Computing debt cuts leading to global zero-equity

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Abstract

In this paper we present a method for computing a set of loan cuts, which, once applied, lead to a global zero-equity state, i.e., each and every party in a financial network may forget all liabilities.

1 Basic definitions

Before we proceed to defining the structures needed in discussing the method, we have to impose some definitions: by $\mathfrak{R}_?$, we denote the set of real numbers x such that $x\,?\,0$ holds. We work on a directed graph $G = (V, A)$, $A \subset V \times V$, for which we define a weight function $w_G \colon G.A \to \mathcal{P}(\mathfrak{R}_> \times \mathfrak{R}_\geq \times (\mathfrak{R}_> \cup \{\infty\}) \times \mathfrak{R})$. If $(K, r, n, t) \in \mathcal{P}(\mathfrak{R}_> \times \mathfrak{R}_\geq \times (\mathfrak{R}_> \cup \{\infty\}) \times \mathfrak{R})$, K is the initial principal investment, r is the annual interest rate, n the amount of compounding periods per year (the value of ∞ is allowed, which denotes continuous compounding), and t is the time point at which the loan was granted. Together, the four parameters comprise a *contract*. (Note that the "weight" of an edge is a **set** of contracts as this is physically possible in real-world banking.)

The most fundamental function in this paper is the equity function $e_G \colon G.V \times \mathfrak{R} \to \mathfrak{R}$ defined as

$$
e_G(u, \tau) = \sum_{(u,v)\in G.A} \left(\sum_{(K,r,n,t)\in w_G(u,v)} \mathfrak{C}_\tau(K,r,n,t) \right)
$$
$$
- \sum_{(v,u)\in G.A} \left(\sum_{(K,r,n,t)\in w_G(v,u)} \mathfrak{C}_\tau(K,r,n,t) \right),
$$

where

$$
\mathfrak{C}_\tau(K,r,n,t) = \begin{cases} K\left(1 + \frac{r}{n}\right)^{\lfloor n(\tau-t)\rfloor} & \text{if } n \in \mathfrak{R}_> \\ Ke^{r(\tau-t)} & \text{if } n = \infty, \end{cases}
$$

and τ is no less than the time stamp of the most recent loan of u. Also, we assume we are given a function $\mathfrak{f}_G \colon G.V \times \mathfrak{R}_> \times \mathfrak{R}_\geq \times (\mathfrak{R}_> \cup \infty) \times \mathfrak{R} \to \mathfrak{R}$ mapping every tuple $(u, \mathfrak{K})$ (a node and a contract) in the financial graph to a time point at which u is ready to pay back at most all of its debts on behalf of a contract $\mathfrak{K}$. (By "at most" we mean that we will try to minimize the magnitude of the nodes' debt cuts for every contract, yet it is not possible for a node, say v,

having no loans to the other nodes, to have a debt cut for a contract $\mathfrak{K}$ any less than the value of $\mathfrak{K}$ at time $\mathfrak{f}_G(v, \mathfrak{K})$, which implies that such contract $\mathfrak{K}$ will have to be cut in its entirety.)

As the concept of equilibria in this paper is a global phenomenon, we demand that a time point T_G is given; especially, we demand that $T_G \geq \max\{\mathfrak{f}_G(*, *)\}$. Whenever a party, say $u \in G.V$, is ready to raise C amount of resources for the debt cut to v against a contract $\mathfrak{K}$ (which is supposed to happen at $\mathfrak{f}_G(u, \mathfrak{K})$), with $\mathfrak{K} = (K, r, n, t)$, the contract being cut becomes

$$\mathfrak{C}_\tau\big(\mathfrak{C}_{\mathfrak{f}_G(u,\mathfrak{K})}(K, r, n, t) - C, r, n, \mathfrak{f}_G(u, \mathfrak{K})\big),$$

where $\tau \geq \mathfrak{f}_G(u, \mathfrak{K})$. There is, however, a small issue with debt contracts with non-continuous compounding scheme. Let us define a function of time and a contract:

$$g(\tau, \mathfrak{K}) = \begin{cases} 0 & \text{if } \mathfrak{K}.n = \infty, \\ g'(\mathfrak{K}.n(\tau - \mathfrak{K}.t)) & \text{otherwise,} \end{cases}$$

where $g'(x) = x - \lfloor x \rfloor$. Consider a contract $\mathfrak{K} = (K, r, n, t)$ given from v to u with $n \in \mathfrak{R}_>$. It is easy to see that after cutting $\mathfrak{K}$, it "shifts" forward in time by $g(\mathfrak{f}_G(u, \mathfrak{K}), \mathfrak{K})$. As to keep compounding at moments $t + n^{-1}, t + 2n^{-1}, t + 3n^{-1}, \ldots$, we should set the cut contract to

$$\mathfrak{C}_\tau\left(\mathfrak{C}_{\mathfrak{f}_G(u,\mathfrak{K})}(K, r, n, t) - \Xi(v, u, \mathfrak{K}), r, n, g(\mathfrak{f}_G(u, \mathfrak{K}), \mathfrak{K})\right),$$

Next, we define the concept of equilibrium.

Definition 1 *The financial graph G is said to be in equilibrium at time point τ if and only if $e_G(u, \tau) = 0$ for all $u \in G.V$.*

Once given G, $\mathfrak{f}_G$ and T_G, we attempt to compute a function $\Xi_G \colon G.A \times \mathfrak{R}_> \times \mathfrak{R}_\geq \times (\mathfrak{R}_> \cup \{\infty\}) \times \mathfrak{R} \to \mathfrak{R}_\geq$ such that after applying a debt cut from v to u of magnitude $\Xi(u, v, \mathfrak{K})$ against the contract $\mathfrak{K}$ at time point $\mathfrak{f}_G(v, \mathfrak{K})$ for all $(u, v) \in G.A$, G obtains such a state that it evolves towards equilibrium at time point T_G.

2 Solution

Whenever a node u has incoming contracts from a set of parent nodes (lenders) L, outgoing contracts to a set of children (debtors) D, the equilibrium equation for u is

$$\sum_{v \in D} \sum_{\mathfrak{K} \in w_G(u,v)} \mathfrak{C}_{T_G}\big(\mathfrak{C}_{\mathfrak{f}_G(v,\mathfrak{K})}(\mathfrak{K}.K, \mathfrak{K}.r, \mathfrak{K}.n, \mathfrak{K}.t) - \Xi(u, v, \mathfrak{K}), \mathfrak{K}.r, \mathfrak{K}.n, \mathfrak{f}_G(v, \mathfrak{K})\big) -$$
$$\sum_{v \in L} \sum_{\mathfrak{K} \in w_G(v,u)} \mathfrak{C}_{T_G}\big(\mathfrak{C}_{\mathfrak{f}_G(u,\mathfrak{K})}(\mathfrak{K}.K, \mathfrak{K}.r, \mathfrak{K}.n, \mathfrak{K}.t) - \Xi(v, u, \mathfrak{K}), \mathfrak{K}.r, \mathfrak{K}.n, \mathfrak{f}_G(u, \mathfrak{K})\big) = 0.$$

2

The above equation is equivalent to

$$
\begin{aligned}
&\sum_{v\in D}\sum_{\mathfrak{K}\in w_G(u,v)} \mathfrak{C}_{T_G}\big(\Xi(u,v,\mathfrak{K}),\mathfrak{K}.r,\mathfrak{K}.n,\mathfrak{f}_G(v,\mathfrak{K})\big)- \\
&\sum_{v\in L}\sum_{\mathfrak{K}\in w_G(v,u)} \mathfrak{C}_{T_G}\big(\Xi(v,u,\mathfrak{K}),\mathfrak{K}.r,\mathfrak{K}.n,\mathfrak{f}_G(u,\mathfrak{K})\big)= \\
&\sum_{v\in D}\sum_{\mathfrak{K}\in w_G(u,v)} \mathfrak{C}_{T_G}\big(\mathfrak{C}_{\mathfrak{f}_G(v,\mathfrak{K})}(\mathfrak{K}.K,\mathfrak{K}.r,\mathfrak{K}.n,\mathfrak{K}.t),\mathfrak{K}.r,\mathfrak{K}.n,\mathfrak{f}_G(v,\mathfrak{K})\big)- \\
&\sum_{v\in L}\sum_{\mathfrak{K}\in w_G(v,u)} \mathfrak{C}_{T_G}\big(\mathfrak{C}_{\mathfrak{f}_G(u,\mathfrak{K})}(\mathfrak{K}.K,\mathfrak{K}.r,\mathfrak{K}.n,\mathfrak{K}.t),\mathfrak{K}.r,\mathfrak{K}.n,\mathfrak{f}_G(u,\mathfrak{K})\big).
\end{aligned}
\tag{1}
$$

Now if we write down equilibrium equations for all nodes $v \in G.V$, we obtain a linear system, which is guaranteed to have a solution as we can choose for each $\mathfrak{K} \in w_G(u,v)$ a debt cut of magnitude $\mathfrak{C}_{\mathfrak{f}_G(v,\mathfrak{K})}(\mathfrak{K})$, which results in trivial equilibrium. All the terms $\Xi(\star,\star,\star)$ are to be determined, yet everything else in (1) is known before-hand. We rewrite all equilibrium equations as a $\mathfrak{R}^{|G.V|\times C'_G}$ - matrix, where

$$
C'_G = 1 + \sum_{(u,v)\in G.A} |w_G(u,v)|.
$$

(We add 1 as the matrix is supposed to be augmented.) Having the matrix filled with entries, we reduce it to reduced row echelon form, which has at least one solution (the trivial one). We will obtain two sets: a set of independent variables $S_i = \{s_1,\ldots,s_n\}$ and a set of dependent variables $S_d = \{s'_1,\ldots,s'_m\colon s'_i = f_i(s_1,\ldots,s_n)$ for some linear $f_i\}$. (In case of finite solution set containing only trivial solution, we have $|S_d| = 0$; in such a case, S_i will contain all the debt cuts.) In case of infinite solution set, we choose one that minimizes

$$
\sum_{i=1}^{n} s_i + \sum_{i=1}^{m} s'_i = \sum_{i=1}^{n} s_i + \sum_{i=1}^{m} f_i(s_1,\ldots,s_n)
$$

subject to constraint of not exceeding a contract with a debt cut, which is a linear program that, once solved, yields the desired Ξ.