(1991) The Minimum Steiner Tree Problem

Peter Gordon, Casey Cao-Son, & Brennan Truong

California State University, Fullerton

Mathematical Modeling, Semester Project

peter.gordon@csu.fullerton.edu
dangkhoa27@csu.fullerton.edu
brennantruong@gmail.com

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Introduction / Goal

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- **Goal:** Minimize cost and delay by reducing amount of cabling.

Definition: Weighted Graph / Adjacency

Weighted Graph

- A *weighted graph* $G = (V, E)$ is a pair of a set of points (called *vertices*) V and the set of *edges*, E, where each edge in E is a 3-tuple consisting of two vertices in V combined with a nonnegative real number *weight*.
  That is, each edge is an element of the form $(v_a, v_b, w_{a,b})$ with v_a, $v_b \in V$ and $w_{a,b} \in \{x \in \mathbb{R} \,|\, x > 0\}$.

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Adjacent Vertices

- Given a weighted graph $G = (V, E)$, two vertices v_1 and v_2 in V are *adjacent* provided that there exists an edge between them.

Definition: Connectedness / Vertex Order

Connected Vertices

- Any two vertices v_a and v_b are *connected* provided that there exists an ordered sequence P of vertices $(v_{P_0} = v_a, v_{P_1}, v_{P_2}, \ldots, v_{P_{n-1}}, v_{P_n} = v_b)$ such that every sequential pair of vertices is adjacent. That is, v_a is adjacent to v_{P_1}, v_{P_1} is adjacent to v_{P_2}, and so on.

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Vertex Order

Given a weighted graph $G = (V, E)$, The *order (degree) of a vertex* v is the number of edges which connect v to another vertex in G.

Definition: MST / Rectilinear Distance

Minimum Spanning Tree

- Given a connected graph $G = (E, V)$, the *spanning tree* T of G is a connected graph with the same vertices as G, whose edges form a subset of E, and in which each pair of vertices is connected through exactly one path.

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Rectilinear Distance

The *rectilinear distance*, d_1, between two points is the sum of the absolute values of the difference of their like coordinates.

$$d_1(p, q) = |p_x - q_x| + |p_y - q_y|$$

Problem Statement

The General Minimum Rectilinear Steiner Tree Problem

(#3) Given a set of Cartesian points $S = \{p_1, p_2, \ldots, p_n\}$, find the minimum spanning tree, using rectilinear distances, which connects all points in S using only horizontal and vertical line segments, adding any intermediary points as needed.

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Specific Data

(#1) Find the minimal-cost rectilinear spanning tree for a network with the following nine stations:

$$a(0,15), \quad b(5,20), \quad c(16,24), \quad d(20,20), \quad e(33,25)$$

$$f(23,11), \quad g(35,7), \quad h(25,0), \quad i(10,3) \quad .$$

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- Questions answered out of their given order?
- Creating a general solution first makes the other two "plug-and-chug"!

Hanan's Theorem

Hanan Grid

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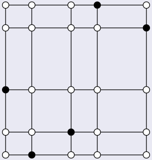

Figure: Hanan grid generated for 5 points, by Jeffrey Sharkey, on Wikipedia.

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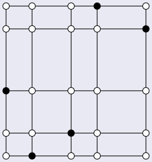

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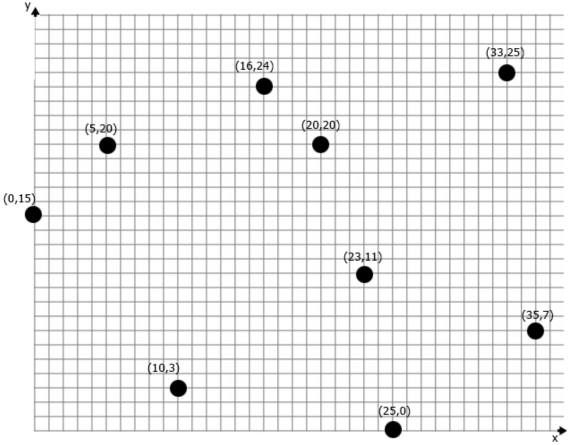
In 1966, Maurice Hanan proved: any MRST of a set of points *must* have vertices on its Hanan grid.

Manual MRST Construction via Hanan Grid



Desmonstrate Given Set of Points

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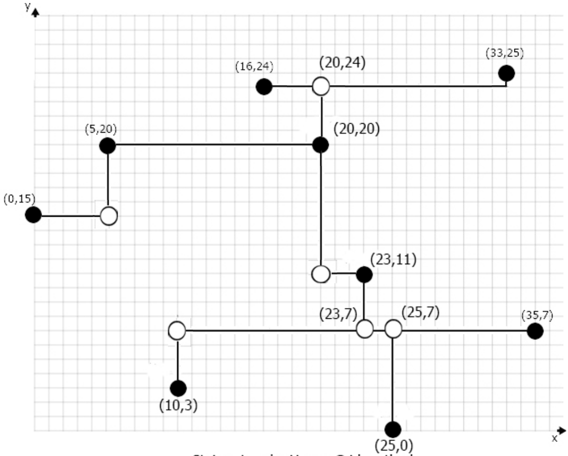

Steiner tree by Hanan Grid method

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- Step 3: Repeat step 2 until all vertices are in the tree.

Using MSTs to Find the Minimum Steiner Tree

(Naïve) Hanan/Prim Hybridization

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  (This reduces the Hanan Grid by removing "outside" points.)
- Step 3: Using Prim's Algorithm, find the MST of this resulting graph.

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- In step 2: When adding the edge, augment the growing tree with a Steiner point, if needed.
  - If near an existing path, insert Steiner point on that path.
  - Otherwise, choose the more "central" Steiner point.

Results: Hanan/Prim Hybridization

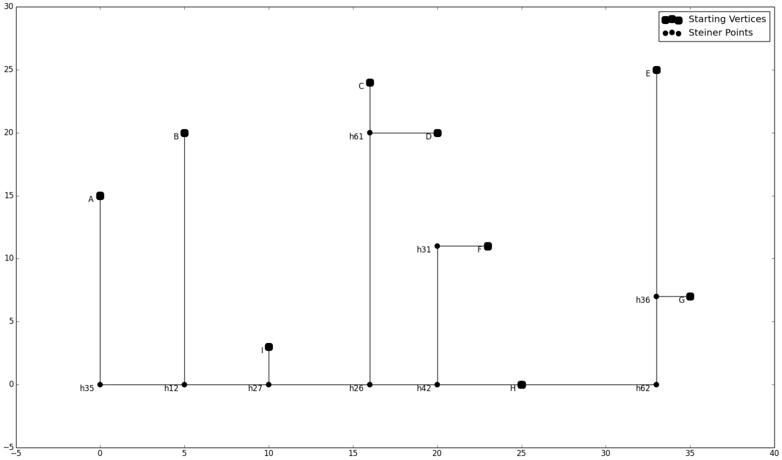
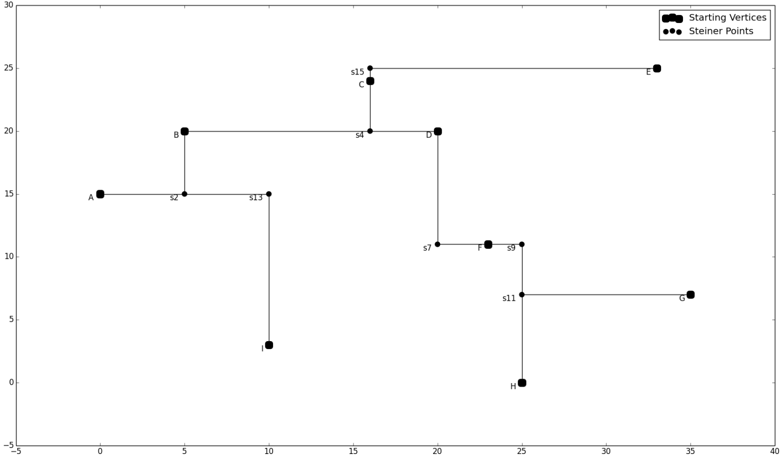


Figure: Result MRST – Total Weight: 140 & 187.59

Results: Modified Prim's Algorithm



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- (−) Constructs entire Hanan Grid: **very slow** for large data sets!
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- (+) Holistic algorithm: MRST constructed with full knowledge of graph
- (−) Constructs entire Hanan Grid: **very slow** for large data sets!
  e.g.: Several minutes running time for $n \approx 50$, hours for $n \gtrsim 200$
- (−) Likely to use more Steiner points for larger sets.

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- (+) Only adds Steiner points on an as-needed basis: **much faster**
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- (+) Attempts to minimize the number of Steiner points to add.

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- Cables have uniform cost (i.e., copper coaxial, some fiber optic, etc.)
- Cables can only go in straight horizontal/vertical lines. (i.e., Steiner point not necessary only to change direction)

Works Cited

- Dreyer,D., and Overton,M. (1998). *Two heuristics for the Euclidean Steiner tree problem*. Journal of Global Optimization, 13(1), 95-106.
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- Hanan, M., *On Steiner's problem with rectilinear distance*. SIAM J. Applied Math., 14:255 – 265, 1966.
- Image: Hanan grid generated for a 5-terminal case. (C) 2007 Jeffrey Sharkey, retrieved from `https://en.wikipedia.org/wiki/File:Hanan5.svg`; licensed under the Creative Commons Attribution-ShareAlike license.

Thank You

Questions?