

An example of computing flux across a moving surface in a time-dependent vector field

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Suppose we are given a vector field $\mathbf{F}(x, y, z, t)$, $\mathbf{F}: \mathbb{R}^4 \rightarrow \mathbb{R}^3$ describing the motion of fluid at any particular time t , and a parametric surface \mathcal{S} parametrized by $\mathbf{r}(u, v, t)$, $\mathbf{r}: A \times \mathbb{R} \rightarrow \mathbb{R}^3$, that, just like the vector field, evolves with time. The question we want to answer is how to compute the amount of fluid going across the moving surface within a particular time range. First of all we need to define a normal unit vector to the surface \mathbf{r} :

$$\hat{\mathbf{N}} = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \cdot \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right|^{-1}.$$

Taking motion into account, we have that the surface element at time point t passes the space at rate

$$\left\langle \hat{\mathbf{N}}, \frac{\partial \mathbf{r}}{\partial t} \right\rangle.$$

Next we need to consider the actual vector field \mathbf{F} and the flux it generates across an area element. As there is no any dependence between the vector field and the

surface, we obtain that the flux across the surface \mathcal{S} at moment t is

$$\begin{aligned}
\Phi(t) &= \iint_{\mathcal{S}} \langle \mathbf{F}, \hat{\mathbf{N}} \rangle - \left\langle \frac{\partial \mathbf{r}}{\partial t}, \hat{\mathbf{N}} \right\rangle dS \\
&= \iint_{\mathcal{S}} \left\langle \mathbf{F} - \frac{\partial \mathbf{r}}{\partial t}, \hat{\mathbf{N}} \right\rangle dS \\
&= \iint_A \left\langle \mathbf{F} - \frac{\partial \mathbf{r}}{\partial t}, \hat{\mathbf{N}} \right\rangle \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| du dv \\
&= \iint_A \left\langle \mathbf{F} - \frac{\partial \mathbf{r}}{\partial t}, \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \cdot \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right|^{-1} \right\rangle \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| du dv \\
&= \iint_A \left\langle \mathbf{F} - \frac{\partial \mathbf{r}}{\partial t}, \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\rangle du dv \\
&= \iint_A \left\langle F(r(u, v, t), t), \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\rangle du dv,
\end{aligned}$$

since

$$dS = \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| du dv.$$

In order to find out the actual amount of fluid going through a moving surface \mathcal{S} , we integrate $\Phi(t)$ over a desired time range $[t_a, t_b]$:

$$\hat{\Phi} = \int_{t_a}^{t_b} \Phi(t) dt.$$

Example

Suppose $\mathbf{F}(x, y, z, t) = (0, 0, t)$ and $\mathbf{r}(u, v, t) = (u, v, 2t)$, $A = [0, 1]^2$. We have that each fluid particle has direction upwards, and its velocity grows linearly with time; the surface \mathbf{r} is a plane parallel to x, y - plane and it moves with constant speed of

two length units per a time unit upwards. Now we have

$$\begin{aligned}\frac{\partial \mathbf{r}}{\partial u} &= (1, 0, 0), \\ \frac{\partial \mathbf{r}}{\partial v} &= (0, 1, 0), \\ \frac{\partial \mathbf{r}}{\partial t} &= (0, 0, 2).\end{aligned}$$

$$\begin{aligned}\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} \\ &= \mathbf{i} \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \\ &= \mathbf{k} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \\ &= \mathbf{k} \\ &= (0, 0, 1).\end{aligned}$$

Now $\mathbf{F}(r(u, v, t), t) = (0, 0, t)$, $\frac{\partial \mathbf{r}}{\partial t} = (0, 0, 2)$ and $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = (0, 0, 1)$, so

$$\begin{aligned}\Phi(t) &= \iint_{\mathcal{S}} \langle \hat{\mathbf{N}}, \mathbf{F} - \frac{\partial \mathbf{r}}{\partial t} \rangle dS \\ &= \iint_A \langle \mathbf{F}(r(u, v, t), t) - \frac{\partial \mathbf{r}}{\partial t}, \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \rangle du dv \\ &= \iint_A \langle (0, 0, t) - (0, 0, 2), (0, 0, 1) \rangle du dv \\ &= \iint_A \langle (0, 0, t - 2), (0, 0, 1) \rangle du dv \\ &= \iint_A t - 2 du dv.\end{aligned}$$

Since $A = [0, 1]^2$,

$$\begin{aligned}\Phi(t) &= \iint_A t - 2 du dv \\ &= t - 2.\end{aligned}$$

Observe that at $t = 0$, the flux is negative (-2), since at that moment \mathbf{F} is stationary and \mathcal{S} is moving upwards with constant speed of two units. Also, at $t = 2$ the flux is zero, since the surface at that moment is moving exactly with the same speed as the particles at \mathcal{S} .

Now the amount of fluid flowing across the surface \mathcal{S} during time interval $[0, 10]$ is

$$\begin{aligned}\hat{\Phi} &= \int_0^{10} t - 2 \, dt \\ &= \left[\frac{1}{2}t^2 - 2t \right]_{t=0}^{t=10} \\ &= 30.\end{aligned}$$

If we aim to calculate the amount of particles going through \mathcal{S} in **any** direction, we should have substituted $t - 2$ with $|t - 2|$ in the equation above, which would have produced 34.