

# Kvanttilaskenta, kevät 2015 – Viikko 2

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## edx Problem 1

No, its 0 for  $|+\rangle$  and  $|-\rangle$  are orthogonal to each other.

## edx Problem 2

Yes.

## edx Problem 3

Yes, but the implicit multiplication operator is in fact the tensor product  $\otimes$ .

## edx Problem 4

False.

## edx Problem 5

False.

## edx Problem 6

True.

## edx Problem 7

$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ .  $|\psi\rangle = \frac{3}{5}|0\rangle - \frac{4}{5}|1\rangle$ . Now we wish to compute

$$\begin{aligned}\langle\psi, +\rangle &= \frac{3}{5} \frac{1}{\sqrt{2}} - \frac{4}{5} \frac{1}{\sqrt{2}} \\ &= -\frac{1}{\sqrt{2}} \frac{1}{5} \\ &= -\frac{1}{5\sqrt{2}}.\end{aligned}$$

## edx Problem 8

$$-\frac{1}{5\sqrt{2}}|+\rangle + \frac{7}{5\sqrt{2}}|-\rangle.$$

## edx Problem 9

Forget the first standard basis measurement as it is not relevant here. At second measurement the chance of getting  $|u\rangle$  is  $\cos^2\theta$ , where  $\theta$  is the angle between  $|u\rangle$  and  $|\phi\rangle$ . Since

$$\cos\theta = \langle\phi|u\rangle = ab + \sqrt{1-a^2}\sqrt{1-b^2},$$

the desired probability is

$$(ab + \sqrt{1-a^2}\sqrt{1-b^2})^2.$$

## edx Problem 10

The third topmost alternative seems suspicious as it mixes sign and bit bases.

## edx Problem 11

$$\left(\frac{\sqrt{2}}{\sqrt{3}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle\right) \otimes \left(\frac{1}{\sqrt{3}}|0\rangle + \frac{\sqrt{2}}{\sqrt{3}}|1\rangle\right) = \frac{\sqrt{2}}{3}|00\rangle + \frac{2}{3}|01\rangle + \frac{1}{3}|10\rangle + \frac{\sqrt{2}}{3}|11\rangle$$

## edx Problem 12

$$\begin{aligned}(a|0\rangle + b|1\rangle)(c|0\rangle + d|1\rangle) &= ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle \\ &= \frac{1}{2\sqrt{2}}|00\rangle - \frac{1}{2\sqrt{2}}|01\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|10\rangle - \frac{\sqrt{3}}{2\sqrt{2}}|11\rangle\end{aligned}$$

So we have that

$$\begin{aligned}ac &= \frac{1}{2\sqrt{2}}, \\ad &= -\frac{1}{2\sqrt{2}}, \\bc &= \frac{\sqrt{3}}{2\sqrt{2}}, \\bd &= -\frac{\sqrt{3}}{2\sqrt{2}}.\end{aligned}$$

The way to factorize is to assign  $a = \frac{1}{2}$ ,  $b = \frac{\sqrt{3}}{2}$ ,  $c = \frac{1}{\sqrt{2}}$ ,  $d = -\frac{1}{\sqrt{2}}$ , so  $|a| = \frac{1}{2}$ .

### edx Problem 13

(a)

The probability is  $\frac{1}{2}$ .

(b)

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle.$$

### edx Problem 14

$$\left(\frac{4}{5}\right)^2 + \left(-\frac{2}{5}\right)^2 = \frac{16}{25} + \frac{4}{25} = \frac{20}{25}.$$

### edx Problem 15

$$|0+\rangle, |0-\rangle, |1+\rangle, |1-\rangle.$$

### edx Problem 16

There is no way to entangle two qubits by a partial measurement.

### edx Problem 17

As

$$|0\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle, |1\rangle = \frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle,$$

we have that

$$\begin{aligned}
|\psi\rangle &= \frac{1}{\sqrt{2}} |0\rangle + \frac{e^{i\theta}}{\sqrt{2}} |1\rangle \\
&= \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{2}} |+\rangle + \frac{1}{\sqrt{2}} |-\rangle \right) + \frac{e^{i\theta}}{\sqrt{2}} \left( \frac{1}{\sqrt{2}} |+\rangle - \frac{1}{\sqrt{2}} |-\rangle \right) \\
&= \frac{1}{2} |+\rangle + \frac{1}{2} |-\rangle + \frac{e^{i\theta}}{2} |+\rangle - \frac{e^{i\theta}}{2} |-\rangle \\
&= \frac{1+e^{i\theta}}{2} |+\rangle + \frac{1-e^{i\theta}}{2} |-\rangle.
\end{aligned}$$

## edx Problem 18

$$\frac{1 + \cos \theta}{2}.$$

### QCE 3.1

$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$ .  $X = |0\rangle \langle 1| + |1\rangle \langle 0|$ .  $Y = -i |0\rangle \langle 1| + i |1\rangle \langle 0|$ . Now

$$\begin{aligned}
X |\psi\rangle &= (|0\rangle \langle 1| + |1\rangle \langle 0|)(\alpha |0\rangle + \beta |1\rangle) \\
&= \alpha(|0\rangle \langle 1| + |1\rangle \langle 0|) |0\rangle + \beta(|0\rangle \langle 1| + |1\rangle \langle 0|) |1\rangle \\
&= \alpha(|0\rangle \langle 1|0\rangle + |1\rangle \langle 0|0\rangle) + \beta(|0\rangle \langle 1|1\rangle + |1\rangle \langle 0|1\rangle) \\
&= \alpha |0\rangle \times 0 + \alpha |1\rangle \times 1 + \beta |0\rangle \times 1 + \beta |1\rangle \times 0 \\
&= \alpha |1\rangle + \beta |0\rangle.
\end{aligned}$$

$$\begin{aligned}
Y |\psi\rangle &= (-i |0\rangle \langle 1| + i |1\rangle \langle 0|)(\alpha |0\rangle + \beta |1\rangle) \\
&= \alpha(-i |0\rangle \langle 1| + i |1\rangle \langle 0|) |0\rangle + \beta(-i |0\rangle \langle 1| + i |1\rangle \langle 0|) |1\rangle \\
&= \alpha(-i |0\rangle \langle 1|0\rangle + i |1\rangle \langle 0|0\rangle) + \beta(-i |0\rangle \langle 1|1\rangle + i |1\rangle \langle 0|1\rangle) \\
&= \alpha i |1\rangle + \beta(-i) |0\rangle \\
&= \alpha i |1\rangle - \beta i |0\rangle.
\end{aligned}$$

### QCE 3.2

Suppose we have a qubit

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

Now

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \times \alpha + 1 \times \beta \\ 1 \times \alpha + 0 \times \beta \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}.$$

### QCE 3.3

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

Now

$$\begin{aligned} X|+\rangle &= X \frac{|0\rangle + |1\rangle}{\sqrt{2}} \\ &= X \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \\ &= \begin{pmatrix} 1 \times \frac{1}{\sqrt{2}} + 0 \times \frac{1}{\sqrt{2}} \\ 0 \times \frac{1}{\sqrt{2}} - 1 \times \frac{1}{\sqrt{2}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \\ &= \frac{|0\rangle - |1\rangle}{\sqrt{2}} \\ &= |-\rangle. \end{aligned}$$

Also

$$\begin{aligned} X|-\rangle &= X \frac{|0\rangle - |1\rangle}{\sqrt{2}} \\ &= X \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \\ &= \begin{pmatrix} 1 \times \frac{1}{\sqrt{2}} + 0 \times -\frac{1}{\sqrt{2}} \\ 0 \times \frac{1}{\sqrt{2}} - 1 \times \left(-\frac{1}{\sqrt{2}}\right) \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \\ &= \frac{|0\rangle + |1\rangle}{\sqrt{2}} \\ &= |+\rangle. \end{aligned}$$

### QCE 3.4

The operator is  $\hat{A} = i|1\rangle\langle 1| + \frac{\sqrt{3}}{2}|1\rangle\langle 2| + 2|2\rangle\langle 1| - |2\rangle\langle 3|$ . Now,

$$\begin{aligned}\hat{A}^\dagger &= -i\langle 1|1\rangle + \frac{\sqrt{3}}{2}\langle 1|2\rangle + 2\langle 2|1\rangle - \langle 2|3\rangle \\ &= -i.\end{aligned}$$

### QCE 3.5

The X operator is given by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and we wish to find  $\lambda, (a \ b)^T$  such that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}.$$

Now

$$\begin{pmatrix} b \\ a \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix},$$

so  $b = \lambda a$  and  $a = \lambda b$ . If  $\lambda = 1$ , the corresponding eigenvector is  $(1, 1)^T$ . If  $\lambda = 0$ , the eigenvector is  $(0, 0)^T$ .

### QCE 3.6

As the matrix of Y-operator is

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

it is evident that its trace is 0.

### QCE 3.7

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 4 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a + 2c \\ 3b + 4c \\ a + 2c \end{pmatrix} = \begin{pmatrix} \lambda a \\ \lambda b \\ \lambda c \end{pmatrix}.$$

Its clear that  $a = b$ . Also  $3b + 4c = \lambda b$ , which implies  $4c = (\lambda - 3)b$ , and  $c = \frac{1}{4}(\lambda - 3)b$ . It follows that  $\lambda \in \mathbb{R}$ .

### QCE 3.8

Suppose

$$A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix}, B = \begin{pmatrix} b_{1,1} & \cdots & b_{1,n} \\ \vdots & \ddots & \vdots \\ b_{n,1} & \cdots & b_{n,n} \end{pmatrix}.$$

Now, it is easy to see that

$$\begin{aligned} Tr(A + B) &= (a_{1,1} + b_{1,1}) + \cdots + (a_{n,n} + b_{n,n}) \\ &= (a_{1,1} + \cdots + a_{n,n}) + (b_{1,1} + \cdots + b_{n,n}) \\ &= Tr(A) + Tr(B). \end{aligned}$$

Also

$$\begin{aligned} Tr(\lambda A) &= \lambda a_{1,1} + \cdots + \lambda a_{n,n} \\ &= \lambda Tr(A). \end{aligned}$$

Also-also

$$\begin{aligned} Tr(AB) &= a_{1,1}b_{1,1} + \cdots + a_{n,n}b_{n,n} \\ &= b_{1,1}a_{1,1} + \cdots + b_{n,n}a_{n,n} \\ &= Tr(BA), \end{aligned}$$

as real multiplication commutes.

### QCE 3.9

$$\begin{aligned} P_+ &= \frac{1}{2}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|). \\ P_- &= \frac{1}{2}(|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|). \end{aligned}$$

Now

$$P_+ - P_- = |0\rangle\langle 1| + |1\rangle\langle 0| = X.$$

### QCE 3.10

We have

$$\begin{aligned} P_0 &= |0\rangle\langle 0| = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} (1 \ 0 \ 0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ P_1 &= |1\rangle\langle 1| = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} (0 \ 1 \ 0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

$$P_2 = |2\rangle \langle 2| = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} (0 \ 0 \ 1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now

$$Pr(0) = |P_0 |\psi\rangle|^2 = \left| \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{i}{\sqrt{2}} \end{pmatrix} \right|^2 = \left| \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \right|^2 = \left( \frac{1}{2} \right)^2 = \frac{1}{4},$$

$$Pr(1) = |P_1 |\psi\rangle|^2 = \left| \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{i}{\sqrt{2}} \end{pmatrix} \right|^2 = \left| \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \right|^2 = \left( \frac{1}{2} \right)^2 = \frac{1}{4},$$

$$Pr(2) = |P_2 |\psi\rangle|^2 = \left| \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{i}{\sqrt{2}} \end{pmatrix} \right|^2 = \left| \begin{pmatrix} 0 \\ 0 \\ -\frac{i}{\sqrt{2}} \end{pmatrix} \right|^2 = \left( \frac{1}{\sqrt{2}} \right)^2 = \frac{1}{2}.$$

### QCE 3.11

The matrix representations of the Pauli operators are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

**0.1**  $[\sigma_2, \sigma_3] = 2i\sigma_1$

$$\sigma_2\sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

$$\sigma_3\sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

Now the commutator is

$$[\sigma_2, \sigma_3] = \sigma_2\sigma_3 - \sigma_3\sigma_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix} = 2i\sigma_1.$$

**0.2**  $[\sigma_3, \sigma_1] = 2i\sigma_2$

$$\sigma_3\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\sigma_1\sigma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Now the commutator is

$$[\sigma_3, \sigma_1] = \sigma_3\sigma_1 - \sigma_1\sigma_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} = 2i\sigma_2.$$



### QCE 3.12

We need to show that if  $i \neq j$ , then  $\{\sigma_i, \sigma_j\} = 0$ . First of all,  $\{\sigma_i, \sigma_j\} = \sigma_i \sigma_j + \sigma_j \sigma_i$ . By the definition of anticommutator we have that

$$\{\sigma_i, \sigma_j\} = \sigma_i \sigma_j + \sigma_j \sigma_i = \sigma_j \sigma_i + \sigma_i \sigma_j = \{\sigma_j, \sigma_i\},$$

because the real + commutes. Now we don't need to compute all permutations of Pauli operators, but only combinations will suffice, so we need to calculate

- $\{\sigma_1, \sigma_2\}$ ,
- $\{\sigma_1, \sigma_3\}$ ,
- $\{\sigma_2, \sigma_3\}$ .

So

$$\begin{aligned} \{\sigma_1, \sigma_2\} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \\ &= 0. \end{aligned}$$

$$\begin{aligned} \{\sigma_1, \sigma_3\} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= 0. \end{aligned}$$

The last, but not least

$$\begin{aligned} \{\sigma_2, \sigma_3\} &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \\ &= 0. \end{aligned}$$

### QCE 4.1

The basis for  $H$  is

$$\begin{aligned} |w_1\rangle &= |0\rangle \otimes |0\rangle, \\ |w_2\rangle &= |0\rangle \otimes |1\rangle, \\ |w_3\rangle &= |1\rangle \otimes |0\rangle, \\ |w_4\rangle &= |1\rangle \otimes |1\rangle. \end{aligned}$$

First we show that  $|w_i\rangle$  has unit length:

$$\begin{aligned}\langle w_1|w_1\rangle &= (\langle 0| \otimes \langle 0|)(|0\rangle \otimes |0\rangle) \\ &= \langle 0|0\rangle \langle 0|0\rangle \\ &= 1 \times 1 \\ &= 1.\end{aligned}$$

$$\begin{aligned}\langle w_2|w_2\rangle &= (\langle 0| \otimes \langle 1|)(|0\rangle \otimes |1\rangle) \\ &= \langle 0|0\rangle \langle 1|1\rangle \\ &= 1 \times 1 \\ &= 1.\end{aligned}$$

$$\begin{aligned}\langle w_3|w_3\rangle &= (\langle 1| \otimes \langle 0|)(|1\rangle \otimes |0\rangle) \\ &= \langle 1|1\rangle \langle 0|0\rangle \\ &= 1 \times 1 \\ &= 1.\end{aligned}$$

$$\begin{aligned}\langle w_4|w_4\rangle &= (\langle 1| \otimes \langle 1|)(|1\rangle \otimes |1\rangle) \\ &= \langle 1|1\rangle \langle 1|1\rangle \\ &= 1 \times 1 \\ &= 1.\end{aligned}$$

Next we need to check the orthogonality of  $|w_i\rangle$ :

$$\begin{aligned}\langle w_1|w_2\rangle &= (\langle 0| \otimes \langle 0|)(|0\rangle \otimes |1\rangle) \\ &= \langle 0|0\rangle \langle 0|1\rangle \\ &= 1 \times 0 \\ &= 0,\end{aligned}$$

$$\begin{aligned}\langle w_1|w_3\rangle &= (\langle 0| \otimes \langle 0|)(|1\rangle \otimes |0\rangle) \\ &= \langle 0|1\rangle \langle 0|0\rangle \\ &= 0 \times 1 \\ &= 0,\end{aligned}$$

$$\begin{aligned}\langle w_1|w_4\rangle &= (\langle 0| \otimes \langle 0|)(|1\rangle \otimes |1\rangle) \\ &= \langle 0|1\rangle \langle 0|1\rangle \\ &= 0 \times 0 \\ &= 0,\end{aligned}$$

$$\begin{aligned}
\langle w_2|w_3\rangle &= (\langle 0|\otimes\langle 1|)(|1\rangle\otimes|0\rangle) \\
&= \langle 0|1\rangle\langle 1|0\rangle \\
&= 0\times 0 \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
\langle w_2|w_4\rangle &= (\langle 0|\otimes\langle 1|)(|1\rangle\otimes|1\rangle) \\
&= \langle 0|1\rangle\langle 1|1\rangle \\
&= 0\times 1 \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
\langle w_3|w_4\rangle &= (\langle 1|\otimes\langle 0|)(|1\rangle\otimes|1\rangle) \\
&= \langle 1|1\rangle\langle 0|1\rangle \\
&= 1\times 0 \\
&= 0.
\end{aligned}$$

Now we see that  $\{w_1, w_2, w_3, w_4\}$  is an orthonormal basis.

## QCE 4.2

In previous exercise we have seen that  $\langle w_3|w_4\rangle = 0$ . Now, let us verify that  $\langle w_4|w_3\rangle = 0$ :

$$\begin{aligned}
\langle w_4|w_3\rangle &= (\langle 1|\otimes\langle 1|)(|1\rangle\otimes|0\rangle) \\
&= \langle 1|1\rangle\langle 1|0\rangle \\
&= 1\times 0 \\
&= 0.
\end{aligned}$$

## QCE 4.3

$\langle a|b\rangle = 1/2$ ,  $\langle c|d\rangle = 3/4$ .

$$\begin{aligned}
\langle\psi|\phi\rangle &= \langle a|\otimes\langle c|\otimes|b\rangle\otimes|d\rangle\rangle \\
&= (\langle a|\otimes\langle c|)(|b\rangle\otimes|d\rangle) \\
&= \langle a|b\rangle\langle c|d\rangle \\
&= \frac{1}{2}\times\frac{3}{4} \\
&= \frac{3}{8}.
\end{aligned}$$

#### QCE 4.4

$$|\psi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |\phi\rangle = \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}.$$

Now the tensor product  $\psi \otimes \phi$  is

$$\begin{aligned} \left( \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \right) \otimes \left( \frac{1}{2} |0\rangle + \frac{\sqrt{3}}{2} |1\rangle \right) &= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2\sqrt{2}} \\ \frac{\sqrt{3}}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \\ \frac{\sqrt{3}}{2\sqrt{2}} \end{pmatrix} \\ &= \frac{1}{2\sqrt{2}} |0\rangle |0\rangle + \frac{\sqrt{3}}{2\sqrt{2}} |0\rangle |1\rangle + \frac{1}{2\sqrt{2}} |1\rangle |0\rangle + \frac{\sqrt{3}}{2\sqrt{2}} |1\rangle |1\rangle. \end{aligned}$$

#### QCE 4.5

$$\begin{aligned} |\psi\rangle &= \frac{1}{2} (|0\rangle |0\rangle - |0\rangle |1\rangle - |1\rangle |0\rangle + |1\rangle |1\rangle) \\ &= \frac{|0\rangle - |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}. \end{aligned}$$

#### QCE 4.6

$$\begin{aligned} |\psi\rangle &= \frac{|0\rangle |0\rangle + |1\rangle |1\rangle}{\sqrt{2}} \\ &= \begin{pmatrix} \alpha |0\rangle + \beta |1\rangle \end{pmatrix} \otimes \begin{pmatrix} \gamma |0\rangle + \delta |1\rangle \end{pmatrix} \\ &= \alpha\gamma |0\rangle |0\rangle + \alpha\delta |0\rangle |1\rangle + \beta\gamma |1\rangle |0\rangle + \beta\delta |1\rangle |1\rangle. \end{aligned}$$

Now we have

$$\begin{aligned} \alpha\gamma &= \beta\delta \\ &= \frac{1}{\sqrt{2}}, \\ \alpha\delta &= \beta\gamma \\ &= 0. \end{aligned}$$

Obviously, either  $\alpha = 0$  and/or  $\delta = 0$ . In any case, either  $\alpha\gamma = 0$  and/or  $\beta\delta = 0$ , when they should be non-zero, so it is not possible to write  $|\psi\rangle$  as a product state.

## QCE 4.7

Let

$$|\psi\rangle = \frac{|0\rangle|1\rangle - |1\rangle|0\rangle}{\sqrt{2}} = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}.$$

Since

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$X \otimes Y = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}.$$

Now,

$$\begin{aligned} X \otimes Y |\psi\rangle &= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ -\frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \\ 0 \end{pmatrix} \\ &= -\frac{-i|0\rangle|1\rangle - i|1\rangle|0\rangle}{\sqrt{2}}. \end{aligned}$$

## QCE 4.8

We are asked to show that  $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$ . Now,

$$\begin{aligned}
(A \otimes B)^\dagger &= \left( \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix} \otimes \begin{pmatrix} b_{1,1} & \cdots & b_{1,n} \\ \vdots & \ddots & \vdots \\ b_{n,1} & \cdots & b_{n,n} \end{pmatrix} \right)^\dagger \\
&= \begin{pmatrix} a_{1,1}b_{1,1} & \cdots & a_{1,1}b_{1,n} & \cdots & a_{1,n}b_{1,1} & \cdots & a_{1,n}b_{1,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{1,1}b_{n,1} & \cdots & a_{1,1}b_{n,n} & \cdots & a_{1,n}b_{n,1} & \cdots & a_{1,n}b_{n,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n,1}b_{1,1} & \cdots & a_{n,1}b_{1,n} & \cdots & a_{n,n}b_{1,1} & \cdots & a_{n,n}b_{1,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n,1}b_{n,1} & \cdots & a_{n,1}b_{n,n} & \cdots & a_{n,n}b_{n,1} & \cdots & a_{n,n}b_{n,n} \end{pmatrix}^\dagger \\
&= \begin{pmatrix} (a_{1,1}b_{1,1})^* & \cdots & (a_{1,1}b_{1,n})^* & \cdots & (a_{1,n}b_{1,1})^* & \cdots & (a_{1,n}b_{1,n})^* \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ (a_{1,1}b_{n,1})^* & \cdots & (a_{1,1}b_{n,n})^* & \cdots & (a_{1,n}b_{n,1})^* & \cdots & (a_{1,n}b_{n,n})^* \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ (a_{n,1}b_{1,1})^* & \cdots & (a_{n,1}b_{1,n})^* & \cdots & (a_{n,n}b_{1,1})^* & \cdots & (a_{n,n}b_{1,n})^* \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ (a_{n,1}b_{n,1})^* & \cdots & (a_{n,1}b_{n,n})^* & \cdots & (a_{n,n}b_{n,1})^* & \cdots & (a_{n,n}b_{n,n})^* \end{pmatrix} \\
&= \begin{pmatrix} a_{1,1}^*b_{1,1}^* & \cdots & a_{1,1}^*b_{1,n}^* & \cdots & a_{1,n}^*b_{1,1}^* & \cdots & a_{1,n}^*b_{1,n}^* \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{1,1}^*b_{n,1}^* & \cdots & a_{1,1}^*b_{n,n}^* & \cdots & a_{1,n}^*b_{n,1}^* & \cdots & a_{1,n}^*b_{n,n}^* \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n,1}^*b_{1,1}^* & \cdots & a_{n,1}^*b_{1,n}^* & \cdots & a_{n,n}^*b_{1,1}^* & \cdots & a_{n,n}^*b_{1,n}^* \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n,1}^*b_{n,1}^* & \cdots & a_{n,1}^*b_{n,n}^* & \cdots & a_{n,n}^*b_{n,1}^* & \cdots & a_{n,n}^*b_{n,n}^* \end{pmatrix} \\
&= \begin{pmatrix} a_{1,1}^* & \cdots & a_{1,n}^* \\ \vdots & \ddots & \vdots \\ a_{n,1}^* & \cdots & a_{n,n}^* \end{pmatrix} \otimes \begin{pmatrix} b_{1,1}^* & \cdots & b_{1,n}^* \\ \vdots & \ddots & \vdots \\ b_{n,1}^* & \cdots & b_{n,n}^* \end{pmatrix} \\
&= \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix}^\dagger \otimes \begin{pmatrix} b_{1,1} & \cdots & b_{1,n} \\ \vdots & \ddots & \vdots \\ b_{n,1} & \cdots & b_{n,n} \end{pmatrix}^\dagger \\
&= A^\dagger \otimes B^\dagger.
\end{aligned}$$

### QCE 4.9

Let

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

As

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

we have that

$$I \otimes Y = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix},$$

so

$$\begin{aligned} I \otimes Y |\psi\rangle &= \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ \frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \\ 0 \end{pmatrix}. \end{aligned}$$

### QCE 4.10

We have

$$\begin{aligned} X \otimes Y &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$