1 Paper Title

An Adventure in the Nth Dimension

2 Summary

This piece by Prof Brian Hayes is a very candid account of his experiments with high dimensional objects particularly the n-cube and n-sphere. The author states the counterintuitive nature of high dimensional space and attempts to understand it from a beginner's perspective. The author does not introduce any new mathematics that already hasn't been done, but brings an experimental attitude by finding out the graph characteristics of volume and surface area of n-sphere and how that relates to n=5 and n=7 respectively, which is seldom discussed in related works.

3 Detailed Analysis

At first, the author asks the reader to do a small experiment - fit a unit n-ball inside a n-cube of side length 2. As we go progressively to higher n from n=1, the ratio of ball's volume to the cube's volume decreases. The volume of a n-sphere of radius r is:

$$V(n,r) = \frac{\pi^{\frac{n}{2}} r^n}{\Gamma(\frac{n}{2}+1)} \tag{1}$$

And the volume of a cube is just 2^n Now as n increases, this means there is less and less volume available for the sphere to occupy in the n-cube as n goes higher which is intuitive as we expect number of corners to increase and most of the volume is left out at the corners. Then the author analyses the actual volume of a ball as n increases and discovers a counterintuitive idea - that the volume of the ball itself is decreasing as n goes higher as shown in the figure:

n	$V(n,1)$	
1	2	
2	π	≈ 3.1416
3	$\frac{4}{3}\pi$	≈ 4.1888
4	$\frac{1}{2}\pi^2$	≈ 4.9348
5	$\frac{8}{15}\pi^2$	≈ 5.2638
6	$\frac{1}{6}\pi^3$	≈ 5.1677
7	$\frac{16}{105}\pi^3$	≈ 4.7248
8	$\frac{1}{24}\pi^4$	≈ 4.0587
9	$\frac{32}{945}\pi^4$	≈ 3.2985
10	$\frac{1}{120}\pi^5$	≈ 2.5502

Figure 1: Volume of sphere of radius 1 as n increases from 1 to 10.

So, from the figure 1, at n=5 the sphere itself is shrinking as n goes higher. As n approaches infinity, the ball occupies zero volume (reaches limit zero). This can be explained as follows. The ball (n-sphere) is still the largest that can be fit inside the cube. It is touching the faces of the cube and there cannot be any bigger ball that fits inside the cube. It's diameter is still 2 and the cube is still of side 2. This counterintuitive is explanied by counting the faces vs corners of a n-cube. The cube has $2n$ faces but it has 2^n corners which really tells that the cube itself doesn't have much volume in the middle, volume that the ball occupies. Another way to look at it is to compare the lengths of the side of the cube to its longest diameter. The side is still 2 (s in general case), but the longest diameter is $2\sqrt{n}$ which increases with dimension. Hence a n-cube has kind-of poking skewers coming out of its corners and most of the volume is in the corners.

Another experiment that piques interest is the nature of the curve of volume in n-dimensions with respect to n. The volume increase first, up to n approx= 5 and then starts decreasing. This can be explained by basic math and looking at the trade-offs between numerator and denominator of the Volume equation.
An intuitive way of thinking about volume of an n-ball can be found by summing up infinite (n-1) ball's volumes and then recursively repeating till n becomes 1. It's like slicing an n-ball into n-1 balls, then the n-1 ball into n-2 balls, and so on.

4 Your critiques

4.1 Assumptions made

One of the assumptions made by the author are that analyzing a n-sphere volume with respect to a n-cube can give a proper view of what's happening in the higher dimensions. The author does not discuss the implication of Euclidean distance between points in higher dimensions. Nor does he mention what happens to angles between vectors (points) in higher dimensions.

4.2 Pros and cons

The pros of this piece are many. This paper encourages the reader to venture into the high-dimesnional world and carry out small experiments like the author has done. Also, the author explicitly states results which makes the curse of dimensionality very accessible to readers. As per the techniques used, the author contributes the 5-ball theory by calculating exactly where (at which dimension), the volume of a sphere reaches maximum, and provides an intutive explanation for the same.
But, the demerits of this work include not having a proper treatment of the higher dimension data points and how Euclidean distances get affected which is a very fundamental property to be considered. Also, as we move to higher dimensions, the angles between vectors lie in a very narrow range around 90 degrees, meaning all vectors are almost orthogonal to each other. This can be a very fundamental property to consider when analyzing higher dimensions.
Also, after stating the problem with higher dimensions - the counterintutive state of volumes, the author doesn't give any practical examples where this might be an issue. Practical implications of the curse of dimensionality are a very importat aspect which are not mentioned here but are necessary for a comprehensive analysis.
We also don't see any recommendations from the author to use alternative techniques like dimensionality reduction.

4.3 Possible future extensions

A very interesting implication of the "curse of dimensionality" is how it affects probability distributions.
In Figure 2, we see that as the dimensionality D of a Gaussian distribution [1] increases, most of the probability mass lies within a thin shell at a specific radius, and the probability is no longer highest in the middle of Gaussian. A Gaussian loses its properties in higher dimensions and any Bayesian analysis we do will get affected. The extension of the paper would be to further analyze such probability distributions.

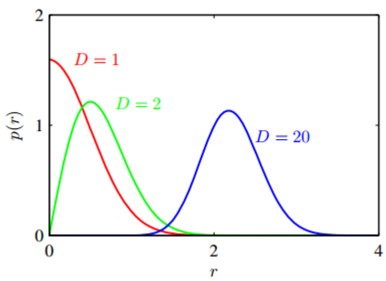

Figure 2: Probability density with respect to radius r of a Gaussian distribution for various values of dimensionality D.

5 Closing Remarks

5.1 What you have learned from reading this paper ?

1. I learned about the need to carefully and critically think about higher dimensions
2. I also learned how to make an article accessible and intutive even if it involves a lot of Mathematics.

References

[1] Bishop, C. M. *Pattern Recognition and Machine Learning*. Springer, 2006.
[1]