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Computational Methods

Review of Taylor Series:

Familiar (and useful) examples of Taylor series are the following:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad (|x| < \infty)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \quad (|x| < \infty)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \quad (|x| < \infty)$$

$$\frac{1}{(1-x)} = 1 + x + x^2 + x^3 + \cdots = \sum_{k=0}^{\infty} x^k \quad (|x| < 1)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k} \quad (-1 < x \leq 1)$$

All the above series are the examples of Taylor series of the given function about the point $c = 0$.

A Taylor series expanded about $c = 1$ is

$$\ln(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(x-1)^k}{k}$$

where $0 < x \leq 2$ $(\because -1 < x-1 \leq 1)$

Formal Taylor series for f about c:

$$f(x) \sim f(c) + (x-c) f'(c) + \frac{(x-c)^2}{2!} f''(c) + \frac{(x-c)^3}{3!} f'''(c) + \cdots$$

or $$f(x) \sim \sum_{k=0}^{\infty} \frac{(x-c)^k}{k!} f^{(k)}(c)$$

is called the "Taylor series of f at the point c."

In the special case $c=0$,

$$f(x) \sim f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

or $f(x) \sim \sum_{k=0}^{\infty} \frac{x^k}{k!} f^{(k)}(0)$

it is also called a Maclaurin series.

Note: (1) Here, rather than using $=$, we have written $\sim$ to indicate that we are not allowed to assume that $f(x)$ equals the series on the right.

(2) Taylor series of f at the point c exists provided the successive derivatives $f', f'', f''', \ldots$ exist at the point c.

Que Write the Taylor series of the function

$$f(x) = 3x^5 - 2x^4 + 15x^3 + 13x^2 - 12x - 5$$ at the point $c = 2$?

Sol We know that the Taylor series of the function f at the point c is given by

$$f(x) \sim f(c) + (x-c) f'(c) + \frac{(x-c)^2}{2!} f''(c) + \frac{(x-c)^3}{3!} f'''(c) + \cdots$$

Given $C = 2$

and $f(x) = 3x^5 - 2x^4 + 15x^3 + 13x^2 - 12x - 5$ $\qquad f(2) = 207$

$\therefore\ f'(x) = 15x^4 - 8x^3 + 45x^2 + 26x - 12$ $\qquad f'(2) = 396$

$f''(x) = 60x^3 - 24x^2 + 90x + 26$ $\qquad f''(2) = 590$

$f'''(x) = 180x^2 - 48x + 90$ $\qquad f'''(2) = 714$

$f^{(4)}(x) = 360x - 48$ $\qquad f^{(4)}(2) = 672$

$f^{(5)}(x) = 360$ $\qquad f^{(5)}(2) = 360$

$f^{(k)}(x) = 0 \quad \forall k \geq 6, k \in N$ $\qquad f^{(k)}(2) = 0 \ \forall k \geq 6, k \in N$

∴ The Taylor series of the given function at the point $c = 2$ is

$$f(x) \sim 207 + (x-2)\cdot 396 + \frac{(x-2)^2}{2!}\cdot 590 + \frac{(x-2)^3}{3!}\cdot 714$$
$$+ \frac{(x-2)^4}{4!}\cdot 672 + \frac{(x-2)^5}{5!}\cdot 360$$

i.e., $$f(x) \sim 207 + 396(x-2) + 295(x-2)^2 + 119(x-2)^3 + 28(x-2)^4 + 3(x-2)^5$$

Ans

Note: In this example, $\sim$ may be replaced by $=$ but it is not possible in general.

Que Using the complete Horner's algorithm, find the Taylor expansion of the function

$$f(x) = 3x^5 - 2x^4 + 15x^3 + 13x^2 - 12x - 5 \text{ at the point } c = 2?$$

Sol The work can be arranged as follows:

2)	3	−2	15	13	−12	−5
		6	8	46	118	212
	3	4	23	59	106	207
		6	20	86	290	
	3	10	43	145	396	
		6	32	150		
	3	16	75	295		
		6	44			
	3	22	119			
		6				
	3	28				

$\therefore$ By Horner's algorithm, the Taylor series of the given function at the point $c = 2$ is

$$f(x) = 3(x-2)^5 + 28(x-2)^4 + 119(x-2)^3 + 295(x-2)^2 + 396(x-2) + 207$$

A

Note: We can use Horner's algorithm for finding the Taylor expansion of a polynomial about any point. So, we can replace $\sim$ by $=$ in the Taylor expansion.

Taylor's theorem for $f(x)$:

If the function f possesses continuous derivatives of order $0, 1, 2, \ldots, (n+1)$ in a closed interval $I = [a, b]$, then for any c and x in I,

$$f(x) = \sum_{k=0}^{n} \frac{(x-c)^k}{k!} f^{(k)}(c) + E_{n+1} = f(c) + \frac{(x-c)}{1!}f'(c) + \frac{(x-c)^2}{2!}f''(c) + \cdots + \frac{(x-c)^n}{n!}f^{(n)}(c) + E_{n+1}$$

where E_{n+1} is called the remainder or error term and is given by

$$E_{n+1} = \frac{(x-c)^{n+1}}{(n+1)!} f^{(n+1)}(\xi) \quad \text{(Lagrange's form)}$$

Here ξ is a point that lies between c and x and depends on both.

Note: The explicit assumption in this theorem is that $f(x), f'(x), f''(x), \ldots, f^{(n+1)}(x)$ are all continuous functions in the interval $I = [a, b]$ and the formula for E_{n+1} is valid when $f^{(n+1)}$ exists at each point of the open interval (a, b). Here the point ξ is in the open interval (c, x) or (x, c).

Other form of Taylor's theorem

Taylor's Theorem for $f(x+h)$:

If the function f possesses continuous derivatives of order $0, 1, 2, ---, (n+1)$ in a closed interval $I = [a, b]$, then for any x in I,

$$f(x+h) = \sum_{k=0}^{n} \frac{h^k}{k!} f^{(k)}(x) + E_{n+1} = f(x) + \frac{h}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \cdots + \frac{h^n}{n!}f^{(n)}(x) + E_{n+1}$$

where h is any value such that $x+h$ is in I and where

$$E_{n+1} = \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(\xi)$$

for some ξ between x and $x+h$.

Note: (1) This form can be obtained from the previous form by replacing x by $x+h$ and replacing c by x.

(2) The requirement on ξ means $x < \xi < x+h$ if $h > 0$
or $x+h < \xi < x$ if $h < 0$

(3) The error term E_{n+1} depends on h in two ways: First h^{n+1} is explicitly present; second the point ξ generally depends on h. As $h \to 0$, $E_{n+1} \to 0$ with essentially the same rapidity with which h^{n+1} converges to zero. For large n, this is quite rapid. To express the qualitative fact, we write

$$E_{n+1} = O(h^{n+1})$$

as $h \to 0$. This is called big O notation.

Roughly speaking, $E_{n+1} = O(h^{n+1})$ means that the behavior of E_{n+1} is similar to the much simpler expression h^{n+1}.

Que Derive the Taylor series for e^x at $c=0$ and prove that it converges to e^x by using Taylor's Theorem.

Sol Let $f(x) = e^x$. Then

$$f^{(k)}(x) = e^x \quad \text{for } k \geq 0$$

$$\therefore f^{(k)}(c) = f^{(k)}(0) = e^0 = 1 \quad \forall\, k \geq 0$$

$\therefore$ By Taylor's Theorem for $f(x)$,

$$f(x) = \sum_{k=0}^{n} \frac{(x-c)^k}{k!} f^{(k)}(c) + E_{n+1}$$

where $E_{n+1} = \dfrac{(x-c)^{n+1}}{(n+1)!} f^{(n+1)}(\xi)$ (ξ is a point that lies between c and x)

$\therefore$ We have $$e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + \frac{x^{n+1}}{(n+1)!} e^{\xi} \qquad (1)$$

Now, let us consider all the values of x in some symmetric interval around the origin, for example, $-s \leq x \leq s$. Then $|x| \leq s$, $|\xi| \leq s$, and $e^{\xi} \leq e^s$. Hence

$$\lim_{n\to\infty} |E_{n+1}| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} e^{\xi} \right| \leq \lim_{n\to\infty} \frac{s^{n+1}}{(n+1)!} e^s = 0$$

$$\Rightarrow \lim_{n\to\infty} E_{n+1} = 0 \qquad \left(\because \lim_{n\to\infty} \frac{x^n}{n!} = 0\right)$$

$\therefore$ If we take the limit as $n \to \infty$ on both sides of equ. (1), we get

$$e^x = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{x^k}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Note: The above example shows that in specific cases, a formal Taylor series actually represents the function.

Now, we examine another example to see how the formal series can fail to represent the function.

Example Derive the formal Taylor series for $f(x) = \ln(1+x)$ at $c=0$, and determine the range of positive x for which the series represents the function.

Sol Given $f(x) = \ln(1+x)$, $c=0$ $\qquad f(0)=0$

$$\therefore f'(x) = \frac{1}{(1+x)} \qquad f'(0) = 1$$

$$f''(x) = -\frac{1}{(1+x)^2} \qquad f''(0) = -1$$

$$f'''(x) = \frac{2}{(1+x)^3} \qquad f'''(0) = 2$$

$$f^{(4)}(x) = \frac{-2\cdot 3}{(1+x)^4} \qquad f^{(4)}(0) = -6$$

$$\vdots \qquad\qquad \vdots$$

$$f^{(k)}(x) = \frac{(-1)^{k-1}(k-1)!}{(1+x)^k} \qquad f^{(k)}(0) = (-1)^{k-1}(k-1)!$$

Hence by Taylor's Theorem, we get $f(x) = f(0) + x$

$$\ln(1+x) = \sum_{k=1}^{n} \left\{(-1)^{k-1}(k-1)!\right\} \frac{x^k}{k!} + E_{n+1} = \sum_{k=1}^{n} \frac{(-1)^{k-1} x^k}{k} + E_{n+1}$$

$$\text{where } E_{n+1} = \frac{(-1)^n n!}{(1+\xi)^{n+1}} \frac{x^{n+1}}{(n+1)!} = \frac{(-1)^n}{(n+1)} \left(\frac{x}{1+\xi}\right)^{n+1}$$

For the infinite series to represent $\ln(1+x)$, it is necessary and sufficient that the error term converge to 0 as $n \to \infty$.

Let us assume that $0 \le x \le 1$. Then $0 \le \xi \le x$ (because 0 is the point of expansion).

$$\therefore \quad 0 \le \frac{x}{(1+\xi)} \le 1$$

$$\therefore \lim_{n\to\infty} |E_{n+1}| = \lim_{n\to\infty} \left| \frac{(-1)^n}{(n+1)} \left(\frac{x}{1+\xi}\right)^{n+1} \right|$$

$$\leq \lim_{n\to\infty} \frac{1}{(n+1)} = 0$$

$$\Rightarrow \lim_{n\to\infty} E_{n+1} = 0$$

But if $x > 1$, the terms in the series do not approach 0, and the series does not converge.

Hence, the series represents $\ln(1+x)$ if $0 \leq x \leq 1$ but not if $x > 1$.

Note: The series also represents $\ln(1+x)$ for $-1 < x < 0$ but not if $x \leq -1$.

Que Evaluate $\sqrt{1+h}$ in powers of h. Then compute $\sqrt{1.00001}$ and $\sqrt{0.99999}$

Sol Let $f(x) = \sqrt{x}$. Then by Taylor's theorem

$$f(x+h) = f(x) + \frac{h}{1!} f'(x) + \frac{h^2}{2!} f''(x) + E_3$$ (by taking $n=2$ for illustration)

where $E_3 = \frac{h^3}{3!} f'''(\xi)$ for some ξ between x and $x+h$

By taking $x=1$, we have

$$f(1+h) = \sqrt{1+h} = f(1) + \frac{h}{1!} f'(1) + \frac{h^2}{2!} f''(1) + \frac{h^3}{3!} f'''(\xi) \quad \text{——} (1)$$

where $1 < \xi < 1+h$ if $h > 0$

Now, $f(x) = \sqrt{x} = x^{\frac{1}{2}}$ $\qquad f(1) = 1$

$\Rightarrow f'(x) = \frac{1}{2} x^{-1/2}$ $\qquad f'(1) = \frac{1}{2}$

$f''(x) = -\frac{1}{4} x^{-3/2}$ $\qquad f''(1) = -\frac{1}{4}$

$f'''(x) = \frac{3}{8} x^{-5/2}$ $\qquad f'''(\xi) = \frac{3}{8} \xi^{-5/2}$

$\therefore$ By equ.(1),

$$\sqrt{1+h} = 1+\frac{1}{2}h-\frac{1}{8}h^2+\frac{1}{16}h^3\xi^{-5/2} \quad \text{where } 1<\xi<1+h \text{ if } h>0$$

Let $h = 0.00001 = 10^{-5}$. Then

$$\sqrt{1.00001} \approx 1+0.000005-0.125\times 10^{-10} = 1.000004999987500$$

By substituting $-h$ for h in the series, we obtain

$$\sqrt{1-h} = 1-\frac{1}{2}h-\frac{1}{8}h^2-\frac{1}{16}h^3\xi^{-5/2}$$

Hence, by taking $h = 0.00001$, we have

$$\sqrt{0.99999} \approx 1-0.000005-0.125\times 10^{-10} = 0.999994999987500$$

Now, $\frac{1}{16}h^3\xi^{-5/2} < \frac{1}{16}10^{-15}$ $\left(\because 1<\xi<1+h \Rightarrow \xi^{-5/2}<1\right)$

$= 0.0625\times 10^{-15}$

$= 0.0000000000000000625$

$\therefore$ Both numerical values are correct to all 15 decimal places shown.

Que Use five terms in Taylor series for $f(x) = \ln(1+x)$ about $x = 0$ to approximate $\ln(1.1)$.

Sol We have, $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{(4)}(0) + \cdots$ ——— (1)

Here $f(x) = \ln(1+x)$ $\qquad f(0) = 0$

$f'(x) = \frac{1}{(1+x)}$ $\qquad f'(0) = 1$

$f''(x) = \frac{-1}{(1+x)^2}$ $\qquad f''(0) = -1$

$f'''(x) = +\frac{2}{(1+x)^3}$ $\qquad f'''(0) = 2$

$f^{(4)}(x) = \dfrac{-6}{(1+x)^4}$ $\qquad f^{(4)}(0) = -6$

$f^{(5)}(x) = \dfrac{24}{(1+x)^5}$ $\qquad f^{(5)}(0) = 24$

$\therefore$ The Taylor series for $f(x) = \ln(1+x)$ upto five terms is

$$\ln(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + \frac{x^5}{5!}(24)$$

$$\Rightarrow \ln(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$$

By taking $x = 0.1$,

$$\ln(1.1) \approx 0.1 - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} + \frac{(0.1)^5}{5}$$

$$= 0.1 - \frac{0.01}{2} + \frac{0.001}{3} - \frac{0.0001}{4} + \frac{0.00001}{5}$$

$$= 0.1 - 0.05 + 0.00033333 - 0.000025 + 0.000002$$

$$= 0.0953103333 \ldots$$

This value is correct to six decimal places of accuracy.