Simplifying loan graphs

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Abstract

We choose to tackle a problem of optimizing loan graphs in terms of edge amount. Whenever we have a directed, weighted graph, edges denote the direction of “resource flow” and the weight of each denotes the “velocity” of the flow. Assuming each party honors its debts, and returning a debt requires time and energy, it might be sensible to produce a loan graph “equivalent” to the original, but having less edges.

1 Stating the problem

A loan graph is a directed, weighted graph $G = (V, E)$, where $E \subseteq V^2$. Whenever there is a loan from node u to node v, $(u, v) \in E$. The amount lent is given by the weight function $w \colon E \to \mathfrak{R}_{>}$. The equity function $e_G \colon G.V \to \mathfrak{R}$ is defined as follows:

$$e_G(u) = \sum_{(u,v) \in G.E} w(u, v) - \sum_{(v,u) \in G.E} w(v, u).$$

Definition 1. *Two loan graphs, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be* ***equivalent*** *if and only if there exists a bijection $f \colon V_1 \to V_2$ such that for all $v \in V_1$, $e_{G_1}(v) = e_{G_2}(f(v))$ and we denote such a fact by stating $G_1 \sim G_2$.*

Given an input loan graph G, we wish to find a loan graph G' such that

$$G' \in \underset{G'' \sim G}{\arg\min}\, |G''.E|.$$

Definition 2. *A* ***group*** *is any vertex set $\mathfrak{G} \subseteq V$ such that $\sum_{u \in \mathfrak{G}} e(u) = 0$. A group $\mathfrak{G}$ is said to be a* ***proper group*** *if and only if it cannot be partitioned in two sets S and $\mathfrak{G} \setminus S$ such that S and $\mathfrak{G} \setminus S$ are groups.*

It follows from the definition of a loan graph that for any loan graph G, $G.V$ is a group. Also it is clear that a proper group $\mathfrak{G}$ can be reconnected with exactly $|\mathfrak{G}| - 1$ edges such that equities are preserved. In order to minimize the edge amount, we need to maximize the amount of groups. Whenever we have $V_+ = \{u \in V : e(u) > 0\}$ and $V_- = \{u \in V : e(u) < 0\}$, the amount of proper groups in V is no more than $\min\{|V_+|, |V_-|\}$ as any non-trivial group contains at least one node from V_+ and at least one node from V_-.

Definition 3. *The **order** of a group $\mathfrak{G}$ is simply the amount of vertices in a group, i.e., $|\mathfrak{G}|$.*

Definition 4. *A **binary group** $\{u, v\}$ from G, is a proper group of order 2 ($e_G(u) = -e_G(v) \neq 0$).*

Definition 5. *A **trivial** group is any vertex u with $e(u) = 0$.*

Theorem 1 (Binary group theorem)**.** *In any loan graph G with N proper groups, $k \leq N$ of which are binary, resolving all binary groups in greedy fashion before computing the higher-order groups does not necessarily lead to suboptimal solution.*

Proof. Suppose the contrary: resolving all k binary groups leaves in the graph $N-k$ proper groups. Now, by definition of suboptimality we have $k+(N-k) < N$, which is absurd. $\square$

2 Exact solutions

Suppose we are given V_+ and V_-. One solution is to generate all partitions of V_+, and for each partition of V_+ generate all possible partitions of V_-. Once generating a single partition requires $O(n)$ time, it is obvious that running time of an exact, yet brute-force algorithm to simplify a graph $G = (V, E)$ is

$$O\left(|V_+| \sum_{i=1}^{|V_+|} \left\{ {|V_+| \atop i} \right\} \left(|V_-| \sum_{j=1}^{|V_-|} \left\{ {|V_-| \atop j} \right\} \right) \right).$$

We , nonetheless, can do better. Suppose we have along the general partition generator (capable to generate all the partitions) a specialized generator generating only all k-partitions. Assuming the latter can do it in linear time, the running time for the entire algorithm reduces to

$$O\left(|V_+||V_-| \sum_{i=1}^{\min\{|V_+|,|V_-|\}} \left\{ {\min\{|V_+|, |V_-|\} \atop i} \right\} \left\{ {\max\{|V_+|, |V_-|\} \atop i} \right\} \right).$$