Dr. Nishu Gupta

Location of roots of an equation:

Algebraic and Transcendental Equations

An equation $f(x)=0$ is called an algebraic equation of degree n, if $f(x)$ is a polynomial of degree n.

If $f(x)$ contains some other functions such as trigonometric, logarithmic, exponential, etc. then $f(x)$ is called a transcendental equation.

(I.V.P.) **Intermediate Value Property:** If $f(x)$ is continuous in $[a,b]$ and $f(a).f(b)<0$ then $f(x)=0$ has one real root in (a,b).

Convergence: Let $x_1, x_2, x_3, ---, x_{n+1}$ be successive approximations of root r of an equation. If there exists a constant C such that

$$|r-x_{n+1}| \leq C|r-x_n|^m \quad (n \geq 1) \text{ or } |e_{n+1}| \leq C|e_n|^m$$

then convergence is said to be of order m. (where e_{n+1} and e_n are errors at $(n+1)^{th}$ and n^{th} step)

Bisection Method or Bolzano Method or Halving Method:

This method is based on the repeated application of I.V.P. Suppose $f(x)$ is a continuous function of x and we are to find real root of $f(x)=0$

Let a and b be real numbers such that $f(a).f(b)<0$

then I^{st} approximation is $x_1 = \frac{1}{2}(a+b)$

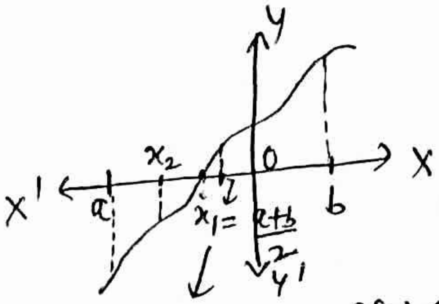
If $f(x_1)=0$, then x_1 is a root.

If $f(x_1) \neq 0$ then

either $f(a).f(x_1)<0$ in which case II^{nd} approximation $x_2 = \frac{a+x_1}{2}$

or $f(x_1).f(b)<0$ in which case II^{nd} approximation $x_2 = \frac{x_1+b}{2}$

Now, replace a or b by x_1 as the case be, then next approximation will be $x_3 = \frac{x_1 + x_2}{2}$ and so on.



$f(x_1) > 0,\ f(a) < 0$

$\therefore x_2 = \frac{a + x_1}{2}$

Convergence of Bisection method:

Suppose f is a continuous function in $[a, b]$ and $f(a).f(b) < 0$. Then there is a root r in $[a, b]$. If we use

$x_1 = \frac{a+b}{2}$ as 1^{st} approximation, then

$$|r - x_1| \leq \frac{b-a}{2}$$

Now, we choose the next approximation $x_2 = \frac{a + x_1}{2}$ or $x_2 = \frac{b + x_1}{2}$ as the case may be, then

$$|r - x_2| \leq \frac{b-a}{2^2}$$ ($\because$ length of the interval at each step is $\frac{1}{2}$ the length of interval in previous step in which root lies)

$\vdots$

$$\therefore |r - x_n| \leq \frac{b-a}{2^n} \quad \text{——— (1)}$$

$\therefore$ At the end of n steps, when we obtain x_n, the root will lie in an interval of length $\frac{b-a}{2^n}$.

[Note: If we use $x_0 = \frac{a+b}{2}$ as initial estimate of r, and from next onward $x_1, x_2, \ldots x_n$, then $|r - x_n| \leq \frac{b-a}{2^{n+1}}$.

Both are correct.]

Now, as the length of interval at each step is $\frac{1}{2}$ the length of the interval in the previous step in which root r lies, so

$$|r - x_{n+1}| = \frac{1}{2} |r - x_n|$$ (this shows that error at $(n+1)^{th}$ step is $\frac{1}{2}$ of the error at n^{th} step)

$\therefore$ Convergence is linear.

Hence the process is slow but must converge.

Number of iterations required to reach accuracy ε

By equation (1), the no. of iterations n required to reach accuracy ε, we must have

$$\frac{b-a}{2^n} \leq \varepsilon$$

or $\log(b-a) - n\log 2 \leq \log\varepsilon$

or $n \geq \dfrac{\log(b-a) - \log\varepsilon}{\log 2}$ ——— (2)

$\therefore$ Smallest natural no. n satisfying this inequality gives the no. of iterations required to reach accuracy ε.

Bisection Method theorem If the bisection algorithm is applied to a continuous function f on an interval $[a, b]$, where $f(a).f(b) < 0$, then after n steps, an approximate root will have been computed with error at most $(b-a)/2^n$. (where we are considering first step as $x_1 = \frac{a+b}{2}$)

Que How many steps of the bisection algorithm are needed to compute a root of f to full machine single precision on a 32-bit word-length computer if $a = 16$ and $b = 17$?

Sol By equation (2), the no. of steps n required is given by

$$n \geq \frac{\log(b-a) - \log\varepsilon}{\log 2}$$

Here $a = 16$ and $b = 17$

$\therefore\ n \geq \dfrac{\log 1 - \log \varepsilon}{\log 2} \Rightarrow n \geq \dfrac{-\log \varepsilon}{\log 2}$ ——— (1)

Now, the root is between the two binary numbers $a = (10000.0)_2$ and $b = (10001.0)_2$. Thus, we already know five of the binary digits in the answer. Since we can use 23 bits for mantissa f in $(1.f)_2$ form and 4 digits for f are given that leaves 19 bits to determine. We want the last one to be correct, so we want the error to be less than 2^{-19} or 2^{-20} (being conservative). $\therefore\ \varepsilon = 2^{-20}$

$\therefore$ By equ. (1) above $n \geq -\dfrac{\log 2^{-20}}{\log 2} \Rightarrow \boxed{n \geq 20}$ Δ

Que How many steps of the bisection method are needed to find a root of the equation $xe^x = 1$ correct to three decimal places in the interval $[0, 1]$.

Sol Given $a = 0$, $b = 1$, $\varepsilon = 0.0005$

$\therefore$ No. of steps n are given by

$$n \geq \frac{\log(b-a) - \log \varepsilon}{\log 2}$$

$$\Rightarrow n \geq \frac{\log 1 - \log(0.0005)}{\log 2}$$

$$\Rightarrow n \geq 10.97$$

$\therefore$ Minimum steps are required $n = 11$ Δ

Note: It will be verified by solving the problem in next question.

Que Find a real root of the equation $xe^x = 1$ correct to three decimal places using bisection method.

Sol Let $f(x) = xe^x - 1 = 0$, Then

Clearly f is continuous as

$f(0) = -1$, $f(1) = e - 1 = 1.71828$

i.e, $f(0).f(1) < 0 \Rightarrow$ root lies between 0 and 1.

$\therefore \; x_1 = \frac{0+1}{2} = 0.5$

Approximate root	$f(x)$	Root lies between	Next Approximation
$x_1 = 0.5$	$-$ive	0.5 and 1	$\frac{0.5+1}{2} = 0.75$
$x_2 = 0.75$	$+$ive	0.5 and 0.75	$\frac{0.5+0.75}{2} = 0.625$
$x_3 = 0.625$	$+$ive	0.5 and 0.625	$\frac{0.5+0.625}{2} = 0.5625$
$x_4 = 0.5625$	$-$ive	0.5625 and 0.625	$\frac{0.5625+0.625}{2} = 0.59375$
$x_5 = 0.59375$	$+$ive	0.5625 and 0.59375	$\frac{0.5625+0.59375}{2} = 0.57812$
$x_6 = 0.57812$	$+$ive	0.5625 and 0.57812	$\frac{0.5625+0.57812}{2} = 0.57031$
$x_7 = 0.57031$	$+$ive	0.5625 and 0.57031	$\frac{0.5625+0.57031}{2} = 0.56640$
$x_8 = 0.56640$	$-$ive	0.56640 and 0.57031	$\frac{0.56640+0.57031}{2} = 0.56836$
$x_9 = 0.56836$	$+$ive	0.56640 and 0.56836	$\frac{0.56640+0.56836}{2} = 0.56738$
$x_{10} = 0.56738$	$+$ive	0.56640 and 0.56738	$\frac{0.56640+0.56738}{2} = 0.56689$
$x_{11} = 0.56689$			

Since $x_{10} \approx x_{11}$ (correct to three decimal places)

$\therefore$ root $= 0.567$ (correct to 3D) A

Newton method or Newton-Raphson method or Method of Tangents:

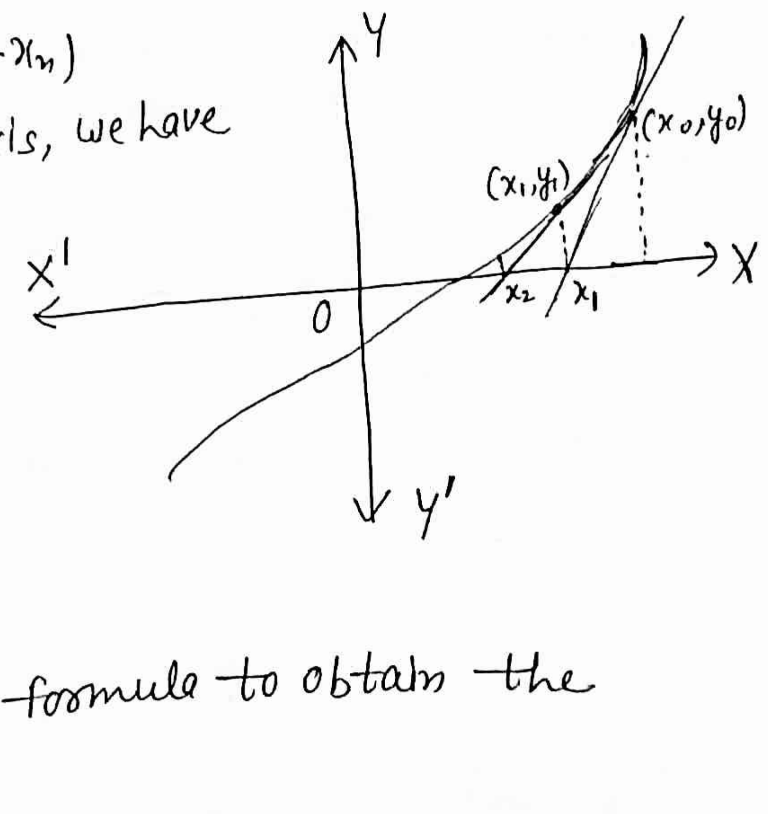
Let x_0 be an approximation to the root of $f(x)=0$. We find the equation of tangent at (x_0, y_0) to the graph of curve $y=f(x)$ where $y_0 = f(x_0)$.

Let this tangent meets x-axis at x_1 then x_1 will be next approximation and we find (x_1, y_1) on the graph and draw tangent at (x_1, y_1) to the curve $y=f(x)$. Its intersection with x-axis will be x_2.

Proceeding in this way, when approximation x_n is found then intersection of tangent at (x_n, y_n) to $y=f(x)$ with x-axis will give next approximation x_{n+1}.

Now, equation of tangent at (x_n, y_n) to $y=f(x)$ is

$$y - y_n = f'(x_n)(x - x_n)$$



For its intersection with x-axis, we have

$y=0$ and $x = x_{n+1}$

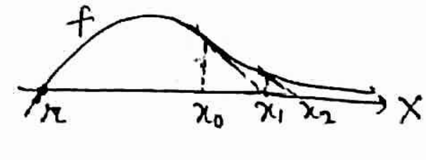
$$\therefore -y_n = f'(x_n)(x_{n+1} - x_n)$$

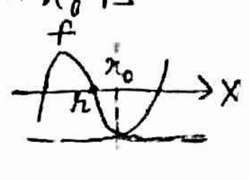
$$\Rightarrow x_{n+1} = x_n - \frac{y_n}{f'(x_n)}$$

or $$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

It is Newton's iterative formula to obtain the approximations.

Note: The method may fail if the initial approximation x_0 is far away from the root.





or the tangent at (x_0, y_0) does not intersect the x-axis.

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Convergence of Newton Raphson method:

Let r be exact root of $f(x)$. Let x_n and x_{n+1} be its two successive approximations. Then by Newton Raphson iterative formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Substituting $x_n = r + e_n$ and $x_{n+1} = r + e_{n+1}$

$$\cancel{r} + e_{n+1} = \cancel{r} + e_n - \frac{f(r+e_n)}{f'(r+e_n)}$$

$$\therefore e_{n+1} = \frac{e_n f'(r+e_n) - f(r+e_n)}{f'(r+e_n)}$$

$$= \frac{e_n\left[\cancel{f'(r)} + \frac{e_n}{1!} f''(r) + \frac{e_n^2}{2!} f'''(r) + \cdots\right] - \left[f(r) + \cancel{e_n f'(r)} + \frac{e_n^2}{2!} f''(r) + \frac{e_n^3}{3!} f'''(r) + \cdots\right]}{f'(r) + e_n f''(r) + \frac{e_n^2}{2!} f'''(r) + \cdots}$$

(by Taylor series expansion)

But $f(r) = 0$ ($\because r$ is a root of $f(x) = 0$)

$$\therefore e_{n+1} = \frac{\frac{e_n^2}{2} f''(r) + e_n^3 f'''(r) \cdot \left(\frac{1}{2} - \frac{1}{3!}\right) + \cdots}{f'(r) + e_n f''(r) + \frac{e_n^2}{2!} f'''(r) + \cdots}$$

$$= \frac{1}{f'(r)}\left\{\frac{e_n^2}{2} f''(r) + \frac{e_n^3}{3} f'''(r) + \cdots\right\}\left\{1 + \left(e_n \frac{f''(r)}{f'(r)} + \frac{e_n^2 f'''(r)}{2!\, f'(r)} + \cdots\right)\right\}^{-1}$$

$$= \frac{1}{f'(r)}\left\{\frac{e_n^2}{2} f''(r) + \frac{e_n^3}{3} f'''(r) + \cdots\right\}\left\{1 - e_n \frac{f''(r)}{f'(r)} - \frac{e_n^2 f'''(r)}{2 f'(r)} - \cdots\right\}$$

$$\therefore e_{n+1} = \frac{1}{2} e_n^2 \frac{f''(r)}{f'(r)} + \cdots$$

$$\Rightarrow e_{n+1} = \frac{1}{2} e_n^2 \frac{f''(r)}{f'(r)}$$ (If remaining terms are neglected)

$$\Rightarrow |e_{n+1}| \le C \cdot |e_n|^2 \text{ where } C = \frac{1}{2}\left|\frac{f''(r)}{f'(r)}\right|$$

Hence the convergence is of order 2 i.e, quadratic.

Que Using Newton-Raphson method evaluate $\sqrt[3]{41}$ correct to four places of decimals.

Sol Let $f(x) = x^3 - 41 = 0$

$\therefore f'(x) = 3x^2$

$\therefore$ Newton-Raphson iterative formula $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

becomes, $x_{n+1} = x_n - \dfrac{x_n^3 - 41}{3x_n^2} = \dfrac{2x_n^3 + 41}{3x_n^2} = \dfrac{1}{3}\left(2x_n + \dfrac{41}{x_n^2}\right)$

Since $3^3 = 27$, $4^3 = 64$

$\therefore \sqrt[3]{27} = 3$, $\sqrt[3]{64} = 4$

Take $x_0 = 3.4$

n	x_n	$x_{n+1} = \frac{1}{3}\left(2x_n + \frac{41}{x_n^2}\right)$
0	3.4	3.4489
1	3.4489	3.44822
2	3.44822	3.44822

$\therefore \sqrt[3]{41} = 3.4482$ (correct to four places of decimals)

A

Que Use Newton-Raphson method to solve the equation

$$3x - \cos x - 1 = 0$$

Sol Let $f(x) = 3x - \cos x - 1 = 0$ $\therefore f(0) = -2$, $f(1) = 1.4597$

$\therefore f'(x) = 3 + \sin x$ $\therefore$ We take $x_0 = 0.6$

$\therefore$ Newton-Raphson iterative formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow x_{n+1} = x_n - \frac{(3x_n - \cos x_n - 1)}{3+\sin x_n} = \frac{x_n \sin x_n + \cos x_n + 1}{3+\sin x_n}$$

n	x_n	$x_n \sin x_n + \cos x_n + 1$	$3+\sin x_n$	x_{n+1}
0	0.6	2.1641	3.5646	0.6071
1	0.6071	2.1676	3.5705	0.607099

$\therefore$ Root to four decimal places $= 0.6071$ Ans

Multiplicity of the zero r of $f(x)=0$ is the least m such that $f^{(k)}(r)=0$ for $0 \leq k < m$ but $f^{(m)}(r) \neq 0$

For ex: $f(x) = x^2 - 2x + 1 = 0$ has a root at 1 of multiplicity 2 $(\because f(x) = (x-1)^2)$

If we already know in advance that there is a zero of $f(x)=0$ with multiplicity m, then we can find it by modifying the Newton's method as

$$x_{n+1} = x_n - \frac{m f(x_n)}{f'(x_n)}$$

and problem can be solved in a similar manner.

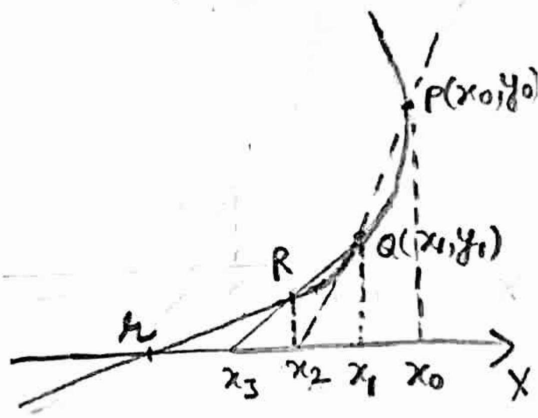
—x—x—

Secant Method: Let x_0, x_1 be two approximations of root of $y=f(x)=0$. Then $P(x_0,y_0)$ and $Q(x_1,y_1)$ are two points on the curve $y=f(x)$ where $y_0=f(x_0)$, $y_1=f(x_1)$. Join PQ. We approximate the curve by secant (chord) PQ and take ~~secant~~ the point of intersection of PQ with x-axis as the next approximation x_2 of the root. Then we take secant joining $Q(x_1,y_1)$ and $R(x_2,y_2)$ and repeat the same process to get the next approximation x_3.

Proceeding in this way, curve is approximated by secant joining (x_{n-1},y_{n-1}) and (x_n,y_n) and its point of intersection with x-axis as the approximation x_{n+1} of the root.

Equation of secant joining (x_{n-1},y_{n-1}) and (x_n,y_n) is

$$y-y_n = \frac{y_n-y_{n-1}}{x_n-x_{n-1}}(x-x_n) \qquad \text{———— (1)}$$



Now, $y=0$ and $x=x_{n+1}$

$$\Rightarrow \quad -y_n = \frac{y_n-y_{n-1}}{x_n-x_{n-1}}(x_{n+1}-x_n)$$

$$\therefore \quad x_{n+1} = x_n - \frac{(x_n-x_{n-1})\,y_n}{(y_n-y_{n-1})}$$

$$\Rightarrow \quad \boxed{x_{n+1} = x_n - \frac{x_n-x_{n-1}}{f(x_n)-f(x_{n-1})}f(x_n)}$$

Equation of secant joining (x_{n-1},y_{n-1}) and (x_n,y_n) can also be written as

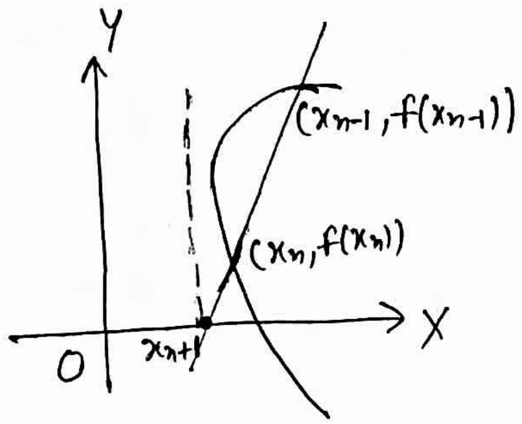
$$y-y_{n-1} = \frac{y_n-y_{n-1}}{x_n-x_{n-1}}(x-x_{n-1})$$

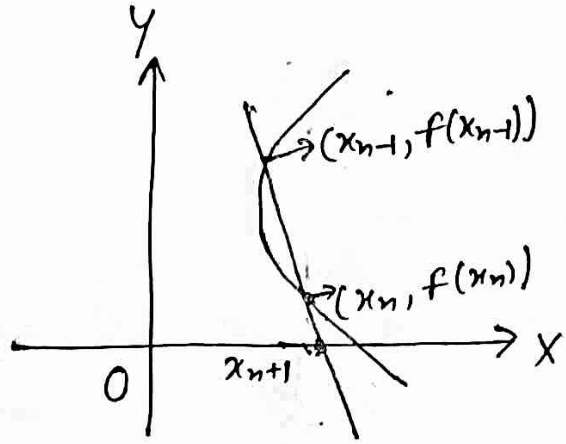
Now $y = 0$ and $x = x_{n+1}$

$$\Rightarrow \quad -y_{n-1} = \frac{y_n - y_{n-1}}{x_n - x_{n-1}}(x_{n+1} - x_{n-1})$$

$$\Rightarrow \quad \boxed{x_{n+1} = x_{n-1} - \frac{(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} f(x_{n-1})}$$

which is the iterative formula to find the approximations.





In figure (1), $f(x_{n+1})$ cannot be found and hence iteration process diverges but in figure (2) iteration process converges to root.

Note:
- It does not require the condition $f(x_0).f(x_1) < 0$.
- Two most recent approximations to the root are used to find the next approximation
- Also it is not necessary that the iteration process converge i.e, contain the root in (x_n, x_{n+1}).

Convergence of secant method: Let r be exact root of $f(x)$.

Let x_{n-1}, x_n and x_{n+1} be its successive approximations. Then by the iterative formula of secant method

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$$

Substituting $x_n = r + e_n$, we have

$$\cancel{r} + e_{n+1} = \cancel{r} + e_n - \frac{(\cancel{r}+e_n) - (\cancel{r}+e_{n-1})}{f(r+e_n) - f(r+e_{n-1})} f(r+e_n)$$

$$\Rightarrow e_{n+1} = \frac{e_n f(r+e_n) - e_n f(r+e_{n-1}) - (e_n - e_{n-1}) f(r+e_n)}{f(r+e_n) - f(r+e_{n-1})}$$

$$= \frac{e_{n-1} f(r+e_n) - e_n f(r+e_{n-1})}{f(r+e_n) - f(r+e_{n-1})}$$

$$= \frac{e_{n-1}\left[f(r) + \cancel{e_n f'(r)} + \frac{e_n^2}{2!} f''(r) + \cdots\right] - e_n\left[f(r) + \cancel{e_{n-1} f'(r)} + \frac{e_{n-1}^2}{2!} f''(r) + \cdots\right]}{\left[f(r) + e_n f'(r) + \frac{e_n^2}{2!} f''(r) + \cdots\right] - \left[f(r) + e_{n-1} f'(r) + \frac{e_{n-1}^2}{2!} f''(r) + \cdots\right]}$$

(using Taylor series)

$$= \frac{\frac{e_n e_{n-1}(e_n - e_{n-1}) f''(r)}{2} + \cdots}{(e_n - e_{n-1}) f'(r) + \frac{(e_n^2 - e_{n-1}^2)}{2} f''(r) + \cdots} \qquad (\because f(r) = 0)$$

$$\therefore e_{n+1} \equiv \frac{e_n e_{n-1}}{2} \frac{f''(r)}{f'(r)} + \cdots$$

$$\Rightarrow e_{n+1} = A e_n e_{n-1} \quad \text{--- (1)} \qquad \text{where } A = \frac{f''(r)}{f'(r)}$$

(if the remaining terms are neglected)

Let m be the order of convergence, then we can find k such that $|e_n| = k|e_{n-1}|^m$ for some k ——— (2)

$\therefore$ From (1) and (2),

$$|e_{n+1}| = |A||e_n||e_{n-1}|$$

and $$|e_{n-1}| = \left(\frac{|e_n|}{k}\right)^{1/m}$$

$$\Rightarrow \quad |e_{n+1}| = \frac{|A|}{k^{1/m}}|e_n|^{1+\frac{1}{m}}$$

But order of convergence is m

$\therefore$ From this we get $m = 1 + \frac{1}{m}$

or $$m^2 - m - 1 = 0$$

$$\Rightarrow \quad m = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

But $m > 0$

$$\therefore \quad m = \frac{1+\sqrt{5}}{2} = 1.62$$

$\therefore$ order of convergence is 1.62.

Que Find a root of the equation $x^3-2x-5=0$ using secant method correct to three decimal places.

Sol Let $f(x) = x^3 - 2x - 5 = 0$

$\therefore f(2) = -1, \quad f(3) = 27-6-5 = 16$

$\therefore$ Taking initial approximations $x_0 = 2$ and $x_1 = 3$, by secant method, we have

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n) \qquad \text{——— (1)}$$

$$\therefore x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1) = 3 - \frac{(3-2).16}{16+1} = 3 - \frac{16}{17} = 2.058823$$

Now, $f(x_2) = -0.390799$

$$\therefore x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) = 2.058823 - \frac{2.058823-3}{-0.390799-16} (-0.390799)$$

$\Rightarrow x_3 = 2.081263$

$f(x_3) = -0.147204$

$$\therefore x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3) = 2.094824$$

and $f(x_4) = 0.003042$

$$\therefore x_5 = x_4 - \frac{x_4 - x_3}{f(x_4) - f(x_3)} f(x_4) = 2.094549$$

Hence the root is 2.095 correct to 3 decimal places.