An example of computing flux across a moving surface in a time-dependent vector field

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Suppose we are given a vector field $\mathbf{F}(x, y, z, t)$, $\mathbf{F}\colon \mathbb{R}^4 \to \mathbb{R}^3$, describing the motion of fluid at any particular time t, and a parametric surface $\mathscr{S}$ parametrized by $\mathbf{r}(u, v, t), \mathbf{r}\colon A \times \mathbb{R} \to \mathbb{R}^3$, that, just like the vector field, evolves with time. The question we want to answer is how to compute the amount of fluid going across the moving surface within a particular time range. First of all we need to define a normal unit vector to the surface $\mathscr{S}$:

$$\hat{\mathbf{N}} = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \cdot \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right|^{-1}.$$

Taking motion into account, we have that the surface element at time t passes the space at rate

$$\left\langle \hat{\mathbf{N}}, \frac{\partial \mathbf{r}}{\partial t} \right\rangle.$$

Next we need to consider the actual vector field $\mathbf{F}$ and the flux it generates across an area element. As there is no any dependence between the vector field and the surface, we obtain that the flux across the surface $\mathscr{S}$ at moment t is

$$\begin{aligned}
\Phi(t) &= \iint_{\mathscr{S}} \langle \mathbf{F}, \hat{\mathbf{N}} \rangle - \langle \frac{\partial \mathbf{r}}{\partial t}, \hat{\mathbf{N}} \rangle \, \mathrm{d}S \\
&= \iint_{\mathscr{S}} \langle \mathbf{F} - \frac{\partial \mathbf{r}}{\partial t}, \hat{\mathbf{N}} \rangle \, \mathrm{d}S \\
&= \iint_{A} \langle \mathbf{F} - \frac{\partial \mathbf{r}}{\partial t}, \hat{\mathbf{N}} \rangle \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| \mathrm{d}u \, \mathrm{d}v \\
&= \iint_{A} \left\langle \mathbf{F} - \frac{\partial \mathbf{r}}{\partial t}, \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \cdot \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right|^{-1} \right\rangle \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| \mathrm{d}u \, \mathrm{d}v \\
&= \iint_{A} \left\langle \mathbf{F} - \frac{\partial \mathbf{r}}{\partial t}, \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\rangle \mathrm{d}u \, \mathrm{d}v \\
&= \iint_{A} \left\langle \mathbf{F}(r(u,v,t),t) - \frac{\partial \mathbf{r}}{\partial t}, \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\rangle \mathrm{d}u \, \mathrm{d}v,
\end{aligned}$$

since

$$\mathrm{d}S = \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| \mathrm{d}u \, \mathrm{d}v.$$

In order to find out the actual amount of fluid going through a moving surface $\mathscr{S}$, we integrate $\Phi(t)$ over a desired time range $[t_a, t_b]$:

$$V = \int_{t_a}^{t_b} \Phi(t) \, \mathrm{d}t.$$

Example

Suppose $\mathbf{F}(x,y,z,t) = (0,0,t)$ and $\mathbf{r}(u,v,t) = (u,v,2t)$, $A = [0,1]^2$. We have that each fluid particle has direction upwards, and its velocity grows linearly with time; the surface $\mathbf{r}$ is a plane parallel to x, y - plane and it moves with constant speed of two length units per a time unit upwards. Now we have

$$\begin{aligned}\frac{\partial \mathbf{r}}{\partial u} &= (1,0,0),\\ \frac{\partial \mathbf{r}}{\partial v} &= (0,1,0),\\ \frac{\partial \mathbf{r}}{\partial t} &= (0,0,2).\end{aligned}$$

$$\begin{aligned}\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}\\ &= \mathbf{i}\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\\ &= \mathbf{k}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\\ &= \mathbf{k}\\ &= (0,0,1).\end{aligned}$$

Now $\mathbf{F}(r(u,v,t),t) = (0,0,t)$, $\frac{\partial \mathbf{r}}{\partial t} = (0,0,2)$ and $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = (0,0,1)$, so

$$\begin{aligned}\Phi(t) &= \iint_{\mathscr{S}} \langle \mathbf{F} - \frac{\partial \mathbf{r}}{\partial t}, \hat{\mathbf{N}} \rangle \,\mathrm{d}S\\ &= \iint_{A} \langle \mathbf{F}(r(u,v,t),t) - \frac{\partial \mathbf{r}}{\partial t}, \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \rangle \,\mathrm{d}u\,\mathrm{d}v\\ &= \iint_{A} \langle (0,0,t) - (0,0,2), (0,0,1) \rangle \,\mathrm{d}u\,\mathrm{d}v\\ &= \iint_{A} \langle (0,0,t-2), (0,0,1) \rangle \,\mathrm{d}u\,\mathrm{d}v\\ &= \iint_{A} t - 2 \,\mathrm{d}u\,\mathrm{d}v.\end{aligned}$$

Since $A = [0,1]^2$,

$$\begin{aligned}\Phi(t) &= \iint_{A} t - 2 \,\mathrm{d}u\,\mathrm{d}v\\ &= t - 2.\end{aligned}$$

Observe that at $t = 0$, the flux is negative (-2), since at that moment $\mathbf{F}$ is stationary and $\mathscr{S}$ is moving upwards with constant speed of two units. Also, at $t = 2$ the flux is zero, since the surface at that moment is moving exactly with the same speed as the particles at $\mathscr{S}$.

Now the amount of fluid flowing across the surface $\mathscr{S}$ during time interval $[0, 10]$ is

$$\begin{aligned} V &= \int_0^{10} t - 2 \,\mathrm{d}t \\ &= \left[\frac{1}{2}t^2 - 2t\right]_{t=0}^{t=10} \\ &= 30. \end{aligned}$$

If we aim to calculate the amount of particles going through $\mathscr{S}$ in **any** direction, we should have substituted $t - 2$ with $|t - 2|$ in the equation above, which would have produced 34.