An example on computing flux through a moving surface in a time-dependent vector field

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Suppose we are given a vector field $\mathbf{F}(x, y, z, t)$ describing motion of fluid at any particular time t, and a parametric curve $\mathbf{r}(u, v, t)$ that, just like the vector field, evolves with time. The question we want to answer is how to compute the amount of fluid going through the surface within a particular time range using a simple example. Let assume that the time interval is $[t_1, t_2]$ and $u \in [0, a], v \in [0, b]$. If

$$\Phi(t) = \iint\limits_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$$

is the rate of flux at a particular moment, the amount of fluid flowing throught the surface between the moments t_1 and t_2 is

$$\int\limits_{t_1}^{t_2} \Phi(t)\,\mathrm{d}t.$$

Now, let us define $\mathbf{F}(x, y, z, t) = (xt, y^2, x + t)$ and $\mathbf{r}(u, v, t) = (\overbrace{u - t}^{x}, \overbrace{vt}^{y}, \overbrace{uvt}^{z})$. Next, we have

$$\frac{\partial \mathbf{r}}{\partial u} = \frac{\partial}{\partial u}(u - t, vt, uvt) = (1, 0, vt) \qquad \frac{\partial \mathbf{r}}{\partial v} = \frac{\partial}{\partial v}(u - t, vt, uvt) = (0, t, ut),$$

so

$$\begin{aligned}
\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial u} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & vt \\ 0 & t & ut \end{vmatrix} \\
&= \mathbf{i}\begin{vmatrix} 0 & vt \\ t & ut \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & vt \\ 0 & ut \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & 0 \\ 0 & t \end{vmatrix} \\
&= -vt^2\mathbf{i} - ut\mathbf{j} + t\mathbf{k} \\
&= (-vt^2, -ut, t).
\end{aligned}$$

Also

$$\mathbf{F} = (xt, y^2, z + t) = (ut - t^2, v^2t^2, uvt + t),$$

which leads us to

$$
\begin{aligned}
\Phi(t) &= \iint\limits_{\mathcal{S}} \mathbf{F} \cdot \mathrm{d}\mathbf{S} \\
&= \int_0^b \int_0^a (ut - t^2, v^2t^2, uvt + t) \cdot (-vt^2, -ut, t) \, \mathrm{d}u \, \mathrm{d}v \\
&= \int_0^b \int_0^a vt^4 - uvt^3 - uv^2t^3 + uvt^2 + t^2 \, \mathrm{d}u \, \mathrm{d}v \\
&= \int_0^b \left[uvt^4 - \frac{1}{2}u^2vt^3 - \frac{1}{2}u^2v^2t^3 + \frac{1}{2}u^2vt^2 + ut^2 \right]_{u=0}^{u=a} \mathrm{d}v \\
&= \int_0^b avt^4 - \frac{1}{2}a^2vt^3 - \frac{1}{2}a^2v^2t^3 + \frac{1}{2}a^2vt^2 + at^2 \, \mathrm{d}v \\
&= \left[\frac{1}{2}av^2t^4 - \frac{1}{4}a^2v^2t^3 - \frac{1}{6}a^2v^3t^3 + \frac{1}{4}a^2v^2t^2 + avt^2 \right]_{v=0}^{v=b} \\
&= \frac{1}{2}ab^2t^4 - \frac{1}{4}a^2b^2t^3 - \frac{1}{6}a^2b^3t^3 + \frac{1}{4}a^2b^2t^2 + abt^2 \\
&= \frac{ab^2}{2}t^4 - \left(\frac{a^2b^3}{6} + \frac{a^2b^2}{4} \right) t^3 + \left(\frac{a^2b^2}{4} + ab \right) t^2.
\end{aligned}
$$

Now let us assign $a = 1$ and $b = 3$; we obtain

$$
\begin{aligned}
\Phi(t) &= \frac{9}{2}t^4 - \left(\frac{27}{6} + \frac{9}{4} \right) t^3 + \left(\frac{9}{4} + 3 \right) t^2 \\
&= \frac{9t^4}{2} - \frac{27t^3}{4} + \frac{21t^2}{4}.
\end{aligned}
$$

Let us choose $t_1 = 1$ and $t_2 = 3$; the total amount of fluid flowed through the surface is

$$
\begin{aligned}
\int_{t_1}^{t_2} \Phi(t) \, \mathrm{d}t &= \int_1^3 \frac{9t^4}{2} - \frac{27t^3}{4} + \frac{21t^2}{4} \, \mathrm{d}t \\
&= \left[\frac{9}{10}t^5 - \frac{27}{16}t^4 + \frac{21}{12}t^3 \right]_{t=1}^{t=3} \\
&= \left[\frac{2187}{10} - \frac{2187}{16} + \frac{567}{12} \right] - \left[\frac{9}{10} - \frac{27}{16} + \frac{21}{12} \right] \\
&= \frac{2178}{10} - \frac{2160}{16} + \frac{546}{12} \\
&= 128.3.
\end{aligned}
$$