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# Unconstrained one variable function minimization

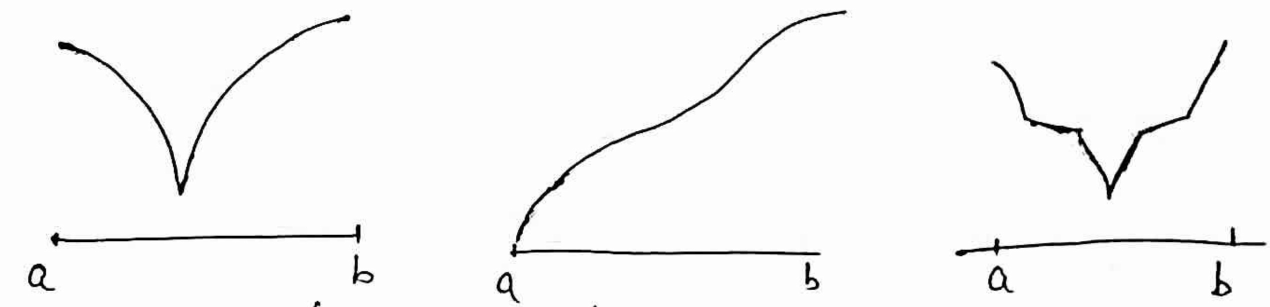
In an uncontrained one variable minimization problem, a function $f: R \to R$ is defined and a point $z \in R$ is sought with the property that $f(z) \leq f(x) \ \forall \ x \in R$

Note that if no assumptions are made about $f$, this problem is insoluble in its general form.

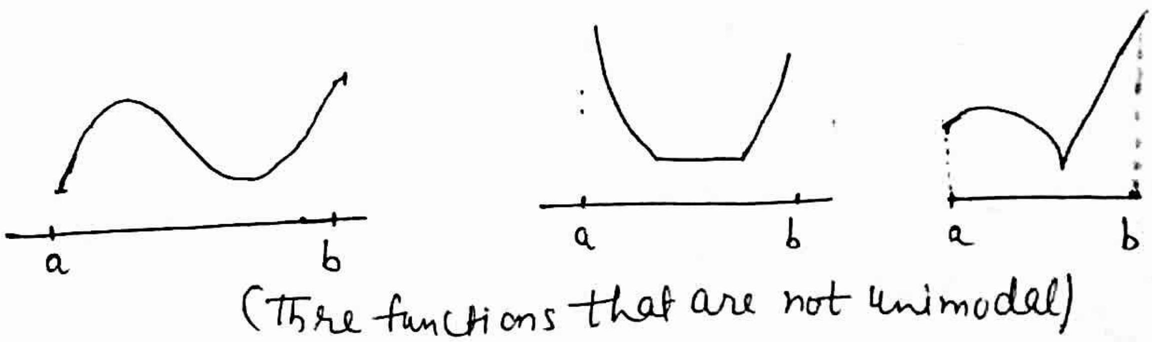
In attaking a minimization problem, one reasonable assumption is that on some interval $[a, b]$ given to us in advance, $f$ has only a single local minimum. This property is often expressed by saying that $f$ is unimodal on $[a, b]$.

Note:
- A point $z$ is a local minimum point of a function $f$ there is some neighbourhood of $z$ in which all points satisfy
$$f(z) \leq f(x).$$
- An important property of a continuous unimodal function is that it is strictly decreasing up to the minimum point and strictly increasing thereafter.

Examples:




(Three unimodal functions)




(Three functions that are not unimodal)

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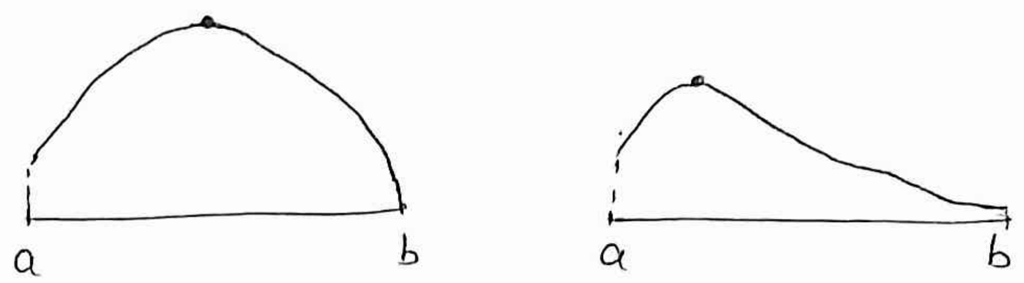
## Unimodal function for maximization problem:

A function $f: R \to R$ that has only one local maximum in a given interval $[a, b]$ is also called a unimodal function.

Ex:




**Fibonacci numbers:** Fibonacci numbers are defined as

$$F_0 = F_1 = 1$$

$$F_n = F_{n-1} + F_{n-2} \; ; \; n \geq 2$$

$\therefore$ Fibonacci sequence $\{F_n\} = \{F_0, F_1, F_2, F_3, F_4, F_5, F_6, F_7, \ldots\}$
$= \{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$

## Fibonacci search method:

- This method is an elimination technique and makes the use of Fibonacci numbers.
- Here we want to minimize a continuous unimodal function $f$ over $[a, b]$ i.e., find $x \in [a, b]$ which minimize $f(x)$.

**Note:** The problem to maximize a continuous unimodal function $f$ over $[a, b]$ can also be solved by this method.

- $L_0 = b - a$ is the length of the initial interval.
- $L_n$ denote the length of the interval of uncertainty after $n$ experiments.

- The number of steps $n$ must be given in advance or the desired accuracy (tolerance) $\varepsilon$ must be given in advance to find $n$.

- Divide the initial interval $[a, b]$ equally into $F_n$ subintervals and hence the length of each subinterval is $\frac{1}{F_n}(b-a)$.

$$\therefore \quad L_n = \frac{1}{F_n}(b-a) = \frac{1}{F_n} L_0$$

$$\Rightarrow \boxed{\frac{L_n}{L_0} = \frac{1}{F_n}} \qquad \text{———— (1)}$$

- For a given tolerance $\varepsilon$ with exact value of $x$, we can find $n$ such that

$$\boxed{\frac{L_n}{2} \leq \varepsilon}$$

$\longleftarrow L_n \longrightarrow$

at mid-point we get the optimal value of $x$ (i.e, $\hat{x}$) and exact value of $x$ lies in the final interval of length $L_n$ either on the left of $\hat{x}$ or right of $\hat{x}$ or exactly at $\hat{x}$.

and using (1), we get

$$\frac{L_0}{F_n} \leq 2\varepsilon$$

$$\Rightarrow F_n \geq \frac{L_0}{2\varepsilon}$$

$$\Rightarrow F_n \geq \frac{b-a}{2\varepsilon}$$

the smallest value of $n$ satisfying this inequality can be used as no. of steps $n$. $(n \in \mathbb{N})$

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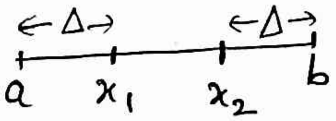
## Fibonacci search algorithm

After fixing the no. of steps as $n$, we define a sequence of intervals starting with the given interval $[a, b]$ of length $L_0 = b-a$ and for $k = n, n-1, ----3$ use these formulas for updating

$$\Delta = \left(\frac{F_{k-2}}{F_k}\right)(b-a)$$




$$x_1 = a + \Delta, \quad x_2 = b - \Delta$$

$$\begin{cases} a = x_1, & \text{if } f(x_1) \geq f(x_2) \\ b = x_2 & \text{if } f(x_1) < f(x_2) \end{cases}$$

At the step $k = 2$,

$$x_1 = \frac{1}{2}(a+b) - 2\delta \qquad (\text{Take } 2\delta < \varepsilon)$$

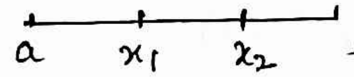
$$x_2 = \frac{1}{2}(a+b) + 2\delta$$

$$\begin{cases} a = x_1, & \text{if } f(x_1) \geq f(x_2) \\ b = x_2, & \text{if } f(x_1) < f(x_2) \end{cases}$$

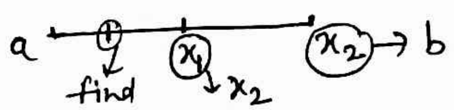
and we have the final interval $[a, b]$ from which we compute $\hat{x} = \frac{1}{2}(a+b)$.

This algorithm requires only one function evaluation per step after the initial step.

**Note:** (1) $x_1 + x_2 = a + b$ (always)




(2) If $f(x_1) < f(x_2)$ in $[a, b]$, then next interval of uncertainty $= [a, x_2]$




(3) If $f(x_1) \geq f(x_2)$ in $[a, b]$, then next interval of uncertainty $= [x_1, b]$

**Que** (1) Find the Minimum of $f(x) = x^2 - 6x + 2$ on $[0,10]$ using Fibonacci search algorithm. Obtain the optimal value with tolerance $\varepsilon = \frac{1}{4}$.

or

(2) Find the minimum of $f(x) = x^2 - 6x + 2$ on $[0,10]$ using Fibonacci search algorithm by taking $n = 7$.

or

(3) Find the minimum of $f(x) = x^2 - 6x + 2$ on $[0,10]$ using Fibonacci search algorithm. Locate the value of $x$ within 2.5% of exact value.

**Sol** Let $L_0$ be the length of the initial interval $[a,b] = [0,10]$ and $L_n$ be the length of final interval of uncertainty after $n$ experiments.

Here $a_0 = 0$, $b = 10$, $L_0 = b - a = 10$

(1) For given $\varepsilon = \frac{1}{4}$, we have $\frac{L_n}{2} \leq \frac{1}{4} \Rightarrow L_n \leq \frac{1}{2}$

$$\Rightarrow \frac{L_0}{F_n} \leq \frac{1}{2} \qquad \left(\because \frac{L_n}{L_0} = \frac{1}{F_n}\right)$$

$$\Rightarrow F_n \geq 10 \times 2$$

$$\Rightarrow F_n \geq 20$$

$\therefore$ Smallest $n$ for which this inequality is satisfied is $n = 7$. $(\because F_7 = 21)$

(3) Given $\frac{L_n}{2} \leq 2.5$ of $L_0$

$$\Rightarrow L_n \leq \frac{5}{100} \times 10 \Rightarrow L_n \leq \frac{1}{2} \Rightarrow F_n \geq 20 \text{ (as above)}$$

$$\Rightarrow n = 7$$

So, in all, we find $n = 7$.

- If $f(x_1) < f(x_2)$ in $[a,b]$ new interval of uncertainty $= [a, x_2]$
- If $f(x_1) \geq f(x_2)$ in $[a,b]$ then $[x_1, b]$

$a \quad x_1 \quad x_2 \quad b$

Here $n=7$, $a=0$, $b=10$ (for initial interval)

| $k$ | $\frac{F_{k-2}}{F_k}$ | $a$ | $b$ | $x_1 = a + \frac{F_{k-2}}{F_k}(b-a)$ | $x_2 = b - \frac{F_{k-2}}{F_k}(b-a)$ | $f(x_1) = x_1^2 - 6x_1 + 2$ | $f(x_2) = x_2^2 - 6x_2 + 2$ | Min at |
|---|---|---|---|---|---|---|---|---|
| $k=7$ | $\frac{F_5}{F_7} = \frac{8}{21}$ | 0 | 10 | 3.810 | 6.190 | −6.344 | 3.176 | $x_1$ |
| $k=6$ | $\frac{F_4}{F_6} = \frac{5}{13}$ | 0 | 6.190 | 2.380 | 3.810 | −6.616 | −6.344 | $x_1$ |
| $k=5$ | $\frac{F_3}{F_5} = \frac{3}{8}$ | 0 | 3.810 | 1.43 | 2.380 | −4.535 | −6.616 | $x_2$ |
| $k=4$ | $\frac{F_2}{F_4} = \frac{2}{5}$ | 1.43 | 3.810 | 2.380 | 2.860 | −6.616 | −6.980 | $x_2$ |
| $k=3$ | $\frac{F_1}{F_3} = \frac{1}{3}$ | 2.380 | 3.810 | 2.860 | 3.330 | −6.980 | −6.891 | $x_1$ |
| $k=2$ | $\frac{F_0}{F_2} = \frac{1}{2}$ | 2.380 | 3.330 | Here $x_1 = \frac{1}{2}(a+b) - 0.2 = 2.655$ (by taking $2\delta = 0.2 < \varepsilon = 0.25$) | 2.860 | −6.881 | −6.980 | $x_2$ |

$\therefore$ Final interval of uncertainty $= [x_1, b] = [2.655, 3.330]$

$$\therefore \quad \hat{x} = \frac{2.655 + 3.330}{2} = \frac{5.985}{2} = 2.99$$

$\therefore$ for $x = 2.89$, Optimal value $= (2.99)^2 - 6 \times 2.99 + 2 = -6.999 \approx -7$

Note: If the problem is of maximization, then write last column as Max. at and make the changes accordingly.

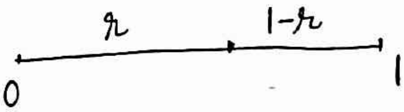
## Golden section search method:

In Fibonacci method, the ratio for the reduction of intervals is not constant and the number of subintervals (iterations) is predetermined which are based on the specified tolerance while in golden section search the ratio of intervals is constant i.e., it depends on a ratio $\rho$ known as the golden section ratio.

## Golden Ratio:

The ratio of the smaller part of a line segment to the larger part is the same as the ratio of the larger part to the whole line segment.

For a line segment of length 1, denote the larger part by $r$ and the smaller part by $1-r$ as shown here:




Hence, we have the ratios $\frac{1-r}{r} = \frac{r}{1}$ and we obtain the quadratic equation $r^2 = 1-r$ or $r^2+r=1$

The equation $r^2+r-1=0$ has two roots as

$$r = \frac{-1+\sqrt{5}}{2} \approx 0.61803\text{----} \quad \text{and} \quad r = \frac{-1-\sqrt{5}}{2} \approx -1.61803\text{----}$$

The reciprocal of the positive root is the golden ratio $\rho$ i.e.,

$$\rho = \frac{1}{r} \quad (r>0)$$

$$= \frac{2}{\sqrt{5}-1} = \frac{2(\sqrt{5}+1)}{5-1} = \frac{\sqrt{5}+1}{2} \approx 1.61803\text{-----}$$

## The Golden Section Search Algorithm:

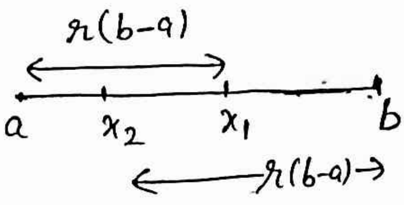
Our problem is $\text{Min } f(x)$
s.t. $x \in [a, b]$ where $f(x)$ is continuous and unimodal.

We can follow the following algorithm:

**Step 1** Find two intermediate points $x_1$ and $x_2$ such that

$$x_1 = a + r(b-a)$$
$$x_2 = b - r(b-a)$$

where $r = \frac{\sqrt{5}-1}{2} \approx 0.61803.....$




**Step 2** Evaluate $f(x_1)$ and $f(x_2)$.

If $f(x_1) > f(x_2)$, then interval of uncertainty is $[a, x_1]$ and

$$a = a$$
$$b = x_1$$
$$x_1 = x_2$$
$$x_2 = b - r(b-a)$$

If $f(x_1) \leq f(x_2)$, then interval of uncertainty is $[x_2, b]$ and

$$b = b$$
$$a = x_2$$
$$x_2 = x_1$$
$$x_1 = a + r(b-a)$$

**Step 3** If $b - a < \varepsilon$ (a sufficienty smaller number according to desire accuracy), then minimum occurs at $\frac{a+b}{2}$ and stop iterating, else go to step 2.

**Note:** If problem is of maximization, then choose the interval of uncertainty according to that and make the changes on the same way.

For eg: For maximization problem, if $f(x_1) > f(x_2)$ in step 2, then interval is $[x_2, b]$ and then $a = x_2$, $b = b$, $x_2 = x_1$, $x_1 = a + r(b-a)$.

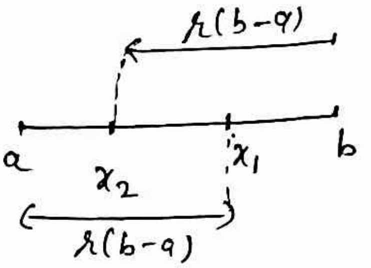
**Que** Use the golden section search to find the value of $x$ that minimizes $f(x) = x^2 - 6x + 2$ in the range $[0, 10]$. Locate this value of $x$ to within a range of $0.25$.

**Sol** Given $f(x) = x^2 - 6x + 2$, $a = 0$, $b = 10$ (for initial interval)

$x_1$ and $x_2$ are two intermediate points in $[a, b]$ such that

$$\left.\begin{aligned} x_1 &= a + r(b-a) \\ x_2 &= b - r(b-a) \end{aligned}\right\} \Rightarrow x_1 + x_2 = a + b$$

where $r = \dfrac{\sqrt{5}-1}{2} \approx 0.61803$




| no. of steps (n) | a | b | $x_1 = a + 0.61803(b-a)$ | $x_2 = b - 0.61803(b-a)$ | $f(x_1) = x_1^2 - 6x_1 + 2$ | $f(x_2) = x_2^2 - 6x_2 + 2$ | Min at $x_1/x_2$ |
|---|---|---|---|---|---|---|---|
| n=1 | 0 | 10 | 6.1803 | 3.8197 | 3.1143 | −6.3281 | $x_2$ |
| n=2 | 0 | 6.1803 | 3.8197 | 2.3606 | −6.3281 | −6.5912 | $x_2$ |
| n=3 | 0 | 3.8197 | 2.3606 | 1.4591 | −6.5912 | −4.6256 | $x_1$ |
| n=4 | 1.4591 | 3.8197 | 2.9182 | 2.3606 | −6.9933 | −6.5912 | $x_1$ |
| n=5 | 2.3606 | 3.8197 | 3.2621 | 2.9182 | −6.9313 | −6.9933 | $x_2$ |
| n=6 | 2.3602 | 3.2621 | 2.9182 | 2.7041 | −6.9933 | −6.9124 | $x_1$ |
| n=7 | 2.7041 | 3.2621 | 3.048 | 2.9182 | −6.9977 | −6.9933 | $x_1$ |
| n=8 | 2.9182 | 3.2621 | 3.1323 | 3.048 | −6.9825 | −6.9977 | $x_2$ |
| | 2.9182 | 3.1323 | | | | | |

Since $(3.1323 - 2.9182) = 0.2141 < 0.25$, we stop here.

$\therefore\ x^* = \dfrac{2.9182 + 3.1323}{2} = 3.025$ and $f(x^*) = -6.999 \approx 7$

Number of steps required to reach accuracy $\varepsilon$ by the golden section search method (where $\varepsilon$ is the range for final interval)

After step 1, we get two evaluations and the length of new reduced interval $= r(b-a)$

After step 2, we get three evaluations and the length of new reduced interval $= r^2(b-a)$

$\vdots$

After step $n$, we get $(n+1)$ evaluations and the length of new reduced interval $= r^n(b-a)$

To reach the accuracy $\varepsilon$, $r^n(b-a) \leq \varepsilon$ where $r = 0.61803$ the smallest value of $n$ satisfying this inequality gives the no. of steps required. $(n \in N)$

Que Find the no. of steps required to reach the value of $x$ within a range of 0.25 to minimize the function

$$f(x) = x^2 - 2x + 10 \,;\; x \in [0, 10]$$

by using golden section search method.

Sol Let the no. of steps required be $n$.

Given $a = 0$, $b = 10$, $\varepsilon = 0.25$. To reach the accuracy $\varepsilon$,

$$r^n(b-a) \leq \varepsilon \quad \text{where } r = 0.61803$$

$$\Rightarrow (0.61803)^n \times 10 \leq 0.25 \Rightarrow (0.61803)^n \leq 0.025$$

$$\Rightarrow n \log(0.61803) \leq \log(0.025)$$

$$\Rightarrow n \times (-0.48122) \leq (-3.68888)$$

$$\Rightarrow n \geq \frac{3.68888}{0.48122} = 7.6657 \Rightarrow \boxed{n = 8} \quad \text{Ans}$$

Note: It is already verified in the previous question.

## Newton's method for unconstrained one variable minimization:

Minimize $f(x)$ when $x_0$, the initial guess for $x$ is given or some interval is given in which $x_0$ lies.

- We assume $f'(x)$ and $f''(x)$ exists for each measurement point $x_n$.
- We can fit a quadratic function through $x_n$ that matches its first and second derivatives with that of the function $f$.

$$q(x) = f(x_n) + (x - x_n) f'(x_n) + \frac{1}{2}(x - x_n)^2 f''(x_n)$$

Clearly $q(x_n) = f(x_n)$, $q'(x_n) = f'(x_n)$ and $q''(x_n) = f''(x_n)$

- Instead of minimizing $f$, we minimize its approximation $q$.
- For minimizing $q$, the necessary condition is

$$q'(x) = 0$$

$$\Rightarrow f'(x_n) + (x - x_n) f''(x_n) = 0$$

$$\Rightarrow x = x_n - \frac{f'(x_n)}{f''(x_n)}$$

Setting $x = x_{n+1}$, we obtain

$$\boxed{x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)}}, \quad n = 0, 1, 2, 3, \ldots$$

Note: • We stop when $|x_{n+1} - x_n| < \varepsilon$ for a given $\varepsilon$ (accuracy).

- Newton's method work well if $f''(x) > 0$ everywhere. However, if $f''(x) < 0$ for some $x$, Newton's method may fail to converge to the minimizer.

**Que** Using Newton's method, find the minimum of

$$f(x) = \frac{1}{2}x^2 - \sin x \;;\; x_0 = 0.5$$

when the accuracy required $\varepsilon = 10^{-5} = 0.00001$ for $x$.

**Sol** Given $f(x) = \frac{1}{2}x^2 - \sin x$, $x_0 = 0.5$

$\therefore f'(x) = x - \cos x$

$f''(x) = 1 + \sin x$

According to Newton's method,

$$x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)}, \quad n = 0, 1, 2, 3, \ldots$$

$$\Rightarrow x_{n+1} = x_n - \frac{(x_n - \cos x_n)}{(1 + \sin x_n)}$$

$$= \frac{\cancel{x_n} + x_n \sin x_n - \cancel{x_n} + \cos x_n}{1 + \sin x_n}$$

$$\therefore x_{n+1} = \frac{x_n \sin x_n + \cos x_n}{1 + \sin x_n}$$

| $n$ | $x_n$ | $x_n \sin x_n + \cos x_n$ | $1 + \sin x_n$ | $x_{n+1}$ |
|---|---|---|---|---|
| 0 | 0.5 | 1.117295 | 1.479426 | 0.755222 |
| 1 | 0.755222 | 1.245787 | 1.685450 | 0.739274 |
| 2 | 0.739274 | 1.237045 | 1.673752 | 0.739085 |
| 3 | 0.739085 | 1.236942 | 1.673612 | 0.739085 |

From last two iterations $|x_3 - x_4| = 0 < \varepsilon = 10^{-5}$

$$\left.\begin{aligned} \therefore\ & x^* = 0.739085 \\ \text{and}\ & f(x^*) = -0.400489 \end{aligned}\right\} \text{Ans}$$