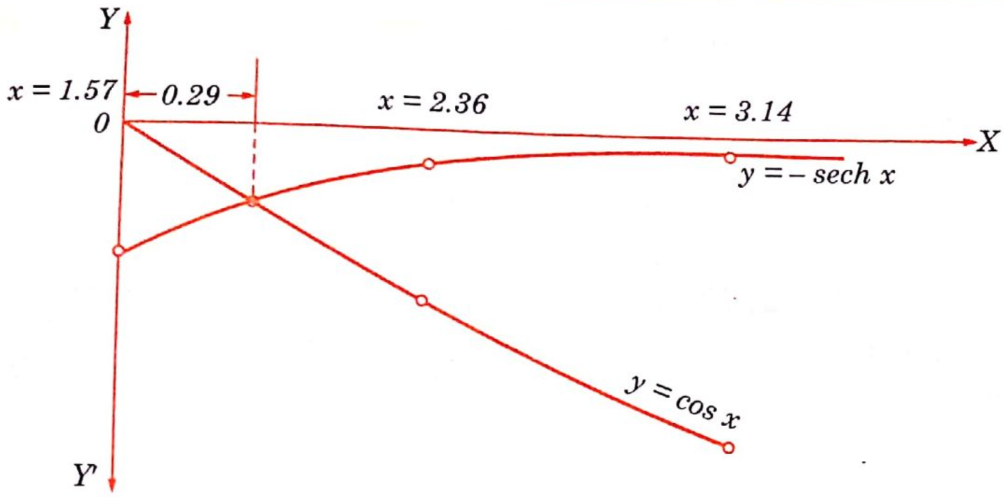

Fig. 2.4

To draw (iii), we form the following table :

x	1.57	2.36	3.14
$\cosh x$	2.51	5.34	11.57
$y = -\operatorname{sech} x$	– 0.4	– 0.19	– 0.09

Then we plot the curve (*iii*) to the same scale with the same axes.

From the above figure, we get the lowest root to be approximately $x = 1.57 + 0.29 = 1.86$.

PROBLEMS 2.3

Find the approximate value of the root of the following equations graphically (1—4) :

1. $x^3 - x - 1 = 0$ (*Madras B.E., 2000 S*) **2.** $x^3 - 6x^2 + 9x - 3 = 0$

3. $\tan x = 1.2\, x$ **4.** $x = 3 \cos (x - \pi/4)$.

2.7. RATE OF CONVERGENCE

Let $x_0, x_1, x_2, \ldots\ldots$ be the values of a root (α) of an equation at the 0th, 1st, 2nd ...... iterations while its actual value is 3.5567. The values of this root calculated by three different methods, are as given below :

Root	*1st method*	*2nd method*	*3rd method*
x_0	5	5	5
x_1	5.6	3.8527	3.8327
x_2	6.4	3.5693	3.56834
x_3	8.3	3.55798	3.55743
x_4	9.7	3.55687	3.55672
x_5	10.6	3.55676	
x_6	11.9	3.55671	

The values in the 1st method do not converge towards the root 3.5567. In the 2nd and 3rd methods, the values converge to the root after 6th and 4th iterations respectively. Clearly

3rd method converges faster than the 2nd method. This *fastness of convergence in any method is represented by its* **rate of convergence.**

If e be the error then $e_i = \alpha - x_i = x_{i+1} - x_i$.

If e_{i+1}/e_i *is almost constant, convergence is said to be* **linear** *i.e. slow.*

If e_{i+1}/e_i^p *is nearly constant, convergence is said to be of order* p *i.e. faster.*

2.8. (1) BISECTION METHOD

This method is based on the repeated application of the *intermediate value property*. Let the function $f(x)$ be continuous between a and b. For definiteness, let $f(a)$ be negative and $f(b)$ be positive. Then the first approximation to the root is $x_1 = \frac{1}{2}(a + b)$.

If $f(x_1) = 0$, then x_1 is a root of $f(x) = 0$. Otherwise, the root lies between a and x_1 or x_1 and b according as $f(x_1)$ is positive or negative. Then we bisect the interval as before and continue the process until the root is found to desired accuracy.

In the Fig. 2.4, $f(x_1)$ is + ve, so that the root lies between a and x_1. Then the second approximation to the root is $x_2 = \frac{1}{2}(a + x_1)$. If $f(x_2)$ is – ve, the root lies between x_1 and x_2. Then the third approximation to the root is $x_3 = \frac{1}{2}(x_1 + x_2)$ and so on.

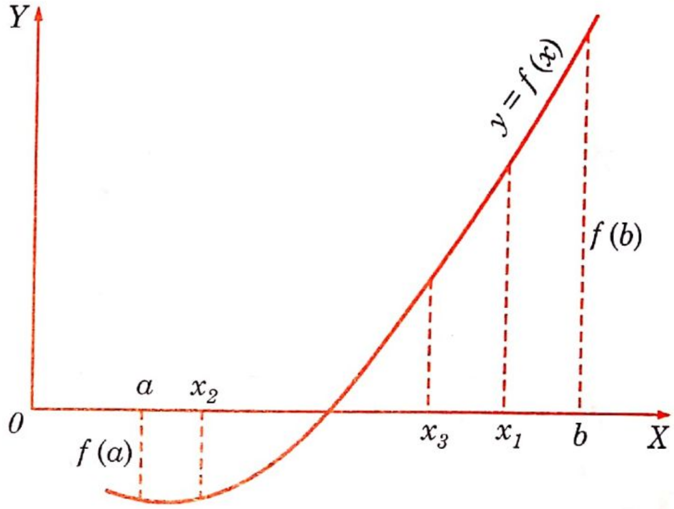

Fig. 2.5

Obs. 1. Since the new interval containing the root, is exactly half the length of the previous one, the interval width is reduced by a factor of $\frac{1}{2}$ at each step. At the end of the nth step, the new interval will therefore, be of length $(b - a)/2^n$. If on repeating this process n times, the latest interval is as small as given ε, then $(b - a)/2^n \le \varepsilon$

or $$n \ge [\log (b - a) - \log \varepsilon]/\log 2$$

This gives the number of iterations required for achieving an accuracy ε.

In particular, the minimum number of iterations required for converging to a root in the interval (0, 1) for a given ε are as under :

ε :	10^{-2}	10^{-3}	10^{-4}
n :	7	10	14

(2) Rate of Convergence. As the error decreases with each step by a factor of $\frac{1}{2}$, (*i.e.* $e_{n+1}/e_n = \frac{1}{2}$), *the convergence in the bisection method is* **linear.**

Example 2.15. *(a) Find a root of the equation $x^3 - 4x - 9 = 0$, using the bisection method correct to three decimal places.* (*Madras, B. Tech., 2003*)

(b) Using bisection method, find the negative root of the equation $x^3 - 4x + 9 = 0$. (*J.N.T.U., B. Tech., 2009*)

Sol. (*a*) Let $$f(x) = x^3 - 4x - 9$$

Since $f(2)$ is – ve and $f(3)$ is + ve, a root lies between 2 and 3.

$\therefore$ First approximation to the root is

$$x_1 = \frac{1}{2}(2 + 3) = 2.5.$$

Thus $\quad f(x_1) = (2.5)^3 - 4(2.5) - 9 = -3.375$ *i.e.* –ve.

$\therefore$ The root lies between x_1 and 3. Thus the second approximation to the root is

$$x_2 = \frac{1}{2}(x_1 + 3) = 2.75.$$

Then $\quad f(x_2) = (2.75)^3 - 4(2.75) - 9 = 0.7969$ *i.e.* +ve.

$\therefore$ The root lies between x_1 and x_2. Thus the third approximation to the root is

$$x_3 = \frac{1}{2}(x_1 + x_2) = 2.625.$$

Then $\quad f(x_3) = (2.625)^3 - 4(2.625) - 9 = -1.4121$ *i.e.* –ve.

The root lies between x_2 and x_3. Thus the fourth approximation to the root is

$$x_4 = \frac{1}{2}(x_2 + x_3) = 2.6875.$$

Repeating this process, the successive approximations are

$$x_5 = 2.71875, \quad x_6 = 2.70313, \quad x_7 = 2.71094$$
$$x_8 = 2.70703, \quad x_9 = 2.70508, \quad x_{10} = 2.70605$$
$$x_{11} = 2.70654, \quad x_{12} = 2.70642$$

Hence the root is 2.7064.

(*b*) If α, β, γ are the roots of the given equation, then $-\alpha, -\beta, -\gamma$ are the roots of $(-x)^3 - 4(-x) + 9 = 0$ $\therefore$ The negative root of the given equation is the positive root of $x^3 - 4x - 9 = 0$ which we have found above to be 2.7064.

Hence the negative root the given equation – 2.7064.

Example 2.16. *Using the bisection method, find an approximate root of the equation $\sin x = 1/x$, that lies between $x = 1$ and $x = 1.5$ (measured in radians). Carry out computations upto the 7th stage.* *(V.T.U., B.E., 2003 S)*

Sol. Let $f(x) = x \sin x - 1$. We know that $1^r = 57.3°$.

Since $\quad f(1) = 1 \times \sin(1) - 1 = \sin(57.3°) - 1 = -0.15849$

and $\quad f(1.5) = 1.5 \times \sin(1.5)^r - 1 = 1.5 \times \sin(85.95)° - 1 = 0.49625$;

a root lies between 1 and 1.5.

$\therefore$ First approximation to the root is $x_1 = \frac{1}{2}(1 + 1.5) = 1.25.$

Then $f(x_1) = (1.25)\sin(1.25) - 1 = 1.25 \sin(71.625°) - 1 = 0.18627$ and $f(1) < 0$.

$\therefore$ A root lies between 1 and $x_1 = 1.25$.

Thus the second approximation to the root is $x_2 = \frac{1}{2}(1 + 1.25) = 1.125.$

Then $f(x_2) = 1.125 \sin(1.125) - 1 = 1.125 \sin(64.46)° - 1 = 0.01509$ and $f(1) < 0$.

$\therefore$ A root lies between 1 and $x_2 = 1.125$.

Thus the third approximation to the root is $x_3 = \frac{1}{2}(1 + 1.125) = 1.0625.$

Then $f(x_3) = 1.0625 \sin(1.0625) - 1 = 1.0625 \sin(60.88) - 1 = -0.0718 < 0$ and $f(x_2) > 0$, *i.e.* now the root lies between $x_3 = 1.0625$ and $x_2 = 1.125$.

$\therefore$ Fourth approximation to the root is $x_4 = \frac{1}{2}(1.0625 + 1.125) = 1.09375$

Then $f(x_4) = -0.02836 < 0$ and $f(x_2) > 0$,

i.e. The root lies between $x_4 = 1.09375$ and $x_2 = 1.125$.

$\therefore$ Fifth approximation to the root is $x_5 = \frac{1}{2}(1.09375 + 1.125) = 1.10937$

Then $f(x_5) = -0.00664 < 0$ and $f(x_2) > 0$.

$\therefore$ The root lies between $x_5 = 1.10937$ and $x_2 = 1.125$.

Thus the sixth approximation to the root is

$$x_6 = \frac{1}{2}(1.10937 + 1.125) = 1.11719$$

Then $f(x_6) = 0.00421 > 0$. But $f(x_5) < 0$.

$\therefore$ The root lies between $x_5 = 1.10937$ and $x_6 = 1.11719$.

Thus the seventh approximation to the root is $x_7 = \frac{1}{2}(1.10937 + 1.11719) = 1.11328$

Hence the desired approximation to the root is 1.11328.

Example 2.17. *Find the root of the equation $\cos x = xe^x$ using the bisection method correct to four decimal places.* *(Mumbai, B. Tech., 2004)*

Sol. Let $f(x) = \cos x - xe^x$.

Since $f(0) = 1$ and $f(1) = -2.18$, so a root lies between 0 and 1.

$\therefore$ First approximation to the root is $x_1 = \frac{1}{2}(0 + 1) = 0.5$

Now $f(x_1) = 0.05$ and $f(1) = -2.18$, therefore the root lies between 1 and $x_1 = 0.5$.

$\therefore$ Second approximation to the root is $x_2 = \frac{1}{2}(0.5 + 1) = 0.75$

Now $f(x_2) = -0.86$ and $f(0.5) = 0.05$, therefore the root lies between 0.5 and 0.75.

$\therefore$ Third approximation to the root is $x_3 = \frac{1}{2}(0.5 + 0.75) = 0.625$

Now $f(x_3) = -0.36$ and $f(0.5) = 0.05$, therefore the root lies between 0.5 and 0.625.

$\therefore$ Fourth approximation to the root is $x_4 = \frac{1}{2}(0.5 + 0.625) = 0.5625$

Now $f(x_4) = -0.14$ and $0.5 = 0.05$, therefore the root lies between 0.5 and 0.5625

$\therefore$ Fifth approximation is $x_5 = \frac{1}{2}(0.5 + 0.5625) = 0.5312$

Now $f(x_5) = -0.04$ and $f(0.5) = 0.05$, therefore the root lies between 0.5 and 0.5312.

$\therefore$ Sixth approximation is $x_6 = \frac{1}{2}(0.5 + 0.5312) = 0.5156$

Hence the desired approximation to the root is 0.5156.

Example 2.18. *Find a positive real root of $x \log_{10} x = 1.2$ using the bisection method.*

Sol. Let $f(x) = x \log_{10} x - 1.2$.

Since $f(2) = -0.598$ and $f(3) = 0.231$, so a root lies between 2 and 3.

$\therefore$ First approximation to the root is $x_1 = \frac{1}{2}(2 + 3) = 2.5$.

Now $f(2.5) = -0.205$ and $f(3) = 0.231$, therefore a root lies between 2.5 and 3.

$\therefore$ Second approximation to the root is $x_2 = \frac{1}{2}(2.5 + 3) = 2.75$.

Now $f(2.75) = 0.008$ and $f(2.5) = -0.205$, therefore, a root lies between 2.5 and 2.75.

$\therefore$ Third approximation to the root is $x_3 = \frac{1}{2}(2.5 + 2.75) = 2.625$

Now $f(2.625) = -0.1$ and $f(2.75) = 0.008$, therefore a root lies between 2.625 and 2.75.

$\therefore$ Fourth approximation to the root is $x_4 = \frac{1}{2}(2.625 + 2.75) = 2.687$

Hence the desired root is 2.687.

2.9. (1) METHOD OF FALSE POSITION *or* REGULA-FALSI METHOD *or* INTERPOLATION METHOD

This is the oldest method of finding the real root of an equation $f(x) = 0$ and closely resembles the bisection method.

Here we choose two points x_0 and x_1 such that $f(x_0)$ and $f(x_1)$ are of opposite signs *i.e.* the graph of $y = f(x)$ crosses the x-axis between these points (Fig. 2.6). This indicates that a root lies between x_0 and x_1 and consequently $f(x_0) f(x_1) < 0$.

Equation of the chord joining the points $A[x_0, f(x_0)]$ and $B[x_1, f(x_1)]$ is

$$y - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

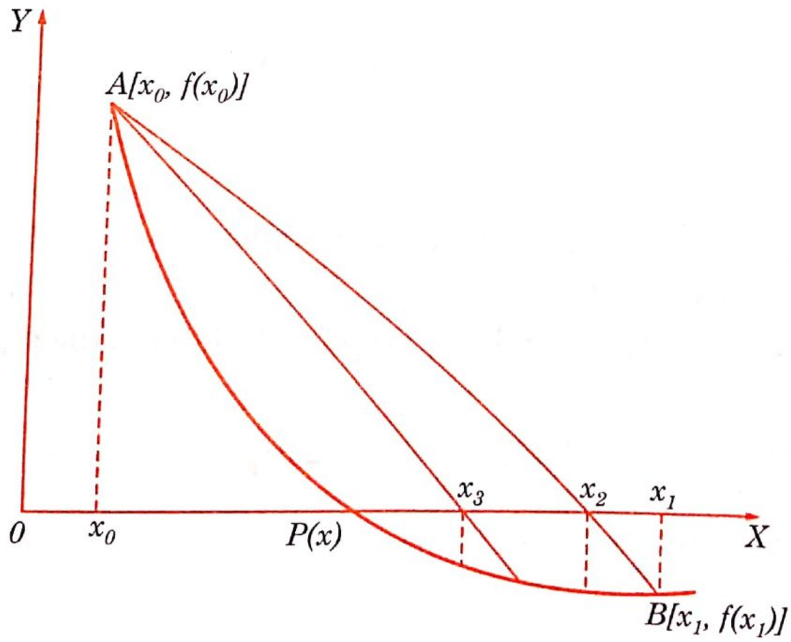

Fig. 2.6

The method consists in replacing the curve AB by means of the chord AB and taking the point of intersection of the chord with the x-axis as an approximation to the root. So the abscissa of the point where the chord cuts the x-axis ($y = 0$) is given by

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \quad \text{...(1)}$$

which is an approximation to the root.

If now $f(x_0)$ and $f(x_2)$ are of opposite signs, then the root lies between x_0 and x_2. So replacing x_1 by x_2 in (1), we obtain the next approximation x_3. (The root could as well lie between x_1 and x_2 and we would obtain x_3 accordingly). This procedure is repeated till the root is found to the desired accuracy. The iteration process based on (1) is known as the *method of false position.*

(2) Rate of Convergence. This method has linear rate of convergence which is faster than that of the bisection method.

Example 2.19. *Find a real root of the equation $x^3 - 2x - 5 = 0$ by the method of false position correct to three decimal places.* *(Manipal, B.E., 2005)*

Sol. Let $f(x) = x^3 - 2x - 5$

so that $f(2) = -1$ and $f(3) = 16$,

i.e. A root lies between 2 and 3.

$\therefore$ Taking $x_0 = 2, x_1 = 3, f(x_0) = -1, f(x_1) = 16$, in the method of false position, we get

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 2 + \frac{1}{17} = 2.0588 \quad \text{...}(i)$$

Now $f(x_2) = f(2.0588) = -0.3908$

i.e., the root lies between 2.0588 and 3.

$\therefore$ Taking $x_0 = 2.0588, x_1 = 3, f(x_0) = -0.3908, f(x_1) = 16$, in (i), we get

$$x_3 = 2.0588 - \frac{0.9412}{16.3908}(-0.3908) = 2.0813$$

Repeating this process, the successive approximations are

$$x_4 = 2.0862, \quad x_5 = 2.0915, x_6 = 2.0934,$$
$$x_7 = 2.0941, \quad x_8 = 2.0943 \text{ etc.}$$

Hence the root is 2.094 correct to 3 decimal places.

Example 2.20. *Find the root of the equation $\cos x = xe^x$ using the regula-falsi method correct to four decimal places.* *(Delhi B. Tech., 2013)*

Sol. Let $f(x) = \cos x - xe^x = 0$

so that $f(0) = 1, f(1) = \cos 1 - e = -2.17798$

i.e., the root lies between 0 and 1.

$\therefore$ Taking $x_0 = 0, x_1 = 1, f(x_0) = 1$ and $f(x_1) = -2.17798$ in the regula-falsi method, we get

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 0 + \frac{1}{3.17798} \times 1 = 0.31467 \quad \text{...}(i)$$

Now $f(0.31467) = 0.51987$ *i.e.,* the root lies between 0.31467 and 1.

$\therefore$ Taking $x_0 = 0.31467$, $x_1 = 1$, $f(x_0) = 0.51987$, $f(x_1) = -2.17798$ in (*i*), we get

$$x_3 = 0.31467 + \frac{0.68533}{2.69785} \times 0.51987 = 0.44673$$

Now $f(0.44673) = 0.20356$ *i.e.*, the root lies between 0.44673 and 1.

$\therefore$ Taking $x_0 = 0.44673$, $x_1 = 1$, $f(x_0) = 0.20356$, $f(x_1) = -2.17798$ in (*i*), we get

$$x_4 = 0.44673 + \frac{0.55327}{2.38154} \times 0.20356 = 0.49402$$

Repeating this process, the successive approximations are

$$x_5 = 0.50995,\ x_6 = 0.51520,\ x_7 = 0.51692$$
$$x_8 = 0.51748,\ x_9 = 0.51767,\ x_{10} = 0.51775 \text{ etc.}$$

Hence the root is 0.5177 correct to 4 decimal places.

Example 2.21. *Find a real root of the equation $x \log_{10} x = 1.2$ by regula-falsi method correct to four decimal places.* *(V.T.U., B. Tech., 2010)*

Sol. Let $f(x) = x \log_{10} x - 1.2$

so that $f(1) = -$ ve, $f(2) = -$ ve and $f(3) = +$ ve.

$\therefore$ A root lies between 2 and 3.

Taking $x_0 = 2$ and $x_1 = 3$, $f(x_0) = -0.59794$ and $f(x_1) = 0.23136$, in the method of false position, we get

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 2.72102 \quad \ldots(i)$$

Now $f(x_2) = f(2.72102) = -0.01709$

i.e. the root lies between 2.72102 and 3.

$\therefore$ Taking $x_0 = 2.72102$, $x_1 = 3$, $f(x_0) = -0.01709$ and $f(x_1) = 0.23136$ in (*i*), we get

$$x_3 = 2.72102 + \frac{0.27898}{0.23136 + 0.01709} \times 0.01709 = 2.74021$$

Repeating this process, the successive approximations are

$$x_4 = 2.74024,\ x_5 = 2.74063 \text{ etc.}$$

Hence the root is 2.7406 correct to 4 decimal places.

Example 2.22. *Use the method of false position, to find the fourth root of 32 correct to three decimal places.*

Sol. Let $x = (32)^{1/4}$ so that $x^4 - 32 = 0$

Take $f(x) = x^4 - 32$. Then $f(2) = -16$ and $f(3) = 49$, *i.e.* a root lies between 2 and 3.

$\therefore$ Taking $x_0 = 2$, $x_1 = 3$, $f(x_0) = -16$, $f(x_1) = 49$ in the method of false position, we get

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 2 + \frac{16}{65} = 2.2462 \quad \ldots(i)$$

Now $f(x_2) = f(2.2462) = -6.5438$ *i.e.* the root lies between 2.2462 and 3.

$\therefore$ Taking $x_0 = 2.2462$, $x_1 = 3$, $f(x_0) = -6.5438$, $f(x_1) = 49$ in (*i*), we get

$$x_3 = 2.2462 - \frac{3 - 2.2462}{49 + 6.5438}(-6.5438) = 2.335$$

Now $f(x_3) = f(2.335) = -2.2732$ *i.e.* the root lies between 2.335 and 3.

$\therefore$ Taking $x_0 = 2.335$ and $x_1 = 3$, $f(x_0) = -2.2732$ and $f(x_1) = 49$ in (*i*), we obtain

$$x_4 = 2.335 - \frac{3 - 2.335}{49 + 2.2732}(-2.2732) = 2.3645$$

Repeating this process, the successive approximations are $x_5 = 2.3770$, $x_6 = 2.3779$ etc.
Since $x_5 = x_6$ upto 3 decimal places, we take $(32)^{1/4} = 2.378$.

2.10. (1) SECANT METHOD

This method is an improvement over the method of false position as it does not require the condition $f(x_0) f(x_1) < 0$ of that method (Fig. 2.5). Here also the graph of the function $y = f(x)$ is approximated by a secant line but at each iteration, two most recent approximations to the root are used to find the next approximation. Also it is not necessary that the interval must contain the root.

Taking x_0, x_1 as the initial limits of the interval, we write the equation of the chord joining these as

$$y - f(x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_1)$$

Then the abscissa of the point where it crosses the x-axis ($y = 0$) is given by

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1)$$

which is an approximation to the root. The general formula for successive approximations is, therefore, given by

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n),\ n \geq 1.$$

(2) Rate of Convergence. If at any interation $f(x_n) = f(x_{n-1})$, this method fails and shows that it does not converge necessarily. *This is a drawback of secant method over the method of false position which always converges. But* ***if the secant method once converges, its rate of convergence is 1.6 which is faster than that of the method of false position.***

Example 2.23. *Find a root of the equation $x^3 - 2x - 5 = 0$ using secant method correct to three decimal places.*

Sol. Let $f(x) = x^3 - 2x - 5$ so that $f(2) = -1$ and $f(3) = 16$.

$\therefore$ Taking initial approximations $x_0 = 2$ and $x_1 = 3$, by secant method, we have

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1) = 3 - \frac{3 - 2}{16 + 1} 16 = 2.058823$$

Now $f(x_2) = -0.390799$

$$\therefore \quad x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) = 2.081263$$

and $f(x_3) = -0.147204$

$$\therefore \qquad x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3) = 2.094824$$

and $\qquad f(x_4) = 0.003042$

$$\therefore \qquad x_5 = x_4 - \frac{x_4 - x_3}{f(x_4) - f(x_3)} f(x_4) = 2.094549$$

Hence the root is 2.094 correct to 3 decimal places.

Example 2.24. *Find the root of the equation $xe^x = \cos x$ using the secant method correct to four decimal places.* *(U.P.T.C., MCA., 2009)*

Sol. Let $\qquad f(x) = \cos x - xe^x = 0.$

Taking the initial approximations $x_0 = 0$, $x_1 = 1$ so that $f(x_0) = 1$, $f(x_1) = \cos 1 - e = -2.17798$

Then by secant method, we have

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1) = 1 + \frac{1}{3.17798}(-2.17798) = 0.31467$$

Now $\qquad f(x_2) = 0.51987$

$$\therefore \qquad x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) = 0.44673 \quad \text{and} \quad f(x_3) = 0.20354$$

$$\therefore \qquad x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3) = 0.53171.$$

Repeating this process, the successive approximations are $x_5 = 0.51690$, $x_6 = 0.51775$, $x_7 = 0.51776$ etc.

Hence the root is 0.5177 correct to 4 decimal places.

Obs. Comparing Examples 2.18 and 2.21, we notice that the rate of convergence in secant method is definitely faster than that of the method of false position.

2.11. (1) ITERATION METHOD

To find the roots of the equation $\quad f(x) = 0 \qquad$ ...(*i*)

by successive approximations, we rewrite (*i*) in the form $x = \phi(x) \qquad$ ...(*ii*)

The roots of (*i*) are the same as the points of intersection of the straight line $y = x$ and the curve representing $y = \phi(x)$. Fig. 2.6 illustrates the working of the iteration method which provides a spiral solution.

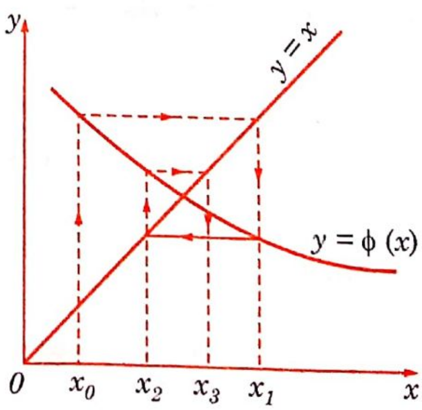

Fig. 2.6

Let $x = x_0$ be an initial approximation of the desired root α. Then the first approximation x_1 is given by

$$x_1 = \phi(x_0)$$

Now treating x_1 as the initial value, the second approximation is $\qquad x_2 = \phi(x_1)$

Proceeding in this way, the nth approximation is given by $\qquad x_n = \phi(x_{n-1})$

(2) Sufficient condition for convergence of iterations. Now it is not sure whether the sequence of approximations $x_1, x_2, ..., x_n$ always converges to the same number which is a root of (1) or not. As such, we have to choose the initial approximation x_0 suitably so that the successive approximations $x_1, x_2, ..., x_n$ converge to the root α. The following theorem helps in making the right choice of x_0 :

Theorem. *If (i) α be a root of $f(x) = 0$ which is equivalent to $x = \phi(x)$,*

(ii) I, be any interval containing the point $x = \alpha$,

(iii) $|\phi'(x)| < 1$ for all x in I,

then the sequence of approximations $x_0, x_1, x_2, ..., x_n$ will converge to the root α provided the initial approximation x_0 is chosen in I.

Proof. Since α is a root of $x = \phi(x)$, we have $\alpha = \phi(\alpha)$

If x_{n-1} and x_n be 2 successive approximations to α, we have $x_n = \phi(x_{n-1})$

$$\therefore \qquad x_n - \alpha = \phi(x_{n-1}) - \phi(\alpha) \qquad ...(1)$$

By mean value theorem, $\dfrac{\phi(x_{n-1}) - \phi(\alpha)}{x_{n-1} - \alpha} = \phi'(\xi)$ where $x_{n-1} < \xi < \alpha$

Hence (1) becomes $x_n - \alpha = (x_{n-1} - \alpha)\, \phi'(\xi)$

If $|\phi'(x_i)| \le k < 1$ for all i, then

$$|x_n - \alpha| \le k\, |x_{n-1} - \alpha| \qquad ...(2)$$

Similarly $\qquad |x_{n-1} - \alpha| \le k\, |x_{n-2} - \alpha|$

i.e., $\qquad |x_n - \alpha| \le k^2\, |x_{n-2} - \alpha|$

Proceeding in this way, $\qquad |x_n - \alpha| \le k^n\, |x_0 - \alpha|$

As $n \to \infty$, the R.H.S. tends to zero, therefore, the sequence of approximations converges to the root α.

Obs. 1. The smaller the value of $\phi'(x)$, the more rapid will be the convergence.

2. This method of iteration is particularly useful for finding the real roots of an equation given in the form of an infinite series.

(3) Acceleration of convergence. From (2), we have

$$|x_n - \alpha| \le k\, |x_{n-1} - \alpha|, \; k < 1.$$

It is clear from this relation that the iteration method is **linearly convergent**. *This slow rate of convergence can be improved by using the following method :*

(4) Aitken's Δ^2 method. Let x_{i-1}, x_i, x_{i+1} be three successive approximations to the desired root α of the equation $x = \phi(x)$. Then we know that

$$\alpha - x_i = k(\alpha - x_{i-1}), \quad \alpha - x_{i+1} = k(\alpha - x_i)$$

Dividing, we get $\dfrac{\alpha - x_i}{\alpha - x_{i+1}} = \dfrac{\alpha - x_{i-1}}{\alpha - x_i}$

whence
$$\alpha = x_{i+1} - \frac{(x_{i+1} - x_i)^2}{x_{i+1} - 2x_i + x_{i-1}} \qquad ...(3)$$

But in the sequence of successive approximations, we have

$$\Delta x_i = x_{i+1} - x_i$$
$$\Delta^2 x_i = \Delta(\Delta x_1) = \Delta(x_{i+1} - x_i) = \Delta x_{i+1} - \Delta x_i$$
$$= x_{i+2} - x_{i+1} - (x_{i+1} - x_i) = x_{i+2} - 2x_{i+1} + x_i$$

$\therefore \qquad \Delta^2 x_{i-1} = x_{i+1} - 2x_i + x_{i-1}$

Hence (3) can be written as $\alpha = x_{i+1} - \dfrac{(\Delta x_i)^2}{\Delta^2 x_{i-1}}$ ...(4)

which yields successive approximations to the root α.

Example 2.25. *Find a real root of the equation $\cos x = 3x - 1$ correct to three decimal places using*

(i) Iteration method

(ii) Aitken's Δ^2 method. *(Anna, B. Tech., 2013)*

Sol. (*i*) We have $\quad f(x) = \cos x - 3x + 1 = 0$

$f(0) = 2 = +$ ve and $f(\pi/2) = -3\pi/2 + 1 = -$ ve

$\therefore$ A root lies between 0 and $\pi/2$.

Rewriting the given equation as $x = \frac{1}{3}(\cos x + 1) = \phi(x)$, we have

$$\phi'(x) = \frac{\sin x}{3} \quad \text{and} \quad |\phi'(x)| = \tfrac{1}{3}|\sin x| < 1 \text{ in } (0, \pi/2).$$

Hence the iteration method can be applied and we start with $x_0 = 0$. Then the successive approximations are,

$$x_1 = \phi(x_0) = \tfrac{1}{3}(\cos 0 + 1) = 0.6667$$
$$x_2 = \phi(x_1) = \tfrac{1}{3}(\cos 0.6667 + 1) = 0.5953$$
$$x_3 = \phi(x_2) = \tfrac{1}{3}(\cos 0.5953 + 1) = 0.6093$$
$$x_4 = \phi(x_3) = \tfrac{1}{3}(\cos 0.6093 + 1) = 0.6067$$
$$x_5 = \phi(x_4) = \tfrac{1}{3}(\cos 0.6067 + 1) = 0.6072$$
$$x_6 = \phi(x_5) = \tfrac{1}{3}(\cos 0.6072 + 1) = 0.6071$$

Hence x_5 and x_6 being almost the same, the root is 0.607 correct to 3 decimal places.

(*ii*) We calculate x_1, x_2, x_3 as above. To use Aitken's method, we have

x	Δx	$\Delta^2 x$
$x_1 = 0.667$		
	-0.0714	
$x_2 = 0.5953$		$\underline{0.0854}$
	$\underline{0.014}$	
$x_3 = \underline{0.6093}$		

Hence $\qquad x_4 = x_3 - \dfrac{(\Delta x_2)^2}{\Delta^2 x_1} = 0.6093 - \dfrac{(0.014)^2}{0.0854} = 0.607$

which corresponds to six iterations in normal form.

Thus the required root is 0.607.

Example 2.26. *Using iteration method, find a root of the equation $x^3 + x^2 - 1 = 0$ correct to four decimal places.* *(U.P.T.U., B.Tech., 2006)*

Sol. We have $f(x) = x^3 + x^2 - 1 = 0$

Since $f(0) = -1$ and $f(1) = 1$, a root lies between 0 and 1.

Rewriting the given equation as $x = (x + 1)^{-1/2} = \phi(x)$, we have $\phi'(x) = -\frac{1}{2}(x + 1)^{-3/2}$ and $|\phi'(x)| < 1$ for $x < 1$. Hence the iteration method can be applied. Starting with $x_0 = 0.75$, the successive approximations are

$$x_1 = \phi(x_0) = \frac{1}{\sqrt{(x_0 + 1)}} = 0.7559$$

$$x_2 = \phi(x_1) = \frac{1}{\sqrt{(0.7559 + 1)}} = 0.75466$$

$$x_3 = 0.75492,\ x_4 = 0.75487,\ x_5 = 0.75488$$

Hence x_4 and x_5 being almost the same, the root is 0.7548 correct to 4 decimal places.

Example 2.27. *Apply iteration method to find the negative root of the equation $x^3 - 2x + 5 = 0$ correct to four decimal places.*

Sol. If α, β, γ are the roots of the given equation, then $-\alpha, -\beta, -\gamma$ are the roots of

$$(-x)^3 - 2(-x) + 5 = 0$$

$\therefore$ The negative root of the given equation is the positive root of

$$f(x) = x^3 - 2x - 5 = 0. \qquad ...(i)$$

Since $f(2) = -1$ and $f(3) = 16$, a root lies between 2 and 3.

Rewriting (i) as $x = (2x + 5)^{1/3} = \phi(x)$,

we have $\phi'(x) = \frac{1}{3}(2x + 5)^{-2/3}.\ 2$ and $|\phi'(x)| < 1$ for $x < 3$.

$\therefore$ The iteration method can be applied :

Starting with $x_0 = 2$. The successive approximations are

$$x_1 = \phi(x_0) = (2x_0 + 5)^{1/3} = 2.08008$$

$$x_2 = \phi(x_1) = 2.09235, \quad x_3 = 2.09422$$

$$x_4 = 2.09450, \quad x_5 = 2.09454$$

Since x_4 and x_5 being almost the same, the root of (i) is 2.0945 correct to 4 decimal places.

Hence the negative root of the given equation is -2.0945.

Example 2.28. *Find a real root of $2x - \log_{10} x = 7$ correct to four decimal places using iteration method.*

Sol. We have $f(x) = 2x - \log_{10} x - 7$

$$f(3) = 6 - \log_{10} 3 - 7 = 6 - 0.4771 - 7 = -1.4471$$

$$f(4) = 8 - \log_{10} 4 - 7 = 8 - 0.602 - 7 = 0.398$$

$\therefore$ A root lies between 3 and 4.

Rewriting the given equation as $x = \frac{1}{2}(\log_{10} x + 7) = \phi(x)$, we have

$$\phi'(x) = \frac{1}{2}\left(\frac{1}{x}\log_{10} e\right)$$

$\therefore$ $|\phi'(x)| < 1$ when $3 < x < 4$

$[\because \log_{10} e = 0.4343]$

Since $|f(4)| < |f(3)|$, the root is near to 4.

Hence the iteration method can be applied. Taking $x_0 = 3.6$, the successive approximations are

$$x_1 = \phi(x_0) = \frac{1}{2}(\log_{10} 3.6 + 7) = 3.77815$$

$$x_2 = \phi(x_1) = \frac{1}{2}(\log_{10} 3.77815 + 7) = 3.78863$$

$$x_3 = \phi(x_2) = \frac{1}{2}(\log 3.78863 + 7) = 3.78924$$

$$x_4 = \phi(x_3) = \frac{1}{2}(\log 3.78924 + 7) = 3.78927$$

Hence x_3 and x_4 being almost equal, the root is 3.7892 correct to 4 decimal places.

Example 2.29. *Find the smallest root of the equation*

$$1 - x + \frac{x^2}{(2!)^2} - \frac{x^3}{(3!)^2} + \frac{x^4}{(4!)^2} - \frac{x^5}{(5!)^2} + \ldots = 0$$

Sol. Writing the given equation as

$$x = 1 + \frac{x^2}{(2!)^2} - \frac{x^3}{(3!)^2} + \frac{x^4}{(4!)^2} - \frac{x^5}{(5!)^2} + \ldots = \phi(x)$$

Omitting x^2 and higher powers of x, we get $x = 1$ approximately.

Taking $x_0 = 1$, we obtain

$$x_1 = \phi(x_0) = 1 + \frac{1}{(2!)^2} - \frac{1}{(3!)^2} + \frac{1}{(4!)^2} - \frac{1}{(5!)^2} + \ldots = 1.2239$$

$$x_2 = \phi(x_1) = 1 + \frac{(1.2239)^2}{(2!)^2} - \frac{(1.2239)^3}{(3!)^2} + \frac{(1.2239)^4}{(4!)^2} - \frac{(1.2239)^5}{(5!)^2} + \ldots = 1.3263$$

Similarly $x_3 = 1.38$, $x_4 = 1.409$, $x_5 = 1.425$

$x_6 = 1.434$, $x_7 = 1.439$, $x_8 = 1.442$.

The values of x_7 and x_8 indicate that the root is 1.44 correct to 2 decimal places.

PROBLEMS 2.4

1. Find a root of the following equations, using the bisection method correct to three decimal places :

   (*i*) $x^3 - x - 1 = 0$ (*U.P.T.U, MCA, 2009*) (*ii*) $x^3 - x^2 - 1 = 0$ (*J.N.T.U., B. Tech., 2009*)

   (*iii*) $2x^3 + x^2 - 20x + 12 = 0$ (*iv*) $x^4 - x - 10 = 0$.

2. Evaluate a real root of the following equations by bisection method:

   (*i*) $x - \cos x = 0$ (*Mumbai, B.E., 2004*) (*ii*) $e^{-x} - x = 0$

   (*iii*) $e^x = 4 \sin x$. (*J.N.T.U., B. Tech., 2015*)

3. Find a real root of the following equations correct to three decimal places, by the method of false position :

(*i*) $x^3 - 5x + 1 = 0$ (*U.P.T.U., B. Tech., 2012*) (*ii*) $x^3 - 4x - 9 = 0$ (*V.T.U., B. Tech., 2007*)

(*iii*) $x^6 - x^4 - x^3 - 1 = 0$. (*Delhi, B. Tech., 2013*)

4. Using Regula falsi method, compute the real root of the following equations correct to three decimal places :

(*i*) $xe^x = 2$ (*J.N.T.U., B. Tech., 2015*) (*ii*) $\cos x = 3x - 1$

(*iii*) $xe^x = \sin x$ (*iv*) $x \tan x = -1$

(*v*) $2x - \log x = 7$ (*J.N.T.U., B. Tech., 2006*)

(*vi*) $3x + \sin x = e^x$. (*U.P.T.U., B. Tech., 2010*)

5. Find the fourth root of 12 correct to three decimal places by interpolation method.

6. Locate the root of $f(x) = x^{10} - 1 = 0$, between 0 and 1.3 using bisection method and method of false position. Comment on which method is preferable. (*Pune BVP., B. Tech., 2004*)

7. Find a root of the following equations correct to three decimal places by the secant method :

(*i*) $x^3 + x^2 + x + 7 = 0$ (*ii*) $x - e^{-x} = 0$

(*iii*) $x \log_{10} x = 1.9$.

8. Use the iteration method to find a root of the equations to four decimal places :

(*i*) $x^3 + x^2 - 100 = 0$ (*ii*) $x^3 - 9x + 1 = 0$ (*Madras, B. Tech., 2006*)

(*iii*) $x = \frac{1}{2} + \sin x$ (*iv*) $\tan x = x$

(*v*) $e^x - 3x = 0$ (*Anna, B. Tech., 2012*) (*vi*) $2^x - x - 3 = 0$ which lies between $(-3, -2)$.

9. Evaluate $\sqrt{30}$ by (*i*) secant method (*ii*) iteration method correct to four decimal places.

10. Find the root of the equation $2x = \cos x + 3$ correct to three decimal places using (*i*) Iteration method, (*ii*) Aitken's Δ^2 method.

11. Find the real root of the equation $x - \dfrac{x^3}{3} + \dfrac{x^5}{10} - \dfrac{x^7}{42} + \dfrac{x^9}{216} - \dfrac{x^{11}}{1320} + \ldots = 0.443$

correct to three decimal places using iteration method.

2.12. (1) NEWTON-RAPHSON METHOD

Let x_0 be an approximate root of the equation $f(x) = 0$. If $x_1 = x_0 + h$ be the exact root, then $f(x_1) = 0$.

$\therefore$ Expanding $f(x_0 + h)$ by Taylor's series $f(x_0) + h f'(x_0) + \dfrac{h^2}{2!} f''(x_0) + \ldots\ldots = 0$

Since h is small, neglecting h^2 and higher powers of h, we get $f(x_0) + h f'(x_0) = 0$

or
$$h = -\frac{f(x_0)}{f'(x_0)} \qquad \ldots(1)$$

$\therefore$ A closer approximation to the root is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Similarly starting with x_1, a still better approximation x_2 is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

In general, $$\mathbf{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}} \quad (n = 0, 1, 2 \ldots) \qquad \ldots(2)$$

which is known as the *Newton-Raphson formula* or *Newton's iteration formula.*

Obs. 1. *Newton's method is useful in cases of large values of $f'(x)$ i.e.,* when the graph of $f(x)$ while crossing the x-axis is nearly vertical.

For if $f'(x)$ is small in the vicinity of the root, then by (1), h will be large and the computation of the root is slow or may not be possible. *Thus this method is not suitable in those cases where the graph of $f(x)$ is nearly horizontal while crossing the x-axis.*

Obs. 2. Geometrical interpretation. Let x_0 be a point near the root α of the equation $f(x) = 0$ (Fig. 2.7). Then the equation of the tangent at A_0 $[x_0, f(x_0)]$ is

$$y - f(x_0) = f'(x_0)(x - x_0).$$

It cuts the x-axis at $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

which is a first approximation to the root α. If A_1 is the point corresponding to x_1 on the curve, then the tangent at A_1 will cut the x-axis at x_2 which is nearer to α and is, therefore, a second approximation to the root. Repeating this process, we approach the root α quite rapidly. Hence *the method consists in replacing the part of the curve between the point A_0 and the x-axis by means of the tangent to the curve at A_0.*

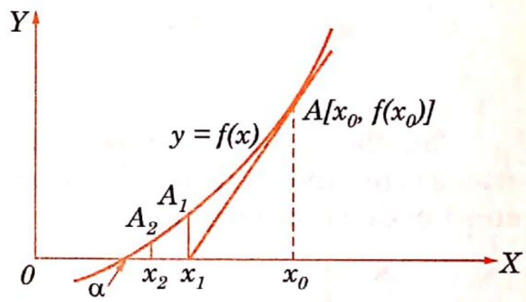

Fig. 2.7

Obs. 3. Newton's method is generally used to improve the result obtained by other methods. *It is applicable to the solution of both algebraic and transcendental equations.*

(2) Convergence of Newton-Raphson Method. *Newton's formula converges provided the initial approximation x_0 is chosen sufficiently close to the root.*

If it is not near the root, the procedure may lead to an endless cycle. A bad initial choice will lead one astray. *Thus a proper choice of the initial guess is very important for the success of the Newton's method.*

Comparing (2) with the relation $x_{n+1} = \phi(x_n)$ of the iteration method, we get

$$\phi(x_n) = x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

In general, $$\phi(x) = x - \frac{f(x)}{f'(x)} \quad \text{which gives} \quad \phi'(x) = \frac{f(x)f''(x)}{[f'(x)]^2}.$$

Since the iteration method (§ 2.10) converges if $|\phi'(x)| < 1$

$\therefore$ Newton's formula will converge if $|f(x)f''(x)| < |f'(x)|^2$ in the interval considered.

Assuming $f(x)$, $f'(x)$ and $f''(x)$ to be continuous, we can select a small interval in the vicinity of the root α, in which the above condition is satisfied. Hence the result.

Newtons method converges conditionally while Regula-falsi method always converges. However when once Newton-Raphson method converges, it converges faster and is preferred.

(3) Newton's method has a quadratic convergence

Suppose x_n differs from the root α by a small quantity ε_n so that

$$x_0 = \alpha + \varepsilon_n \text{ and } x_{n+1} = \alpha + \varepsilon_{n+1}.$$

Then (2) becomes $$\alpha + \varepsilon_{n+1} = \alpha + \varepsilon_n - \frac{f(\alpha + \varepsilon_n)}{f'(\alpha + \varepsilon_n)}$$

i.e. $$\varepsilon_{n+1} = \varepsilon_n - \frac{f(\alpha + \varepsilon_n)}{f'(\alpha + \varepsilon_n)}$$

$$= \varepsilon_n - \frac{f(\alpha) + \varepsilon_n f'(\alpha) + \frac{1}{2!}\varepsilon_n^2 f''(\alpha) + \ldots\ldots}{f'(\alpha) + \varepsilon_n f''(\alpha) + \ldots\ldots} \text{ by Taylor's expansion.}$$

$$= \varepsilon_n - \frac{\varepsilon_n f'(\alpha) + \frac{1}{2}\varepsilon_n^2 f''(\alpha) + \ldots\ldots}{f'(\alpha) + \varepsilon_n f''(\alpha) + \ldots\ldots} = \frac{\varepsilon_n^2}{2} \frac{f''(\alpha)}{f'(\alpha)}. \qquad [\because f(\alpha) = 0]$$

This shows that the subsequent error at each step, is proportional to the square of the previous error and as such the convergence is quadratic. Thus Newton-Raphson method has **second order convergence**.

Example 2.30. *Find the positive root of $x^4 - x = 10$ correct to three decimal places, using Newton-Raphson method.* (*Anna, B. Tech., 2010*)

Sol. Let $f(x) = x^4 - x - 10$

so that $f(1) = -10 = -$ ve, $f(2) = 16 - 2 - 10 = 4 = +$ ve.

$\therefore$ A root of $f(x) = 0$ lies between 1 and 2.

Let us take $x_0 = 2$

Also $f'(x) = 4x^3 - 1$

Newton-Raphson's formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \ldots(i)$$

Putting $n = 0$, the first approximation x_1 is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)}$$

$$= 2 - \frac{4}{4 \times 2^3 - 1} = 2 - \frac{4}{31} = 1.871$$

Putting $n = 1$ in (i), the second approximation is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.871 - \frac{f(1.871)}{f'(1.871)}$$

$$= 1.871 - \frac{(1.871)^4 - (1.871) - 10}{4(1.871)^3 - 1}$$

$$= 1.871 - \frac{0.3835}{25.199} = 1.856$$

Putting $n = 2$ in (*ii*), the third approximation is

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.856 - \frac{(1.856)^4 - (1.856) - 10}{4(1.856)^3 - 1}$$

$$= 1.856 - \frac{0.010}{24.574} = 1.856$$

Here $x_2 = x_3$. Hence the desired root is 1.856 correct to three decimal places.

Example 2.31. *Find by Newton's method, the real root of the equation $3x = \cos x + 1$, correct to four decimal places.* *(Anna, B. Tech., 2016)*

Sol. Let $f(x) = 3x - \cos x - 1$

$f(0) = -2 = -$ ve, $f(1) = 3 - 0.5403 - 1 = 1.4597 = +$ ve.

So a root of $f(x) = 0$ lies between 0 and 1. It is nearer to 1. Let us take $x_0 = 0.6$.

Also $f'(x) = 3 + \sin x$

∴ Newton's iteration formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{3x_n - \cos x_n - 1}{3 + \sin x_n}$$

$$= \frac{x_n \sin x_n + \cos x_n + 1}{3 + \sin x_n} \quad ...(i)$$

Putting $n = 0$, the first approximation x_1 is given by

$$x_1 = \frac{x_0 \sin x_0 + \cos x_0 + 1}{3 + \sin x_0} = \frac{(0.6) \sin (0.6) + \cos (0.6) + 1}{3 + \sin (0.6)}$$

$$= \frac{0.6 \times 0.5729 + 0.82533 + 1}{3 + 0.5729} = 0.6071$$

Putting $n = 1$ in (*i*), the second approximation is

$$x_2 = \frac{x_1 \sin x_1 + \cos x_1 + 1}{3 + \sin x_1} = \frac{0.6071 \sin (0.6071) + \cos (0.6071) + 1}{3 + \sin (0.6071)}$$

$$= \frac{0.6071 \times 0.57049 + 0.8213 + 1}{3 + 0.57049} = 0.6071$$

Here $x_1 = x_2$. Hence the desired root is 0.6071 correct to four decimal places.

Example 2.32. *Using Newton's iterative method, find the real root of $x \log_{10} x = 1.2$ correct to five decimal places.* *(Anna, B. Tech., 2015)*

Sol. Let $f(x) = x \log_{10} x - 1.2$

$f(1) = -1.2 = -$ ve, $f(2) = 2 \log_{10} 2 - 1.2 = 0.59794 = -$ ve

and $f(3) = 3 \log_{10} 3 - 1.2 = 1.4314 - 1.2 = 0.23136 = +$ ve.

So a root of $f(x) = 0$ lies between 2 and 3. Let us take $x_0 = 2$.

Also $f'(x) = \log_{10} x + x \cdot \frac{1}{x} \log_{10} e = \log_{10} x + 0.43429.$

∴ Newton's iteration formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = \frac{0.43429 x_n + 1.2}{\log_{10} x_n + 0.43429} \quad ...(i)$$

Putting $n = 0$, the first approximation is

$$x_1 = \frac{0.43429 \times x_0 + 1.2}{\log_{10} x_0 + 0.43429} = \frac{0.43429 \times 2 + 1.2}{\log_{10} 2 + 0.43429}$$

$$= \frac{0.86858 \times 1.2}{0.30103 + 0.43429} = 2.81$$

Similarly putting $n = 1, 2, 3, 4$ in (i), we get

$$x_2 = \frac{0.43429 \times 2.81 + 1.2}{\log_{10} 2.81 + 0.43429} = 2.741$$

$$x_3 = \frac{0.43429 \times 2.741 + 1.2}{\log_{10} 2.741 + 0.43429} = 2.74064$$

$$x_4 = \frac{0.43429 \times 2.741 + 1.2}{\log_{10} 2.74064 + 0.43429} = 2.74065$$

$$x_5 = \frac{0.43429 \times 2.74065 + 1.2}{\log_{10} 2.74065 + 0.43429} = 2.74065$$

Here $x_4 = x_5$. Hence the required root is 2.74065 correct to five decimal places.

2.13. SOME DEDUCTIONS FROM NEWTON-RAPHSON FORMULA

We can derive the following useful results from the Newton's iteration formula :

(1) *Iterative formula to find 1/N is* $\mathbf{x_{n+1} = x_n(2 - Nx_n)}$

(2) *Iterative formula to find* $\sqrt{N}$ *is* $\mathbf{x_{n+1} = \frac{1}{2}(x_n + N/x_n)}$

(3) *Iterative formula to find* $1/\sqrt{N}$ *is* $\mathbf{x_{n+1} = \frac{1}{2}(x_n + 1/Nx_n)}$

(4) *Iterative formula to find* $\sqrt[k]{N}$ *is* $\mathbf{x_{n+1} = \frac{1}{k}[(k-1)\, x_n + N/x_n^{k-1})]}$

Proofs. (1) Let $x = 1/N$ or $1/x - N = 0$

Taking $f(x) = 1/x - N$, we have $f'(x) = -x^{-2}$.

Then Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{(1/x_n - N)}{-x_n^{-2}} = x_n + \left(\frac{1}{x_n} - N\right)x_n^2.$$

$$= x_n + x_n - Nx_n^2 = x_n(2 - Nx_n)$$

(*Anna, B. Tech., 2013*)

(2) Let $x = \sqrt{N}$ or $x^2 - N = 0$.

Taking $f(x) = x^2 - N$, we have $f'(x) = 2x$.

Then Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - N}{2x_n} = \frac{1}{2}(x_n + N/x_n)$$

(*Anna, B. Tech., 2015*)

(3) Let $x = \frac{1}{\sqrt{N}}$ or $x^2 - \frac{1}{N} = 0$

Taking $f(x) = x^2 - 1/N$, we have $f'(x) = 2x$.

Then Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 1/N}{2x_n} = \frac{1}{2}\left(x_n + \frac{1}{Nx_n}\right)$$

(4) Let $x = \sqrt[k]{N}$ or $x^k - N = 0$

Taking $f(x) = x^k - N$, we have $f'(x) = kx^{k-1}$

Then Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n \frac{x_n^k - N}{kx_n^{k-1}} = \frac{1}{k}\left[(k-1)x_n + \frac{N}{x_n^{k-1}}\right].$$

Example 2.33. *Evaluate the following (correct to four decimal places) by Newton's iteration method :*

(i) $1/31$ (ii) $\sqrt{5}$ *(Anna, B.E., 2012)*

(iii) $1/\sqrt{14}$ (iv) $\sqrt[3]{24}$

(v) $(30)^{-1/5}$.

Sol. (i) Taking $N = 31$, the above formula (1) becomes

$$x_{n+1} = x_n(2 - 31x_n)$$

Since an approximate value of $1/31 = 0.03$, we take $x_0 = 0.03$.

Then
$$x_1 = x_0(2 - 31x_0) = 0.03(2 - 31 \times 0.03) = 0.0321$$
$$x_2 = x_1(2 - 31x_1) = 0.0321(2 - 31 \times 0.0321) = 0.032257$$
$$x_3 = x_2(2 - 31x_2) = 0.032257(2 - 31 \times 0.032257) = 0.03226$$

Since $x_2 = x_3$ upto 4 decimal places, we have $1/31 = 0.0323$.

(ii) Taking $N = 5$, the above formula (2), becomes $x_{n+1} = \frac{1}{2}(x_n + 5/x_n)$.

Since an approximate value of $\sqrt{5} = 2$, we take $x_0 = 2$.

Then
$$x_1 = \frac{1}{2}(x_0 + 5/x_0) = \frac{1}{2}(2 + 5/2) = 2.25$$
$$x_2 = \frac{1}{2}(x_1 + 5/x_1) = 2.2361$$
$$x_3 = \frac{1}{2}(x_2 + 5/x_2) = 2.2361$$

Since $x_2 = x_3$ upto 4 decimal places, we have $\sqrt{5} = 2.2361$.

(iii) Taking $N = 14$, the above formula (3), becomes $x_{n+1} = \frac{1}{2}[x_n + 1/(14x_n)]$

Since an approximate value of $1/\sqrt{14} = 1/\sqrt{16} = \frac{1}{4} = 0.25$, we take $x_0 = 0.25$,

Then
$$x_1 = \frac{1}{2}[x_0 + (14x_0)^{-1}] = \frac{1}{2}[0.25 + (14 \times 0.25)^{-1}] = 0.26785$$
$$x_2 = \frac{1}{2}[x_1 + (14x_1)^{-1}] = \frac{1}{2}[0.26785 + (14 \times 0.26785)^{-1}] = 0.2672618$$
$$x_3 = \frac{1}{2}[x_2 + (14x_2)^{-1}] = \frac{1}{2}[0.2672618 + (14 \times 0.2672618)^{-1}] = 0.2672612$$

Since $x_2 = x_3$ upto 4 decimal places, we take $1/\sqrt{14} = 0.2673$.

(*iv*) Taking $N = 24$ and $k = 3$, the above formula (4) becomes $x_{n+1} = \frac{1}{3}[2x_n + 24/x_n^2]$

Since an approximate value of $(24)^{1/3} = (27)^{1/3} = 3$, we take $x_0 = 3$.

Then $$x_1 = \frac{1}{3}(2x_0 + 24x_0^2) = \frac{1}{3}(6 + 24/9) = 2.88889$$

$$x_2 = \frac{1}{3}(2x_1 + 24/x_1^2) = \frac{1}{3}[(2 \times 2.88889) + 24/(2.88889)^2] = 2.88451$$

$$x_3 = \frac{1}{3}(2x_2 + 24/x_2^2) = \frac{1}{3}[2 \times 2.88451 + 24/(2.88451)^2] = 2.8845$$

Since $x_2 = x_3$ upto 4 decimal places, we take $(24)^{1/3} = 2.8845$.

(*v*) Taking $N = 30$ and $k = -5$, the above formula (4) becomes

$$x_{n+1} = \frac{1}{-5}(6x_n + 30/x_n^{-6}) = \frac{x_n}{5}(6 - 30x_n^5)$$

Since an approximate value of $(30)^{-1/5} = (32)^{-1/5} = 1/2$, we take $x_0 = 1/2$

Then $$x_1 = \frac{x_0}{5}(6 - 30x_0^5) = \frac{1}{10}(6 - 30/2^5) = 0.50625$$

$$x_2 = \frac{x_1}{5}(6 - 30x_1^5) = \frac{0.50625}{5}[6 - 30(0.50625)^5] = 0.506495$$

$$x_3 = \frac{x_2}{5}(6 - 30x_2^5) = \frac{0.506495}{5}[6 - 30(0.506495)^5] = 0.506496$$

Since $x_2 = x_3$ upto 4 decimal places, we take $(30)^{-1/5} = 0.5065$.

PROBLEMS 2.5

1. Find by Newton-Raphson method, a root of the following equations correct to 3 decimal places :
   (*i*) $x^3 - 3x + 1 = 0$ (*Bhopal, B.E., 2009*) (*ii*) $x^3 - 2x - 5 = 0$ (*P.T.U., B. Tech., 2005*)
   (*iii*) $x^3 - 5x + 3 = 0$ (*iv*) $3x^3 - 9x^2 + 8 = 0$.
2. Using Newton's iterative method, find a root of the following equations correct to 4 decimal places :
   (*i*) $x^4 + x^3 - 7x^2 - x + 5 = 0$ which lies between 2 and 3.
   (*ii*) $x^5 - 5x^2 + 3 = 0$.
3. Find the negative root of the equation $x^3 - 21x + 3500 = 0$ correct to 2 decimal places by Newton's method.
4. Using Newton-Raphson method, find a root of the following equations correct to 3 decimal places :
   (*i*) $x^2 + 4 \sin x = 0$
   (*ii*) $x \sin x + \cos x = 0$ or $x \tan x + 1 = 0$ (*Delhi, B. Tech., 2015*)
   (*iii*) $e^x = x^3 + \cos 25x$ which is near 4.5. (*Anna, B. Tech., 2012*)
   (*iv*) $x \log_{10} x = 12.34$, start with $x_0 = 10$. (*V.T.U., B. Tech., 2007*)
   (*v*) $\cos x = xe^x$ (*J.N.T.U., B. Tech., 2009*) (*vi*) $10^x + x - 4 = 0$.

5. The equation $2e^{-x} = \frac{1}{x+2} + \frac{1}{x+1}$ has two roots greater than – 1. Calculate these roots correct to five decir . places. (*U.P.T.U., B. Tech., 2012*)

6. The bacteria concentration in a reservoir varies as $C = 4e^{-2t} + e^{-0.1t}$. Using Newton Raphson method, calculate the time required for the bacteria concentration to be 0.5.

7. Use Newton's method to find the smallest root of the equation $e^x \sin x = 1$ to four places of decimal.

8. The current i in an electric circuit is given by $i = 10e^{-t} \sin 2\pi t$ where t is in seconds. Using Newton's method, find the value of t correct to 3 decimal places for $i = 2$ amp.

9. Find the iterative formulae for finding $\sqrt{N}$, $\sqrt[3]{N}$ where N is a real number, using Newton-Raphson formula.

   Hence evaluate : (*a*) $\sqrt{15}$. (*Anna, B. Tech., 2014*)

   (*b*) $\sqrt{21}$ (*U.P.T.U., MCA, 2009*)

   (*c*) the cube-root of 17 to three places of decimal. (*M.T.U., B. Tech., 2013*)

10. Develop an algorithm using N.R. method, to find the fourth root of a positive number N. (*Anna, B. Tech., 2012*)

    Hence find $\sqrt[4]{32}$. (*W.B.T.U., B. Tech., 2005*)

11. Evaluate the following (correct to 3 decimal places) by using the Newton-Raphson method.

    (*i*) 1/18 (*J.N.T.U., B. Tech., 2004*) (*ii*) $1/\sqrt{15}$ (*iii*) $(28)^{-1/4}$.

12. *Obtain Newton-Raphson extended formula*

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{1}{2}\frac{|f(x_0)|^2 f''(x_0)}{\{f'(x_0)\}^2}$$

for the root of the equation f(x) = 0.

Hence find the root of the equation $\cos x = xe^x$ correct to five decimal places.

[**Sol.** Expanding $f(x)$ in the neighbourhood of x_0 by Taylor's series ; we have

$$0 = f(x) = f(x_0 + \overline{x - x_0}) = f(x_0) + (x - x_0) f'(x_0) \text{ to first approx.}$$

Hence the first approximation to the root is given by

$$x_1 - x_0 = -f(x_0)/f'(x_0) \quad ...(i)$$

Again by Taylor's series to the second approximation, we get

$$f(x_1) = f(x_0) + (x_1 - x_0) f'(x_0) + \frac{1}{2!}(x_1 - x_0)^2 f''(x_0)$$

Since x_1 is an approximation to the root, $f(x_1) = 0$

$$\therefore \quad f(x_0) + (x_1 - x_0) f'(x_0) + \tfrac{1}{2}(x_1 - x_0)^2 f''(x_0) = 0$$

or
$$x_1 - x_0 = -\frac{f(x_0)}{f'(x_0)} - \frac{1}{2}\left\{\frac{-f(x_0)}{f'(x_0)}\right\} f''(x_0) \quad \text{[by } (i)]$$

whence follows the desired formula. This is known as **Chebyshev formula** of third order.]

2.14. MULLER'S METHOD

(1) This method is a generalisation of the secant method as it doesn't require the derivative of the function. It is an iterative method that requires three starting points. Here, $y = f(x)$ is approximated by a second degree parabola passing through these three points (x_{i-2},

(V.T.U., B. Tech., 2015)

22. $2x + y + 4z = 12$; $8x - 3y + 2z = 20$; $4x + 11y - z = 33$.

23. $10x + y + 2z = 13$; $3x + 10y + z = 14$; $2x + 3y + 10z = 15$.

24. $2x_1 - x_2 + x_3 = -1$; $2x_2 - x_3 + x_4 = 1$; $x_1 + 2x_3 - x_4 = -1$; $x_1 + x_2 + 2x_4 = 5$.

3.5. ITERATIVE METHODS OF SOLUTION

The preceding methods of solving simultaneous linear equations are known as *direct methods*, as these methods yield the solution after a certain amount of fixed computation. On the other hand, an iterative method is that in which we start from an approximation to the true solution and obtain better and better approximations from a computation cycle repeated as often as may be necessary for achieving a desired accuracy. Thus in an iterative method, the amount of computation depends on the degree of accuracy required.

For large systems, iterative methods may be faster than the direct methods. Even the round-off errors in iterative methods are smaller. In fact, iteration is a self correcting process and any error made at any stage of computation gets automatically corrected in the subsequent steps.

Simple iterative methods can be devised for systems in which the coefficients of the leading diagonal are large as compared to others. We now describe three such methods :

(1) Jacobi's iteration method. Consider the equations

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \quad \text{...(1)}$$

If a_1, b_2, c_3 are large as compared to other coefficients, solve for x, y, z respectively. Then the system can be written as

$$\left.\begin{aligned} x &= \frac{1}{a_1}(d_1 - b_1y - c_1z) \\ y &= \frac{1}{b_2}(d_2 - a_2x - c_2z) \\ z &= \frac{1}{c_3}(d_3 - a_3x - b_3y) \end{aligned}\right\} \quad \text{...(2)}$$

Let us start with the initial approximations x_0, y_0, z_0 for the values of x, y, z respectively. Substituting these on the right sides of (2), the first approximations are given by

$$x_1 = \frac{1}{a_1}(d_1 - b_1y_0 - c_1z_0)$$

$$y_1 = \frac{1}{b_2}(d_2 - a_2x_0 - c_2z_0)$$

$$z_1 = \frac{1}{c_3}(d_3 - a_3x_0 - b_3y_0)$$

Substituting the values x_1, y_1, z_1 on the right sides of (2), the second approximations are given by

$$x_2 = \frac{1}{a_1}(d_1 - b_1 y_1 - c_1 z_1)$$

$$y_2 = \frac{1}{b_2}(d_2 - a_2 x_1 - c_2 z_1)$$

$$z_2 = \frac{1}{c_3}(d_3 - a_3 x_1 - b_3 y_1)$$

This process is repeated till the difference between two consecutive approximations is negligible.

Obs. In the absence of any better estimates for x_0, y_0, z_0, these may each be taken as zero.

Example 3.25. *Solve, by Jacobi's iteration method, the equations*

$20x + y - 2z = 17$; $3x + 20y - z = -18$; $2x - 3y + 20z = 25$. *(Anna, B. Tech., 2016)*

Sol. We write the given equations in the form

$$\left.\begin{aligned} x &= \frac{1}{20}(17 - y + 2z) \\ y &= \frac{1}{20}(-18 - 3x + z) \\ z &= \frac{1}{20}(25 - 2x + 3y) \end{aligned}\right\} \quad ...(i)$$

We start from an approximation $x_0 = y_0 = z_0 = 0$.

Substituting these on the right sides of the equations (i), we get

$$x_1 = \frac{17}{20} = 0.85, \quad y_1 = -\frac{18}{20} = -0.9, \quad z_1 = \frac{25}{20} = 1.25$$

Putting these values on the right sides of the equations (i), we obtain

$$x_2 = \frac{1}{20}(17 - y_1 + 2z_1) = 1.02$$

$$y_2 = \frac{1}{20}(-18 - 3x_1 + z_1) = -0.965$$

$$z_2 = \frac{1}{20}(25 - 2x_1 + 3y_1) = 1.03$$

Substituting these values on the right sides of the equations (i), we have

$$x_3 = \frac{1}{20}(17 - y_2 + 2z_2) = 1.00125$$

$$y_3 = \frac{1}{20}(-18 - 3x_2 + z_2) = -1.0015$$

$$z_3 = \frac{1}{20}(25 - 2x_2 + 3y_2) = 1.00325$$

Substituting these values, we get

$$x_4 = \frac{1}{20}(17 - y_3 + 2z_3) \approx 1.0004$$

$$y_4 = \frac{1}{20}(-18 - 3x_3 + z_3) = -1.000025$$

$$z_4 = \frac{1}{20}(25 - 2x_3 + 3y_3) = 0.9965$$

Putting these values, we have

$$x_5 = \frac{1}{20}(-17 - y_4 + 2z_4) = 0.999966$$

$$y_5 = \frac{1}{20}(-18 - 3x_4 + z_4) = -1.000078$$

$$z_5 = \frac{1}{20}(25 - 2x_4 + 3y_4) = 0.999956$$

Again substituting these values, we get

$$x_6 = \frac{1}{20}(-17 - y_5 + 2z_5) = 1.0000$$

$$y_6 = \frac{1}{20}(-18 - 3x_5 + z_5) = -0.999997$$

$$z_6 = \frac{1}{20}(25 - 2x_5 + 3y_5) = 0.999992$$

The values in the 5th and 6th iterations being practically the same, we can stop. Hen the solution is

$$x = 1, y = -1, z = 1.$$

Example 3.26. *Solve by Jacobi's iteration method, the equations $10x + y - z = 11.$ $x + 10y + z = 28.08$, $-x + y + 10z = 35.61$, correct to two decimal places.*

(Anna, B.Tech., 20

Sol. Rewriting the given equations as

$$x = \frac{1}{10}(11.19 - y + z),\ y = \frac{1}{10}(28.08 - x - z),\ z = \frac{1}{10}(35.61 + x - y)$$

We start from an approximation, $x_0 = y_0 = z_0 = 0$.

First iteration

$$x_1 = \frac{11.19}{10} = 1.119,\ y_1 = \frac{28.08}{10} = 2.808,\ z_1 = \frac{35.61}{10} = 3.561$$

Second iteration

$$x_2 = \frac{1}{10}(11.19 - y_1 + z_1) = 1.19$$

$$y_2 = \frac{1}{10}(28.08 - x_1 - z_1) = 2.34$$

$$z_2 = \frac{1}{10}(35.61 + x_1 - y_1) = 3.39$$

Third iteration

$$x_3 = \frac{1}{10}(11.19 - y_2 + z_2) = 1.22$$

$$y_3 = \frac{1}{10}(28.08 - x_2 - z_2) = 2.35$$

$$z_3 = \frac{1}{10}(35.61 + x_2 - y_2) = 3.45$$

Fourth iteration

$$x_4 = \frac{1}{10}(11.19 - y_3 + z_3) = 1.23$$

$$y_4 = \frac{1}{10}(28.08 - x_3 - z_3) = 2.34$$

$$z_4 = \frac{1}{10}(35.61 + x_3 - y_3) = 3.45$$

Fifth iteration

$$x_5 = \frac{1}{10}(11.19 - y_4 + z_4) = 1.23$$

$$y_5 = \frac{1}{10}(28.08 - x_4 - z_4) = 2.34$$

$$z_5 = \frac{1}{10}(35.61 + x_4 - y_4) = 3.45$$

Hence $x = 1.23$, $y = 2.34$, $z = 3.45$

Example 3.27. *Solve the equations*

$$10x - 2x_2 - x_3 - x_4 = 3$$
$$-2x_1 + 10x_2 - x_3 - x_4 = 15$$
$$-x_1 - x_2 + 10x_3 - 2x_4 = 27$$
$$-x_1 - x_2 - 2x_3 + 10x_4 = -9,$$ *by Gauss-Jacobi iteration method.*

Sol. Rewriting the given equation as

$$x_1 = \frac{1}{10}(3 + 2x_2 + x_3 + x_4)$$

$$x_2 = \frac{1}{10}(15 + 2x_1 + x_3 + x_4)$$

$$x_3 = \frac{1}{10}(27 + x_1 + x_2 + 2x_4)$$

$$x_4 = \frac{1}{10}(-9 + x_1 + x_2 + 2x_3)$$

We start from an approximation $x_1 = x_2 = x_3 = x_4 = 0$.

First iteration

$$x_1 = 0.3,\ x_2 = 1.5,\ x_3 = 2.7,\ x_4 = -0.9.$$

Second iteration

$$x_1 = \frac{1}{10}\ (3 + 2\ (1.5) + 2.7 + (-\ 0.9)] = 0.78$$

$$x_2 = \frac{1}{10}\ [15 + 2\ (0.3) + 2.7 + (-\ 0.9)] = 1.74$$

$$x_3 = \frac{1}{10}\ [27 + 0.3 + 1.5 + 2\ (-\ 0.9)] = 2.7$$

$$x_4 = \frac{1}{10}\ [-\ 9 + 0.3 + 1.5 + 2\ (-\ 0.9)] = -\ 0.18$$

Proceeding in this way, we get

Third iteration

$$x_1 = 0.9,\ x_2 = 1.908,\ x_3 = 2.916,\ x_4 = -\ 0.108$$

Fourth iteration

$$x_1 = 0.9624,\ x_2 = 1.9608,\ x_3 = 2.9592,\ x_4 = -\ 0.036$$

Fifth iteration

$$x_1 = 0.9845,\ x_2 = 1.9848,\ x_3 = 2.9851,\ x_4 = -\ 0.0158$$

Sixth iteration

$$x_1 = 0.9939,\ x_2 = 1.9938,\ x_3 = 2.9938,\ x_4 = -\ 0.006$$

Seventh iteration

$$x_1 = 0.9939,\ x_2 = 1.9975,\ x_3 = 2.9976,\ x_4 = -\ 0.0025$$

Eighth iteration

$$x_1 = 0.999,\ x_2 = 1.999,\ x_3 = 2.999,\ x_4 = -\ 0.001$$

Ninth iteration

$$x_1 = 0.9996,\ x_2 = 1.9996,\ x_3 = 2.9996,\ x_4 = -\ 0.004$$

Tenth iteration

$$x_1 = 0.9998,\ x_2 = 1.9998,\ x_3 = 2.9998,\ x_4 = -\ 0.0001$$

Hence $x_1 = 1,\ x_2 = 2,\ x_3 = 3,\ x_4 = 0$.

(2) Gauss-Seidal iteration method. This is a modification of Jacobi's method. As before the system of equations :

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \quad ...(1)$$

is written as

$$\left.\begin{aligned} x &= \frac{1}{a_1}\ (d_1 - b_1y - c_1z) \\ y &= \frac{1}{b_2}\ (d_2 - a_2x - c_2z) \\ z &= \frac{1}{c_3}\ (d_3 - a_3x - b_3y) \end{aligned}\right\} \quad ...(2)$$

Here also we start with the initial approximations $x_0,\ y_0,\ z_0$ for $x,\ y,\ z$ respectively which may each be taken as zero. Substituting $y = y_0,\ z = z_0$ in the first of the equations (2), we get

$$x_1 = \frac{1}{a_1}\ (d_1 - b_1y_0 - c_1z_0)$$

Then putting $x = x_1, z = z_0$ in the second of the equations (2), we have

$$y_1 = \frac{1}{b_2}(d_2 - a_2x_1 - c_2z_0)$$

Next substituting $x = x_1, y = y_1$ in the third of the equations (2), we obtain

$$z_1 = \frac{1}{c_3}(d_3 - a_3x_1 - b_3y_1)$$

and so on *i.e.* as soon as a new approximation for an unknown is found, it is immediately used in the next step.

This process of iteration is repeated till the values of x, y, z are obtained to desired degree of accuracy.

Obs. 1. Since the most recent approximations of the unknowns are used while proceeding to the next step, the *convergence in the Gauss-Seidal method is twice as fast as in Jacobi's method.*

Obs. 2. *Jacobi and Gauss-Seidal methods converge for any choice of the initial approximations if in each equation of the system,* **the absolute value of the largest co-efficient is almost equal to or in atleast one equation greater than the sum of the absolute values of all the remaining coefficients.**

Example 3.28. *Apply Gauss-Seidal iteration method to solve the equations $20x + y - 2z = 17$; $3x + 20y - z = -18$; $2x - 3y + 20z = 25$.* (V.T.U., B. Tech., 2016) (cf. Example 3.25)

Sol. We write the given equations in the form

$$x = \frac{1}{20}(17 - y + 2z) \quad ...(i)$$

$$y = \frac{1}{20}(-18 - 3x + z) \quad ...(ii)$$

$$z = \frac{1}{20}(25 - 2x + 3y) \quad ...(iii)$$

First iteration

Putting $y = y_0, z = z_0$ in (*i*), we get $\quad x_1 = \frac{1}{20}(17 - y_0 + 2z_0) = 0.8500$

Putting $x = x_1, z = z_0$ in (*ii*), we have $\quad y_1 = \frac{1}{20}(-18 - 3x_1 + z_0) = -1.0275$

Putting $x = x_1, y = y_1$ in (*iii*), we obtain $\quad z_1 = \frac{1}{20}(25 - 2x_1 + 3y_1) = 1.0109$

Second iteration

Putting $y = y_1, z = z_1$ in (*i*), we get $\quad x_2 = \frac{1}{20}(17 - y_1 + 2z_1) = 1.0025$

Putting $x = x_2, z = z_1$ in (*ii*), we obtain $\quad y_2 = \frac{1}{20}(-18 - 3x_2 + z_1) = -0.9998$

Putting $x = x_2, y = y_2$ in (*iii*), we get $\quad z_2 = \frac{1}{20}(25 - 2x_2 + 3y_2) = 0.9998$

Third iteration, we get

$$x_3 = \frac{1}{20}(17 - y_2 + 2z_2) = 1.0000$$

$$y_3 = \frac{1}{20}(-18 - 3x_3 + z_2) = -1.0000$$

$$z_3 = \frac{1}{20}(25 - 2x_3 + 3y_3) = 1.0000$$

The values in the 2nd and 3rd iterations being practically the same, we can stop.

Hence the solution is $x = 1, y = -1, z = 1$.

Example 3.29. *Solve the equations $27x + 6y - z = 85$, $x + y + 54z = 110$; $6x + 15y + 2z = 72$ by Gauss-Jacobi method and Gauss-Seidel method.* (*Anna, B.Tech.*, 2013)

Sol. Rewriting the given equations as

$$x = \frac{1}{27}(85 - 6y + z) \quad ...(i)$$

$$y = \frac{1}{15}(72 - 6x - 2z) \quad ...(ii)$$

$$z = \frac{1}{54}(110 - x - y) \quad ...(iii)$$

(*a*) *Gauss-Jacobi's method*

We start from an approximation $x_0 = y_0 = z_0 = 0$

First iteration

$$x_1 = \frac{85}{27} = 3.148,\ y_1 = \frac{72}{15} = 4.8,\ z_1 = \frac{110}{54} = 2.037$$

Second iteration

$$x_2 = \frac{1}{27}(85 - 6y_1 + z_1) = 2.157$$

$$y_2 = \frac{1}{15}(72 - 6x_1 - 2z_1) = 3.269$$

$$z_2 = \frac{1}{54}(110 - x_1 - y_1) = 1.890$$

Third iteration

$$x_3 = \frac{1}{27}(85 - 6y_2 + z_2) = 2.492$$

$$y_3 = \frac{1}{15}(72 - 6x_2 - 2z_2) = 3.685$$

$$z_3 = \frac{1}{54}(110 - x_2 - y_2) = 1.937$$

Fourth iteration

$$x_4 = \frac{1}{27}(85 - 6y_3 + z_3) = 2.401$$

$$y_4 = \frac{1}{15}(72 - 6x_3 - 2z_3) = 3.545$$

$$z_4 = \frac{1}{54}(110 - x_3 - y_3) = 1.923$$

Fifth iteration

$$x_5 = \frac{1}{27}(85 - 6y_4 + z_4) = 2.432$$

$$y_5 = \frac{1}{15}(72 - 6x_4 - 2z_4) = 3.583$$

$$z_4 = \frac{1}{54}(110 - x_3 - y_3) = 1.927$$

Repeating this process, the successive iterations are.

$$x_6 = 2.423,\ y_6 = 3.570,\ z_6 = 1.926$$
$$x_7 = 2.426,\ y_7 = 3.574,\ z_7 = 1.926$$
$$x_8 = 2.425,\ y_8 = 3.573,\ z_8 = 1.926$$
$$x_9 = 2.426,\ y_9 = 3.573,\ z_9 = 1.926$$

Hence $x = 2.426$, $y = 3.573$, $z = 1.926$

(*b*) *Gauss-Seidal method*

First iteration

Putting $y = y_0 = 0$, $z = z_0 = 0$ in (*i*), $\quad x_1 = \frac{1}{27}(85 - 6y_0 + z_0) = 3.14$

Putting $x = x_1$, $z = z_0$ in (*ii*), $\quad y_1 = \frac{1}{15}(72 - 6x_1 - 2z_0) = 3.541$

Putting $x = x_1$, $y = y_1$ in (*iii*), $\quad z_1 = \frac{1}{54}(110 - x_1 - y_1) = 1.913$

Second iteration

$$x_2 = \frac{1}{27}(85 - 6y_1 + z_1) = 2.432$$

$$y_2 = \frac{1}{15}(72 - 6x_2 - 2z_1) = 3.572$$

$$z_2 = \frac{1}{54}(110 - x_2 - y_2) = 1.926$$

Third iteration

$$x_3 = \frac{1}{27}(85 - 6y_2 + z_2) = 2.426$$

$$y_3 = \frac{1}{15}(72 - 6x_3 - 2z_2) = 3.573$$

$$z_3 = \frac{1}{54}(110 - x_3 - y_3) = 1.926$$

Fourth iteration

$$x_4 = \frac{1}{27}(85 - 6y_3 + z_3) = 2.426$$

$$y_4 = \frac{1}{15}(72 - 6x_4 - 2z_3) = 3.573$$

$$z_4 = \frac{1}{54}(110 - x_4 - y_4) = 1.926.$$

Hence $x = 2.426$, $y = 3.573$, $z = 1.926$.

Obs. We have seen that *the convergence is quite fast in Gauss-Seidal method as compared to Gauss-Jacobi method.*

Example 3.30. *Apply Gauss-Seidal iteration method to solve the equations : $10x_1 - 2x_2 - x_3 - x_4 = 3$; $-2x_1 + 10x_2 - x_3 - x_4 = 15$; $-x_1 - x_2 + 10x_3 + 2x_4 = 27$; $-x_1 - x_2 - 2x_3 + 10x_4 = -9$.*

(*Bhopal, B.E. 2009*) (*cf.* Example 3.27)

Sol. Rewriting the given equations as

$$x_1 = 0.3 + 0.2x_2 + 0.1x_3 + 0.1x_4 \quad ...(i)$$

$$x_2 = 1.5 + 0.2x_1 + 0.1x_3 + 0.1x_4 \quad ...(ii)$$

$$x_3 = 2.7 + 0.1x_1 + 0.1x_2 + 0.2x_4 \quad ...(iii)$$

$$x_4 = -0.9 + 0.1x_1 + 0.1x_2 + 0.2x_3 \quad ...(iv)$$

First iteration

Putting $x_2 = 0$, $x_3 = 0$, $x_4 = 0$ in (*i*), we get $x_1 = 0.3$

Putting $x_1 = 0.3$, $x_3 = 0$, $x_4 = 0$ in (*ii*), we obtain $x_2 = 1.56$

Putting $x_1 = 0.3$, $x_2 = 1.56$, $x_4 = 0$ in (*iii*), we obtain $x_3 = 2.886$

Putting $x_1 = 0.3$, $x_2 = 1.56$, $x_3 = 2.886$ in (*iv*), we get $x_4 = -0.1368$

Second iteration

Putting $x_2 = 1.56$, $x_3 = 2.886$, $x_4 = -0.1368$ in (*i*), we obtain $x_1 = 0.8869$

Putting $x_1 = 0.8869$, $x_3 = 2.886$, $x_4 = -0.1368$ in (*ii*), we obtain $x_2 = 1.9523$

Putting $x_1 = 0.8869$, $x_2 = 1.9523$, $x_4 = -0.1368$ in (*iii*), we have $x_3 = 2.9566$

Putting $x_1 = 0.8869$, $x_2 = 1.9523$, $x_3 = 2.9566$ in (*iv*), we get $x_4 = -0.0248$.

Third iteration

Putting $x_2 = 1.9523$, $x_3 = 2.9566$, $x_4 = -0.0248$ in (*i*), we obtain $x_1 = 0.9836$

Putting $x_1 = 0.9836$, $x_3 = 2.9566$, $x_4 = -0.0248$ in (*ii*), we obtain $x_2 = 1.9899$

Putting $x_1 = 0.9836$, $x_2 = 1.9899$, $x_4 = -0.0248$ in (*iii*), we get $x_3 = 2.9924$

Putting $x_1 = 0.9836$, $x_2 = 1.9899$, $x_3 = 2.9924$ in (*iv*), we get $x_4 = -0.0042$.

Fourth iteration. Proceeding as above

$$x_1 = 0.9968,\ x_2 = 1.9982,\ x_3 = 2.9987,\ x_4 = -0.0008.$$

Fifth iteration is $x_1 = 0.9994$, $x_2 = 1.9997$, $x_3 = 2.9997$, $x_4 = -0.0001$.

Sixth iteration is $x_1 = 0.9999$, $x_2 = 1.9999$, $x_3 = 2.9999$, $x_4 = -0.0001$

Hence the solution is $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, $x_4 = 0$.

Table 6.1. Forward Difference Table

Value of x	*Value of y*	*1st diff.*	*2nd diff.*	*3rd diff.*	*4th diff.*	*5th diff.*
x_0	y_0					
		Δy_0				
$x_0 + h$	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
$x_0 + 2h$	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
$x_0 + 3h$	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
$x_0 + 4h$	y_4		$\Delta^2 y_3$			
		Δy_4				
$x_0 + 5h$	y_5					

Obs. 1. *Any higher order forward difference can be expressed in terms of the entries.*

We have $\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0$

$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = (y_3 - 2y_2 + y_1) - (y_2 - 2y_1 + y_0) = y_3 - 3y_2 + 3y_1 - y_0$

$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0 = (y_4 - 3y_3 + 3y_2 - y_1) - (y_3 - 3y_2 + 3y_1 - y_0)$

$= y_4 - 4y_3 + 6y_2 - 4y_1 + y_0$

The coefficients occurring on the right hand side being the binomial coefficients, we have in general, $\Delta^n y_0 = y_n - {}^nc_1\, y_{n-1} + {}^nc_2\, y_{n-2} - \ldots\ldots + (-1)^n\, y_0$.

Obs. 2. *The operator Δ obeys the distributive, commutative and index laws*

i.e. (*i*) $\Delta[f(x) \pm \phi(x)] = \Delta f(x) \pm \Delta\phi(x)$

(*ii*) $\Delta[cf(x)] = c\,\Delta f(x)$, c being a constant.

(*iii*) $\Delta^m\, \Delta^n\, f(x) = \Delta^{m+n}\, f(x)$, m and n being positive integers. In view of (*i*) and (*ii*), Δ is a *linear operator*.

But $\Delta[f(x)\,.\,\phi(x)] \neq f(x)\,.\,\Delta\phi(x)$.

(2) Backward differences. The differences $y_1 - y_0, y_2 - y_1, \ldots\ldots, y_n - y_{n-1}$ when denoted by $\nabla y_1, \nabla y_2, \ldots\ldots, \nabla y_n$ respectively, are called the *first backward differences* where ∇

Table 6.2. Backward Difference Table

Value of x	*Value of y*	*1st diff.*	*2nd diff.*	*3rd diff.*	*4th diff.*	*5th diff.*
x_0	y_0					
		∇y_1				
$x_0 + h$	y_1		$\nabla^2 y_2$			
		∇y_2		$\nabla^3 y_3$		
$x_0 + 2h$	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$	
		∇y_3		$\nabla^3 y_4$		$\nabla^5 y_5$
$x_0 + 3h$	y_3		$\nabla^2 y_4$		$\nabla^4 y_5$	
		∇y_4		$\nabla^3 y_5$		
$x_0 + 4h$	y_4		$\nabla^2 y_5$			
		∇y_5				
$x_0 + 5h$	y_5					

6.8. OTHER DIFFERENCE OPERATORS

We have already introduced the operators Δ, ∇ and δ. Besides these, there are the operators E and μ, which we define below :

(1) Shift operator E *is the operation of increasing the argument x by h so that* $Ef(x) = f(x + h)$, $E^2 f(x) = f(x + 2h)$, $E^3 f(x) = f(x + 3h)$ etc.

The inverse operator E^{-1} is defined by $E^{-1} f(x) = f(x - h)$

If y_x is the function $f(x)$, then $Ey_x = y_{x+h}$, $E^{-1}y_x = y_{x-h}$, $E^n y_x = y_{x+nh}$, where n may be any real number.

(2) Averaging operator μ is defined by the equation $\mu y_x = \frac{1}{2}(y_{x+\frac{1}{2}h} + y_{x-\frac{1}{2}h})$

Obs. *In the difference calculus E is regarded as the fundamental operator and $\Delta, \nabla, \delta, \mu$ can be expressed in terms of E.*

6.9. RELATIONS BETWEEN THE OPERATORS

(1) We shall now establish the following identities :

(*i*) $\Delta = \mathbf{E} - \mathbf{1}$

(*ii*) $\nabla = \mathbf{1} - \mathbf{E}^{-1}$

(*iii*) $\delta = \mathbf{E}^{1/2} - \mathbf{E}^{-1/2}$

(*iv*) $\mu = \frac{1}{2}(\mathbf{E}^{1/2} + \mathbf{E}^{-1/2})$

(*v*) $\Delta = \mathbf{E}\nabla = \nabla\mathbf{E} = \delta\mathbf{E}^{1/2}$

(*vi*) $\mathbf{E} = \mathbf{e}^{\mathbf{hD}}$.

Proofs. (*i*) $\Delta y_x = y_{x+h} - y_x = Ey_x - y_x = (E - 1)y_x$

This shows that the operators Δ and E are connected by the symbolic relation

$$\Delta = E - 1 \qquad \text{or} \qquad \mathbf{E} = 1 + \Delta.$$

Obs. *These relations imply that the effect of operator E on y_x is the same as that of the operator $(1 + \Delta)$ on y_x. The operator's E and Δ do not have any existence as separate entities.*

(*ii*) $$\nabla y_x = y_x - y_{x-h} = y_x - E^{-1} y_x = (1 - E^{-1})y_x$$

$$\therefore \quad \nabla = 1 - E^{-1}$$

(*iii*) $$\delta y_x = y_{x+\frac{1}{2}h} - y_{x-\frac{1}{2}h} = E^{1/2} y_x - E^{-1/2}y_x = (E^{1/2} - E^{-1/2})y_x$$

$$\delta = E^{1/2} - E^{-1/2}.$$

(*iv*) $$\mu y_x = \frac{1}{2}(y_{x+\frac{1}{2}h} + y_{x-\frac{1}{2}h}) = \frac{1}{2}(E^{1/2}y_x + E^{-1/2}y_x) = \frac{1}{2}(E^{1/2} + E^{-1/2})y_x$$

$$\therefore \quad \mu = \frac{1}{2}(E^{1/2} + E^{-1/2}).$$

(*v*) $$E\nabla y_x = E(y_x - y_{x-h}) = Ey_x - Ey_{x-h} = y_{x+h} - y_x = \Delta y_x$$

$$\therefore \quad E\nabla = \Delta$$

$$\nabla E y_x = \nabla y_{x+h} = y_{x+h} - y_x = \Delta y_x$$

$$\therefore \quad \nabla E = \Delta$$

$$\delta E^{1/2} y_x = \delta y_{x+\frac{1}{2}h} = y_{x+\frac{1}{2}h+\frac{1}{2}h} - y_{x+\frac{1}{2}h-\frac{1}{2}h} = y_{x+h} - y_x = \Delta y_x$$

$$\therefore \quad \delta E^{1/2} = \Delta$$

Hence $$\Delta = E\nabla = \nabla E = \delta E^{1/2}.$$

(*vi*) $$Ef(x) = f(x + h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \ldots\ldots \qquad \text{[by Taylor's series]}$$

$$= f(x) + hDf(x) + \frac{h^2}{2!} D^2f(x) + \ldots\ldots$$

$$= \left(1 + hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + \ldots\ldots\right) f(x) = e^{hD} f(x)$$

$$\therefore \quad E = e^{hD}$$

Cor. $E = 1 + \Delta = e^{hD}.$

Note. *A table showing the symbolic relations between the various operators is given below for ready reference. To prove such relations between the operators, always express each operator in terms of the fundamental operator E.*

(2) Relations between the various operators

In terms of	E	Δ	∇	δ	hD
E	—	$\Delta + 1$	$(1-\nabla)^{-1}$	$1 + \frac{1}{2}\delta^2 + \delta\sqrt{(1+\delta^2/4)}$	e^{hD}
Δ	$E - 1$	—	$(1-\nabla)^{-1} - 1$	$\frac{1}{2}\delta^2 + \delta\sqrt{(1+\delta^2/4)}$	$e^{hD} - 1$
∇	$1 - E^{-1}$	$1 - (1+\Delta)^{-1}$	—	$-\frac{1}{2}\delta^2 + \delta\sqrt{(1+\delta^2/4)}$	$1 - e^{-hD}$
δ	$E^{1/2} - E^{-1/2}$	$\Delta(1+\Delta)^{-1/2}$	$\nabla(1-\nabla)^{-1/2}$	—	$2\sinh(hD/2)$
μ	$\frac{1}{2}(E^{1/2} + E^{-1/2})$	$(1+\Delta/2)(1+\Delta)^{-1/2}$	$(1+\nabla/2)(1+\nabla)^{-1/2}$	$\sqrt{(1+\delta^2/4)}$	$\cosh(hD/2)$
hD	$\log E$	$\log(1+\Delta)$	$\log(1-\nabla)^{-1}$	$2\sinh^{-1}(\delta/2)$	—

Example 6.13. *Prove that* $e^x = \left(\frac{\Delta^2}{E}\right) e^x \cdot \frac{Ee^x}{\Delta^2 e^x}$, *the interval of differencing being h.*

(Bhopal, B.E., 2009)

Sol. Since $\left(\frac{\Delta^2}{E}\right) e^x = \Delta^2 . E^{-1} e^x = \Delta^2 e^{x-h} = \Delta^2 e^x . e^{-h} = e^{-h} \Delta^2 e^x$

$\therefore$ R.H.S. $= e^{-h} \Delta^2 e^x . \frac{Ee^x}{\Delta^2 e^x} = e^{-h} Ee^x = e^{-h} . e^{x+h} = e^x.$

Example 6.14. *Prove with the usual notations, that*

(i) $hD = \log(1+\Delta) = -\log(1-\nabla) = \sinh^{-1}(\mu\delta)$ (M.T.U., B. Tech., 2013)

(ii) $(E^{1/2} + E^{-1/2})(1+\Delta)^{1/2} = 2 + \Delta$ (Bhopal, B. Tech., 2009)

(iii) $\Delta - \nabla = \Delta\nabla = \delta^2$ (Delhi, B. Tech., 2015)

(iv) $\Delta^3 y_2 = \nabla^3 y_5$.

Sol. (i) We know that $e^{hD} = E = 1 + \Delta$

$\therefore \quad hD = \log(1+\Delta)$

Also $\quad hD = \log E = -\log(E^{-1}) = -\log(1-\nabla)$ $\quad [\because E^{-1} = 1 - \nabla]$

We have proved that $\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$

and $\delta = E^{1/2} - E^{-1/2}$

$\therefore$ $\mu\delta = \frac{1}{2}(E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2})$

$= \frac{1}{2}(E - E^{-1}) = \frac{1}{2}(e^{hD} - e^{-hD}) = \sinh(hD)$

i.e. $hD = \sinh^{-1}(\mu\delta)$.

Hence $hD = \log(1 + \Delta) = -\log(1 - \Delta) = \sinh^{-1}(\mu\delta)$.

(*ii*) $(E^{1/2} + E^{-1/2})(1 + \Delta)^{1/2}$

$= (E^{1/2} + E^{-1/2})E^{1/2} = E + 1 = 1 + \Delta + 1 = 2 + \Delta.$

(*iii*) We know that $\Delta = E - 1$, $\nabla = 1 - E^{-1}$ and $\delta = E^{1/2} - E^{-1/2}$

$\therefore$ $\Delta - \nabla = E - 2 + E^{-1} = (E^{1/2} - E^{-1/2})^2 = \delta^2$

Also $\Delta\nabla = (E - 1)(1 - E^{-1}) = E + E^{-1} - 2$

$= (E^{1/2} - E^{-1/2})^2 = \delta^2.$

Hence $\Delta - \nabla = \Delta\nabla = \delta^2.$

(*iv*) $\Delta^3 y_2 = (E - 1)^3 y_2$ $[\because \ \Delta = E - 1$

$= (E^3 - 3E^2 + 3E - 1)y_2$

$= y_5 - 3y_4 + 3y_3 - y_2$ ...(1)

$\nabla^3 y_5 = (1 - E^{-1})^3 y_5$ $[\because \ \nabla = 1 - E^{-1}$

$= (1 - 3E^{-1} + 3E^{-2} - E^{-3})y_5$

$= y_5 - 3y_4 + 3y_3 - y_2$ ...(2)

From (1) and (2), $\Delta^3 y_2 = \nabla^3 y_5$.

Example 6.15. *Prove that*

(*i*) $\Delta = \frac{1}{2}\delta^2 + \delta\sqrt{\left(1 + \frac{\delta^2}{4}\right)}$ *(Mumbai, B.E., 2004)*

(*ii*) $1 + \delta^2\mu^2 = \left(1 + \frac{1}{2}\delta^2\right)$

(*iii*) $\mu = \frac{2 + \Delta}{2\sqrt{(1 + \Delta)}} + \sqrt{\left(1 + \frac{1}{4}\delta^2\right)}$

Sol. (*i*) $\frac{1}{2}\delta^2 + \delta\sqrt{\left(1 + \frac{\delta^2}{4}\right)}$ $[\because \ \delta = E^{1/2} - E^{-1/2}]$

$$= \frac{1}{2}(E^{1/2} - E^{-1/2})^2 + (E^{1/2} - E^{-1/2})\sqrt{[1 + (E^{1/2} - E^{-1/2})^2/4]}$$

$$= \frac{1}{2}(E + E^{-1} - 2) + (E^{1/2} - E^{-1/2})\sqrt{[(E + E^{-1} + 2)/4]}$$

$$= \frac{1}{2}(E + E^{-1} - 2) + (E^{1/2} - E^{-1/2})(E^{1/2} + E^{-1/2})/2$$

$$= \frac{1}{2}[(E + E^{-1} - 2) + (E - E^{-1})] = E - 1 = \Delta.$$

(ii) We know that $\delta = E^{1/2} - E^{-1/2}$ and $\mu = (E^{1/2} + E^{-1/2})/2$.

$\therefore$ L.H.S. $= 1 + \delta^2\mu^2 = 1 + (E^{1/2} - E^{-1/2})^2 (E^{1/2} + E^{-1/2})^2/4$

$$= \frac{1}{4}[4 + (E - E^{-1})^2] = \frac{1}{4}(E^2 + E^{-2} + 2) = \frac{1}{4}(E + E^{-1})^2$$

R.H.S. $= (1 + \frac{1}{2}\delta^2)^2 = [1 + \frac{1}{2}(E^{1/2} - E^{-1/2})^2]^2 = [1 + \frac{1}{2}(E + E^{-1} - 2)]^2$

$$= \frac{1}{4}(E + E^{-1})^2$$

Hence $\quad 1 + \delta^2\mu^2 = \left(1 + \frac{1}{2}\delta^2\right)^2.$

(iii) Since $\Delta = E - 1$, $\delta = E^{1/2} - E^{-1/2}$ and $\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$

$$\therefore \quad \frac{2+\Delta}{2\sqrt{(1+\Delta)}} = \frac{2+E-1}{2\sqrt{(1+E-1)}} = \frac{E+1}{2\sqrt{E}}$$

$$= \frac{1}{2}(E^{1/2} + E^{-1/2}) = \mu$$

Also $\sqrt{\left(1 + \frac{1}{4}\delta^2\right)} = \sqrt{\left[1 + \frac{1}{4}(E^{1/2} - E^{-1/2})^2\right]} = \sqrt{\left[1 + \frac{1}{4}(E + E^{-1} - 2)\right]}$

$$= \frac{1}{2}\sqrt{(E + E^{-1} + 2)} = \frac{1}{2}(E^{1/2} + E^{-1/2}) = \mu$$

Hence from (1) and (2), we get

$$\mu = \frac{2+\Delta}{2\sqrt{(1+\Delta)}} = \sqrt{\left(1 + \frac{1}{4}\delta^2\right)}.$$

Example 6.16. *Prove that* $\nabla y_{n+1} = h\left(1 + \frac{1}{2}\nabla + \frac{5}{12}\nabla^2 + \ldots\ldots\right)y'_n$

Sol. We have $\nabla_{y+1} = y_{n+1} - y_n = (E - 1)\, y_n$

$$= (e^{hD} - 1)\, y_n = \left(1 + hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + \ldots\ldots - 1\right) y_n$$

$$= hD\left(1 + \frac{hD}{2!} + \frac{h^2D^2}{3!} + \ldots\ldots\right) y_n$$

$$= h\left(1 + \frac{hD}{2!} + \frac{h^2D^2}{3!} + \ldots\ldots\right) Dy_n$$

Since $E^{-1} = 1 - \nabla = e^{-hD}$, $\quad\therefore\quad hD = -\log(1 - \nabla) = \nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \ldots\ldots$

$$\therefore \quad \nabla y_{n+1} = h\left\{1 + \frac{1}{2}\left(\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \ldots\ldots\right) + \frac{1}{6}\left(\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \ldots\ldots\right)^2 + \ldots\ldots\right\} y'_n$$

Hence $\quad \nabla y_{n+1} = h\left(1 + \frac{1}{2}\nabla + \frac{5}{12}\nabla^2 + \ldots\ldots\right) y'_n$.

7.2. NEWTON'S FORWARD INTERPOLATION FORMULA

Let the function $y = f(x)$ take the values $y_0, y_1, \ldots, y_n$ corresponding to the values $x_0, x_1, \ldots, x_n$ of x. Let these values of x be equi-spaced such that $x_i = x_0 + ih$ $(i = 0, 1, \ldots)$. Assuming $y(x)$ to be a polynomial of the nth degree in x such that $y(x_0) = y_0, y(x_1) = y_1, \ldots, y(x_n) = y_n$. We can write

$$y(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) + \ldots + a_n(x - x_0)(x - x_1) \ldots (x - x_{n-1}) \quad \ldots(1)$$

Putting $x = x_0, x_1, \ldots, x_n$ successively in (1), we get

$$y_0 = a_0, y_1 = a_0 + a_1 (x_1 - x_0), y_2 = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

and so on.

From these, we find that $a_0 = y_0$, $\Delta y_0 = y_1 - y_0 = a_1(x_1 - x_0) = a_1 h$

$$\therefore \quad a_1 = \frac{1}{h} \Delta y_0$$

Also $\quad \Delta y_1 = y_2 - y_1 = a_1(x_2 - x_1) + a_2(x_2 - x_0)(x_2 - x_1) = a_1 h + a_2.2h.h = \Delta y_0 + 2h^2 a_2$

$$\therefore \quad a_2 = \frac{1}{2h^2} (\Delta y_1 - \Delta y_0) = \frac{1}{2!\,h^2} \Delta^2 y_0$$

Similarly $a_3 = \dfrac{1}{3!\,h^3} \Delta^3 y_0$ and so on.

Substituting these values in (1), we obtain

$$y(x) = y_0 + \frac{\Delta y_0}{h} (x - x_0) + \frac{\Delta^2 y_0}{2!\,h^2} (x - x_0)(x - x_1) + \frac{\Delta^3 y_0}{3!\,h^3} (x - x_0)(x - x_1)(x - x_2) + \ldots \quad \ldots(2)$$

Now if it is required to evaluate y for $x = x_0 + ph$, then

$$x - x_0 = ph, \; x - x_1 = x - x_0 - (x_1 - x_0) = ph - h = (p - 1)h,$$
$$x - x_2 = x - x_1 - (x_2 - x_1) = (p - 1)h - h = (p - 2)h \text{ etc.}$$

Hence, writing $y(x) = y(x_0 + ph) = y_p$, (2) becomes

$$\mathbf{y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \ldots + \frac{p(p-1)\ldots(p-\overline{n-1})}{n!} \Delta^n y_0} \quad \ldots(3)$$

It is called *Newton's forward interpolation formula* as (3) contains y_0 and the forward differences of y_0.

Otherwise : Let the function $y = f(x)$ take the values $y_0, y_1, y_2, \ldots$ corresponding to the values $x_0, x_0 + h, x_0 + 2h, \ldots$ of x. Suppose it is required to evaluate $f(x)$ for $x = x_0 + ph$, where p is any real number.

For any real number p, we have defined E such that

$$E^p f(x) = f(x + ph)$$

$$y_p = f(x_0 + ph) = E^p f(x_0) = (1 + \Delta)^p y_0 \qquad [\because E = 1 + \Delta$$

$$= \left\{1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 + \ldots\ldots\right\} y_0 \qquad \ldots(4)$$

[using Binomial theorem

i.e.
$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots\ldots$$

If $y = f(x)$ is a polynomial of the nth degree, then $\Delta^{n+1} y_0$ and higher differences will be zero. Hence (4) will become

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots\ldots + \frac{p(p-1)(p-\overline{n-1})}{n!}\Delta^n y_0$$

which is same as (3).

Obs. 1. *This formula is used for interpolating the values of y near the beginning of a set of tabulated values and extrapolating values of y a little backward (i.e. to the left) of y_0.*

Obs. 2. *The first two terms of this formula give the linear interpolation while the first three terms give a parabolic interpolation and so on.*

7.3. NEWTON'S BACKWARD INTERPOLATION FORMULA

Let the function $y = f(x)$ take the values $y_0, y_1, y_2, \ldots..$ corresponding to the values x_0, $x_0 + h, x_0 + 2h, \ldots..$ of x. Suppose it is required to evaluate $f(x)$ for $x = x_n + ph$, where p is any real number. Then we have

$$y_p = f(x_n + ph) = E^p f(x_n) = (1 - \nabla)^{-p} y_n \qquad [\because E^{-1} = 1 - \nabla$$

$$= \left[1 + p\nabla + \frac{p(p+1)}{2!}\nabla^2 + \frac{p(p+1)(p+2)}{3!}\nabla^3 + \ldots\ldots\right] y_n$$

[using Binomial theorem

i.e.
$$\mathbf{y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \ldots.} \qquad \ldots(1)$$

It is called *Newton's backward interpolation formula* as (1) contains y_n and backward differences of y_n.

Obs. *This formula is used for interpolating the values of y near the end of a set of tabulated values and also for extrapolating values of y a little ahead* (to the right) *of y_n.*

Example 7.1. *The table gives the distance in nautical miles of the visible horizon for the given heights in feet above the earth's surface :*

x = height :	*100*	*150*	*200*	*250*	*300*	*350*	*400*
y = distance :	*10.63*	*13.03*	*15.04*	*16.81*	*18.42*	*19.90*	*21.27*

Find the values of y when

(i) x = 160 ft. (V.T.U., B.E., 2013)

(ii) x = 410.

Sol. The difference table is as under :

x	y	Δ	Δ^2	Δ^3	Δ^4
100	10.63				
		2.40			
			– 0.39		
150	**13.03**			0.15	
		2.01			
			– 0.24		– 0.07
200	15.04			**0.08**	
		1.77			
			– 0.16		**– 0.05**
250	16.81			0.03	
		1.61			
			– 0.13		– 0.01
300	18.42			**0.02**	
		1.48			
			– 0.11		
350	19.90				
		1.37			
400	**21.27**				

(*i*) If we take $x_0 = 160$, then $y_0 = 13.03$, $\Delta y_0 = 2.01$, $\Delta^2 y_0 = -0.24$, $\Delta^3 = 0.08$, $\Delta^4 y_0 = -0.05$

Since $x = 160$ and $h = 50$, $\therefore \quad p = \dfrac{x - x_0}{h} = \dfrac{10}{50} = 0.2$

$\therefore$ Using Newton's forward interpolation formula, we get

$$y_{218} = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \ldots$$

$$y_{160} = 13.03 + 0.402 + 0.192 + 0.0384 + 0.00168 = 13.46 \text{ nautical miles}$$

(*ii*) Since $x = 410$ is near the end of the table, we use Newton's backward interpolation formula.

$\therefore$ Taking $\quad x_n = 400, p = \dfrac{x - x_n}{h} = \dfrac{10}{50} = 0.2$

Using the line of backward difference

$$y_n = 21.27, \nabla y_n = 1.37, \nabla^2 y_n = -0.11, \nabla^3 y_n = 0.02 \text{ etc.}$$

$\therefore$ Newton's backward formula gives

$$y_{410} = y_{400} + p\nabla y_{400} + \frac{p(p+1)}{2!}\nabla^2 y_{400} + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_{400} + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_{400}$$

$$= 21.27 + 0.2(1.37) + \frac{0.2(1.2)}{2!}(-0.11) + \frac{0.2(1.2)(2.2)}{3!}(0.02) + \frac{0.2(1.2)(2.2)(3.2)}{4!}(-0.01)$$

$$= 21.27 + 0.274 - 0.0132 + 0.0018 - 0.0007$$

$$= 21.53 \text{ nautical miles.}$$

Example 7.2. *From the following table, estimate the number of students who obtained marks between 40 and 45 :*

Marks :	*30—40*	*40—50*	*50—60*	*60—70*	*70—80*
No. of students :	*31*	*42*	*51*	*35*	*31*

(V.T.U., B. Tech., 2011)

Sol. First we prepare the cumulative frequency table, as follows :

Marks less than (x) :	40	50	60	70	80
No. of students (y_x) :	31	73	124	159	190

Now the difference table is

x	y_x	Δy_x	$\Delta^2 y_x$	$\Delta^3 y_x$	$\Delta^4 y_x$
40	**31**				
		42			
50	73		**9**		
		51		**– 25**	
60	124		**– 16**		**37**
		35		**12**	
70	159		**– 4**		
		31			
80	190				

We shall find y_{45} *i.e.* number of students with marks less than 45. Taking $x_0 = 40$, $x = 45$, we have

$$p = \frac{x - x_0}{h} = \frac{5}{10} = 0.5 \qquad [\because \; h = 10$$

$\therefore$ Using Newton's forward interpolation formula, we get

$$y_{45} = y_{40} + p\,\Delta y_{40} + \frac{p(p-1)}{2!}\Delta^2 y_{40} + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_{40} + \frac{p(p-1)(p-2)(p-3)}{3!}\Delta^4 y_{40}$$

$$= 31 + 0.5 \times 42 + \frac{0.5(-0.5)}{2} \times 9 + \frac{0.5(-0.5)(-1.5)}{6} \times (-25) + \frac{0.5(-0.5)(-1.5)(-2.5)}{24} \times 37$$

$$= 31 + 21 - 1.125 - 1.5625 - 1.4453$$

$$= 47.87, \text{ on simplification.}$$

The number of students with marks less than 45 is 47.87 *i.e.*, 48.

But the number of students with marks less than 40 is 31.

Hence the number of students getting marks between 40 and 45 = 48 – 31 = 17.

Example 7.3. *Find the cubic polynomial which takes the following values :*

x :	0	1	2	3
$f(x)$:	1	2	1	10

Hence or otherwise evaluate f(4). *(Anna, B. Tech., 2013)*

Sol. The difference table is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	**1**			
		1		
			– 2	
1	2			**12**
		– 1		
			10	
2	1			
		9		
3	**10**			

We take $x_0 = 0$ and $p = \dfrac{x - 0}{h} = x$ [$\because h = 1$

$\therefore$ Using Newton's forward interpolation formula, we get

$$f(x) = f(0) + \frac{x}{1}\Delta f(0) + \frac{x(x-1)}{1.2}\Delta^2 f(0) + \frac{x(x-1)(x-2)}{1.2.3}\Delta^3 f(0)$$

$$= 1 + x(1) + \frac{x(x-1)}{2}(-2) + \frac{x(x-1)(x-2)}{6}(12)$$

$$= 2x^3 - 7x^2 + 6x + 1,$$

which is the required polynomial.

To compute $f(4)$, we take $x_n = 3$, $x = 4$ so that $p = \dfrac{x - x_n}{h} = 1$ [$\because h = 1$

Obs. Using Newton's backward interpolation formula, we get

$$f(4) = f(3) + p\nabla f(3) + \frac{p(p+1)}{1.2}\nabla^2 f(3) + \frac{p(p+1)(p+2)}{1.2.3}\nabla^3 f(3)$$

$$= 10 + 9 + 10 + 12 = 41$$

which is the same value as that obtained by substituting $x = 4$ in the cubic polynomial above.

The above example shows that *if a tabulated function is a polynomial, then interpolation and extrapolation give the same values.*

Example 7.4. *Using Newton's backward difference formula, construct an interpolating polynomial of degree 3 for the data : $f(-0.75) = -0.0718125$, $f(-0.5) = -0.02475$, $f(-0.25) = 0.3349375$, $f(0) = 1.10100$. Hence find $f(-1/3)$.*

Sol. The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
– 0.75	– 0.0718125			
		0.0470625		
– 0.5	– 0.02475		0.312625	
		0.3596875		**0.09375**
– 0.25	0.3349375		**0.400375**	
		0.7660625		
0	**1.10100**			

We use Newton's backward difference formula

$$y(x) = y_3 + \frac{p}{1!}\nabla y_3 + \frac{p(p+1)}{2!}\nabla^2 y_3 + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_3$$

taking $x_3 = 0,\ p = \dfrac{x-0}{h} = \dfrac{x}{0.25} = 4x,$ $[\because\ h = 0.25]$

$$\therefore\quad y(x) = 1.10100 + 4x\,(0.7660625) + \frac{4x(4x+1)}{2}(0.400375) + \frac{4x(4x+1)(4x+2)}{6}(0.09375)$$

$$= 1.101 + 3.06425x + 3.251x^2 + 0.81275x + x^3 + 0.75x^2 + 0.125x$$

$$= x^3 + 4.001x^2 + 4.002x + 1.101.$$

Put $x = -\dfrac{1}{3}$, so that

$$y\left(-\frac{1}{3}\right) = \left(-\frac{1}{3}\right)^3 + 4.001\left(-\frac{1}{3}\right)^2 + 4.002\left(-\frac{1}{3}\right) + 1.101$$

$$= 0.1745$$

Example 7.5. *In the table below, the values of y are consecutive terms of a series of which 23.6 is the 6th term. Find the first and tenth terms of the series :*

x :	*3*	*4*	*5*	*6*	*7*	*8*	*9*
y :	*4.8*	*8.4*	*14.5*	*23.6*	*36.2*	*52.8*	*73.9*

(*Anna, B.E., 2007*)

Sol. The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
3	4.8				
		3.6			
4	8.4		2.5		
		6.1		0.5	
5	14.5		3.0		0
		9.1		0.5	
6	23.6		3.5		0
		12.6		0.5	
7	36.2		4.0		0
		16.6		0.5	
8	52.8		4.5		
		21.1			
9	73.9				

To find the first term, use Newton's forward interpolation formula with $x_0 = 3, x = 1,$ $h = 1$ and $p = -2$. We have

$$y(1) = 4.8 + \frac{(-2)}{1} \times 3.6 + \frac{(-2)(-3)}{1.2} \times 2.5 + \frac{(-2)(-3)(-4)}{1.2.3} \times 0.5 = 3.1$$

To obtain the tenth term, use Newton's backward interpolation formula with $x_n = 9$, $x = 10$, $h = 1$ and $p = 1$. This gives

$$y(10) = 73.9 + \frac{1}{1} \times 21.1 + \frac{1(2)}{1.2} \times 4.5 + \frac{1(2)(3)}{1.2.3} \times 0.5 = 100.$$

Example 7.6. *Using Newton's forward interpolation formula, show that*

$$\Sigma n^3 = \left\{\frac{n(n+1)}{2}\right\}^2.$$

Sol. If $s_n = sn^3$, then $s_{n+1} = \Sigma(n+1)^3$

$\therefore$ $\quad \Delta s_n = s_{n+1} - s_n = \Sigma(n+1)^3 - \Sigma n^3 = (n+1)^3.$

Then $\quad \Delta^2 s_n = \Delta s_{n+1} - \Delta s_n = (n+2)^3 - (n+1)^3 = 3n^2 + 9n + 7$

$$\Delta^3 s_n = \Delta^2 s_{n+1} - \Delta^2 s_n$$
$$= [3(n+1)^2 + 9(n+1) + 7] - (3n^2 + 9n + 7) = 6n + 12.$$
$$\Delta^4 s_n = \Delta^3 s_{n+1} - \Delta^3 s_n = [6(n+1) + 12] - [6n + 12] = 6$$

and $\quad \Delta^5 s_n = \Delta^5 \Delta_n = \ldots\ldots = 0.$

Since the first term of the given series is 1, therefore taking $n = 1$, $s_1 = 1$, $\Delta s_1 = 8$, $\Delta^2 s_1 = 19$, $\Delta^3 s_1 = 18$, $\Delta^4 s_1 = 6$.

Substituting these in the Newton's forward interpolation formula *i.e.*

$$s_n = s_1 + (n-1)\Delta s_1 + \frac{(n-1)(n-2)}{2!}\Delta^2 s_1 + \frac{(n-1)(n-2)(n-3)}{3!}\Delta^3 s_1 + \frac{(n-1)(n-2)(n-3)(n-4)}{4!}\Delta^4 s_1,$$

$$s_n = 1 + 8(n-1) + \frac{19}{2}(n-1)(n-2) + 3(n-1)(n-2)(n-3) + \frac{1}{4}(n-1)(n-2)(n-3)(n-4) = \frac{1}{4}(n^4 + 2n^3 + n^2) = \left\{\frac{n(n+1)}{2}\right\}^2.$$

PROBLEMS 7.1

1. Using Newton's forward formula, find the value of $f(1.6)$, if

x :	1	1.4	1.8	2.2
$f(x)$:	3.49	4.82	5.96	6.5

(*J.N.T.U., B. Tech.,* 2006)

2. From the following table, find y when $x = 1.85$ and 2.4 by Newton's interpolation formulae :

x :	1.7	1.8	1.9	2.0	2.1	2.2	2.3
$y = e^x$:	5.474	6.050	6.686	7.389	8.166	9.025	9.974

(*Kottayam, B.E.,* 2005)

3. Express the value of θ interms of x using the following data :

x :	40	50	60	70	80	90
θ :	184	204	226	250	276	304

Also find θ at $x = 43$ and $x = 84$.

(*Anna, B. Tech.,* 2015)

7.11. INTERPOLATION WITH UNEQUAL INTERVALS

The various interpolation formulae derived so far possess the disadvantage of being applicable only to equally spaced values of the argument. It is, therefore, desirable to develop interpolation formulae for unequally spaced values of x. Now we shall study two such formulae :

(i) Lagrange's interpolation formula

(ii) Newton's general interpolation formula with divided differences.

7.12. LAGRANGE'S INTERPOLATION FORMULA

If $y = f(x)$ takes the value $y_0, y_1, ..., y_n$ corresponding to $x = x_0, x_1, ..., x_n$, then

$$f(x) = \frac{(x - x_1)(x - x_2).....(x - x_n)}{(x_0 - x_1)(x_0 - x_2).....(x_0 - x_n)} y_0 + \frac{(x - x_0)(x - x_2).....(x - x_n)}{(x_1 - x_0)(x_1 - x_2).....(x_1 - x_n)} y_1$$

$$+ ...... + \frac{(x - x_0)(x - x_1).....(x - x_{n-1})}{(x_n - x_0)(x_n - x_1).....(x_n - x_{n-1})} y_n \qquad ...(1)$$

This is known as *Lagrange's interpolation formula for unequal intervals.*

Proof. Let $y = f(x)$ be a function which takes the values $(x_0, y_0), (x_1, y_1), ..., (x_n, y_n)$. Since there are $n + 1$ pairs of values of x and y, we can represent $f(x)$ by a polynomial in x of degree n. Let this polynomial be of the form

$$y = f(x) = a_0(x - x_1)(x - x_2) ...... (x - x_n) + a_1(x - x_0)(x - x_2) ...... (x - x_n)$$
$$+ a_2(x - x_0)(x - x_1)(x - x_3) ...... (x - x_n) + ... + a_n(x - x_0)(x - x_1) ...... (x - x_{n-1}) \qquad ...(2)$$

Putting $x = x_0$, $y = y_0$, in (2), we get

$$y_0 = a_0(x_0 - x_1)(x_0 - x_2) ...... (x_0 - x_n)$$
$$a_0 = y_0/[(x_0 - x_1)(x_0 - x_2) ...... (x_0 - x_n)]$$

Similarly putting $x = x_1$, $y = y_1$ in (2), we have $a_1 = y_1/[(x_1 - x_0)(x_1 - x_2) ...... (x_1 - x_n)]$

Proceeding the same way, we find $a_2, a_3 ...... a_n$.

Substituting the values of $a_0, a_1, ..., a_n$ in (2), we get (1).

Obs. Lagranges interpolation formula (1) for n points is a polynomial of degree $(n - 1)$ which is known as *Lagrangian polynomial* and is very simple to implement on a computer.

This formula can also be used to split the given function into partial fractions.

For on dividing both sides of (1) by $(x - x_0)(x - x_1) ...... (x - x_n)$, we get

$$\frac{f(x)}{(x - x_0)(x - x_1).....(x - x_n)} = \frac{y_0}{(x_0 - x_1)(x_0 - x_2).....(x_0 - x_n)} \cdot \frac{1}{x - x_0}$$

$$+ \frac{y_1}{(x_1 - x_0)(x_1 - x_2).....(x_1 - x_n)} \cdot \frac{1}{x - x_1} + ...$$

$$+ \frac{y_n}{(x_n - x_0)(x_n - x_1).....(x_n - x_{n-1})} \cdot \frac{1}{x - x_n}.$$

Example 7.17. *Given the values*

x :	5	7	11	13	17
f(x) :	150	392	1452	2366	5202,

evaluate f(9), using Lagrange's formula (Anna, B. Tech., 2006)

Sol. (*i*) Here $x_0 = 5, \quad x_1 = 7, \quad x_2 = 11, \quad x_3 = 13, \quad x_4 = 17$

and $y_0 = 150, y_1 = 392, y_2 = 1452, y_3 = 2366, y_4 = 5202.$

Putting $x = 9$ and substituting the above values in Lagrange's formula, we get

$$f(9) = \frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)} \times 150 + \frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)} \times 392$$
$$+ \frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)} \times 1452$$
$$+ \frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)} \times 2366$$
$$+ \frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)} \times 5202$$

$$= -\frac{50}{3} + \frac{3136}{15} + \frac{3872}{3} - \frac{2366}{3} + \frac{578}{5} = 810$$

Example 7.18. *Find the polynomial f (x) by using Lagrange's formula and hence find f (3) for*

x :	0	1	2	5
f (x) :	2	3	12	147

(Anna, B. Tech., 2012)

Sol. Here $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 5$

and $y_0 = 2, y_1 = 3, y_2 = 12, y_3 = 147.$

Lagrange's formula is

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$
$$= \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)} (2) + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)} (3)$$
$$+ \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)} (12) + \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)} (147)$$

Hence $f(x) = x^3 + x^2 - x + 2$

$\therefore$ $f(3) = 27 + 9 - 3 + 2 = 35.$

Example 7.19. *A curve passes through the points (0, 18), (1, 10), (3, – 18) and (6, 90). Find the slope of the curve at x = 2* (J.N.T.U., B.Tech., 2008)

Sol. Here $x_0 = 0, x_1 = 1, x_2 = 3, x_3 = 6$ and $y_0 = 18, y_1 = 10, y_2 = -18, y_3 = 90$

Since the values of x are unequally spaced, we use the Lagrange's formula :

$$y = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} y_0 + \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} y_1$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} y_2 + \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} y_3$$

$$= \frac{(x-1)(x-3)(x-6)}{(0-1)(0-3)(0-6)}(18) + \frac{(x-0)(x-3)(x-6)}{(1-0)(1-3)(1-6)}(10)$$

$$+ \frac{(x-0)(x-1)(x-6)}{(3-0)(3-1)(3-6)}(-18) + \frac{(x-0)(x-1)(x-3)}{(6-0)(6-1)(6-3)}(90)$$

$$= (-x^3 + 10x^2 - 27x + 18) + (x^3 - 9x^2 + 18x) + (x^3 - 7x^2 + 6x) + (x^3 - 4x^2 + 3x)$$

i.e., $$y = 2x^3 - 10x^2 + 18$$

Thus the slope of the curve at $x = 2 = \left(\frac{dy}{dx}\right)_{x=2}$

$$= (6x^2 - 20x)_{x=2} = -16.$$

Example 7.20. *Using Lagrange's formula, express the function* $\frac{3x^2 + x + 1}{(x-1)(x-2)(x-3)}$ *as a sum of partial fractions.*

Sol. Let us evaluate $y = 3x^2 + x + 1$ for $x = 1$, $x = 2$ and $x = 3$

These values are

x :	$x_0 = 1$	$x_1 = 2$	$x_2 = 3$
y :	$y_0 = 5$	$y_1 = 15$	$y_2 = 31$

Lagrange's formula is

$$y = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} y_0 + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} y_1 + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} y_2$$

Substituting the above values, we get

$$y = \frac{(x-2)(x-3)}{(1-2)(1-3)}(5) + \frac{(x-1)(x-3)}{(2-1)(2-3)}(15) + \frac{(x-1)(x-2)}{(3-1)(3-2)}(31)$$

$$= 2.5(x-2)(x-3) - 15(x-1)(x-3) + 15.5(x-1)(x-2)$$

Thus $$\frac{3x^2 + x + 1}{(x-1)(x-2)(x-3)} = \frac{2.5(x-2)(x-3) - 15(x-1)(x-3) + 15.5(x-1)(x-2)}{(x-1)(x-2)(x-3)}$$

$$= \frac{2.5}{x-1} - \frac{15}{x-2} + \frac{15.5}{x-3}$$

9. Find the missing term in the following table:

x :	1	2	4	5	6
y :	14	15	5	...	9.

10. Using Lagrange's formula, express the function $\dfrac{x^2+x-3}{x^3-2x^2-x+2}$ as sum of partial fractions.

11. Using Lagrange's formula, express the function $\dfrac{x^2+6x-1}{(x^2-1)(x-4)(x-6)}$ as a sum of partial fractions. (J.N.T.U., B. Tech., 2015)

[**Hint.** Tabulate the values of $f(x) = x^2 + 6x - 1$ for $x = -1, 1, 4, 6$ and apply Lagrange's formula.]

12. Using Lagrange's formula, prove that $y_0 = \dfrac{1}{2}(y_1 + y_{-1}) - \dfrac{1}{8}\left\{\dfrac{1}{2}(y_3 - y_1) - \dfrac{1}{2}(y_{-1} - y_{-3})\right\}$.

[**Hint:** Here $x_0 = -3, x_1 = -1, x_2 = 1, x_3 = 3$.]

7.13 (1) DIVIDED DIFFERENCES

The Lagrange's formula has the drawback that if another interpolation value were inserted, then the interpolation coefficients are required to be recalculated. This labour of recomputing the interpolation coefficients is saved by using Newton's general interpolation formula which employs what are called '**divided differences**'. Before deriving this formula, we shall first define these differences.

If $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots\ldots$ be given points, then the *first divided difference* for the arguments x_0, x_1 is defined by the relation $[x_0, x_1]$ or $\underset{x_1}{\Delta} y_0 = \dfrac{y_1 - y_0}{x_1 - x_0}$.

Similarly $[x_1, x_2]$ or $\underset{x_2}{\Delta} y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}$ and $[x_2, x_3]$ or $\underset{x_3}{\Delta} y_2 = \dfrac{y_3 - y_2}{x_3 - x_2}$ etc.

The *second divided difference* for x_0, x_1, x_2 is defined as

$$[x_0, x_1, x_2] \text{ or } \underset{x_1, x_2}{\Delta^2} y_0 = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}.$$

The *third divided difference* for x_0, x_1, x_2, x_3 is defined as

$$[x_0, x_1, x_2, x_3] \text{ or } \underset{x_1, x_2, x_3}{\Delta^3} y_0 = \frac{[x_1, x_2, x_3] - [x_0, x_1, x_2]}{x_3 - x_0} \text{ and so on.}$$

(2) Properties of Divided Differences

I. *The divided differences are symmetrical in their arguments i.e. independent of the order of the arguments.* For it is easy to write $[x_0, x_1] = \dfrac{y_0}{x_0 - x_1} + \dfrac{y_1}{x_1 - x_0} = [x_1, x_0]$, $[x_0, x_1, x_2]$

$$= \frac{y_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{y_1}{(x_1 - x_0)(x_1 - x_2)} + \frac{y_2}{(x_2 - x_0)(x_2 - x_1)}$$

$= [x_1, x_2, x_0]$ or $[x_2, x_0, x_1]$ and so on.

II. *The nth divided differences of a polynomial of the nth degree are constant.*

Let the arguments be equally spaced so that $x_1 - x_0 = x_2 - x_1 = \ldots = x_n - x_{n-1} = h$. Then

$$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h}$$

$$[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0 - x_1]}{x_2 - x_0} = \frac{1}{2h}\left\{\frac{\Delta y_1}{h} - \frac{\Delta y_0}{h}\right\}$$

$$= \frac{1}{2!\,h^2}\Delta^2 y_0 \text{ and in general, } [x_0, x_1, x_2, \ldots\ldots, x_n] = \frac{1}{n!\,h^n}\Delta^n y_0.$$

If the tabulated function is a nth degree polynomial, then $\Delta^n y_0$ will be constant. Hence the nth divided differences will also be constant.

III. *The divided difference operator* Δ *is linear i.e.,* $\Delta\{au_x + bv_x\} = a\ \Delta\ u_x + b\,\Delta v_x$.

We have $\underset{x_1}{\Delta}\ (au_{x_0} + bv_{x_0}) = \dfrac{(au_{x_1} + bv_{x_1}) - (au_{x_0} + bv_{x_0})}{x_1 - x_0}$

$$= a\left\{\frac{u_{x_1} - u_{x_0}}{x_1 - x_0}\right\} + b\left\{\frac{v_{x_1} - v_{x_0}}{x_1 - x_0}\right\}$$

$$= a\underset{x_1}{\Delta}\ u_{x_0} + b\underset{x_0}{\Delta}\ v_{x_0}$$

In general $\Delta(au_x + bv_x) = a\,\Delta u_x + b\,\Delta v_x$. This property is also true for higher order differences.

7.14. NEWTON'S DIVIDED DIFFERENCE FORMULA

Let $y_0, y_1, \ldots\ldots, y_n$ be the values of $y = f(x)$ corresponding to the arguments $x_0, x_1, \ldots\ldots, x_n$. Then from the definition of divided differences, we have

$$[x, x_0] = \frac{y - y_0}{x - x_0}$$

so that $$y = y_0 + (x - x_0)[x, x_0] \qquad \ldots(1)$$

Again $$[x, x_0, x_1] = \frac{[x, x_0] - [x_0, x_1]}{x - x_1}$$

which gives $$[x, x_0] = [x_0, x_1] + (x - x_1)[x, x_0, x_1]$$

Substituting this value of $[x, x_0]$ in (1), we get

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x, x_0, x_1] \qquad \ldots(2)$$

Also $$[x, x_0, x_1, x_2] = \frac{[x, x_0, x_1] - [x_0\ x_1, x_2]}{x - x_2}$$

which gives $$[x, x_0, x_1] = [x_0, x_1, x_2] + (x - x_2)[x, x_0, x_1, x_2]$$

Substituting this value of $[x, x_0, x_1]$ in (2), we obtain

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)[x, x_0, x_1, x_2]$$

Proceeding in this manner, we get

$$\begin{aligned} y = f(x) = y_0 &+ (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] \\ &+ (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3] + \ldots\ldots \\ &+ (x - x_0)(x - x_1)\ \ldots\ldots\ (x - x_n)[x, x_0, x_1, \ldots\ldots\ x_n] \qquad \ldots(3) \end{aligned}$$

which is called *Newton's general interpolation formula with divided differences.*

7.15. RELATION BETWEEN DIVIDED AND FORWARD DIFFERENCES

If $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots\ldots$ be the given points, then

$$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0}$$

Also $$\Delta y_0 = y_1 - y_0$$

If $x_0, x_1, x_2, \ldots\ldots$ are equispaced, then $x_1 - x_0 = h$, so that

$$[x_0, x_1] = \frac{\Delta y_0}{h}$$

Similarly $$[x_1, x_2] = \frac{\Delta y_1}{h}$$

Now $$[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}$$

$$= \frac{\Delta y_1/h - \Delta y_0/h}{2h} \qquad [\because \ x_2 - x_0 = 2h]$$

$$= \frac{\Delta y_1 - \Delta y_0}{2h^2} = \frac{\Delta^2 y_0}{2h^2}$$

Thus $$[x_0, x_1, x_2] = \frac{\Delta^2 y_0}{2!\,h^2}$$

Similarly $$[x_1, x_2, x_3] = \frac{\Delta^2 y_1}{2!\,h^2}$$

$$\therefore \quad [x_0, x_1, x_2, x_3] = \frac{\Delta^2 y_1/2h^2 - \Delta^2 y_0/2h^2}{x_3 - x_0} = \frac{\Delta^2 y_1 - \Delta^2 y_0}{2h^2(3h)} \qquad [\because \ x_3 - x_0 = 3h]$$

Thus $$[x_0, x_1, x_2, x_3] = \frac{\Delta^3 y_0}{3!\,h^3}$$

In general, $[x_0, x_1, \ldots\ldots, x_n] = \dfrac{\Delta^n y_0}{n!\,h^n}$.

This is the relation between divided and forward differences.

Example 7.23. *Given the values*

x :	5	7	11	13	17
f(x) :	150	392	1452	2366	5202,

evaluate f(9), using Newton's divided difference formula. (V.T.U., B. Tech., 2010)

Sol. The divided differences table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
5	150			
		$\frac{392-150}{7-5} = 121$		
7	392		$\frac{265-121}{11-5} = 24$	
		$\frac{1452-392}{11-7} = 265$		$\frac{32-24}{13-5} = 1$
11	1452		$\frac{457-265}{13-7} = 32$	
		$\frac{2366-1452}{13-11} = 457$		$\frac{42-32}{17-7} = 1$
13	2366		$\frac{709-457}{17-11} = 42$	
		$\frac{5202-2366}{17-13} = 709$		
17	5202			

Taking $x = 9$ in the Newton's divided difference formula, we obtain

$$f(9) = 150 + (9-5) \times 121 + (9-5)(9-7) \times 24 + (9-5)(9-7)(9-11) \times 1$$
$$= 150 + 484 + 192 - 16 = 810.$$

Example 7.24. *Using Newton's divided differences formula, evaluate f(8) and f(15) given :*

x :	4	5	7	10	11	13
$y = f(x)$:	48	100	294	900	1210	2028

(Anna, B. Tech., 2014)

Sol. The divided differences table is

x	$f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
4	48				
		52			
5	100		15		
		97		1	
7	294		21		0
		202		1	
10	900		27		0
		310		1	
11	1210		33		
		409			
13	2028				

Taking $x = 8$ in the Newton's divided difference formula, we obtain

$$f(8) = 48 + (8-4)\,52 + (8-4)(8-5)\,15 + (8-4)(8-5)(8-7)\,1$$
$$= \mathbf{448.}$$

Similarly $f(15) = 3150$.

Example 7.25. *Determine f(x) as a polynomial in x for the following data :*

x :	-4	-1	0	2	5
$y = f(x)$:	1245	33	5	9	1335.

(V.T.U., B. Tech., 2007)

Sol. The divided differences table is

x	$f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
-4	1245				
		-404			
			94		
-1	33			-14	
		-28			3
			10		
0	5			13	
		2			
			88		
2	9				
		442			
5	1335				

Applying Newton's divided difference formula

$$f(x) = f(x_0) + (x - x_0)\,[x_0, x_1] + (x - x_0)(x - x_1)\,[x_0, x_1, x_2] + \ldots$$

$$= 1245 + (x + 4)(-404) + (x + 4)(x + 1)(94)$$

$$+ (x + 4)(x + 1)(x - 0)(-14) + (x + 4)(x + 1)x(x - 2)(3)$$

$$= 3x^4 - 5x^3 + 6x^2 - 14x + 5.$$

Example 7.26. *Using Newton's divided difference formula, find the missing value from the table :*

x :	1	2	4	5	6
y :	14	15	5	...	9

Sol. The divided difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1	14			
		$\dfrac{15-14}{2-1} = 1$		
2	15		$\dfrac{-5-1}{4-1} = -2$	
		$\dfrac{5-15}{4-2} = -5$		$\dfrac{7/4+2}{6-1} = \dfrac{3}{4}$
4	5		$\dfrac{2+5}{6-2} = \dfrac{7}{4}$	
		$\dfrac{9-5}{6-4} = 2$		
6	9			

Newton's divided difference formula is

$$y = y_0 + (x - x_0)\,[x_0, x_1] + (x - x_0)(x - x_1)\,[x_0, x_1, x_2]$$
$$+ (x - x_0)(x - x_1)(x - x_2)\,[x_0, x_1, x_2, x_3] + \ldots\ldots$$
$$= 14 + (x - 1)(1) + (x - 1)(x - 2)(-2) + (x - 1)(x - 2)(x - 4)\left(\frac{3}{4}\right)$$

Putting $x = 5$, we get

$$y(5) = 14 + 4 + (4)(3)(-2) + (4)(3)(1)\left(\frac{3}{4}\right) = 3.$$

Hence missing value is 3.

PROBLEMS 7.4

1. Show that $\dfrac{\overset{3}{\Delta}}{bcd}\left(\dfrac{1}{a}\right) = -\dfrac{1}{abcd}$. *(Anna, B.E., 2015)*

2. Obtain the Newton's divided difference interpolating polynomial and hence find $f(6)$:

x :	3	7	9	10
$f(x)$:	160	120	72	63

(U.P.T.U., B. Tech., 2012)

3. Using Newton's divided differences interpolation, find $u(3)$, given that $u(1) = -26$, $u(2) = 12$, $u(4) = 256$, $u(6) = 844$. *(Anna, B.E., 2004)*

4. A thermocouple gives the following output for rise in temperature

Temp (°C)	0	10	20	30	40	50
Output (mV)	0.0	0.4	0.8	1.2	1.6	2.0

Find the output of thermocouple for 37°C temperature using Newton's divided difference formula.

5. Using Newton's divided difference interpolation, find the cubic polynomial of the given data :

x :	0	1	2	5
$f(x)$:	2	3	12	147

Hence find $f(4)$. *(Anna, B.E., 2016)*

6. For the following table, find $f(x)$ as a polynomial in x using Newton's divided difference formula :

x :	5	6	9	11
$f(x)$:	12	13	14	16

7. Using the following data, find $f(x)$ as a polynomial in x :

x :	-1	0	3	6	7
$f(x)$:	3	-6	39	822	1611

(U.P.T.U., B. Tech., 2012)

8. The observed values of a function are respectively 168, 120, 72 and 63 at the four positions 3, 7, 9 and 10 of the independent variable. What is the best estimate for the value of the function at the position 6.

9. Find the equation of the cubic curve which passes through the points $(4, -43)$, $(7, 83)$, $(9, 327)$ and $(12, 1053)$.

10. Find the missing term in the following table using Newton's divided difference formula

x :	0	1	2	3	4
y :	1	3	9	...	81

8.5. NEWTON-COTES QUADRATURE FORMULA

Let $$I = \int_a^b f(x)\, dx$$

where $f(x)$ takes the values $y_0, y_1, y_2, \ldots.. y_n$ for $x = x_0, x_1, x_2, \ldots. x_n$.

Let us divide the interval (a, b) into n sub-intervals of width h so that $x_0 = a$, $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, ... $x_n = x_0 + nh = b$. Then

$$I = \int_{x_0}^{x_0+nh} f(x)\, dx = h \int_0^n f(x_0 + rh)\, dr, \quad \text{Putting } x = x_0 + rh,\ dx = h\,dr$$

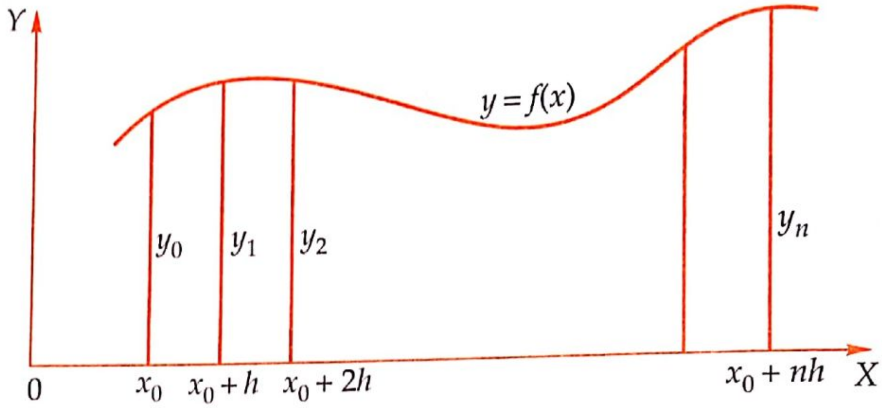

Fig. 8.1

$$= h \int_0^n \left[y_0 + r\,\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_0 + \frac{r(r-1)(r-2)(r-3)(r-4)}{5!}\Delta^5 y_0 + \frac{r(r-1)(r-2)(r-3)(r-4)(r-5)}{6!}\Delta^6 y_0 + \ldots\ldots \right] dr$$

[by Newton's forward interpolation formula]

Integrating term by term, we obtain

$$\int_{x_0}^{x_0+nh} f(x)\, dx = nh \left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \left(\frac{n^4}{5} - \frac{3n^3}{2} + \frac{11n^2}{3} - 3n\right)\frac{\Delta^4 y_0}{4!} + \left(\frac{n^5}{6} - 2n^4 + \frac{35n^3}{4} - \frac{50n^2}{3} + 12n\right)\frac{\Delta^5 y_0}{5!} + \left(\frac{n^6}{7} - \frac{15n^5}{6} + 17n^4 - \frac{225n^3}{4} + \frac{274n^2}{3} - 60n\right)\frac{\Delta^6 y_0}{6!} + \ldots \right] \quad \ldots(1)$$

This is known as *Newton-Cotes quadrature formula*. From this general formula, we deduce the following important quadrature rules by taking $n = 1, 2, 3, \ldots\ldots$

I. Trapezoidal rule. Putting $n = 1$ in (1) and taking the curve through (x_0, y_0) and (x_1, y_1) as a straight line (Fig. 8.2) *i.e.* a polynomial of first order so that differences of order higher than first become zero, we get

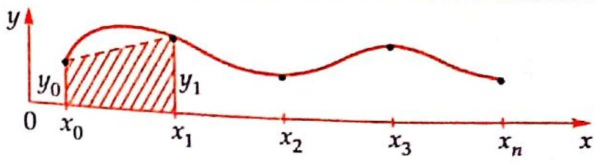

Fig. 8.2

$$\int_{x_0}^{x_0+h} f(x)\,dx = h\left(y_0 + \frac{1}{2}\Delta y_0\right) = \frac{h}{2}(y_0 + y_1)$$

Similarly $$\int_{x_0+h}^{x_0+2h} f(x)\,dx = h\left(y_1 + \frac{1}{2}\Delta y_1\right) = \frac{h}{2}(y_1 + y_2)$$

....

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx = \frac{h}{2}(y_{n-1} + y_n)$$

Adding these n integrals, we obtain

$$\int_{x_0}^{x_0+nh} \mathbf{f(x)\,dx} = \frac{\mathbf{h}}{\mathbf{2}}\,[(\mathbf{y_0 + y_n}) + \mathbf{2(y_1 + y_2 + \dots + y_{n-1})}] \qquad ...(2)$$

This is known as the *trapezoidal rule*.

Obs. The area of each strip (trapezium) is found separately. Then the area under the curve and the ordinates at x_0 and x_n is approximately equal to the sum of the areas of the n trapeziums.

II. Simpson's one-third rule. Putting $n = 2$ in (1) above and taking the curve through (x_0, y_0), (x_1, y_1) and (x_2, y_2) as a parabola (Fig. 8.3) *i.e.* a polynomial of second order so that differences of order higher than second vanish, we get

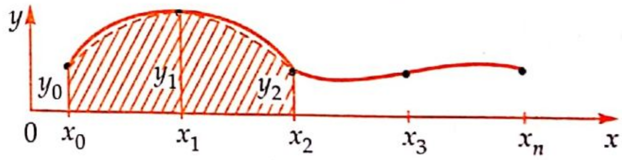

Fig. 8.3

$$\int_{x_0}^{x_0+2h} f(x)\,dx = 2h(y_0 + \Delta y_0 + \tfrac{1}{6}\Delta^2 y_0) = \frac{h}{3}(y_0 + 4y_1 + y_2)$$

Similarly $$\int_{x_0+2h}^{x_0+4h} f(x)\,dx = \frac{h}{3}(y_2 + 4y_3 + y_4)$$

....

$$\int_{x_0+(n-2)h}^{x_0+nh} f(x)\,dx = \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n), \; n \text{ being even.}$$

Adding all these integrals, we have when n is even

$$\int_{x_0}^{x_0+nh} \mathbf{f(x)\,dx} = \frac{\mathbf{h}}{\mathbf{3}}\,[(\mathbf{y_0 + y_n}) + \mathbf{4(y_1 + y_3 + \dots + y_{n-1})} + \mathbf{2(y_2 + y_4 + \dots + y_{n-2})}] \qquad ...(3)$$

This is known as the *Simpson's one-third rule* or simply *Simpson's rule* and is most commonly used.

Obs. While applying (3), *the given interval must be divided into even number of equal sub-intervals*, since we find the area of two strips at a time.

III. Simpson's three-eighth rule. Putting $n = 3$ in (1) above and taking the curve through $(x_i, y_i) : i = 0, 1, 2, 3$ as a polynomial of third order (Fig. 8.4) so that differences above the third order vanish, we get

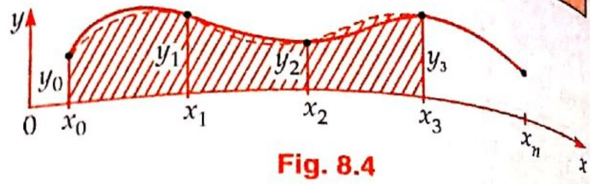

Fig. 8.4

$$\int_{x_0}^{x_0+3h} f(x)\,dx = 3h\left(y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right)$$

$$= \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + y_3)$$

Similarly,

$$\int_{x_0+3h}^{x_0+5h} f(x)\,dx = \frac{3h}{8}(y_3 + 3y_4 + 3y_5 + y_6) \text{ and so on.}$$

Adding all such expressions from x_0 to $x_0 + nh$, where n is a multiple of 3, we obtain

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \ldots + y_{n-1}) + 2(y_3 + y_6 + \ldots + y_{n-3})] \quad \ldots(4)$$

which is known as *Simpson's three-eighth rule.*

Obs. While applying (4), *the number of sub-intervals should be taken as multiple of* 3.

IV. Boole's rule. Putting $n = 4$ in (1) above and taking the curve (x_i, y_i), $i = 0, 1, 2, 3, 4$ as a polynomial of fourth order (Fig. 8.5) and neglecting all differences above the fourth, we obtain

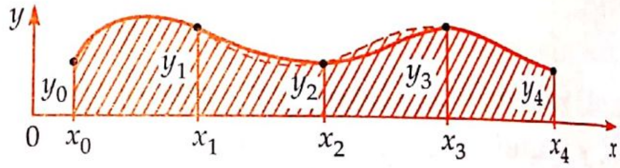

Fig. 8.5

$$\int_{x_0}^{x_0+4h} f(x)\,dx = 4h\left(y_0 + 2\Delta y_0 + \frac{5}{3}\Delta^2 y_0 + \frac{2}{3}\Delta^3 y_0 + \frac{7}{90}\Delta^4 y_0\right)$$

$$= \frac{2h}{45}(7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4)$$

Similarly

$$\int_{x_0+4h}^{x_0+8h} f(x)\,dx = \frac{2h}{45}(7y_4 + 32y_5 + 12y_6 + 32y_7 + 7y_8) \text{ and so on.}$$

Adding all these integrals from x_0 to $x_0 + nh$, where n is a multiple of 4, we get

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{2h}{45}(7y_0 + 32y_1 + 12y_2 + 32y_3 + 14y_4 + 32y_5 + 12y_6 + 32y_7 + 14y_8 + \ldots \quad \ldots(5)$$

This is known as *Boole's rule.*

Obs. While applying (5), the number of sub-intervals should be taken as a multiple of 4.

V. Weddle's rule. Putting $n = 6$ in (1) above and neglecting all differences above the sixth, we obtain

$$\int_{x_0}^{x_0+6h} f(x)\,dx = 6h\left(y_0 + 3\Delta y_0 + \frac{9}{2}\Delta^2 y_0 + 4\Delta^3 y_0 + \frac{123}{60}\Delta^4 y_0 + \frac{11}{20}\Delta^5 x_0 + \frac{1}{6}\cdot\frac{41}{140}\Delta^6 y_0\right)$$

If we replace $\frac{41}{140}\Delta^6 y_0$ by $\frac{3}{10}\Delta^6 y_0$, the error made will be negligible.

$$\therefore \quad \int_{x_0}^{x_0+6h} f(x)\,dx = \frac{3h}{10}(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6)$$

Similarly

$$\int_{x_0+6h}^{x_0+12h} f(x)\,dx = \frac{3h}{10}(y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}) \text{ and so on.}$$

Adding all these integrals from x_0 to $x_0 + nh$, where n is a multiple of 6, we get

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{10}(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 + y_8 + \ldots) \quad \ldots(6)$$

This is known as *Weddle's rule*.

Obs. While applying (6), *the number of sub-intervals should be taken as a multiple of 6.*

Weddle's rule is generally more accurate than any of the others. Of the two Simpson rules, the 1/3 rule is better.

Example 8.10. *Evaluate* $\int_0^6 \frac{dx}{1+x^2}$ *by using (i) Trapezoidal rule,*

(ii) Simpson's 1/3 rule, (Mumbai, B.E., 2005)

(iii) Simpson's 3/8 rule, (J.N.T.U., B. Tech., 2008)

(iv) Weddle's rule and compare the results with its actual value. (V.T.U., B.E., 2013)

Sol. Divide the interval (0, 6) into six parts each of width $h = 1$. The values of $f(x) = \frac{1}{1+x^2}$ are given below :

x	0	1	2	3	4	5	6
$f(x)$ $= y$	1 y_0	0.5 y_1	0.2 y_2	0.1 y_3	0.0588 y_4	0.0385 y_5	0.027 y_6

(*i*) By Trapezoidal rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{2}[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$= \frac{1}{2}[(1 + 0.027) + 2(0.5 + 0.2 + 0.1 + 0.0588 + 0.0385)] = 1.4108.$$

(*ii*) By Simpson's 1/3 rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{1}{3}[(1 + 0.027) + 4(0.5 + 0.1 + 0.0385) + 2(0.2 + 0.0588)] = 1.3662.$$

(*iii*) By Simpson's 3/8 rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3}{8}[(1 + 0.027) + 3(0.5 + 0.2 + 0.0588 + 0.0385) + 2(0.1)] = 1.3571.$$

(iv) By Weddle's rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{10}\ [y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6]$$

$$= 0.3[1 + 5(0.5) + 0.2 + 6(0.1) + 0.0588 + 5(0.0385) + 0.027] = 1.3735$$

Also $\displaystyle\int_0^6 \frac{dx}{1+x^2} = \left| \tan^{-1} x \right|_0^6 = \tan^{-1} 6 = 1.4056$

This shows that the value of the integral found by Weddle's rule is the nearest to the actual value followed by its value given by Simpson's 1/3 rule.

Example 8.11. *Evaluate the integral $\int_0^1 \frac{x^2}{1+x^3}\,dx$ using Simpson's 1/3rd rule. Compare the error with the exact value $\log 2^{1/3}$.* (Anna, B. Tech., 2016)

Sol. Let us divide the interval (0, 1) into 4 equal parts so that $h = 0.25$. Taking $y = \frac{x^2}{(1+x^3)}$, we have

x :	0	0.25	0.50	0.75	1.00
y :	0	0.06153	0.22222	0.39560	0.5
	y_0	y_1	y_2	y_3	y_4

By Simpson's 1/3rd rule, we have

$$\int_0^1 \frac{x^2}{1+x^3}\,dx = \frac{h}{3}\ [(y_0 + y_4) + 2(y_2) + 4(y_1 + y_3)]$$

$$= \frac{0.25}{3}\ [(0 + 0.5) + 2\ (0.22222) + 4\ (0.06153 + 0.3956)]$$

$$= \frac{0.25}{3}\ [0.5 + 0.44444 + 1.82852] = 0.23108$$

Also $\displaystyle\int_0^1 \frac{x^2}{1+x^3}\,dx = \frac{1}{3}\left| \log(1+x^3) \right|_0^1 = \frac{1}{3}\log_e 2 = 0.23105$

Thus the error $= 0.23108 - 0.23105 = -0.00003$ and $\log 2^{1/3} = 0.23105$.

Example 8.12. *Use the Trapezoidal rule to estimate the integral $\int_0^2 e^{x^2}\,dx$ taking the number 10 intervals.* (U.P.T.U., B. Tech., 2008)

Sol. Let $y = e^{x^2}$, $h = 0.2$ and $n = 10$.

The values of x and y are as follows :

x :	0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
y :	1	1.0408	1.1735	1.4333	1.8964	2.1782	4.2206	7.0993	12.9358	25.5337	54.5981
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}