String Processing Algorithms 2015 - Week 2 Exercises

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Exercise 1

Outline algorithms that find the most frequent symbol in a give string.

(a) for ordered alphabet, and

(b) for integer alphabet.

The algorithms should be as fast as possible. What are their (worst case) time complexities? Consider also the case where $\sigma \gg n$.

Solution

Algorithm 1: MostFrequentSymbol(S)

```
let f be an empty map f : Σ → ℕ
μ = nil
L_μ = 0
for i = 1 to |S| do
    if S[i] is not mapped in f then
        f(S[i]) = 1
        if L_μ = 0 then
            L_μ = 1
            μ = S[i]
    else
        f(S[i]) = f(S[i]) + 1
        if L_μ < f(S[i]) then
            L_μ = f(S[i])
            μ = S[i]
return μ
```

Exercise 2

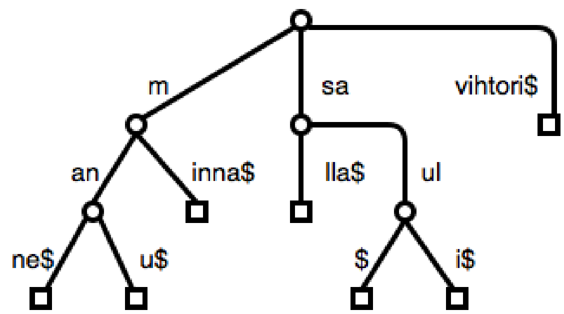
Let $\mathcal{R} = \{$`manne`, `manu`, `minna`, `salla`, `saul`, `sauli`, `vihtori`$\}$.

(a) Give the compact trie of $\mathcal{R}$.

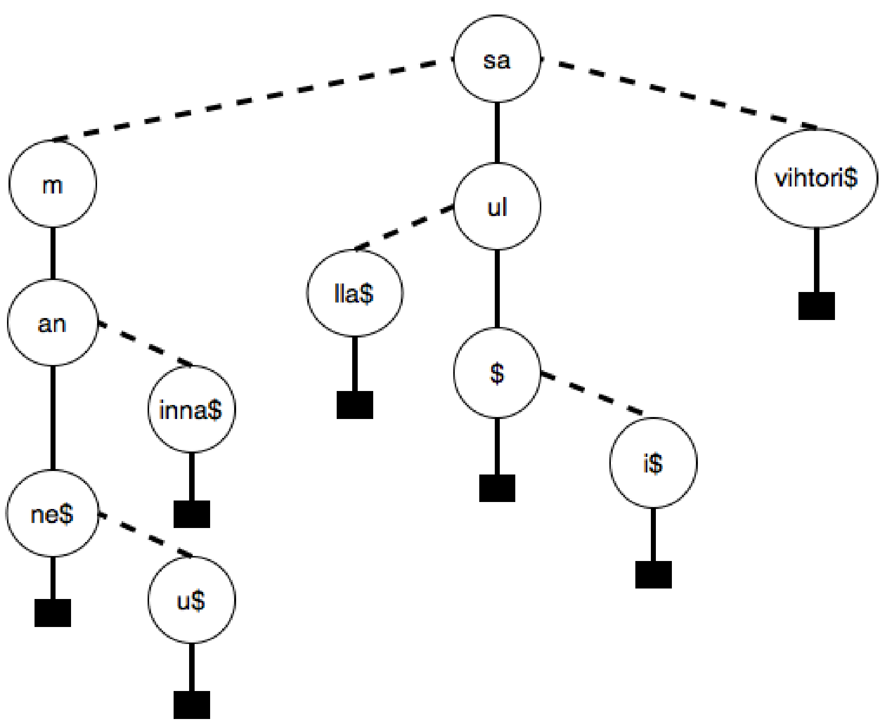
(b) Give the balanced compact ternary trie of $\mathcal{R}$.

Solution

(a)



(b)



Exercise 3

What is the time complexity of prefix queries for

(a) trie with constant alphabet

(b) compact trie with constant alphabet

(c) compact trie with ordered alphabet and binary tree implementation of the child function

(d) balanced compact ternary trie?

The queries should return the resulting strings as a list of pointers or other identifiers rather than the full strings.

Solution

(a)

Assuming a hash table implementation for the child function, the complexity is $\mathcal{O}(p + \mathcal{R})$, where p is the length of the prefix and $\mathcal{R}$ is the amount of distinct strings sharing the prefix.

(b)

In the worst case, asymptotically no faster than $\mathcal{O}(p + \mathcal{R})$ as above.

(c)

$\mathcal{O}((p + \mathcal{R}) \log n)$.

(d)

$\mathcal{O}(p + \mathcal{R})$.

Exercise 4

Prove

(a) Lemma 1.14: For $i \in [2..n]$, $LCP_{\mathcal{R}}[i] = lcp(S_i, \{S_1, \ldots, S_{i-1}\})$.

(b) Lemma 1.15: $\Sigma LCP(\mathcal{R}) \leq \Sigma lcp(\mathcal{R}) \leq 2 \cdot \Sigma LCP(\mathcal{R})$.

Solution

(a)

We need an auxiliary lemma first:

Lemma 1 (Neighborhood lemma)**.** *If $S_1 < S_2 < S_3$, $lcp(S_1, S_3) \leq lcp(S_2, S_3)$.*

Proof. The proof is by contradiction: Let $l_{13} = lcp(S_1, S_3)$ and $l_{23} = lcp(S_2, S_3)$. Assume the opposite that $l_{13} > l_{23}$. Now

$$\begin{aligned}
S_1 &= a_1 a_2 \ldots a_{l_{23}} \ldots a_{l_{13}} b_1 b_2 \ldots, \\
S_2 &= a_1 a_2 \ldots a_{l_{23}} c_1 c_2 \ldots, \\
S_3 &= a_1 a_2 \ldots a_{l_{23}} \ldots a_{l_{13}} d_1 d_2 \ldots.
\end{aligned}$$

Since $S_1 < S_2$, $c_1 > a_{l_{23}+1}$ and we must also have that $S_2 > S_3$, which is a contradiction. Analogous proof can be used to deduce that $lcp(S_1, S_3) \leq lcp(S_1, S_2)$. □

Example:

$$\begin{aligned}
S_1 :& aaaab \\
S_2 :& aaaba \\
S_3 :& aabba
\end{aligned}$$

Now assume that $i \in [2...n]$. We have that

$$lcp(S_i, \{S_1, \ldots, S_{i-1}\}) = \max\{lcp(S_i, S_{i-1}),\ lcp(S_i, \{S_1, \ldots, S_{i-2}\})\}.$$

By neighborhood lemma, for any $j = 1, 2, \ldots, i-2$, $lcp(S_i, S_j)$ cannot exceed $lcp(S_i, S_{i-1})$ and we must have that

$$lcp(S_i, \{S_1, \ldots, S_{i-1}\}) = lcp(S_i, S_{i-1}) = LCP_{\mathcal{R}}[i],$$

as expected.

(b)

$$
\begin{aligned}
\Sigma lcp(\mathcal{R}) &= \sum_{S \in \mathcal{R}} lcp(S, \mathcal{R} \setminus \{S\}) \\
&\leq \sum_{i \in [1..n-1]} lcp(S_i, S_{i+1}) + \sum_{i \in [2..n]} lcp(S_{i-1}, S_i) && \text{(by neighborhood lemma)} \\
&= \sum_{i \in [2..n]} lcp(S_{i-1}, S_i) + \sum_{i \in [2..n]} lcp(S_{i-1}, S_i) \\
&= 2 \sum_{i \in [2..n]} lcp(S_{i-1}, S_i) \\
&= 2 \sum_{i \in [2..n]} LCP_{\mathcal{R}}[i] \\
&= 2 \sum_{i \in [1..n]} LCP_{\mathcal{R}}[i] && \text{(since } LCP_{\mathcal{R}}[1] = 0\text{)} \\
&= 2 \cdot \Sigma LCP(\mathcal{R}).
\end{aligned}
$$

What comes to the lower bound of $\Sigma lcp(\mathcal{R})$, we have

$$
\begin{aligned}
\Sigma lcp(\mathcal{R}) &= lcp(S_1, S_2) + lcp(S_{n-1}, S_n) \\
&\quad + \sum_{i \in [2..n-1]} \max\{lcp(S_i, S_{i-1}), lcp(S_i, S_{i+1})\} && \text{(by neighborhood lemma)} \\
&\geq \sum_{i \in [2..n]} lcp(S_{i-1}, S_i) \\
&= \sum_{i \in [2..n]} LCP_{\mathcal{R}}[i] \\
&= \sum_{i \in [1..n]} LCP_{\mathcal{R}}[i] && \text{(since } LCP_{\mathcal{R}}[1] = 0\text{)} \\
&= \Sigma LCP(\mathcal{R}),
\end{aligned}
$$

which concludes the proof.

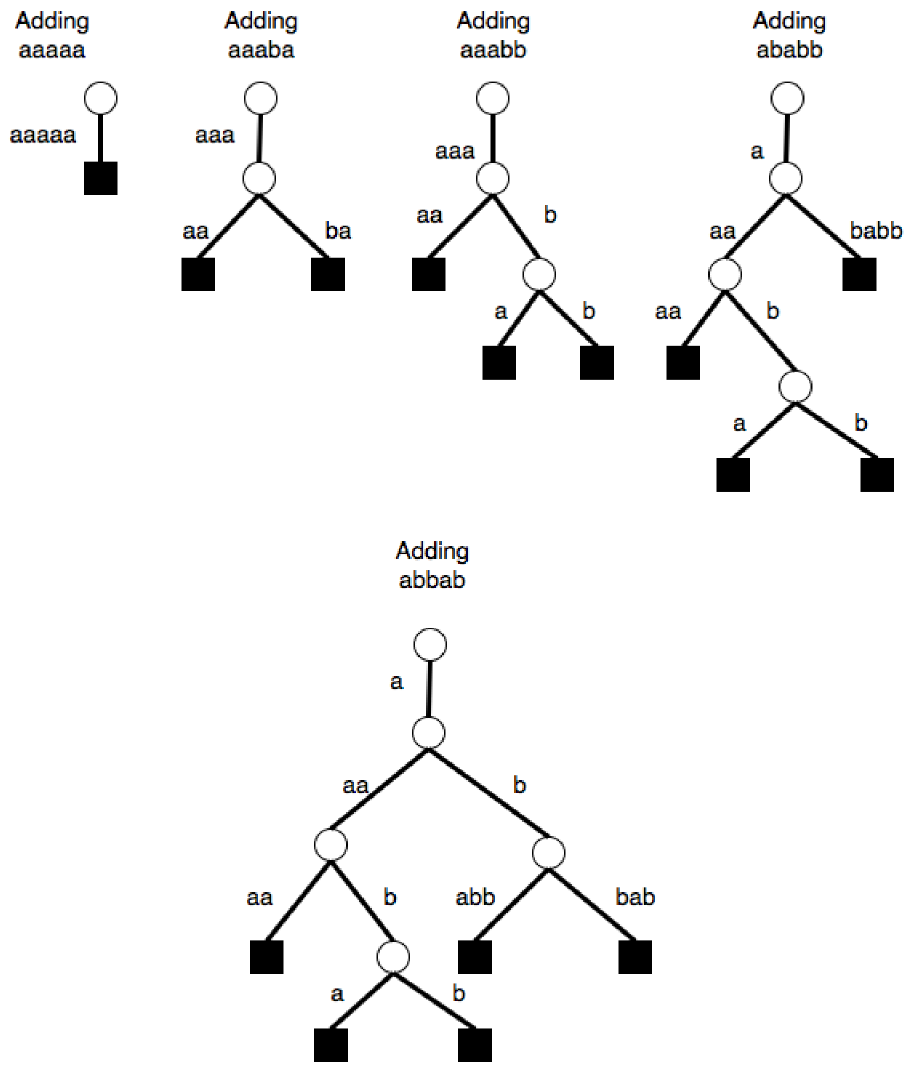
Example:

$\mathcal{R}$	$LCP_{\mathcal{R}}$	$lcp(S, \mathcal{R})$
aaaa	0	3
aaab	3	3
abba	1	1
baab	0	0
$\sum$	4	7

Exercise 5

Show how to construct the compact trie for a set $\mathcal{R}$ in $\mathcal{O}(|\mathcal{R}|)$ time (rather than $\mathcal{O}(||\mathcal{R}||)$ time) given the string set $\mathcal{R}$ in lexicographic order and the LCP array $LCP_{\mathcal{R}}$.

Solution



The LCP array is

i	S_i	$LCP_{\mathcal{R}}[i]$
1	$aaaaa$	0
2	$aaaba$	3
3	$aaabb$	4
4	$ababb$	1
5	$abbab$	2

It would seem that the algorithm must keep track of the last added edge. If the current LCP value ($l_i = LCP_{\mathcal{R}}[i]$) is no less than the previous one ($l_{i-1} = LCP_{\mathcal{R}}[i-1]$), take the last created edge and split it after $l_i - l_{i-1}$ characters into two edges: the left one completing the previously added string, and the right one completing current string. Set the right edge as the last one. If, however, $l_i < l_{i-1}$, we must restart from the root of the trie.