

# Kvanttilaskenta, kevät 2015 – Viikko 4

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## edx Problem 1

False. By definition of reversibility we should have  $x$  at the output of  $R_f$ .

## edx Problem 2

False. The quantum circuit should modify each  $\alpha_x$ .

## edx Problem 3

True. Straight from the slides.

## edx Problem 4

$$\begin{aligned} H^{\otimes 2} \frac{1}{\sqrt{2}} |00\rangle &= \frac{1}{\sqrt{2}} |++\rangle \\ &= \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \right) \left( \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \right) \\ &= \frac{1}{2\sqrt{2}} \left( |00\rangle + |01\rangle + |10\rangle + |11\rangle \right). \end{aligned}$$

Also

$$\begin{aligned} H^{\otimes 2} \frac{1}{\sqrt{2}} |11\rangle &= \frac{1}{\sqrt{2}} |--\rangle \\ &= \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle \right) \left( \frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle \right) \\ &= \frac{1}{2\sqrt{2}} \left( |00\rangle - |01\rangle - |10\rangle + |11\rangle \right), \end{aligned}$$

so

$$\begin{aligned}
H^{\otimes 2}\psi &= H^{\otimes 2}\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle\right) \\
&= \frac{1}{2\sqrt{2}}\left(2|00\rangle + 2|11\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \\
&= \psi.
\end{aligned}$$

## edx Problem 5

Let

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle).$$

$$\begin{aligned}
H^{\otimes 2}\frac{1}{\sqrt{2}}|01\rangle &= \frac{1}{\sqrt{2}}|+-\rangle \\
&= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right) \\
&= \frac{1}{2\sqrt{2}}\left(|00\rangle - |01\rangle + |10\rangle - |11\rangle\right).
\end{aligned}$$

$$\begin{aligned}
H^{\otimes 2}\frac{1}{\sqrt{2}}|10\rangle &= \frac{1}{\sqrt{2}}|-+\rangle \\
&= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \\
&= \frac{1}{2\sqrt{2}}\left(|00\rangle + |01\rangle - |10\rangle - |11\rangle\right).
\end{aligned}$$

Now we see that  $H^{\otimes 2}|\psi\rangle$  is

$$\begin{aligned}
\frac{1}{2\sqrt{2}}\left(|00\rangle - |01\rangle + |10\rangle - |11\rangle\right) + \frac{1}{2\sqrt{2}}\left(|00\rangle + |01\rangle - |10\rangle - |11\rangle\right) &= \frac{1}{2\sqrt{2}}(2|00\rangle - 2|11\rangle) \\
&= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle).
\end{aligned}$$

## edx Problem 6

Yes. Both may yield  $|00\rangle$  with probability  $\frac{1}{2}$  and  $|11\rangle$  with probability  $\frac{1}{2}$ .

## edx Problem 7

Apply circuit A and then D.

## edx Problem 8

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

## edx Problem 9

Suppose

$$H^{\otimes 3} |\psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle).$$

Since  $H^{\otimes 3}$  is reversible, if we apply it again to  $H^{\otimes 3} |\psi\rangle$ , we will obtain  $|\psi\rangle$ . Let us calculate that ket by ket:

$$\begin{aligned} H^{\otimes 3} \frac{1}{\sqrt{2}} |000\rangle &= \frac{1}{\sqrt{2}} |+++ \rangle \\ &= \frac{1}{4} \left( |0\rangle + |1\rangle \right)^3 \\ &= \frac{1}{4} \left( |0\rangle + |1\rangle \right) \left( |00\rangle + |01\rangle + |10\rangle + |11\rangle \right) \\ &= \frac{1}{4} \left( |000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle \right). \end{aligned}$$

$$\begin{aligned} H^{\otimes 3} \frac{1}{\sqrt{2}} |111\rangle &= \frac{1}{\sqrt{2}} |-- - \rangle \\ &= \frac{1}{4} \left( |0\rangle - |1\rangle \right)^3 \\ &= \frac{1}{4} \left( |0\rangle - |1\rangle \right) \left( |00\rangle - |01\rangle - |10\rangle + |11\rangle \right) \\ &= \frac{1}{4} \left( |000\rangle - |001\rangle - |010\rangle + |011\rangle - |100\rangle + |101\rangle + |110\rangle - |111\rangle \right). \end{aligned}$$

Now

$$\begin{aligned}
 |\psi\rangle &= H^{\otimes 3} H^{\otimes 3} |\psi\rangle \\
 &= H^{\otimes 3} \frac{1}{\sqrt{2}} |000\rangle + H^{\otimes 3} \frac{1}{\sqrt{2}} |11\rangle \\
 &= \frac{1}{4} (2 |000\rangle + 2 |011\rangle + 2 |101\rangle + 2 |110\rangle) \\
 &= \frac{1}{2} (|000\rangle + |011\rangle + |101\rangle + |110\rangle).
 \end{aligned}$$

## edx Problem 10

(a)

$$\frac{1}{2^{n-1}}.$$

(b) We see a uniformly random string  $y \in \{0, 1\}^n$ .

## edx Problem 11

$$\frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{x \cdot f(x)} |x\rangle |f(x)\rangle.$$

## edx Problem 12

$$\frac{1}{\sqrt{2^n}} \sum_x |x\rangle |0101\rangle.$$

## edx Problem 13

1111.

## edx Problem 14

Suppose Alice starts with two qubits in the Bell state  $\frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle$  and teleports these qubits to Bob by applying the quantum teleportation protocol to each qubit separately.

As we are speaking about teleportation, Bob sees the same state and receives exactly 2 bits of information as there is only 4 Bell states.

### QCE 7.1

$$\vec{\sigma} = \sigma_x \hat{x} + \sigma_y \hat{y} + \sigma_z \hat{z}, \quad \vec{n} = \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z}.$$

Now

$$\begin{aligned} \vec{\sigma} \cdot \vec{n} &= (\sigma_x \hat{x} + \sigma_y \hat{y} + \sigma_z \hat{z})(\sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z}) \\ &= \end{aligned}$$

### QCE 7.2

Let

$$|\psi\rangle = \frac{|0\rangle|1\rangle - |1\rangle|0\rangle}{\sqrt{2}}.$$

Now we have

$$|0\rangle = \frac{|+\rangle + i|-\rangle}{\sqrt{2}}, \quad |1\rangle = \frac{|+\rangle - i|-\rangle}{\sqrt{2}},$$

so

$$\begin{aligned} |\psi\rangle &= \frac{|0\rangle|1\rangle - |1\rangle|0\rangle}{\sqrt{2}} \\ &= \frac{1}{2\sqrt{2}} \left( (|+\rangle + i|-\rangle)(|+\rangle - i|-\rangle) - (|+\rangle - i|-\rangle)(|+\rangle + i|-\rangle) \right) \\ &= \frac{1}{2\sqrt{2}} \left( |++\rangle - i|+-\rangle + i|-+\rangle + |--\rangle + |++\rangle + i|+-\rangle - i|-+\rangle + |--\rangle \right) \\ &= \frac{1}{2\sqrt{2}} (2|++\rangle + 2|--\rangle) \\ &= \frac{1}{\sqrt{2}} (|++\rangle + |--\rangle). \end{aligned}$$

### QCE 7.3

Let

$$|\beta_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad |\beta_{01}\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

Also we know that

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

so

$$\begin{aligned} Z \otimes Z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Now

$$\begin{aligned} Z \otimes Z |\beta_{00}\rangle &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\ &= \beta_{00} \\ &= (-1)^y |\beta_{xy}\rangle, \end{aligned}$$

where  $y = 0$  and  $x = 0$ . Also

$$\begin{aligned} Z \otimes Z |\beta_{01}\rangle &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ -1 \\ 0 \end{pmatrix} \\ &= -\beta_{01} \\ &= (-1)^y |\beta_{xy}\rangle, \end{aligned}$$

where  $y = 1$  and  $x = 0$ .

## QCE 7.4

Show that  $X \otimes X |\beta_{xy}\rangle = (-1)^x |\beta_{xy}\rangle$ . From the book we know that

$$|\beta_{xy}\rangle = \frac{|0y\rangle + (-1)^x |1\bar{y}\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{y} \\ y \\ (-1)^x y \\ (-1)^x \bar{y} \end{pmatrix}.$$

Also

$$\begin{aligned} X \otimes X &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

so

$$\begin{aligned} X \otimes X |\beta_{xy}\rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{y} \\ y \\ (-1)^x y \\ (-1)^x \bar{y} \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} (-1)^x \bar{y} \\ (-1)^x y \\ y \\ \bar{y} \end{pmatrix} \\ &= \frac{(-1)^x |0y\rangle + |1\bar{y}\rangle}{\sqrt{2}} \\ &= (-1)^x \beta_{xy}. \end{aligned}$$

## QCE 7.5

Show that  $Y \otimes Y |\beta_{xy}\rangle = (-1)^{x+y} |\beta_{xy}\rangle$ . From the book we know that

$$|\beta_{xy}\rangle = \frac{|0y\rangle + (-1)^x |1\bar{y}\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{y} \\ y \\ (-1)^x y \\ (-1)^x \bar{y} \end{pmatrix}.$$

Also

$$\begin{aligned} Y \otimes Y &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

so

$$\begin{aligned}
Y \otimes Y |\beta_{xy}\rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{y} \\ y \\ (-1)^x y \\ (-1)^x \bar{y} \end{pmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} -(-1)^x \bar{y} \\ (-1)^x y \\ y \\ -\bar{y} \end{pmatrix} \\
&= (-1)^{x+\bar{y}} |\beta_{xy}\rangle.
\end{aligned}$$

The book has a typo...

## QCE 7.6

Show that  $X \otimes X$  commutes with  $Z \otimes Z$ . They commute if and only if  $X \otimes XZ \otimes Z = X \otimes XZ \otimes Z$ :

$$\begin{aligned}
X \otimes X &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\otimes 2} \\
&= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

$$\begin{aligned}
Z \otimes Z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{\otimes 2} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\end{aligned}$$

Now

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Also

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

which proves that the two matrices commute.



## QCE 7.7

Consider the eigenvectors in Example 7.4. Show that  $[H_I, \vec{\sigma}_A \cdot \vec{\sigma}_B] = 0$ , and hence show that the eigenvectors of the Hamiltonian are eigenvectors of the  $\vec{\sigma}_A \cdot \vec{\sigma}_B$  operator. In particular, show that  $\vec{\sigma}_A \cdot \vec{\sigma}_B |\phi_i\rangle = |\phi_i\rangle$  for  $i = 1, 2, 3$  and  $\vec{\sigma}_A \cdot \vec{\sigma}_B |\phi_4\rangle = -3 |\phi_4\rangle$ .

From the Example 7.4 we have that

$$H_I = \frac{\mu^2}{r^3} \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}, \quad \vec{\sigma}_A \cdot \vec{\sigma}_B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Condition  $[H_I, \vec{\sigma}_A \cdot \vec{\sigma}_B] = 0$  means that the two matrices commute and that is the case as a routine calculation may show. Wolframalpha tells me that the eigenvectors of  $H_I$  are

$$\begin{pmatrix} 0 \\ -1 \\ -1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix},$$

and that is not in accord with the eigenvectors of  $\vec{\sigma}_A \cdot \vec{\sigma}_B$ , thank you, book. :(

Now

$$\begin{aligned} \vec{\sigma}_A \cdot \vec{\sigma}_B |\phi_1\rangle &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \\ &= |\phi_1\rangle, \end{aligned}$$

$$\begin{aligned} \vec{\sigma}_A \cdot \vec{\sigma}_B |\phi_2\rangle &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\ &= |\phi_2\rangle, \end{aligned}$$

$$\begin{aligned}
\vec{\sigma}_A \cdot \vec{\sigma}_B |\phi_3\rangle &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
&= |\phi_3\rangle,
\end{aligned}$$

$$\begin{aligned}
\vec{\sigma}_A \cdot \vec{\sigma}_B |\phi_4\rangle &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -3 \\ 3 \\ 0 \end{pmatrix} \\
&= -3 |\phi_4\rangle.
\end{aligned}$$

## QCE 7,8

Is the state  $X \otimes Z |\beta_{00}\rangle$  entangled?

$$\begin{aligned}
X \otimes Z |00\rangle &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \\
&= \frac{-|01\rangle + |10\rangle}{\sqrt{2}} \\
&= -|\beta_{11}\rangle,
\end{aligned}$$

so the state is entangled but is not a Bell state.

## QCE 7.9

Find the Pauli representation of

$$\rho = \begin{pmatrix} \sin^2 \theta & e^{-i\phi} \sin \theta \cos \theta \\ e^{i\phi} \sin \theta \cos \theta & \cos^2 \theta \end{pmatrix}.$$

First we need to compute

$$\begin{aligned} c_0 &= \text{Tr}(\rho\sigma_0) \\ &= \text{Tr}(\rho) \\ &= \sin^2 \theta + \cos^2 \theta \\ &= 1, \end{aligned}$$

$$\begin{aligned} c_1 &= \text{Tr}(\rho\sigma_1) \\ &= \text{Tr}(\rho\sigma_x) \\ &= \text{Tr} \begin{pmatrix} \sin^2 \theta & e^{-i\phi} \sin \theta \cos \theta \\ e^{i\phi} \sin \theta \cos \theta & \cos^2 \theta \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \text{Tr} \begin{pmatrix} e^{-i\phi} \sin \theta \cos \theta & \sin^2 \theta \\ \cos^2 \theta & e^{i\phi} \sin \theta \cos \theta \end{pmatrix} \\ &= e^{-i\phi} \sin \theta \cos \theta + e^{i\phi} \sin \theta \cos \theta \\ &= (\sin \theta \cos \theta)(\cos \phi - i \sin \phi + \cos \phi + i \sin \phi) \\ &= 2 \sin \theta \cos \theta \cos \phi, \end{aligned}$$

$$\begin{aligned} c_2 &= \text{Tr}(\rho\sigma_2) \\ &= \text{Tr}(\rho\sigma_y) \\ &= \text{Tr} \begin{pmatrix} \sin^2 \theta & e^{-i\phi} \sin \theta \cos \theta \\ e^{i\phi} \sin \theta \cos \theta & \cos^2 \theta \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ &= \text{Tr} \begin{pmatrix} ie^{-i\phi} \sin \theta \cos \theta & -i \sin^2 \theta \\ i \cos^2 \theta & -ie^{i\phi} \sin \theta \cos \theta \end{pmatrix} \\ &= ie^{-i\phi} \sin \theta \cos \theta - ie^{i\phi} \sin \theta \cos \theta \\ &= \sin \theta \cos \theta (ie^{-i\phi} - ie^{i\phi}) \\ &= \sin \theta \cos \theta (i(\cos \phi - i \sin \phi) - i(\cos \phi + i \sin \phi)) \\ &= \sin \theta \cos \theta (i \cos \phi - i^2 \sin \phi - i \cos \phi - i^2 \sin \phi) \\ &= 2 \sin \theta \cos \theta \sin \phi. \end{aligned}$$

$$\begin{aligned}
c_3 &= \text{Tr}(\rho\sigma_3) \\
&= \text{Tr}(\rho\sigma_z) \\
&= \text{Tr} \begin{pmatrix} \sin^2 \theta & e^{-i\phi} \sin \theta \cos \theta \\ e^{i\phi} \sin \theta \cos \theta & \cos^2 \theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= \text{Tr} \begin{pmatrix} \sin^2 \theta & -e^{-i\phi} \sin \theta \cos \theta \\ e^{i\phi} \sin \theta \cos \theta & -\cos^2 \theta \end{pmatrix} \\
&= \sin^2 \theta - \cos^2 \theta.
\end{aligned}$$

The Pauli representation for the system is

$$\rho = \sum_{i=0}^3 \text{Tr}(\sigma_i)\sigma_i.$$

## QCE 7.10

Use (7.36) to show that  $|\beta_{10}\rangle$  is entangled. Apply the same criterion to test  $X \otimes Z |\beta_{00}\rangle$ .

Here we have

$$|\beta_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \quad |\beta_{10}\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}.$$

The density operator for  $|\beta_{10}\rangle$  is

$$\rho = \frac{1}{2}(|00\rangle - |11\rangle)(\langle 00| - \langle 11|) = \frac{1}{2}(|00\rangle\langle 00| - |00\rangle\langle 11| - |11\rangle\langle 00| + |11\rangle\langle 11|),$$

and its density matrix is

$$\frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

Next we find

$$\begin{aligned}
c_{11} &= \text{Tr}(\rho X \otimes X) = \text{Tr} \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\
&= \frac{1}{2} \text{Tr} \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \\
&= -1
\end{aligned}$$

$$\begin{aligned}
c_{22} &= Tr(\rho Y \otimes Y) = Tr \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \\
&= \frac{1}{2} Tr \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \\
&= 1
\end{aligned}$$

$$\begin{aligned}
c_{33} &= Tr(\rho Z \otimes Z) = Tr \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \frac{1}{2} Tr \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \\
&= 1.
\end{aligned}$$

Now we see that  $|c_{11}| + |c_{22}| + |c_{33}| = 3$ , so the state is entangled.

$$\begin{aligned}
X \otimes Z |\beta_{00}\rangle &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \\
&= \frac{-|01\rangle + |10\rangle}{\sqrt{2}}.
\end{aligned}$$

Now the density operator/matrix for the above state is

$$\begin{aligned}
\rho &= \frac{1}{2}(-|01\rangle + |10\rangle)(-\langle 01| + \langle 10|) \\
&= \frac{1}{2}(|01\rangle\langle 01| - |01\rangle\langle 10| - |10\rangle\langle 01| + |10\rangle\langle 10|) \\
&= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

Next we find

$$\begin{aligned}
c_{11} &= \text{Tr}(\rho X \otimes X) = \frac{1}{2} \text{Tr} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\
&= \frac{1}{2} \text{Tr} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = -1,
\end{aligned}$$

$$\begin{aligned}
c_{22} &= \text{Tr}(\rho Y \otimes Y) = \frac{1}{2} \text{Tr} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \\
&= \frac{1}{2} \text{Tr} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = -1,
\end{aligned}$$

$$\begin{aligned}
c_{33} &= \text{Tr}(\rho Z \otimes Z) = \frac{1}{2} \text{Tr} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \frac{1}{2} \text{Tr} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = -1.
\end{aligned}$$

Once again we have  $|c_{11}| + |c_{22}| + |c_{33}| = 3$  so this state is entangled as well.

## QCE 7.11

Derive

$$|\beta_{00}\rangle\langle\beta_{00}| = \frac{1}{4}(I \otimes I + X \otimes X - Y \otimes Y + Z \otimes Z).$$

We have that

$$|\beta_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}},$$

so

$$\begin{aligned} |\beta_{00}\rangle \langle \beta_{00}| &= \frac{1}{2}(|00\rangle + |11\rangle)(\langle 00| + \langle 11|) \\ &= \frac{1}{2}(|00\rangle \langle 00| + |00\rangle \langle 11| + |11\rangle \langle 00| + |11\rangle \langle 11|) \\ &= \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Now

$$\begin{aligned} I \otimes I + X \otimes X - Y \otimes Y + Z \otimes Z &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\ &\quad - \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 2 \end{pmatrix} \\ &= 4 |\beta_{00}\rangle \langle \beta_{00}|, \end{aligned}$$

which yields the desired result.

## QCE 7.12

Can the following state be written in diagonal form in terms of the Bell basis?

$$\rho = \begin{pmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{8} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{8} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

Using (7.43), determine if this state is a separable state.

Now

$$\begin{aligned}
\rho &= \frac{1}{2} \left( |00\rangle \langle 00| + |11\rangle \langle 11| \right) - \frac{1}{8} \left( |00\rangle \langle 11| + |11\rangle \langle 00| \right) \\
&= \frac{1}{2} \left( |\beta_{00}\rangle \langle \beta_{00}| + |\beta_{10}\rangle \langle \beta_{10}| \right) - \frac{1}{8} \left( |\beta_{00}\rangle \langle \beta_{00}| - |\beta_{10}\rangle \langle \beta_{10}| \right) \\
&= \frac{3}{8} \left( |\beta_{00}\rangle \langle \beta_{00}| \right) + \frac{5}{8} \left( |\beta_{10}\rangle \langle \beta_{10}| \right).
\end{aligned}$$

Since  $c_{00} = \frac{3}{8} \leq \frac{1}{2}$ , this state is separable.

## QCE 7.13

Verify that

$$|\psi\rangle = \left( \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \otimes \left( \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)$$

is a product state using (7.36).

A product state is also called separable. Now

$$\begin{aligned}
|\psi\rangle &= \left( \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \otimes \left( \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \\
&= \frac{1}{2} (|00\rangle - |01\rangle - |10\rangle + |11\rangle).
\end{aligned}$$

Now

$$\begin{aligned}
\rho &= |\psi\rangle \langle \psi| \\
&= \frac{1}{4} (|00\rangle - |01\rangle - |10\rangle + |11\rangle)(\langle 00| - \langle 01| - \langle 10| + \langle 11|) \\
&= \frac{1}{4} \left( |00\rangle \langle 00| - |00\rangle \langle 01| - |00\rangle \langle 10| + |00\rangle \langle 11| \right. \\
&\quad - |01\rangle \langle 00| + |01\rangle \langle 01| + |01\rangle \langle 10| - |01\rangle \langle 11| \\
&\quad - |10\rangle \langle 00| + |10\rangle \langle 01| + |10\rangle \langle 10| - |10\rangle \langle 11| \\
&\quad \left. + |11\rangle \langle 00| - |11\rangle \langle 01| - |11\rangle \langle 10| + |11\rangle \langle 11| \right) \\
&= \frac{1}{4} \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.
\end{aligned}$$



The first term is

$$\begin{aligned}
c_{11} &= \text{Tr}(\rho X \otimes X) \\
&= \frac{1}{4} \text{Tr} \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\
&= \frac{1}{4} \text{Tr} \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \\
&= 1.
\end{aligned}$$

$$\begin{aligned}
c_{22} &= \text{Tr}(\rho Y \otimes Y) \\
&= \frac{1}{4} \text{Tr} \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \\
&= \frac{1}{4} \text{Tr} \begin{pmatrix} -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 \end{pmatrix} \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
c_{33} &= \text{Tr}(\rho Z \otimes Z) \\
&= \frac{1}{4} \text{Tr} \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \frac{1}{4} \text{Tr} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \\
&= 0.
\end{aligned}$$

As  $|c_{11}| + |c_{22}| + |c_{33}| = 1 \leq 1$ , it is a product state in question.

## QCE 7.14

Verify that the state

$$|\psi\rangle = \frac{1}{\sqrt{2}} |\beta_{00}\rangle - \frac{1}{\sqrt{2}} |\beta_{01}\rangle$$

is entangled by calculating the Schmidt number.

First, let us rewrite the state:

$$\begin{aligned} |\psi\rangle &= \frac{1}{\sqrt{2}} |\beta_{00}\rangle - \frac{1}{\sqrt{2}} |\beta_{01}\rangle \\ &= \frac{1}{2}(|00\rangle + |11\rangle) - \frac{1}{2}(|01\rangle + |10\rangle) \\ &= \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle). \end{aligned}$$

Now

$$\begin{aligned} \rho &= |\psi\rangle \langle\psi| \\ &= \frac{1}{4} \left( |00\rangle - |01\rangle - |10\rangle + |11\rangle \right) \left( \langle 00| - \langle 01| - \langle 10| + \langle 11| \right) \\ &= \frac{1}{4} \left( |00\rangle \langle 00| - |00\rangle \langle 01| - |00\rangle \langle 10| + |00\rangle \langle 11| \right. \\ &\quad - |01\rangle \langle 00| + |01\rangle \langle 01| + |01\rangle \langle 10| - |01\rangle \langle 11| \\ &\quad - |10\rangle \langle 00| + |10\rangle \langle 01| + |10\rangle \langle 10| - |10\rangle \langle 11| \\ &\quad \left. + |11\rangle \langle 00| - |11\rangle \langle 01| - |11\rangle \langle 10| + |11\rangle \langle 11| \right) \\ &= \frac{1}{4} \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}. \end{aligned}$$

Now

$$\begin{aligned} \rho' &= Tr(|\psi\rangle \langle\psi|) \\ &= \langle 0|\psi\rangle \langle\psi|0\rangle + \langle 1|\psi\rangle \langle\psi|1\rangle \\ &= \frac{1}{4}(|0\rangle \langle 0| - |0\rangle \langle 1| - |1\rangle \langle 0| + |1\rangle \langle 1|) \\ &\quad + \frac{1}{4}(|0\rangle \langle 0| - |0\rangle \langle 1| - |1\rangle \langle 0| + |1\rangle \langle 1|) \\ &= \frac{1}{2}(|0\rangle \langle 0| - |0\rangle \langle 1| - |1\rangle \langle 0| + |1\rangle \langle 1|) \\ &= \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \end{aligned}$$

On page 169 of the book, it is told that the above matrix has eigenvalues  $\lambda_1 = 1, \lambda_2 = 0$ , so the Schmidt number is one