Kvanttilaskenta, kevät 2015 – Viikko 5

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edx Problem 1

We have

$$QFT_M = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 \\ 1 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & \omega^{10} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & \omega^{15} \\ 1 & \omega^4 & \omega^8 & \omega^{12} & \omega^{16} & \omega^{20} \\ 1 & \omega^5 & \omega^{10} & \omega^{15} & \omega^{20} & \omega^{25} \end{pmatrix}$$
$$= \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 \\ 1 & \omega^2 & \omega^4 & 1 & \omega^2 & \omega^4 \\ 1 & \omega^3 & 1 & \omega^3 & 1 & \omega^3 \\ 1 & \omega^4 & \omega^2 & 1 & \omega^4 & \omega^2 \\ 1 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega^1 \end{pmatrix}$$

Now $\omega = e^{2\pi i j/M}$, for $j = 0, 1, \ldots, M-1$.

edx Problem 2

What is QFT_6 of $\frac{1}{\sqrt{2}}(|0\rangle + |3\rangle)$?

Now the matrix representation of the input state is

$$|\psi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

so the QFT of $|\psi\rangle$ is

$$\frac{1}{\sqrt{12}}\begin{pmatrix} 2 \\ 1+\omega^3 \\ 2 \\ 1+\omega^3 \\ 2 \\ 1+\omega^3 \end{pmatrix},$$

where $\omega^3 = e^{2\pi i\times 3/6} = e^{\pi i} = \cos\pi + i\sin\pi = -1$, so

$$|\psi\rangle = \frac{1}{\sqrt{12}}\begin{pmatrix} 2 \\ 0 \\ 2 \\ 0 \\ 2 \\ 0 \end{pmatrix}.$$

The probability of measuring $|0\rangle, |2\rangle, |4\rangle$ is $\frac{1}{3}$ each and other probabilities are 0.

edx Problem 3

What is QFT_6 of $\frac{1}{\sqrt{2}}(|1\rangle + |4\rangle)$? Now we have a state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

so the result is

$$\frac{1}{\sqrt{12}}\begin{pmatrix} 2 \\ \omega+\omega^4 \\ 2\omega^2 \\ 1+\omega^3 \\ 2\omega^4 \\ \omega^5+\omega^2 \end{pmatrix} = \frac{1}{\sqrt{12}}\begin{pmatrix} 2 \\ 0 \\ 2\omega^2 \\ 0 \\ 2\omega^4 \\ 0 \end{pmatrix}.$$

Now it is easy to see that the probability of measuring $|0\rangle, |2\rangle, |4\rangle$ is $\frac{1}{3}$ each and zero probability of seeing other.

edx Problem 4

What is QFT_6 of $\frac{1}{\sqrt{3}}(|0\rangle + |2\rangle + |4\rangle)$? Now the input state is

$$|\psi\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

so that the result state is

$$\frac{1}{\sqrt{18}} \begin{pmatrix} 3 \\ 1+\omega^2+\omega^4 \\ 1+\omega^4+\omega^2 \\ 3 \\ 1+\omega^2+\omega^4 \\ 1+\omega^4+\omega^2 \end{pmatrix} = \frac{1}{\sqrt{18}} \begin{pmatrix} 3 \\ 0 \\ 0 \\ 3 \\ 0 \\ 0 \end{pmatrix},$$

so the probability of measuring $|0\rangle$ or $|3\rangle$ is $\frac{1}{2}$ each and the other states has probability 0.

edx Problem 5

What is QFT_6 of $\frac{1}{\sqrt{3}}(|1\rangle + |3\rangle + |5\rangle)$? The input state is

$$\frac{1}{\sqrt{3}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix},$$

so the result state is

$$\frac{1}{\sqrt{18}} \begin{pmatrix} 3 \\ \omega+\omega^3+\omega^5 \\ \omega^2+1+\omega^4 \\ 3\omega^3 \\ \omega^4+1+\omega^2 \\ \omega^5+\omega^3+\omega \end{pmatrix} = \frac{1}{\sqrt{18}} \begin{pmatrix} 3 \\ 0 \\ 0 \\ -3 \\ 0 \\ 0 \end{pmatrix}.$$

The probability of measuring $|0\rangle$ or $|3\rangle$ is $\frac{1}{2}$ each, and other probabilities are zero.

edx Problem 7

Consider the periodic superposition $|a\rangle = \sqrt{\frac{k}{M}} \sum_{j=0}^{M/k-1} |jk\rangle$. Let $\beta = \sum_j \beta_j |j\rangle$ be its QFT_M.

(a) Derive an expression for β_j. The desired expression is

$$\frac{\sqrt{k}}{M} \sum_{l=0}^{M/k-1} e^{2\pi jlki/M}.$$

(b) If j is a multiple of M/k, what is the value of β_j? So we have that $j = aM/k$ for some nonnegative integer a. Now

$$\begin{aligned}
\frac{\sqrt{k}}{M} \sum_{l=0}^{M/k-1} e^{2\pi jlki/M} &= \frac{\sqrt{k}}{M} \sum_{l=0}^{M/k-1} e^{2\pi ali} \\
&= \frac{\sqrt{k}}{M} \sum_{l=0}^{M/k-1} 1 \\
&= \frac{\sqrt{k}}{M} M/k \\
&= \frac{1}{\sqrt{k}}.
\end{aligned}$$

(c) If j is not a multiple of M/k, what is the value of β_j? Now suppose $j = aM/k + r$ for some nonnegative integer a and an integer $r \in [1, a)$. We have

$$\begin{aligned}
\frac{\sqrt{k}}{M} \sum_{l=0}^{M/k-1} e^{2\pi jlki/M} &= \frac{\sqrt{k}}{M} \sum_{l=0}^{M/k-1} e^{2\pi lki(aM/k+r)/M} \\
&= \frac{\sqrt{k}}{M} \sum_{l=0}^{M/k-1} e^{2\pi lki(aM/k)/M + 2\pi lkir/M} \\
&= \frac{\sqrt{k}}{M} \sum_{l=0}^{M/k-1} e^{2\pi lia} e^{2i\pi lkr/M} \\
&= \frac{\sqrt{k}}{M} \sum_{l=0}^{M/k-1} e^{2i\pi lkr/M} \\
&=
\end{aligned}$$

QCE 7.1

$$\vec{\sigma} = \sigma_x \hat{x} + \sigma_y \hat{y} + \sigma_z \hat{z}, \qquad \vec{n} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}.$$

Now

$$
\begin{aligned}
\vec{\sigma} \cdot \vec{n} &= (\sigma_x \hat{x} + \sigma_y \hat{y} + \sigma_z \hat{z})(\sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}) \\
&=
\end{aligned}
$$

QCE 7.2

Let

$$|\psi\rangle = \frac{|0\rangle|1\rangle - |1\rangle|0\rangle}{\sqrt{2}}.$$

Now we have

$$|0\rangle = \frac{|+\rangle + i|-\rangle}{\sqrt{2}},\ |1\rangle = \frac{|+\rangle - i|-\rangle}{\sqrt{2}},$$

so

$$
\begin{aligned}
|\psi\rangle &= \frac{|0\rangle|1\rangle - |1\rangle|0\rangle}{\sqrt{2}} \\
&= \frac{1}{2\sqrt{2}}\Bigg((|+\rangle + i|-\rangle)(|+\rangle - i|-\rangle) - (|+\rangle - i|-\rangle)(|+\rangle + i|-\rangle)\Bigg) \\
&= \frac{1}{2\sqrt{2}}\Bigg(|{++}\rangle - i|{+-}\rangle + i|{-+}\rangle + |{--}\rangle + |{++}\rangle + i|{+-}\rangle - i|{-+}\rangle + |{--}\rangle\Bigg) \\
&= \frac{1}{2\sqrt{2}}(2|{++}\rangle + 2|{--}\rangle) \\
&= \frac{1}{\sqrt{2}}(|{++}\rangle + |{--}\rangle).
\end{aligned}
$$

QCE 7.3

Let

$$|\beta_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix},\ |\beta_{01}\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}.$$

Also we know that

$$Z = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix},$$

so

$$
\begin{aligned}
Z \otimes Z &= \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} \\
&= \begin{pmatrix}1 & 0 & 0 & 0\\0 & -1 & 0 & 0\\0 & 0 & -1 & 0\\0 & 0 & 0 & 1\end{pmatrix}.
\end{aligned}
$$

Now

$$
\begin{aligned}
Z \otimes Z\left|\beta_{00}\right\rangle &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\
&= \beta_{00} \\
&= (-1)^y \left|\beta_{xy}\right\rangle,
\end{aligned}
$$

where $y = 0$ and $x = 0$. Also

$$
\begin{aligned}
Z \otimes Z\left|\beta_{01}\right\rangle &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ -1 \\ 0 \end{pmatrix} \\
&= -\beta_{01} \\
&= (-1)^y \left|\beta_{xy}\right\rangle,
\end{aligned}
$$

where $y = 1$ and $x = 0$.

QCE 7.4

Show that $X \otimes X\left|\beta_{xy}\right\rangle = (-1)^x \left|\beta_{xy}\right\rangle$. From the book we know that

$$
\left|\beta_{xy}\right\rangle = \frac{|0y\rangle + (-1)^x |1\bar{y}\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{y} \\ y \\ (-1)^x y \\ (-1)^x \bar{y} \end{pmatrix}.
$$

Also

$$
\begin{aligned}
X \otimes X &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},
\end{aligned}
$$

so

$$
\begin{aligned}
X \otimes X \left|\beta_{xy}\right\rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{y} \\ y \\ (-1)^x y \\ (-1)^x \bar{y} \end{pmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} (-1)^x \bar{y} \\ (-1)^x y \\ y \\ \bar{y} \end{pmatrix} \\
&= \frac{(-1)^x \left|0y\right\rangle + \left|1\bar{y}\right\rangle}{\sqrt{2}} \\
&= (-1)^x \beta_{xy}.
\end{aligned}
$$

QCE 7.5

Show that $Y \otimes Y \left|\beta_{xy}\right\rangle = (-1)^{x+y} \left|\beta_{xy}\right\rangle$. From the book we know that

$$
\left|\beta_{xy}\right\rangle = \frac{\left|0y\right\rangle + (-1)^x \left|1\bar{y}\right\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{y} \\ y \\ (-1)^x y \\ (-1)^x \bar{y} \end{pmatrix}.
$$

Also

$$
\begin{aligned}
Y \otimes Y &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},
\end{aligned}
$$

so

$$
\begin{aligned}
Y \otimes Y \left|\beta_{xy}\right\rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{y} \\ y \\ (-1)^x y \\ (-1)^x \bar{y} \end{pmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} -(-1)^x \bar{y} \\ (-1)^x y \\ y \\ -\bar{y} \end{pmatrix} \\
&= (-1)^{x+\bar{y}} \left|\beta_{xy}\right\rangle.
\end{aligned}
$$

The book has a typo as in the above formula in the term $x + y$ y should be $\bar{y}$.

QCE 7.6

Show that $X \otimes X$ commutes with $Z \otimes Z$. They commute if and only if $X \otimes XZ \otimes Z = X \otimes XZ \otimes Z$:

$$\begin{aligned} X \otimes X &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\otimes 2} \\ &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

$$\begin{aligned} Z \otimes Z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{\otimes 2} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Now

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Also

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

which proves that the two matrices commute.

QCE 7.7

Consider the eigenvectors in Example 7.4. Show that $[H_I, \vec{\sigma_A} \cdot \vec{\sigma_B}] = 0$, and hence show that the eigenvectors of the Hamiltonian are eigenvectors of the $\vec{\sigma_A} \cdot \vec{\sigma_B}$ operator. In particular, show that $\vec{\sigma_A} \cdot \vec{\sigma_B} \left|\phi_i\right\rangle = \left|\phi_i\right\rangle$ for $i = 1, 2, 3$ and $\vec{\sigma_A} \cdot \vec{\sigma_B} \left|\phi_4\right\rangle = -3 \left|\phi_4\right\rangle$.

From the Example 7.4 we have that

$$H_I = \frac{\mu^2}{r^3} \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}, \ \vec{\sigma_A} \cdot \vec{\sigma_B} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Condition $[H_I, \vec{\sigma_A} \cdot \vec{\sigma_B}] = 0$ means that the two matrices commute and that is the case as a routine calculation may show. Wolframalpha tells me that the eigenvectors of H_I are

$$\begin{pmatrix} 0 \\ -1 \\ -1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix},$$

and that is not in accord with the eigenvectors of $\vec{\sigma_A} \cdot \vec{\sigma_B}$, thank you, book. :(

Now

$$\begin{aligned} \vec{\sigma_A} \cdot \vec{\sigma_B} \left|\phi_1\right\rangle &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \\ &= \left|\phi_1\right\rangle, \end{aligned}$$

$$\begin{aligned} \vec{\sigma_A} \cdot \vec{\sigma_B} \left|\phi_2\right\rangle &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\ &= \left|\phi_2\right\rangle, \end{aligned}$$

$$\begin{aligned} \vec{\sigma_A} \cdot \vec{\sigma_B} \left|\phi_3\right\rangle &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ &= \left|\phi_3\right\rangle, \end{aligned}$$

$$
\begin{aligned}
\vec{\sigma_A} \cdot \vec{\sigma_B} |\phi_4\rangle &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -3 \\ 3 \\ 0 \end{pmatrix} \\
&= -3 |\phi_4\rangle .
\end{aligned}
$$

QCE 7,8

Is the state $X \otimes Z |\beta_{00}\rangle$ entangled?

$$
\begin{aligned}
X \otimes Z |00\rangle &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \\
&= \frac{-|01\rangle + |10\rangle}{\sqrt{2}} \\
&= -|\beta_{11}\rangle ,
\end{aligned}
$$

so the state is entangled but is not a Bell state.

QCE 7.9

Find the Pauli representation of

$$
\rho = \begin{pmatrix} \sin^2\theta & e^{-i\phi}\sin\theta\cos\theta \\ e^{i\phi}\sin\theta\cos\theta & \cos^2\theta \end{pmatrix} .
$$

First we need to compute

$$\begin{aligned} c_0 &= Tr(\rho\sigma_0) \\ &= Tr(\rho) \\ &= \sin^2\theta + \cos^2\theta \\ &= 1, \end{aligned}$$

$$\begin{aligned} c_1 &= Tr(\rho\sigma_1) \\ &= Tr(\rho\sigma_x) \\ &= Tr\begin{pmatrix} \sin^2\theta & e^{-i\phi}\sin\theta\cos\theta \\ e^{i\phi}\sin\theta\cos\theta & \cos^2\theta \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= Tr\begin{pmatrix} e^{-i\phi}\sin\theta\cos\theta & \sin^2\theta \\ \cos^2\theta & e^{i\phi}\sin\theta\cos\theta \end{pmatrix} \\ &= e^{-i\phi}\sin\theta\cos\theta + e^{i\phi}\sin\theta\cos\theta \\ &= (\sin\theta\cos\theta)(\cos\phi - i\sin\phi + \cos\phi + i\sin\phi) \\ &= 2\sin\theta\cos\theta\cos\phi, \end{aligned}$$

$$\begin{aligned} c_2 &= Tr(\rho\sigma_2) \\ &= Tr(\rho\sigma_y) \\ &= Tr\begin{pmatrix} \sin^2\theta & e^{-i\phi}\sin\theta\cos\theta \\ e^{i\phi}\sin\theta\cos\theta & \cos^2\theta \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ &= Tr\begin{pmatrix} ie^{-i\phi}\sin\theta\cos\theta & -i\sin^2\theta \\ i\cos^2\theta & -ie^{i\phi}\sin\theta\cos\theta \end{pmatrix} \\ &= ie^{-i\phi}\sin\theta\cos\theta - ie^{i\phi}\sin\theta\cos\theta \\ &= \sin\theta\cos\theta(ie^{-i\phi} - ie^{i\phi}) \\ &= \sin\theta\cos\theta(i(\cos\phi - i\sin\phi) - i(\cos\phi + i\sin\phi)) \\ &= \sin\theta\cos\theta(i\cos\phi - i^2\sin\phi - i\cos\phi - i^2\sin\phi) \\ &= 2\sin\theta\cos\theta\sin\phi. \end{aligned}$$

$$\begin{aligned} c_3 &= Tr(\rho\sigma_3) \\ &= Tr(\rho\sigma_z) \\ &= Tr\begin{pmatrix} \sin^2\theta & e^{-i\phi}\sin\theta\cos\theta \\ e^{i\phi}\sin\theta\cos\theta & \cos^2\theta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= Tr\begin{pmatrix} \sin^2\theta & -e^{-i\phi}\sin\theta\cos\theta \\ e^{i\phi}\sin\theta\cos\theta & -\cos^2\theta \end{pmatrix} \\ &= \sin^2\theta - \cos^2\theta. \end{aligned}$$

The Pauli representation for the system is

$$\rho = \sum_{i=0}^{3} Tr(\sigma_i)\sigma_i.$$

QCE 7.10

Use (7.36) to show that $|\beta_{10}\rangle$ is entangled. Apply the same criterion to test $X \otimes Z\,|\beta_{00}\rangle$.

Here we have

$$|\beta_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \qquad |\beta_{10}\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}.$$

The density operator for $|\beta_{10}\rangle$ is

$$\rho = \frac{1}{2}(|00\rangle - |11\rangle)(\langle 00| - \langle 11|) = \frac{1}{2}(|00\rangle\langle 00| - |00\rangle\langle 11| - |11\rangle\langle 00| + |11\rangle\langle 11|),$$

and its density matrix is

$$\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

Next we find

$$\begin{aligned} c_{11} = Tr(\rho X \otimes X) &= Tr\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\ &= \frac{1}{2}Tr\begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \\ &= -1 \end{aligned}$$

$$\begin{aligned} c_{22} = Tr(\rho Y \otimes Y) &= Tr\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \\ &= \frac{1}{2}Tr\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \\ &= 1 \end{aligned}$$

$$
\begin{aligned}
c_{33} = Tr(\rho Z \otimes Z) = Tr\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
= \frac{1}{2}Tr\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \\
= 1.
\end{aligned}
$$

Now we see that $|c_{11}| + |c_{22}| + |c_{33}| = 3$, so the state is entangled.

$$
\begin{aligned}
X \otimes Z |\beta_{00}\rangle &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \\
&= \frac{-|01\rangle + |10\rangle}{\sqrt{2}}.
\end{aligned}
$$

Now the density operator/matrix for the above state is

$$
\begin{aligned}
\rho &= \frac{1}{2}(-|01\rangle + |10\rangle)(-\langle 01| + \langle 10|) \\
&= \frac{1}{2}(|01\rangle\langle 01| - |01\rangle\langle 10| - |10\rangle\langle 01| + |10\rangle\langle 10|) \\
&= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}
$$

Next we find

$$c_{11} = Tr(\rho X \otimes X) = \frac{1}{2} Tr \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$
$$= \frac{1}{2} Tr \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = -1,$$

$$c_{22} = Tr(\rho Y \otimes Y) = \frac{1}{2} Tr \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$
$$= \frac{1}{2} Tr \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = -1,$$

$$c_{33} = Tr(\rho Z \otimes Z) = \frac{1}{2} Tr \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$= \frac{1}{2} Tr \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = -1.$$

Once again we have $|c_{11}| + |c_{22}| + |c_{33}| = 3$ so this state is entangled as well.

QCE 7.11

Derive

$$|\beta_{00}\rangle \langle\beta_{00}| = \frac{1}{4}(I \otimes I + X \otimes X - Y \otimes Y + Z \otimes Z).$$

We have that

$$|\beta_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}},$$

so

$$
\begin{aligned}
|\beta_{00}\rangle\langle\beta_{00}| &= \frac{1}{2}(|00\rangle + |11\rangle)(\langle 00| + \langle 11|) \\
&= \frac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|) \\
&= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.
\end{aligned}
$$

Now

$$
\begin{aligned}
I \otimes I + X \otimes X - Y \otimes Y + Z \otimes Z &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\
&\quad - \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 2 \end{pmatrix} \\
&= 4\,|\beta_{00}\rangle\langle\beta_{00}|,
\end{aligned}
$$

which yields the desired result.

QCE 7.12

Can the following state be written in diagonal form in terms of the Bell basis?

$$
\rho = \begin{pmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{8} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{8} & 0 & 0 & \frac{1}{2} \end{pmatrix}
$$

Using (7.43), determine if this state is a separable state.

Now

$$\begin{aligned}
\rho &= \frac{1}{2}\Bigg(|00\rangle\langle 00| + |11\rangle\langle 11| \Bigg) - \frac{1}{8}\Bigg(|00\rangle\langle 11| + |11\rangle\langle 00| \Bigg) \\
&= \frac{1}{2}\Bigg(|\beta_{00}\rangle\langle \beta_{00}| + |\beta_{10}\rangle\langle \beta_{10}| \Bigg) - \frac{1}{8}\Bigg(|\beta_{00}\rangle\langle \beta_{00}| - |\beta_{10}\rangle\langle \beta_{10}| \Bigg) \\
&= \frac{3}{8}\Bigg(|\beta_{00}\rangle\langle \beta_{00}| \Bigg) + \frac{5}{8}\Bigg(|\beta_{10}\rangle\langle \beta_{10}| \Bigg).
\end{aligned}$$

Since $c_{00} = \frac{3}{8} \leq \frac{1}{2}$, this state is separable.

QCE 7.13

Verify that

$$|\psi\rangle = \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

is a product state using (7.36).

A product state is also called separable. Now

$$\begin{aligned}
|\psi\rangle &= \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) \\
&= \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle).
\end{aligned}$$

Now

$$\begin{aligned}
\rho &= |\psi\rangle\langle\psi| \\
&= \frac{1}{4}(|00\rangle - |01\rangle - |10\rangle + |11\rangle)(\langle 00| - \langle 01| - \langle 10| + \langle 11|) \\
&= \frac{1}{4}\Bigg(|00\rangle\langle 00| - |00\rangle\langle 01| - |00\rangle\langle 10| + |00\rangle\langle 11| \\
&- |01\rangle\langle 00| + |01\rangle\langle 01| + |01\rangle\langle 10| - |01\rangle\langle 11| \\
&- |10\rangle\langle 00| + |10\rangle\langle 01| + |10\rangle\langle 10| - |10\rangle\langle 11| \\
&+ |11\rangle\langle 00| - |11\rangle\langle 01| - |11\rangle\langle 10| + |11\rangle\langle 11| \Bigg) \\
&= \frac{1}{4}\begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.
\end{aligned}$$

The first term is

$$\begin{aligned}
c_{11} &= Tr(\rho X \otimes X) \\
&= \frac{1}{4} Tr \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\
&= \frac{1}{4} Tr \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \\
&= 1.
\end{aligned}$$

$$\begin{aligned}
c_{22} &= Tr(\rho Y \otimes Y) \\
&= \frac{1}{4} Tr \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \\
&= \frac{1}{4} Tr \begin{pmatrix} -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 \end{pmatrix} \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
c_{33} &= Tr(\rho Z \otimes Z) \\
&= \frac{1}{4} Tr \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \frac{1}{4} Tr \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \\
&= 0.
\end{aligned}$$

As $|c_{11}| + |c_{22}| + |c_{33}| = 1 \leq 1$, it is a product state in question.

QCE 7.14

Verify that the state

$$|\psi\rangle = \frac{1}{\sqrt{2}} |\beta_{00}\rangle - \frac{1}{\sqrt{2}} |\beta_{01}\rangle$$

is entangled by calculating the Schmidt number.

First, let us rewrite the state:

$$
\begin{aligned}
|\psi\rangle &= \frac{1}{\sqrt{2}}|\beta_{00}\rangle - \frac{1}{\sqrt{2}}|\beta_{01}\rangle \\
&= \frac{1}{2}(|00\rangle + |11\rangle) - \frac{1}{2}(|01\rangle + |10\rangle) \\
&= \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle).
\end{aligned}
$$

Now

$$
\begin{aligned}
\rho &= |\psi\rangle\langle\psi| \\
&= \frac{1}{4}\Bigg(|00\rangle - |01\rangle - |10\rangle + |11\rangle\Bigg)\Bigg(\langle 00| - \langle 01| - \langle 10| + \langle 11|\Bigg) \\
&= \frac{1}{4}\Bigg(|00\rangle\langle 00| - |00\rangle\langle 01| - |00\rangle\langle 10| + |00\rangle\langle 11| \\
&- |01\rangle\langle 00| + |01\rangle\langle 01| + |01\rangle\langle 10| - |01\rangle\langle 11| \\
&- |10\rangle\langle 00| + |10\rangle\langle 01| + |10\rangle\langle 10| - |10\rangle\langle 11| \\
&+ |11\rangle\langle 00| - |11\rangle\langle 01| - |11\rangle\langle 10| + |11\rangle\langle 11|\Bigg) \\
&= \frac{1}{4}\begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.
\end{aligned}
$$

Now

$$
\begin{aligned}
\rho' &= Tr(|\psi\rangle\langle\psi|) \\
&= \langle 0|\psi\rangle\langle\psi|0\rangle + \langle 1|\psi\rangle\langle\psi|1\rangle \\
&= \frac{1}{4}(|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|) \\
&+ \frac{1}{4}(|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|) \\
&= \frac{1}{2}(|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|) \\
&= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.
\end{aligned}
$$

On page 169 of the book, it is told that the above matrix has eigenvalues $\lambda_1 = 1, \lambda_2 = 0$, so the Schmidt number is one and this is a separable state.

QCE 8.1

Describe the action of the Y gate in terms of the Bloch sphere picture.

We have

$$|\psi\rangle = \cos\theta\,|0\rangle + e^{i\phi}\sin\theta\,|1\rangle = \begin{pmatrix} \cos\theta \\ e^{i\phi}\sin\theta \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Now

$$\begin{aligned} Y\,|\psi\rangle &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \cos\theta \\ e^{i\phi}\sin\theta \end{pmatrix} \\ &= \begin{pmatrix} -ie^{i\phi}\sin\theta \\ i\cos\theta \end{pmatrix} \\ &= \begin{pmatrix} -i(\cos\phi + i\sin\phi)\sin\theta \\ i\cos\theta \end{pmatrix} \\ &= \begin{pmatrix} -i\cos\phi\sin\theta + \sin\phi\sin\theta \\ i\cos\theta \end{pmatrix} \end{aligned}$$

QCE 8.2

$$X^{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad X^{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad X^{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad X^{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The Hadamard basis is

$$|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad |-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

so

$$X^{11}\,|+\rangle = \frac{1}{\sqrt{2}}\,|0\rangle\,,\ X^{12}\,|+\rangle = \frac{1}{\sqrt{2}}\,|0\rangle\,,\ X^{21}\,|+\rangle = \frac{1}{\sqrt{2}}\,|1\rangle\,,\ X^{22}\,|+\rangle = \frac{1}{\sqrt{2}}\,|1\rangle\,,$$

and

$$X^{11}\,|-\rangle = \frac{1}{\sqrt{2}}\,|0\rangle\,,\ X^{12}\,|-\rangle = -\frac{1}{\sqrt{2}}\,|0\rangle\,,\ X^{21}\,|-\rangle = \frac{1}{\sqrt{2}}\,|1\rangle\,,\ X^{22}\,|-\rangle = -\frac{1}{\sqrt{2}}\,|1\rangle\,.$$

QCE 8.3

Find a way to write the Pauli operators X, Y and Z in terms of the Hubbard operators.

This is a basic matrix shit:

$$\begin{aligned} X &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X^{12} + X^{21}, \\ Y &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -iX^{12} + iX^{21} = i(X^{21} - X^{12}), \\ Z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = X^{11} - X^{22}. \end{aligned}$$

QCE 8.4

Show that the controlled NOT gate is Hermitian and unitary.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{\dagger},$$

so the matrix is Hermitian. Also

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

so the matrix is unitary.

QCE 8.5

Let $|a\rangle = |1\rangle$, and consider the circuit shown in Figure 8.5. Determine which Bell states are generated as output when $|b\rangle = |0\rangle$, $|b\rangle = |1\rangle$.

The Hadamard gate converts $|a\rangle = |1\rangle$ to $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. Also the matrix representation of CNOT is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

For the first case, we have $|b\rangle = |0\rangle$, so the input qubit is

$$|a\rangle\,|b\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\,|0\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |10\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix},$$

and

$$\begin{aligned} CNOT\,|ab\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \\ &= |\beta_{10}\rangle\,. \end{aligned}$$

For the second case, we have $|b\rangle = |1\rangle$, so the input qubit is

$$|a\rangle |b\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) |1\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |11\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix},$$

and

$$\begin{aligned} CNOT\,|ab\rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \\ &= |\beta_{11}\rangle . \end{aligned}$$

QCE 8.6

Write down the matrix representation for the controlled Z gate. Then write down its representation using Dirac notation.

The matrix representation is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

and in Dirac notation this is $|00\rangle \langle 00| + |01\rangle \langle 01| + |10\rangle \langle 10| - |11\rangle \langle 11|$.

QCE 8.7

$$\begin{aligned} X^2 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= I, \end{aligned}$$

$$\begin{aligned} Y^2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= I, \end{aligned}$$

$$
\begin{aligned}
Z^2 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= I,
\end{aligned}
$$

$$
\begin{aligned}
S^2 &= \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= Z,
\end{aligned}
$$

$$
\begin{aligned}
T^2 &= \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} e^{i\pi/4} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & e^{2i\pi/4} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \\
&= S.
\end{aligned}
$$

QCE 8.10

By using the tensor product methods developed in chapter 4, show that the controlled-NOT matrix can be generated from $P_0 \otimes I + P_1 \otimes X$.

$$
\begin{aligned}
P_0 \otimes I + P_1 \otimes X &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\
&= CNOT.
\end{aligned}
$$

QCE 8.11

The operator matrix of the left circuit is that of CNOT:

$$
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.
$$

The two "parallel" Hadamard gates may be represented as

$$
H^{\otimes 2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix},
$$

so

$$
\begin{aligned}
H^{\otimes 2} \times CNOT \times H^{\otimes 2} &= \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \\
&= \frac{1}{4} \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \end{pmatrix} \\
&= CNOT.
\end{aligned}
$$

QCE 8.12

$$\begin{aligned} V &= (1-i)\frac{I+iX}{2} \\ &= \frac{(1-i)}{2}\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}0 & i\\ i & 0\end{pmatrix}\right) \\ &= \frac{(1-i)}{2}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix}1-i & 1+i\\ 1+i & 1-i\end{pmatrix} \end{aligned}$$

Now, the conditional CV gate is

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1-i & 1+i\\ 0 & 0 & 1+i & 1-i\end{pmatrix},$$

and the conditional $CV^\dagger$ is

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1+i & 1-i\\ 0 & 0 & 1-i & 1+i\end{pmatrix}.$$

Now the entire circuit can be represented as

$$(I\otimes CV)(CNOT\otimes I)(I\otimes CV^\dagger)(CNOT\otimes I)(I\otimes CV).$$

$$\begin{aligned} I\otimes CV &= \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\otimes\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1-i & 1+i\\ 0 & 0 & 1+i & 1-i\end{pmatrix} \\ &= \left(\begin{array}{cccccccc}1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1-i & 1+i & 0 & 0 & 0 & 0\\ 0 & 0 & 1+i & 1-i & 0 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1-i & 1+i\\ 0 & 0 & 0 & 0 & 0 & 0 & 1+i & 1-i\end{array}\right). \end{aligned}$$

$$CNOT \otimes I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \left(\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right).$$

$$I \otimes CV^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1+i & 1-i \\ 0 & 0 & 1-i & 1+i \end{pmatrix}$$

$$= \left(\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1+i & 1-i & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-i & 1+i & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1+i & 1-i \\ 0 & 0 & 0 & 0 & 0 & 0 & 1-i & 1+i \end{array}\right).$$

Now

$$(I \otimes CV)(CNOT \otimes I) = \left(\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-i & 1+i & 0 & 0 & 0 & 0 \\ 0 & 0 & 1+i & 1-i & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1-i & 1+i \\ 0 & 0 & 0 & 0 & 0 & 0 & 1+i & 1-i \end{array}\right) \left(\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right)$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-i & 1+i & 0 & 0 & 0 & 0 \\ 0 & 0 & 1+i & 1-i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1-i & 1+i & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+i & 1-i & 0 & 0 \end{pmatrix}$$

$$
(CNOT \otimes I)(I \otimes CV) = \left(\begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}\right)
\left(\begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1-i & 1+i & 0 & 0 & 0 & 0 \\
0 & 0 & 1+i & 1-i & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1-i & 1+i \\
0 & 0 & 0 & 0 & 0 & 0 & 1+i & 1-i
\end{array}\right)
$$

$$
= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1-i & 1+i & 0 & 0 & 0 & 0 \\
0 & 0 & 1+i & 1-i & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1-i & 1+i \\
0 & 0 & 0 & 0 & 0 & 0 & 1+i & 1-i \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}.
$$

$$
(I \otimes CV)(CNOT \otimes I)(I \otimes CV^{\dagger}) =
$$

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1-i & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1-i & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1-i & 1+i & 0 & 0 \\
0 & 0 & 0 & 0 & 1+i & 1-i & 0 & 0
\end{pmatrix}
\left(\begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1+i & 1-i & 0 & 0 & 0 & 0 \\
0 & 0 & 1-i & 1+i & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1+i & 1-i \\
0 & 0 & 0 & 0 & 0 & 0 & 1-i & 1+i
\end{array}\right)
$$

$$
= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & -2i & 0 & 0 & 0 & 0 \\
0 & 0 & -2i & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1+i & 1-i \\
0 & 0 & 0 & 0 & 0 & 0 & 1-i & 1+i \\
0 & 0 & 0 & 0 & 1-i & 1+i & 0 & 0 \\
0 & 0 & 0 & 0 & 1+i & 1-i & 0 & 0
\end{pmatrix}.
$$

Finally,

$$
\begin{aligned}
&(I\otimes CV)(CNOT\otimes I)(I\otimes CV^{\dagger})(CNOT\otimes I)(I\otimes CV)\\
&=\begin{pmatrix}
1&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
0&0&2&-2i&0&0&0&0\\
0&0&-2i&2&0&0&0&0\\
0&0&0&0&0&0&1+i&1-i\\
0&0&0&0&0&0&1-i&1+i\\
0&0&0&0&1-i&1+i&0&0\\
0&0&0&0&1+i&1-i&0&0
\end{pmatrix}
\begin{pmatrix}
1&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
0&0&1-i&1+i&0&0&0&0\\
0&0&1+i&1-i&0&0&0&0\\
0&0&0&0&0&0&1-i&1+i\\
0&0&0&0&0&0&1+i&1-i\\
0&0&0&0&1&0&0&0\\
0&0&0&0&0&1&0&0
\end{pmatrix}=
\end{aligned}
$$

$$
\begin{pmatrix}
1&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
0&0&2(1-i)-2i(1+i)&2(1+i)-2i(1-i)&0&0&0&0\\
0&0&-2i(1-i)+2(1+i)&-2i(1+i)+2(1-i)&0&0&0&0\\
0&0&0&0&1+i&1-i&0&0\\
0&0&0&0&1-i&1+i&0&0\\
0&0&0&0&0&0&(1-i)^2+(1+i)^2&2(1-i)(1+i)\\
0&0&0&0&0&0&2(1+i)(1-i)&(1+i)^2+(1-i)^2
\end{pmatrix}
$$

$$
=\begin{pmatrix}
1&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
2-2i-2i+2=4-4i0&0&4(1-i)&0&0&0&0&0
\end{pmatrix}
$$

Problems not solved

- QCE 7.1
- (Ask about QCE 7.9)
- (Ask about QCE 7.14)
- (Ask about QCE 8.1)
- QCE 8.12