Kvanttilaskenta, kevät 2015 – Viikko 4

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edx Problem 1

False. By definition of reversibility we should have x at the output of R_f.

edx Problem 2

False. The quantum circuit should modify each α_x.

edx Problem 3

True. Straight from the slides.

edx Problem 4

$$\begin{aligned} H^{\otimes 2}\frac{1}{\sqrt{2}}|00\rangle &= \frac{1}{\sqrt{2}}|{+}{+}\rangle \\ &= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \\ &= \frac{1}{2\sqrt{2}}\Bigg(|00\rangle + |01\rangle + |10\rangle + |11\rangle\Bigg). \end{aligned}$$

Also

$$\begin{aligned} H^{\otimes 2}\frac{1}{\sqrt{2}}|11\rangle &= \frac{1}{\sqrt{2}}|{-}{-}\rangle \\ &= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right) \\ &= \frac{1}{2\sqrt{2}}\Bigg(|00\rangle - |01\rangle - |10\rangle + |11\rangle\Bigg), \end{aligned}$$

so

$$\begin{aligned}
H^{\otimes 2}\psi &= H^{\otimes 2}\left(\frac{1}{\sqrt{2}}\left|00\right\rangle + \frac{1}{\sqrt{2}}\left|11\right\rangle\right) \\
&= \frac{1}{2\sqrt{2}}\left(2\left|00\right\rangle + 2\left|11\right\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left(\left|00\right\rangle + \left|11\right\rangle\right) \\
&= \psi.
\end{aligned}$$

edx Problem 5

Let

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle).$$

$$\begin{aligned}
H^{\otimes 2}\frac{1}{\sqrt{2}}\left|01\right\rangle &= \frac{1}{\sqrt{2}}\left|+-\right\rangle \\
&= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\left|0\right\rangle + \frac{1}{\sqrt{2}}\left|1\right\rangle\right)\left(\frac{1}{\sqrt{2}}\left|0\right\rangle - \frac{1}{\sqrt{2}}\left|1\right\rangle\right) \\
&= \frac{1}{2\sqrt{2}}\left(\left|00\right\rangle - \left|01\right\rangle + \left|10\right\rangle - \left|11\right\rangle\right).
\end{aligned}$$

$$\begin{aligned}
H^{\otimes 2}\frac{1}{\sqrt{2}}\left|10\right\rangle &= \frac{1}{\sqrt{2}}\left|-+\right\rangle \\
&= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\left|0\right\rangle - \frac{1}{\sqrt{2}}\left|1\right\rangle\right)\left(\frac{1}{\sqrt{2}}\left|0\right\rangle + \frac{1}{\sqrt{2}}\left|1\right\rangle\right) \\
&= \frac{1}{2\sqrt{2}}\left(\left|00\right\rangle + \left|01\right\rangle - \left|10\right\rangle - \left|11\right\rangle\right).
\end{aligned}$$

Now we see that $H^{\otimes 2}\left|\psi\right\rangle$ is

$$\begin{aligned}
\frac{1}{2\sqrt{2}}\left(\left|00\right\rangle - \left|01\right\rangle + \left|10\right\rangle - \left|11\right\rangle\right) + \frac{1}{2\sqrt{2}}\left(\left|00\right\rangle + \left|01\right\rangle - \left|10\right\rangle - \left|11\right\rangle\right) &= \frac{1}{2\sqrt{2}}(2\left|00\right\rangle - 2\left|11\right\rangle) \\
&= \frac{1}{\sqrt{2}}(\left|00\right\rangle - \left|11\right\rangle).
\end{aligned}$$

edx Problem 6

Yes. Both may yield $\left|00\right\rangle$ with probability $\frac{1}{2}$ and $\left|11\right\rangle$ with probability $\frac{1}{2}$.

edx Problem 7

Apply circuit A and then D.

edx Problem 8

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

edx Problem 9

Suppose

$$H^{\otimes 3} |\psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle).$$

Since $H^{\otimes 3}$ is reversible, if we apply it again to $H^{\otimes 3} |\psi\rangle$, we will obtain $|\psi\rangle$. Let us calculate that ket by ket:

$$\begin{aligned} H^{\otimes 3} \frac{1}{\sqrt{2}} |000\rangle &= \frac{1}{\sqrt{2}} |+++\rangle \\ &= \frac{1}{4} \Bigg(|0\rangle + |1\rangle \Bigg)^3 \\ &= \frac{1}{4} \Bigg(|0\rangle + |1\rangle \Bigg) \Bigg(|00\rangle + |01\rangle + |10\rangle + |11\rangle \Bigg) \\ &= \frac{1}{4} \Bigg(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle \Bigg). \end{aligned}$$

$$\begin{aligned} H^{\otimes 3} \frac{1}{\sqrt{2}} |111\rangle &= \frac{1}{\sqrt{2}} |---\rangle \\ &= \frac{1}{4} \Bigg(|0\rangle - |1\rangle \Bigg)^3 \\ &= \frac{1}{4} \Bigg(|0\rangle - |1\rangle \Bigg) \Bigg(|00\rangle - |01\rangle - |10\rangle + |11\rangle \Bigg) \\ &= \frac{1}{4} \Bigg(|000\rangle - |001\rangle - |010\rangle + |011\rangle - |100\rangle + |101\rangle + |110\rangle - |111\rangle \Bigg). \end{aligned}$$

Now

$$\begin{aligned}
|\psi\rangle &= H^{\otimes 3} H^{\otimes 3} |\psi\rangle \\
&= H^{\otimes 3} \frac{1}{\sqrt{2}} |000\rangle + H^{\otimes 3} \frac{1}{\sqrt{2}} |11\rangle \\
&= \frac{1}{4}(2\,|000\rangle + 2\,|011\rangle + 2\,|101\rangle + 2\,|110\rangle) \\
&= \frac{1}{2}(|000\rangle + |011\rangle + |101\rangle + |110\rangle).
\end{aligned}$$

edx Problem 10

(a)

$$\frac{1}{2^{n-1}}.$$

(b) We see a uniformly random string $y \in \{0,1\}^n$.

edx Problem 11

$$\frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{x \cdot f(x)} |x\rangle\, |f(x)\rangle\,.$$

edx Problem 12

$$\frac{1}{\sqrt{2^n}} \sum_{x} |x\rangle\, |0101\rangle\,.$$

edx Problem 13

1111.

edx Problem 14

Suppose Alice starts with two qubits in the Bell state $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$ and teleports these qubits to Bob by applying the quantum teleportation protocol to each qubit separately.

As we are speaking about teleportation, Bob sees the same state and receives exactly 2 bits of information as there is only 4 Bell states.

QCE 7.1

$$\vec{\sigma} = \sigma_x \hat{x} + \sigma_y \hat{y} + \sigma_z \hat{z}, \qquad \vec{n} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}.$$

Now

$$\begin{aligned}\vec{\sigma}\cdot\vec{n} &= (\sigma_x \hat{x} + \sigma_y \hat{y} + \sigma_z \hat{z})(\sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}) \\ &= \end{aligned}$$

QCE 7.2

Let

$$|\psi\rangle = \frac{|0\rangle|1\rangle - |1\rangle|0\rangle}{\sqrt{2}}.$$

Now we have

$$|0\rangle = \frac{|+\rangle + i|-\rangle}{\sqrt{2}},\ |1\rangle = \frac{|+\rangle - i|-\rangle}{\sqrt{2}},$$

so

$$\begin{aligned}|\psi\rangle &= \frac{|0\rangle|1\rangle - |1\rangle|0\rangle}{\sqrt{2}} \\ &= \frac{1}{2\sqrt{2}}\Big((|+\rangle + i|-\rangle)(|+\rangle - i|-\rangle) - (|+\rangle - i|-\rangle)(|+\rangle + i|-\rangle)\Big) \\ &= \frac{1}{2\sqrt{2}}\Big(|{+}{+}\rangle - i|{+}{-}\rangle + i|{-}{+}\rangle + |{-}{-}\rangle + |{+}{+}\rangle + i|{+}{-}\rangle - i|{-}{+}\rangle + |{-}{-}\rangle\Big) \\ &= \frac{1}{2\sqrt{2}}(2|{+}{+}\rangle + 2|{-}{-}\rangle) \\ &= \frac{1}{\sqrt{2}}(|{+}{+}\rangle + |{-}{-}\rangle).\end{aligned}$$

QCE 7.3

Let

$$|\beta_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix},\ |\beta_{01}\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}.$$

Also we know that

$$Z = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix},$$

so

$$
\begin{aligned}
Z \otimes Z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\end{aligned}
$$

Now

$$
\begin{aligned}
Z \otimes Z \left|\beta_{00}\right\rangle &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\
&= \beta_{00} \\
&= (-1)^y \left|\beta_{xy}\right\rangle,
\end{aligned}
$$

where $y = 0$ and $x = 0$. Also

$$
\begin{aligned}
Z \otimes Z \left|\beta_{01}\right\rangle &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ -1 \\ 0 \end{pmatrix} \\
&= -\beta_{01} \\
&= (-1)^y \left|\beta_{xy}\right\rangle,
\end{aligned}
$$

where $y = 1$ and $x = 0$.

QCE 7.4

Show that $X \otimes X \left|\beta_{xy}\right\rangle = (-1)^x \left|\beta_{xy}\right\rangle$. From the book we know that

$$
\left|\beta_{xy}\right\rangle = \frac{\left|0y\right\rangle + (-1)^x \left|1\bar{y}\right\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{y} \\ y \\ (-1)^x y \\ (-1)^x \bar{y} \end{pmatrix}.
$$

Also

$$
\begin{aligned}
X \otimes X &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},
\end{aligned}
$$

so

$$
\begin{aligned}
X \otimes X \left|\beta_{xy}\right\rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{y} \\ y \\ (-1)^x y \\ (-1)^x \bar{y} \end{pmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} (-1)^x \bar{y} \\ (-1)^x y \\ y \\ \bar{y} \end{pmatrix} \\
&= \frac{(-1)^x \left|0y\right\rangle + \left|1\bar{y}\right\rangle}{\sqrt{2}} \\
&= (-1)^x \beta_{xy}.
\end{aligned}
$$

QCE 7.5

Show that $Y \otimes Y \left|\beta_{xy}\right\rangle = (-1)^{x+y} \left|\beta_{xy}\right\rangle$. From the book we know that

$$
\left|\beta_{xy}\right\rangle = \frac{\left|0y\right\rangle + (-1)^x \left|1\bar{y}\right\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{y} \\ y \\ (-1)^x y \\ (-1)^x \bar{y} \end{pmatrix}.
$$

Also

$$
\begin{aligned}
Y \otimes Y &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},
\end{aligned}
$$

so

$$
\begin{aligned}
Y \otimes Y \left|\beta_{xy}\right\rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{y} \\ y \\ (-1)^x y \\ (-1)^x \bar{y} \end{pmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} -(-1)^x \bar{y} \\ (-1)^x y \\ y \\ -\bar{y} \end{pmatrix} \\
&= (-1)^{x+\bar{y}} \left|\beta_{xy}\right\rangle .
\end{aligned}
$$

The book has a typo...

QCE 7.6

Show that $X \otimes X$ commutes with $Z \otimes Z$. They commute if and only if $X \otimes XZ \otimes Z = X \otimes XZ \otimes Z$:

$$
\begin{aligned}
X \otimes X &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\otimes 2} \\
&= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} .
\end{aligned}
$$

$$
\begin{aligned}
Z \otimes Z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{\otimes 2} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .
\end{aligned}
$$

Now

$$
\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} .
$$

Also

$$
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} ,
$$

which proves that the two matrices commute.

QCE 7.7

Consider the eigenvectors in Example 7.4. Show that $[H_I, \vec{\sigma_A} \cdot \vec{\sigma_B}] = 0$, and hence show that the eigenvectors of the Hamiltonian are eigenvectors of the $\vec{\sigma_A} \cdot \vec{\sigma_B}$ operator. In particular, show that $\vec{\sigma_A} \cdot \vec{\sigma_B} |\phi_i\rangle = |\phi_i\rangle$ for $i = 1, 2, 3$ and $\vec{\sigma_A} \cdot \vec{\sigma_B} |\phi_4\rangle = -3 |\phi_4\rangle$.

From the Example 7.4 we have that

$$H_I = \frac{\mu^2}{r^3} \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}, \ \vec{\sigma_A} \cdot \vec{\sigma_B} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Condition $[H_I, \vec{\sigma_A} \cdot \vec{\sigma_B}] = 0$ means that the two matrices commute and that is the case as a routine calculation may show. Wolframalpha tells me that the eigenvectors of H_I are

$$\begin{pmatrix} 0 \\ -1 \\ -1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix},$$

and that is not in accord with the eigenvectors of $\vec{\sigma_A} \cdot \vec{\sigma_B}$, thank you, book. :(

Now

$$\begin{aligned} \vec{\sigma_A} \cdot \vec{\sigma_B} |\phi_1\rangle &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \\ &= |\phi_1\rangle, \end{aligned}$$

$$\begin{aligned} \vec{\sigma_A} \cdot \vec{\sigma_B} |\phi_2\rangle &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\ &= |\phi_2\rangle, \end{aligned}$$

$$
\begin{aligned}
\vec{\sigma_A} \cdot \vec{\sigma_B} \left|\phi_3\right\rangle &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
&= \left|\phi_3\right\rangle,
\end{aligned}
$$

$$
\begin{aligned}
\vec{\sigma_A} \cdot \vec{\sigma_B} \left|\phi_4\right\rangle &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -3 \\ 3 \\ 0 \end{pmatrix} \\
&= -3 \left|\phi_4\right\rangle.
\end{aligned}
$$

QCE 7,8

Is the state $X \otimes Z \left|\beta_{00}\right\rangle$ entangled?

$$
\begin{aligned}
X \otimes Z \left|00\right\rangle &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \\
&= \frac{-\left|01\right\rangle + \left|10\right\rangle}{\sqrt{2}} \\
&= -\left|\beta_{11}\right\rangle,
\end{aligned}
$$

so the state is entangled but is not a Bell state.