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Abstract

The Novikov equation is a nonlinear partial differential equation (PDE) that admits peakon solutions. This thesis focuses on the high-frequency limit of the Novikov equation, which results in a new PDE. We derive the corresponding system of ordinary differential equations (ODEs) for piecewise linear solutions and provide a detailed solution for the case when the number of peakons $N = 2$. We also explore the periodic case for $N = 2$ and perform numerical simulations for higher values of N. Additionally, we investigate the Hamiltonian structure of the high-frequency Novikov equation and identify several constants of motion using both direct search and Lax pair methods.

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Contents

1 Introduction

The objective of this thesis is to study the following nonlinear partial differential equation (PDE):

$$u_{xxt} = -u^2 u_{xxx} - 3uu_x u_{xx}, \tag{1}$$

where $u = u(x,t)$. This equation is obtained by taking the high-frequency limit of the Novikov equation, which will be explained in this section. It can be written in a more compact form as

$$m_t + ((um)_x + 2u_x m)u = 0, \quad m = u_{xx}. \tag{2}$$

We will study solutions to this equation on the form,

$$u(x,t) = \sum_{k=1}^{N} m_k(t)|x - x_k(t)|. \tag{3}$$

Solutions of this form give rise to a system of ordinary differential equations (ODEs). The ODEs are given by

$$\dot{x}_k = u(x_k)^2, \quad \dot{m}_k = -m_k u_x(x_k) u(x_k). \tag{4}$$

In the next section, we provide a brief historical background and then discuss the high-frequency limit of the Novikov equation and two closely related PDEs. After that, we present an outline of the thesis.

1.1 A brief history of solitons

In 1834, John Scott Russell observed a solitary wave in a canal. The wave was created by a boat traveling along the canal. The wave was solitary in that it maintained its shape and speed as it traveled along the canal.

Diederik Johannes Korteweg and Gustav de Vries conducted further studies of solitary waves, and they were the first to derive a partial differential equation (PDE) that describes solitary waves. In 1895, they discovered the now famous Korteweg–de Vries (KdV) equation

$$u_t + u_{xxx} - 6uu_x = 0, \tag{5}$$

that describes solitary waves in shallow water.

It was not until much later in 1965 that Zabusky and Kruskal [10] discovered numerically that the KdV equation has solutions of solitary travelling waves that can interact with each other without changing shape. They called these solitary waves solitons. Solitons have since been studied in many fields such as fluid dynamics, optics, and quantum mechanics.

In 1993[2], while working on an integrable model of one-dimensional dispersive waves in shallow water Camassa and Holm discovered the first PDE which admit peakon (peaked soliton) solutions. The Camassa–Holm (CH) equation

$$m_t + (um)_x + u_x m = 0, \quad m = u - u_{xx}, \tag{6}$$

opened up the door for many other PDEs with peakon solutions. The peakon traveling wave moves at a speed based on its maximum height, at which it has a sharp peak (jump in derivative). These peakon solutions have the following form

$$u(x,t) = \sum_{k=1}^{N} m_k(t) e^{-|x-x_k(t)|}. \tag{7}$$

Later the Degasperis–Procesi (DP) equation

$$m_t + (um)_x + 2u_x m = 0, \quad m = u - u_{xx}. \tag{8}$$

was studied by Degasperis and Procesi in 1999 [3]. Both the CH and DP equations share similar properties with the Novikov equation. For a more detailed discussion of the peakon equations and their properties, see the comprehensive overview by Lundmark and Szmigielski [8].

This brings us back to the Novikov equation

$$m_t + ((um)_x + 2u_x m)u = 0, \quad m = u - u_{xx}. \tag{9}$$

It was discovered by Vladimir Novikov in a classification of cubically nonlinear PDEs admitting infinitely many symmetries [9], with Hone and Wang [4] later providing a Lax pair for it. Lax pairs will be discussed in more detail later in the thesis.

1.2 High-frequency limit

As mentioned at the start the focus of the thesis is on the high-frequency limit of the Novikov equation. The high-frequency limit is obtained by substitution of $x \mapsto \epsilon x$, $t \mapsto \epsilon t$, $m \mapsto -\epsilon^{-2} m$, and letting $\epsilon \to 0$. This limit gives the high-frequency Novikov equation (9).

Both the CH and DP equations have been studied in the high-frequency limit. The high frequency limit of the Camassa–Holm equation yields the Hunter–Saxton equation [?, ?] for nematic liquid crystals. The high frequency limit of the Degasperis–Procesi equation yields the derivative Burgers equation [5, 7]. The only difference between all three of the original equations and their respective high-frequency limit is that $m = u_{xx}$ instead of $m = u - u_{xx}$.

Write as can be seen in this table:

	$m = u - u_{xx}$	$m = u_{xx}$
$m_t + (um)_x + u_x m = 0$	Camassa–Holm	Hunter–Saxton
$m_t + (um)_x + 2u_x m = 0$	Degasperis–Procesi	Derivative Burgers
$m_t + ((um)_x + 2u_x m)u = 0$	Novikov	HF Novikov

All three high-frequency limits also admits piecewise linear solutions on the following form

$$u(x,t) = \sum_{k=1}^{N} m_k(t)|x - x_k(t)|. \tag{10}$$

For the HF Novikov equation the time derivative of x_k and m_k will be governed by the following ODEs

$$\dot{x}_k = u(x_k)^2, \quad \dot{m}_k = -m_k u_x(x_k)u(x_k). \tag{11}$$

Dots denote $\frac{d}{dt}$ as usual. $u(x_k)$ and $u(x = x_k, t)$ is given by

$$u(x_k) = \sum_{i=1}^{N} m_i(t)|x_k - x_i(t)|, \tag{12}$$

$$u_x(x_k) = \sum_{i=1}^{N} m_i(t)\,\mathrm{sgn}(x_k - x_i). \tag{13}$$

The u_x isn't defined at $x = x_i$ but we define it to be the average $\langle u_x(x_i)\rangle$ of the left and right limits and we will see that this is consistent with the PDE. See Appendix (A) for the proof that the HF Novikov equation admits piecewise linear solutions on the form of equation (10) and that the ODE-system is compatible with the PDE.

The peakon equations and their respective high-frequency limits share many similarities. The ODEs for both equations share the same form, but the $u(x_k)$ and $u_x(x_k)$ are different. Both the regular and HF Novikov equations only have x_i's that increase due to the $u(x_k)^2$ term in the ODEs.

	$u = \sum_{i=1}^{n} m_i e^{-\lvert x-x_i\rvert}$	$u = \sum_{i=1}^{n} m_i \lvert x - x_i\rvert$
$\dot{x}_i = u(x_k)$, $\dot{m}_i = -\ m_i u_x(m_i)$	Camassa–Holm	Hunter–Saxton
$\dot{x}_i = u(x_k)$, $\dot{m}_i = -2m_i u_x(m_i)$	Degasperis–Procesi	Derivative Burgers
$\dot{x}_i = u(x_k)^2$, $\dot{m}_i = -\ m_i u_x(m_i) u(x_i)$	Novikov	HF Novikov

1.3 Outline

The initial objective of this thesis was to solve the HF Novikov equation explicitly for any N. However, it was found that solving the equation presents significant challenges and the solutions method used for similar equations such as the derivative Burgers equation and the Novikov equation did not work. The thesis will instead focus on other aspects of the HF Novikov equation.

In the next section of the thesis there will be a detailed solution for the case when $N = 2$. The solution will be given for two cases, both the regular and a periodic case which will be discussed in the next section. For higher values of N a numerical analysis will be given in the third section. Then the Hamiltonian structure of the HF Novikov equation will be discuss in section four.

After that in the sections that follow we will examine the constants of motion for the HF Novikov equation. First a short survey of the constants of motion discovered by direct search will be given. Then a more systematic approach will be taken to find the constants of motion for the HF Novikov equation. Next a short theorem of the constants of motion for a generalized system of ODEs will be given. It connects the constant of motion of the HF Novikov equation to the regular Novikov equation. After that the Lax pair will be discussed and the constants of motion derived from it. The thesis will conclude with appendices that provide additional mathematical proofs and verifications.

2 Solution for N = 2

For $N = 1$, the solution to the HF Novikov equation is trivial since $u(x_k) = 0$. Therefore, both $\dot{x}_k$ and $\dot{m}_k$ are equal to zero, resulting in a constant solution. For the more interesting case when $N = 2$, we will give a detailed solution to the system of ODEs for the HF Novikov equation. We will also address the periodic case for $N = 2$.

2.1 HF Novikov equation

For $N = 2$ the system of ODEs is given by

$$\begin{aligned} \dot{x}_k &= u(x_k)^2, \\ \dot{m}_k &= -m_k u_x(x_k) u(x_k). \end{aligned} \tag{14}$$

The assumption that $x_1 < \cdots < x_N$ can be made without loss of generality. For $N = 1$, the solution is trivial since $u(x_k) = 0$, which appears in both $\dot{x}_k$ and $\dot{m}_k$, so both $\dot{x}_k$ and $\dot{m}_k$ are equal to zero and the solution for is constant. For the more interesting case when $N = 2$ we get the following system of ODEs from equation (11):

$$\dot{x}_1 = m_2^2 (x_2 - x_1)^2, \tag{15}$$

$$\dot{x}_2 = m_1^2 (x_2 - x_1)^2, \tag{16}$$

$$\dot{m}_1 = m_1 m_2^2 (x_2 - x_1), \tag{17}$$

$$\dot{m}_2 = -m_1^2 m_2 (x_2 - x_1). \tag{18}$$

We will also assume that $m_k \neq 0$ since in the case where it is zero it will remain identically zero. To solve this system, we first identify two conserved quantities

$$M = m_1^2 + m_2^2, \tag{19}$$

$$K = m_1 m_2 (x_2 - x_1). \tag{20}$$

We show that these quantities are conserved by making sure that their time derivatives are zero

$$\begin{aligned} (m_1^2 + m_2^2)_t &= 2m_1 \dot{m}_1 + 2m_2 \dot{m}_2 \\ &= 2m_1^2 m_2^2 (x_2 - x_1) - 2m_1^2 m_2^2 (x_2 - x_1) = 0, \end{aligned} \tag{21}$$

$$\begin{aligned} &(m_1 m_2 (x_2 - x_1))_t \\ &\quad = \dot{m}_1 m_2 (x_2 - x_1) + m_1 \dot{m}_2 (x_2 - x_1) - m_1 m_2 \dot{x}_1 + m_1 m_2 \dot{x}_2 \\ &\quad = \quad m_1 m_2^3 (x_2 - x_1)^2 - m_1^3 m_2 (x_2 - x_1)^2 \\ &\qquad - m_1 m_2^3 (x_2 - x_1)^2 + m_1^3 m_2 (x_2 - x_1)^2 \\ &\quad = 0. \end{aligned} \tag{22}$$

Leveraging these conserved quantities, we derive expressions for $\dot{m}_1$ and $\dot{m}_2$:

$$\begin{aligned} \dot{m}_1 &= \quad m_2 K, \\ \dot{m}_2 &= -m_1 K. \end{aligned} \tag{23}$$

The solutions to these equations take the form

$$\begin{aligned} m_1 &= \sqrt{M}\sin(Kt+\phi), \\ m_2 &= \sqrt{M}\cos(Kt+\phi). \end{aligned} \tag{24}$$

Now we can solve for $\dot{x}_1$ and $\dot{x}_2$:

$$\dot{x}_1 m_1^2 = m_1^2 m_2^2 (x_2 - x_1)^2 = K^2 \tag{25}$$

$$\implies \dot{x}_1 = \frac{K^2}{m_1^2} = \frac{K^2}{M\sin^2(Kt+\phi)}, \tag{26}$$

$$\dot{x}_2 m_2^2 = m_1^2 m_2^2 (x_2 - x_1)^2 = K^2 \tag{27}$$

$$\implies \dot{x}_2 = \frac{K^2}{m_2^2} = \frac{K^2}{M\cos^2(Kt+\phi)}. \tag{28}$$

Integration yields the positions:

$$\begin{aligned} x_1 &= -\frac{K}{M}\cot(Kt+\phi) + C, \\ x_2 &= \frac{K}{M}\tan(Kt+\phi) - D. \end{aligned} \tag{29}$$

The K quantity implies that $D = -C$:

$$\begin{aligned} K &= m_1 m_2 (x_2 - x_1) \\ &= M\sin(Kt+\phi)\cos(Kt+\phi) \\ &\quad \Big(\frac{K}{M}\Big(\frac{\sin(Kt+\phi)}{\cos(Kt+\phi)} + \frac{\cos(Kt+\phi)}{\sin(Kt+\phi)}\Big) - D - C\Big) \\ &= K(\sin^2(Kt+\phi) + \cos^2(Kt+\phi)) \\ &\qquad - M\sin(Kt+\phi)\cos(Kt+\phi)(C+D) \\ &= K - M\sin(Kt+\phi)\cos(Kt+\phi)(C+D). \end{aligned} \tag{30}$$

Since it has to hold for all t, we get that $C + D = 0$. The constant C is determined by the inital conditions

$$C = \frac{m_2(0)^2 x_2(0) + m_1(0)^2 x_1(0)}{M}. \tag{31}$$

In conclusion the piecewise linear solution looks like this:

$$\begin{aligned} u(x,t) &= m_1|x-x_1| + m_2|x-x_2|, && (32)\\ m_1 &= \sqrt{M}\sin(Kt+\phi), && (33)\\ m_2 &= \sqrt{M}\cos(Kt+\phi), && (34)\\ x_1 &= -\frac{K}{M}\cot(Kt+\phi)+C, && (35)\\ x_2 &= \frac{K}{M}\tan(Kt+\phi)+C. && (36) \end{aligned}$$

There are four constants M, K, ϕ and C, which is to be expected since there are four initial quantities, $m_1(0)$, $m_2(0)$, $x_1(0)$ and $x_2(0)$. We made the assumption that $x_1 < x_2$, will be valid for all t.

$$\begin{aligned} x_2 - x_1 &= \frac{K}{M}(\tan(Kt+\phi)+\cot(Kt+\phi))\\ &= \frac{K}{M}\frac{2}{\sin(2Kt+2\phi)} \end{aligned} \tag{37}$$

The solution is valid only for a finite time, since x_2 will go to infinity in a finite time. As x_2 goes to infinity, $u(x,t)$ will go to

$$\begin{aligned} \lim_{x_2\to\infty} u(x,t) &= \lim_{x_2\to\infty} m_1|x-x_1| + \lim_{x_2\to\infty} m_2|x-x_2|\\ &= m_1|x-C| + \lim_{x_2\to\infty} m_2x_2\\ &= m_1|x-C| + \lim_{x_2\to\infty} \frac{K}{\sqrt{M}}\sin(Kt+\phi)\\ &\quad + \lim_{x_2\to\infty} \sqrt{M}\cos(Kt+\phi)C. \end{aligned} \tag{38}$$

Here we have that

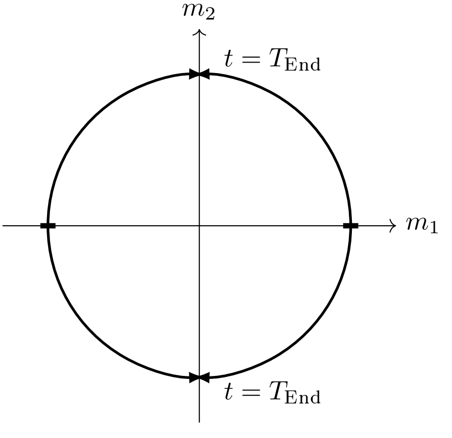
$$x_2 = \frac{K}{M}\tan(Kt+\phi)+C \to \infty \implies \begin{cases} \sin(Kt+\phi)\to 1\\ \cos(Kt+\phi)\to 0 \end{cases}, \tag{39}$$

thus the limit of $u(x,t)$ as x_2 goes to infinity is

$$m_1|x-C| + \frac{K}{\sqrt{M}} \tag{40}$$

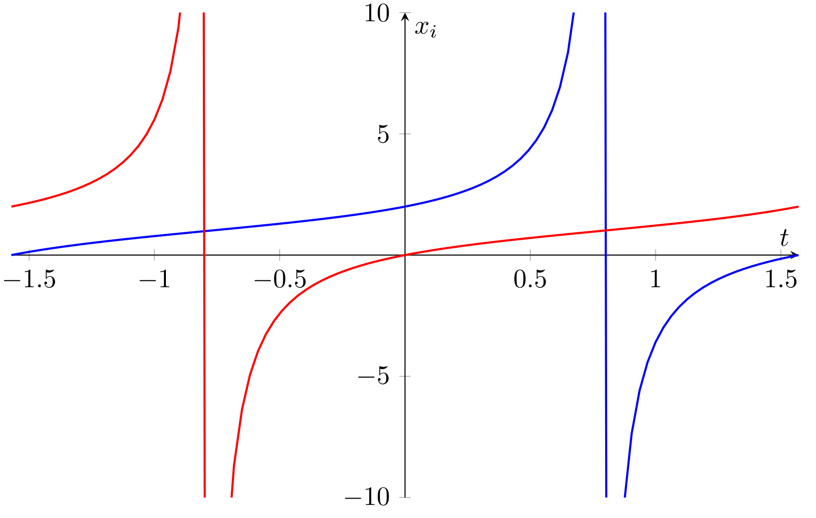
It is possible to create a new system from the old one that will conserve the constants of motion as well as keep $\dot{u}(x,t)$ continuous. We define the new system by keeping x_1 and m_1 as they are and setting x_2 to be $-\infty$ and keeping m_2 equal to 0. The difference now is that x_2 will not always be greater than x_1. With this genealization of the solutions the real axis can be thought of as a circle and when one of the x_i's reaches ∞ it comes back at $-\infty$. The solution for the new system is then given by the same equations as before. Thus it seems possible to create a weak solution for $u(x,t)$ that is valid for all time as long as we define $u(x,t)$ at the points where x_i goes to ∞ to be the limit of the solution as x_i goes to ∞.

Looking at the asymptotic behavior of the solution we see that m_1 always goes to $\pm M$ as m_2 goes to zero. The asymptoticics can be visualized by looking at how the momenta move on the circle with radius M_1:



Here T_{End} is the time when x_2 goes to infinity. Which T_{End} it goes to depends on the initial conditions. The continuatin of the solution can then be visualized by continuing travelling along the circle in the same direction as before.

While the x_2 term goes to infinity the x_1 term goes to C. For the continuation of the solution it is always true that the smaller x_i will go to C. Here is a graph of the values of x_1 and x_2 over time:



2.2 Periodic HF Novikov equation

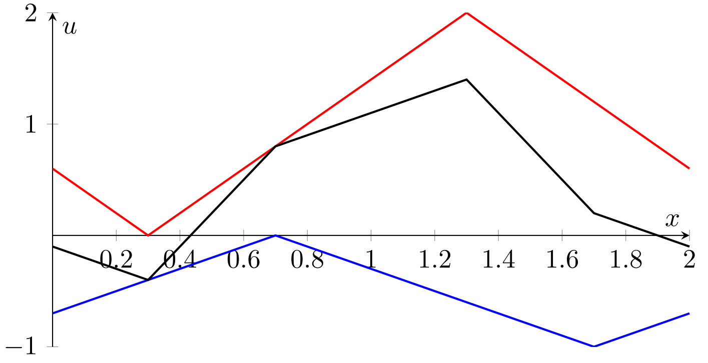
The u and u_x functions have been adjusted to for the periodic boundary conditions

$$
\begin{aligned}
u(x,t) &= \sum_{k=1}^{N} m_k(t)|[x - x_k(t) - L]_{2L} - L|, \\
u_x(x,t) &= \sum_{k=1}^{N} m_k(t)\,\mathrm{sgn}([x - x_k(t) - L]_{2L} - L),
\end{aligned}
\tag{41}
$$

where

$$
[x]_{2L} = x - 2L\left\lfloor \frac{x}{2L} \right\rfloor. \tag{42}
$$

Here is an example of the function u (shown in black) with its two contributing terms (shown in red and blue):



This leads to the following ODEs for the peakon positions and momenta

$$\dot{x}_1 = m_2^2([x_2 - x_1 - L]_{2L} - L)^2, \tag{43}$$
$$\dot{x}_2 = m_1^2([x_2 - x_1 - L]_{2L} - L)^2, \tag{44}$$
$$\dot{m}_1 = m_1 m_2^2([x_2 - x_1 - L]_{2L} - L), \tag{45}$$
$$\dot{m}_2 = -m_1^2 m_2([x_2 - x_1 - L]_{2L} - L). \tag{46}$$

To solve this system, we first identify two conserved quantities

$$M = m_1^2 + m_2^2, \tag{47}$$
$$K = m_1 m_2 |[x_2 - x_1 - L]_{2L} - L|. \tag{48}$$

We show that these quantities are conserved by making sure that their time derivatives are zero

$$\begin{aligned}(m_1^2 + m_2^2)_t &= 2m_1\dot{m}_1 + 2m_2\dot{m}_2 \\ &= (2m_1^2m_2^2 - 2m_1^2m_2^2)([x_2 - x_1 - L]_{2L} - L) = 0,\end{aligned} \tag{49}$$

$$\begin{aligned}&(m_1 m_2 |[x_2 - x_1 - L]_{2L} - L|)_t \\ &\quad = (\dot{m}_1 m_2 + m_1 \dot{m}_2)|[x_2 - x_1 - L]_{2L} - L| \\ &\quad + m_1 m_2(\dot{x}_2 - \dot{x}_1)\operatorname{sgn}([x_2 - x_1 - L]_{2L} - L) \\ &\quad = (m_1 m_2^3 - m_1^3 m_2 + m_1^3 m_2 - m_1 m_2^3) \\ &\quad |[x_2 - x_1 - L]_{2L} - L|([x_2 - x_1 - L]_{2L} - L) = 0.\end{aligned} \tag{50}$$

Leveraging these conserved quantities, we derive expressions for $\dot{m}_1$ and $\dot{m}_2$

$$\begin{aligned}\dot{m}_1 &= \quad m_2 K \operatorname{sgn}([x_2 - x_1 - L]_{2L} - L), \\ \dot{m}_2 &= -m_1 K \operatorname{sgn}([x_2 - x_1 - L]_{2L} - L).\end{aligned} \tag{51}$$

Let

$$S(t_0) = \operatorname{sgn}([x_2 - x_1 - L]_{2L} - L). \tag{52}$$

The $S(t_0)$ function can only take on the values -1, 0, or 1. For a moment, let's assume that $S(t_0)$ is equal to 1. Then we can solve the ODEs for m_1 and m_2 around t_0

$$\begin{aligned} m_1 &= \sqrt{M}\sin(Kt+\phi), \\ m_2 &= \sqrt{M}\cos(Kt+\phi), \\ \phi &= \operatorname{atan2}(m_1(0), m_2(0)). \end{aligned} \tag{53}$$

The solution functions for m_1 and m_2 are the same as for the non-periodic case as long as the assumption that $S(t_0) = 1$ holds. The same is true for the x_1 and x_2 solution functions since

$$\dot{x}_1 m_1^2 = m_1^2 m_2^2 (x_2 - x_1)^2 = K^2 \tag{54}$$

$$\implies \dot{x}_1 = \frac{K^2}{m_1^2} = \frac{K^2}{M\sin^2(Kt+\phi)}, \tag{55}$$

$$\dot{x}_2 m_2^2 = m_1^2 m_2^2 (x_2 - x_1)^2 = K^2 \tag{56}$$

$$\implies \dot{x}_2 = \frac{K^2}{m_2^2} = \frac{K^2}{M\cos^2(Kt+\phi)}. \tag{57}$$

And just as before we can integrate these equations to get the positions

$$\begin{aligned} x_1 &= -\frac{K}{M}\cot(Kt+\phi) + C, \\ x_2 &= \frac{K}{M}\tan(Kt+\phi) - D, \end{aligned} \tag{58}$$

which results in the same solution as for the non periodic case

$$m_1 = \sqrt{M}\sin(Kt+\phi), \tag{59}$$

$$m_2 = \sqrt{M}\cos(Kt+\phi), \tag{60}$$

$$x_1 = -\frac{K}{M}\cot(Kt+\phi) + C, \tag{61}$$

$$x_2 = \frac{K}{M}\tan(Kt+\phi) + D, \tag{62}$$

with the only difference that C and D doesn't need to be equal. The constants

C and D are determined by the initial conditions

$$
\begin{aligned}
C &= x_1(0) + \frac{K}{M}\frac{m_2(0)}{m_1(0)}, \\
D &= x_2(0) - \frac{K}{M}\frac{m_1(0)}{m_2(0)}.
\end{aligned} \tag{63}
$$

It's important to note that since we used that $S(t_0) = 1$, we only obtained a local solution. The $S(t_0)$ function will change sign if x_2 and x_1 swap places in the expression for $S(t_0)$

$$
\begin{aligned}
\mathrm{sgn}([x_1 - x_2 - L]_{2L} - L) &= \mathrm{sgn}([x_2 - x_1 - L + 2L]_{2L} - L) \\
&= \mathrm{sgn}([-(x_2 - x_1 - L)]_{2L} - L) \\
&= \mathrm{sgn}(-[x_1 - x_2 - L]_{2L} + L) \\
&= -\, S(t_0).
\end{aligned} \tag{64}
$$

This tells us that the solution for when $S(t_0) = -1$ can be obtained by swapping the solution functions for m_1 and m_2 as well as for x_1 and x_2.

For the case where $S(t_0) = 0$ we have that m_1 and m_2 are constant and from the ODEs for x_1 and x_2 we see that the time derivative of x_1 and x_2 will be the same only if $m_1 = m_2$. This means that if $S(t_0) = 0$ and $m_1(t_0) = m_2(t_0)$ is going to be constant and x_1 and x_2 will both travel with constant speed with a distance of L between them. In the case where $S(t_0)$ and $m_1(t_0) \neq m_2(t_0)$ the solution will be only be valid for $t = t_0$. At any other t close to t_0 the $S(t)$ function will be either -1 or 1.

As soon as the local solution becomes invalid, it's possible to continue the solution by creating a new solution with new values for C, D, and ϕ. Both M and K will be conserved for the continuation of the solution. The non-periodic solution can be obtained by letting $L \to \infty$, since that will make the periodic boundary conditions irrelevant.

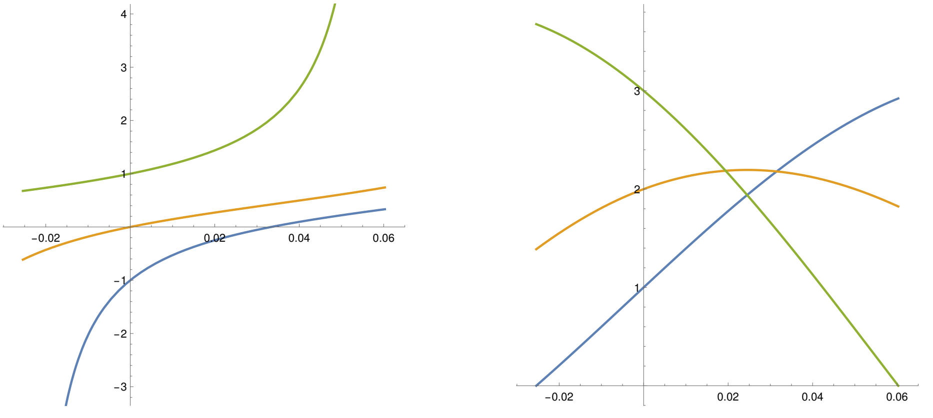
3 Numerical analysis

Numerical simulations have been done. Just need to decide what's worth mentioning. Maybe confimation of the N=2 solution, confimation of the constants of motion, collisions, periodic?

Both in the non periodic and periodic cases the numerical simulations are consistent with the analytical solution for $N = 2$ and the constants of motion for larger N.

3.1 HF Novikov equation

First, we examine the evolution of x_i and m_i for $N = 3$.



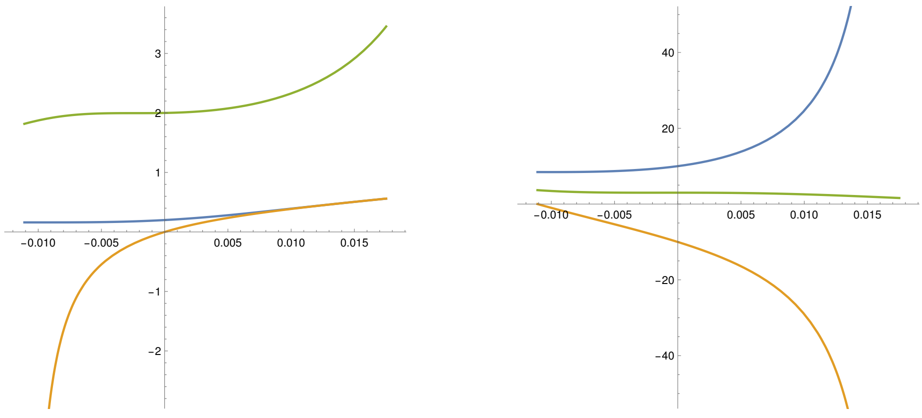
(a) First graph

(b) First graph2

Figure 1: $m_1 = 1$, $x_1 = 0$, $m_1 = 2$, $x_2 = 3$

It follows the same pattern as the $N = 2$ case were it quickly diverges to infinity.

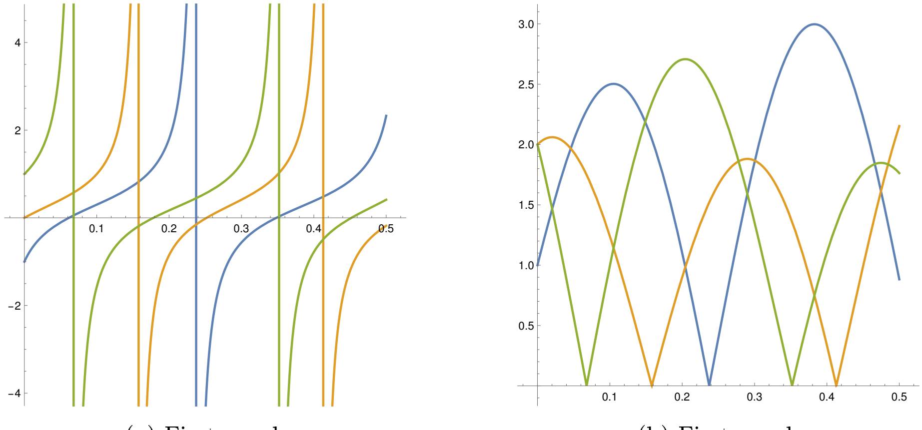
If $m_k < 0$ and $N > 2$ we have that the peakons can collide. This can be seen in the following graph



(a) First graph (b) First graph

Figure 2: $m_1 = 1$, $x_1 = 0$, $m_1 = 2$, $x_2 = 3$

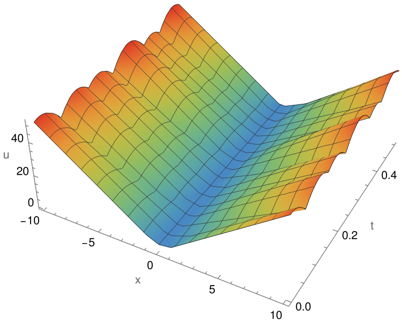
Next, we will take a look at the case where we continue the solution after the peakons have diverged to infinity.



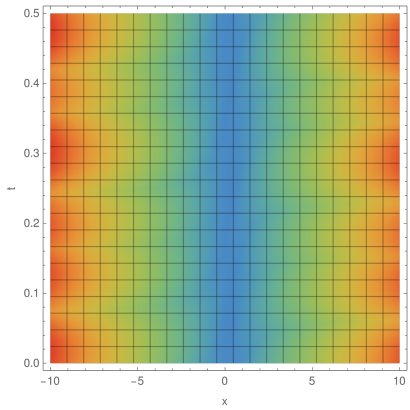
(a) First graph (b) First graph

Figure 3: $m_1 = 1$, $x_1 = 0$, $m_1 = 2$, $x_2 = 3$

Here, in figure (3), it can be observed that the x_i and m_i behaves very closely to the $N = 2$ case. The x_i is still the same shape as $\tan(t)$ and m_i is still the same shape as $|\sin(t)|$.



(a) First graph



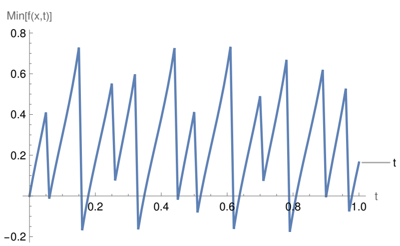
(b) First graph

Figure 4: $m_1 = 1$, $x_1 = 0$, $m_1 = 2$, $x_2 = 3$

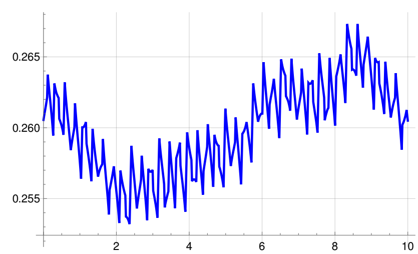
In figure (4) it can be observed the function behaves like the absolute function with increasing fluctuations away from a center position. The fluctuations scale linearly with the distance from the center. The behavior of $u(x,t)$ for x far from the center is

$$\lim_{x\to\infty} u(x,t) = \lim_{x\to\infty} \sum_{k=1}^{N} m_k|x - x_k| = \sum_{k=1}^{N} m_k|x| + \text{sgn}(x)x_k \tag{65}$$

What is interesting here is that the center seems to not be moving from it's inital position.



(a) First graph



(b) First graph

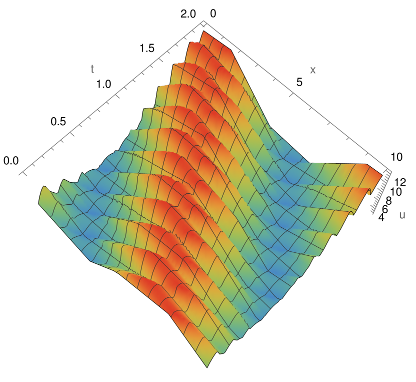
Figure 5: $m_1 = 1$, $x_1 = 0$, $m_1 = 2$, $x_2 = 3$

Even though the minimum of the function $u(x,t)$ varies it fluctuates around a constant value as can be seen in figure (**??**).

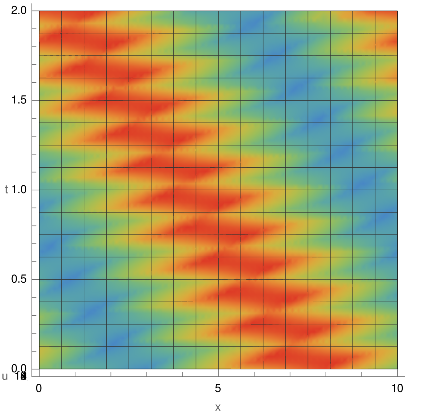
3.2 Periodic HF Novikov equation

For the periodic graphs the boundary conditions are that $u(x,t)$ is periodic with period L. L is set to 10 for all simulations.

First we will look at the graph of the function $u(x,t)$.



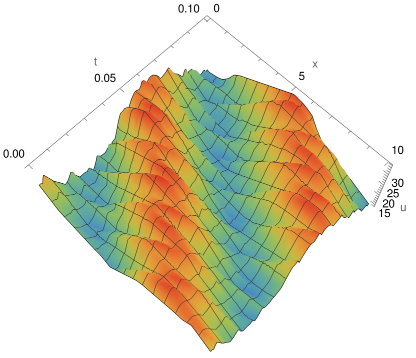
(a) First graph



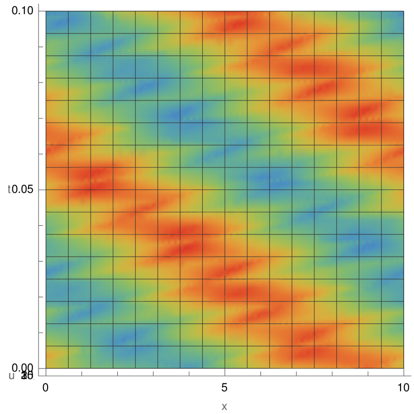
(b) First graph

Figure 6: $m_1 = 1$, $x_1 = 0$, $m_1 = 2$, $x_2 = 3$

In figure (6) it can be observed the function behaves like a wave of constant speed with internal fluctuations. This have been observed for all N and all initial conditions where $m_k > 0$.

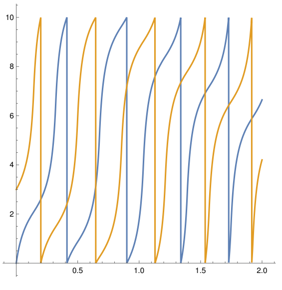


(a) First graph

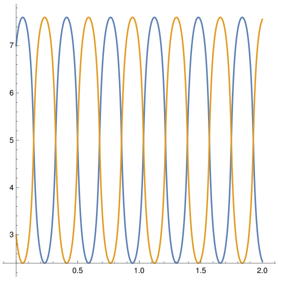


(b) First graph

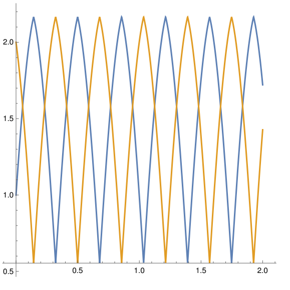
Figure 7: $m_1 = 2, x_1 = 1, m_1 = 0.5, x_2 = 3, m_3 = 5, x_3 = 4, m_4 = 2, x_4 = 8.5$



(a) First graph

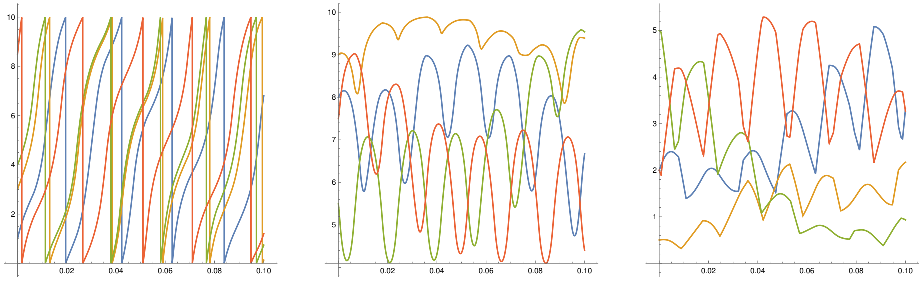


(b) First graph



(c) First graph

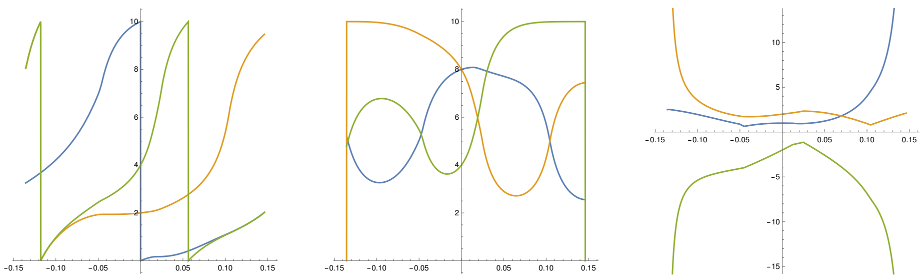
Figure 8: $m_1 = 2, x_1 = 1, m_1 = 0.5, x_2 = 3, m_3 = 5, x_3 = 4, m_4 = 2, x_4 = 8.5$



(a) First graph (b) First graph (c) First graph

Figure 9: $m_1 = 2, x_1 = 1, m_1 = 0.5, x_2 = 3, m_3 = 5, x_3 = 4, m_4 = 2, x_4 = 8.5$

Just as for the non periodic case when $m_k < 0$ and $N > 2$ the peakons can collide.



(a) First graph (b) First graph (c) First graph

Figure 10: $m_1 = 2, x_1 = 1, m_1 = 0.5, x_2 = 3, m_3 = 5, x_3 = 4, m_4 = 2, x_4 = 8.5$

4 Hamiltonian structure

Todo: Add references

The Hamiltonian formalism is a fundamental framework that offers a powerful method for analyzing dynamical systems. The state of the system is described by a set of canonical coordinates

$$(q_i, p_i), \qquad i = 1, \ldots, n. \tag{66}$$

In the context of peakon ODEs, the canonical coordinates are the peakon positions x_i and the peakon momenta m_i. The Hamiltonian $H(q, p, t)$ is a function representing the total energy or another conserved quantity of the system. It is expressed in terms of the canonical coordinates and possibly time. The evolution of the system is governed by Hamilton's equations, which take the form

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \Pi(x) \begin{pmatrix} \partial H/\partial q \\ \partial H/\partial p \end{pmatrix}. \tag{67}$$

Or as

$$\dot{x} = \Pi(x)\nabla H(x), \tag{68}$$

where $x = (q,p)$, $\nabla H(x) = (\partial H/\partial q, \partial H/\partial p)$. The Poisson matrix has to be skew-symmetric, $\Pi^T = -\Pi$, and satisfy the Jacobi identity

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0. \tag{69}$$

where the Poisson brackets is defined as

$$\{f,g\} = (\nabla f)^T \Pi \nabla g. \tag{70}$$

For the canonical coordinates it's defined as

$$\{x_i, x_j\} = \Pi^{ij}(x), \tag{71}$$

$$\{\{x_i, x_j\}, x_k\} = \sum_{l=1}^{n} \Pi^{lk}(x) \frac{\partial}{\partial x_l} \Pi^{ij}(x). \tag{72}$$

A common way to construct the Hamiltonian structure is to use the canonical Poisson matrix

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \partial H/\partial q \\ \partial H/\partial p \end{pmatrix}, \tag{73}$$

or as

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}. \tag{74}$$

where I is the $n \times n$ identity matrix. A more general form of the Hamiltonian structure is with the $2n \times 2n$ Poisson matrix Π, where the equations of motion are instead written as

A finite dimensional Hamiltonian system is integrable if it possesses as many conserved quantities F_i as degrees of freedom, and these quantities are in involution

$$\{F_i, F_j\} = 0. \tag{75}$$

These conserved quantities allow the system's equations of motion to in principle be solved explicitly.

4.1 Comparison to other soliton equations

Here we will show the Hamiltonian structure for the CH, DP, and Novikov peakon ODEs as well as the Hunter Saxton peakon ODE, which is the only HF peakon ODE that has a known Hamiltonian structure. The non HF peakon ODEs Hamiltonian structure can be written on the common form

$$\{x_i,\ x_j\} = \ G(x_i - x_j), \tag{76}$$

$$\{x_i,\, m_j\} = \ G'(x_i - x_j)m_j(b-1), \tag{77}$$

$$\{m_i, m_j\} = -G''(x_i - x_j)m_i m_j(b-1)^2 \tag{78}$$

with the paramters and Hamilton function given in the table below:

	b	H	G
Camassa–Holm	2	$\sum m_i$	$\operatorname{sgn}(x)(1-e^{-\lvert x\rvert})$
Degasperis–Procesi	3	$\sum m_i$	$1/2\operatorname{sgn}(x)(1-e^{-\lvert x\rvert})$
Novikov	3/2	$\sum m_i m_j e^{-\lvert x_i-x_j\rvert}$	$\operatorname{sgn}(x)(1-e^{-2\lvert x\rvert})$
Hunter Saxton	2	$\sum m_i$	$\operatorname{sgn}(x)(1-e^{-2\lvert x\rvert})$
Derivative Burgers	3	$\sum m_i$	$\operatorname{sgn}(x)(1-e^{-2\lvert x\rvert})$

The Hunter Saxton's Hamiltonian structure is given by the canonical Poisson matrix

$$\{m_i, x_i\} = \delta_{ij} \quad \{m_i, m_j\} = 0 = \{x_i, x_j\} \tag{79}$$

and the Hamiltonian

$$H = \frac{1}{2}\sum m_i m_j |x_i - x_j|, \tag{80}$$

which can be related to another similar Hamiltonian structure of Camassa–Holm that also has the canonical Poisson matrix but with the Hamiltonian

$$H = \frac{1}{2}\sum m_i m_j e^{-|x_i - x_j|}. \tag{81}$$

We wanted to find a Hamiltonian structure for the HF Novikov peakon ODEs but it was harder than expected. The search for a Hamiltonian structure was done by trying to start from the Novikov Hamiltonian structure and try to modify it to fit the HF Novikov peakon ODEs. If the search is continued in the future, a more structured approach should be taken. Finding a Hamiltonian for the Novikov PDE could be a good starting point and has been the method used to find the other Hamiltonians for the peakon ODEs.

5 Constants of motion by direct search

Here we will outline the exhaustive search algorithm used to find constants of motion for the HF Novikov equation and the results of the search.

Since the system's time evolution is governed by polynomial expressions of a certain degree, we restrict our search to polynomials functions of a certain degree in the variables m_k and x_k. Each polynomial will be a sum of monomials of the form

$$\prod_{i=1}^{N} m_i^{a_i} x_i^{b_i}, \tag{82}$$

where the sum of a_i is some constant A and the sum of b_i is some constant B. The notation for the constants of motion will be M_{AB}, where A is the sum of the m_i and B is the sum of the x_i.

Example for $N = 2$ and M_{20}

To give an example of the algorithm, we will look at the case for $N = 2$. The algorithm will then list all monomials for a given A and B, in this case $A = 2$ and $B = 0$. The monomials are then

$$m_1^2, m_2^2, m_1 m_2 \tag{83}$$

Then we will take the time derivative of the linear combination of the monomials

$$\begin{aligned}
&\partial_t(\alpha_1 m_1^2 + \alpha_2 m_2^2 + \alpha_3 m_1 m_2) \\
&= \alpha_1(2m_1^2 m_2^2 x_2 - 2m_1^2 m_2^2 x_1) \\
&+ \alpha_2(-2m_1^2 m_2^2 x_1 + 2m_1^2 m_2^2 x_2) \\
&+ \alpha_3(m_1 m_2^3 x_2 - m_1 m_2^3 x_1 + m_1^3 m_2 x_2 - m_1^3 m_2 x_1).
\end{aligned} \tag{84}$$

The monomials are then indexed resulting in the following expression:

$$\alpha_1(2w_1 - 2w_2) + \alpha_2(-2w_1 + 2w_2) + \alpha_3(w_3 - w_4 + w_5 - w_6) \tag{85}$$

Writing the linear combination of the monomials as a matrix, we get that the constants of motion are then the null space of the matrix

$$\begin{pmatrix}
2 & -2 & 0 \\
-2 & 2 & 0 \\
0 & 0 & 1 \\
0 & 0 & -1 \\
0 & 0 & 1 \\
0 & 0 & -1
\end{pmatrix} \tag{86}$$

Which is this case is $\alpha_1 = \alpha_2, \alpha_3 = 0$, which is the same as the known constant of motion $m_1^2 + m_2^2$.

This is done repeatedly for growing A and B. For multiple of known constants of motion the algorithm will need to take into account that the same constants of motion will be found again. For example, at $A = 4$ and $B = 0$ the algorithm will find $(m_1^2 + m_2^2)^2$.

The solving of the null space have been done both symbolically in Mathematica and numerically in C++ using LU decomposition method from the Eigen library. The numerical methods broke down at $N > 4$ due to numerical precision. The symbolic method was able to find the constants of motion for $N = 5$ but took a long time.

Jacobian method

The methods described above have been been successful in finding constants of motion for the HF Novikov equation. However, it is not guaranteed that the constants of motion are independent. Two or more constants of motion are said to be functionally independent if none of them can be expressed as a function of the others. In other words, they provide unique information about the system.

To determine if a set of constants of motion are independent, the Jacobian matrix is used. Suppose there are a set of n constants of motion $M_1, M_2, \ldots, M_m$. These constants are functions of the variables $x_1, x_2, \ldots, x_n, m_1, m_2, \ldots, m_n$.

The Jacobian matrix J of the functions $M_1, M_2, \ldots, M_m$ with respect to the variables $x_1, x_2, \ldots, x_n, m_1, m_2, \ldots, m_n$ is given by:

$$j = \begin{pmatrix} \frac{\partial M_1}{\partial x_1} & \frac{\partial M_1}{\partial x_2} & \cdots & \frac{\partial M_1}{\partial x_n} & \frac{\partial M_1}{\partial m_1} & \frac{\partial M_1}{\partial m_2} & \cdots & \frac{\partial M_1}{\partial m_n} \\ \frac{\partial M_2}{\partial x_1} & \frac{\partial M_2}{\partial x_2} & \cdots & \frac{\partial M_2}{\partial x_n} & \frac{\partial M_2}{\partial m_1} & \frac{\partial M_2}{\partial m_2} & \cdots & \frac{\partial M_2}{\partial m_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial M_m}{\partial x_1} & \frac{\partial M_m}{\partial x_2} & \cdots & \frac{\partial M_m}{\partial x_n} & \frac{\partial M_m}{\partial m_1} & \frac{\partial M_m}{\partial m_2} & \cdots & \frac{\partial M_m}{\partial m_n} \end{pmatrix} \tag{87}$$

Each row of the Jacobian matrix is the gradient of one of the constants of motion. If the gradients are linearly dependent it means that one of the constants of motion can be expressed as a function of the others. Thus the rank of the Jacobian matrix is the number of independent constants of motion.

N = 2

The algorithm has found the following four conserved quantities for $N = 2$:

$$
\begin{aligned}
M_{21A} &= m_1 m_2 (x_2 - x_1), && (88)\\
M_{20} &= m_1^2 \quad + m_2^2 \quad , && (89)\\
M_{21B} &= m_1^2 x_1 + m_2^2 x_2, && (90)\\
M_{22} &= m_1^2 x_1^2 + m_2^2 x_2^2. && (91)\\
& && (92)
\end{aligned}
$$

We observe that M_{20} and M_{21A} are equal the constants M and K from the solution of the ODEs (Eqs. 19 & 20). The system of ODEs for $N = 2$ only has four variables so the system should have at most three functionally independent constants of motion. It can be seen that

$$
M_{21A}^2 = M_{20} M_{22} - M_{21B}^2. \tag{93}
$$

N = 3

The algorithm has found the following five conserved quantities for $N = 3$. The first two are

$$
\begin{aligned}
M_{21} &= m_1 m_2 (x_2 - x_1) + m_1 m_3 (x_3 - x_1) + m_2 m_3 (x_3 - x_2), && (94)\\
M_{63} &= m_1^2 m_2^2 m_3^2 (x_2 - x_1)(x_3 - x_1)(x_3 - x_1). && (95)
\end{aligned}
$$

Let

$$
\begin{aligned}
A &= m_1^2 (M_{21} + 2 m_2 m_3 (x_3 - x_2)), && (96)\\
B &= m_2^2 (M_{21} + 2 m_1 m_3 (x_3 - x_1)), && (97)\\
C &= m_3^2 (M_{21} + 2 m_1 m_2 (x_2 - x_1)) && (98)
\end{aligned}
$$

then the three other conserved quantities can be written as

$$
\begin{aligned}
M_{41} &= A \quad + B \quad + C && (99)\\
M_{42} &= A x_1 + B x_2 + C x_3 && (100)\\
M_{43} &= A x_1^2 + B x_2^2 + C x_3^2 && (101)
\end{aligned}
$$

These conserved quantities aren't functionally independent. The Jacobian method shows that there are only four functionally independent conserved quantities. The first constant can be written as

$$
M_{21}^4 = M_{41} M_{43} - M_{42}^2 + 2 M_{21} M_{63}. \tag{102}
$$

N = 4

For $N = 4$, six conserved quantities have been found. The first two follow the same pattern as for $N = 3$:

$$M_{21} = m_1m_2(x_2 - x_1) + m_1m_3(x_3 - x_1) + m_1m_4(x_4 - x_1) \tag{103}$$
$$+ m_2m_3(x_3 - x_2) + m_2m_4(x_4 - x_2) + m_3m_4(x_4 - x_3), \tag{104}$$
$$M_{84} = m_1^2m_2^2m_3^2m_4^2(x_2 - x_1)(x_3 - x_2)(x_4 - x_3)(x_4 - x_1). \tag{105}$$

For the next three, let's define the expression W_i to be the sum of all the terms in M_{21} that don't contain m_i. Define A, B, C, D as:

$$A = m_1^2(M_{21} + 2W_1), \tag{106}$$
$$B = m_2^2(M_{21} + 2W_2), \tag{107}$$
$$C = m_3^2(M_{21} + 2W_3), \tag{108}$$
$$D = m_4^2(M_{21} + 2W_4). \tag{109}$$

The next three conserved quantities can then be written as

$$\begin{aligned} M_{41} =& A \quad + B \quad + C \quad + D \\ &+ 6m_1m_2m_3m_4(x_1 - x_2 + x_3 - x_4) \end{aligned} \tag{110}$$
$$\begin{aligned} M_{42} =& Ax_1 + Bx_2 + Cx_3 + Dx_4 \\ &+ 6m_1m_2m_3m_4(x_1x_3 - x_2x_4) \end{aligned} \tag{111}$$
$$\begin{aligned} M_{43} =& Ax_1^2 + Bx_2^2 + Cx_3^2 + Dx_4^2 \\ &+ 6m_1m_2m_3m_4(x_1x_2x_3 - x_1x_2x_4 + x_1x_3x_4 - x_2x_3x_4) \end{aligned} \tag{112}$$

Interestingly, it follows very close to the pattern of the $N = 3$ case. The only difference is that there is some extra terms that all contain $m_1m_2m_3m_4$. The last found conserved quantity is

$$\begin{aligned} M_{63} =& m_1^2m_2^2m_3^2(x_2 - x_1)(x_3 - x_1)(x_3 - x_2) \\ &+ m_1^2m_2^2m_4^2(x_2 - x_1)(x_4 - x_1)(x_4 - x_2) \\ &+ m_1^2m_3^2m_4^2(x_3 - x_1)(x_4 - x_1)(x_4 - x_3) \\ &+ m_2^2m_3^2m_4^2(x_3 - x_2)(x_4 - x_2)(x_4 - x_3) \\ &+ 2m_1^2m_2^2m_3m_4(x_4 - x_1)(x_2 - x_1)(x_3 - x_2) \\ &+ 2m_1m_2^2m_3^2m_4(x_2 - x_1)(x_3 - x_2)(x_4 - x_3) \\ &+ 2m_1m_2m_3^2m_4^2(x_3 - x_2)(x_4 - x_3)(x_4 - x_1) \\ &+ 2m_1^2m_2m_3m_4^2(x_4 - x_3)(x_4 - x_1)(x_2 - x_1). \end{aligned} \tag{113}$$

Here we also find that all the conserved quantities aren't functionally independent. The Jacobian method showed that there are only five functionally independent conserved quantities, but the relation between them is still unknown.

6 Constants of motion for a generalized system of ODEs

The M_{21} constants in the previous section followed a pattern. Here we will show that the pattern is not unique to the HF Novikov equation and applies for all N.

Theorem 6.1. *For a general system of ODEs*

$$\dot{x}_i = u(x_i)^2, \quad \dot{m}_i = -m_i^2 u_{x_i}(x_i)u(x_i), \tag{114}$$

where $x_i < x_j$ and the function u is of the form

$$u(x_i) = \sum_{i=1}^{N} m_j g(|x_i - x_j|), \tag{115}$$

there exists a constant of motion of the form

$$\sum_{i=1}^{N}\sum_{j=1}^{N} m_i m_j g(|x_i - x_j|) \tag{116}$$

Proof. The derivative of u with respect to x_i is

$$u_{x_i}(x_i) = \sum_j m_j \operatorname{sgn}(i-j) g'(|x_i - x_j|). \tag{117}$$

We will omit the bounds of the summation for clarity, and all bounds will be understood to be from 1 to N. We introduce the following shorthands

$$\begin{aligned} g(i,j) &= g(|x_i - x_j|), \\ g'(i,j) &= g'(|x_i - x_j|), \\ s(i,j) &= \operatorname{sgn}(i-j). \end{aligned} \tag{118}$$

The time derivative of x_i and m_i is thus

$$\begin{aligned} \dot{x}_i &= \sum_{k,l} m_j^2 g(i,k) g(i,l), \\ \dot{m}_i &= -m_i \sum_{k,l} m_k m_l s(i,k) g'(i,k) g(i,l). \end{aligned} \tag{119}$$

Taking the time derivative of the constant of motion we get

$$\begin{aligned}
&\sum_{i,j} m_i m_j g(i,j) \\
=&\sum_{i,j} (\dot{m}_i m_j + \dot{m}_j m_i) g(i,j) + m_i m_j s(i,j) g'(i,j)(\dot{x}_i - \dot{x}_j) \\
=&-\sum_{i,j,k,l} m_i m_j m_k m_l g(i,j)(s(i,k)g'(i,k)g(i,l) + s(j,k)g'(j,k)g(j,l)) \\
&+\sum_{i,j,k,l} m_i m_j m_k m_l s(i,j) g'(i,j)(g(i,j)g(i,l) - g(j,k)g(j,l)).
\end{aligned} \tag{120}$$

Since the bounds on the sum is the same for all terms we are free to change the indices of the sums. Since all terms have $m_i m_j m_k m_l$ we can factor out this term and get

$$\begin{aligned}
&-\sum_{i,j,k,l} g'(i,k)s(i,k)g(i,j)g(i,l) \\
&-\sum_{i,j,k,l} g'(j,k)s(j,k)g(i,j)g(j,l) \\
&+\sum_{i,j,k,l} g'(i,j)s(i,j)g(i,k)g(i,l) \\
&-\sum_{i,j,k,l} g'(i,j)s(i,j)g(j,k)g(j,l)
\end{aligned} \tag{121}$$

We can write all on the same form by changing the indices of the sums

$$\sum_{i,j,k,l} g'(i,k)g(i,j)g(i,l)(-s(i,k) - s(i,k) + s(i,k) - s(k,i)) = 0. \tag{122}$$

All the $s(i,k)$ terms will cancel out and the constant of motion is conserved. $\square$

This means that when the function u is of the form

$$u(x_i) = \sum_{i=1}^{N} m_j |x_i - x_j|, \tag{123}$$

there exists a constant of motion of the form

$$\sum_{i=1}^{N} \sum_{j=1}^{N} m_i m_j |x_i - x_j|. \tag{124}$$

We see here that the constant of motion is similar to the one you get for the Novikov equation

$$\sum_{i=1}^{N}\sum_{j=1}^{N} m_i m_j e^{-|x_i - x_j|}, \tag{125}$$

where one of the differences is that the Novikov equation gets conserved quantities with terms of m_i^2 since the exponential function is equal to one at zero.

7 Constants of motion from the Lax pair

7.1 Preliminaries

Lax pairs

Lax pairs are a mathematical framework used to analyze and solve certain types of integrable systems. They are the main tool used to solve peakon equations. A Lax pair consists of two operators, U and V,

$$\partial_x \psi = U\psi, \quad \partial_t \psi = V\psi, \tag{126}$$

that satisfy the zero curvature condition

$$\partial_t U - \partial_x V + [U, V] = 0, \tag{127}$$

where U and V are often square matrices. The condition comes from equating the mixed partials $\partial_t \partial_x \psi = \partial_x \partial_t \psi$. The existence of a Lax pair for a nonlinear PDE is a strong indicator of the equation's integrability. It allows the application of powerful analytical methods to find exact solutions and conservation laws.

Example 7.1. The KdV equation

$$u_t + u_{xxx} - 6uu_x = 0, \tag{128}$$

has the following Lax pair

$$\partial x \psi = \begin{pmatrix} 0 & 1 \\ u - \lambda & 0 \end{pmatrix} \psi, \tag{129a}$$

$$\partial t \psi = \begin{pmatrix} -u_x & 2u + 4\lambda \\ 2u^2 - u_{xx} + 2u\lambda - 4\lambda^2 & u_x \end{pmatrix} \psi. \tag{129b}$$

By taking the mixed partials of the Lax pair we get

$$\begin{aligned} & \partial_t\partial_x\psi = \partial_x\partial_t\psi \\ \implies \quad & \partial_t U\psi = \partial_x V\psi \\ \implies \quad & U_t\psi + U\psi_t = V_x\psi + V\psi_x \\ \implies \quad & U_t\psi + UV\psi = V_x\psi + VU\psi \\ \implies \quad & (U_t - V_x + UV - VU)\psi = 0 \end{aligned} \tag{130}$$

This has to be true for all ψ, which implies that $U_t - V_x + [U, V] = 0$. With

$$\partial_t U = \begin{pmatrix} 0 & 0 \\ u_t & 0 \end{pmatrix} \quad \text{and} \quad \partial_x V = \begin{pmatrix} -u_{xt} & 2u_t \\ 4uu_t - u_{xxt} + 2u_t\lambda & u_{xt} \end{pmatrix}, \tag{131}$$

the zero curvature condition becomes

$$\begin{pmatrix} 0 & 0 \\ u_t + u_{xxx} - 6uu_x & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{132}$$

which is only satisfied if the KdV equation is satisfied.

Monodromy Matrix

The Lax pair can be used to construct the monodromy matrix [1, 6], which is a matrix that encodes the dynamics of the system. We first need to construct the transition matrix T. It is defined as the fundamental solution to the linear system

$$\partial_x T(x, a, t) = U(x, t)T(x, a, t), \quad T(a, a, t) = I, \tag{133}$$

where U is the matrix occuring in spatial part of the Lax pair. It describes the evolution of the system along the x-axis, $\psi(x, t) = T(x, a, t)\psi(a, t)$. The monodromy matrix is then defined as

$$M(t) = \lim_{a\to-\infty} \lim_{x\to\infty} T(x, a, t), \tag{134}$$

which exists under the assumption that

$$\int_{-\infty}^{\infty} ||U(x, t)||dx < \infty. \tag{135}$$

where $||\cdot||$ is some matricial norm. It can be viewed as the transition matrix of the system along the x-axis. The monodromy matrix is very useful for finding conserved quantities of the system.

Theorem 7.2.

$$\partial_t T(x,a,t) = V(x,t)T(x,a,t) - T(x,a,t)V(a,t). \tag{136}$$

Proof. Expanding the time derivative of $T(x,a,t)$ and plugging in the zero curvature condition we get

$$\begin{aligned}\partial_x \partial_t T &= \partial_t(U)T + U\partial_t T \\ &= \partial_t(V)T + VUT - UVT + U\partial_t T \\ &= \partial_a(VT) + U(\partial_t T - VT)\end{aligned} \tag{137}$$

which is the same as

$$\partial_a(\partial_t T - VT) = U(\partial_t T - VT). \tag{138}$$

This means that $\partial_t T - VT$ is a solution to equation (136). This implies the existence of a non-singular matrix C, independent of a, such that

$$\partial_t T - VT = TC, \tag{139}$$

letting $a = x$ we get

$$C(x,t) = -V(x,t), \tag{140}$$

since $T(a,a,t) = I$. □

This leads to the following useful corollary

Corollary 7.3. *If* $\lim_{a\to\pm\infty} V(a,t) = V_0$, *then*

$$\partial_t M = [V_0, M]. \tag{141}$$

And since the trace of a commutator is zero, the time derivative of the trace of M is zero.

7.2 Constants of motion

The Lax pair for the Novikov equation [8] is

$$\frac{\partial}{\partial x}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix} = \begin{pmatrix}0 & zm & 1\\ 0 & 0 & zm\\ 1 & 0 & 0\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix}, \tag{142a}$$

$$\frac{\partial}{\partial t}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix} = \begin{pmatrix}-uu_x & \frac{u_x}{z} - u^2 mz & u_x^2\\ \frac{u}{z} & -\frac{1}{z^2} & -\frac{u_x}{z} - u^2 mz\\ -u^2 & \frac{u}{z} & uu_x\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix}, \tag{142b}$$

The zero curvature condition only holds when the Novikov equation (**??**) is satisfied. The high frequency limit of the Novikov equation is obtained by substitution $x \mapsto \epsilon x$, $t \mapsto \epsilon t$, and $m \mapsto \epsilon^{-2} m$ and letting $\epsilon \to 0$. We are left to find the correct substitution for z. We can do this by looking at the spatial derivative of the Lax pair and since we don't know the x and t factors for ψ_1, ψ_2, and ψ_3 we call them A, B, and C respectively

$$\begin{pmatrix} \epsilon^{-1} A \\ \epsilon^{-1} B \\ \epsilon^{-1} C \end{pmatrix} = \begin{pmatrix} 0 & zm\epsilon^{-2} & 1 \\ 0 & 0 & zm\epsilon^{-2} \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix}. \tag{143}$$

Simplifying this we get

$$A = z^2 m^2 \epsilon^{-1} A + \epsilon A, \tag{144}$$

and since we don't want A to go to 0 or ∞ in the limit we get that z should be substituted with $\epsilon^{1/2} z$. Finally doing the substitutions $x \mapsto \epsilon x$, $t \mapsto \epsilon t$, $m \mapsto \epsilon^{-2} m$ and $z \mapsto \epsilon^{1/2} z$, and letting $\epsilon \to 0$ we get rid of the 1 in the $(1,3)$ entry in the Lax pair

$$\frac{\partial}{\partial x} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} 0 & zm & 0 \\ 0 & 0 & zm \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \tag{145}$$

The time derivative is left unchanged. We can simplify the system even further with the following substitutions

$$\begin{cases} \varphi_1 = & \psi_1, \\ \varphi_2 = & z\psi_2, \\ \varphi_3 = & z^2 \psi_3. \end{cases} \tag{146}$$

By also setting $z^2 = -\lambda$ we get the following Lax pair

$$\frac{\partial}{\partial x} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} = \begin{pmatrix} 0 & m & 0 \\ 0 & 0 & m \\ -\lambda & 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}, \tag{147a}$$

$$\frac{\partial}{\partial t} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} = \begin{pmatrix} -uu_x & -\frac{u_x}{\lambda} - u^2 m & -\frac{u_x^2}{\lambda} \\ u & \frac{1}{\lambda} & \frac{u_x}{\lambda} - u^2 m \\ u^2 \lambda & u & uu_x \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}. \tag{147b}$$

With

$$u_x = \sum_{k=1}^{n} m_k \operatorname{sgn}(x - x_k), \tag{148}$$

it means that m will be

$$m = u_{xx} = \sum_{k=1}^{n} 2m_k\delta(x - x_k), \tag{149}$$

since the sign function have a jump discontinuity at x_k of size 2 and the derivative of the sign function is the Dirac delta function. This means that φ_1 and φ_2 will have jump discontinuities at x_k because of the Dirac delta function in their spatial derivative. φ_3 doesn't have any Dirac delta function in its spatial derivative so it will be continuous. We will assume that $x_1 < \cdots < x_n$, which will stay true at least for a time after $t = 0$ if it's true at $t = 0$. We will also use the convention that $x_0 = -\infty$ and $x_{n+1} = \infty$. Since $m = 0$ in the intervals $x_k < x < x_{k+1}$, the spatial Lax pair simplifies to $\delta_x\varphi_1 = 0$, $\delta_x\varphi_2 = 0$, and $\delta_x\varphi_3 = \varphi_1$. This means that φ_1, φ_2 and φ_3 have to be

$$\begin{pmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\end{pmatrix} = \begin{pmatrix}A_i\\ B_i\\ -\lambda A_i x + C_i\end{pmatrix} \text{for } x_k < x < x_{k+1}. \tag{150}$$

We can also see from the Lax pair that φ_3 is going to be continuous, while φ_1 and φ_2 is constant in the intervals with jump discontinuities at x_k. Let's rewrite the Lax pair in terms of A, B, C instead of $\varphi_1, \varphi_2, \varphi_3$:

$$\frac{\partial}{\partial x}\begin{pmatrix}A\\ B\\ C\end{pmatrix} = \begin{pmatrix}0 & m & 0\\ -\lambda m x & 0 & m\\ 0 & \lambda m x & 0\end{pmatrix}\begin{pmatrix}A\\ B\\ C\end{pmatrix} =: U(x)\begin{pmatrix}A\\ B\\ C\end{pmatrix}, \tag{151a}$$

$$\frac{\partial}{\partial t}\begin{pmatrix}A\\ B\\ C\end{pmatrix} = \begin{pmatrix}u_x X & -\frac{u_x}{\lambda} - u^2 m & -\frac{u_x^2}{\lambda}\\ -X + \lambda u^2 m x & \frac{1}{\lambda} & \frac{u_x}{\lambda} - u^2 m\\ \lambda X^2 & -X - \lambda u^2 m x & -u_x X\end{pmatrix}\begin{pmatrix}A\\ B\\ C\end{pmatrix} \tag{151b}$$

$$=: V(x)\begin{pmatrix}A\\ B\\ C\end{pmatrix}, \tag{151c}$$

where $X = u_x x - u$. This also fulfills the zero curvature condition. Call $F(x;t)$ the vector of A, B, C and let T be the transition matrix that takes us from x to y as $F(y;t) = T(y, x;t)F(x;t)$. F is mostly constant with jump discontinuities at x_k so we can write the jump matrix S_k at x_k as

$$\begin{aligned}\begin{pmatrix}A_k\\B_k\\C_k\end{pmatrix} &= \begin{pmatrix}1-2\lambda m_k^2x_k & 2m_k & 2m_k^2\\ -2\lambda m_kx_k & 1 & 2m_k\\ -2\lambda^2m_k^2x_k^2 & 2\lambda m_kx_k & 1+2\lambda m_k^2x_k\end{pmatrix}\begin{pmatrix}A_{k-1}\\B_{k-1}\\C_{k-1}\end{pmatrix}\\ &=: S_k(t)\begin{pmatrix}A_{k-1}\\B_{k-1}\\C_{k-1}\end{pmatrix}\end{aligned} \tag{152}$$

Let $T_a(t)=T(a,-a;t)$. Then at large enough a we can write

$$T_a(t)=S_n(t)S_{n-1}(t)\cdots S_1(t). \tag{153}$$

Since U is bounded we can now write the monodromy matrix as

$$M(t)=\lim_{a\to\infty}T_a(t). \tag{154}$$

From corollary 7.3 we know that the time derivative of M is given by $[V_0,M]$ if $V(x;t)$ goes to the same thing at $\pm\infty$. At the limits, we can know that m will be zero. Our V can then be split into one even and one odd part

$$V(x;t)=\begin{pmatrix}u_xX & 0 & -\frac{u_x^2}{\lambda}\\ 0 & \frac{1}{\lambda} & 0\\ \lambda X^2 & 0 & -u_xX\end{pmatrix}+\begin{pmatrix}0 & -\frac{u_x}{\lambda} & 0\\ -X & 0 & \frac{u_x}{\lambda}\\ 0 & -X & 0\end{pmatrix}. \tag{155}$$

So we almost have that $V(-\infty)=V(\infty)$. If

$$D=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}, \tag{156}$$

we can write $V(a)=DV(-a)D$. This means that

$$\partial_tM(t)=DV(-a;t)DM(t)-M(t)V(-a;t). \tag{157}$$

Multiplying by D from the left, it can be seen that the trace of $DM(t)$ will be constant in time

$$\begin{aligned}\partial_t\mathrm{tr}(DM(t)) &= \mathrm{tr}(D\partial_tM(t))\\ &= \mathrm{tr}(DDV(-a;t)DM(t)-DM(t)V(-a;t))\\ &= \mathrm{tr}(V(-a;t)DM(t)-DM(t)V(-a;t))\\ &= \mathrm{tr}([V(-a;t),DM(t)])\\ &= 0.\end{aligned} \tag{158}$$

Thus the sum $M(t;\lambda)[1,1] - M(t;\lambda)[2,2] + M(t;\lambda)[3,3]$ forms a polynomial in λ on the form

$$C_0 + C_1\lambda + C_2\lambda^2 + \cdots + C_n\lambda^n. \tag{159}$$

where every coefficient C_i is a constant of motion.

The jump matrix can be written as

$$S_k = D + \hat{S}_k = D + \begin{pmatrix} m_k \\ 1 \\ \lambda m_k x_k \end{pmatrix} \begin{pmatrix} -2\lambda m_k x_k & 2 & 2m_k \end{pmatrix}. \tag{160}$$

Then the Monodromy matrix will be

$$M = (D + \hat{S}_n)\cdots(D + \hat{S}_2)(D + \hat{S}_1). \tag{161}$$

The product of all $(D + \hat{S}_k)$ matrices will be a sum of 2^n terms, where each term is a product involving combinations of D and $\hat{S}_k$. To establish notation, we introduce the following

- $P = \{p_1, \ldots, p_k\}$, an ordered subset of $\{1, \ldots, N\}$ with $p_1 < p_2 < \cdots < p_k$.
- $A_{ij} = \lambda m_i m_j (x_j - x_i)$.
- $S_{ij}^{\pm} = 2\big(\pm 1 + \lambda m_i m_j (x_j - x_i)\big) = 2(\pm 1 + A_{ij})$.

The product of any two $\hat{S}_k$ matrices will result in a dot product between the inner two vectors

$$\begin{aligned}
\hat{S}_i\hat{S}_j &= \begin{pmatrix} m_i \\ 1 \\ \lambda m_i x_i \end{pmatrix} \begin{pmatrix} -2\lambda m_i x_i & 2 & 2m_i \end{pmatrix} \begin{pmatrix} m_j \\ 1 \\ \lambda m_j x_j \end{pmatrix} \begin{pmatrix} -2\lambda m_j x_j & 2 & 2m_j \end{pmatrix} \\
&= 2(1 + \lambda m_i m_j (x_j - x_i)) \begin{pmatrix} m_i \\ 1 \\ \lambda m_i x_i \end{pmatrix} \begin{pmatrix} -2\lambda m_j x_j & 2 & 2m_j \end{pmatrix} \\
&= 2(1 + A_{ij}) \begin{pmatrix} m_i \\ 1 \\ \lambda m_i x_i \end{pmatrix} \begin{pmatrix} -2\lambda m_j x_j & 2 & 2m_j \end{pmatrix}
\end{aligned} \tag{162}$$

Any D matrices that appear between the S^k matrices will just flip the sign of the constant term like this:

$$\hat{S}_i D^n \hat{S}_j = 2((-1)^n + A_{ij}) \begin{pmatrix} m_i \\ 1 \\ \lambda m_i x_i \end{pmatrix} \begin{pmatrix} -2\lambda m_j x_j & 2 & 2m_j \end{pmatrix} \tag{163}$$

The last outer product that remains will also become a dot product after taking the trace.

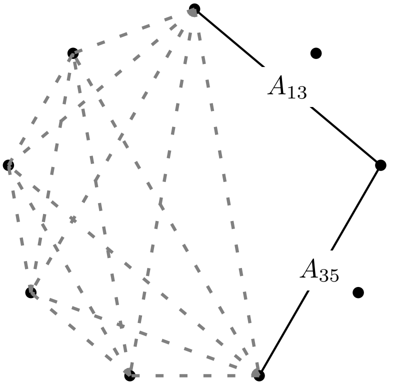
$$\mathrm{tr}\left(D\begin{pmatrix} m_i \\ 1 \\ \lambda m_i x_i \end{pmatrix}\begin{pmatrix} -2\lambda m_j x_j & 2 & 2m_j \end{pmatrix}\right) = -2(1 + A_{ij}). \tag{164}$$

It's important to note that the order of the indices in the A_{ij} is always the same. This means that after taking the trace of DM we will get a sum of products of $2(\pm 1 + A_{ij})$ factors. Thus the trace of the term in DM involving only the S_k matrices with $k \in P$ will be

$$\prod_{i=1}^{k} A^{\pm}_{p_i p_{i+1}}. \tag{165}$$

Next we will focus on the ways to get a term involving only a specific A_{ij} factors.

The terms involving the A_{ij} factors can be visualized as a noncrossing path in a complete planar graph where the nodes are the S_k and the edges are the S_{ij} factors. The path has to start and end at the same node. Here is an example of the ways to get the term involving only $A_{13}A_{35}$ times a constant with $N = 9$:



The term can only come form a matrix product involving $\hat{S}_1 D \hat{S}_3 D \hat{S}_5$. The D matrices between the S_1, S_3, and S_5 doesn't contribute anything to the final product since we only focus on the A_{ij} factors and the D matrices will only flip the sign of the constant term.

To determine the constant term we get from the other factors, we will first consider if there were no nodes between S_5 and S_1. Then the only factor would be $S_{51} = (2 + A_{51})$ which would result in a constant term of 2, since we don't want to add more A_{ij} factors.

Adding another node between any nodes will result in a multiplication with a term on the form $D+(2+A_{ij})$. In the case were we only care about the constant terms we can see the we will be left with $-1+2=1$, where the -1 comes from that the D matrix will flip the sign of the constant term. So the constant factor we get whenever there is a gap between two any A_{ij} and A_{kl} factors is always 2.

7.3 Discussion of the constants of motion

Here we will discuss the constants of motion we get from the monodromy matrix. We will see that we seem to get $N-1$ independent constants of motion for each N and that the coefficient of the lowest and highest term of λ yield the constants of motion

$$C_1 = \sum_{i=1}^{N}\sum_{j=1}^{N} m_i m_j |x_i - x_j|, \quad \text{and} \quad C_N = \prod_{i=1}^{N} m_i^2 (x_i - x_{i-1}). \tag{166}$$

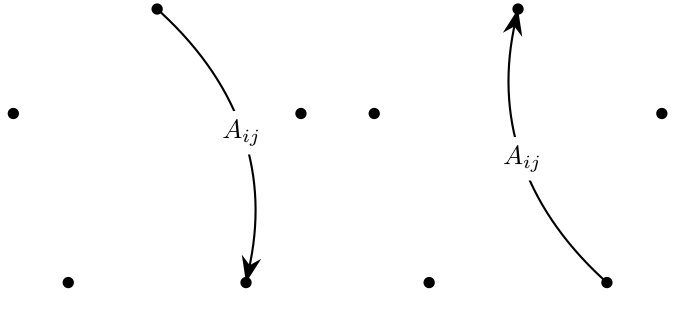
We will also see that the coefficient of the second highest term is equal to the square of the coefficient of the lowest term

$$C_2 = \sum_{i=1}^{N}\sum_{j=1}^{N} (m_i m_j (x_i - x_j))^2 = C_1^2. \tag{167}$$

The rest of the terms seem to be functionally independent. We will now derive these constants of motion from the monodromy matrix.

Remember that the terms involving the A_{ij} factors can be visualized as a non-crossing path in a complete planar graph where the nodes are the S_k and the edges are the A_{ij} factors.

The terms with only one A_{ij} factor will come from the paths



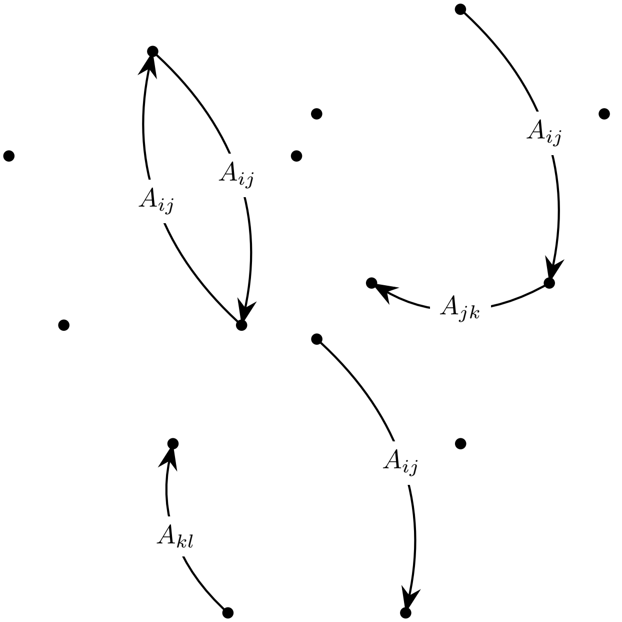
This results in the following terms

$$\sum_{i=1}^{N}\sum_{j=i+1}^{N} -8A_{ij} = -8C_1 \tag{168}$$

The C_N coefficient comes from the product of all the A_{ij} factors

$$-\prod_{i=1}^{N} 2^N A_{i,i+1} = -2^N C_N \tag{169}$$

Now let's prove that $C_2 = C_1^2$. The C_2 coefficient will be the sum of all the terms involving two A_{ij} factors. The different configurations of factors A_{ij} is A_{ij}^2, $A_{ij}A_{jk}$, and $A_{ij}A_{kl}$, where $i < j < k < l$.



Each combination will have different factors of 2 depending on the number of gaps between the paths. The terms will have the following form

$$4A_{ij}^2, \quad 8A_{ij}A_{jk}, \quad 16A_{ij}A_{kl} \tag{170}$$

Taking the square of C_1 we get

$$C_1^2 = \left(\sum_{i<j} A_{ij}\right)^2 = \sum_{i<j} A_{ij}^2 + \sum_{i<j<k} 2A_{ij}A_{jk} + \sum_{i<j<k<l} 2(A_{ij}A_{kl} + A_{ik}A_{jl}). \tag{171}$$

The $A_{ik}A_{jl}$ configuration can't be directly from the graph since we can't have crossing paths. But using the following identity

$$2(A_{ij}A_{kl} + A_{il}A_{jk}) = A_{ij}A_{kl} + A_{il}A_{jk} + A_{ik}A_{jl} + A_{il}A_{jk} \tag{172}$$

$$S_k = D + \hat{S}_k = D + \begin{pmatrix} m_k \\ 1 \\ \lambda m_k x_k \end{pmatrix} \begin{pmatrix} -2\lambda m_k x_k & 2 & 2m_k \end{pmatrix}. \tag{173}$$

Then the T_a matrix will be

$$T_a = (D + \hat{S}_n) \cdots (D + \hat{S}_2)(D + \hat{S}_1). \tag{174}$$

The product of all $(D + \hat{S}_k)$ matrices will be a sum of 2^n terms where each term is a product of D and $\hat{S}_k$ matrices. We will take a look at the term that involves only a specific subset of the $\hat{S}_k$ matrices.

Call the subset of matricies Σ, and let $\sigma_1, \sigma_2, \cdots, \sigma_q$, where $\sigma_1 < \sigma_2 < \cdots < \sigma_q$, be the indices of the matrices in Σ. Then the term involving only these matrices and D will be on the form

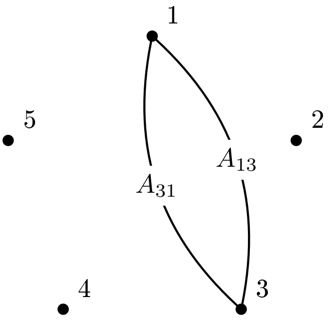
$$T(\Sigma) = 2^q \prod_1^q (\lambda m_{\sigma_i} m_{\sigma_{i+1}} (x_{\sigma_{i+1}} - x_{\sigma_i}) + (-1)^{\sigma_{i+1} - \sigma_i}) \tag{175}$$

where we define the special cases $\sigma_{q+1} := \sigma_1$ and $\sigma_1 - \sigma_q := \sigma_1 - 1 + n - \sigma_q$.

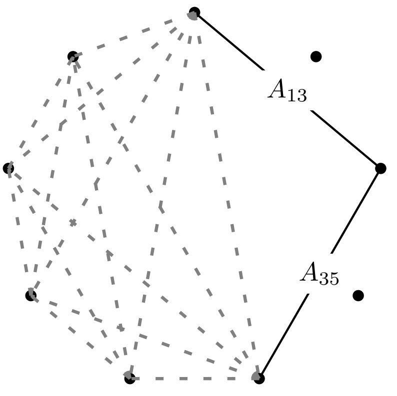
The only way to get a term with only A_{ij}^2 is when there are only two matrices in the subset Σ. Then the final product will be

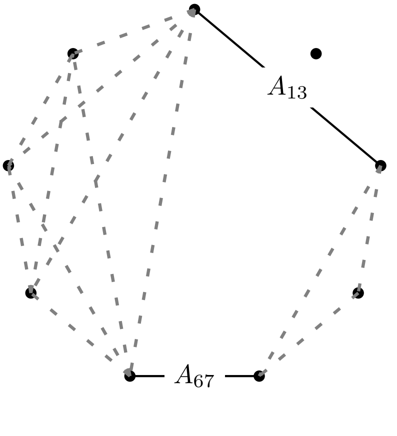
$$4(\lambda A_{ij} + (-1)^{j-i})(\lambda A_{ji} + (-1)^{i-1+n-j}) = 4\lambda^2 A_{ij}A_{ji} + \dots, \tag{176}$$

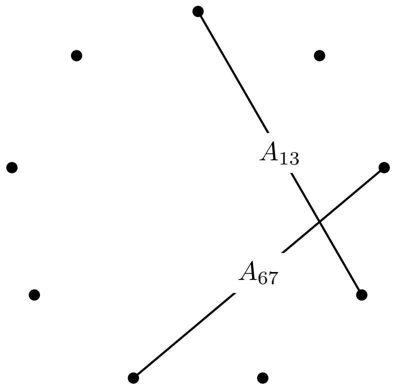
and since $A_{ij} = A_{ji}$ we get $4\lambda^2 A_{ij}^2$.



Let's first look at the case with $A_{ij}A_{jl}$. Here is an example of the term ways to get the term $A_{13}A_{35}$ with $N = 9$:







Then you have the case that will not be a direct term from the S_k matrices from the way we are writing them. That is the case where you have $B_{ik}B_{jl}$ which would give crossing paths in the graph.

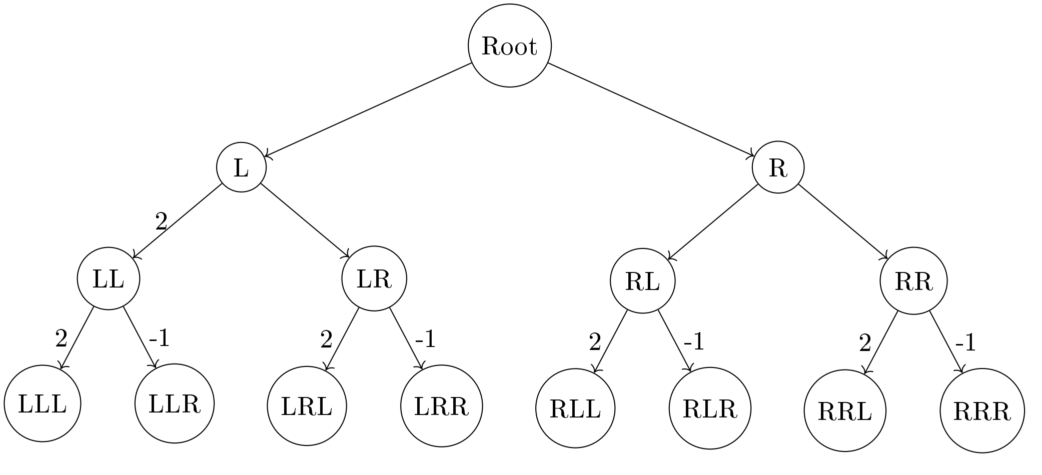
We can however rewrite any $2B_{ij}Bkl + 2B_{li}B_{jk}$ as $B_{ij}B_{kl} + B_{li}B_{jk} + B_{ik}B_{jl}$.

The term can only come from a matrix product involving $\hat{S}_5 D\hat{S}_3 D\hat{S}_1$. Then there is a choice of the other four matrices that can be either D or $\hat{S}_k$.

Choosing a $\hat{S}_k$ matrix will result in a multiplication with a term on the form

$$2\lambda A_{ij} + 2. \tag{177}$$

Since we don't want to add more A_{ij} factors we only care about the multiplication with the 2. If we choose a D matrix it will flip the sign of the next 2 term.



We want to show that the constant of motion with factor λ^2 is equal to the square of the constant of motion with factor λ times a constant factor.

We introduce the following notation

$$A_{ij} = m_i m_j |x_j - x_i|. \tag{178}$$

The term with factor λ is just the sum

$$\sum_{1 \leq i < j \leq N} m_i m_j |x_i - x_j|. \tag{179}$$

So the term with factor λ^2 will be

$$\begin{aligned} &\sum_{i=1}^{N}\sum_{j=1}^{N} A_{ij}^2 \\ +2&\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} A_{ij}A_{jk} \\ +2\sum_{i=1}^{N}&\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{N} A_{ij}A_{kl}. \end{aligned} \tag{180}$$

with $i \neq j \neq k \neq l$ in all sums. So what we want to show is that for all N the terms on the form $A_{ij}A_{jk}$ and $A_{ij}A_{kl}$ will have a constant coefficient twice as big as the coefficient of A_{ij}^2.

So what we want to show is that the terms of $tr(DT_a)$ that have a factor of λ^2 will be equal to the square of the terms with a factor of λ times a constant factor. All terms with a factor of λ^2 will be of the rorm $\lambda^2 A_{ij}A_{kl}$ and all terms with a factor of λ will be of the form λA_{ij}. We just need to show that the coefficient of $\lambda^2 A_{ij}A_{kl}, i \neq j \neq k \neq l$, and $\lambda^2 A_{ij}A_{jk}, i \neq j \neq k$ is twice the coefficient of λA_{ij}^2.

If we look at all the ways to generate terms on that form we see that it can be represented as a loop on a graph. Every A_{ij} will be an edge between the nodes i and j. The edges needs to form a closed loop. All edges will be the factor

$$2\lambda m_i m_j (x_j - x_i) + 2. \tag{181}$$

To get the required factors of A_{ij} only the

A Derivation of the ODE system

This section provides proof of that the ODE-system (11) is obtained from the PDE by plugging in the ansatz $u = \sum m_k |x - x_k|$.

Preliminaries

The jump and the average of f at x_k will be defined as

$$[f(x_k)] := f(x_k^+) - f(x_k^-) \quad \text{and} \quad \langle f(x_k) \rangle := \frac{f(x_k^+) + f(x_k^-)}{2}, \tag{182}$$

respectively. They satisfy the product rules

$$[fg] = \langle f \rangle [g] + [f] \langle g \rangle, \qquad \langle fg \rangle = \langle f \rangle \langle g \rangle + \frac{1}{4}[f][g]. \tag{183}$$

For a continuous function f and a piecewise continuous function g the product rule for jumps simplifies to

$$[fg] = f[g]. \tag{184}$$

In the way the HF-Novikov equation is written here

$$u_{xxt} = -u^2 u_{xxx} - 3uu_x u_{xx}, \tag{185}$$

there is an issue with term $3uu_x u_{xx}$, since u_x isn't defined at x_k and u_{xx} is a Dirac delta function. One way to get around that is to write the HF-Novikov equation as

$$-\partial_x^2 u_t - \partial_x^2 u^2 u_x + \partial_x \frac{3}{2} u u_x^2 + \frac{1}{2} u_x^3 = 0. \tag{186}$$

The first term is

$$-\partial_x^2 u_t = -\sum \dot{m}_k 2\delta_{x_k} + \sum 2\dot{x}_k m_k \delta'_{x_k}. \tag{187}$$

The second term $-\partial_x^2 u^2 u_x$ will have the two singular terms

$$-\sum [(u^2 u_x)_x] \delta_{x_k} - \sum [u^2 u_x] \delta'_{x_k}. \tag{188}$$

Let's first look at the δ' term at x_k

$$-[u^2 u_x]_{x_k} = -u^2(x_k) \underbrace{[u_x(x_k)]}_{2m_k} = -2m_k u^2(x_k). \tag{189}$$

Next let's look at the δ term at x_k

$$\begin{aligned}
-[(u^2u_x)_x]_{x_k} &= -[2uu_x^2 + u^2u_{xx}]_{x_k} \\
&= -2u(x_k)\underbrace{[u_x^2(x_k)]}_{2\langle u_x\rangle[u_x]} - u^2(x_k)\underbrace{[u_{xx}(x_k)]}_{=0} \\
&= -4u(x_k)\langle u_x(x_k)\rangle\underbrace{[u_x(x_k)]}_{2m_k} = -8m_k u(x_k)u_x(x_k).
\end{aligned} \tag{190}$$

The third term $\partial_x \frac{3}{2}uu_x^2$ will have one singular term

$$\sum[\frac{3}{2}uu_x^2]\delta_{x_k}, \tag{191}$$

which at x_k will be

$$[\frac{3}{2}uu_x^2]_{x_k} = \frac{3}{2}u(x_k)\underbrace{[u_x^2(x_k)]}_{2\langle u_x\rangle[u_x]} = 3\langle u_x(x_k)\rangle\underbrace{[u_x(x_k)]}_{2m_k} = 6m_k u(x_k)u_x(x_k). \tag{192}$$

The fourth term will not have any singular terms. The δ and δ' distributions are linearly independent so the terms can be split up into separate equations. At x_k the sum from all the terms for δ and δ' is

$$0 = -2\dot{m}_k - [(u^2u_x)_x]_{x_k} + [\frac{3}{2}uu_x^2]_{x_k} = -2\dot{m}_k - 2m_k u(x_k)u_x(x_k), \tag{193}$$

respectively

$$0 = 2\dot{x}_k m_k - [u^2u_x]_{x_k} = 2\dot{x}_k m_k - 2m_k u^2(x_k). \tag{194}$$

The $\dot{m}_k$ equation can be written as

$$\dot{m}_k = -m_k u_x(x_k)u(x_k). \tag{195}$$

It can be assumed that $m_k \neq 0$ since $\dot{m}_k = m_k \cdot (\text{something})$ implies that either $m_k(t) = 0$ for all t or $m_k(t) \neq 0$ for all t. If $m_k = 0$ it doesn't contribute anything to u and can therefore be ignored. With $m_k \neq 0$ the expression can be divided by m_k to get

$$\dot{x}_k = u(x_k)^2. \tag{196}$$

The time derivatives you get from periodic ansatz (41) is of the same form as above. The only difference is that the functions $u(x,t)$ and $u_x(x,t)$ will be the periodic versions of the functions. This is because

$$u_{xx} = \sum 2m_k(\delta_x - \delta_{[x_k+L]_{2L}}), \tag{197}$$

so every singular term at x_k will have a corresponding singular term at $[x_k+L]_{2L}$ that will be the same but with a different sign. These terms will give the same form of equations for $\dot{x}_k$ and $\dot{m}_k$ as above.

B Verification of the Lax pair for the ansatz

Mostly just a copy from the Novikov paper. Is it enough to reference that paper or it good to include again here too?
Todo: Need to double check that it's correct.

Generally a function needs to be continuous for the it's multiplication with a Dirac delta function to be well defined. Which creates a problem with the multiplication of $u_x\delta_{x_k}$ since u_x is not continuous at x_k. Therefore we will define $u_x\delta_{x_k} := \langle u_x(x_k)\rangle\delta_{x_k}$. The Lax pair formulation is

$$\partial_x\Psi = \hat{U}\Psi, \qquad \partial_t\Psi = \hat{V}\Psi. \tag{198}$$

In this case $\Psi = (\psi_1, \psi_2, \psi_3)^T$ and $\hat{U}$ and $\hat{V}$ are matrices. Both $\hat{U}$ and $\hat{V}$ can be split up into a regular and a singular part

$$\hat{U} = U + zmN, \qquad \hat{V} = V - zu^2mN, \tag{199}$$

where

$$U = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, V = \begin{pmatrix} -uu_x & \frac{u_x}{z} & u_x^2 \\ \frac{u}{z} & -\frac{1}{z^2} & -\frac{u_x}{z} \\ -u^2 & \frac{u}{z} & uu_x \end{pmatrix}, N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{200}$$

We also have that

$$u = \sum m_k|x - x_k|, \quad u_x = \sum m_k \operatorname{sgn}(x - x_k), \quad m = u_{xx} = \sum 2m_k\delta(x - x_k). \tag{201}$$

Next we do the mixed partials

$$\begin{aligned} \partial_t\partial_x\Psi &= \partial_t(U\Psi + zmN\Psi) \\ &= (\underbrace{U_t}_{=0} + U\hat{V})\Psi + (zmN\Psi)_t \\ &= UV\Psi - 2zu^2mUN\Psi + 2zN\sum((m_k\Psi(x_k))_t - m_k\dot{x}_k\Psi(x_k)\dot{x}_k\delta'_{x_k}), \\ \partial_x\partial_t\Psi &= \partial_t(V\Psi + zu^2mN\Psi) \\ &= (V_x + VU)\Psi + \sum[V\Psi]\delta_{x_k} + 2zN\sum(u(x_k)^2m_k\Psi(x_k)\delta'_{x_k}). \end{aligned} \tag{202}$$

The mixed partials should be equal. The terms involving δ'_{x_k} are equal since $\dot{x}_k = u(x_k)^2$. The regular of the mixed partials is

$$UV - VU - V_x = \begin{pmatrix} uu_{xx} & -\frac{u_{xx}}{z} & -2u_xu_{xx} \\ 0 & 0 & \frac{u_{xx}}{z} \\ 0 & 0 & -uu_{xx} \end{pmatrix}, \tag{203}$$

which is identically zero, since the regular part of u_{xx} is zero. Thus the compatibility condition reduces to an equality between the coefficients of δ_{x_k},

$$-2zm_k u(x_k)^2 UN\Psi(x_k) + (zmN\Psi(x_k))_t - \sum[V\Psi(x_k)] = 0. \tag{204}$$

Using the produce rule (183) and that $[\Psi(x_k)] = 2zm_k N\Psi(x_k)$ we get that

$$\begin{aligned}\sum[V\Psi(x_k)]\delta_{x_k} &= \sum\langle V(x_k)\rangle 2zmN\Psi(x_k) + [V(x_k)]\langle\Psi(x_k)\rangle \\ &= 2zm_k \begin{pmatrix} 0 & -u\langle u_x\rangle & \langle u_x\rangle/z \\ 0 & u/z & -1/z^2 \\ 0 & -u^2 & u/z \end{pmatrix}_{x_k} \Psi(x_k) + 2m_k \begin{pmatrix} u & -1/z & -2\langle u_x\rangle \\ 0 & 0 & 1/z \\ 0 & 0 & -u \end{pmatrix}_{x_k} \langle\Psi(x_k)\rangle.\end{aligned} \tag{205}$$

The $(3,2)$ element u^2 in the first term above is going to cancel out the first term in (204) since UN's only non zero element is the $(3,2)$ element which is one. What's then left of the compatibility condition is

$$\begin{aligned}&\dot{m}_k N\Psi(x_k) + m_k N\partial_t\Psi(x_k) \\ &= m_k \begin{pmatrix} 0 & -u\langle u_x\rangle & \langle u_x\rangle/z \\ 0 & u/z & -1/z^2 \\ 0 & 0 & u/z \end{pmatrix}_{x_k} \Psi(x_k) + m_k \begin{pmatrix} u & -1/z^2 & -2\langle u_x\rangle/z \\ 0 & 0 & 1/z^2 \\ 0 & 0 & -u/z \end{pmatrix}_{x_k} \langle\Psi(x_k)\rangle.\end{aligned} \tag{206}$$

Next let's compute $N\partial_t\langle\Psi(x_k)\rangle$

$$\begin{aligned}N\partial_t\langle\Psi(x_k)\rangle &= N(\langle U\Psi(x_k)\rangle\dot{x}_k + \langle V\Psi(x_k)\rangle) \\ &= N(Uu(x_k)^2\langle A(x_k)\rangle)\langle\Psi(x_k)\rangle + N\frac{1}{4}[A(x_k)][\Psi(x_k)] \\ &= \begin{pmatrix} u/z & -1/z^2 & -\langle u_x\rangle/z \\ 0 & u/z & u\langle u_x\rangle \\ 0 & 0 & 0 \end{pmatrix}_{x_k} \langle\Psi(x_k)\rangle + \frac{1}{4}\underbrace{N[A(x_k)]N}_{=0} 2zm_k\Psi(x_k).\end{aligned} \tag{207}$$

After a bit of manipulation and using that $\langle\psi_3\rangle(x_k) = \psi_3(x_k)$, we can show that the compatibility condition can be written as

$$\begin{aligned}m_k N\partial_t(\Psi(x_k) - \langle\Psi(x_k)\rangle) + (\dot{m}_k + m_k u(x_k)\langle u_x(x_k)\rangle)N\Psi(x_k) \\ = m_k \begin{pmatrix} 0 & 0 & 0 \\ 0 & u/z & 0 \\ 0 & 0 & 0 \end{pmatrix}_{x_k} (\Psi(x_k) - \langle\Psi(x_k)\rangle).\end{aligned} \tag{208}$$

The third row is zero and the first two rows says that

$$\begin{aligned}(\dot{m}_k + m_k u(x_k)\langle u_x(x_k)\rangle)\psi_2(x_k) &= -m_k\partial_t(\psi_2(x_k) - \langle\psi_2(x_k)\rangle), \\ (\dot{m}_k + m_k u(x_k)\langle u_x(x_k)\rangle)\psi_3(x_k) &= \frac{1}{z}m_k u(x_k)(\psi_2(x_k) - \langle\psi_2(x_k)\rangle).\end{aligned} \tag{209}$$

If we choose to assign $\psi_2(x_k) = \langle\psi_2(x_k)\rangle$ then it's clear that the compatibility condition is satisfied only if

$$\dot{m}_k = -m_k u(x_k)\langle u_x(x_k)\rangle. \tag{210}$$

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