

Kvanttilaskenta, kevät 2015 – Viikko 2

Rodion “rodde” Efremov

January 29, 2015

edx Problem 1

Let

$$U = \begin{pmatrix} -2i & 5i \\ 5 & 1-i \end{pmatrix}.$$

Now

$$U^\dagger = \begin{pmatrix} 2i & 5 \\ -5i & 1+i \end{pmatrix}.$$

edx Problem 2

All unitary matrices are self-inverse.

edx Problem 3

We are given a U that maps $|0\rangle$ to $\frac{1}{\sqrt{2}}|0\rangle + \frac{1+i}{2}|1\rangle$ and $|1\rangle$ to $\frac{1-i}{2}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$. So we have that

$$\begin{aligned} U|0\rangle &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} a \\ c \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1+i}{2} \end{pmatrix}. \end{aligned}$$

Also, we have that

$$\begin{aligned} U|1\rangle &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} b \\ d \end{pmatrix} \\ &= \begin{pmatrix} \frac{1-i}{2} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}. \end{aligned}$$

It follows immediately that

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & 1-i \\ \frac{1+i}{2} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

edx Problem 4

Suppose we have a one-qubit unitary U that maps $|0\rangle$ to $\frac{-3}{5}|0\rangle + \frac{4i}{5}|1\rangle$ and $|+\rangle$ to $\frac{-3-4i}{5\sqrt{2}}|0\rangle + \frac{3+4i}{5\sqrt{2}}|1\rangle$.

We have that

$$\begin{aligned} U|0\rangle &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} a \\ c \end{pmatrix} \\ &= \begin{pmatrix} \frac{-3}{5} \\ \frac{4i}{5} \end{pmatrix}. \end{aligned}$$

As $|+\rangle = (\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}})^T$, we have that

$$\begin{aligned} U|+\rangle &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\sqrt{2}}(a+b) \\ \frac{1}{\sqrt{2}}(c+d) \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\sqrt{2}}(-\frac{3}{5}+b) \\ \frac{1}{\sqrt{2}}(\frac{4i}{5}+d) \end{pmatrix} \\ &= \begin{pmatrix} \frac{-3-4i}{5\sqrt{2}} \\ \frac{3+4i}{5\sqrt{2}} \end{pmatrix}. \end{aligned}$$

Now for b we have an equality

$$-\frac{3}{5} + b = \frac{-3-4i}{5},$$

which has solution $b = -\frac{4i}{5}$. For d we have an equality

$$\frac{4i}{5} + d = \frac{3+4i}{5},$$

which has solution $d = \frac{3}{5}$.

So the matrix in question is

$$U = \begin{pmatrix} -\frac{3}{5} & -\frac{4i}{5} \\ \frac{4i}{5} & \frac{3}{5} \end{pmatrix}.$$

edx Problem 4 once again - Thanks, Tomi!

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

edx Problem 5

$$\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle).$$

edx Problem 6

True.

edx Problem 5

What is ZX applied to $|0\rangle$? We are given Pauli operators

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

so

$$\begin{aligned} ZX|0\rangle &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\ &= -|1\rangle. \end{aligned}$$

edx Problem 6

What is ZX applied to $H|0\rangle$?

From the exercises above we have that

$$ZX = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Also $H|0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$, so

$$\begin{aligned} ZXH|0\rangle &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \\ &= |-\rangle. \end{aligned}$$

edx Problem 7

We are given a qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ and we know that $|\alpha|^2 = \frac{2}{9}$ so $|\beta|^2 = \frac{7}{9}$. Also we know that

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

so

$$\begin{aligned} H|\psi\rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha + \beta \\ \alpha - \beta \end{pmatrix} \end{aligned}$$

Also we know that $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Now the probability of measuring $+$ is

$$\begin{aligned} \cos^2 \theta &= |\langle H\psi|+\rangle|^2 \\ &= \left| \frac{1}{\sqrt{2}}(\alpha + \beta)\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}(\alpha - \beta)\left(\frac{1}{\sqrt{2}}\right) \right|^2 \\ &= \left| \frac{1}{2}(\alpha + \beta) + \frac{1}{2}(\alpha - \beta) \right|^2 \\ &= |\alpha|^2 \\ &= \frac{2}{9}. \end{aligned}$$

edx Problem 8

All pairs commute except CNOT and X applied to the target qubit.

edx Problem 9

Apply the 3rd (last) circuit from above.

edx Problem 10

(a) The resulting state ain't entangled. (b) First qubit: $b|0\rangle + a|1\rangle$, second qubit: $|0\rangle$.

edx Problem 11

No circuit exists by no cloning theorem.

edx Problem 12

I think in general case the correct alternative is $[0, \frac{\pi}{2}]$, yet unitary matrices preserve angles, so for any unitary U $U|\psi\rangle$ $U|\psi'\rangle$ have the same angle as $|\psi\rangle$ and $|\psi'\rangle$.

edx Problem 13

When Alice's outcome was 0, apply I . When Alice's outcome was 1, apply Z .

QCE 5.1

Consider the following state vector:

$$|\psi\rangle = \sqrt{\frac{5}{6}}|0\rangle + \frac{1}{\sqrt{6}}|1\rangle.$$

(A) Is the state normalized? As

$$\left(\sqrt{\frac{5}{6}}\right)^2 + \left(\frac{1}{\sqrt{6}}\right)^2 = \frac{5}{6} + \frac{1}{6} = 1,$$

the state vector is normalized.

(B) What is the probability that the system is found to be in state $|0\rangle$ if Z is measured?

After applying the Z -gate, we obtain

$$\begin{aligned} Z|\psi\rangle &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \sqrt{\frac{5}{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{\frac{5}{6}} \\ -\frac{1}{\sqrt{6}} \end{pmatrix}, \end{aligned}$$

which implies that the probability in question is $\frac{5}{5}$.

Lets do this the adult way: As $P_0 = |0\rangle\langle 0|$ and $P_1 = |1\rangle\langle 1|$, we have that

$$P_0 = \begin{pmatrix} \langle 0|P_0|0\rangle & \langle 0|P_0|1\rangle \\ \langle 1|P_0|0\rangle & \langle 1|P_0|1\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$P_1 = \begin{pmatrix} \langle 0|P_1|0\rangle & \langle 0|P_1|1\rangle \\ \langle 1|P_1|0\rangle & \langle 1|P_1|1\rangle \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Next we need the density operator, which is given by

$$\begin{aligned}\rho &= |\psi\rangle\langle\psi| \\ &= \left(\sqrt{\frac{5}{6}}|0\rangle + \frac{1}{\sqrt{6}}|1\rangle \right) \left(\sqrt{\frac{5}{6}}\langle 0| + \frac{1}{\sqrt{6}}\langle 1| \right) \\ &= \frac{5}{6}|0\rangle\langle 0| + \frac{\sqrt{5}}{6}|0\rangle\langle 1| + \frac{\sqrt{5}}{6}|1\rangle\langle 0| + \frac{1}{6}|1\rangle\langle 1|,\end{aligned}$$

so the density matrix in the $\{|0\rangle, |1\rangle\}$ basis is

$$\begin{pmatrix} \frac{5}{6} & \frac{\sqrt{5}}{6} \\ \frac{\sqrt{5}}{6} & \frac{1}{6} \end{pmatrix}.$$

Now the probability of finding the system in state $|0\rangle$ is

$$p(0) = \text{Tr}(P_0\rho) = \text{Tr} \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{5}{6} & \frac{\sqrt{5}}{6} \\ \frac{\sqrt{5}}{6} & \frac{1}{6} \end{pmatrix} \right] = \text{Tr} \begin{pmatrix} \frac{5}{6} & \frac{\sqrt{5}}{6} \\ 0 & 0 \end{pmatrix} = \frac{5}{6}.$$

(C) Write down the density operator. See above.

(D) Find the density matrix in the $\{|0\rangle, |1\rangle\}$ basis, and show that $\text{Tr}(\rho) = 1$. See above.

QCE 5.2

$$|\psi\rangle = \begin{pmatrix} \cos\theta \\ i\sin\theta \end{pmatrix},$$

so

$$|\langle\psi|\psi\rangle| = |(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)| = |\cos^2\theta + \sin^2\theta| = |1| = 1$$

and the state is normalized. Now

$$\begin{aligned}\rho &= |\psi\rangle\langle\psi| \\ &= (\cos\theta|0\rangle + i\sin\theta|1\rangle)(\cos\theta\langle 0| + i\sin\theta\langle 1|) \\ &= \cos^2\theta|0\rangle\langle 0| + i\sin\theta\cos\theta|0\rangle\langle 1| + i\sin\theta\cos\theta|1\rangle\langle 0| - \sin^2\theta|1\rangle\langle 1| \\ &= \begin{pmatrix} \cos^2\theta & i\sin\theta\cos\theta \\ i\sin\theta\cos\theta & \sin^2\theta \end{pmatrix}.\end{aligned}$$

Obviously, $\text{Tr}(\rho) = \cos^2\theta + \sin^2\theta = 1$. Also

$$\begin{aligned}\rho^\dagger &= \begin{pmatrix} \cos^2\theta & i\sin\theta\cos\theta \\ i\sin\theta\cos\theta & \sin^2\theta \end{pmatrix}^\dagger \\ &= \begin{pmatrix} \cos^2\theta & i\sin\theta\cos\theta \\ i\sin\theta\cos\theta & \sin^2\theta \end{pmatrix}^* \\ &= \begin{pmatrix} \cos^2\theta & -i\sin\theta\cos\theta \\ -i\sin\theta\cos\theta & \sin^2\theta \end{pmatrix} \\ &\neq \rho,\end{aligned}$$

so the operator is not Hermitian, and, thus, not a density operator.

QCE 5.3

Let

$$|\psi\rangle = \sqrt{\frac{3}{7}}|0\rangle + \frac{2}{\sqrt{7}}|1\rangle.$$

(A) The density matrix in the $\{|0\rangle, |1\rangle\}$ basis is

$$\begin{aligned}\rho &= |\psi\rangle\langle\psi| \\ &= \left(\sqrt{\frac{3}{7}}|0\rangle + \frac{2}{\sqrt{7}}|1\rangle\right)\left(\sqrt{\frac{3}{7}}\langle 0| + \frac{2}{\sqrt{7}}\langle 1|\right) \\ &= \frac{3}{7}|0\rangle\langle 0| + \frac{2\sqrt{3}}{7}|0\rangle\langle 1| + \frac{2\sqrt{3}}{7}|1\rangle\langle 0| + \frac{4}{7}|1\rangle\langle 1| \\ &= \begin{pmatrix} \frac{3}{7} & \frac{2\sqrt{3}}{7} \\ \frac{2\sqrt{3}}{7} & \frac{4}{7} \end{pmatrix}.\end{aligned}$$

Next we need ρ^2 which is given by

$$\begin{aligned}&\begin{pmatrix} \frac{3}{7} & \frac{2\sqrt{3}}{7} \\ \frac{2\sqrt{3}}{7} & \frac{4}{7} \end{pmatrix} \begin{pmatrix} \frac{3}{7} & \frac{2\sqrt{3}}{7} \\ \frac{2\sqrt{3}}{7} & \frac{4}{7} \end{pmatrix} = \\ &\frac{1}{49} \begin{pmatrix} 3 & 2\sqrt{3} \\ 2\sqrt{3} & 4 \end{pmatrix} \begin{pmatrix} 3 & 2\sqrt{3} \\ 2\sqrt{3} & 4 \end{pmatrix} = \\ &\frac{1}{49} \begin{pmatrix} 9+12 & 6\sqrt{3}+8\sqrt{3} \\ 6\sqrt{3}+8\sqrt{3} & 12+16 \end{pmatrix} = \\ &\frac{1}{49} \begin{pmatrix} 21 & 14\sqrt{3} \\ 14\sqrt{3} & 28 \end{pmatrix}.\end{aligned}$$

As $\text{Tr}(\rho^2) = 1$, this is a pure state.

(C) Write down the density matrix in the $\{|+\rangle, |-\rangle\}$ basis, show that $\text{Tr}(\rho) = 1$ still holds, and determine if you still obtain the same result as in part (b).

Here we have

$$\begin{aligned}|\psi\rangle &= \sqrt{\frac{3}{7}}|0\rangle + \frac{2}{\sqrt{7}}|1\rangle \\ &= \sqrt{\frac{3}{7}}\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) + \frac{2}{\sqrt{7}}\frac{1}{\sqrt{2}}(|+\rangle - |-\rangle) \\ &= \sqrt{\frac{3}{14}}(|+\rangle + |-\rangle) + \sqrt{\frac{4}{14}}(|+\rangle - |-\rangle) \\ &= \left(\sqrt{\frac{3}{14}} + \sqrt{\frac{4}{14}}\right)|+\rangle + \left(\sqrt{\frac{3}{14}} - \sqrt{\frac{4}{14}}\right)|-\rangle\end{aligned}$$

Now

$$\begin{aligned}
\rho &= |\psi\rangle\langle\psi| \\
&= \left(\left(\sqrt{\frac{3}{14}} + \sqrt{\frac{4}{14}} \right) |+\rangle + \left(\sqrt{\frac{3}{14}} - \sqrt{\frac{4}{14}} \right) |-\rangle \right) \left(\left(\sqrt{\frac{3}{14}} + \sqrt{\frac{4}{14}} \right) \langle+| + \left(\sqrt{\frac{3}{14}} - \sqrt{\frac{4}{14}} \right) \langle-| \right) \\
&= \left(\sqrt{\frac{3}{14}} + \sqrt{\frac{4}{14}} \right)^2 |+\rangle\langle+| \\
&\quad + \left(\sqrt{\frac{3}{14}} + \sqrt{\frac{4}{14}} \right) \left(\sqrt{\frac{3}{14}} - \sqrt{\frac{4}{14}} \right) |+\rangle\langle-| \\
&\quad + \left(\sqrt{\frac{3}{14}} + \sqrt{\frac{4}{14}} \right) \left(\sqrt{\frac{3}{14}} - \sqrt{\frac{4}{14}} \right) |-\rangle\langle+| \\
&\quad + \left(\sqrt{\frac{3}{14}} - \sqrt{\frac{4}{14}} \right)^2 |-\rangle\langle-| \\
&= \left(\sqrt{\frac{3}{14}} + \sqrt{\frac{4}{14}} \right)^2 |+\rangle\langle+| \\
&\quad + \left(\frac{3}{14} - \frac{4}{14} \right) |+\rangle\langle-| \\
&\quad + \left(\frac{3}{14} - \frac{4}{14} \right) |-\rangle\langle+| \\
&\quad + \left(\sqrt{\frac{3}{14}} - \sqrt{\frac{4}{14}} \right)^2 |-\rangle\langle-| \\
&= \left(\sqrt{\frac{3}{14}} + \sqrt{\frac{4}{14}} \right)^2 |+\rangle\langle+| - \frac{1}{14} |+\rangle\langle-| - \frac{1}{14} |-\rangle\langle+| + \left(\sqrt{\frac{3}{14}} - \sqrt{\frac{4}{14}} \right)^2 |-\rangle\langle-| \\
&= \begin{pmatrix} \left(\sqrt{\frac{3}{14}} + \sqrt{\frac{4}{14}} \right)^2 & -\frac{1}{14} \\ -\frac{1}{14} & \left(\sqrt{\frac{3}{14}} - \sqrt{\frac{4}{14}} \right)^2 \end{pmatrix}.
\end{aligned}$$

Now

$$\begin{aligned}
\text{Tr}(\rho) &= \left(\sqrt{\frac{3}{14}} + \sqrt{\frac{4}{14}} \right)^2 + \left(\sqrt{\frac{3}{14}} - \sqrt{\frac{4}{14}} \right)^2 \\
&= \frac{3}{14} + \frac{4}{14} + 2\frac{\sqrt{12}}{14} + \frac{3}{14} + \frac{4}{14} - 2\frac{\sqrt{12}}{14} \\
&= 1.
\end{aligned}$$

Also

$$\begin{aligned}\rho^2 &= \begin{pmatrix} \left(\sqrt{\frac{3}{14}} + \sqrt{\frac{4}{14}}\right)^2 & -\frac{1}{14} \\ -\frac{1}{14} & \left(\sqrt{\frac{3}{14}} - \sqrt{\frac{4}{14}}\right)^2 \end{pmatrix} \begin{pmatrix} \left(\sqrt{\frac{3}{14}} + \sqrt{\frac{4}{14}}\right)^2 & -\frac{1}{14} \\ -\frac{1}{14} & \left(\sqrt{\frac{3}{14}} - \sqrt{\frac{4}{14}}\right)^2 \end{pmatrix} \\ &= \begin{pmatrix} \left(\sqrt{\frac{3}{14}} + \sqrt{\frac{4}{14}}\right)^4 + \frac{1}{196} & \dots \\ \dots & \left(\sqrt{\frac{3}{14}} - \sqrt{\frac{4}{14}}\right)^4 + \frac{1}{196} \end{pmatrix}.\end{aligned}$$

According to Wolframalpha, $Tr(\rho^2) = 1$, so this state is pure.

QCE 5.4

Let

$$|\psi\rangle = \sqrt{\frac{2}{3}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle.$$

Now

$$\begin{aligned}\rho &= |\psi\rangle\langle\psi| \\ &= \left(\sqrt{\frac{2}{3}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle\right)\left(\sqrt{\frac{2}{3}}\langle 0| + \frac{1}{\sqrt{3}}\langle 1|\right) \\ &= \frac{2}{3}|0\rangle\langle 0| + \frac{\sqrt{2}}{3}|0\rangle\langle 1| + \frac{\sqrt{2}}{3}|1\rangle\langle 0| + \frac{1}{3}|1\rangle\langle 1| \\ &= \begin{pmatrix} \frac{2}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix}.\end{aligned}$$

It is obvious that $Tr(\rho) = 1$.

$$\begin{aligned}\rho^2 &= \begin{pmatrix} \frac{2}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} \frac{2}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix} \\ &= \begin{pmatrix} \frac{4}{9} + \frac{2}{9} & \frac{2\sqrt{2}}{9} + \frac{\sqrt{2}}{9} \\ \frac{2\sqrt{2}}{9} + \frac{\sqrt{2}}{9} & \frac{2}{9} + \frac{1}{9} \end{pmatrix} \\ &= \begin{pmatrix} \frac{6}{9} & \frac{3\sqrt{2}}{9} \\ \frac{3\sqrt{2}}{9} & \frac{3}{9} \end{pmatrix}.\end{aligned}$$

$Tr(\rho^2) = 1$ so the state is pure.

In order to compute $\langle X \rangle$ we can fall down to equation $\langle X \rangle = \text{Tr}(\rho X)$. Now

$$\begin{aligned}\rho X &= \begin{pmatrix} \frac{2}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{\sqrt{2}}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{\sqrt{2}}{3} \end{pmatrix},\end{aligned}$$

$$\text{so } \langle X \rangle = \text{Tr}(\rho X) = \frac{2\sqrt{2}}{3}.$$

QCE 5.5

Suppose that

$$\rho = \begin{pmatrix} \frac{1}{3} & \frac{i}{4} \\ -\frac{i}{4} & \frac{2}{3} \end{pmatrix}.$$

As $\text{Tr}(\rho) = 1$ and $\rho = \rho^\dagger$, ρ is a valid density matrix. Now

$$\begin{aligned}\rho^2 &= \begin{pmatrix} \frac{1}{3} & \frac{i}{4} \\ -\frac{i}{4} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{i}{4} \\ -\frac{i}{4} & \frac{2}{3} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{9} + \frac{1}{16} & \frac{i}{12} + \frac{2i}{12} \\ -\frac{i}{12} - \frac{2i}{12} & \frac{1}{16} + \frac{4}{9} \end{pmatrix} \\ &= \begin{pmatrix} \frac{25}{144} & \frac{3i}{72} \\ -\frac{3i}{72} & \frac{17}{36} \end{pmatrix}.\end{aligned}$$

Now, it is obvious that this is not a pure state as $\text{Tr}(\rho^2) \neq 1$.

QCE 5.6

Let

$$\rho = \frac{1}{5} \begin{pmatrix} 3 & 1-i \\ 1+i & 2 \end{pmatrix}.$$

Now

$$\begin{aligned}\rho^2 &= \frac{1}{25} \begin{pmatrix} 3 & 1-i \\ 1+i & 2 \end{pmatrix} \begin{pmatrix} 3 & 1-i \\ 1+i & 2 \end{pmatrix} \\ &= \frac{1}{25} \begin{pmatrix} 9 + (1-i)(1+i) & 3 - 3i + 2 - 2i \\ 3 + 3i + 2 + 2i & (1-i)(1+i) + 4 \end{pmatrix} \\ &= \begin{pmatrix} 9 + 1 - i^2 & 5 - 5i \\ 5 + 5i & 1 - i^2 + 4 \end{pmatrix} \\ &= \begin{pmatrix} 11 & 5 - 5i \\ 5 + 5i & 6 \end{pmatrix},\end{aligned}$$

so the state is mixed as $Tr(\rho^2) = \frac{17}{25} \neq 1$. Next, let us compute $\langle X \rangle, \langle Y \rangle, \langle Z \rangle$:

$$\begin{aligned}\rho X &= \frac{1}{5} \begin{pmatrix} 3 & 1-i \\ 1+i & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} 1-i & 3 \\ 2 & 1+i \end{pmatrix},\end{aligned}$$

so $\langle X \rangle = Tr(\rho X) = \frac{2}{5}$.

$$\begin{aligned}\rho Y &= \frac{1}{5} \begin{pmatrix} 3 & 1-i \\ 1+i & 2 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} i-i^2 & -3i \\ 2i & -i-i^2 \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} 1+i & -3i \\ 2i & 1-i \end{pmatrix},\end{aligned}$$

so $\langle Y \rangle = Tr(\rho Y) = \frac{2}{5}$.

$$\begin{aligned}\rho Z &= \frac{1}{5} \begin{pmatrix} 3 & 1-i \\ 1+i & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} 3 & i-1 \\ 1+i & -2 \end{pmatrix},\end{aligned}$$

so $\langle Z \rangle = Tr(\rho Z) = \frac{1}{5}$.

QCE 5.7

Let

$$|\psi\rangle = \frac{2}{\sqrt{5}} |0\rangle + \frac{1}{\sqrt{5}} |1\rangle.$$

Now

$$\begin{aligned}\rho_\psi &= |\psi\rangle \langle \psi| \\ &= \left(\frac{2}{\sqrt{5}} |0\rangle + \frac{1}{\sqrt{5}} |1\rangle \right) \left(\frac{2}{\sqrt{5}} \langle 0| + \frac{1}{\sqrt{5}} \langle 1| \right) \\ &= \frac{4}{5} |0\rangle \langle 0| + \frac{2}{5} |0\rangle \langle 1| + \frac{2}{5} |1\rangle \langle 0| + \frac{1}{5} |1\rangle \langle 1| \\ &= \begin{pmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix},\end{aligned}$$

so

$$\begin{aligned}\rho_\psi^2 &= \frac{1}{25} \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \\ &= \frac{1}{25} \begin{pmatrix} 20 & 10 \\ 10 & 5 \end{pmatrix},\end{aligned}$$

so $Tr(\rho_\psi^2) = 1$ and the state is pure.

Let

$$|\phi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle.$$

Now

$$\begin{aligned}\rho_\phi &= |\phi\rangle\langle\phi| \\ &= \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)\left(\frac{1}{\sqrt{2}}\langle 0| + \frac{1}{\sqrt{2}}\langle 1|\right) \\ &= \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|0\rangle\langle 1| + \frac{1}{2}|1\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| \\ &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\end{aligned}$$

so

$$\begin{aligned}\rho_\phi^2 &= \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{4}\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix},\end{aligned}$$

so $Tr(\rho_\phi^2) = 1$ and the state is pure.

The density operator for the ensemble is given by

$$\begin{aligned}\rho &= \frac{1}{4}\rho_\psi + \frac{3}{4}\rho_\phi \\ &= \frac{1}{4}\left(\frac{4}{5}|0\rangle\langle 0| + \frac{2}{5}|0\rangle\langle 1| + \frac{2}{5}|1\rangle\langle 0| + \frac{1}{5}|1\rangle\langle 1|\right) + \frac{3}{4}\left(\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|0\rangle\langle 1| + \frac{1}{2}|1\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|\right) \\ &= \left(\frac{1}{5} + \frac{3}{8}\right)|0\rangle\langle 0| + \left(\frac{1}{10} + \frac{3}{8}\right)|0\rangle\langle 1| + \left(\frac{1}{10} + \frac{3}{8}\right)|1\rangle\langle 0| + \left(\frac{1}{20} + \frac{3}{8}\right)|1\rangle\langle 1|,\end{aligned}$$

and it is easy to see that $Tr(\rho) = 1$.

Upon measurement, $|\psi\rangle$ is found in state $|0\rangle$ with probability $4/5$ and in state $|1\rangle$ with probability $1/5$. Upon measurement, $|\phi\rangle$ is found in state $|0\rangle$ with probability $1/2$ and in state $|1\rangle$ with probability $1/2$.

The probability of measuring $|0\rangle$ within the ensemble is

$$\begin{aligned}p(0) &= \langle 0|\rho|0\rangle \\ &= \langle 0|\left(\left(\frac{1}{5} + \frac{3}{8}\right)|0\rangle\langle 0| + \left(\frac{1}{10} + \frac{3}{8}\right)|0\rangle\langle 1| + \left(\frac{1}{10} + \frac{3}{8}\right)|1\rangle\langle 0| + \left(\frac{1}{20} + \frac{3}{8}\right)|1\rangle\langle 1|\right)|0\rangle \\ &= \left(\frac{1}{5} + \frac{3}{8}\right) \\ &= \frac{23}{40},\end{aligned}$$

and the probability of measuring $|+\rangle$ within the ensemble is

$$\begin{aligned}
 p(1) &= \langle 1 | \rho | 1 \rangle \\
 &= \langle 1 | \left(\left(\frac{1}{5} + \frac{3}{8} \right) |0\rangle \langle 0| + \left(\frac{1}{10} + \frac{3}{8} \right) |0\rangle \langle 1| + \left(\frac{1}{10} + \frac{3}{8} \right) |1\rangle \langle 0| + \left(\frac{1}{20} + \frac{3}{8} \right) |1\rangle \langle 1| \right) |1\rangle \\
 &= \left(\frac{1}{20} + \frac{3}{8} \right) \\
 &= \frac{68}{160} \\
 &= \frac{34}{80} \\
 &= \frac{17}{40}.
 \end{aligned}$$