Sets Theory

Set: A set is a well-defined collection of objects. The objects are called the elements or members of the set. These elements may be anything: numbers, people, letters of the alphabet etc.

e.g. The vocoels of alphabets.

Representation of a set:

(i) Roster or Tabular form → $A = \{a, e, i, o, u\}$

(ii) Rule method form or set Builder form →

e.g. The set of a, e, i, o, u can be written as

$A = \{x : x$ is a vocel of english alphabet$\}$

Null set: A set which contains no elements. Also called as empty set or void set.

$A = \{x : x^2 + 4 = 0, x$ is a real no.$\}$

It is denoted by symbol ϕ.

Subset: $A \subseteq B$ then if $x \in A$ then $x \in B$.

If A and B are sets such that every element of A is also an element of B, then A is said to be subset of B and is denoted by $A \subseteq B$. In other words,

$A \subseteq B$ if $x \in A$ and $x \in B$.

Equal set: Two sets A and B are said be equal iff every element of A is an element of B and consequently every element of B is an element of A. i.e. $A \subseteq B$ and $B \subseteq A$.

It is written as $A = B$.

Union: The union of two sets A and B denoted by A ∪ B.

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

Intersection: The intersection of two sets A and B is denoted by $A \cap B$.

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

Complements: $A^c = A' = U - A$

$$A' = \{x : x \in U \text{ and } x \notin A\}$$

Algebra of Sets:

(1) Idempotent laws. (a) $A \cup A = A$

(b) $A \cap A = A$

Proof → Let $x \in A \cup A$. Then

$x \in A \cup A \Rightarrow x \in A$ or $x \in A$

$x \in A \cup A \Rightarrow x \in A$ $\quad A \cup A \subseteq A$ — (I)

Conversely, let $x \in A$ Then

$x \in A \Rightarrow x \in A$ or $x \in A$

$\Rightarrow x \in A \cup A$

Thus $A \subseteq A \cup A$ — (II)

from eqn (I) & (II) we have —

$$\boxed{A \cup A = A}$$

Demorgan's law:

(a) $(A \cup B)' = A' \cap B'$

(b) $(A \cap B)' = A' \cup B'$

Proof: $x \notin (A \cup B) \Rightarrow x \notin A$ and $x \notin B$.

$x \notin (A \cap B) \Rightarrow x \notin A$ or $x \notin B$.

Let $x \in (A \cup B)'$ Then

$x \in (A \cup B)' \Rightarrow x \notin (A \cup B)$

$\Rightarrow x \notin A$ and $x \notin B$.

$\Rightarrow x \in A'$ and $x \in B'$.

$x \in (A \cup B)' \Rightarrow x \in (A' \cap B')$

$\Rightarrow (A \cup B)' \subseteq A' \cap B'$ — (I)

Conversely let $x \in A' \cap B'$

$x \in A' \cap B' \Rightarrow x \in A'$ and $x \in B'$

$\Rightarrow x \notin A$ and $x \notin B$.

$\Rightarrow x \notin (A \cup B)$

$x \in A' \cap B' \Rightarrow x \in (A \cup B)'$

$A' \cap B' \subseteq (A \cup B)'$ — (II)

Thus, from (I) & (II) we obtain $(A \cup B)' = A' \cap B'$ Proved.

(4.) Prove that $A - (B \cap C) = (A - B) \cup (A - C)$.

Proof: Let $x \in A - (B \cap C)$ Then

$x \in A - (B \cap C) = x \in A$ and $x \notin (B \cap C)$

$= x \in A$ and $(x \notin B$ or $x \notin C)$

$= x \in A$ and $x \notin B$ or $x \in A$ and $x \notin C$

$= x \in (A - B)$ or $x \in (A - C)$

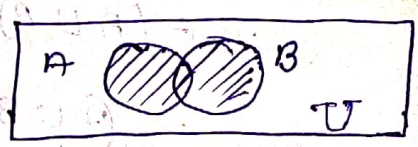
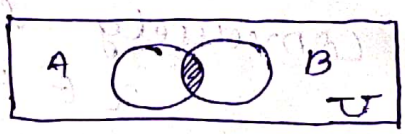
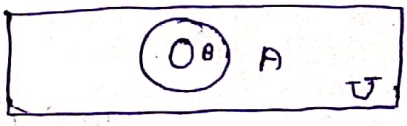
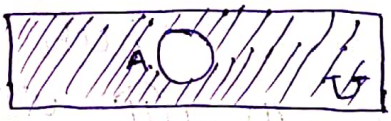
$= x \in (A - B) \cup (A - C)$

Also, $x \in (A - B) \cup (A - C)$

Thus, $A - (B \cap C) = (A - B) \cup (A - C)$

Venn Diagrams

A venn diagram is a pictorial representation of sets which are used to show relationship between sets & and operations on them.

Set operations	Symbol	Venn Diagram
1. The union of set A and set B.	$A \cup B$	
2. The intersection of set A and set B.	$A \cap B$	
3. Set B is proper subset of A.	$B \subset A$	
4. The complement of set A.	A' or A^c $A' = U - A$	

Example: Let $A = \{1,2,3,4,5,6\}$, $B = \{3,6,8,12,17,18\}$ and $C = \{2,3,5,6,12,8\}$

Then $A \cup B = \{1,2,3,4,5,6,8,12,17,18\}$

$A \cap B = \{3,6\}$ $\quad A - B = \{1,2,4,5\}$

For any two sets A and B we have the following formula

(1) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(2) $n(A - B) + n(A \cap B) = n(A)$

or

$n(B - A) + n(A \cap B) = n(B)$

Eg. In a class of 25 students, 12 have taken mathematics, 8 have taken mathematics but not biology. Find the no. of students who have taken mathematics and biology and those who have taken biology but not mathematics.

Solu: $n(A \cup B) = 25$ $n(A) = 12$ $n(A - B) = 8$ $n(A \cap B) = ?$ $n(B - A) = ?$

$n(A - B) + n(A \cap B) = n(A)$

$8 + n(A \cap B) = 12$

$\boxed{n(A \cap B) = 4}$

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$25 = 12 + n(B) - 4$

$\boxed{17 = n(B)}$

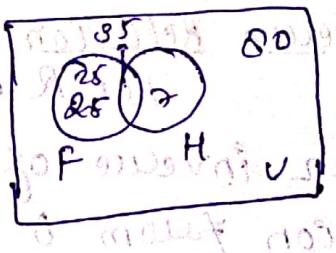
$n(B - A) + n(A \cap B) = n(B)$

$n(B - A) + 4 = 17$

$\boxed{n(B - A) = 13}$

Eg. Out of 80 students in a class 60 play football, 53 play hockey and 25 both the games. How many students do not play these games?

Solu.



Total no. of students who play both games: $80 - (25 + 35 + 18)$

$= 80 - 78 = 2$

(Relation)

Cautesian product — ordered pair (a, b)

A and B

$A \times B = \{(a, b) : a \in A, b \in B\}$

Eg. $A = \{a, b\}$ $B = \{1\}$

then $A \times B = \{(a, 1), (b, 1)\}$ i.e. $A \times B \neq B \times A$

$B \times A = \{(1, a), (1, b)\}$

⇒ Let A and B be two sets. A relation from A to B is a subset of the cartesian product $A \times B$.

aRb read as 'a' is related to b by R i.e. $(a,b) \in R$.

$(a,b) \notin R$ i.e. a is not related to b.

e.g. Let $A = \{2,3,4\}$ and $B = \{3,4,5\}$. List the elements of each relation R defined below and domain & Range.

Solve (a) $a \in A$ is related to $b \in B$, i.e. aRb iff $a < b$.

(b) aRb iff a and b are both odd numbers.

Solve (a) $R = \{(2,3), (2,4), (2,5), (3,4), (3,5), (4,5)\}$

$\text{domain}(R) = \{2,3,4\}$ $\quad \text{Range}(R) = \{3,4,5\}$

(b) $R = \{(3,3), (3,5)\}$ $\quad \text{domain}(R) = \{3\}$ $\quad \text{Range}(R) = \{3,5\}$

Types of Relations in a set

(1) Inverse Relation:

Let R be any relation from a set A to a set B then inverse of R is denoted by R^{-1} and is defined by relation from B to A

eg. $R = \{(a,b), (c,d), (e,f)\}$

Then $R^{-1} \{(b,a), (d,c), (f,e)\}$

$$\boxed{(R^{-1})^{-1} = R}$$

Properties of relations :

(1) Reflexive Relation →

A relation R on a set 'A' is reflexive if aRa, for every $a \in A$ i.e. $(a,a) \in R \ \forall \ a \in A$.

eg. $R_1 = \{(x,y) \in R^2 : x \leq y\}$ is a reflexive because $x \leq x \ \forall \ x \in R$

(4) Irreflexive Relation → A relation R on a set A is irreflexive if $\forall a \in A$, $(a,a) \notin R$.

eg. The relation $R = \{(x,y) \in R^2, x<y\}$ is irreflexive since $x<x$ for no $x \in R$.

(5) Symmetric Relation →

A relation R on a set A is symmetric if whenever $(a,b) \in R$ then $(b,a) \in R$ i.e. $aRb \Rightarrow bRa$.

eg. $R = \{(x,y) \in R^2 : x^2+y^2=1\}$ is a symmetric relation on R ∵ if $x^2+y^2=1$, then $y^2+x^2=1$ i.e. $xRy \Rightarrow yRx$

(6) Antisymmetric Relation →

If aRb and $b \neq a$, $\Rightarrow a=b$

eg. $R = \{(x,y) \in R^2 : x \leq y\}$

(7) Transitive Relation →

If aRb and bRc $\Rightarrow$ aRc

eg. $a<b$ & $b<c \Rightarrow a<c$

Equivalence Relation :

A relation on a set A is called an equivalence relation if it is reflexive, symmetric and transitive. That is R is an equivalence relation on if has the following properties –

(i) $(a,a) \in R \ \forall a \in A$ (reflexive)

(ii) $(a,b) \in R$ implies $(b,a) \in R$ (symmetric)

(iii) (a,b) and $(b,c) \in R$ implies $(a,c) \in R$ (transitive.

eg. If R be a relation in the set of integers Z defined by $R = \{(x,y) : x \in Z, y \in Z, (x-y) \text{ is divisible by } 6\}$

Sol. Let $x \in Z$. Then $x - x = 0$ and 0 is divisible by 6.

Therefore, $xRx \ \forall \ x \in Z$. Hence, R is reflexive.

Again, $xRy \Rightarrow (x-y)$ is divisible by 6.

$\Rightarrow -(y-x)$ " " " 6.

$\Rightarrow (y-x)$ " " " 6.

Hence R is symmetric.

xRy and $yRz \Rightarrow (x-y)$ and $(y-z)$ are divisible by 6.

$\Rightarrow [(x-y)+(y-z)]$ is divisible by 6.

$\Rightarrow (x-z)$ is divisible by 6.

$\Rightarrow xRz$

Hence, R is transitive. Thus R is an equivalence relation.

Theorem :

Let R and S be relation from A to B, show that

(i) if $R \subseteq S$ then $R^{-1} \subseteq S^{-1}$.

(ii) $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$

(iii) $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$

Proof: (i) Let $R \subseteq S$ if $(a,b) \in R^{-1}$ then $(b,a) \in R$ ... ①

$\because R \subseteq S$ so, $(b,a) \in S$

Then $(a,b) \in S^{-1}$ — ②

from ① & ②,

$$\boxed{R^{-1} \subseteq S^{-1}}$$

(ii) Let $(a,b) \in (R \cap S)^{-1}$ Then
Then, $(b,a) \in R \cap S$. So, $(b,a) \in R$ and $(b,a) \in S$
Next, $(a,b) \in R^{-1}$ and $(a,b) \in S^{-1}$
So, $(a,b) \in (R^{-1} \cap S^{-1})$
So $(R^{-1} \cap S^{-1}) \subseteq R^{-1} \cap S^{-1}$ — (1)
$(R \cap S)^{-1} = R^{-1} \cap S^{-1}$
Conversely, let $(a,b) \in R^{-1} \cap S^{-1}$ Then $(a,b) \in R^{-1}$ & $(a,b) \in S^{-1}$
Hence $(b,a) \in R$ and $(b,a) \in S$. So, $(b,a) \in R \cap S$. Now we have $(a,b) \in (R \cap S)^{-1}$. So $R^{-1} \cap S^{-1} \subseteq (R \cap S)^{-1}$ — (2)

R ∩ S

* Theorem: Suppose R and S are relations on a set A. Prove that
(i) if R and S are reflexive then $R \cup S$ and $R \cap S$ are reflexive.
(ii) If R and S are symmetric then $R \cup S$ and $R \cap S$ are symmetric

Composition of Relations:
Let A, B, C be sets. Let R be a relation from A to B and let S be a relation from B to C. Then composition of R and S, denoted by $R \circ S$, is the relation consisting of ordered pairs (a,c) where $a \in A$, $c \in C$ and for which $b \in B$. such that $(a,b) \in R$ and $(b,c) \in S$ Thus.

$$R \circ S = \{(a,c) \in A \times C : \text{for some } b \in B, (a,b) \in R, (b,c) \in S\}$$

$R \circ R = R^2$ $\quad R \circ R \circ R = R^3$ $\quad a (R \circ S) c$

Example: Let $A = \{1,2,3\}$, $B = \{p,q,r\}$, $C = \{x,y,z\}$ and
let $R = \{(1,p), (1,r), (2,q), (3,q)\}$ and
$S = \{(p,y), (q,x), (r,z)\}$ compute $R \circ S$.

Soln

The ordered pair $(1,p)$ in R and (p,y) in S produce the ordered pair $(1,y)$ in ROS.

$(1,2) \in R$ & $(2,3) \in S$ then $(1,3) \in ROS$

$(2,4) \in R$ & $(4,2) \in S$ then $(2,2) \in ROS$

$(3,4) \in R$ & $(4,2) \in R$ then $(3,2) \in ROS$

Thus, $ROS = \{(1,3), (2,2), (1,4), (3,2)\}$

Theorem:

Let A, B, C and D be four non-empty sets. Let R_1 be a relation from A to B, R_2 is a relation from B to C and R_3 is a relation from C to D. Then

(i) $(R_1 \circ R_2)^{-1} = R_2^{-1} \circ R_1^{-1}$

(ii) $(R_1 \circ R_2) \circ R_3 = R_1 \circ (R_2 \circ R_3)$

Proof:

(i) Since R_1 is a relation from A to B & R_2 is a relation from B to C, then $R_1 \circ R_2$ is a relation from A to C. Hence $(R_1 \circ R_2)^{-1}$ is a relation from C to A. — (1)

Again, let R_1^{-1} is a relation from B to A

R_2^{-1} is a relation from C to B

Then $(R_2^{-1} \circ R_1^{-1})$ is a relation from C to A. — (2)

Hence from (1) & (2),

for $z \in C$ & $x \in A$

$z\,(R_1 \circ R_2)^{-1}\,x \iff x\,(R_1 \circ R_2)\,z$

$\iff$ iff $y \in B$ s.t. $(x\,R_1\,y \wedge y\,R_2\,z)$

$\iff$ iff $y \in B$ s.t. $(y\,R_2\,z \wedge x\,R_1\,y)$

$\iff \exists\, y \in B$ s.t. $(z\,R_2^{-1}\,y \wedge y\,R_1^{-1}\,x)$

$\iff z\,(R_2^{-1} \circ R_1^{-1})\,x$ Hence

$(R_1 \circ R_2)^{-1} = R_2^{-1} \circ R_1^{-1}$

Thm. If R is a relation on set A, then R is transitive iff $R^2 \subseteq R$

Proof: Let R be transitive and $(x,z) \in R^2$. But by def^n of R^2, $R \circ R$ $\exists\, y \in A$ s.t. $(x,y) \in R$ and also $(y,z) \in R$ so $(x,z) \in R$. Hence

$R^2 \subseteq R$.

Conversely, suppose $R^2 \subseteq R$. Let $(x,y) \in R$ $(y,z) \in R$. Then $(x,z) \in R^2$ and hence in R. This proves that R is transitive.

#functions

A function is a relation between sets that associates to every element of a first set exactly one element of the second set.

E.g. The following relation is a function

$\{(-1,0), (0,-3), (2,-3), (3,0), (4,5)\}$

(ii) $\{(6,10), (-7,3), (0,4), (6,-4)\}$ → Not a function

The statement "f is a function from A to B" is represented by

$f: A \to B$ or $A \xrightarrow{f} B$

E.g. Let $A = \{1,2,3,4,5\}$ and $B = \{0,1,2,3,4,5,6,7,8,9,10,11,12,13\}$

(i) $f = \{(1,1), (2,10), (3,7), (4,9), (5,12)\}$

Range $(f) = \{1,0,7,9,12\}$

(ii) $f = \{(1,1), (2,3), (4,7), (5,12)\}$ then f is not a func^n because 3 of A has no image in B.

* Let f and g be two functions. Then f and g are said to be equal if (i) dom f = dom g

(ii) co-domain of f = co-domain of g.

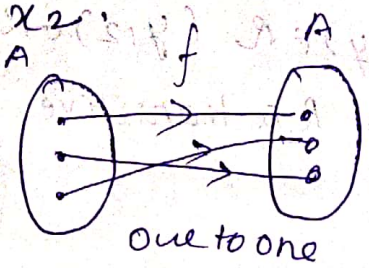
and

(iii) $f(x) = g(x)$ if $x \in$ dom f

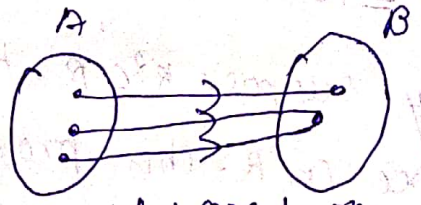
Types of functions

(1) One-to-one function:

A function from A to B is called one-to-one if for all elements $x_1, x_2 \in A$ such that $f(x_1) = f(x_2)$ implies $x_1 = x_2$.



One to one



Not one to one.

Example:

If $f(x) = 3x - 2$, then show that f is one-one

Sol: $f(x_1) = 3x_1 - 2$ $\quad f(x_2) = 3x_2 - 2$

$f(x_1) = f(x_2)$ $\quad 3x_1 - 2 = 3x_2 - 2$

$\boxed{x_1 = x_2}$

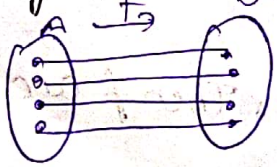
(2) Many-one function:

A function f from A to B is said to be many-one if and only if two or more elements of A have same image in B.

Eg: Let $f(x) = x^2$, x is any real number and $f: R \to R$, then f is many one function.

(3) Onto function:

A function f from A to B is called onto (surjective) if every element of B is the image of some element in A, that is, Range of B = codomain of B.



In order to check whether of $y = f(x)$ from a set A to B is onto or not, write x in terms of y and set if for every $y \in B$, $x \in A$; if so, it is onto. otherwise, it is into

$f: A \to B$

range $f = B$

Perto $\to f(x) = 2x$ $f: N \to N$

Group Theory

If $f: a \times a \to a$

$$a \times a = \{ (a, b) \ , \ a \in a, b \in a \}$$

Binary operations:

If $f: a \times a \to a$, then f is said to be binary operation on a. The symbols $+, \cdot, \circ, *$ are used to denote binary operations on a set.

Thus, $+$ will be a binary operation on a iff $(a+b) \in a$; if $a, b \in a$.

Algebraic Structure:

$(a, *)$, $(R, *, \cdot)$

Laws of binary operations:

1.) **Associative law** → Let a be a non-empty set then

$$a * (b * c) = (a * b) * c \quad \text{if } a, b \in a$$

2.) **Commutative law** →

$$a * b = b * a \quad \text{if } a, b \in a \quad \text{eg. } (N, +)$$

3.) **Identity element** →

Let $e \in a$ Then

$$a * e = e * a = a \quad \forall \ a \in a$$

4.) **Inverse element** →

Consider a set a having identity element e w.r.t. B.O. $*$

Then for each element $a \in a$ $\exists$ $b \in a$

$$a * b = b * a = e$$

 # Group: Let $(a, *)$ be an algebraic structure, where $*$ is a binary operator, then $(a, *)$ is called a Group under this operation if the following conditions are satisfied-

(1) closure law → $a * b \in a \quad \forall \ a, b \in a$

(ii) Associative law → $a*(b*c) = (a*b)*c \ \forall\ a,b,c \in G$

(iii) Identity element → for some $e \in G$, $e*a = a*e = a, a \in G$

(iv) Inverse element → for each $a \in G$ if $a' \in G$ such that $a*a' = a'*a = e$

Note: A Group G is said to be Abelian if $a*b = b*a \ \forall$ $a,b \in G$ (In addition to above prop. it also holds commutative property)

Ques. Show the set $\{1, -1, i, -i\}$ is an abelian group w.r.t. multiplication.

Soln. Identity $= 1$ ✓ $\quad \frac{1}{i} \times \frac{i}{i^2} = \frac{i}{i^2}$, $-i$. Inverse

Theorem:

(Cancellation Law) → If $(G, *)$ is a group and $a,b \in G$ then

(i) $a*b = a*c \Rightarrow b = c$ (left canc. law)

(ii) $b*a = c*a \Rightarrow b = c$ (right canc. law)

Proof:

(i) $\because a \in G$ so from $a*b = a*c$ we can write

$a^{-1}*(a*b) = a^{-1}*(a*c)$ $\quad e*a = a = e*a$

$\Rightarrow (a^{-1}*a)*b = (a^{-1}*a)*c$ $\quad a^{-1}*a = e = a*a^{-1}$

$\Rightarrow e*b = e*c$

$\Rightarrow \boxed{b = c}$

Thm: In a group $(G, *)$ (i) $(a^{-1})^{-1} = a$

(ii) $(ab)^{-1} = b^{-1}a^{-1}$

*Show that if a,b are arbitrary element of a group G then $(ab)^2 = a^2b^2$ iff G is abelian.

Proof: Let $a,b \in G$ suppose

$(ab)^2 = a^2b^2$ — (1)

we have to show that $ab = ba \ \forall\ a,b \in G$

$(ab)^2 = a^2b^2$

$(ab)(ab) = (aa)(bb)$ So by left cancell. of Right cancellation we have

$\Rightarrow a(ba)b = a(ab)b$ $\quad \boxed{ba = ab}$

Hence, G is abelian.

Conversely: suppose G is an abelian group so that $ab = ba \; \forall \; a, b \in G$.

To prove that $(ab)^2 = a^2 b^2$

$\therefore (ab)^2 = (ab)(ab) = a(ba)b$ (from (1))

$= (aa)(bb) = a^2 b^2$

Hence, Proved.

Subgroup:

If the following conditions are satisfied –

(i) $a * b \in S \; \forall \; a, b \in S$

(ii) $a * (b * c) = (a * b) * c \; \forall \; a, b, c \in S$

(iii) for some $e \in S$, $e * a = a * e \; \forall \; a \in S$.

Let $(G, *)$ be a group and H is a subgroup of G. $(H, *)$ is said to be subgroup of G if $(H, *)$ is also group by itself.

Note: Since set is a subset of itself. Therefore, if G is a group of G then G itself is a subgroup of G. The set $\{e\}$ is also a subgroup of G. These two subgroups $(G, *)$ and $(e, *)$ of G is called improper or trivial subgroups others are proper or non-trivial.

E.g. The multiplicative group $\{-1, 1\}$ is a subgroup of the multiplicative group $\{1, -1, i, -i\}$

Theorem:

A non-empty subset of H of a group G is a subgroup of G iff

(i) $a \in H, \; b \in H \Rightarrow a * b \in H$

(ii) $a \in H \Rightarrow a^{-1} \in H$