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Abstract

A summary of the thesis, presenting the important results.

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Contents

Chapter 1

Theory

This whole chapter is very much a work in progress.

1.1 Introduction

The Novikov equation is given by:

$$m_t + ((um)_x + 2u_x m)u = 0, \quad m = u - u_{xx}. \tag{1.1}$$

It was discovered by Vladimir Novikov in a classification of cubically nonlinear PDEs admitting infinitely many symmetries[1]. The Novikov equation has peakon solutions, given by:

$$u(x,t) = \sum_{k=1}^{N} m_k(t) e^{-|x-x_k(t)|}. \tag{1.2}$$

Peakons are a type of soliton wave characterized by a sharp peak. The Novikov equation is closely related to the Camassa-Holm equation and the Degasperis-Procesi equation. These equations are all members of the class of integrable partial differential equations known as peakon equations. The Camassa-Holm equation is given by:

$$m_t + (um)_x + u_x m = 0, \quad m = u - u_{xx}. \tag{1.3}$$

It was discovered by Camassa and Holm in 1993[?], who studied unidirectional flow in shallow water. The Degasperis-Procesi equation is given by:

$$m_t + (um)_x + 2u_x m = 0, \quad m = u - u_{xx}. \tag{1.4}$$

It was discovered by Degasperis and Procesi in 1999[**?**]. These equations share similar properties with the Novikov equation, such as the existence of peakon solutions and the presence of infinitely many symmetries. For a more detailed discussion of the peakon equations and its properties, see the works of Lundmark and Szmigielski[**?**].

The focus of the thesis is on the high-frequency limit of the Novikov equation. The high-frequency limit is obtained by substitution of $x \mapsto \epsilon x$, $t \mapsto \epsilon t$, and letting $\epsilon \to 0$. This limit gives the following equation:

$$m_t + ((um)_x + 2u_x m)u = 0, \quad m = u_{xx}. \tag{1.5}$$

which is the same as the Novikov equation, but with $m = u_{xx}$ instead of $m = u - u_{xx}$. Both the CH and DP equations have been studied in the high-frequency limit, and it has been shown that they yield piecewise linear solutions[**?**]. The high-frequency limit of the Novikov equation is expected to yield similar piecewise linear solutions. The high frequency limit of the Camassa Holm equation yields the Hunter-Saxton equation[**?**, **?**] for nematic liquid crystals. The high frequency limit of the Degasperis-Procesi equation yields the derivative Burgers equation[**?**, **?**].

$m = u - u_{xx}$	$m = u_{xx}$
Camassa–Holm	Hunter–Saxton
Degasperis–Procesi	Derivative Burgers
Novikov	HF Novikov

The CH and DP have had piecewise solutions on the following form:

$$u(x,t) = \sum_{k=1}^{N} m_k(t)|x - x_k(t)|. \tag{1.6}$$

So we will make the assumption that the high frequency limit of the Novikov equation will have a similar piecewise solution.

Theorem 1.1.1. *The time derivate of x_k and m_k will be governed by the following ODEs:*

$$\dot{x}_k = u(x_k)^2, \quad \dot{m}_k = -m_k u_x(x_k)u(x_k). \tag{1.7}$$

Dots denote $\frac{d}{dt}$ as usual.

Proof. If we plug in the peakon solution (1.2) into the Novikov equation (1.1):

$$u_{xxt} = -u^2 u_{xxx} - 3uu_x u_{xx}, \tag{1.8}$$

we get:

$$\sum_{k=1}^{N} \dot{m}_k(t)\delta(x-x_k) - \sum_{k=1}^{N} \dot{x}_k m_k \delta_x(x-x_k) = \\ -\sum_{k=1}^{N} u^2 m_k \delta_x(x-x_k) - 3\sum_{k=1}^{N} u u_x m_k \delta(x-x_k). \tag{1.9}$$

At $x = x_k$ we get:

$$\dot{m}_k = 2m_k u_x(x_k)u(x_k) - 3m_k u_x(x_k)u(x_k) = -m_k u_x(x_k)u(x_k) \tag{1.10}$$

Multiplying by $(x - x_k)$ first and then plugging in $x = x_k$ we get: This feels a bit iffy, but maybe it's okey to do in a distributional sense? It gives the right result at least. (For $m_k \neq 0$...)

$$m_k \dot{x}_k = m_k u(x_k)^2 \tag{1.11}$$

□

The time derivate of x_k and m_k for the original Novikov equation have the same form, but then u is the peakon equation(1.2) instead of the linear equation(1.6).

1.2 Lax Pairs

Lax Pairs are a mathematical framework used to analyze and solve certain types of nonlinear PDEs. The concept of Lax Pairs was introduced by Peter Lax in 1968. A Lax Pair consists of two operators, L and A, that satisfy a specific compatibility condition. The existence of a Lax Pair for a nonlinear PDE is a strong indicator of the equation's integrability. It implies the presence of an infinite number of conservation laws and allows for the application of powerful analytical methods, such as the inverse spectral transform (IST), to find exact solutions.

Definition

A Lax Pair is defined by two differential operators $L(t)$ and $A(t)$, which depend on a temporal parameter t and satisfy the Lax equation:

$$\dot{L} = [L, A] \equiv LA - AL, \tag{1.12}$$

where $[L, A]$ denotes the commutator of L and A. The operator L is associated with a linear eigenvalue problem, and A governs the time evolution of the eigenfunctions, like this:

$$L\psi = \lambda\psi, \quad \dot{\psi} = A\psi. \tag{1.13}$$

In many cases, Lax Pairs are represented in matrix form, enabling a more straightforward application of the IST method. This approach is particularly useful for systems where the operator algebra becomes cumbersome.

Chapter 2

Solution

2.1 N = 2 solution to the high frequency limit Novikov

The assumption that $x_1 < x_2 < ... < x_N$ can be made without loss of generality. We can also assume that $m_k \neq 0$ since if it were equal to zero it wouldn't affect the equation. For $N = 1$, the solution is trivial since $u(x_k) = 0$. For the more interesting case when $N = 2$ we get the following system of ODEs from equation (1.7):

$$\dot{x}_1 = m_2^2(x_2 - x_1)^2, \tag{2.1}$$

$$\dot{x}_2 = m_1^2(x_2 - x_1)^2, \tag{2.2}$$

$$\dot{m}_1 = m_1 m_2^2(x_2 - x_1), \tag{2.3}$$

$$\dot{m}_2 = -m_1^2 m_2(x_2 - x_1). \tag{2.4}$$

To solve this system, we first identify conserved quantities:

$$(m_1^2 + m_2^2)_t = 0, \tag{2.5}$$

$$(m_1 m_2(x_2 - x_1))_t = 0. \tag{2.6}$$

To show why these are conserved, we take the time derivative of the first equation:

$$\begin{aligned}(m_1^2 + m_2^2)_t &= 2m_1\dot{m}_1 + 2m_2\dot{m}_2 \\ &= 2m_1^2m_2^2(x_2 - x_1) - 2m_1^2m_2^2(x_2 - x_1) = 0,\end{aligned} \tag{2.7}$$

$$
\begin{aligned}
(m_1 m_2(x_2 - x_1))_t &= \dot{m}_1 m_2(x_2 - x_1) + m_1 \dot{m}_2(x_2 - x_1) \\
&\quad - m_1 m_2 \dot{x}_1 + m_1 m_2 \dot{x}_2 \\
&= m_1 m_2^3 (x_2 - x_1)^2 - m_1^3 m_2 (x_2 - x_1)^2 \\
&\quad - m_1 m_2^3 (x_2 - x_1)^2 + m_1^3 m_2 (x_2 - x_1)^2 \\
&= 0.
\end{aligned} \tag{2.8}
$$

Denoting these conserved quantities as M_1 and M_2 respectively, we can express the conservation laws as:

$$m_1^2 + m_2^2 = M_1, \tag{2.9}$$

$$m_1 m_2 (x_2 - x_1) = M_2. \tag{2.10}$$

Leveraging these conserved quantities, we derive expressions for $\dot{m}_1$ and $\dot{m}_2$:

$$
\begin{cases}
\dot{m}_1 = m_2 M_2, \\
\dot{m}_2 = -m_1 M_2.
\end{cases} \tag{2.11}
$$

The solutions to these equations take the form:

$$
\begin{cases}
m_1 = A\cos(M_2 t) + B\sin(M_2 t) = \sqrt{M_1}\cos(M_2 t - \phi), \\
m_2 = -A\sin(M_2 t) + B\cos(M_2 t) = -\sqrt{M_1}\sin(M_2 t - \phi), \\
\phi = \operatorname{atan2}(B, A).
\end{cases} \tag{2.12}
$$

Atan2 is a function that returns the angle whose tangent is the quotient of its arguments and is used to determine the correct quadrant of the angle. It is defined as:

$$
\operatorname{atan2}(y, x) =
\begin{cases}
\arctan(y/x) & \text{if } x > 0, \\
\arctan(y/x) + \pi & \text{if } x < 0, y \geq 0, \\
\arctan(y/x) - \pi & \text{if } x < 0, y < 0, \\
\pi/2 & \text{if } x = 0, y > 0, \\
-\pi/2 & \text{if } x = 0, y < 0, \\
\text{undefined} & \text{if } x = 0, y = 0.
\end{cases} \tag{2.13}
$$

Now we can solve for $\dot{x}_1$ and $\dot{x}_2$:

$$
\begin{cases}
\dot{x}_1 m_1^2 = m_1^2 m_2^2 (x_2 - x_1) = M_2^2, \\
\dot{x}_2 m_2^2 = m_1^2 m_2^2 (x_2 - x_1) = M_2^2,
\end{cases} \tag{2.14}
$$

$$
\implies
\begin{cases}
\dot{x}_1 = \dfrac{M_2^2}{m_1^2} = \dfrac{M_2^2}{M_1 \cos^2(M_2 t - \phi)}, \\[2ex]
\dot{x}_2 = \dfrac{M_2^2}{m_2^2} = \dfrac{M_2^2}{M_1 \cos^2(M_2 t - \phi)}.
\end{cases} \tag{2.15}
$$

Integration yields the positions:

$$\begin{cases} x_1 = & \dfrac{M_2}{M_1}\tan(M_2 t-\phi)+C, \\ x_2 = & -\dfrac{M_2}{M_1}\cot(M_2 t-\phi)-D. \end{cases} \tag{2.16}$$

The M_2 quantity implies that $D=-C$:

$$M_2 = m_1 m_2(x_2-x_1) = M_2 + M_1\cos(M_2 t-\phi)\sin(M_2 t-\phi)(C+D)). \tag{2.17}$$

In conclusion the piecewise solution looks like this:

$$u(x,t) = m_1|x-x_1| + m_2|x-x_2|, \tag{2.18}$$

$$m_1 = +\sqrt{M_1}\cos(M_2 t-\phi), \tag{2.19}$$

$$m_2 = -\sqrt{M_1}\sin(M_2 t-\phi), \tag{2.20}$$

$$x_1 = +\frac{M_2}{M_1}\tan(M_2 t-\phi)+C, \tag{2.21}$$

$$x_2 = -\frac{M_2}{M_1}\cot(M_2 t-\phi)+C. \tag{2.22}$$

There are four constants M_1, M_2, ϕ and C, which is to be expected since we had four initial quantities, $m_1(0)$, $m_2(0)$, $x_1(0)$ and $x_2(0)$. The solution is valid only for a finite time, since we made the assumtion that $x_1 < x_2$. The assumption that $x_1 < x_2$ is only valid for finite intervals since tan and cot are periodic functions.

2.2 N = 3 solution to the high frequency limit Novikov

The equations for $N=3$ are:

$$\dot{x}_1 = m_2^2(x_2-x_1)^2 + m_3^2(x_3-x_1)^2 + 2m_2m_3(x_2-x_1)(x_3-x_1), \tag{2.23}$$

$$\dot{x}_2 = m_1^2(x_2-x_1)^2 + m_3^2(x_3-x_2)^2 + 2m_1m_3(x_2-x_1)(x_3-x_2), \tag{2.24}$$

$$\dot{x}_3 = m_1^2(x_3-x_1)^2 + m_2^2(x_3-x_2)^2 + 2m_1m_2(x_3-x_1)(x_3-x_2), \tag{2.25}$$

$$\dot{m}_1 = -m_1(-m_2-m_3)(m_2(x_2-x1)+m_3(x_3-x_1)), \tag{2.26}$$

$$\dot{m}_2 = -m_2(+m_1-m_3)(m_1(x_2-x1)+m_3(x_3-x_2)), \tag{2.27}$$

$$\dot{m}_3 = -m_3(+m_1+m_2)(m_1(x_3-x1)+m_2(x_3-x_2)). \tag{2.28}$$

It's a bit more complicated to solve this system, but we can still identify conserved quantities. The first two are:

$$(m_1m_2(x_2-x_1)+m_1m_3(x_3-x_1)+m_2m_3(x_3-x_2))_t = M_1, \tag{2.29}$$

$$(m_1^2m_2^2m_3^2(x_2-x1)(x3-x1)(x_3-x_1))_t = M_2. \tag{2.30}$$

If we let

$$A = m_1^2(M_1+2m_2m_3(x_3-x_2)), \tag{2.31}$$
$$B = m_2^2(M_1+2m_1m_3(x_3-x_1)), \tag{2.32}$$
$$C = m_3^2(M_1+2m_1m_2(x_2-x_1)) \tag{2.33}$$

then we can write the three other conserved quantities as:

$$A \quad +B \quad +C \quad = M_3 \tag{2.34}$$
$$Ax_1+Bx_2+Cx_3 = M_4 \tag{2.35}$$
$$Ax_1^2+Bx_2^2+Cx_3^2 = M_5. \tag{2.36}$$

2.3 Constants of motion

2.3.1 N = 2

$$m_1^2 \quad +m_2^2 \quad = M_1 \tag{2.37}$$
$$m_1^2x_1+m_2^2x_2 = M_2 \tag{2.38}$$
$$m_1^2x_1^2+m_2^2x_2^2 = M_3 \tag{2.39}$$
$$\tag{2.40}$$

$$P = \begin{pmatrix} 0 & m_1m_2(x_2-x_1) \\ m_1m_2(x_2-x_1) & 0 \end{pmatrix} \tag{2.41}$$

Both H_1 and H_2 are conserved quantities:

$$H_1 = m_1m_2(x_2-x_1) \tag{2.42}$$
$$H_2 = (m_1m_2(x_2-x_1))^2 \tag{2.43}$$
$$H_2 = H_1^2 \tag{2.44}$$
$$H_2 = M_1M_3-M_2^2 \tag{2.45}$$

2.3.2 N = 3

$$P = \begin{pmatrix} 0 & m_1m_2(x_2-x_1) & m_1m_3(x_3-x_1) \\ m_1m_2(x_2-x_1) & 0 & m_2m_3(x_3-x_2) \\ m_1m_3(x_3-x_1) & m_2m_3(x_3-x_2) & 0 \end{pmatrix} \tag{2.46}$$

Only H_1 and H_3 are conserved quantities:

$$H_1 = (m_1m_2(x_2-x_1) + m_1m_3(x_3-x_1) + m_2m_3(x_3-x_2))_t, \tag{2.47}$$

$$H_3 = (m_1^2m_2^2m_3^2(x_2-x1)(x3-x1)(x_3-x_1))_t = M_2. \tag{2.48}$$

If we let

$$A = m_1^2(H_1 + 2m_2m_3(x_3-x_2)), \tag{2.49}$$

$$B = m_2^2(H_1 + 2m_1m_3(x_3-x_1)), \tag{2.50}$$

$$C = m_3^2(H_1 + 2m_1m_2(x_2-x_1)) \tag{2.51}$$

then we can write the three other conserved quantities as:

$$A \quad + B \quad + C \quad = M_3 \tag{2.52}$$

$$Ax_1 + Bx_2 + Cx_3 = M_4 \tag{2.53}$$

$$Ax_1^2 + Bx_2^2 + Cx_3^2 = M_5. \tag{2.54}$$

2.3.3 N = 4

P same form as above. Only H_1, H_3 and H_4 are conserved quantities:

$$H_1 = m_1m_2(x_2-x_1) + m_1m_3(x_3-x_1) + m_1m_4(x_4-x_1) \tag{2.55}$$

$$+ m_2m_3(x_3-x_2) + m_2m_4(x_4-x_2) + m_3m_4(x_4-x_3), \tag{2.56}$$

$$H_3 = \text{Too complicated to write out (I haven't found it's pattern)}, \tag{2.57}$$

$$H_4 = m_1^2m_2^2m_3^2m_4^2(x_2-x1)(x_3-x_2)(x_4-x_3)(x_4-x_1). \tag{2.58}$$

I haven't found any more conserved quantities for $N = 4$. I assume it has more on the same form as $N = 2, 3$ had:

$$\sum_{k=1}^{N} A_k x_k^n \quad \forall n = 0, 1, 2. \tag{2.59}$$

My exhaustive search couldn't find anymore, it could only handle exponents up to order two for $N = 4$.

2.3.4 N = 5

P same form as above. Only H_1, H_3, and H_5 are conserved quantities. I'm not going to write them out as they are quite complicated.

2.3.5 N = 6

Only H_1, H_2, H_3, and H_6 where small enough to be tested and only H_1 and H_6 were conserved quantities.

2.3.6 Conclusion

Conjecture 2.3.1. *H_1 and H_N are always conserved quantities.*

Conjecture 2.3.2. *For all N there is conserved quantities on the form*

$$\sum_{k=1}^{N} A_k x_k^n \quad \forall n = 0, 1, 2. \tag{2.60}$$

Conjecture 2.3.3. *Either the P matrix is incorrect or the there aren't enough conserved quantities to solve the system.*

2.4 Constants

Theorem 2.4.1. *One constant of motion is the sum of on the following form:*

$$\sum_{i=1}^{N} \sum_{j=1}^{N} m_i m_j |x_i - x_j| \tag{2.61}$$

Proof. Since $x_i < x_j$ for $i < j$ we can write the sum as:

$$\sum_{i,j} m_i m_j (x_i - x_j) \mathrm{sgn}(i - j). \tag{2.62}$$

We take summation notation to mean that the sum is over all i and j from 1 to N. Taking the time derivate of this sum we get:

$$\sum_{i,j} \mathrm{sgn}(i - j)[\dot{m}_i m_j (x_i - x_j) + m_i \dot{m}_j (x_i - x_j) + m_i m_j (\dot{x}_i - \dot{x}_j)]. \tag{2.63}$$

Plugging in the derivate equations we get:

$$\begin{aligned}\sum_{i,j} & m_i m_j \mathrm{sgn}(i-j)\{ \\ & [u(x_i)^2 - (x_i - x_j)u_x(x_i)u(x_i)] \\ & [u(x_j)^2 - (x_i - x_j)u_x(x_j)u(x_j)]\}.\end{aligned} \tag{2.64}$$

The terms $u(x_i)^2$ and $u(x_j)^2$ can be written as:

$$u(x_i)^2 = \sum_{k,l} m_k m_l (x_i - x_k)(x_i - x_l)\mathrm{sgn}(i-k)\mathrm{sgn}(i-l), \tag{2.65}$$

$$u_x(x_i)u(x_i) = \sum_{k,l} m_k m_l (x_i - x_l)\mathrm{sgn}(i-k)\mathrm{sgn}(i-l). \tag{2.66}$$

Plugging in these expressions we get:

$$\begin{aligned}\sum_{i,j} & m_i m_j \mathrm{sgn}(i-j)\{ \\ & [\sum_{k,l} m_k m_l (x_i - x_k)(x_i - x_l)\mathrm{sgn}(i-k)\mathrm{sgn}(i-l) \\ & - (x_i - x_j)\sum_{k,l} m_k m_l (x_i - x_l)\mathrm{sgn}(i-k)\mathrm{sgn}(i-l)] \\ & [-\sum_{k,l} m_k m_l (x_j - x_k)(x_j - x_l)\mathrm{sgn}(j-k)\mathrm{sgn}(j-l) \\ & - (x_i - x_j)\sum_{k,l} m_k m_l (x_j - x_l)\mathrm{sgn}(j-k)\mathrm{sgn}(j-l)]\}\end{aligned} \tag{2.67}$$

If we collect the terms we get:

$$\sum_{i,j,k,l} m_i m_j m_k m_l \mathrm{sgn}(i-j)\mathrm{sgn}(i-k)\mathrm{sgn}(i-l)(x_i - x_l)(x_j - x_k) \tag{2.68}$$

$$+ \sum_{i,j,k,l} m_i m_j m_k m_l \mathrm{sgn}(j-i)\mathrm{sgn}(j-k)\mathrm{sgn}(j-l)(x_j - x_l)(x_i - x_k) \tag{2.69}$$

Both terms are symmetric in i and j so we only need to consider one of them. We can write the sum as:

$$\sum_{i,j,k,l} m_i m_j m_k m_l \mathrm{sgn}(i-j)\mathrm{sgn}(i-k)\mathrm{sgn}(i-l)(x_i x_j - x_i x_k + x_l x_k - x_l x_j). \tag{2.70}$$

Factoring out the x terms we get:

$$x_i x_j(\sum_{i,j,k,l} m_i m_j m_k m_l \text{sgn}(i-j)\text{sgn}(i-k)\text{sgn}(i-l)) \tag{2.71}$$

$$-x_i x_l(\sum_{i,j,k,l} m_i m_j m_k m_l \text{sgn}(i-j)\text{sgn}(i-k)\text{sgn}(i-l)) \tag{2.72}$$

$$+x_k x_l(\sum_{i,j,k,l} m_i m_j m_k m_l \text{sgn}(i-j)\text{sgn}(i-k)\text{sgn}(i-l)) \tag{2.73}$$

$$-x_k x_j(\sum_{i,j,k,l} m_i m_j m_k m_l \text{sgn}(i-j)\text{sgn}(i-k)\text{sgn}(i-l)). \tag{2.74}$$

If we look at at a fixed $x_{n_1}x_{n_2}$ term we see that the first and second term will cancel out and the third and fourth term will cancel out. This means that the sum will be zero and the constant of motion is conserved. □

Theorem 2.4.2. *Another constant of motion is the product on the following form:*

$$\prod_{i=1}^{N} m_i^2(x_i - x_{i-1}). \tag{2.75}$$

For the sake of breivity we define $x_0 = x_N$.

Proof. Taking the time derivate of this sum we get:

$$\prod_i [m_i^2(x_i - x_{i-1})] \sum_j (\frac{\dot{m}_j}{m_j} + \frac{\dot{x}_j - \dot{x}_{j-1}}{x_j - x_{j-1}}). \tag{2.76}$$

By dividing by the constant product to left and plugging in the derivate equations we get:

$$\sum_j [-u(u_j)u_x(u_j) + \frac{u(u_j)^2 - u(u_{j-1})^2}{x_j - x_{j-1}}]. \tag{2.77}$$

The terms $u(x_i)^2$ and $u(x_j)^2$ can be written as:

$$u(x_i)^2 = \sum_{k,l} m_k m_l (x_i - x_k)(x_i - x_l)\text{sgn}(i-k)\text{sgn}(i-l), \tag{2.78}$$

$$u_x(x_i)u(x_i) = \sum_{k,l} m_k m_l (x_i - x_l)\text{sgn}(i-k)\text{sgn}(i-l). \tag{2.79}$$

Plugging in these expressions we get:

$$\sum_j \sum_{k,l} m_k m_l (x_j - x_l)\text{sgn}(j-k)\text{sgn}(j-l)(-1 + \frac{x_j - x_k}{x_j - x_{j-1}} - \frac{x_j - x_k}{x_{j+1} - x_j}) \tag{2.80}$$

Let's look at a fixed $m_j m_l$ term and collect all terms with the same denominator. The terms that are interesting are:

$$\begin{aligned}&\sum_j(-1+\frac{x_j-x_k}{x_j-x_{j-1}})(x_j-x_l)\mathrm{sgn}(x_j-x_k)\mathrm{sgn}(x_j-x_l)\\ -&\sum_j\frac{x_{j-1}-x_k}{x_j-x_{j-1}}(x_{j-1}-x_l)\mathrm{sgn}(x_{j-1}-x_k)\mathrm{sgn}(x_{j-1}-x_l)\end{aligned} \tag{2.81}$$

□

2.5 Novikov

The Lax pair for the Novikov equation is given by ADD REF:

$$\frac{\partial}{\partial x}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix}=\begin{pmatrix}0 & zm & 1\\ 0 & 0 & zm\\ 1 & 0 & 0\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix}, \tag{2.82a}$$

$$\frac{\partial}{\partial t}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix}=\begin{pmatrix}-uu_x & \frac{u_x}{z}-u^2mz & u_x^2\\ \frac{u}{z} & -\frac{1}{z^2} & -\frac{u_x}{z}-u^2mz\\ -u^2 & \frac{u}{z} & uu_x\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix}, \tag{2.82b}$$

Doing the high frequency substitution $x\mapsto\epsilon x$, $t\mapsto\epsilon t$, we get that $m\mapsto\epsilon^{-2}m$ since its the second derivate of u and u shouldn't go to 0 or ∞ in the limit. We are left to find the correct substitution for z. We can do this by looking at the spacial derivate of the Lax pair and since we don't know the x and t factors for ψ_1, ψ_2, and ψ_3 we call them A, B, and C respectively

$$\begin{pmatrix}\epsilon^{-1}A\\ \epsilon^{-1}B\\ \epsilon^{-1}C\end{pmatrix}=\begin{pmatrix}0 & zm\epsilon^{-2} & 1\\ 0 & 0 & zm\epsilon^{-2}\\ 1 & 0 & 0\end{pmatrix}\begin{pmatrix}A\\ B\\ C\end{pmatrix}. \tag{2.83}$$

Simplifying this we get

$$A=z^2m^2\epsilon^{-1}A+\epsilon A, \tag{2.84}$$

and since we don't want A to go to 0 or ∞ in the limit we get that z should be substituted with $\epsilon^{1/2}$. Finally doing the substitutions $x\mapsto\epsilon x$, $t\mapsto\epsilon t$, $m\mapsto\epsilon^{-2}m$ and $z\mapsto\epsilon^{1/2}z$, and letting $\epsilon\to 0$ we get rid of the 1 in the Lax pair:

$$\frac{\partial}{\partial x}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix}=\begin{pmatrix}0 & zm & 0\\ 0 & 0 & zm\\ 1 & 0 & 0\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix} \tag{2.85}$$

The time derivate is left unchanged. We can simply the system even further with the following substitutions:

$$\begin{cases} \varphi_1 = & \psi_1, \\ \varphi_2 = & z\psi_2, \\ \varphi_3 = & z^2\psi_3. \end{cases} \tag{2.86}$$

By also setting $z^2 = -\lambda$ we get the following Lax pairs:

$$\frac{\partial}{\partial x}\begin{pmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\end{pmatrix} = \begin{pmatrix}0 & m & 0\\ 0 & 0 & m\\ -\lambda & 0 & 0\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\end{pmatrix}, \tag{2.87a}$$

$$\frac{\partial}{\partial t}\begin{pmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\end{pmatrix} = \begin{pmatrix}-uu_x & -\frac{u_x}{\lambda} - u^2 m & -\frac{u_x^2}{\lambda}\\ u & \frac{1}{\lambda} & \frac{u_x}{\lambda} - u^2 m\\ u^2\lambda & u & uu_x\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\end{pmatrix}. \tag{2.87b}$$

We know that

$$m = u_{xx} = \sum_{k=1}^{n} m_k \delta(x - x_k), \tag{2.88}$$

This means that φ_3 will be continous while φ_1 and φ_2 will have jumps at x_k. We will assume that $x_0 < x_1 < ... < x_n$, which will stay true at least for a time if it's true at $t = 0$. We will also use the convention that $x_0 = -\infty$ and $x_{n+1} = \infty$. Since $m = 0$ in the intervals $x_k < x < x_{k+1}$, the spacial Lax pair simplifies to $\delta_x\varphi_1 = 0$, $\delta_x\varphi_2 = 0$, and $\delta_x\varphi_3 = \varphi_1$. This means that φ_1, φ_2 and φ_3 have to be

$$\begin{pmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\end{pmatrix} = \begin{pmatrix}A_i\\ B_i\\ -\lambda A_i x + C_i\end{pmatrix} \text{ for } x_k < x < x_{k+1}. \tag{2.89}$$

We can also see from the Lax pair that φ_3 is going to be continous, while φ_1 and φ_2 is constant in the intervals with jump discontinuities at x_k.

$$\begin{pmatrix}A_k\\ B_k\\ C_k\end{pmatrix} = \begin{pmatrix}1 - 2\lambda x_k m_k^2 & 2m_k & 2m_k^2\\ -2\lambda m_k x_k & 1 & 2m_k\\ -2\lambda^2 m_k^2 x_k^2 & 2\lambda m_k x_k & 1 + 2\lambda m_k^2 x_k\end{pmatrix}\begin{pmatrix}A_{k-1}\\ B_{k-1}\\ C_{k-1}\end{pmatrix} \tag{2.90}$$

To be consistent with the time evolution for $x < x_1$, we get the following boundary condition:

$$\begin{pmatrix}A_0\\ B_0\\ C_0\end{pmatrix} = \begin{pmatrix}0\\ -u_x\\ 1\end{pmatrix}. \tag{2.91}$$

u_x is a constant and for $x < x_1$ it's given by:

$$u_x = \sum_{k=1}^{n} m_k \mathrm{sgn}(x - x_k) = \sum_{k=1}^{n} -m_k. \tag{2.92}$$

$$\begin{pmatrix} \dot{A} \\ \dot{B} \\ \dot{C} \end{pmatrix} = \begin{pmatrix} M_+ u_x & -\frac{u_x}{\lambda} & -\frac{u_x^2}{\lambda} \\ -M_+ & \frac{1}{\lambda} & \frac{u_x}{\lambda} \\ M_+^2 \lambda & -M_+ & -u_x M_+ \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix}. \tag{2.93}$$

Bibliography

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