Kvanttilaskenta, kevät 2015 – Viikko 2

Rodion "rodde" Efremov

January 28, 2015

edx Problem 1

Let

$$U = \begin{pmatrix} -2i & 5i \\ 5 & 1-i \end{pmatrix}.$$

Now

$$U^\dagger = \begin{pmatrix} 2i & 5 \\ -5i & 1+i \end{pmatrix}.$$

edx Problem 2

All unitary matrices are self-inverse.

edx Problem 3

We are given a U that maps $|0\rangle$ to $\frac{1}{\sqrt{2}}|0\rangle + \frac{1+i}{2}|1\rangle$ and $|1\rangle$ to $\frac{1-i}{2}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$. So we have that

$$\begin{aligned} U|0\rangle &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} a \\ c \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1+i}{2} \end{pmatrix}. \end{aligned}$$

Also, we have that

$$\begin{aligned} U|1\rangle &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} b \\ d \end{pmatrix} \\ &= \begin{pmatrix} 1-i \\ -\frac{1}{\sqrt{2}} \end{pmatrix}. \end{aligned}$$

It follows immediately that

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & 1-i \\ \frac{1+i}{2} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

edx Problem 4

Suppose we have a one-qubit unitary U that maps $|0\rangle$ to $\frac{-3}{5}|0\rangle + \frac{4i}{5}|1\rangle$ and $|+\rangle$ to $\frac{-3-4i}{5\sqrt{2}}|0\rangle + \frac{3+4i}{5\sqrt{2}}|1\rangle$.

We have that

$$\begin{aligned} U|0\rangle &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} a \\ c \end{pmatrix} \\ &= \begin{pmatrix} \frac{-3}{5} \\ \frac{4i}{5} \end{pmatrix}. \end{aligned}$$

As $|+\rangle = (\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}})^T$, we have that

$$\begin{aligned} U|+\rangle &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\sqrt{2}}(a+b) \\ \frac{1}{\sqrt{2}}(c+d) \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\sqrt{2}}(-\frac{3}{5}+b) \\ \frac{1}{\sqrt{2}}(\frac{4i}{5}+d) \end{pmatrix} \\ &= \begin{pmatrix} \frac{-3-4i}{5\sqrt{2}} \\ \frac{3+4i}{5\sqrt{2}} \end{pmatrix}. \end{aligned}$$

Now for b we have an equality

$$-\frac{3}{5} + b = \frac{-3-4i}{5},$$

which has solution $b = -\frac{4i}{5}$. For d we have an equality

$$\frac{4i}{5} + d = \frac{3+4i}{5},$$

which has solution $d = \frac{3}{5}$.

So the matrix in question is

$$U = \begin{pmatrix} -\frac{3}{5} & -\frac{4i}{5} \\ \frac{4i}{5} & \frac{3}{5} \end{pmatrix}.$$

edx Problem 4 once again - Thanks, Tomi!

$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

edx Problem 5

$\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$.

edx Problem 6

True.

edx Problem 5

What is ZX applied to $|0\rangle$? We are given Pauli operators

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

so

$$\begin{aligned} ZX\,|0\rangle &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\ &= -\,|1\rangle\,. \end{aligned}$$

edx Problem 6

What is ZX applied to $H\,|0\rangle$?

From the exerices above we have that

$$ZX = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Also $H\,|0\rangle = \frac{1}{\sqrt{2}}\,|0\rangle + \frac{1}{\sqrt{2}}\,|1\rangle$, so

$$\begin{aligned} ZXH\,|0\rangle &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \\ &= |-\rangle\,. \end{aligned}$$

edx Problem 7

We are given a qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ and we know that $|\alpha|^2 = \frac{2}{9}$ so $|\beta|^2 = \frac{7}{9}$. Also we know that

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

so

$$\begin{aligned} H|\psi\rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha + \beta \\ \alpha - \beta \end{pmatrix} \end{aligned}$$

Also we know that $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Now the probability of measuring + is

$$\begin{aligned} \cos^2\theta &= |\langle H\psi|+\rangle|^2 \\ &= \left| \frac{1}{\sqrt{2}}(\alpha+\beta)(\frac{1}{\sqrt{2}}) + \frac{1}{\sqrt{2}}(\alpha-\beta)(\frac{1}{\sqrt{2}}) \right|^2 \\ &= \left| \frac{1}{2}(\alpha+\beta) + \frac{1}{2}(\alpha-\beta) \right|^2 \\ &= |\alpha|^2 \\ &= \frac{2}{9}. \end{aligned}$$

edx Problem 8

All pairs commute except CNOT and X applied to the target qubit.

edx Problem 9

Apply the 3rd (last) circuit from above.

edx Problem 10

(a) The resulting state ain't entangled. (b) First qubit: $b|0\rangle + a|1\rangle$, second cubit: $|0\rangle$.

edx Problem 11

No circuit exists by no cloning theorem.

edx Problem 12

I think in general case the correct alternative is $[0, \frac{\pi}{2}]$, yet unitary matrices preserve angles, so for any unitary U $U|\psi\rangle$ $U|\psi'\rangle$ have the same angle as $|\psi\rangle$ and $|\psi'\rangle$.

edx Problem 13

When Alice's outcome was 0, apply I. When Alice's outcome was 1, apply Z.

QCE 5.1

Consider the following state vector:

$$|\psi\rangle = \sqrt{\frac{5}{6}}\,|0\rangle + \frac{1}{\sqrt{6}}\,|1\rangle\,.$$

(A) Is the state normalized? As

$$\left(\sqrt{\frac{5}{6}}\right)^2 + \left(\frac{1}{\sqrt{6}}\right)^2 = \frac{5}{6} + \frac{1}{6} = 1,$$

the state vector is normalized. (B) After applying the Z-gate, we obtain

$$\begin{aligned} Z\,|\psi\rangle &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \sqrt{\frac{5}{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix} \\ &= \end{aligned}$$