

Introduction to Machine Learning, Fall 2014 - Exercise session IV

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Problem 1 (6 points)

(a)

We seek to compute

$$P(Y | \mathbf{X}) = \frac{P(\mathbf{X} | Y)P(Y)}{P(\mathbf{X})},$$

where

$$P(\mathbf{X}) = \sum_{i=0}^2 P(\mathbf{X} | Y = i)P(Y = i).$$

(0, 0)

If $\mathbf{X} = (0, 0)$,

$$\begin{aligned} P((0, 0)) &= \sum_{i=0}^2 P((0, 0) | Y = i)P(Y = i) \\ &= 0.2 \times 0.4 + 0.6 \times 0.3 + 0.1 \times 0.3 \\ &= 0.08 + 0.18 + 0.03 \\ &= 0.29. \end{aligned}$$

$$\begin{aligned} P(Y = 0 | (0, 0)) &= \frac{P((0, 0) | Y = 0)P(Y = 0)}{P((0, 0))} \\ &= \frac{0.2 \times 0.4}{0.29} \\ &= \frac{0.08}{0.29} \\ &\approx 0.276, \end{aligned}$$

$$\begin{aligned}
P(Y = 1 | (0, 0)) &= \frac{P((0, 0) | Y = 1)P(Y = 1)}{P((0, 0))} \\
&= \frac{0.6 \times 0.3}{0.29} \\
&= \frac{0.18}{0.29} \\
&\approx 0.621,
\end{aligned}$$

$$\begin{aligned}
P(Y = 2 | (0, 0)) &= \frac{P((0, 0) | Y = 2)P(Y = 2)}{P((0, 0))} \\
&= \frac{0.1 \times 0.3}{0.29} \\
&= \frac{0.03}{0.29} \\
&\approx 0.103.
\end{aligned}$$

(0, 1)

If $\mathbf{X} = (0, 1)$,

$$\begin{aligned}
P((0, 1)) &= \sum_{i=0}^2 P((0, 1) | Y = i)P(Y = i) \\
&= 0.4 \times 0.4 + 0.1 \times 0.3 + 0.3 \times 0.3 \\
&= 0.16 + 0.03 + 0.09 \\
&= 0.28.
\end{aligned}$$

$$\begin{aligned}
P(Y = 0 | (0, 1)) &= \frac{P((0, 1) | Y = 0)P(Y = 0)}{0.28} \\
&= \frac{0.4 \times 0.4}{0.28} \\
&= \frac{0.16}{0.28} \\
&\approx 0.571,
\end{aligned}$$

$$\begin{aligned}
P(Y = 1 | (0, 1)) &= \frac{P((0, 1) | Y = 1)P(Y = 1)}{0.28} \\
&= \frac{0.1 \times 0.3}{0.28} \\
&= \frac{0.03}{0.28} \\
&\approx 0.107,
\end{aligned}$$

$$\begin{aligned}
P(Y = 2 | (0, 1)) &= \frac{P((0, 1) | Y = 2)P(Y = 2)}{0.28} \\
&= \frac{0.3 \times 0.3}{0.28} \\
&= \frac{0.09}{0.28} \\
&\approx 0.321.
\end{aligned}$$

(0, 2)

If $\mathbf{X} = (0, 2)$,

$$\begin{aligned}
P((0, 2)) &= \sum_{i=0}^2 P((0, 2) | Y = i)P(Y = i) \\
&= 0.0 \times 0.4 + 0.1 \times 0.3 + 0.2 \times 0.3 \\
&= 0.03 + 0.06 \\
&= 0.09,
\end{aligned}$$

$$\begin{aligned}
P(Y = 0 | (0, 2)) &= \frac{P((0, 2) | Y = 0)P(Y = 0)}{0.09} \\
&= \frac{0.0 \times 0.4}{0.09} \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
P(Y = 1 | (0, 2)) &= \frac{P((0, 2) | Y = 1)P(Y = 1)}{0.09} \\
&= \frac{0.1 \times 0.3}{0.09} \\
&= \frac{0.03}{0.09} \\
&\approx 0.333,
\end{aligned}$$

$$\begin{aligned}
P(Y = 2 | (0, 2)) &= \frac{P((0, 2) | Y = 2)P(Y = 2)}{0.09} \\
&= \frac{0.2 \times 0.3}{0.09} \\
&= \frac{0.06}{0.09} \\
&\approx 0.666.
\end{aligned}$$

(1, 0)

If $\mathbf{X} = (1, 0)$,

$$\begin{aligned}P((1, 0)) &= \sum_{i=0}^2 P((1, 0) | Y = i)P(Y = i) \\&= 0.1 \times 0.4 + 0.1 \times 0.3 + 0.4 \times 0.3 \\&= 0.04 + 0.03 + 0.12 \\&= 0.19,\end{aligned}$$

$$\begin{aligned}P(Y = 0 | (1, 0)) &= \frac{P((1, 0) | Y = 0)P(Y = 0)}{0.19} \\&= \frac{0.1 \times 0.4}{0.19} \\&= \frac{0.04}{0.19} \\&\approx 0.211,\end{aligned}$$

$$\begin{aligned}P(Y = 1 | (1, 0)) &= \frac{P((1, 0) | Y = 1)P(Y = 1)}{0.19} \\&= \frac{0.1 \times 0.3}{0.19} \\&= \frac{0.03}{0.19} \\&\approx 0.158,\end{aligned}$$

$$\begin{aligned}P(Y = 2 | (1, 0)) &= \frac{P((1, 0) | Y = 2)P(Y = 2)}{0.19} \\&= \frac{0.4 \times 0.3}{0.19} \\&= \frac{0.12}{0.19} \\&\approx 0.632.\end{aligned}$$

(1, 1)

If $\mathbf{X} = (1, 1)$,

$$\begin{aligned}P((1, 1)) &= \sum_{i=0}^2 P((1, 1) | Y = i)P(Y = i) \\&= 0.2 \times 0.4 + 0.1 \times 0.3 + 0.0 \times 0.3 \\&= 0.08 + 0.03 \\&= 0.11,\end{aligned}$$

$$\begin{aligned}
P(Y = 0 | (1, 1)) &= \frac{P((1, 1) | Y = 0)P(Y = 0)}{0.11} \\
&= \frac{0.2 \times 0.4}{0.11} \\
&= \frac{0.08}{0.11} \\
&\approx 0.727,
\end{aligned}$$

$$\begin{aligned}
P(Y = 1 | (1, 1)) &= \frac{P((1, 1) | Y = 1)P(Y = 1)}{0.11} \\
&= \frac{0.1 \times 0.3}{0.11} \\
&= \frac{0.03}{0.11} \\
&\approx 0.273,
\end{aligned}$$

$$\begin{aligned}
P(Y = 2 | (1, 1)) &= \frac{P((1, 1) | Y = 2)P(Y = 2)}{0.11} \\
&= \frac{0.0 \times 0.3}{0.11} \\
&= 0.0.
\end{aligned}$$

(1, 2)

If $\mathbf{X} = (1, 2)$,

$$\begin{aligned}
P((1, 2)) &= \sum_{i=0}^2 P((1, 2) | Y = i)P(Y = i) \\
&= 0.1 \times 0.4 + 0.0 \times 0.3 + 0.0 \times 0.3 \\
&= 0.04,
\end{aligned}$$

$$\begin{aligned}
P(Y = 0 | (1, 2)) &= \frac{P((1, 2) | Y = 0)P(Y = 0)}{0.04} \\
&= \frac{0.1 \times 0.4}{0.04} \\
&= 1.0,
\end{aligned}$$

$$\begin{aligned}
P(Y = 1 | (1, 2)) &= \frac{P((1, 2) | Y = 1)P(Y = 1)}{0.04} \\
&= \frac{0.0 \times 0.3}{0.11} \\
&= 0.0,
\end{aligned}$$

$$\begin{aligned}
P(Y = 2 | (1, 2)) &= \frac{P((1, 2) | Y = 2)P(Y = 2)}{0.04} \\
&= \frac{0.0 \times 0.3}{0.11} \\
&= 0.0.
\end{aligned}$$

Summary

	$Y = 0$	$Y = 1$	$Y = 2$
$(0, 0)$	0.276	0.621	0.103
$(0, 1)$	0.571	0.107	0.321
$(0, 2)$	0.0	0.333	0.667
$(1, 0)$	0.211	0.158	0.631
$(1, 1)$	0.727	0.273	0.0
$(1, 2)$	1.0	0.0	0.0

The optimal Bayes classifier is

$$Y(\mathbf{X}) = \begin{cases} 0 & \text{if } \mathbf{X} \in \{(0, 1), (1, 1), (1, 2)\} \\ 1 & \text{if } \mathbf{X} = (0, 0) \\ 2 & \text{if } \mathbf{X} \in \{(0, 2), (1, 0)\}. \end{cases}$$

(c)

For naïve Bayes classifier in this context, we have

$$P(\mathbf{X} | Y = y) = \prod_{i=1}^2 P(X_i | Y = y) = P(X_1 | Y = y)P(X_2 | Y = y),$$

so we seek to compute

$$P(Y = y | \mathbf{X}) = \frac{P(\mathbf{X} | Y = y)P(Y = y)}{P(\mathbf{X})} = \frac{P(X_1 | Y = y)P(X_2 | Y = y)P(Y = y)}{P(\mathbf{X})},$$

where

$$\begin{aligned}
P(\mathbf{X}) &= \sum_{i=0}^2 P(\mathbf{X} | Y = i)P(Y = i) \\
&= \sum_{i=0}^2 P(X_1 | Y = i)P(X_2 | Y = i)P(Y = i).
\end{aligned}$$

$(0, 0)$

If $\mathbf{X} = (0, 0)$, and

$$\begin{aligned} P(\mathbf{X}) &= \sum_{i=0}^2 P(X_1 | Y = i)P(X_2 | Y = i)P(Y = i) \\ &= (0.2 + 0.4 + 0.0) \times (0.2 + 0.1) \times 0.4 \\ &\quad + (0.6 + 0.1 + 0.1) \times (0.6 + 0.1) \times 0.3 \\ &\quad + (0.1 + 0.3 + 0.2) \times (0.1 + 0.4) \times 0.3 \\ &= 0.6 \times 0.3 \times 0.4 + 0.8 \times 0.7 \times 0.3 + 0.6 \times 0.5 \times 0.3 \\ &= 0.072 + 0.168 + 0.09 \\ &= 0.33. \end{aligned}$$

$$\begin{aligned} P(Y = 0 | (0, 0)) &= \frac{P(X_1 | Y = 0)P(X_2 | Y = 0)P(Y = 0)}{0.33} \\ &= \frac{(0.2 + 0.4 + 0.0) \times (0.2 + 0.1) \times 0.4}{0.33} \\ &= \frac{0.6 \times 0.3 \times 0.4}{0.33} \\ &= \frac{0.072}{0.33} \\ &\approx 0.218, \end{aligned}$$

$$\begin{aligned} P(Y = 1 | (0, 0)) &= \frac{P(X_1 | Y = 1)P(X_2 | Y = 1)P(Y = 1)}{0.33} \\ &= \frac{(0.6 + 0.1 + 0.1) \times (0.6 + 0.1) \times 0.3}{0.33} \\ &= \frac{0.8 \times 0.7 \times 0.3}{0.33} \\ &= \frac{0.168}{0.33} \\ &\approx 0.509, \end{aligned}$$

$$\begin{aligned} P(Y = 2 | (0, 0)) &= \frac{P(X_1 | Y = 2)P(X_2 | Y = 2)P(Y = 2)}{0.33} \\ &= \frac{(0.1 + 0.3 + 0.2) \times (0.1 + 0.4) \times 0.3}{0.33} \\ &= \frac{0.6 \times 0.5 \times 0.3}{0.33} \\ &= \frac{0.09}{0.33} \\ &\approx 0.273. \end{aligned}$$

$(0, 1)$

If $\mathbf{X} = (0, 1)$, and

$$\begin{aligned} P(\mathbf{X}) &= \sum_{i=0}^2 P(X_1 | Y = i)P(X_2 | Y = i)P(Y = i) \\ &= (0.2 + 0.4 + 0.0) \times (0.4 + 0.2) \times 0.4 \\ &\quad + (0.6 + 0.1 + 0.1) \times (0.1 + 0.1) \times 0.3 \\ &\quad + (0.1 + 0.3 + 0.2) \times (0.3 + 0.0) \times 0.3 \\ &= 0.6 \times 0.6 \times 0.4 + 0.8 \times 0.2 \times 0.3 + 0.6 \times 0.3 \times 0.3 \\ &= 0.144 + 0.048 + 0.054 \\ &= 0.246. \end{aligned}$$

$$\begin{aligned} P(Y = 0 | (0, 1)) &= \frac{P(X_1 = 0 | Y = 0)P(X_2 = 1 | Y = 0)P(Y = 0)}{0.246} \\ &= \frac{(0.2 + 0.4 + 0.0) \times (0.4 + 0.2) \times 0.4}{0.246} \\ &= \frac{0.6 \times 0.6 \times 0.4}{0.246} \\ &= \frac{0.144}{0.246} \\ &\approx 0.585, \end{aligned}$$

$$\begin{aligned} P(Y = 1 | (0, 1)) &= \frac{P(X_1 = 0 | Y = 1)P(X_2 = 1 | Y = 1)P(Y = 1)}{0.246} \\ &= \frac{(0.6 + 0.1 + 0.1) \times (0.1 + 0.1) \times 0.3}{0.246} \\ &= \frac{0.8 \times 0.2 \times 0.3}{0.246} \\ &= \frac{0.048}{0.246} \\ &\approx 0.195, \end{aligned}$$

$$\begin{aligned} P(Y = 2 | (0, 1)) &= \frac{P(X_1 = 0 | Y = 2)P(X_2 = 1 | Y = 2)P(Y = 2)}{0.246} \\ &= \frac{(0.1 + 0.3 + 0.2) \times (0.3 + 0.0) \times 0.3}{0.246} \\ &= \frac{0.6 \times 0.3 \times 0.3}{0.246} \\ &= \frac{0.054}{0.246} \\ &\approx 0.219. \end{aligned}$$

$(0, 2)$

If $\mathbf{X} = (0, 2)$, and

$$\begin{aligned} P(\mathbf{X}) &= \sum_{i=0}^2 P(X_1 | Y = i)P(X_2 | Y = i)P(Y = i) \\ &= (0.2 + 0.4 + 0.0) \times (0.0 + 0.1) \times 0.4 \\ &\quad + (0.6 + 0.1 + 0.1) \times (0.1 + 0.0) \times 0.3 \\ &\quad + (0.1 + 0.3 + 0.2) \times (0.2 + 0.0) \times 0.3 \\ &= 0.6 \times 0.1 \times 0.4 + 0.8 \times 0.1 \times 0.3 + 0.6 \times 0.2 \times 0.3 \\ &= 0.024 + 0.024 + 0.036 \\ &= 0.084. \end{aligned}$$

$$\begin{aligned} P(Y = 0 | (0, 2)) &= \frac{P(X_1 = 0 | Y = 0)P(X_2 = 2 | Y = 0)P(Y = 0)}{0.084} \\ &= \frac{(0.2 + 0.4 + 0.0) \times (0.0 + 0.1) \times 0.4}{0.084} \\ &= \frac{0.6 \times 0.1 \times 0.4}{0.084} \\ &= \frac{0.024}{0.084} \\ &\approx 0.286, \end{aligned}$$

$$\begin{aligned} P(Y = 1 | (0, 2)) &= \frac{P(X_1 = 0 | Y = 1)P(X_2 = 2 | Y = 1)P(Y = 1)}{0.084} \\ &= \frac{(0.6 + 0.1 + 0.1) \times (0.1 + 0.0) \times 0.3}{0.084} \\ &= \frac{0.8 \times 0.1 \times 0.3}{0.084} \\ &= \frac{0.024}{0.084} \\ &\approx 0.286, \end{aligned}$$

$$\begin{aligned} P(Y = 2 | (0, 2)) &= \frac{P(X_1 = 0 | Y = 2)P(X_2 = 2 | Y = 2)P(Y = 2)}{0.084} \\ &= \frac{(0.1 + 0.3 + 0.2) \times (0.2 + 0.0) \times 0.3}{0.084} \\ &= \frac{0.6 \times 0.2 \times 0.3}{0.084} \\ &= \frac{0.036}{0.084} \\ &\approx 0.429. \end{aligned}$$

$(1, 0)$

If $\mathbf{X} = (1, 0)$, and

$$\begin{aligned} P(\mathbf{X}) &= \sum_{i=0}^2 P(X_1 | Y = i)P(X_2 | Y = i)P(Y = i) \\ &= (0.1 + 0.2 + 0.1) \times (0.2 + 0.1) \times 0.4 \\ &\quad + (0.1 + 0.1 + 0.0) \times (0.6 + 0.1) \times 0.3 \\ &\quad + (0.4 + 0.0 + 0.0) \times (0.1 + 0.4) \times 0.3 \\ &= 0.4 \times 0.3 \times 0.4 + 0.2 \times 0.7 \times 0.3 + 0.4 \times 0.5 \times 0.3 \\ &= 0.048 + 0.042 + 0.06 \\ &= 0.15. \end{aligned}$$

$$\begin{aligned} P(Y = 0 | (1, 0)) &= \frac{P(X_1 = 1 | Y = 0)P(X_2 = 0 | Y = 0)P(Y = 0)}{0.15} \\ &= \frac{(0.1 + 0.2 + 0.1) \times (0.2 + 0.1) \times 0.4}{0.15} \\ &= \frac{0.4 \times 0.3 \times 0.4}{0.15} \\ &= \frac{0.048}{0.15} \\ &= 0.32, \end{aligned}$$

$$\begin{aligned} P(Y = 1 | (1, 0)) &= \frac{P(X_1 = 1 | Y = 1)P(X_2 = 0 | Y = 1)P(Y = 1)}{0.15} \\ &= \frac{(0.1 + 0.1 + 0.0) \times (0.6 + 0.1) \times 0.3}{0.15} \\ &= \frac{0.2 \times 0.7 \times 0.3}{0.15} \\ &= \frac{0.042}{0.15} \\ &= 0.28, \end{aligned}$$

$$\begin{aligned} P(Y = 2 | (1, 0)) &= \frac{P(X_1 = 1 | Y = 2)P(X_2 = 0 | Y = 2)P(Y = 2)}{0.15} \\ &= \frac{(0.4 + 0.0 + 0.0) \times (0.1 + 0.4) \times 0.3}{0.15} \\ &= \frac{0.4 \times 0.5 \times 0.3}{0.15} \\ &= \frac{0.06}{0.15} \\ &= 0.4. \end{aligned}$$

$(1, 1)$

If $\mathbf{X} = (1, 1)$, and

$$\begin{aligned} P(\mathbf{X}) &= \sum_{i=0}^2 P(X_1 | Y = i)P(X_2 | Y = i)P(Y = i) \\ &= (0.1 + 0.2 + 0.1) \times (0.4 + 0.2) \times 0.4 \\ &\quad + (0.1 + 0.1 + 0.0) \times (0.1 + 0.1) \times 0.3 \\ &\quad + (0.4 + 0.0 + 0.0) \times (0.3 + 0.0) \times 0.3 \\ &= 0.4 \times 0.6 \times 0.4 + 0.2 \times 0.2 \times 0.3 + 0.4 \times 0.3 \times 0.3 \\ &= 0.096 + 0.012 + 0.036 \\ &= 0.144. \end{aligned}$$

$$\begin{aligned} P(Y = 0 | (1, 1)) &= \frac{P(X_1 = 1 | Y = 0)P(X_2 = 1 | Y = 0)P(Y = 0)}{0.144} \\ &= \frac{(0.1 + 0.2 + 0.1) \times (0.4 + 0.2) \times 0.4}{0.144} \\ &= \frac{0.4 \times 0.6 \times 0.4}{0.144} \\ &= \frac{0.096}{0.144} \\ &\approx 0.667, \end{aligned}$$

$$\begin{aligned} P(Y = 1 | (1, 1)) &= \frac{P(X_1 = 1 | Y = 1)P(X_2 = 1 | Y = 1)P(Y = 1)}{0.144} \\ &= \frac{(0.1 + 0.1 + 0.0) \times (0.1 + 0.1) \times 0.3}{0.144} \\ &= \frac{0.2 \times 0.2 \times 0.3}{0.144} \\ &= \frac{0.012}{0.144} \\ &\approx 0.083, \end{aligned}$$

$$\begin{aligned} P(Y = 2 | (1, 1)) &= \frac{P(X_1 = 1 | Y = 2)P(X_2 = 1 | Y = 2)P(Y = 2)}{0.144} \\ &= \frac{(0.4 + 0.0 + 0.0) \times (0.3 + 0.0) \times 0.3}{0.144} \\ &= \frac{0.4 \times 0.3 \times 0.3}{0.144} \\ &= \frac{0.036}{0.144} \\ &= 0.25. \end{aligned}$$

(1, 2)

If $\mathbf{X} = (1, 2)$, and

$$\begin{aligned} P(\mathbf{X}) &= \sum_{i=0}^2 P(X_1 | Y = i)P(X_2 | Y = i)P(Y = i) \\ &= (0.1 + 0.2 + 0.1) \times (0.0 + 0.1) \times 0.4 \\ &\quad + (0.1 + 0.1 + 0.0) \times (0.1 + 0.0) \times 0.3 \\ &\quad + (0.4 + 0.0 + 0.0) \times (0.2 + 0.0) \times 0.3 \\ &= 0.4 \times 0.1 \times 0.4 + 0.2 \times 0.1 \times 0.3 + 0.4 \times 0.2 \times 0.3 \\ &= 0.016 + 0.006 + 0.024 \\ &= 0.046. \end{aligned}$$

$$\begin{aligned} P(Y = 0 | (1, 2)) &= \frac{P(X_1 = 1 | Y = 0)P(X_2 = 2 | Y = 0)P(Y = 0)}{0.046} \\ &= \frac{(0.1 + 0.2 + 0.1) \times (0.0 + 0.1) \times 0.4}{0.046} \\ &= \frac{0.4 \times 0.1 \times 0.4}{0.046} \\ &= \frac{0.016}{0.046} \\ &\approx 0.348, \end{aligned}$$

$$\begin{aligned} P(Y = 1 | (1, 1)) &= \frac{P(X_1 = 1 | Y = 1)P(X_2 = 2 | Y = 1)P(Y = 1)}{0.046} \\ &= \frac{(0.1 + 0.1 + 0.0) \times (0.1 + 0.0) \times 0.3}{0.046} \\ &= \frac{0.2 \times 0.1 \times 0.3}{0.046} \\ &= \frac{0.006}{0.046} \\ &\approx 0.130, \end{aligned}$$

$$\begin{aligned} P(Y = 2 | (1, 1)) &= \frac{P(X_1 = 1 | Y = 2)P(X_2 = 2 | Y = 2)P(Y = 2)}{0.046} \\ &= \frac{(0.4 + 0.0 + 0.0) \times (0.2 + 0.0) \times 0.3}{0.046} \\ &= \frac{0.4 \times 0.2 \times 0.3}{0.046} \\ &= \frac{0.024}{0.046} \\ &= 0.522. \end{aligned}$$

Summary

	$Y = 0$	$Y = 1$	$Y = 2$
$(0, 0)$	0.218	0.509	0.273
$(0, 1)$	0.585	0.195	0.219
$(0, 2)$	0.286	0.286	0.429
$(1, 0)$	0.32	0.28	0.4
$(1, 1)$	0.667	0.083	0.25
$(1, 2)$	0.348	0.130	0.522

The Naïve Bayes classifier is

$$Y(\mathbf{X}) = \begin{cases} 0 & \text{if } \mathbf{X} \in \{(0, 1), (1, 1)\} \\ 1 & \text{if } \mathbf{X} = (0, 0) \\ 2 & \text{if } \mathbf{X} \in \{(0, 2), (1, 0), (1, 2)\}. \end{cases}$$

Problem 2 (3 points)

Problem 3 (15 points)