Introduction to Machine Learning, Fall 2014 - Exercise session IV

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Problem 1 (6 points)

(a)

We seek to compute

$$P(Y \mid \mathbf{X}) = \frac{P(\mathbf{X} \mid Y)P(Y)}{P(\mathbf{X})},$$

where

$$P(\mathbf{X}) = \sum_{i=0}^{2} P(\mathbf{X} \mid Y = i)P(Y = i).$$

(0, 0)

If $\mathbf{X} = (0,0)$,

$$
\begin{aligned}
P((0,0)) &= \sum_{i=0}^{2} P((0,0) \mid Y = i)P(Y = i) \\
&= 0.2 \times 0.4 + 0.6 \times 0.3 + 0.1 \times 0.3 \\
&= 0.08 + 0.18 + 0.03 \\
&= 0.29.
\end{aligned}
$$

$$
\begin{aligned}
P(Y = 0 \mid (0,0)) &= \frac{P((0,0) \mid Y = 0)P(Y = 0)}{P((0,0))} \\
&= \frac{0.2 \times 0.4}{0.29} \\
&= \frac{0.08}{0.29} \\
&\approx 0.276,
\end{aligned}
$$

$$P(Y = 1 \,|\, (0,0)) = \frac{P((0,0) \,|\, Y = 1)P(Y = 1)}{P((0,0))}$$
$$= \frac{0.6 \times 0.3}{0.29}$$
$$= \frac{0.18}{0.29}$$
$$\approx 0.621,$$

$$P(Y = 2 \,|\, (0,0)) = \frac{P((0,0) \,|\, Y = 2)P(Y = 2)}{P((0,0))}$$
$$= \frac{0.1 \times 0.3}{0.29}$$
$$= \frac{0.03}{0.29}$$
$$\approx 0.103.$$

(0, 1)

If $\mathbf{X} = (0,1)$,

$$P((0,1)) = \sum_{i=0}^{2} P((0,1) \,|\, Y = i)P(Y = i)$$
$$= 0.4 \times 0.4 + 0.1 \times 0.3 + 0.3 \times 0.3$$
$$= 0.16 + 0.03 + 0.09$$
$$= 0.28.$$

$$P(Y = 0 \,|\, (0,1)) = \frac{P((0,1) \,|\, Y = 0)P(Y = 0)}{0.28}$$
$$= \frac{0.4 \times 0.4}{0.28}$$
$$= \frac{0.16}{0.28}$$
$$\approx 0.571,$$

$$P(Y = 1 \,|\, (0,1)) = \frac{P((0,1) \,|\, Y = 1)P(Y = 1)}{0.28}$$
$$= \frac{0.1 \times 0.3}{0.28}$$
$$= \frac{0.03}{0.28}$$
$$\approx 0.107,$$

$$P(Y = 2 \,|\, (0, 1)) = \frac{P((0, 1) \,|\, Y = 2)P(Y = 2)}{0.28}$$
$$= \frac{0.3 \times 0.3}{0.28}$$
$$= \frac{0.09}{0.28}$$
$$\approx 0.321.$$

(0, 2)

If $\mathbf{X} = (0, 2)$,

$$P((0, 2)) = \sum_{i=0}^{2} P((0, 2) \,|\, Y = i)P(Y = i)$$
$$= 0.0 \times 0.4 + 0.1 \times 0.3 + 0.2 \times 0.3$$
$$= 0.03 + 0.06$$
$$= 0.09,$$

$$P(Y = 0 \,|\, (0, 2)) = \frac{P((0, 2) \,|\, Y = 0)P(Y = 0)}{0.09}$$
$$= \frac{0.0 \times 0.4}{0.09}$$
$$= 0,$$

$$P(Y = 1 \,|\, (0, 2)) = \frac{P((0, 2) \,|\, Y = 1)P(Y = 1)}{0.09}$$
$$= \frac{0.1 \times 0.3}{0.09}$$
$$= \frac{0.03}{0.09}$$
$$\approx 0.333,$$

$$P(Y = 2 \,|\, (0, 2)) = \frac{P((0, 2) \,|\, Y = 2)P(Y = 2)}{0.09}$$
$$= \frac{0.2 \times 0.3}{0.09}$$
$$= \frac{0.06}{0.09}$$
$$\approx 0.666.$$

(1, 0)

If $\mathbf{X} = (1, 0)$,

$$
\begin{aligned}
P((1,0)) &= \sum_{i=0}^{2} P((1,0) \,|\, Y = i)P(Y = i) \\
&= 0.1 \times 0.4 + 0.1 \times 0.3 + 0.4 \times 0.3 \\
&= 0.04 + 0.03 + 0.12 \\
&= 0.19,
\end{aligned}
$$

$$
\begin{aligned}
P(Y = 0 \,|\, (1,0)) &= \frac{P((1,0) \,|\, Y = 0)P(Y = 0)}{0.19} \\
&= \frac{0.1 \times 0.4}{0.19} \\
&= \frac{0.04}{0.19} \\
&\approx 0.211,
\end{aligned}
$$

$$
\begin{aligned}
P(Y = 1 \,|\, (1,0)) &= \frac{P((1,0) \,|\, Y = 1)P(Y = 1)}{0.19} \\
&= \frac{0.1 \times 0.3}{0.19} \\
&= \frac{0.03}{0.19} \\
&\approx 0.158,
\end{aligned}
$$

$$
\begin{aligned}
P(Y = 2 \,|\, (1,0)) &= \frac{P((1,0) \,|\, Y = 2)P(Y = 2)}{0.19} \\
&= \frac{0.4 \times 0.3}{0.19} \\
&= \frac{0.12}{0.19} \\
&\approx 0.632.
\end{aligned}
$$

(1, 1)

If $\mathbf{X} = (1, 1)$,

$$
\begin{aligned}
P((1,1)) &= \sum_{i=0}^{2} P((1,1) \,|\, Y = i)P(Y = i) \\
&= 0.2 \times 0.4 + 0.1 \times 0.3 + 0.0 \times 0.3 \\
&= 0.08 + 0.03 \\
&= 0.11,
\end{aligned}
$$

$$P(Y = 0 \,|\, (1,1)) = \frac{P((1,1) \,|\, Y = 0)P(Y = 0)}{0.11}$$
$$= \frac{0.2 \times 0.4}{0.11}$$
$$= \frac{0.08}{0.11}$$
$$\approx 0.727,$$

$$P(Y = 1 \,|\, (1,1)) = \frac{P((1,1) \,|\, Y = 1)P(Y = 1)}{0.11}$$
$$= \frac{0.1 \times 0.3}{0.11}$$
$$= \frac{0.03}{0.11}$$
$$\approx 0.273,$$

$$P(Y = 2 \,|\, (1,1)) = \frac{P((1,1) \,|\, Y = 2)P(Y = 2)}{0.11}$$
$$= \frac{0.0 \times 0.3}{0.11}$$
$$= 0.0.$$

(1, 2)

If $\mathbf{X} = (1,2)$,

$$P((1,2)) = \sum_{i=0}^{2} P((1,2) \,|\, Y = i)P(Y = i)$$
$$= 0.1 \times 0.4 + 0.0 \times 0.3 + 0.0 \times 0.3$$
$$= 0.04,$$

$$P(Y = 0 \,|\, (1,2)) = \frac{P((1,2) \,|\, Y = 0)P(Y = 0)}{0.04}$$
$$= \frac{0.1 \times 0.4}{0.04}$$
$$= 1.0,$$

$$P(Y = 1 \,|\, (1,2)) = \frac{P((1,2) \,|\, Y = 1)P(Y = 1)}{0.04}$$
$$= \frac{0.0 \times 0.3}{0.11}$$
$$= 0.0,$$

$$P(Y = 2\,|\,(1,2)) = \frac{P((1,2)\,|\,Y = 2)P(Y = 2)}{0.04}$$
$$= \frac{0.0 \times 0.3}{0.11}$$
$$= 0.0.$$

Summary

	$Y = 0$	$Y = 1$	$Y = 2$
$(0,0)$	0.276	0.621	0.103
$(0,1)$	0.571	0.107	0.321
$(0,2)$	0.0	0.333	0.667
$(1,0)$	0.211	0.158	0.631
$(1,1)$	0.727	0.273	0.0
$(1,2)$	1.0	0.0	0.0

The optimal Bayes classifier is

$$Y(\mathbf{X}) = \begin{cases} 0 & \text{if } \mathbf{X} \in \{(0,1), (1,1), (1,2)\} \\ 1 & \text{if } \mathbf{X} = (0,0) \\ 2 & \text{if } \mathbf{X} \in \{(0,2), (1,0)\}. \end{cases}$$

(c)

For naïve Bayes classifier in this context, we have

$$P(\mathbf{X}\,|\,Y = y) = \prod_{i=1}^{2} P(X_i\,|\,Y = y) = P(X_1\,|\,Y = y)P(X_2\,|\,Y = y),$$

so we seek to compute

$$P(Y = y\,|\,\mathbf{X}) = \frac{P(\mathbf{X}\,|\,Y = y)P(Y = y)}{P(\mathbf{X})} = \frac{P(X_1\,|\,Y = y)P(X_2\,|\,Y = y)P(Y = y)}{P(\mathbf{X})},$$

where

$$P(\mathbf{X}) = \sum_{i=0}^{2} P(\mathbf{X}\,|\,Y = i)P(Y = i)$$
$$= \sum_{i=0}^{2} P(X_1\,|\,Y = i)P(X_2\,|\,Y = i)P(Y = i).$$

$(0,0)$

If $\mathbf{X} = (0,0)$, and

$$P(\mathbf{X}) = \sum_{i=0}^{2} P(X_1 \,|\, Y = i)P(X_2 \,|\, Y = i)P(Y = i)$$

$$= (0.2 + 0.4 + 0.0) \times (0.2 + 0.1) \times 0.4$$
$$+ (0.6 + 0.1 + 0.1) \times (0.6 + 0.1) \times 0.3$$
$$+ (0.1 + 0.3 + 0.2) \times (0.1 + 0.4) \times 0.3$$
$$= 0.6 \times 0.3 \times 0.4 + 0.8 \times 0.7 \times 0.3 + 0.6 \times 0.5 \times 0.3$$
$$= 0.072 + 0.168 + 0.09$$
$$= 0.33.$$

$$P(Y = 0 \,|\, (0,0)) = \frac{P(X_1 \,|\, Y = 0)P(X_2 \,|\, Y = 0)P(Y = 0)}{0.33}$$
$$= \frac{(0.2 + 0.4 + 0.0) \times (0.2 + 0.1) \times 0.4}{0.33}$$
$$= \frac{0.6 \times 0.3 \times 0.4}{0.33}$$
$$= \frac{0.072}{0.33}$$
$$\approx 0.218,$$

$$P(Y = 1 \,|\, (0,0)) = \frac{P(X_1 \,|\, Y = 1)P(X_2 \,|\, Y = 1)P(Y = 1)}{0.33}$$
$$= \frac{(0.6 + 0.1 + 0.1) \times (0.6 + 0.1) \times 0.3}{0.33}$$
$$= \frac{0.8 \times 0.7 \times 0.3}{0.33}$$
$$= \frac{0.168}{0.33}$$
$$\approx 0.509,$$

$$P(Y = 2 \,|\, (0,0)) = \frac{P(X_1 \,|\, Y = 2)P(X_2 \,|\, Y = 2)P(Y = 2)}{0.33}$$
$$= \frac{(0.1 + 0.3 + 0.2) \times (0.1 + 0.4) \times 0.3}{0.33}$$
$$= \frac{0.6 \times 0.5 \times 0.3}{0.33}$$
$$= \frac{0.09}{0.33}$$
$$\approx 0.273.$$

$(0, 1)$

If $\mathbf{X} = (0, 1)$, and

$$
\begin{aligned}
P(\mathbf{X}) &= \sum_{i=0}^{2} P(X_1 \mid Y = i)P(X_2 \mid Y = i)P(Y = i) \\
&= (0.2 + 0.4 + 0.0) \times (0.4 + 0.2) \times 0.4 \\
&\quad + (0.6 + 0.1 + 0.1) \times (0.1 + 0.1) \times 0.3 \\
&\quad + (0.1 + 0.3 + 0.2) \times (0.3 + 0.0) \times 0.3 \\
&= 0.6 \times 0.6 \times 0.4 + 0.8 \times 0.2 \times 0.3 + 0.6 \times 0.3 \times 0.3 \\
&= 0.144 + 0.048 + 0.054 \\
&= 0.246.
\end{aligned}
$$

$$
\begin{aligned}
P(Y = 0 \mid (0, 1)) &= \frac{P(X_1 = 0 \mid Y = 0)P(X_2 = 1 \mid Y = 0)P(Y = 0)}{0.246} \\
&= \frac{(0.2 + 0.4 + 0.0) \times (0.4 + 0.2) \times 0.4}{0.33} \\
&= \frac{0.6 \times 0.6 \times 0.4}{0.33} \\
&= \frac{0.144}{0.33} \\
&\approx 0.436,
\end{aligned}
$$

$$
\begin{aligned}
P(Y = 1 \mid (0, 1)) &= \frac{P(X_1 = 0 \mid Y = 1)P(X_2 = 1 \mid Y = 1)P(Y = 1)}{0.246} \\
&= \frac{(0.6 + 0.1 + 0.1) \times (0.1 + 0.1) \times 0.3}{0.33} \\
&= \frac{0.8 \times 0.2 \times 0.3}{0.33} \\
&= \frac{0.048}{0.33} \\
&\approx 0.146,
\end{aligned}
$$

$$
\begin{aligned}
P(Y = 2 \mid (0, 1)) &= \frac{P(X_1 = 0 \mid Y = 2)P(X_2 = 1 \mid Y = 2)P(Y = 2)}{0.246} \\
&= \frac{(0.1 + 0.3 + 0.2) \times (0.3 + 0.0) \times 0.3}{0.33} \\
&= \frac{0.6 \times 0.3 \times 0.3}{0.33} \\
&= \frac{0.054}{0.33} \\
&\approx 0.164.
\end{aligned}
$$

$(0, 2)$

$(1, 0)$

$(1, 1)$

$(1, 2)$

Problem 2 (3 points)

Problem 3 (15 points)