Till infinity and beyond!

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Abstract

In this paper, we will consider a space ship approaching the speed of light.

1 Introduction

Suppose we are given a space ship. While stationary, its mass is m_0 and its power output is P_0. At $t = 0$, the ship takes off and starts accelerating. We want to calculate the speed function $v(t)$, taking time dilation and relativistic mass into account. Whenever the speed of the ship is $v(t)$, its relativistic mass is

$$m(t) = \frac{m_0}{\sqrt{1 - \frac{v^2(t)}{c^2}}},$$

and, due to time dilation, its power output is

$$P(t) = \frac{P_0}{\frac{1}{\sqrt{1-\frac{v^2(t)}{c^2}}}} = P_0\sqrt{1 - \frac{v^2(t)}{c^2}}.$$

Since

$$P = \frac{W}{t} = \frac{Fs}{t} = Fv = mav,$$

we have that

$$P(t) = m(t)\frac{\mathrm{d}v(t)}{\mathrm{d}t}v(t).$$

Now,

$$P_0\sqrt{1-\frac{v^2(t)}{c^2}} = \frac{m_0}{\sqrt{1-\frac{v^2(t)}{c^2}}}\frac{\mathrm{d}v(t)}{\mathrm{d}t}v(t),$$

which leads to

$$\begin{aligned}\frac{\mathrm{d}v(t)}{\mathrm{d}t} &= \frac{P_0}{m_0}\frac{1-\frac{v^2(t)}{c^2}}{v(t)}\\ &= \frac{P_0}{m_0}\left(\frac{1}{v(t)}-\frac{v(t)}{c^2}\right).\end{aligned}$$

Rewriting to conventional notation for differential equations, we obtain

$$y(t)' = \frac{P_0}{m_0}\left(\frac{1}{y(t)}-\frac{y(t)}{c^2}\right).$$

The solution is

$$v(t) = y(t) = \sqrt{c^2 - e^{C-2P_0t/c^2m_0}},$$

where C is the integration constant. Since $v(0)=0$, we must have

$$\sqrt{c^2-e^{C-2P_0\cdot 0/c^2m_0}} = \sqrt{c^2-e^C} = 0,$$

which leads to

$$\begin{aligned}e^C &= c^2\\ \ln e^C &= \ln c^2\\ C &= 2\ln c.\end{aligned}$$

Finally,

$$\begin{aligned}v(t) &= \sqrt{c^2-e^{2\ln c-\frac{2P_0t}{c^2m_0}}}\\ &= \sqrt{c^2-c^2e^{-\frac{2P_0t}{c^2m_0}}}\\ &= c\sqrt{1-e^{-\frac{2P_0t}{c^2m_0}}}.\end{aligned}$$

Above, we have

$$\begin{aligned}
v^2(t) &= c^2\left(1 - e^{-\frac{2P_0 t}{c^2 m_0}}\right), \\
\frac{v^2(t)}{c^2} &= 1 - e^{-\frac{2P_0 t}{c^2 m_0}}, \\
e^{-\frac{2P_0 t}{c^2 m_0}} &= 1 - \frac{v^2(t)}{c^2}. \qquad (1)
\end{aligned}$$

Also,

$$\begin{aligned}
\frac{\mathrm{d}v(t)}{\mathrm{d}t} &= c\frac{1}{2}\left(1 - e^{-\frac{2P_0 t}{c^2 m_0}}\right)^{-\frac{1}{2}} \frac{\mathrm{d}}{\mathrm{d}t}\left(1 - e^{-\frac{2P_0 t}{c^2 m_0}}\right) \\
&= \frac{c}{2}\left(1 - e^{-\frac{2P_0 t}{c^2 m_0}}\right)^{-\frac{1}{2}} \left(- e^{-\frac{2P_0 t}{c^2 m_0}}\right)\left(-\frac{2P_0}{c^2 m_0}\right) \\
&= \frac{c}{2}\frac{2P_0 e^{-\frac{2P_0 t}{c^2 m_0}}}{c^2 m_0\sqrt{1 - e^{-\frac{2P_0 t}{c^2 m_0}}}} \\
&= \frac{P_0 e^{-\frac{2P_0 t}{c^2 m_0}}}{c m_0\sqrt{1 - e^{-\frac{2P_0 t}{c^2 m_0}}}},
\end{aligned}$$

so

$$\begin{aligned}
m(t)v(t)\frac{\mathrm{d}v(t)}{\mathrm{d}t} &= \frac{m_0}{\sqrt{1 - \frac{v^2(t)}{c^2}}} c\sqrt{1 - e^{-\frac{2P_0 t}{c^2 m_0}}} \frac{P_0 e^{-\frac{2P_0 t}{c^2 m_0}}}{c m_0\sqrt{1 - e^{-\frac{2P_0 t}{c^2 m_0}}}} \\
&= \frac{P_0 e^{-\frac{2P_0 t}{c^2 m_0}}}{\sqrt{1 - \frac{v^2(t)}{c^2}}} \\
&\overset{(1)}{=} \frac{P_0\left(1 - \frac{v^2(t)}{c^2}\right)}{\sqrt{1 - \frac{v^2(t)}{c^2}}} \\
&= P_0\sqrt{1 - \frac{v^2(t)}{c^2}} \\
&= P(t),
\end{aligned}$$

as expected.