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Rolle's Theorem: If

(i) f is a continuous function on the closed interval $[a,b]$

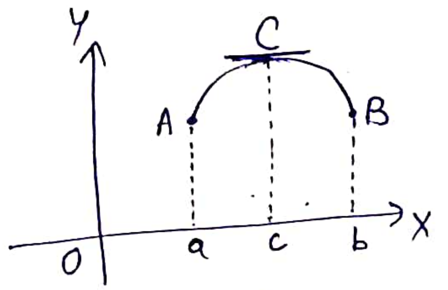
(ii) f' exists at each point of the open interval (a,b)

(iii) $f(a) = f(b)$,

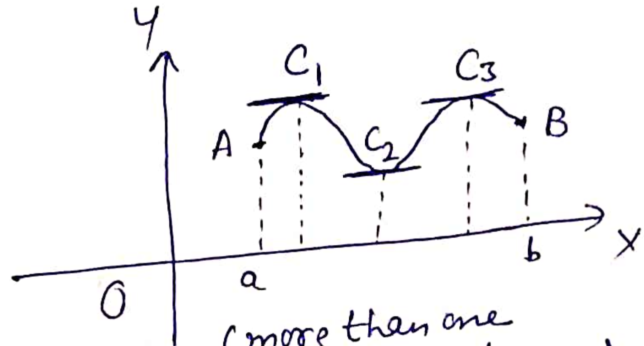
then there is at least one value $c \in (a,b)$ such that $f'(c) = 0$

Geometrical Interpretation:

Geometrically we can say that there is at least one point C (may be more) of the curve at which the tangent is parallel to the x-axis.



(exactly one point when $f'(c) = 0$)



(more than one point when $f'(c) = 0$)

Que: Verify Rolle's theorem for

(i) $\frac{\sin x}{e^x}$ in $[0, \pi]$ (ii) $(x-a)^m (x-b)^n$ where m, n are positive integers in $[a,b]$

Sol (i) Let $f(x) = \frac{\sin x}{e^x}$

(i) Clearly f is a continuous function in $[0,\pi]$ because $\sin x$ and e^x both are continuous in $[0,\pi]$ and $e^x \neq 0$ for any x.

(ii) Also, f is differentiable in $(0,\pi)$ as $\sin x$ and e^x both are differentiable in $(0,\pi)$ and $e^x \neq 0$ for any x.

(iii) $f(0) = \frac{\sin 0}{e^0} = \frac{0}{1} = 0$, $f(\pi) = \frac{\sin \pi}{e^\pi} = \frac{0}{e^\pi} = 0$

$\therefore f(0) = f(\pi)$

Hence the conditions of Rolle's theorem are satisfied.

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Now, $f'(x) = \dfrac{e^x\cos x - e^x \sin x}{e^{2x}} = \dfrac{(\cos x - \sin x)e^x}{e^{2x}} = \dfrac{\cos x - \sin x}{e^x}$

$(\because e^x \neq 0)$

$\therefore$ $f'(x) = 0$ when $(\cos x - \sin x) = 0$

i.e, $\tan x = 1$

$\Rightarrow x = \frac{\pi}{4} \in (0, \pi)$

So, there exists a point $c = \frac{\pi}{4} \in (0,\pi)$ such that

$f'(\frac{\pi}{4}) = 0$

Hence Rolle's theorem is verified.

(ii) Let $f(x) = (x-a)^m (x-b)^n$ where m, n are positive integers in $[a,b]$.

Since every polynomial is continuous and differentiable for all values of x.

$\therefore$ f is continuous function in $[a,b]$ and differentiable in (a,b).

Also, $f(a) = f(b) = 0$

Hence the conditions of Rolle's theorem are satisfied.

Now, $f'(x) = m(x-a)^{m-1}(x-b)^n + (x-a)^m . n(x-b)^{n-1}$

$= (x-a)^{m-1}(x-b)^{n-1}[m(x-b) + n(x-a)]$

$= (x-a)^{m-1}(x-b)^{n-1}[(m+n)x - (mb+na)]$

$\therefore$ $f'(x) = 0$ when $x = \dfrac{mb+na}{m+n}$

So, there exists a point $c = \dfrac{mb+na}{m+n} \in (a,b)$ such that

$f'(c) = 0$

Hence Rolle's theorem is verified.

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Mean Value Theorem: If

(i) f is a continuous function on the closed interval $[a, b]$

(ii) f' exists at each point of the open interval (a, b),

then there is at least one value $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Que Verify Mean value theorem for

$$f(x) = x(x-1)(x-2) \text{ in } [0, \tfrac{1}{2}].$$

Sol Since every polynomial is continuous and differentiable for all values of x, therefore

$$f(x) = x^3 - 3x^2 + 2x$$

$\Rightarrow f$ is continuous in $[0, \frac{1}{2}]$ and f' exists in (a, b).

Now, $f'(x) = 3x^2 - 6x + 2$

$$f(b) = f\left(\tfrac{1}{2}\right) = \left(\tfrac{1}{2}\right)^3 - 3\cdot\left(\tfrac{1}{2}\right)^2 + 2\cdot\left(\tfrac{1}{2}\right) = \frac{1}{8} - \frac{3}{4} + 1 = \frac{9-6}{8} = \frac{3}{8}$$

$$f(a) = f(0) = 0$$

$$\therefore \frac{f(b) - f(a)}{b - a} = \frac{\frac{3}{8} - 0}{\frac{1}{2} - 0} = \frac{3}{8} \times 2 = \frac{3}{4}$$

Now, $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

When $3c^2 - 6c + 2 = \dfrac{3}{4}$

$$\Rightarrow 12c^2 - 24c + 5 = 0$$

$$\Rightarrow c = \frac{24 \pm \sqrt{(24)^2 - 4 \times 12 \times 5}}{2 \times 12} = \frac{24 \pm \sqrt{576 - 240}}{24}$$

$$\Rightarrow c = \frac{24 \pm \sqrt{336}}{24} = \frac{24 \pm 18.33}{24} = 1.764,\ 0.236$$

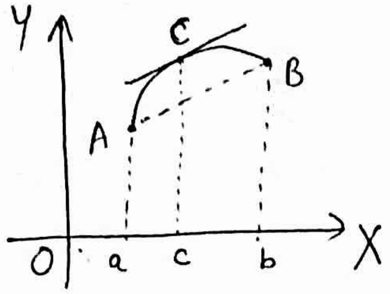
So, there exists $c = 0.236 \in (0, \frac{1}{2})$ such that

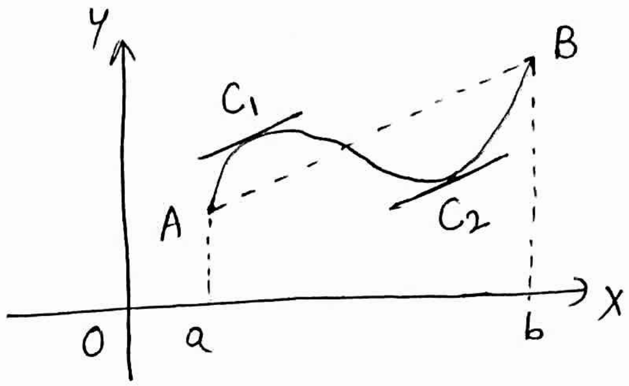
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Hence Mean value theorem is verified.

Geometrical Interpretation:

Geometrically, by mean value theorem we can say that there exists at least one point C (may be more) of the curve at which the tangent is parallel to the chord AB where $A = (a, f(a))$ and $B = (b, f(b))$.





<u>Note</u>: The special case $n = 0$ in Taylor's Theorem is known as the Mean-Value Theorem.