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Abstract

A summary of the thesis, presenting the important results.

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Contents

Chapter 1

Theory

This whole chapter is very much a work in progress.

1.1 Novikov Equation

The Novikov equation is given by:

$$m_t + ((um)_x + 2u_x m)u = 0, \quad m = u - u_{xx}. \tag{1.1}$$

It was discovered by Vladimir Novikov in a classification of cubically nonlinear PDEs admitting infinitely many symmetries[1]. The Novikov equation has peakon solutions, given by:

$$u(x,t) = \sum_{k=1}^{N} m_k(t) e^{-|x - x_k(t)|}, \tag{1.2}$$

which has weak peakon solutions. Peakons are a type of soliton wave characterized by a sharp peak. The dynamics of these peakons are governed by a set of ordinary differential equations (ODEs):

$$\dot{x}_k = u(x_k)^2, \quad \dot{m}_k = -m_k u_x(u_k) u(x_k). \tag{1.3}$$

Where u_x is the derivative of u with respect to x and $\dot{x}_k$ and $\dot{m}_k$ are the time derivatives of x_k and m_k respectively. Our focus is on the high-frequency limit of the Novikov equation, which will be demonstrated to yield a piecewise solution of the form:

$$u(x,t) = \sum_{k=1}^{N} m_k(t) |x - x_k(t)|. \tag{1.4}$$

It maintains the same form of ODE equations (1.3), but with with the linear equation (1.4) instead of the peakon equation (1.2). The piecewise linear form can be obtained by taking the second term of the Maclaurin expansion of (1.2). This is a VERY rough explanation of the high-frequency limit. It will be expanded upon.

1.2 Lax Pairs

Lax Pairs are a mathematical framework used to analyze and solve certain types of nonlinear PDEs. The concept of Lax Pairs was introduced by Peter Lax in 1968. A Lax Pair consists of two operators, L and A, that satisfy a specific compatibility condition. The existence of a Lax Pair for a nonlinear PDE is a strong indicator of the equation's integrability. It implies the presence of an infinite number of conservation laws and allows for the application of powerful analytical methods, such as the inverse spectral transform (IST), to find exact solutions.

Definition

A Lax Pair is defined by two differential operators $L(t)$ and $A(t)$, which depend on a temporal parameter t and satisfy the Lax equation:

$$\dot{L} = [L, A] \equiv LA - AL, \tag{1.5}$$

where $[L, A]$ denotes the commutator of L and A. The operator L is associated with a linear eigenvalue problem, and A governs the time evolution of the eigenfunctions, like this:

$$L\psi = \lambda\psi, \quad \dot{\psi} = A\psi. \tag{1.6}$$

In many cases, Lax Pairs are represented in matrix form, enabling a more straightforward application of the IST method. This approach is particularly useful for systems where the operator algebra becomes cumbersome.

Chapter 2

Solution

2.1 N = 2 solution to the high frequency limit Novikov

The assumption that $x_1 < x_2 < ... < x_N$ can be made without loss of generality. We can also assume that $m_k \neq 0$ since if it were equal to zero it wouldn't affect the equation. For $N = 1$, the solution is trivial since $u(x_k) = 0$. For the more interesting case when $N = 2$ we get the following system of ODEs from equation (1.3):

$$\dot{x}_1 = m_2^2(x_2 - x_1)^2, \tag{2.1}$$

$$\dot{x}_2 = m_1^2(x_2 - x_1)^2, \tag{2.2}$$

$$\dot{m}_1 = m_1 m_2^2(x_2 - x_1), \tag{2.3}$$

$$\dot{m}_2 = -m_1^2 m_2(x_2 - x_1). \tag{2.4}$$

To solve this system, we first identify conserved quantities:

$$(m_1^2 + m_2^2)_t = 0, \tag{2.5}$$

$$(m_1 m_2(x_2 - x_1))_t = 0. \tag{2.6}$$

To show why these are conserved, we take the time derivative of the first equation:

$$\begin{aligned}(m_1^2 + m_2^2)_t &= 2m_1\dot{m}_1 + 2m_2\dot{m}_2 \\ &= 2m_1^2 m_2^2(x_2 - x_1) - 2m_1^2 m_2^2(x_2 - x_1) = 0,\end{aligned} \tag{2.7}$$

$$
\begin{aligned}
(m_1 m_2(x_2 - x_1))_t &= \dot{m}_1 m_2(x_2 - x_1) + m_1 \dot{m}_2(x_2 - x_1) \\
&- m_1 m_2 \dot{x}_1 + m_1 m_2 \dot{x}_2 \\
&= m_1 m_2^3 (x_2 - x_1)^2 - m_1^3 m_2 (x_2 - x_1)^2 \\
&- m_1 m_2^3 (x_2 - x_1)^2 + m_1^3 m_2 (x_2 - x_1)^2 \\
&= 0.
\end{aligned} \tag{2.8}
$$

Denoting these conserved quantities as M_1 and M_2 respectively, we can express the conservation laws as:

$$
m_1^2 + m_2^2 = M_1, \tag{2.9}
$$

$$
m_1 m_2 (x_2 - x_1) = M_2. \tag{2.10}
$$

Leveraging these conserved quantities, we derive expressions for $\dot{m}_1$ and $\dot{m}_2$:

$$
\begin{cases}
\dot{m}_1 = m_2 M_2, \\
\dot{m}_2 = -m_1 M_2.
\end{cases} \tag{2.11}
$$

The solutions to these equations take the form:

$$
\begin{cases}
m_1 = A\cos(M_2 t) + B\sin(M_2 t) = \sqrt{M_1}\cos(M_2 t - \phi), \\
m_2 = -A\sin(M_2 t) + B\cos(M_2 t) = -\sqrt{M_1}\sin(M_2 t - \phi), \\
\quad \phi = \operatorname{atan2}(B, A).
\end{cases} \tag{2.12}
$$

Atan2 is a function that returns the angle whose tangent is the quotient of its arguments and is used to determine the correct quadrant of the angle. It is defined as:

$$
\operatorname{atan2}(y, x) = \begin{cases}
\arctan(y/x) & \text{if } x > 0, \\
\arctan(y/x) + \pi & \text{if } x < 0, y \geq 0, \\
\arctan(y/x) - \pi & \text{if } x < 0, y < 0, \\
\quad \pi/2 & \text{if } x = 0, y > 0, \\
-\pi/2 & \text{if } x = 0, y < 0, \\
\text{undefined} & \text{if } x = 0, y = 0.
\end{cases} \tag{2.13}
$$

Now we can solve for $\dot{x}_1$ and $\dot{x}_2$:

$$
\begin{cases}
\dot{x}_1 m_1^2 = m_1^2 m_2^2 (x_2 - x_1) = M_2^2, \\
\dot{x}_2 m_2^2 = m_1^2 m_2^2 (x_2 - x_1) = M_2^2,
\end{cases} \tag{2.14}
$$

$$
\implies \begin{cases}
\dot{x}_1 = \dfrac{M_2^2}{m_1^2} = \dfrac{M_2^2}{M_1 \cos^2(M_2 t - \phi)}, \\[2ex]
\dot{x}_2 = \dfrac{M_2^2}{m_2^2} = \dfrac{M_2^2}{M_1 \cos^2(M_2 t - \phi)}.
\end{cases} \tag{2.15}
$$

Integration yields the positions:

$$\begin{cases} x_1 = \dfrac{M_2}{M_1}\tan(M_2 t - \phi) + C, \\ x_2 = -\dfrac{M_2}{M_1}\cot(M_2 t - \phi) - D. \end{cases} \tag{2.16}$$

The M_2 quantity implies that $D = -C$:

$$M_2 = m_1 m_2 (x_2 - x_1) = M_2 + M_1 \cos(M_2 t - \phi)\sin(M_2 t - \phi)(C + D)). \tag{2.17}$$

In conclusion the piecewise solution looks like this:

$$u(x,t) = m_1|x - x_1| + m_2|x - x_2|, \tag{2.18}$$

$$m_1 = +\sqrt{M_1}\cos(M_2 t - \phi), \tag{2.19}$$

$$m_2 = -\sqrt{M_1}\sin(M_2 t - \phi), \tag{2.20}$$

$$x_1 = +\frac{M_2}{M_1}\tan(M_2 t - \phi) + C, \tag{2.21}$$

$$x_2 = -\frac{M_2}{M_1}\cot(M_2 t - \phi) + C. \tag{2.22}$$

There are four constants M_1, M_2, ϕ and C, which is to be expected since we had four initial quantities, $m_1(0)$, $m_2(0)$, $x_1(0)$ and $x_2(0)$. The solution is valid only for a finite time, since we made the assumtion that $x_1 < x_2$. The assumption that $x_1 < x_2$ is only valid for finite intervals since tan and cot are periodic functions.

Bibliography

[1] Vladimir Novikov. Generalizations of the Camassa–Holm equation. *Journal of Physics A: Mathematical and Theoretical*, 42(34):342002, Aug 2009.