

# Kvanttilaskenta, kevät 2015 – Viikko 2

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## edx Problem 1

Let

$$U = \begin{pmatrix} -2i & 5i \\ 5 & 1-i \end{pmatrix}.$$

Now

$$U^\dagger = \begin{pmatrix} 2i & 5 \\ -5i & 1+i \end{pmatrix}.$$

## edx Problem 2

All unitary matrices are self-inverse.

## edx Problem 3

We are given a  $U$  that maps  $|0\rangle$  to  $\frac{1}{\sqrt{2}}|0\rangle + \frac{1+i}{2}|1\rangle$  and  $|1\rangle$  to  $\frac{1-i}{2}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$ . So we have that

$$\begin{aligned} U|0\rangle &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} a \\ c \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1+i}{2} \end{pmatrix}. \end{aligned}$$

Also, we have that

$$\begin{aligned} U|1\rangle &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} b \\ d \end{pmatrix} \\ &= \begin{pmatrix} \frac{1-i}{2} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}. \end{aligned}$$

It follows immediately that

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & 1-i \\ \frac{1+i}{2} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

## edx Problem 4

Suppose we have a one-qubit unitary  $U$  that maps  $|0\rangle$  to  $\frac{-3}{5}|0\rangle + \frac{4i}{5}|1\rangle$  and  $|+\rangle$  to  $\frac{-3-4i}{5\sqrt{2}}|0\rangle + \frac{3+4i}{5\sqrt{2}}|1\rangle$ .

We have that

$$\begin{aligned} U|0\rangle &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} a \\ c \end{pmatrix} \\ &= \begin{pmatrix} \frac{-3}{5} \\ \frac{4i}{5} \end{pmatrix}. \end{aligned}$$

As  $|+\rangle = (\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}})^T$ , we have that

$$\begin{aligned} U|+\rangle &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\sqrt{2}}(a+b) \\ \frac{1}{\sqrt{2}}(c+d) \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\sqrt{2}}(-\frac{3}{5}+b) \\ \frac{1}{\sqrt{2}}(\frac{4i}{5}+d) \end{pmatrix} \\ &= \begin{pmatrix} \frac{-3-4i}{5\sqrt{2}} \\ \frac{3+4i}{5\sqrt{2}} \end{pmatrix}. \end{aligned}$$

Now for  $b$  we have an equality

$$-\frac{3}{5} + b = \frac{-3-4i}{5},$$

which has solution  $b = -\frac{4i}{5}$ . For  $d$  we have an equality

$$\frac{4i}{5} + d = \frac{3+4i}{5},$$

which has solution  $d = \frac{3}{5}$ .

So the matrix in question is

$$U = \begin{pmatrix} -\frac{3}{5} & -\frac{4i}{5} \\ \frac{4i}{5} & \frac{3}{5} \end{pmatrix}.$$

## edx Problem 4 once again - Thanks, Tomi!

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

## edx Problem 5

$$\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle).$$

## edx Problem 6

True.

## edx Problem 5

What is  $ZX$  applied to  $|0\rangle$ ? We are given Pauli operators

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

so

$$\begin{aligned} ZX|0\rangle &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\ &= -|1\rangle. \end{aligned}$$

## edx Problem 6

What is  $ZX$  applied to  $H|0\rangle$ ?

From the exercises above we have that

$$ZX = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Also  $H|0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ , so

$$\begin{aligned} ZXH|0\rangle &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \\ &= |-\rangle. \end{aligned}$$

## edx Problem 7

We are given a qubit  $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$  and we know that  $|\alpha|^2 = \frac{2}{9}$  so  $|\beta|^2 = \frac{7}{9}$ . Also we know that

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

so

$$\begin{aligned} H|\psi\rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha + \beta \\ \alpha - \beta \end{pmatrix} \end{aligned}$$

Also we know that  $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ . Now the probability of measuring  $+$  is

$$\begin{aligned} \cos^2 \theta &= |\langle H\psi|+\rangle|^2 \\ &= \left| \frac{1}{\sqrt{2}}(\alpha + \beta)\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}(\alpha - \beta)\left(\frac{1}{\sqrt{2}}\right) \right|^2 \\ &= \left| \frac{1}{2}(\alpha + \beta) + \frac{1}{2}(\alpha - \beta) \right|^2 \\ &= |\alpha|^2 \\ &= \frac{2}{9}. \end{aligned}$$

## edx Problem 8

All pairs commute except CNOT and X applied to the target qubit.

## edx Problem 9

Apply the 3rd (last) circuit from above.

## edx Problem 10

(a) The resulting state ain't entangled. (b) First qubit:  $b|0\rangle + a|1\rangle$ , second qubit:  $|0\rangle$ .

## edx Problem 11

No circuit exists by no cloning theorem.

## edx Problem 12

I think in general case the correct alternative is  $[0, \frac{\pi}{2}]$ , yet unitary matrices preserve angles, so for any unitary  $U$   $U|\psi\rangle$   $U|\psi'\rangle$  have the same angle as  $|\psi\rangle$  and  $|\psi'\rangle$ .

## edx Problem 13

When Alice's outcome was 0, apply  $I$ . When Alice's outcome was 1, apply  $Z$ .

### QCE 5.1

Consider the following state vector:

$$|\psi\rangle = \sqrt{\frac{5}{6}}|0\rangle + \frac{1}{\sqrt{6}}|1\rangle.$$

(A) Is the state normalized? As

$$\left(\sqrt{\frac{5}{6}}\right)^2 + \left(\frac{1}{\sqrt{6}}\right)^2 = \frac{5}{6} + \frac{1}{6} = 1,$$

the state vector is normalized.

(B) What is the probability that the system is found to be in state  $|0\rangle$  if  $Z$  is measured?

After applying the  $Z$ -gate, we obtain

$$\begin{aligned} Z|\psi\rangle &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \sqrt{\frac{5}{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{\frac{5}{6}} \\ -\frac{1}{\sqrt{6}} \end{pmatrix}, \end{aligned}$$

which implies that the probability in question is  $\frac{5}{5}$ .

Lets do this the adult way: As  $P_0 = |0\rangle\langle 0|$  and  $P_1 = |1\rangle\langle 1|$ , we have that

$$P_0 = \begin{pmatrix} \langle 0|P_0|0\rangle & \langle 0|P_0|1\rangle \\ \langle 1|P_0|0\rangle & \langle 1|P_0|1\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$P_1 = \begin{pmatrix} \langle 0|P_1|0\rangle & \langle 0|P_1|1\rangle \\ \langle 1|P_1|0\rangle & \langle 1|P_1|1\rangle \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Next we need the density operator, which is given by

$$\begin{aligned}\rho &= |\psi\rangle\langle\psi| \\ &= \left( \sqrt{\frac{5}{6}}|0\rangle + \frac{1}{\sqrt{6}}|1\rangle \right) \left( \sqrt{\frac{5}{6}}\langle 0| + \frac{1}{\sqrt{6}}\langle 1| \right) \\ &= \frac{5}{6}|0\rangle\langle 0| + \frac{\sqrt{5}}{6}|0\rangle\langle 1| + \frac{\sqrt{5}}{6}|1\rangle\langle 0| + \frac{1}{6}|1\rangle\langle 1|,\end{aligned}$$

so the density matrix in the  $\{|0\rangle, |1\rangle\}$  basis is

$$\begin{pmatrix} \frac{5}{6} & \frac{\sqrt{5}}{6} \\ \frac{\sqrt{5}}{6} & \frac{1}{6} \end{pmatrix}.$$

Now the probability of finding the system in state  $|0\rangle$  is

$$p(0) = \text{Tr}(P_0\rho) = \text{Tr} \left[ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{5}{6} & \frac{\sqrt{5}}{6} \\ \frac{\sqrt{5}}{6} & \frac{1}{6} \end{pmatrix} \right] = \text{Tr} \begin{pmatrix} \frac{5}{6} & \frac{\sqrt{5}}{6} \\ 0 & 0 \end{pmatrix} = \frac{5}{6}.$$

(C) Write down the density operator. See above.

(D) Find the density matrix in the  $\{|0\rangle, |1\rangle\}$  basis, and show that  $\text{Tr}(\rho) = 1$ . See above.

## QCE 5.2

$$|\psi\rangle = \begin{pmatrix} \cos\theta \\ i\sin\theta \end{pmatrix},$$

so

$$|\langle\psi|\psi\rangle| = |(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)| = |\cos^2\theta + \sin^2\theta| = |1| = 1$$

and the state is normalized. Now

$$\begin{aligned}\rho &= |\psi\rangle\langle\psi| \\ &= (\cos\theta|0\rangle + i\sin\theta|1\rangle)(\cos\theta\langle 0| + i\sin\theta\langle 1|) \\ &= \cos^2\theta|0\rangle\langle 0| + i\sin\theta\cos\theta|0\rangle\langle 1| + i\sin\theta\cos\theta|1\rangle\langle 0| - \sin^2\theta|1\rangle\langle 1| \\ &= \begin{pmatrix} \cos^2\theta & i\sin\theta\cos\theta \\ i\sin\theta\cos\theta & \sin^2\theta \end{pmatrix}.\end{aligned}$$

Obviously,  $\text{Tr}(\rho) = \cos^2\theta + \sin^2\theta = 1$ . Also

$$\begin{aligned}\rho^\dagger &= \begin{pmatrix} \cos^2\theta & i\sin\theta\cos\theta \\ i\sin\theta\cos\theta & \sin^2\theta \end{pmatrix}^\dagger \\ &= \begin{pmatrix} \cos^2\theta & i\sin\theta\cos\theta \\ i\sin\theta\cos\theta & \sin^2\theta \end{pmatrix}^* \\ &= \begin{pmatrix} \cos^2\theta & -i\sin\theta\cos\theta \\ -i\sin\theta\cos\theta & \sin^2\theta \end{pmatrix} \\ &\neq \rho,\end{aligned}$$

so the operator is not Hermitian, and, thus, not a density operator.

### QCE 5.3

Let

$$|\psi\rangle = \sqrt{\frac{3}{7}}|0\rangle + \frac{2}{\sqrt{7}}|1\rangle.$$

(A) The density matrix in the  $\{|0\rangle, |1\rangle\}$  basis is

$$\begin{aligned}\rho &= |\psi\rangle\langle\psi| \\ &= \left(\sqrt{\frac{3}{7}}|0\rangle + \frac{2}{\sqrt{7}}|1\rangle\right)\left(\sqrt{\frac{3}{7}}\langle 0| + \frac{2}{\sqrt{7}}\langle 1|\right) \\ &= \frac{3}{7}|0\rangle\langle 0| + \frac{2\sqrt{3}}{7}|0\rangle\langle 1| + \frac{2\sqrt{3}}{7}|1\rangle\langle 0| + \frac{4}{7}|1\rangle\langle 1| \\ &= \begin{pmatrix} \frac{3}{7} & \frac{2\sqrt{3}}{7} \\ \frac{2\sqrt{3}}{7} & \frac{4}{7} \end{pmatrix}.\end{aligned}$$

Next we need  $\rho^2$  which is given by

$$\begin{aligned}&\begin{pmatrix} \frac{3}{7} & \frac{2\sqrt{3}}{7} \\ \frac{2\sqrt{3}}{7} & \frac{4}{7} \end{pmatrix} \begin{pmatrix} \frac{3}{7} & \frac{2\sqrt{3}}{7} \\ \frac{2\sqrt{3}}{7} & \frac{4}{7} \end{pmatrix} = \\ &\frac{1}{49} \begin{pmatrix} 3 & 2\sqrt{3} \\ 2\sqrt{3} & 4 \end{pmatrix} \begin{pmatrix} 3 & 2\sqrt{3} \\ 2\sqrt{3} & 4 \end{pmatrix} = \\ &\frac{1}{49} \begin{pmatrix} 9 + 12 & 6\sqrt{3} + 8\sqrt{3} \\ 6\sqrt{3} + 8\sqrt{3} & 12 + 16 \end{pmatrix} = \\ &\frac{1}{49} \begin{pmatrix} 21 & 14\sqrt{3} \\ 14\sqrt{3} & 28 \end{pmatrix}.\end{aligned}$$

As  $\text{Tr}(\rho^2) = 1$ , this is a pure state.

(C) Write down the density matrix in the  $\{|+\rangle, |-\rangle\}$  basis, show that  $\text{Tr}(\rho) = 1$  still holds, and determine if you still obtain the same result as in part (b).

Here we have

$$\begin{aligned}|\psi\rangle &= \sqrt{\frac{3}{7}}|0\rangle + \frac{2}{\sqrt{7}}|1\rangle \\ &= \sqrt{\frac{3}{7}}\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) + \frac{2}{\sqrt{7}}\frac{1}{\sqrt{2}}(|+\rangle - |-\rangle) \\ &= \sqrt{\frac{3}{14}}(|+\rangle + |-\rangle) + \sqrt{\frac{4}{14}}(|+\rangle - |-\rangle) \\ &= \left(\sqrt{\frac{3}{14}} + \sqrt{\frac{4}{14}}\right)|+\rangle + \left(\sqrt{\frac{3}{14}} - \sqrt{\frac{4}{14}}\right)|-\rangle\end{aligned}$$

Now

$$\begin{aligned}
\rho &= |\psi\rangle\langle\psi| \\
&= \left( \left( \sqrt{\frac{3}{14}} + \sqrt{\frac{4}{14}} \right) |+\rangle + \left( \sqrt{\frac{3}{14}} - \sqrt{\frac{4}{14}} \right) |-\rangle \right) \left( \left( \sqrt{\frac{3}{14}} + \sqrt{\frac{4}{14}} \right) \langle+| + \left( \sqrt{\frac{3}{14}} - \sqrt{\frac{4}{14}} \right) \langle-| \right) \\
&= \left( \sqrt{\frac{3}{14}} + \sqrt{\frac{4}{14}} \right)^2 |+\rangle\langle+| \\
&\quad + \left( \sqrt{\frac{3}{14}} + \sqrt{\frac{4}{14}} \right) \left( \sqrt{\frac{3}{14}} - \sqrt{\frac{4}{14}} \right) |+\rangle\langle-| \\
&\quad + \left( \sqrt{\frac{3}{14}} + \sqrt{\frac{4}{14}} \right) \left( \sqrt{\frac{3}{14}} - \sqrt{\frac{4}{14}} \right) |-\rangle\langle+| \\
&\quad + \left( \sqrt{\frac{3}{14}} - \sqrt{\frac{4}{14}} \right)^2 |-\rangle\langle-| \\
&= \left( \sqrt{\frac{3}{14}} + \sqrt{\frac{4}{14}} \right)^2 |+\rangle\langle+| \\
&\quad + \left( \frac{3}{14} - \frac{4}{14} \right) |+\rangle\langle-| \\
&\quad + \left( \frac{3}{14} - \frac{4}{14} \right) |-\rangle\langle+| \\
&\quad + \left( \sqrt{\frac{3}{14}} - \sqrt{\frac{4}{14}} \right)^2 |-\rangle\langle-| \\
&= \left( \sqrt{\frac{3}{14}} + \sqrt{\frac{4}{14}} \right)^2 |+\rangle\langle+| - \frac{1}{14} |+\rangle\langle-| - \frac{1}{14} |-\rangle\langle+| + \left( \sqrt{\frac{3}{14}} - \sqrt{\frac{4}{14}} \right)^2 |-\rangle\langle-| \\
&= \begin{pmatrix} \left( \sqrt{\frac{3}{14}} + \sqrt{\frac{4}{14}} \right)^2 & -\frac{1}{14} \\ -\frac{1}{14} & \left( \sqrt{\frac{3}{14}} - \sqrt{\frac{4}{14}} \right)^2 \end{pmatrix}.
\end{aligned}$$

Now

$$\begin{aligned}
Tr(\rho) &= \left( \sqrt{\frac{3}{14}} + \sqrt{\frac{4}{14}} \right)^2 + \left( \sqrt{\frac{3}{14}} - \sqrt{\frac{4}{14}} \right)^2 \\
&= \frac{3}{14} + \frac{4}{14} + 2\frac{\sqrt{12}}{14} + \frac{3}{14} + \frac{4}{14} - 2\frac{\sqrt{12}}{14} \\
&= 1.
\end{aligned}$$



Also

$$\begin{aligned}\rho^2 &= \begin{pmatrix} \left(\sqrt{\frac{3}{14}} + \sqrt{\frac{4}{14}}\right)^2 & -\frac{1}{14} \\ -\frac{1}{14} & \left(\sqrt{\frac{3}{14}} - \sqrt{\frac{4}{14}}\right)^2 \end{pmatrix} \begin{pmatrix} \left(\sqrt{\frac{3}{14}} + \sqrt{\frac{4}{14}}\right)^2 & -\frac{1}{14} \\ -\frac{1}{14} & \left(\sqrt{\frac{3}{14}} - \sqrt{\frac{4}{14}}\right)^2 \end{pmatrix} \\ &= \begin{pmatrix} \left(\sqrt{\frac{3}{14}} + \sqrt{\frac{4}{14}}\right)^4 + \frac{1}{196} & \dots \\ \dots & \left(\sqrt{\frac{3}{14}} - \sqrt{\frac{4}{14}}\right)^4 + \frac{1}{196} \end{pmatrix}.\end{aligned}$$

According to Wolframalpha,  $Tr(\rho^2) = 1$ , so this state is pure.

## QCE 5.4

Let

$$|\psi\rangle = \sqrt{\frac{2}{3}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle.$$

Now

$$\begin{aligned}\rho &= |\psi\rangle\langle\psi| \\ &= \left(\sqrt{\frac{2}{3}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle\right)\left(\sqrt{\frac{2}{3}}\langle 0| + \frac{1}{\sqrt{3}}\langle 1|\right) \\ &= \frac{2}{3}|0\rangle\langle 0| + \frac{\sqrt{2}}{3}|0\rangle\langle 1| + \frac{\sqrt{2}}{3}|1\rangle\langle 0| + \frac{1}{3}|1\rangle\langle 1| \\ &= \begin{pmatrix} \frac{2}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix}.\end{aligned}$$

It is obvious that  $Tr(\rho) = 1$ .

$$\begin{aligned}\rho^2 &= \begin{pmatrix} \frac{2}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} \frac{2}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix} \\ &= \begin{pmatrix} \frac{4}{9} + \frac{2}{9} & \frac{2\sqrt{2}}{9} + \frac{\sqrt{2}}{9} \\ \frac{2\sqrt{2}}{9} + \frac{\sqrt{2}}{9} & \frac{2}{9} + \frac{1}{9} \end{pmatrix} \\ &= \begin{pmatrix} \frac{6}{9} & \frac{3\sqrt{2}}{9} \\ \frac{3\sqrt{2}}{9} & \frac{3}{9} \end{pmatrix}.\end{aligned}$$

$Tr(\rho^2) = 1$  so the state is pure.

In order to compute  $\langle X \rangle$  we can fall down to equation  $\langle X \rangle = Tr(\rho X)$ . Now

$$\begin{aligned}\rho X &= \begin{pmatrix} \frac{2}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{\sqrt{2}}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{\sqrt{2}}{3} \end{pmatrix},\end{aligned}$$

$$\text{so } \langle X \rangle = Tr(\rho X) = \frac{2\sqrt{2}}{3}.$$

### QCE 5.5

Suppose that

$$\rho = \begin{pmatrix} \frac{1}{3} & \frac{i}{4} \\ -\frac{i}{4} & \frac{2}{3} \end{pmatrix}.$$

As  $Tr(\rho) = 1$  and  $\rho = \rho^\dagger$ ,  $\rho$  is a valid density matrix. Now

$$\begin{aligned}\rho^2 &= \begin{pmatrix} \frac{1}{3} & \frac{i}{4} \\ -\frac{i}{4} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{i}{4} \\ -\frac{i}{4} & \frac{2}{3} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{9} + \frac{1}{16} & \frac{i}{12} + \frac{2i}{12} \\ -\frac{i}{12} - \frac{2i}{12} & \frac{1}{16} + \frac{4}{9} \end{pmatrix} \\ &= \begin{pmatrix} \frac{25}{144} & \frac{3i}{72} \\ \frac{-3i}{72} & \frac{17}{36} \end{pmatrix}.\end{aligned}$$

Now, it is obvious that this is not a pure state as  $Tr(\rho^2) \neq 1$ .

### QCE 5.6

Let

$$\rho = \frac{1}{5} \begin{pmatrix} 3 & 1-i \\ 1+i & 2 \end{pmatrix}.$$

Now

$$\begin{aligned}\rho^2 &= \frac{1}{25} \begin{pmatrix} 3 & 1-i \\ 1+i & 2 \end{pmatrix} \begin{pmatrix} 3 & 1-i \\ 1+i & 2 \end{pmatrix} \\ &= \frac{1}{25} \begin{pmatrix} 9 + (1-i)(1+i) & 3 - 3i + 2 - 2i \\ 3 + 3i + 2 + 2i & (1-i)(1+i) + 4 \end{pmatrix} \\ &= \begin{pmatrix} 9 + 1 - i^2 & 5 - 5i \\ 5 + 5i & 1 - i^2 + 4 \end{pmatrix} \\ &= \begin{pmatrix} 11 & 5 - 5i \\ 5 + 5i & 6 \end{pmatrix},\end{aligned}$$

so the state is mixed as  $Tr(\rho^2) = \frac{17}{25} \neq 1$ . Next, let us compute  $\langle X \rangle, \langle Y \rangle, \langle Z \rangle$ :

$$\begin{aligned}\rho X &= \frac{1}{5} \begin{pmatrix} 3 & 1-i \\ 1+i & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} 1-i & 3 \\ 2 & 1+i \end{pmatrix},\end{aligned}$$

so  $\langle X \rangle = Tr(\rho X) = \frac{2}{5}$ .

$$\begin{aligned}\rho Y &= \frac{1}{5} \begin{pmatrix} 3 & 1-i \\ 1+i & 2 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} i-i^2 & -3i \\ 2i & -i-i^2 \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} 1+i & -3i \\ 2i & 1-i \end{pmatrix},\end{aligned}$$

so  $\langle Y \rangle = Tr(\rho Y) = \frac{2}{5}$ .

$$\begin{aligned}\rho Z &= \frac{1}{5} \begin{pmatrix} 3 & 1-i \\ 1+i & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} 3 & i-1 \\ 1+i & -2 \end{pmatrix},\end{aligned}$$

so  $\langle Z \rangle = Tr(\rho Z) = \frac{1}{5}$ .

## QCE 5.7

Let

$$|\psi\rangle = \frac{2}{\sqrt{5}} |0\rangle + \frac{1}{\sqrt{5}} |1\rangle.$$

Now

$$\begin{aligned}\rho_\psi &= |\psi\rangle \langle \psi| \\ &= \left( \frac{2}{\sqrt{5}} |0\rangle + \frac{1}{\sqrt{5}} |1\rangle \right) \left( \frac{2}{\sqrt{5}} \langle 0| + \frac{1}{\sqrt{5}} \langle 1| \right) \\ &= \frac{4}{5} |0\rangle \langle 0| + \frac{2}{5} |0\rangle \langle 1| + \frac{2}{5} |1\rangle \langle 0| + \frac{1}{5} |1\rangle \langle 1| \\ &= \begin{pmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix},\end{aligned}$$

so

$$\begin{aligned}\rho_\psi^2 &= \frac{1}{25} \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \\ &= \frac{1}{25} \begin{pmatrix} 20 & 10 \\ 10 & 5 \end{pmatrix},\end{aligned}$$

so  $Tr(\rho_\psi^2) = 1$  and the state is pure.

Let

$$|\phi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle.$$

Now

$$\begin{aligned}\rho_\phi &= |\phi\rangle\langle\phi| \\ &= \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)\left(\frac{1}{\sqrt{2}}\langle 0| + \frac{1}{\sqrt{2}}\langle 1|\right) \\ &= \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|0\rangle\langle 1| + \frac{1}{2}|1\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| \\ &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\end{aligned}$$

so

$$\begin{aligned}\rho_\phi^2 &= \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{4}\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix},\end{aligned}$$

so  $Tr(\rho_\phi^2) = 1$  and the state is pure.

The density operator for the ensemble is given by

$$\begin{aligned}\rho &= \frac{1}{4}\rho_\psi + \frac{3}{4}\rho_\phi \\ &= \frac{1}{4}\left(\frac{4}{5}|0\rangle\langle 0| + \frac{2}{5}|0\rangle\langle 1| + \frac{2}{5}|1\rangle\langle 0| + \frac{1}{5}|1\rangle\langle 1|\right) + \frac{3}{4}\left(\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|0\rangle\langle 1| + \frac{1}{2}|1\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|\right) \\ &= \left(\frac{1}{5} + \frac{3}{8}\right)|0\rangle\langle 0| + \left(\frac{1}{10} + \frac{3}{8}\right)|0\rangle\langle 1| + \left(\frac{1}{10} + \frac{3}{8}\right)|1\rangle\langle 0| + \left(\frac{1}{20} + \frac{3}{8}\right)|1\rangle\langle 1|,\end{aligned}$$

and it is easy to see that  $Tr(\rho) = 1$ .

Upon measurement,  $|\psi\rangle$  is found in state  $|0\rangle$  with probability  $4/5$  and in state  $|1\rangle$  with probability  $1/5$ . Upon measurement,  $|\phi\rangle$  is found in state  $|0\rangle$  with probability  $1/2$  and in state  $|1\rangle$  with probability  $1/2$ .

The probability of measuring  $|0\rangle$  within the ensemble is

$$\begin{aligned}p(0) &= \langle 0|\rho|0\rangle \\ &= \langle 0|\left(\left(\frac{1}{5} + \frac{3}{8}\right)|0\rangle\langle 0| + \left(\frac{1}{10} + \frac{3}{8}\right)|0\rangle\langle 1| + \left(\frac{1}{10} + \frac{3}{8}\right)|1\rangle\langle 0| + \left(\frac{1}{20} + \frac{3}{8}\right)|1\rangle\langle 1|\right)|0\rangle \\ &= \left(\frac{1}{5} + \frac{3}{8}\right) \\ &= \frac{23}{40},\end{aligned}$$

and the probability of measuring  $|+\rangle$  within the ensemble is

$$\begin{aligned}
p(1) &= \langle 1 | \rho | 1 \rangle \\
&= \langle 1 | \left( \left( \frac{1}{5} + \frac{3}{8} \right) |0\rangle \langle 0| + \left( \frac{1}{10} + \frac{3}{8} \right) |0\rangle \langle 1| + \left( \frac{1}{10} + \frac{3}{8} \right) |1\rangle \langle 0| + \left( \frac{1}{20} + \frac{3}{8} \right) |1\rangle \langle 1| \right) |1\rangle \\
&= \left( \frac{1}{20} + \frac{3}{8} \right) \\
&= \frac{68}{160} \\
&= \frac{34}{80} \\
&= \frac{17}{40}.
\end{aligned}$$

## QCE 5.8

Let

$$\begin{aligned}
|a\rangle &= \sqrt{\frac{2}{5}} |+\rangle - \sqrt{\frac{3}{5}} |-\rangle \\
&= \sqrt{\frac{2}{5}} \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) - \sqrt{\frac{3}{5}} \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \\
&= \frac{1}{\sqrt{5}} (|0\rangle + |1\rangle) - \sqrt{\frac{3}{10}} (|0\rangle - |1\rangle) \\
&= \left( \frac{1}{\sqrt{5}} - \sqrt{\frac{3}{10}} \right) |0\rangle + \left( \frac{1}{\sqrt{5}} + \sqrt{\frac{3}{10}} \right) |1\rangle \\
&= x_a |0\rangle + y_a |1\rangle
\end{aligned}$$

with probability 0.6 and

$$\begin{aligned}
|b\rangle &= \sqrt{\frac{5}{8}} |+\rangle + \sqrt{\frac{3}{8}} |-\rangle \\
&= \sqrt{\frac{5}{8}} \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) + \sqrt{\frac{3}{8}} \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \\
&= \sqrt{\frac{5}{16}} (|0\rangle + |1\rangle) + \sqrt{\frac{3}{16}} (|0\rangle - |1\rangle) \\
&= \left( \sqrt{\frac{5}{16}} + \sqrt{\frac{3}{16}} \right) |0\rangle + \left( \sqrt{\frac{5}{16}} - \sqrt{\frac{3}{16}} \right) |1\rangle \\
&= x_b |0\rangle + y_b |1\rangle
\end{aligned}$$

with probability 0.4. Now

$$\begin{aligned}\rho_a &= |a\rangle \langle a| \\ &= (x_a |0\rangle + y_a |1\rangle)(x_a \langle 0| + y_a \langle 1|) \\ &= x_a^2 |0\rangle \langle 0| + x_a y_a |0\rangle \langle 1| + y_a x_a |1\rangle \langle 0| + y_a^2 |1\rangle \langle 1|,\end{aligned}$$

and

$$\begin{aligned}\rho_b &= |b\rangle \langle b| \\ &= (x_b |0\rangle + y_b |1\rangle)(x_b \langle 0| + y_b \langle 1|) \\ &= x_b^2 |0\rangle \langle 0| + x_b y_b |0\rangle \langle 1| + y_b x_b |1\rangle \langle 0| + y_b^2 |1\rangle \langle 1|,\end{aligned}$$

so

$$\begin{aligned}p(0) &= \langle 0|\rho|0\rangle \\ &= \langle 0|\frac{3}{5}\rho_a + \frac{2}{5}\rho_b|0\rangle \\ &= \frac{3}{5}x_a^2 + \frac{2}{5}x_b^2 \\ &= \frac{3}{5}\left(\frac{1}{\sqrt{5}} - \sqrt{\frac{3}{10}}\right)^2 + \frac{2}{5}\left(\sqrt{\frac{5}{16}} + \sqrt{\frac{3}{16}}\right)^2 \\ &= \frac{3}{5}\left(\frac{1}{5} + \frac{3}{10} - \frac{2}{\sqrt{5}}\sqrt{\frac{3}{10}}\right)^2 + \frac{2}{5}\left(\frac{5}{16} + \frac{3}{16} + 2\sqrt{\frac{15}{256}}\right)^2 \\ &\approx 0.387.\end{aligned}$$

## QCE 5.9

Suppose that Alice and Bob share the entangled state

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}.$$

(A) Write down the density operator for this state. (B) Compute the density matrix. Verify that  $\text{Tr}(\rho) = 1$ , and determine if this is a pure state. (C) Find the density matrix that represents the reduced density operator as seen by Alice. (D) Show that the reduced density operator as seen by Alice is a completely mixed state.

Let

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}},$$

so

$$\begin{aligned}
\rho &= |\psi\rangle \langle \psi| \\
&= \left( \frac{|00\rangle + |11\rangle}{\sqrt{2}} \right) \left( \frac{\langle 00| + \langle 11|}{\sqrt{2}} \right) \\
&= \frac{1}{2} \left( |00\rangle \langle 00| + |00\rangle \langle 11| + |11\rangle \langle 00| + |11\rangle \langle 11| \right) \\
&= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.
\end{aligned}$$

Now  $Tr(\rho) = 1$  and  $\rho^2 = \rho$  so the state is pure.

$$\rho_a = \langle 0| (|\psi\rangle \langle \psi|) |0\rangle + \langle 1| (|\psi\rangle \langle \psi|) |1\rangle$$

Now

$$\begin{aligned}
\langle 0| (|\psi\rangle \langle \psi|) |0\rangle &= \langle 0| \left( \frac{|0\rangle \langle 0| + |0\rangle \langle 1| + |1\rangle \langle 0| + |1\rangle \langle 1|}{2} \right) |0\rangle \\
&= \frac{|0\rangle \langle 0|}{2}.
\end{aligned}$$

Also, it is easy to see that

$$\langle 1| (|\psi\rangle \langle \psi|) |1\rangle = \frac{|1\rangle \langle 1|}{2},$$

so the density operator for Alice is

$$\begin{aligned}
\rho_a &= \frac{|0\rangle \langle 0| + |1\rangle \langle 1|}{2} \\
&= \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \frac{1}{2} I,
\end{aligned}$$

so

$$\begin{aligned}
Tr(\rho_a^2) &= Tr(\rho_a^2) \\
&= Tr\left(\frac{1}{4} I^2\right) \\
&= Tr\left(\frac{1}{4} I\right) \\
&= \frac{1}{2}.
\end{aligned}$$

## QCE 5.10

Let

$$\rho = \begin{pmatrix} \frac{2}{5} & \frac{-i}{5} \\ \frac{i}{5} & \frac{3}{5} \end{pmatrix}.$$

Now as

$$\begin{aligned} \rho^\dagger &= \begin{pmatrix} \frac{2}{5} & \frac{i}{5} \\ \frac{-i}{5} & \frac{3}{5} \end{pmatrix}^T \\ &= \begin{pmatrix} \frac{2}{5} & \frac{-i}{5} \\ \frac{i}{5} & \frac{3}{5} \end{pmatrix} \\ &= \rho, \end{aligned}$$

so the matrix is Hermitian.

In order to verify that given values are eigenvalues, we need to verify that  $\det|\rho - \lambda I| = 0$ . Now

$$\begin{aligned} \det|\rho - \lambda_1 I| &= \det \left| \begin{pmatrix} \frac{2}{5} & \frac{-i}{5} \\ \frac{i}{5} & \frac{3}{5} \end{pmatrix} - \lambda_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| \\ &= \det \left| \begin{pmatrix} \frac{2}{5} & \frac{-i}{5} \\ \frac{i}{5} & \frac{3}{5} \end{pmatrix} - \begin{pmatrix} \frac{20+\sqrt{41}}{40} & 0 \\ 0 & \frac{20+\sqrt{41}}{40} \end{pmatrix} \right| \\ &= \det \left| \begin{array}{cc} \frac{16}{40} - \frac{20+\sqrt{41}}{40} & -\frac{i}{8} \\ \frac{i}{8} & \frac{24}{40} - \frac{20+\sqrt{41}}{40} \end{array} \right| \\ &= \det \left| \begin{array}{cc} -\frac{4}{40} - \frac{\sqrt{41}}{40} & -\frac{5i}{40} \\ \frac{5i}{40} & \frac{4}{40} - \frac{\sqrt{41}}{40} \end{array} \right| \\ &= \left( -\frac{4}{40} - \frac{\sqrt{41}}{40} \right) \left( \frac{4}{40} - \frac{\sqrt{41}}{40} \right) - \left( -\frac{25i^2}{40^2} \right) \\ &= -\frac{16}{40^2} + \frac{4\sqrt{41}}{40^2} - \frac{4\sqrt{41}}{40^2} + \frac{41}{40^2} - \left( \frac{25}{40^2} \right) \\ &= -\frac{16}{40^2} + \frac{41}{40^2} - \frac{25}{40^2} \\ &= 0, \end{aligned}$$



so  $\lambda_1$  is an eigenvalue of  $\rho$ . Next let us check  $\lambda_2$ :

$$\begin{aligned}
\det |\rho - \lambda_2 I| &= \det \begin{vmatrix} \frac{16}{40} - \frac{20-\sqrt{41}}{40} & -\frac{i}{8} \\ \frac{i}{8} & \frac{24}{40} - \frac{20-\sqrt{41}}{40} \end{vmatrix} \\
&= \det \begin{vmatrix} -\frac{4}{40} + \frac{\sqrt{41}}{40} & -\frac{5i}{40} \\ \frac{5i}{40} & \frac{4}{40} + \frac{\sqrt{41}}{40} \end{vmatrix} \\
&= \left( -\frac{4}{40} + \frac{\sqrt{41}}{40} \right) + \left( \frac{4}{40} + \frac{\sqrt{41}}{40} \right) - \left( \frac{-25i^2}{40^2} \right) \\
&= -\frac{16}{40^2} + \frac{41}{40^2} - \frac{25}{40^2} \\
&= 0,
\end{aligned}$$

so  $\lambda_2$  is an eigenvalue as well. As  $\rho$  is Hermitian,  $Tr(\rho) = 1$  and has nonnegative eigenvalues, it is a valid density matrix.

Next, let us kick with the Bloch vector:

$$\begin{aligned}
S_x &= Tr(X\rho) \\
&= Tr \left[ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{5} & \frac{-i}{5} \\ \frac{i}{5} & \frac{3}{5} \end{pmatrix} \right] \\
&= Tr \begin{bmatrix} \frac{i}{5} & \frac{3}{5} \\ \frac{2i}{5} & -\frac{i}{8} \end{bmatrix} \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
S_y &= Tr(Y\rho) \\
&= Tr \left[ \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{5} & \frac{-i}{5} \\ \frac{i}{5} & \frac{3}{5} \end{pmatrix} \right] \\
&= Tr \begin{bmatrix} \frac{1}{5} & -\frac{3i}{5} \\ \frac{8i}{5} & \frac{1}{8} \end{bmatrix} \\
&= \frac{1}{4},
\end{aligned}$$

$$\begin{aligned}
S_z &= Tr(Z\rho) \\
&= Tr \left[ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{2}{5} & \frac{-i}{5} \\ \frac{i}{5} & \frac{3}{5} \end{pmatrix} \right] \\
&= Tr \begin{bmatrix} \frac{2}{5} & -\frac{i}{5} \\ -\frac{i}{8} & -\frac{3}{5} \end{bmatrix} \\
&= -\frac{1}{5},
\end{aligned}$$

so the Bloch vector  $\vec{S} = \frac{1}{4}\hat{y} - \frac{1}{5}\hat{z}$  and

$$\begin{aligned}
 |\vec{S}| &= \sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{1}{5}\right)^2} \\
 &= \sqrt{\frac{1}{16} + \frac{1}{25}} \\
 &= \sqrt{\frac{41}{400}} \\
 &= \frac{\sqrt{41}}{20} \\
 &\approx 0.32 < 1,
 \end{aligned}$$

so the state is mixed.