Kvanttilaskenta, kevät 2015 – Viikko 2

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edx Problem 1

No, its 0 for $|+\rangle$ and $|-\rangle$ are orthogonal to each other.

edx Problem 2

Yes.

edx Problem 3

Yes, but the implicit multiplication operator is in fact the tensor product $\otimes$.

edx Problem 4

False.

edx Problem 5

False.

edx Problem 6

True.

edx Problem 7

$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$. $|\psi\rangle = \frac{3}{5}|0\rangle - \frac{4}{5}|1\rangle$. Now we wish to compute

$$\begin{aligned}\langle\psi,+\rangle &= \frac{3}{5}\frac{1}{\sqrt{2}} - \frac{4}{5}\frac{1}{\sqrt{2}}\\ &= -\frac{1}{\sqrt{2}}\frac{1}{5}\\ &= -\frac{1}{5\sqrt{2}}.\end{aligned}$$

edx Problem 8

$$-\frac{1}{5\sqrt{2}}|+\rangle + \frac{7}{5\sqrt{2}}|-\rangle.$$

edx Problem 9

Forget the first standard basis measurement as it is not relevant here. At second measurement the chance of getting $|u\rangle$ is $\cos^2\theta$, where θ is the angle between $|u\rangle$ and $|\phi\rangle$. Since

$$\cos\theta = \langle\phi|u\rangle = ab + \sqrt{1-a^2}\sqrt{1-b^2},$$

the desired probability is

$$(ab + \sqrt{1-a^2}\sqrt{1-b^2})^2.$$

edx Problem 10

The third topmost alternative seems suspicious as it mixes sign and bit bases.

edx Problem 11

$$\left(\frac{\sqrt{2}}{\sqrt{3}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle\right)\otimes\left(\frac{1}{\sqrt{3}}|0\rangle + \frac{\sqrt{2}}{\sqrt{3}}|1\rangle\right) = \frac{\sqrt{2}}{3}|00\rangle + \frac{2}{3}|01\rangle + \frac{1}{3}|10\rangle + \frac{\sqrt{2}}{3}|11\rangle$$

edx Problem 12

$$\begin{aligned}(a|0\rangle + b|1\rangle)(c|0\rangle + d|1\rangle) &= ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle\\ &= \frac{1}{2\sqrt{2}}|00\rangle - \frac{1}{2\sqrt{2}}|01\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|10\rangle - \frac{\sqrt{3}}{2\sqrt{2}}|11\rangle\end{aligned}$$

So we have that

$$
\begin{aligned}
ac &= \frac{1}{2\sqrt{2}}, \\
ad &= -\frac{1}{2\sqrt{2}}, \\
bc &= \frac{\sqrt{3}}{2\sqrt{2}}, \\
bd &= -\frac{\sqrt{3}}{2\sqrt{2}}.
\end{aligned}
$$

The way to factorize is to assign $a = \frac{1}{2}$, $b = \frac{\sqrt{3}}{2}$, $c = \frac{1}{\sqrt{2}}$, $d = -\frac{1}{\sqrt{2}}$, so $|a| = \frac{1}{2}$.

edx Problem 13

(a)

The probability is $\frac{1}{2}$.

(b)

$|+\rangle = \frac{1}{\sqrt{2}}\,|0\rangle + \frac{1}{\sqrt{2}}\,|1\rangle$.

edx Problem 14

$$
\left(\frac{4}{5}\right)^2 + \left(-\frac{2}{5}\right)^2 = \frac{16}{25} + \frac{4}{25} = \frac{20}{25}.
$$

edx Problem 15

$|0+\rangle\,, |0-\rangle\,, |1+\rangle\,, |1-\rangle$.

edx Problem 16

There si no way to entangle two qubits by a partial measurement.

edx Problem 17

As

$$
|0\rangle = \frac{1}{\sqrt{2}}\,|+\rangle + \frac{1}{\sqrt{2}}\,|-\rangle\,, |1\rangle = \frac{1}{\sqrt{2}}\,|+\rangle - \frac{1}{\sqrt{2}}\,|-\rangle\,,
$$

we have that

$$\begin{aligned}
|\psi\rangle &= \frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\theta}}{\sqrt{2}}|1\rangle \\
&= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle\right) + \frac{e^{i\theta}}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle\right) \\
&= \frac{1}{2}|+\rangle + \frac{1}{2}|-\rangle + \frac{e^{i\theta}}{2}|+\rangle - \frac{e^{i\theta}}{2}|-\rangle \\
&= \frac{1+e^{i\theta}}{2}|+\rangle + \frac{1-e^{i\theta}}{2}|-\rangle .
\end{aligned}$$

edx Problem 18

$$\frac{1+\cos\theta}{2}.$$

QCE 3.1

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. $X = |0\rangle\langle 1| + |1\rangle\langle 0|$. $Y = -i|0\rangle\langle 1| + i|1\rangle\langle 0|$. Now

$$\begin{aligned}
X|\psi\rangle &= (|0\rangle\langle 1| + |1\rangle\langle 0|)(\alpha|0\rangle + \beta|1\rangle) \\
&= \alpha(|0\rangle\langle 1| + |1\rangle\langle 0|)|0\rangle + \beta(|0\rangle\langle 1| + |1\rangle\langle 0|)|1\rangle \\
&= \alpha(|0\rangle\langle 1|0\rangle + |1\rangle\langle 0|0\rangle) + \beta(|0\rangle\langle 1|1\rangle + |1\rangle\langle 0|1\rangle) \\
&= \alpha|0\rangle \times 0 + \alpha|1\rangle \times 1 + \beta|0\rangle \times 1 + \beta|1\rangle \times 0 \\
&= \alpha|1\rangle + \beta|0\rangle .
\end{aligned}$$

$$\begin{aligned}
Y|\psi\rangle &= (-i|0\rangle\langle 1| + i|1\rangle\langle 0|)(\alpha|0\rangle + \beta|1\rangle) \\
&= \alpha(-i|0\rangle\langle 1| + i|1\rangle\langle 0|)|0\rangle + \beta(-i|0\rangle\langle 1| + i|1\rangle\langle 0|)|1\rangle \\
&= \alpha(-i|0\rangle\langle 1|0\rangle + i|1\rangle\langle 0|0\rangle) + \beta(-i|0\rangle\langle 1|1\rangle + i|1\rangle\langle 0|1\rangle) \\
&= \alpha i|1\rangle + \beta(-i)|0\rangle \\
&= \alpha i|1\rangle - \beta i|0\rangle .
\end{aligned}$$

QCE 3.2

Suppose we have a qubit

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

Now

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0\times\alpha + 1\times\beta \\ 1\times\alpha + 0\times\beta \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}.$$

QCE 3.3

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

Now

$$\begin{aligned}
X\,|+\rangle &= X\frac{|0\rangle + |1\rangle}{\sqrt{2}} \\
&= X\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \\
&= \begin{pmatrix} 1 \times \frac{1}{\sqrt{2}} + 0 \times \frac{1}{\sqrt{2}} \\ 0 \times \frac{1}{\sqrt{2}} - 1\frac{1}{\sqrt{2}} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \\
&= \frac{|0\rangle - |1\rangle}{\sqrt{2}} \\
&= |-\rangle\,.
\end{aligned}$$

Also

$$\begin{aligned}
X\,|-\rangle &= X\frac{|0\rangle - |1\rangle}{\sqrt{2}} \\
&= X\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \\
&= \begin{pmatrix} 1 \times \frac{1}{\sqrt{2}} + 0 \times -\frac{1}{\sqrt{2}} \\ 0 \times \frac{1}{\sqrt{2}} - 1\left(-\frac{1}{\sqrt{2}}\right) \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \\
&= \frac{|0\rangle + |1\rangle}{\sqrt{2}} \\
&= |+\rangle\,.
\end{aligned}$$

QCE 3.4

The operator is $\hat{A} = i\,|1\rangle\langle 1| + \frac{\sqrt{3}}{2}\,|1\rangle\langle 2| + 2\,|2\rangle\,|1\rangle - |2\rangle\langle 3|$. Now,

$$\begin{aligned}\hat{A}^{\dagger} &= -i\,\langle 1|1\rangle + \frac{\sqrt{3}}{2}\,\langle 1|2\rangle + 2\,\langle 2|1\rangle - \langle 2|3\rangle \\ &= -i.\end{aligned}$$

QCE 3.5

The X operator is given by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and we wish to find λ, $(a \quad b)^T$ such that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda\begin{pmatrix} a \\ b \end{pmatrix}.$$

Now

$$\begin{pmatrix} b \\ a \end{pmatrix} = \lambda\begin{pmatrix} a \\ b \end{pmatrix},$$

so $b = \lambda a$ and $a = \lambda b$. If $\lambda = 1$, the corresponding eigenvector is $(1,1)^T$. If $\lambda = 0$, the eigenvector is $(0,0)^T$.

QCE 3.6

As the matrix of Y-operator is

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

it is evident that its trace is 0.

QCE 3.7

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 4 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a+2c \\ 3b+4c \\ a+2c \end{pmatrix} = \begin{pmatrix} \lambda a \\ \lambda b \\ \lambda c \end{pmatrix}.$$

Its clear that $a = b$. Also $3b + 4c = \lambda b$, which implies $4c = (\lambda - 3)b$, and $c = \frac{1}{4}(\lambda - 3)b$. It follows that $\lambda \in \mathbb{R}$.

QCE 3.8

Suppose

$$A = \begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{pmatrix}, B = \begin{pmatrix} b_{1,1} & \dots & b_{1,n} \\ \vdots & \ddots & \vdots \\ b_{n,1} & \dots & b_{n,n} \end{pmatrix}.$$

Now, it is easy to see that

$$\begin{aligned} Tr(A+B) &= (a_{1,1}+b_{1,1}) + \cdots + (a_{n,n}+b_{n,n}) \\ &= (a_{1,1} + \cdots + a_{n,n}) + (b_{1,1} + \cdots + b_{n,n}) \\ &= Tr(A) + Tr(B). \end{aligned}$$

Also

$$\begin{aligned} Tr(\lambda A) &= \lambda a_{1,1} + \cdots + \lambda a_{n,n} \\ &= \lambda Tr(A). \end{aligned}$$

Also-also

$$\begin{aligned} Tr(AB) &= a_{1,1}b_{1,1} + \cdots + a_{n,n}b_{n,n} \\ &= b_{1,1}a_{1,1} + \cdots + b_{n,n}a_{n,n} \\ &= Tr(BA), \end{aligned}$$

as real multiplication commutes.

QCE 3.9

$$P_+ = \frac{1}{2}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|).$$

$$P_- = \frac{1}{2}(|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|).$$

Now

$$P_+ - P_- = |0\rangle\langle 1| + |1\rangle\langle 0| = X.$$

QCE 3.10

We have

$$P_0 = |0\rangle\langle 0| = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$P_1 = |1\rangle\langle 1| = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$P_2 = |2\rangle\langle 2| = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \; 0 \; 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now

$$Pr(0) = |P_0 |\psi\rangle|^2 = \left| \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{i}{\sqrt{2}} \end{pmatrix} \right|^2 = \left| \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \right|^2 = \left(\frac{1}{2} \right)^2 = \frac{1}{4},$$

$$Pr(1) = |P_1 |\psi\rangle|^2 = \left| \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{i}{\sqrt{2}} \end{pmatrix} \right|^2 = \left| \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \right|^2 = \left(\frac{1}{2} \right)^2 = \frac{1}{4},$$

$$Pr(2) = |P_2 |\psi\rangle|^2 = \left| \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{i}{\sqrt{2}} \end{pmatrix} \right|^2 = \left| \begin{pmatrix} 0 \\ 0 \\ -\frac{i}{\sqrt{2}} \end{pmatrix} \right|^2 = \left(\frac{1}{\sqrt{2}} \right)^2 = \frac{1}{2}.$$

QCE 3.11

The matrix representations of the Pauli operators are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1. \end{pmatrix}$$

0.1 $[\sigma_2, \sigma_3] = 2i\sigma_1$

$$\sigma_2\sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

$$\sigma_3\sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

Now the commutator is

$$[\sigma_2, \sigma_3] = \sigma_2\sigma_3 - \sigma_3\sigma_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix} = 2i\sigma_1.$$

0.2 $[\sigma_3, \sigma_1] = 2i\sigma_2$

$$\sigma_3\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\sigma_1\sigma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Now the commutator is

$$[\sigma_3, \sigma_1] = \sigma_3\sigma_1 - \sigma_1\sigma_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} = 2i\sigma 2.$$

QCE 3.12

We need to show that if $i \neq j$, then $\{\sigma_i, \sigma_j\} = 0$. First of all, $\{\sigma_i, \sigma_j\} = \sigma_i\sigma_j + \sigma_j\sigma_i$. By the definition of anticommutator we have that

$$\{\sigma_i, \sigma_j\} = \sigma_i\sigma_j + \sigma_j\sigma_i = \sigma_j\sigma_i + \sigma_i\sigma_j = \{\sigma_j, \sigma_i\},$$

because the real + commutes. Now we don't need to compute all permutations of Pauli operators, but only combinations will suffice, so we need to calculate

- $\{\sigma_1, \sigma_2\}$,
- $\{\sigma_1, \sigma_3\}$,
- $\{\sigma_2, \sigma_3\}$.

So

$$\begin{aligned}\{\sigma_1, \sigma_2\} &= \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} + \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\\ &= \begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix} + \begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}\\ &= 0.\end{aligned}$$

$$\begin{aligned}\{\sigma_1, \sigma_3\} &= \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\\ &= \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} + \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\\ &= 0.\end{aligned}$$

The last, but not least

$$\begin{aligned}\{\sigma_2, \sigma_3\} &= \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\\ &= \begin{pmatrix}0 & i\\ i & 0\end{pmatrix} + \begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix}\\ &= 0.\end{aligned}$$

QCE 4.1

The basis for H is

$$\begin{aligned}|w_1\rangle &= |0\rangle \otimes |0\rangle\,,\\ |w_2\rangle &= |0\rangle \otimes |1\rangle\,,\\ |w_3\rangle &= |1\rangle \otimes |0\rangle\,,\\ |w_4\rangle &= |1\rangle \otimes |1\rangle\,.\end{aligned}$$

First we show that $|w_i\rangle$ has unit length:

$$\begin{aligned}
\langle w_1|w_1\rangle &= (\langle 0| \otimes \langle 0|)(|0\rangle \otimes |0\rangle) \\
&= \langle 0|0\rangle \langle 0|0\rangle \\
&= 1 \times 1 \\
&= 1.
\end{aligned}$$

$$\begin{aligned}
\langle w_2|w_2\rangle &= (\langle 0| \otimes \langle 1|)(|0\rangle \otimes |1\rangle) \\
&= \langle 0|0\rangle \langle 1|1\rangle \\
&= 1 \times 1 \\
&= 1.
\end{aligned}$$

$$\begin{aligned}
\langle w_3|w_3\rangle &= (\langle 1| \otimes \langle 0|)(|1\rangle \otimes |0\rangle) \\
&= \langle 1|1\rangle \langle 0|0\rangle \\
&= 1 \times 1 \\
&= 1.
\end{aligned}$$

$$\begin{aligned}
\langle w_4|w_4\rangle &= (\langle 1| \otimes \langle 1|)(|1\rangle \otimes |1\rangle) \\
&= \langle 1|1\rangle \langle 1|1\rangle \\
&= 1 \times 1 \\
&= 1.
\end{aligned}$$

Next we need to check the orthogonality of $|w_i\rangle$:

$$\begin{aligned}
\langle w_1|w_2\rangle &= (\langle 0| \otimes \langle 0|)(|0\rangle \otimes |1\rangle) \\
&= \langle 0|0\rangle \langle 0|1\rangle \\
&= 1 \times 0 \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
\langle w_1|w_3\rangle &= (\langle 0| \otimes \langle 0|)(|1\rangle \otimes |0\rangle) \\
&= \langle 0|1\rangle \langle 0|0\rangle \\
&= 0 \times 1 \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
\langle w_1|w_4\rangle &= (\langle 0| \otimes \langle 0|)(|1\rangle \otimes |1\rangle) \\
&= \langle 0|1\rangle \langle 0|1\rangle \\
&= 0 \times 0 \\
&= 0,
\end{aligned}$$

$$
\begin{aligned}
\langle w_2|w_3\rangle &= (\langle 0| \otimes \langle 1|)(|1\rangle \otimes |0\rangle) \\
&= \langle 0|1\rangle \langle 1|0\rangle \\
&= 0 \times 0 \\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
\langle w_2|w_4\rangle &= (\langle 0| \otimes \langle 1|)(|1\rangle \otimes |1\rangle) \\
&= \langle 0|1\rangle \langle 1|1\rangle \\
&= 0 \times 1 \\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
\langle w_3|w_4\rangle &= (\langle 1| \otimes \langle 0|)(|1\rangle \otimes |1\rangle) \\
&= \langle 1|1\rangle \langle 0|1\rangle \\
&= 1 \times 0 \\
&= 0.
\end{aligned}
$$

Now we see that $\{w_1, w_2, w_3, w_4\}$ is a orthonormal basis.

QCE 4.2

In previous exercise we have seen that $\langle w_3|w_4\rangle = 0$. Now, let us verify that $\langle w_4|w_3\rangle = 0$:

$$
\begin{aligned}
\langle w_4|w_3\rangle &= (\langle 1| \otimes \langle 1|)(|1\rangle \otimes |0\rangle) \\
&= \langle 1|1\rangle \langle 1|0\rangle \\
&= 1 \times 0 \\
&= 0.
\end{aligned}
$$

QCE 4.3

$\langle a|b\rangle = 1/2$, $\langle c|d\rangle = 3/4$.

$$
\begin{aligned}
\langle \psi|\phi\rangle &= \langle |a\rangle \otimes |c\rangle \,|\, |b\rangle \otimes |d\rangle\rangle \\
&= (\langle a| \otimes \langle c|)(|b\rangle \otimes |d\rangle) \\
&= \langle a|b\rangle \langle c|d\rangle \\
&= \frac{1}{2} \times \frac{3}{4} \\
&= \frac{3}{8}.
\end{aligned}
$$

QCE 4.4

$$|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \quad |\phi\rangle = \frac{1}{2}\begin{pmatrix}1\\\sqrt{3}\end{pmatrix}.$$

Now the tensor product $\psi \otimes \phi$ is

$$\begin{aligned}\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \otimes \left(\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle\right) &= \begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{pmatrix} \otimes \begin{pmatrix}\frac{1}{2}\\\frac{\sqrt{3}}{2}\end{pmatrix}\\ &= \begin{pmatrix}\frac{1}{2\sqrt{2}}\\\frac{\sqrt{3}}{2\sqrt{2}}\\\frac{1}{2\sqrt{2}}\\\frac{\sqrt{3}}{2\sqrt{2}}\end{pmatrix}\\ &= \frac{1}{2\sqrt{2}}|0\rangle|0\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|0\rangle|1\rangle + \frac{1}{2\sqrt{2}}|1\rangle|0\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|1\rangle|1\rangle.\end{aligned}$$

QCE 4.5

$$\begin{aligned}|\psi\rangle &= \frac{1}{2}(|0\rangle|0\rangle - |0\rangle|1\rangle - |1\rangle|0\rangle + |1\rangle|1\rangle)\\ &= \frac{|0\rangle - |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}.\end{aligned}$$

QCE 4.6

$$\begin{aligned}|\psi\rangle &= \frac{|0\rangle|0\rangle + |1\rangle|1\rangle}{\sqrt{2}}\\ &= \left(\alpha|0\rangle + \beta|1\rangle\right) \otimes \left(\gamma|0\rangle + \delta|1\rangle\right)\\ &= \alpha\gamma|0\rangle|0\rangle + \alpha\delta|0\rangle|1\rangle + \beta\gamma|1\rangle|0\rangle + \beta\delta|1\rangle|1\rangle.\end{aligned}$$

Now we have

$$\begin{aligned}\alpha\gamma &= \beta\delta\\ &= \frac{1}{\sqrt{2}},\\ \alpha\delta &= \beta\gamma\\ &= 0.\end{aligned}$$

Obviously, either $\alpha = 0$ and/or $\delta = 0$. In any case, either $\alpha\gamma = 0$ and/or $\beta\delta = 0$, when they should be non-zero, so it is not possible to write $|\psi\rangle$ as a product state.

QCE 4.7

Let

$$|\psi\rangle = \frac{|0\rangle|1\rangle - |1\rangle|0\rangle}{\sqrt{2}} = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}.$$

Since

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$X \otimes Y = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}.$$

Now,

$$\begin{aligned} X \otimes Y |\psi\rangle &= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ -\frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \\ 0 \end{pmatrix} \\ &= -\frac{-i|0\rangle|1\rangle - i|1\rangle|0\rangle}{\sqrt{2}}. \end{aligned}$$

QCE 4.8

We are asked to show that $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$. Now,

$$
\begin{aligned}
(A \otimes B)^\dagger &= \left(\begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{pmatrix} \otimes \begin{pmatrix} b_{1,1} & \dots & b_{1,n} \\ \vdots & \ddots & \vdots \\ b_{n,1} & \dots & b_{n,n} \end{pmatrix} \right)^\dagger \\
&= \begin{pmatrix} a_{1,1}b_{1,1} & \dots & a_{1,1}b_{1,n} & \dots & a_{1,n}b_{1,1} & \dots & a_{1,n}b_{1,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{1,1}b_{n,1} & \dots & a_{1,1}b_{n,n} & \dots & a_{1,n}b_{n,1} & \dots & a_{1,n}b_{n,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n,1}b_{1,1} & \dots & a_{n,1}b_{1,n} & \dots & a_{n,n}b_{1,1} & \dots & a_{n,n}b_{1,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n,1}b_{n,1} & \dots & a_{n,1}b_{n,n} & \dots & a_{n,n}b_{n,1} & \dots & a_{n,n}b_{n,n} \end{pmatrix}^\dagger \\
&= \begin{pmatrix} (a_{1,1}b_{1,1})^\star & \dots & (a_{1,1}b_{1,n})^\star & \dots & (a_{1,n}b_{1,1})^\star & \dots & (a_{1,n}b_{1,n})^\star \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ (a_{1,1}b_{n,1})^\star & \dots & (a_{1,1}b_{n,n})^\star & \dots & (a_{1,n}b_{n,1})^\star & \dots & (a_{1,n}b_{n,n})^\star \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ (a_{n,1}b_{1,1})^\star & \dots & (a_{n,1}b_{1,n})^\star & \dots & (a_{n,n}b_{1,1})^\star & \dots & (a_{n,n}b_{1,n})^\star \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ (a_{n,1}b_{n,1})^\star & \dots & (a_{n,1}b_{n,n})^\star & \dots & (a_{n,n}b_{n,1})^\star & \dots & (a_{n,n}b_{n,n})^\star \end{pmatrix} \\
&= \begin{pmatrix} a_{1,1}^\star b_{1,1}^\star & \dots & a_{1,1}^\star b_{1,n}^\star & \dots & a_{1,n}^\star b_{1,1}^\star & \dots & a_{1,n}^\star b_{1,n}^\star \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{1,1}^\star b_{n,1}^\star & \dots & a_{1,1}^\star b_{n,n}^\star & \dots & a_{1,n}^\star b_{n,1}^\star & \dots & a_{1,n}^\star b_{n,n}^\star \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n,1}^\star b_{1,1}^\star & \dots & a_{n,1}^\star b_{1,n}^\star & \dots & a_{n,n}^\star b_{1,1}^\star & \dots & a_{n,n}^\star b_{1,n}^\star \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n,1}^\star b_{n,1}^\star & \dots & a_{n,1}^\star b_{n,n}^\star & \dots & a_{n,n}^\star b_{n,1}^\star & \dots & a_{n,n}^\star b_{n,n}^\star \end{pmatrix} \\
&= \begin{pmatrix} a_{1,1}^\star & \dots & a_{1,n}^\star \\ \vdots & \ddots & \vdots \\ a_{n,1}^\star & \dots & a_{n,n}^\star \end{pmatrix} \otimes \begin{pmatrix} b_{1,1}^\star & \dots & b_{1,n}^\star \\ \vdots & \ddots & \vdots \\ b_{n,1}^\star & \dots & b_{n,n}^\star \end{pmatrix} \\
&= \begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{pmatrix}^\dagger \otimes \begin{pmatrix} b_{1,1} & \dots & b_{1,n} \\ \vdots & \ddots & \vdots \\ b_{n,1} & \dots & b_{n,n} \end{pmatrix}^\dagger \\
&= A^\dagger \otimes B^\dagger.
\end{aligned}
$$

QCE 4.9

Let

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

As

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

we have that

$$I \otimes Y = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix},$$

so

$$\begin{aligned} I \otimes Y \, |\psi\rangle &= \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ \frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \\ 0 \end{pmatrix}. \end{aligned}$$

QCE 4.10

We have

$$\begin{aligned} X \otimes Y &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$