Computing debt cuts leading to global zero-equity

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Abstract

In this paper we present a method for computing a set of loan cuts, which, once applied, lead to a global zero-equity state, i.e., each and every party in the financial network may forget all liabilities.

1 Basic definitions

Before we proceed to defining the structures needed in discussing the method, we have to impose some definitions: by $\mathfrak{R}_?$, we denote the set of real numbers x such that $x \,?\, 0$ holds. We work on a directed graph $G = (V, A)$, $A \subset V \times V$, for which we define a weight function $w_G \colon A \to \mathcal{P}(\mathfrak{R}_> \times \mathfrak{R}_\geq \times (\mathfrak{N} \cup \{\infty\}) \times \mathfrak{R})$. If $(K, r, n, t) \in \mathcal{P}(\mathfrak{R}_> \times \mathfrak{R}_\geq \times (\mathfrak{N} \cup \{\infty\}) \times \mathfrak{R})$, K is the initial principal investment, r is the annual interest rate, n the amount of compounding periods per year (the value of ∞ is allowed, which denotes continuous compounding), and t is the time point at which the loan was admitted. Together the four parameters comprise a *contract*. (Note that the "weight" of an edge is a **set** of contracts as this is physically possible in real-world banking.)

The most fundamental function in this paper is the equity function $e_G \colon V \times \mathfrak{R} \to \mathfrak{R}$ defined as

$$\begin{aligned} e_G(u, \tau) = & \sum_{(u,v) \in G.A} \left(\sum_{(K,r,n,t) \in w_G(u,v)} \mathfrak{C}_\tau(K, r, n, t) \right) \\ & - \sum_{(v,u) \in G.A} \left(\sum_{(K,r,n,t) \in w_G(v,u)} \mathfrak{C}_\tau(K, r, n, t) \right), \end{aligned}$$

where

$$\mathfrak{C}_\tau(K, r, n, t) = \begin{cases} K\left(1 + \frac{r}{n}\right)^{\lfloor n(\tau - t) \rfloor} & \text{if } n \in \mathfrak{N} \\ K e^{r(\tau - t)} & \text{if } n = \infty. \end{cases}$$

Also, we assume we are given a function $\mathfrak{f}_G \colon G.V \to \mathfrak{R}$ mapping every party u in the financial graph to a time point at which u is ready to pay back at most all of its debts. (By "at most" we mean that we will try to minimize the magnitude of the node's debt cuts, yet it is not possible for a node having no loans to the other nodes, which implies that such node u will have to pay all debts at once at the time point $\mathfrak{f}_G(u)$.)

As the concept of equilibria in this paper is a global phenomenon, we demand that a time point T_G is given; especially, we demand that $T_G \geq \max_{u \in G.V}\{\mathfrak{f}_G(u)\}$. Whenever a party, say $u \in G.V$, is ready to raise C amount of resources for the debt cut to v (which is supposed to happen at $\mathfrak{f}(u)$) with $(K, r, n, t) \in w_G(v, u)$ the contract becomes

$$\mathfrak{C}_\tau\big(\mathfrak{C}_{\mathfrak{f}_G(u)}(K, r, n, t) - C, r, n, \mathfrak{f}_G(u)\big),$$

$\tau \geq \mathfrak{f}_G(u)$. Next we define the concept of equilibrium.

Definition 1 *The financial graph G is said to be in equilibrium at time point τ if and only if $e_G(u, \tau) = 0$ for all $u \in G.V$.*

Once given G, $\mathfrak{f}_G$ and T_G, we attempt to compute a function $\Xi_G\colon G.A \times \mathfrak{R}_> \times \mathfrak{R}_\geq \times \mathfrak{R}_> \times \mathfrak{R} \to \mathfrak{R}_\geq$ such that after applying a debt cut from v to u of magnitude $\Xi(u, v, \mathfrak{K})$ against the contract $\mathfrak{K}$ for all $(u, v) \in G.A$, G obtains such a state that it evolves towards equilibrium at time point T_G.

2 Solution

Suppose we are given a directed edge $(u, v) \in G.A$ for which $|w(u, v)| = n$. Now $w(u, v) = \{\mathfrak{K}_1, \ldots, \mathfrak{K}_n\}$. ($\mathfrak{K}_i$ is a contract tuple; e.g. $\mathfrak{K}.K$ is the initial principal of $\mathfrak{K}$.) Suppose also that v (the debtor node with respect to u) managed to raise C amount of resources to be invested as a debt cut to u at time point $\mathfrak{f}_G(v)$. Now we can partition C into $\{C_1, \ldots, C_n\}$ such that $C_i \geq 0$ for all i and

$$C = \sum_{i=1}^{n} C_i. \tag{1}$$

Another condition is that

$$\sum_{i=1}^{n} \mathfrak{C}_{T_G}\big(\mathfrak{C}_{\mathfrak{f}_G(v)}(\mathfrak{K}_i) - C_i, \mathfrak{K}_i.r, \mathfrak{K}_i.n, \mathfrak{f}_G(v)\big) \tag{2}$$

is minimized; basically minimizing (2) reduces to minimizing

$$\sum_{i=1}^{n} \mathfrak{C}_{T_G}\big(-C_i, \mathfrak{K}_i.r, \mathfrak{K}_i.n, \mathfrak{f}_G(v)\big), \tag{3}$$

and further to maximizing

$$\sum_{i=1}^{n} \mathfrak{C}_{T_G}\big(C_i, \mathfrak{K}_i.r\mathfrak{K}_i.n, \mathfrak{f}_G(v)\big) = \sum_{i=1}^{n} C_i D_{\mathfrak{K}_i}, \tag{4}$$

where

$$D_{\mathfrak{K}} = \begin{cases} \left(1 + \frac{\mathfrak{K}.r}{\mathfrak{K}.n}\right)^{\lfloor \mathfrak{K}_i.n(T_G - \mathfrak{f}_G(v))\rfloor} & \text{if } n \in \mathfrak{N} \\ e^{\mathfrak{K}.r\mathfrak{K}.n} & \text{if } n = \infty. \end{cases}$$

Also $C_i \in [0, \mathfrak{C}_{\mathfrak{f}_G(v)}(\mathfrak{K}_i)]$ for all i. Intuitively, it appears that it makes sense to maximize C_i of which the constant factor $D_{\mathfrak{K}_i}$ is the largest, then substract C_i from C and proceed with the procedure in greed fashion until C becomes zero. The following theorem proves this intuition.

Theorem 1 (Theorem) *Given $C \in \mathfrak{R}_>$, $C_1, \ldots, C_n$ such that $\sum_i C_i = C$, $C_i \geq 0$ for all i, and factors $d_1, \ldots, d_n$*

Now we proceed to equilibrium equation. If $\mathfrak{K}$ is a contract (K, r, n, t), by $A(\mathfrak{K})$ we denote the "accumulation function" defined as

$$\begin{cases} \mathfrak{C}_\tau(K, r, n, t)/K = \left(1 + \frac{r}{n}\right)^{\lfloor n(\tau - t)\rfloor} & \text{if } n \in \mathfrak{N} \\ \mathfrak{C}_\tau(K, r, n, t)/K = e^{r(\tau - t)} & \text{if } n = \infty. \end{cases}$$

Whenever a node u has incoming contracts from a set of parent nodes (lenders) L, outgoing contracts to a set of children (debtors) D, the equilibrium equation for u is

$$\begin{aligned} &\sum_{v \in D} \sum_{\mathfrak{K} \in w_G(u,v)} \mathfrak{C}_{T_G}\big(\mathfrak{K}.K - \Xi(u, v, \mathfrak{K}), \mathfrak{K}.r, \mathfrak{K}.n, \mathfrak{f}_G(v)\big) - \\ &\sum_{v \in L} \sum_{\mathfrak{K} \in w_G(v,u)} \mathfrak{C}_{T_G}\big(\mathfrak{K}.K - \Xi(v, u, \mathfrak{K}), \mathfrak{K}.r, \mathfrak{K}.n, \mathfrak{f}_G(u)\big) = 0. \end{aligned} \tag{5}$$

The above equation is equivalent to

$$\begin{aligned} &\sum_{v \in D} \sum_{\mathfrak{K} \in w_G(u,v)} \mathfrak{C}_{T_G}\big(\Xi(u, v, \mathfrak{K}), \mathfrak{K}.r, \mathfrak{K}.n, \mathfrak{f}_G(v)\big) - \\ &\sum_{v \in L} \sum_{\mathfrak{K} \in w_G(v,u)} \mathfrak{C}_{T_G}\big(\Xi(v, u, \mathfrak{K}), \mathfrak{K}.r, \mathfrak{K}.n, \mathfrak{f}_G(u)\big) = \\ &\sum_{v \in D} \sum_{\mathfrak{K} \in w(u,v)} \mathfrak{C}_{T_G}\big(\mathfrak{K}.K, \mathfrak{K}.r, \mathfrak{K}.n, \mathfrak{f}_G(v)\big) - \\ &\sum_{v \in L} \sum_{\mathfrak{K} \in w(v,u)} \mathfrak{C}_{T_G}\big(\mathfrak{K}.K, \mathfrak{K}.r, \mathfrak{K}.n, \mathfrak{f}_G(u)\big). \end{aligned} \tag{6}$$

Now if we write down equilibrium equations for all nodes $v \in G.V$, we obtain a linear system, which is guaranteed to have a solution as we can choose for each $(u, v) \in G.A$ a debt cut of magnitude

$$\sum_{\mathfrak{K} \in w_G(u,v)} \mathfrak{K}.K$$

which satisfies trivially every $(u, v) \in G.A$.

Suppose we are given a debt cut $C \in \mathfrak{R}_{\geq}$, $(u,v) \in G.A$

Suppose that $w_G(u,v) = \{(K,r,n,t),(K',r',n',t')\}$ and $C \in \mathfrak{R}_{>}$, τ are given. Now the sum is

$$S = \left(K(1+\frac{r}{n})^{\lfloor n(\tau-t)\rfloor} - C + x\right)\left(1+\frac{r}{n}\right)^{\lfloor n(T_G-\tau)\rfloor} + \left(K(1+\frac{r}{n})^{\lfloor n(\tau-t)\rfloor} - x\right)\left(1+\frac{r}{n}\right)^{\lfloor n(T_G-t)\rfloor}$$