

# An example of computing flux across a moving surface in a time-dependent vector field

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Suppose we are given a vector field  $\mathbf{F}(x, y, z, t)$ ,  $\mathbf{F}: \mathbb{R}^4 \rightarrow \mathbb{R}^3$  describing the motion of fluid at any particular time  $t$ , and a parametric surface  $\mathcal{S}$  parametrized by  $\mathbf{r}(u, v, t)$ ,  $\mathbf{r}: A \times \mathbb{R} \rightarrow \mathbb{R}^3$ , that, just like the vector field, evolves with time. The question we want to answer is how to compute the amount of fluid going across the moving surface within a particular time range. First of all we need to define a normal unit vector to the surface  $\mathbf{r}$ :

$$\hat{\mathbf{N}} = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \cdot \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right|^{-1}.$$

Taking motion into account, we have that the surface element at time point  $t$  passes the space at rate

$$\left\langle \hat{\mathbf{N}}, \frac{\partial \mathbf{r}}{\partial t} \right\rangle.$$

Next we need to consider the actual vector field  $\mathbf{F}$  and the flux it generates across an area element. As there is no any dependence between the vector field and the

surface, we obtain that the flux across the surface  $\mathcal{S}$  at moment  $t$  is

$$\begin{aligned}
\Phi(t) &= \iint_{\mathcal{S}} \langle \mathbf{F}, \hat{\mathbf{N}} \rangle - \left\langle \frac{\partial \mathbf{r}}{\partial t}, \hat{\mathbf{N}} \right\rangle dS \\
&= \iint_{\mathcal{S}} \left\langle \mathbf{F} - \frac{\partial \mathbf{r}}{\partial t}, \hat{\mathbf{N}} \right\rangle dS \\
&= \iint_A \left\langle \mathbf{F} - \frac{\partial \mathbf{r}}{\partial t}, \hat{\mathbf{N}} \right\rangle \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| du dv \\
&= \iint_A \left\langle \mathbf{F} - \frac{\partial \mathbf{r}}{\partial t}, \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \cdot \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right|^{-1} \right\rangle \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| du dv \\
&= \iint_A \left\langle \mathbf{F} - \frac{\partial \mathbf{r}}{\partial t}, \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\rangle du dv \\
&= \iint_A \left\langle \mathbf{F}(r(u, v, t), t) - \frac{\partial \mathbf{r}}{\partial t}, \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\rangle du dv,
\end{aligned}$$

since

$$dS = \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| du dv.$$

In order to find out the actual amount of fluid going through a moving surface  $\mathcal{S}$ , we integrate  $\Phi(t)$  over a desired time range  $[t_a, t_b]$ :

$$\hat{\Phi} = \int_{t_a}^{t_b} \Phi(t) dt.$$

## Example

Suppose  $\mathbf{F}(x, y, z, t) = (0, 0, t)$  and  $\mathbf{r}(u, v, t) = (u, v, 2t)$ ,  $A = [0, 1]^2$ . We have that each fluid particle has direction upwards, and its velocity grows linearly with time; the surface  $\mathbf{r}$  is a plane parallel to  $x, y$  - plane and it moves with constant speed of

two length units per a time unit upwards. Now we have

$$\begin{aligned}\frac{\partial \mathbf{r}}{\partial u} &= (1, 0, 0), \\ \frac{\partial \mathbf{r}}{\partial v} &= (0, 1, 0), \\ \frac{\partial \mathbf{r}}{\partial t} &= (0, 0, 2).\end{aligned}$$

$$\begin{aligned}\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} \\ &= \mathbf{i} \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \\ &= \mathbf{k} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \\ &= \mathbf{k} \\ &= (0, 0, 1).\end{aligned}$$

Now  $\mathbf{F}(r(u, v, t), t) = (0, 0, t)$ ,  $\frac{\partial \mathbf{r}}{\partial t} = (0, 0, 2)$  and  $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = (0, 0, 1)$ , so

$$\begin{aligned}\Phi(t) &= \iint_{\mathcal{S}} \langle \mathbf{F} - \frac{\partial \mathbf{r}}{\partial t}, \hat{\mathbf{N}} \rangle dS \\ &= \iint_A \langle \mathbf{F}(r(u, v, t), t) - \frac{\partial \mathbf{r}}{\partial t}, \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \rangle du dv \\ &= \iint_A \langle (0, 0, t) - (0, 0, 2), (0, 0, 1) \rangle du dv \\ &= \iint_A \langle (0, 0, t - 2), (0, 0, 1) \rangle du dv \\ &= \iint_A t - 2 du dv.\end{aligned}$$

Since  $A = [0, 1]^2$ ,

$$\begin{aligned}\Phi(t) &= \iint_A t - 2 du dv \\ &= t - 2.\end{aligned}$$

Observe that at  $t = 0$ , the flux is negative ( $-2$ ), since at that moment  $\mathbf{F}$  is stationary and  $\mathcal{S}$  is moving upwards with constant speed of two units. Also, at  $t = 2$  the flux is zero, since the surface at that moment is moving exactly with the same speed as the particles at  $\mathcal{S}$ .

Now the amount of fluid flowing across the surface  $\mathcal{S}$  during time interval  $[0, 10]$  is

$$\begin{aligned}\hat{\Phi} &= \int_0^{10} t - 2 \, dt \\ &= \left[ \frac{1}{2}t^2 - 2t \right]_{t=0}^{t=10} \\ &= 30.\end{aligned}$$

If we aim to calculate the amount of particles going through  $\mathcal{S}$  in **any** direction, we should have substituted  $t - 2$  with  $|t - 2|$  in the equation above, which would have produced 34.