be selected in ${}^9C_5$ ways. Now each way of selecting students can be associated with each way of selecting teachers.

So, the required number ways of selecting committee = ${}^{15}C_4 \times {}^9C_5$.

Example 39. *Out of 5 men and 2 women, a committee of 3 is to be formed. In how many ways can this be done so as to include (i) exactly one woman (ii) at least one woman*

Solution. (*i*) We have to select one woman out of 2 and 2 men out of 5.

The number of ways of selecting 1 woman = ${}^2C_1 = 2$

The number of ways of selecting 2 men = ${}^5C_2 = \dfrac{5.4}{2} = 10$

$\therefore$ The committee can be formed in $2 \times 10 = 20$ ways.

(*ii*) The committee can be formed in the following ways

(*a*) 2 men and 1 woman.

(*b*) 1 man and 2 women.

For (*a*) 2 men out of 5 men and 1 woman out of 2 women can be selected in ${}^5C_2 \times {}^5C_1$ ways

For (*b*) 1 man out of 5 men and 2 women out of 2 women can be selected in ${}^5C_1 \times {}^2C_2$ ways

$\therefore$ Total number of ways of forming the committee = ${}^5C_2 \times {}^2C_1 + {}^5C_1 \times {}^2C_2 = 20 + 5 = 25$

Example 40. *In how many ways can a student choose a course of 5 subjects if 9 subjects are available and 2 subjects are compulsory for all the students.*

Solution. Since two subjects are compulsory, the students are required to choose 3 out of 7 subjects and this can be done in ${}^7C_3 = \dfrac{7!}{3!4!} = 35$ ways.

Example 41. *The question paper of mathematics contains ten questions divided into two groups of five questions each. In how many ways can an examinee answer six questions taking at least two questions from each group.*

Solution. The examinee can answer questions from two groups in the following ways.

(*i*) 2 from first group and 4 from the second group.

(*ii*) 3 from first group and 3 from the second group.

(*iii*) 4 from first group and 2 from the second group.

For (*i*), the number of ways of selecting the questions = ${}^5C_2 \times {}^5C_4 = 10 \times 5 = 50$

For (*ii*), the number of ways of selecting the questions = ${}^5C_3 \times {}^5C_3 = 10 \times 10 = 100$

For (*iii*), the number of ways of selecting the questions = ${}^5C_4 \times {}^5C_2 = 5 \times 10 = 50$

Hence the examinee can answer the questions in either of the three ways.

Therefore, the required number of ways = $50 + 100 + 50 = 200$.

Example 42. *There are 50 students in each of the senior or junior classes. Each class has 25 male and 25 female students. In how many ways can an eight student committee be formed so that there are four females and three junior in the committee?*

Solution. A committee of 8 students can be formed in the following ways.

For (i), the number of ways of selecting 8 students $= {}^{25}C_1 \times {}^{25}C_4 \times {}^{25}C_3$.

For (ii), the number of ways of selecting 8 students $= {}^{25}C_2 \times {}^{25}C_3 \times {}^{25}C_2 \times {}^{25}C_1$.

For (iii), the number of ways of selecting 8 students $= {}^{25}C_3 \times {}^{25}C_2 \times {}^{25}C_1 \times {}^{25}C_2$.

For (iv), the eight students can be selected in ${}^{25}C_4 \times {}^{25}C_1 \times {}^{25}C_3$.

Since the committee of 8 students is formed in each case, the total number of ways of forming ...mmittee

$$= {}^{25}C_1 \times {}^{25}C_4 \times {}^{25}C_3 + {}^{25}C_2 \times {}^{25}C_3 \times {}^{25}C_2 \times {}^{25}C_1 + {}^{25}C_3 \times {}^{25}C_2 \times {}^{25}C_1 \times {}^{25}C_2 + {}^{25}C_4 \times {}^{25}C_1 \times {}^{25}C_3.$$

Example 43. *Find the number of diagonals that can be drawn by joining the angular points ...agon.*

Solution. A hexagon has six angular points and six sides. The join of two angular points is ...r a side or a diagonal.

The number of lines joining the angular points $= {}^6C_2 = \dfrac{6 \times 5}{2} = 15$

But the number of sides is 6, hence the number of diagonals $= 15 - 6 = 9$.

...mbinations taken any number at a time

Total number of combinations of n different things taken any number at a time is $2^n - 1$. Each ...ng may be disposed of in two ways. It may be either accepted or rejected. Corresponding to each ...the two ways of disposing the first, the second can be disposed of in the similar two ways. So, the ...al number of ways of disposing of all the things $= 2 \times 2 \times$ ......n times $= 2^n$. But in this 2^n cases, ...re is one case in which all the things are rejected. Hence the total number of ways in which one ...r more things are taken $= 2^n - 1$.

Example 44. *You have 4 friends; in how many ways can you invite one or more them for ...inner?*

Solution. You may invite 1, 2, 3 or 4 of your friends to dinner. Hence the required number of ...ways $= 2^4 - 1 = 63$.

Example 45. *There are 5 questions in a question paper. In how many was can a boy solve ...e or more questions.*

Solution. The boy may either solve a questions or leave it. Thus the boy disposes of each ...estion in two ways. Thus the number of ways of disposing of all the questions $= 2^5$. But this ...cludes the case in which he has left all the questions unsolved. Hence the required number of ...ways of solving the paper $= 2^5 - 1 = 31$.

...ombinations of things not all different

By taking some or all of $(p + q + r +$ ....$)$ things, where p are alike of one kind, q are ...ike of a second kind, r alike of third kind and so on, the total number of possible selection is

$$[\{(p + 1)(q + 1)(r + 1) ..... \} - 1].$$

Out of p things we may take 0, 1, 2, 3, ... or p. Hence they can be disposed of in $(p + 1)$ ...ways. Similarly, q alike things may be disposed of in $(q + 1)$ ways; r alike things in $(r + 1)$ ways ...and so on. Since each way of disposing p like things can be associated with each way of disposing ...any other thing, the total number of ways in which all the things can be disposed of is $(p + 1)$ ...$(q + 1)(r + 1)$ .... But this includes the case in which all the things are left out. Rejecting this, ...the required number of combinations is $[\{(p + 1)(q + 1)(r + 1) ......\} - 1]$.

Example 46. *In how many ways can a selection be made out of 3 mangoes, 5 oranges and 2 apples?*

Solution. Hence $p = 3$, $q = 5$ and $r = 2$.

So, the required number of combinations $= (3 + 1)(5 + 1)(2 + 1) - 1$

$= 4.6.3 - 1 = 71.$

Division Into Groups

To find the number of ways in which $(p + q)$ things can be divided into two groups containing p and q things respectively.

We can select p things from $(p + q)$ things in ${}^{p+q}C_p$ ways, and each time p things are taken a group of q things are left aside from which q things can be selected in ${}^{q}C_q$, *i.e.*, 1 way. Again q things are selected first from $p + q$ things in ${}^{p+q}C_q$ ways, everytime q things are selected, there remain p things which all can be selected in ${}^{p}C_p$, *i.e.*, 1 way.

As ${}^{p+q}C_p = {}^{p+q}C_{(p+q)-p} = {}^{p+q}C_q = \dfrac{(p+q)!}{p!q!}$, the required number of ways $= \dfrac{(p+q)!}{p!q!}$

Note. If $p = q$, the groups are equal and here it is possible to interchange the two groups without obtaining a new distribution. Hence the number of different ways of division into groups

$$= \frac{(2p)!}{2p!p!} = \frac{(2p)!}{2!(p!)^2}.$$

If the two groups are different, then the number of different ways of division into groups

$$= \frac{(2p)!}{(p!)^2}.$$

Example 47. *In how many ways 14 oranges can be divided equally into 2 groups?*

Solution. The required number of ways $= \dfrac{14!}{2!7!7!}$.

Example 48. *In how many ways 14 oranges can be divided equally among 2 children?*

Solution. The number ways of different division $= \dfrac{14!}{(7!)^2}$ for, in this case a group of 7 oranges alloted for the first boy, if given to the second boy, the two groups are different.

Generalization

The number of ways in which $p + q + r$ things can be divided into three groups containing p, q and r things respectively.

$$= {}^{p+q+r}C_p \times {}^{q+r}C_q \times {}^{r}C_r = \frac{(p+q+r)!}{p!(q+r)!} \times \frac{(q+r)!}{q!r!} \times 1 = \frac{(p+q+r)!}{p!q!r!}$$

Similarly the result can be extended to the case of dividing a given number of things into more than three groups.

Note 1. The number of ways in which $3p$ things can be divided equally into three distinct groups is $\dfrac{(3p)!}{(p!)^3}$ $(q = p, r = p)$

Note 2. The number of ways in which $3p$ things can be divided into three identical groups is $\dfrac{(3p)!}{3!(p!)^3}$.

Example 49. *In how many ways can 14 men be partitioned into 6 teams where the first team ...members, the 2nd team has 2 members, the 3rd team has 3 members, 7th, 4th, 5th, 6th teams ...has 2 members?*

Solution. The required number is $\dfrac{14!}{3!\,2!\,3!\,(2!)^3}$

- The total number of ways to divide n identical things among r persons

$$= {}^{n+r-1}C_{r-1}$$

- The total number of ways to divide n identical things among r persons so that each gets at least one.

$$= {}^{n-1}C_{r-1}$$

Combinations with repetition

We first consider the following example:

Example 50. *Consider the set {a b c d}. In how many ways can we select two of these letters when repetition is allowed.*

Solution. If order matters and repetition is allowed, there are $2^4 = 16$ possible selections and they are

aa	*ba*	*ca*	*da*
ab	*bb*	*cb*	*db*
ac	*bc*	*cc*	*dc*
ad	*bd*	*cd*	*dd*

If order does not mater but repetitions are allowed, there are 10 possibilities and these possibilities are

aa	*bb*	*cc*	*dd*		
ab	*bc*	*cd*	*ad*	*ac*	*bd*

Theorem 10.5. The number of unordered choices of r from n, with repetitions allowed is

$$\binom{n+r-1}{r} = C(n+r-1, r)$$

Proof : Any choice will consist of x_1 choices of the first object, x_2 choices of the second object, and so on, subject to the condition $x_1 + \ldots + x_n = r$. So the required nimber is just the number of solutions of the equation $x_1 + \ldots + x_n = r$ in non-negative integers x_i.

Now we can represent a solution $x_1, \ldots, x_n$ by a binary sequence:

$$x_1 \text{ 0s}, 1, x_2 \text{ 0s}, 1, x_3 \text{ 0s}, 1, \ldots, 1, x_n \text{ 0s}.$$

Think of the 1 s as indicating a move from one object to the next. For example, the solution $x_1 = 2, x_2 = 0, x_3 = 2, x_4 = 1$ of $x_1 + x_2 + x_3 + x_4 = 5$ corresponds to the binary sequence 00110010. Two consecutive 1's mean that there are no objects in that place. Corresponding to $x_1 + \ldots + x_n = r$, there will be $n - 1$, 1s and r 0s, and so each sequence will be of length $n + r - 1$, containing exactly r 0s. Conversely, any such sequence corresponds to a non-negative integer solution of $x_1 + \ldots + x_n = r$. Now the r 0s can be in any of the $n + r - 1$ places, so the number of such sequences, and hence the number of unordered choices, is $C(n + r - 1, r)$, the number of ways of choosing r places out of $n + r - 1$.

This proof also establishes the following result.

Theorem 10.6. The number of solutions of $x_1 + \ldots + x_n = r$ in non-negative integers x_i is $C(n + r - 1, r)$.

The equivalent formulation of the above theorem makes it easier to conceptualize the problem.

The number of non-negative integral solutions of $x_1 + x_2 + ... + x_n = r$ is equal to the number of ways of placing r identical (indistinguishable) balls in n numbered boxes.

At this point, we recognize the equivalence of the followings:

The number of r-combinations of n distinct objects with unlimited repetitions

= the number of non-negative integral solutions (i.e. $x_1 \geq 0, x_2 \geq 0, ..., x_n \geq 0$) of $x_1 + x_2 + ... + x_n = r$,

= the number of ways of distributing r similar balls into n numbered boxes

= the number of binary numbers with $n - 1$ one's and r zeros

$= C(n - 1 + r, r) = C(n - 1 + r, n - 1)$

$= \dfrac{(n + r - 1)!}{[r!(n - 1)!]}$,

Note: (*i*) In terms of distributions, this is valid only when the r objects being distributed are identical and the n boxes are distinct.

(*ii*) If we want to ensure that every box gets at least one ball, we must give a ball to each box, then distribute the remaining $r - n$ balls in n boxes which can be done in $C(n - 1 + r - n, n - 1) = C(r - 1, n - 1)$ ways.

(*iii*) The number of positive integral solutions of $x_1 + x_2 + ... + x_n = r$ where $x_1 \geq 1, x_2 \geq 1, ..., x_n \geq 1$ is the same as the number of ways to distribute n identical things among r persons each getting at least $1 = C(r - 1, n - 1)$

(*iv*) The number of non-negative integral solution of $x_1 + x_2 + ... + x_n = r$ where $x_1 \geq r_1$, $x_2 \geq r_2, ..., x_n \geq r_n$ is

$= C(n - 1 + r - r_1 - r_2 ... r_n, r - r_1 - r_2, ... r_n)$

$= C(n - 1 + r - r_1 - r_2, ... r_n, n - 1)$

Example 51. *In how many different ways can 20 identical apples be distributed among 4 persons?*

Solution. The required number

$$= {}^{20+4-1}C_{4-1} \qquad (r = 20, n = 4)$$

$$= {}^{23}C_3 = \frac{23 \times 22 \times 21}{3!} = 1771$$

Example 52. *Find the number of ways to distribute 20 identical balls in 4 different boxes so that no box remains empty.*

Solution. The required number

$$= {}^{20-1}C_{4-1} \qquad (r = 20, n = 4)$$

$$= {}^{19}C_3 = \frac{19 \times 18 \times 17}{3!} = 969$$

Example 53. *How many solutions does the equation*

$$x + y + z = 17$$

have, where x, y, z are non-negative integers?

Solution. The number of integral solution of the given equation is the same as the number of ways of distributing 17 identical things among 3 persons. Hence the required number of solutions

$$= {}^{17+3-1}C_{3-1} = {}^{19}C_2 = \frac{19 \times 18}{2} = 171$$

Example 54. *How many solutions are there of $x + y + z = 17$ subject to the constraints $x \geq 1, y \geq 2$ and $z \geq 3$.*

Solution. Put $x = 1 + u$, $y = 2 + v$ and $z = 3 + w$. The given equation becomes $u + v + w = 11$ and we seek in non-negative integers u, v, w. The required number of solution is

$$= {}^{11+3-1}C_{3-1} = {}^{13}C_2 = \frac{13 \times 12}{2} = 78.$$

Mixed Problems on Permutation and Combinations

Example 55. *Out of 7 consonants and 4 vowels, how many words of 3 consonants and 2 vowels can be formed?*

Solution. Three consonants from 7 consonants can be chosen in ${}^7C_3$ ways

Two vowels from 4 vowels can be chosen in ${}^4C_2$ ways

$\therefore$ 3 constants and 2 vowels can be chosen in ${}^7C_3 \times {}^4C_2$ ways. Thuse, there are ${}^7C_3 \times {}^4C_2$ groups each containing 3 consonants and 2 vowels.

Since each group contains 5 letters, which can be arranged among themselves in 5! ways.

$\therefore$ The total number of words $= {}^7C_3 \times {}^4C_2 \times 5! = 25200$.

10.6 Number of Onto Function

We know the total number of functions from set A having n elements to a set B of m elements is m^n. If $m \geq n$, then the number of one-one function is $m!/(m - n)!$. The principle of inclusion-exclusion can be used to obtain the number of onto functions from one finite set to another.

Let A and B be two sets having n and $m(\leq n)$ elements respecitively. Then there are

$$m^n - {}^mC_1(m-1)^n + {}^mC_2(m-2)^n + \ldots + (-1)^{m-1}\, {}^mC_{m-1}$$

onto functions from A to B.

Example 56. In how many ways five different jobs can be assigned to four persons if each person is assigned at least one job?

Solution. Let A be the set of five different jobs and B be the set of four persons. Since each person is to be assigned at least one job, this is equivalent to finding onto functions from the set A to B. Here $|A| = 5$ and $|B| = 4$. Then there are

$$\begin{aligned} & 4^5 - {}^4C_1(4-1)^5 + {}^4C_2(4-2)^5 - {}^4C_3(4-3)^5. \\ &= 1024 - 4 \times 243 + 6 \times 32 - 4 \\ &= 1024 - 972 + 192 - 4 \\ &= 1216 - 976 = 240 \text{ ways to assign the jobs.} \end{aligned}$$

10.7 Derangements

A derangements of sequence $a_1, a_2, \ldots, a_n$ of n objects is an arrangement in which no object a_i is left in its original position. For example, if 1, 2, 3, 4, 6 is a sequence then 2, 1, 4, 5, 3 and 4, 1, 5, 2, 3 are derangement of 1, 2, 3, 4, 5.

If n things are arranged in a row, the number of ways in which they can be deranged so that no one of them occupies its original place is

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!}\right]$$

If r things go to wrong place out of n things then $(n - r)$ things go to original place $(r < n)$.

If D_n = number of ways, if all n things go to wrong place and D_r = number of ways, if r things go to wrong place.

If r things go to wrong place out of n, then $(n-r)$ things go to correct places. Then

$$D_n = {}^nC_{n-r}D_r$$

If at least p of them are in the wrong places, then

$$D_n = \sum_{r=p}^{n} {}^nC_{n-r}D_r$$

Example 57. A person writes letters to five friends and addresses on the corresponding envelopes. In how many ways can the letters be placed in the envelopes so that (*i*) all the letters are in the wrong envelopes (*ii*) at least two of them are in the wrong envelopes.

Solution. The number of ways in which all the letters be placed in wrong envelopes.

$$\begin{aligned} D_5 &= 5!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}\right) \\ &= 120\left(\frac{1}{2}-\frac{1}{6}+\frac{1}{24}-\frac{1}{120}\right) \\ &= 60-20+5-1 = 44 \end{aligned}$$

(*ii*) The number of ways in which at least two of the letter are in the wrong envelopes

$$\begin{aligned} &= \sum_{r=2}^{5} {}^nC_{n-r}D_r = {}^5C_3D_2 + {}^5C_2D_3 + {}^5C_1D_4 + {}^5C_0D_5 \\ &= 10\times 2!\left[1-\frac{1}{1!}+\frac{1}{2!}\right] + 10\times 3!\left[1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}\right] + 5\times 4!\left[1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\right] + 1\times 44 \\ &= 10 + (30-10) + (60-20+5) + 44 \\ &= 10 + 20 + 45 + 44 = 119 \end{aligned}$$

10.8 Principle of Inclusion-Exclusion

There is an alternative form of the principle of inclusion-exclusion that is useful in counting problems. Let $P_1, P_2, \ldots, P_n$ be n properties. Let $P_{i_1}, P_{i_2}, \ldots, P_{i_k}$ be k properties among these n properties. The number of elements with all the properties $P_{i_1}, P_{i_2}, \ldots, P_{i_k}$ is denoted by

$$N(P_{i_1} P_{i_2} \ldots P_{i_k})$$

Writing these quantities interims of sets, we have

$$|A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}| = N(P_{i_1} P_{i_2} \ldots P_{i_k})$$

If the number of elements in the set is denoted by N, then

$$N(P_1' P_2' \ldots P_n') = N - |A_1 \cup A_2 \cup \ldots \cup A_n|$$

Where $N(P_1' P_2' \ldots P_n')$ denotes the number of elements with none of the properties $P_1, P_2, \ldots, P_n$.

$$\therefore \quad N(P_1' P_2' \ldots P_n') = N - \sum_{1\le i\le n} N(P_i) + \sum_{1\le i<j\le n} N(P_i P_j) - \sum_{1\le i<j<k\le n} N(P_i P_j P_k) + \ldots + (-1)^n N(P_1 P_2 \ldots P_n)$$

Example 58. *Use the principle of inclusion-exclusion, find the number of primes less then 100.*

Solution. We know that a composite integer is divisible by a prime not exceeding its square root. So the composite integers less than 100 must have a prime factor less than $\sqrt{100} = 10$. The only primes less than 10 are 2, 3, 5 and 7. Hence the primes less than 100 are there four members and those positive integers > 1 and < 100 that are divisible by none of 2, 3, 5 or 7.

Let P_i, $i = 1, 2, 3, 4$ be the property that an integer is divisible by 2, 3, 5 and 7 respectively. Thus the number of primes less than 100 is

$$4 + N(P_1' P_2' P_3' P_4')$$

From the principle of inclusion-exclusion, we have

$$\begin{aligned} N(P_1' P_2' P_3' P_4') = {} & 99 - N(P_1) - N(P_2) - P(P_3) - N(P_4) + N(P_1P_2) \\ & + N(P_1P_3) + N(P_1P_4) + N(P_2P_3) + N(P_2P_4) \\ & + N(P_3P_4) - N(P_1P_2P_3) - N(P_1P_2P_4) - N(P_1P_3P_4) \\ & - N(P_2P_3P_4) + N(P_1P_2P_3P_4). \end{aligned}$$

$$N(P_1) = \left\lfloor \frac{100}{2} \right\rfloor = 50,\ N(P_2) = \left\lfloor \frac{100}{3} \right\rfloor = 33,\ N(P_3) = \left\lfloor \frac{100}{5} \right\rfloor = 20,\ N(P_4) = \left\lfloor \frac{100}{7} \right\rfloor$$

$$N(P_1P_2) = \left\lfloor \frac{100}{2\times 3} \right\rfloor = 16,\ N(P_1P_3) = \left\lfloor \frac{100}{2\times 5} \right\rfloor = 10,\ N(P_1P_4) = \left\lfloor \frac{100}{2\times 7} \right\rfloor = 7$$

$$N(P_2P_3) = \left\lfloor \frac{100}{3\times 5} \right\rfloor = 6,\ N(P_2P_4) = \left\lfloor \frac{100}{3\times 7} \right\rfloor = 4,\ N(P_3P_4) = \left\lfloor \frac{100}{35} \right\rfloor = 2$$

$$N(P_1P_2P_3) = \left\lfloor \frac{100}{2\times 3\times 5} \right\rfloor = 3,\ N(P_1P_2P_4) \doteq \left\lfloor \frac{100}{2\times 3\times 7} \right\rfloor = 2,\ N(P_1P_3P_4) = \left\lfloor \frac{100}{2\times 5\times 7} \right\rfloor$$

$$N(P_2P_3P_4) = \left\lfloor \frac{100}{3\times 5\times 7} \right\rfloor = 0,\ N(P_1P_2P_3P_4) = 0.$$

So,

$$\begin{aligned} N(P_1' P_2' P_3' P_4') &= 99 - 50 - 33 - 20 - 14 + 16 + 10 + 7 + 6 + 4 - 2 - 3 \\ &\quad - 2 - 1 + 0 + 0 \\ &= 21 \end{aligned}$$

Hence the number of primes less than 100 is $4 + 21 = 25$

10.9. Pigeonhole Principle

The Pigeon hole principle (also known as the **Dirichlet Drawer Principle or Shoe Box Principle**) is sometime useful in counting methods.

If n pigeons are assigned to m pigeonholes then at least one pigeonhole contains two or more pigeons $(m < n)$.

Proof : Let m pigeons holes be numbered with the numbers 1 though m. Beginning with the pigeon 1, each pigeon is assigned in order to the pigeonholes with the same number. Since $m < n$ i.e. the number of pigeonhole is less than the number of pigeons, n-m pigeons are left without having assigned a pigeon hole. Thus, at least one pigeonhole will be assigned to a more than one pigeon.

We note that the Pigeonhole principle tells us nothing about how to locate the pigeonhole that contains two or more pigeons. It only asserts the existence of a pigeonhole containing two or more pigeons. To apply the principle one has to decide which objects will play the role of pigeon and which objects will play the role of pigeonholes.

Example 59. *In a group of 13 children, there must be least two children who where born in the same month.*

Solution. The thirteen children can be thought of as the pigeon and 12 month of the year as the pigeonhole. Since $12 < 13$, i.e. the number of pigeonholes less than the number of pigeons, by pigeonhole principle we can conclude that out of 13 at least 2 were born in the same month.

Example 60. *Show that if any five integers from 1 to 8 are chosen, then at least two of them will have a sum 9.*

Solution. Let $A = \{1, 2, ... 8\}$. The different sets, each containing two numbers whose sum is equal to 9 are $A1 = \{1, 8\}$, $A2 = \{2, 7\}$, $A3 = \{3, 6\}$, $A4 = \{4, 5\}$. Each of the five numbers chosen from 1 to 8 must belong to one of these sets. The four sets can be thought of as pigeonhole and the chosen numbers as pigeon.

Since $4 < 5$, *i.e.* the number of pigeonholes less than the number of pigeons, by pigeon hole principle we can conclude that two of the selected numbers must belong to the same set whose sum is 9.

Example 61. *Show that in any room of people who have been doing handshaking there will always be at least two people who have shaken hands the same number of times.*

Solution. Let the pigeonholes be labelled with the different numbers of hands shaken and we put the people (pigeons) into their correct pigeonhole. Supple there are n people, then, since people only shake hands with each person at most once, the labels on the pigeonholes will go from 0 to $n - 1$. That is, we have n holes and n people. It is not possible for the 0th and the $(n - 1)$th holes both to be occupied, because if one person has shaken hands with nobody then there cannot be any one person who has shaken hands with every other person. Thus, we have at most $n -$ holes occupied at any one time. Hence, by the pigeonhole principle at least one of the holes has two occupants, which shows that there are at least two people who have shaken hands the same number of times.

The Principle has several generalisation and can be stated in various ways.

Generalized Pigeonhole Principle: In n pigeonholes are occupied by $Kn + 1$ or more pigeons where K is a positive integer, then at least one pigeon hole is occupied by $K + 1$ or more pigeons.

Example 62. A bag contains black, blue and white socks. Find the minimum number of socks that one needs to choose in order to get two pairs of socks of the same colour.

Solution: Here, colours are pigeonholes and hence $n = 3$, and socks are pigeons and $K + 1 = 4 \therefore K = 3$.

Hence $nK + 1 = 3.\ 3 + 1 = 10$ socks (pigeons) are required so that two pairs (four) of them are of the same colour.

Extended Pigeonhole Principle : If n pigeons are assigned to m pigeonholes and $n > m$, then one of the pigeonholes must contain at least $\lfloor (n-1)/m \rfloor + 1$ pigeons, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x, which is a real number.

Proof. Let us assume that none of the pigeonholes contain more than $\lfloor (n-1)/m \rfloor + 1$ pigeons. Then there are at the most $m\lfloor (n-1)/m \rfloor + 1 \leq m(n-1)/m = n - 1$ pigeons. This contradicts our assumption that there are n pigeons. Therefore, one of the pigeonholes must contain at least $\lfloor (n-1)/m \rfloor + 1$ pigeons.

Example 63 (*a*). *If 9 books are to be kept in 4 shelves, there must be at least one shelf which contain at least 3 books.*

Solution. The nine books can be thought of as pigeons and four shelves as the pigeonholes. Then, $n = 9$ and $m = 4$. So, by pigeonhole principle $\lfloor (n-1)/m \rfloor + 1 = \lfloor (9-1)/4 \rfloor + 1 = 3$, *i.e.* at least one shelf will contain at least 3 books.

Example 63 (*b*). *Show that if any 20 people are selected then we may choose a subset of 3 so that all 3 were born on the same day of the week.*

Solution. We may assign each pigeon to the day of the week on which she/he was born. Then 20 number of people (pigeon) are to be assigned to 7 pigeonhole (day of the week). Here $m = 7$, $n = 20$, by pigeonhole principle $\lfloor (20-1)/7 \rfloor + 1 = 3$. So, at least 3 of the persons have been born on the same day of the week.

Example 64. *Find the minimum number of students in a class to be sure that four out of them are born in the same month.*

Solution. We consider each month as a pigeonhole, then $m = 12$ and we have to find the minimum number of students (pigeons) so that four out of them are born in the same month. Take $\lfloor (n-1)/m \rfloor + 1 = 4$ so that $(n-1)/m = 3$ which implies $n = 37$ which is the required minimum number of students.

We next restate the Pigeonhole Principle in an alternative form.

Pigeonhole Principle (Second Form): *If f is a function from a finite set X to a finite set Y and $|X| > |Y|$, then $f(x_1) = f(x_2)$ for some $x_1, x_2 \in X$, $x_1 \neq x_2$*, i.e. there must be at least two elements in the domain that have the same image in the co-domain.

Our next example illustrate the use of the second form of the Pigeonhole Principle.

Example 65. *Show that in any set of eleven integers, there are two whose difference is divisible by 10.*

Solution. Consider the set A of pigeons is the given set of eleven of integers and the set of pigeonholes is the set B = {0, 1, ..., 9} of possible righthand digits. The relevant function $f: A \to B$ takes each integer to its right hand digit and $|A| > |B|$. Hence, by the pigeonhole principle, two of integers have the same right-hand digit, thus their difference is divisible by 10.

Example 66. *A drawer contains ten black and ten white shocks. What is the least number of socks one must pull out to be sure to get a matched pair?*

Solution. Let the socks pulled out (pigeons) be denoted by $S = \{s_1, s_2, \ldots\ldots, s_n\}$ and consider the function f that maps each sock to its colour (pigeon holes) C. If $n = 2$, f could be one to one correspondence (if the two socks pulled out were of different colours). But if $n > 2$ then the number of elements in the domain S of f is larger than the number of elements in the co-domain of C. Thus, by pigeon hole principle, $f(s_i) = f(s_j)$ for some $s_i \neq s_j$. Thus, if at least three socks are pulled out, then at least two of them have the same colour.

10.10. Binomial Theorem

For any real numbers x, y and any integer $n \geq 0$

$$(x+y)^n = \Sigma\, C(n, r)\, x^{n-r} y^r$$

Proof. (Combinatorial version)

Multiplying out the left hand side, we get

$$(x+y)^n = (x+y)(x+y) \ldots\ldots\ldots\ldots\ldots\ldots (x+y) \qquad [n \text{ brackets}]$$

This gives a sum of terms, each of which is obtained by multiplying together one choice of x or y from each bracket. If y is chosen from exactly r brackets then x must be chosen from the remaining $n-r$ brackets, so the resulting term will be $x^{n-r} y^r$. But this can be done in $C(n, r)$ ways, since $C(n, r)$ counts the number of ways of selecting r things from n items. Thus $x^{n-r} y^r$ appears $C(n, r)$ times. It follows that

$$(x+y)^n = C(n, 0)\, x^n y^0 + C(n, 1)\, x^{n-1} y^1 + C(n, 2)\, x^{n-2} y^2 + \ldots\ldots\ldots\ldots + C(n, n)\, x^0 y^n.$$

$$= \sum_{r=0}^{n} C(n, r)\, x^{n-r} y^r.$$

Binomial Coefficients

The quantity $\dfrac{n!}{r!(n-r)!}$ written as $C(n, r)$ or $\dbinom{n}{r}$ is known as Binomial coefficient. The symbol C (n, r) has two meanings (*i*) combinatorial meaning and (*ii*) algebraic meaning. In the first case it represents the number of ways of choosing r objects from n distinct objects and in the second case C $(n, r) = \dfrac{n!}{r!(n-r)!}$.

Hence all theorems and identities about Binomial coefficients and factorials can be given two kinds of proofs a combinatorial proof and an algebraic proof.

Note : When n is a positive integer

(*a*) $(1 + x)^n = \sum_{r=0}^{n} C(n,r)x^r$

(*b*) $(1 + x)^{-n} = \sum_{r=0}^{\infty} (-1)^r C(n+r-1, r)x^r$

(*c*) $(1 - x)^{-n} = \sum_{r=0}^{\infty} C(n+r-1, r)x^r$.

10.11 Combinatorial Identities

An identity that results from some counting process is called a combinatorial identity. Some identities involving binomial coefficients are given below:

1. $C(n, 0) + C(n, 1) + \ldots\ldots\ldots + C(n, n) = 2^n$
2. $C(n, 1) + C(n, 3) + \ldots\ldots\ldots = C(n, 0) + C(n, 2) + \ldots\ldots\ldots = 2^{n-1}$
3. $C(n, r) = C(n, n-r)$
4. **Newton's Identity** $C(n, r). C(r, k) = C(n, k). C(n-k, r-k)$ for integers $n \geq r \geq k \geq 0$
5. **Pascal Identity** $C(n+1, r) = C(n, r) + C(n, r-1)$
6. **Vandermonde's Identity**

$C(n+m, r) = C(n, 0). C(m, r) + C(n, 1). C(m, r-1) + \ldots\ldots\ldots + C(n, r). C(m, 0)$

$= \sum_{k=0}^{r} C(m, r-k). C(n, k)$ for integers $n \geq r \geq 0$ and $m \geq r \geq 0$.

The combinatorial proofs of (3), (4) and (6) are given below and the remaining identities are left as exercises.

$$C(n, r) = C(n, n-r)$$

Proof (combinational version). If r objects are chosen from n objects there are $n-r$ objects are left. Thus selection of r objects from n objects is the same as to pick out the $n-r$ objects that are not to be selected. Hence to every r combination automatically there is an associated $(n-r)$ combination and conversely. This proves the identity.

Pascal Identity

$C(n+1, r) = C(n, r) + C(n, r-1)$ where n and r are positive integers with $r \leq n$.

Proof (combinatorial version). A choice of r of the $n+1$ objects $x_1, x_2, \ldots\ldots\ldots x_n, x_{n+1}$ may or may not include x_{n+1}. If it does not, then r objects have to be chosen from $x_1, x_2, \ldots\ldots x_n$ and there are C (n, r) such choices. If it does contain x_{n+1}, then $r-1$ further objects have to be chosen from $x_1, x_2, \ldots\ldots x_n$, and there are C $(n, r-1)$ such choices. So by the rule of sum, the total number of choices is C (n, r) + C $(n, r-1)$ which must be equal to C $(n+1, r)$.

Hence $C(n+1, r) = C(n, r) + C(n, r-1)$.

i.e. the total number of combinations of $n+1$ things taken r at a time = combinations that contain a particular thing + combinations that do not contain a particular thing.

Pascal's formula gives a recurrence relation for the computation of Binomial coefficient, given initial data C $(n, 0)$ = C $(n, n) = 1$ for all n. Notice that no multiplication is needed for this computation. One can obtain the numbers by constructing a triangular array using very simple arithmetic. The triangular array is usually called **Pascal's triangle.** One can label the rows of the triangular array by $n = 0, 1, 2, \ldots$..and the positions within the nth row as $k = 0, 1, 2, \ldots\ldots n$. The zero row of the triangle be the single entry 1 and the first row be a pair of entries each equal to 1. This gives the first two rows. The nth row of the triangle, which contains $n + 1$ numbers, can be formed from the preceding row by the following rules

(*a*) The first ($k = 0$) and the last ($k = n$) entries are both equal to 1.

(*b*) For $1 \le k \le n - 1$, the kth entry in the nth row is the sum of the $(k - 1)$th and kth entries of the $(n - 1)$ rows.

The first eight rows of Pascal's trinagle are shown in the following diagram.

$$\begin{array}{ccccccccccccccc}
&&&&&&&1&&&&&&&\\
&&&&&&1&&1&&&&&&\\
&&&&&1&&2&&1&&&&&\\
&&&&1&&3&&3&&1&&&&\\
&&&1&&4&&6&&4&&1&&&\\
&&1&&5&&10&&10&&5&&1&&\\
&1&&6&&15&&20&&15&&6&&1&\\
1&&7&&21&&35&&35&&21&&7&&1
\end{array}$$

(*a*)

$$\begin{array}{c}
\binom{0}{0}\\
\binom{1}{0}\ \binom{1}{1}\\
\binom{2}{0}\ \binom{2}{1}\ \binom{2}{2}\\
\binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3}\\
\binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4}\\
\binom{5}{0}\ \binom{5}{1}\ \binom{5}{2}\ \binom{5}{3}\ \binom{5}{4}\ \binom{5}{5}\\
\binom{6}{0}\ \binom{6}{1}\ \binom{6}{2}\ \binom{6}{3}\ \binom{6}{4}\ \binom{6}{5}\ \binom{6}{6}\\
\binom{7}{0}\ \binom{7}{1}\ \binom{7}{2}\ \binom{7}{3}\ \binom{7}{4}\ \binom{7}{5}\ \binom{7}{6}\ \binom{7}{7}
\end{array}$$

(*b*)

Fig. 10.1 Pascal's triangle

A basic property of binomial coefficients is illustrated by Pascal's triangle. If we evaluate the numbers, we can find that we obtain the same numbers as in the first six rows of Pascal's triangle. Each number in the triangle is the sum of the two numbers above it, *i.e.*, the number just above it and to the right, and the number just above it and to the left. Thus with $n = 5$ and $k = 3$, we have

$$\binom{5}{3} = \binom{4}{3} + \binom{4}{2}$$ which is the particular case of Pascal's identity. This shows that one can compute any desired Binomial coefficients using only addition even though formula requires multiplication and division.

Vandermonde's Identity $C(m + n, r) = \sum_{k=0}^{r} C(m, r-k)\, C(n,k)$.

Proof (Combinatorial version). Suppose that there are n items in one set and m items in a second set. Then the total number of ways to pick r elements from the union of these sets is $C(m + n, r)$. Another way to pick r elements from the union is to pick k elements from the first set and then $r - k$ elements from the second set, where k is an integer with $0 \le k \le r$, this can be done in $C(n, k)\, C(m, r - k)$ ways using the product rule. Hence, the total number of ways to pick r elements from the union also equals, $\sum_{k=0}^{r} C(m, r-k)\, C(n, k)$

$$\therefore \quad C(m + n, r) = \sum_{k=0}^{r} C(m, r - k)\, C(n, k).$$

This proves Vandermonde's Identity.

Now we shall see how combinatorial arguments can be used to evaluate the coefficients in the expansion of $(x + y)^n$. We have

$$(x + y)^2 = (x + y) \cdot (x + y) = xx + xy + yx + yy = x^2 + 2xy + y^2 \quad ...(1)$$

$$(x + y)^3 = (x + y)^2 \cdot (x + y) = (x + y)(x + y)(x + y)$$

$$= xxx + xxy + xyx + xyy + yxy + yyx + yyy$$

$$= x^3 + 3x^2y + 3xy^2 + y^3. \quad ...(2)$$

where the terms are ordered in the usual way in descending powers of x, the coefficients are 1, 2 and 1 in (1) and 1, 3, 3 and 1 in (2). These are the numbers in the $n = 2$ and $n = 3$ row of Pascal's triangle. A typical term like x^2y is obtained by multiplying one of the two terms from the first factor times one of the two terms from the second factor times one of the two terms from the third factor (like $x \cdot x \cdot y$). We can get x's in $C(3, 2) = 3$ ways, and each choice for the x's leaves just one way to choose the y. By the product rule, we get 3×1 ways.

Example 67. *What is the coefficient of $x^2 y^4$ in $(x + y)^6$?*

Solution. Here

$$(x + y)^6 = (x + y) \cdot (x + y) \ldots\ldots\ldots (x + y) \text{ [6 factors]}$$

We have to choose an x from two factors and a y from the remaining four factors. The x's can be chosen in $C(6, 2) = 15$ ways and each choice for the x's leaves just one choice for the y's. Hence the coefficient of $x^2 y^4$ is 15.

10.12. Multimonial Coefficients

The expression in the form $x_1 + x_2$ is a binomial, a multimonial is an expression of the form $x_1 + x_2 + \ldots\ldots + x_n$ with $n \ge 3$. Just as binomial coefficients appear in the expansion of powers of a binomial, multimonial coefficients appear when a power of a multimonial is expanded.

Multinomial Theorem. For real numbers $a_1, \ldots, a_k$ and for $n \in \mathbb{N}$, we have

$$(a_1 + \ldots + a_k)^n = \sum_{n_1 + \ldots + n_k = n} \binom{n}{n_1 \ldots n_k} a_1^{n_1} \ldots a_k^{n_k}.$$

Here $\binom{n}{n_1 \ldots n_k}$ stands for $\dfrac{n!}{n_1! \ldots n_k!}$ is called the multimonial coefficient and the sum is over all possible ways to write n as $n_1 + \ldots + n_k$.

This coefficient denotes the number of distinguishable arrangements of n objects, in which there are n_1 objects of type 1, n_2 objects of type 2, ...n_k objects of type k.

Example 68. *Find the number of arrangement of the letters in the word ACCOUNTANT.*

Solution. Total number of letters in the word ACCOUNTANT is 10. Out of which A occurs twice, C occurs twice, N occurs twice, T occurs twice and the rest are all different. Since some of letters are repeated we need to apply multimonial theorem. Hence the number of arrangements is

$$\frac{10!}{2!\,2!\,2!\,2!} = 226800$$

Example 69. *What is the coefficient of $x^3\, y^2\, z^2$ in $(x + y + z)^9$?*

Solution. This is the same as how many ways one can choose x from three brackets, a y from two brackets and a z from two brackets in the expansion.

$(x + y + z)\,(x + y + z)$.............$(x + y + z)$ [9 factors]

This can be done in $\begin{pmatrix} & 9 & \\ 3 & 2 & 2 \end{pmatrix} = \dfrac{9!}{3!\,2!\,2!} = 15120.$

Example 70. *If $x > 2$, $y > 0$, $z > 0$ then find the number of solution of $x + y + z + \omega = 21$.*

Solution.

$$\begin{aligned} x &> 2 \text{ is equivalent to } x \geq 3 \\ y &> 0 \text{ is equivalent to } y \geq 1 \\ z &> 0 \text{ is equivalent to } z \geq 1 \\ \omega &\geq 0. \end{aligned}$$

and

Put $p = x - 3$, $q = y - 1$, $r = z - 1$, $s = \omega$. The given equation becomes $p + q + r + s = 16$ for which the number of non-negative integer solution is required. Thus we can model this problem as distribution 16 identical balls with repetition among 4 distinct children.

Hence the no. of solution $= C(16 + 4 - 1, 16) = C(19, 16) = C(19, 3) = 969$.

Note: We can apply note (*iii*) of Theorem 10.6 with $r = 21$, $n = 4$, $r_1 = 3$, $r_2 = 1$, $r_3 = 1$, $r_4 =$

Hence the number of solutions is $C(21 + 4 - 1 - 3 - 1 - 1 - 0, 4 - 1) = C(19, 3) = 969$.

Example 71. *Find the number of non-negative integer solution of $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 < 10$?*

Solution. We convert the inequality into an equality by introducing an auxiliary variable $x_7 > 0$.

Then, the given inequality reduces to

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 10 \qquad ...(1)$$

where $x_i > 0$, $i = 1, 2, ..., 6$ and $x_7 > 0$ or $x_7 \geq 1$

Taking $x_i = y_i$, $i = 1, 2, ..., 6$ and $x_7 - 1 = y_7$, then the equation (1) becomes $y_1 + y_2 + ... + y_7 = 10 - 1 = 9$, $y_i \geq 0$ for $1 \leq i \leq 7$

So, the required number of solutions

$= C\,(7 + 9 - 1, 9) = C\,(15, 9) = 5\,0\,0\,5.$

Example 72. *Determine the number of ways possible to wear 5 rings on 4 fingers.*

Solution. Case 1. When all rings are identical and assuming all five rings are worn.

We look at this as the no. of ways to sum 4 non-negative integers is equal to 5

$$\underbrace{f_1 + f_2 + f_3 + f_4}_{n = 4 \text{ fingers}} = \underbrace{5}_{5 \text{ rings}}$$

We can model this problem as the total number of ways of distributing r (= 5) similar items into n (= 4) numbered boxes with repetition. This gives us

$$C(n + r - 1, r) = C(4 + 5 - 1, 5) = C(8, 5) = \frac{8!}{5!\,3!} = 56 \text{ combinations.}$$

Case 2. When all rings are distinct and assuming all five rings are worn.

If all the rings are distinct, then 5 rings can be rearranged in 5! ways. Hence the required number is $5! \times C(8, 5) = 5! \times 56 = 6720$.

Example 73. *Determine the number of integers between 1 and 10,000,000 have the sum of digits equal to 18.*

Solution. Any integer lying between 1 and 10,000,000 will have a maximum of 7 digits. If x_i, $1 \le i \le 7$ denote the digits, then the given problem reduces to find the no. of solutions of the equation

$$x_1 + x_2 + ... + x_7 = 18 \quad \text{where} \quad 0 \le x_i \le 9 \quad (1)$$

If $x_i \ge 0$, the no. of solution is $C(7 + 18 - 1, 18) = C(24, 18) = C(24, 6) = 134596$

As the sum of the digits is 18, then one of the six x_i's can be ≥ 10 but not more than one. If $x_1 \ge 10$ and $u_1 = x_1 - 10$, $u_i = x_i$, $z \le i \le 7$. Then the equation (1) becomes

$$u_1 + u_2 + ... + u_7 = 8 \quad \text{where} \quad u_i \ge 0 \quad (2)$$

The number of solution of (z) is $C(7 + 8 - 1, 8) = C(14, 8) = C(14, 6) = 3003$.

There are 7 ways to choose the digit which is ≥ 10.

Hence, the no. of solutions of the equation $x_1 + x_2 + ... + x_7 = 18$, where any $x_i \ge 10$ is $7 \times C(14, 6) = 7 \times 3003 = 21021$

Hence the required member of solution of (1)

$$= 134596 - 21021 = 113575.$$

Example 74. *Use Binomial theorem to prove that*

(i) $3^n = \sum_{r=0}^{n} C(n, r)\, 2^n$

(ii) $C(n, 1) + 2C(n, 2) + ... + nC(n, n) = n \,.\, 2^{n-1}$

Solution. (*i*) From Binomial theorem, we have

$$(x + y)^n = C(n, 0)\, x^n + C(n, 1)\, x^{n-1} y + C(n, 2)\, x^{n-2} y + ... + C(n, n)\, y^n \quad (1)$$

for real x, y and any integer $n \ge 0$

Put $x = 1$, $y = x$

$$(1 + x)^n = C(n, 0) + C(n, 1)\, x + C(n, 2)x^2 + ... + C(n, n)\, x^n \quad (2)$$

Put $x = 2$ in (2)

$$(1 + 2)^n = C(n, 0) + C(n, 1)2 + 2C(n, 2)2^2 + ... + 2C(n, n)2^n$$

$$3^n = \sum_{r=0} C(n, r)\, 2^r. \text{ Hence proved}$$

(*ii*) Differentiating left and right hand side of (2) w.r.t. x

$$n\,(1 + x)^{n-1} = 0 + C(n, 1) + C(n, 2)\,.\,2x + C(n, 3)\,.\,3x^2 + ... + C(n, n)n \quad (3)$$

Put $x = 1$ m (3)

$$n\,(1 + 1)^{n-1} = C(n, 1) + C(n, z)\,.\,z + C(n, 3)\,.\,3 + ... + (n, n)\, n$$

$$n\,.\,2^{n-1} = C(n, 1) + 2C(n, 2) + 3C(n, 3) + ... + nC(n, n)$$

Hence proved.

Example 75. *Use Pascal's identity, prove that*

$$C(r, r) + C(r + 1, r) + C(r + 2, r) + ... + C(n, r) = C(n + 1, r + 1)$$

Solution. Pascal's identity is

$$C(n, r - 1) + C\,(n, r) = C(n + 1, r)$$

we change n to i and r to $r+1$ in Pascal's identity

$$C(i, r) + C(i, r+1) = C(i+1, r+1)$$
$$C(i, r) = C(i+1, r+1) - C(i, r+1) \quad (1)$$

Putting $i = r, r+1, ..., n$ in (1) and adding we get

$$C(r, r) + C(r+1, r) + ... + C(n, r) = C(n+1, r+1) - C(n, r+1)$$

i.e.
$$\sum_{i=r}^{n} C(i, r) = C(n+1, r+1) \quad [\because C(n, r+1) = 0]$$

Example 76. *In how many ways can a lawn tennis mixed double set be made up from 8 married couples if no husband and wife play in the same set?*

Solution. Two husbands, say H_1 and H_2, can be selected out of 8 husbands in $C(8, 2)$ ways. Excluding their wives, two ladies, say L_1 and L_2 out of remaining 6 wives can be selected in $C(6, 2)$ ways. Thus the number of ways the mixed double set can be selected is $C(8, 2) \times C(6, 2)$

Now H_1 can choose L_1 as partner (as a consequence H_2 will go with L_2) or H_1 can choose L_2 as partner (consequently H_2 will get L_1 as partner). Thus we have 2 choices for every set of the total number of sets.

Hence the required number of ways the mixed double sets can be made up from 8 married couples

$$= C(8, 2) \times C(6, 2) \times 2 = \frac{8 \times 7}{2} \times \frac{6 \times 5}{2} \times 2 = 840.$$

Example 77. *Determine the co-efficient of $x^5 y^{10} z^5 \omega^5$ in $(x - 7y + 3z - \omega)^{25}$.*

Solution. The co-efficient is same as how many ways one can choose x from 5 brackets, $(-7y)$ from ten brackets, $3z$ from five brackets and $(-\omega)$ from 5 brackets in the expansion

$$(x - 7y + 3z - \omega)\,(x - 7y + 3z - \omega) \,.....\, (x - 7y + 3z - \omega) \quad [25 \text{ factors}]$$

This can be done in $\dfrac{25!\,(1)^5\,(-7)^{10}\,3^5\,(-1)^5}{5!\;10!\;5!\;5!} = \dfrac{25!\,7^{10}\,.\,3^5}{5!\;10!\;5!\;5!}$

Example 78. *There are 50 people in a room some of them are acquainted with each other and some not. Prove that there are two persons in the room who have equal number of acquaintances.*

Solution. As given, there are persons who have either 1 or 2 or 3 or ..... or 48 or do not have any acquaintance at all. Thus there $m = 49$ pigeon holes which are to be filled up with $n = 50$ pigeons (persons). Hence from extended pigeonhole principle, the required number is

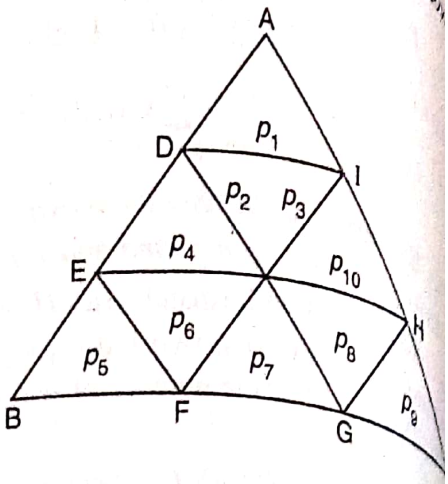
$$\left\lfloor \frac{n-1}{m} \right\rfloor + 1 = \left\lfloor \frac{50-1}{49} \right\rfloor + 1 = 2.$$

Example 79. *Assume that in a group of six people, each pair of individuals consists of two friends or two enemies. Show that there are either three mutual friends or three mutual enemies in the group.*

Solution. Let A be one of the six people. Then the remaining five people are to be divided into two sets labeled "A's friends" and "A's enemies" *i.e.* $m = 2$ sets (pigeonholes) are to be filled up with $n = 5$ people (pigeons). Hence from extended pigeonhole principle, one of the sets has at least $\left\lfloor \frac{5-1}{2} \right\rfloor + 1 = 3$ elements. Suppose B, C and D are friends of A. If any two of these three individuals are friends, then these two and A form a group of three mutual friends. Otherwise B, C and D form a set of three mutual enemies. In either case, we get the required conclusion. The same conclusion is drawn if the room labeled "enemies of A" contain three people.

Example 80. *If we select 10 points in the interior of an equilateral triangle of side 1, show that there must be at least two points whose distance apart is less than 1/3.*

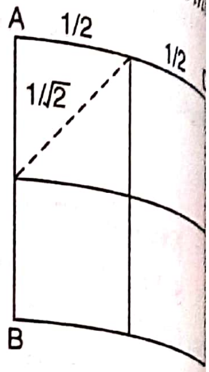
Solution. Let us divide the given equilateral triangle ABC (say) into 9 equal triangles as shown in the figure. The pair of points D, E; F, G and H, I are the points of trisection of the sides AB, BC and AC respectively.



The given 10 points will be placed in these 9 sub triangles. The 9 sub triangles may be regarded as 9 pigeonholes and 10 interior points may be regarded as 10 pigeons. The by the pigeonhole principle, at least two of the interior points belong to one of the nine sub-triangles. Clearly the distance between any two interior points can not exceed the length of the side of the sub triangle which is 1/3.

Example 81. *Given any five points inside a square of side 1, show that there exists two points within a distance of at most $1/\sqrt{2}$.*

Solution. Let us divide the given square into four equal sub squares as shown in the figure.



The given four points will be placed in these four sub-squares. The four sub-squares may be regarded as four pigeonholes and five interior points as pigeons. Then by pigeonhole principle, at least two of the interior points must belong to one of the four squares. The distance between them can not exceed the length of the diagonal of the sub-squares which is $1/\sqrt{2}$.

Hence we conclude that there exists two points withing a distance of at most $1/\sqrt{2}$.

Example 82. *Find the minimum number of students in MCA first semester to be sure that at least six will receive the same grade, if there are five possible grades A, B, C, D and F.*

Solution. Let us consider each grade as pigeonhole so that there are $m = 5$ pigeonholes. Let n be the minimum number of students (pigeons) so that at least 8 of them will receive the same grade. From extended pigeonhole principle,

$$\left\lfloor \frac{n-1}{m} \right\rfloor + 1 = 6 \quad \text{or} \quad \left\lfloor \frac{n-1}{m} \right\rfloor = 5$$

$$\Rightarrow \qquad n - 1 = 5\text{ m} \Rightarrow n = 5 \times 5 + 1 \quad \therefore \quad n = 26$$

Hence the required minimum number of students is 26.

Example 83. *Four dice are thrown simultaneously. In how many ways a total of 14 can be obtained?*

Solution. Let x, y, z, w be the points shown by the four dice respectively. When a die is thrown, one of the six points 1, 2, 3, 4, 5, 6 occur, hence $1 \leq x, y, z, w, \leq 6$. Then the given problem reduces to the problem of finding the non-negative integer solutions of the equation

$$x + y + w + z = 14 \quad \text{subject to the constraints} \quad 1 \leq x, y, z, w \leq 6$$

The required number of ways

= co-efficient of t^{14} in the expansion ...

$= \text{co-efficient of } t^{10} \text{ in the expansion of } \dfrac{(1-t^6)^4}{(1-t)^{-4}}$ (assuming $|t| < 1$)

$= \text{co-efficient of } t^{10} \text{ in the expansion of } (1 - 4t^6 + 6t^{12} - 4t^{18} + t^{24}) \times \Sigma\, C(4 + r - 1, r)\, t^r$

$= C(4 + 10 - 1, 10) - 4 \times C(4 + 4 - 1, 4)$

$= C(13, 10) - 4 \times C(7, 4) = C(13, 3) - 4 \times C(7, 3)$

$= \dfrac{13 \times 12 \times 11}{3 \times 2 \times 1} - 4 \times \dfrac{7 \times 6 \times 5}{3 \times 2 \times 1} = 146$

COMBINATRONICS PROBLEM SET

Problem Set 10.1

1. Evaluate the following
   (i) ${}^5P_3$ (ii) $P(5, 5)$ (iii) ${}^9P_5$ (iv) $P(10, 4)$
2. Find n, if
   (i) ${}^nP_2 = 30$ (ii) $P(n, 4) = 20 \times P(n, 2)$
   (iii) $P(n, 6) = 3P(n, 5)$ (iv) ${}^nP_4 : {}^{n-1}P_5 = 1 : 9$
   (v) ${}^{2n+1}P_{n-1} : {}^{2n-1}P_n = 3 : 5.$ (vi) $P(11, n) = P(12, n-1)$, find n.
3. Show that
   (i) $P(n, n) = 2P(n, n-2)$ (ii) ${}^{n+1}P_{r+1} = (n+1).{}^nP_r$
   (iii) ${}^nP_r = {}^{n-1}P_r + r.{}^{n-1}P_{r-1}$
4. An organizing committee requires one student representative from either the first year or the second year or the third year. If there are 60 first year, 40 second year and 20 third year students, how many different representatives are there ?
5. If E be the event of selecting a prime number less than 10 and F, the event of selecting an even number less than 10. Find the number of ways of happening E or F.
6. A house has 4 doors and 10 windows. In how many ways can a burglar rob the house by entering through a window and exiting through a door.
7. Twelve horses are in a race. The only results that matter are the first three finishers. How many possibilities are there ?
8. A student wishes to take a combination of three courses, one from each of three science departments. There are 4 Physics, 3 Chemistry and 2 Biology courses on offer. How many possible combinations ar[e] there ?
9. A person buying a personal computer system is offered a choice of four models of the basic unit, thr[ee] models of key board, and two models of printer. How many distinct system can be purchased ?
10. There are 6 multiple choice questions in an examination. How many sequences of answers are possi[ble] if the first three question have four choices each and the next three have five each ?

... among 6 contestants so that a contestant