An example on computing flux across a moving surface in a time-dependent vector field

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Suppose we are given a vector field $\mathbf{F}(x, y, z, t)$, $\mathbf{F}\colon \mathbb{R}^4 \to \mathbb{R}^3$ describing the motion of fluid at any particular time t, and a parametric surface $\mathscr{S}$, $\mathbf{r}(u, v, t), \mathbf{r}\colon A \times \mathbb{R} \to \mathbb{R}^3$ that, just like the vector field, evolves with time. The question we want to answer is how to compute the amount of fluid going across the moving surface within a particular time range. First of all we need to define a normal unit vector to the surface $\mathbf{r}$:

$$\hat{\mathbf{N}} = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \cdot \Big| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \Big|^{-1}.$$

Taking motion into account, we have that the surface element at time point t passes the space at rate

$$\langle \hat{\mathbf{N}}, \frac{\partial \mathbf{r}}{\partial t} \rangle.$$

Next we need to consider the actual vector field $\mathbf{F}$ and the flux it generates across an area element. As there is no any dependence between the vector field and the surface, we obtain that the flux across the surface $\mathscr{S}$ at moment t is

$$\begin{aligned} \Phi(t) &= \iint_{\mathscr{S}} \langle \hat{\mathbf{N}}, \mathbf{F} \rangle - \langle \hat{\mathbf{N}}, \frac{\partial \mathbf{r}}{\partial t} \rangle \, \mathrm{d}S \\ &= \iint_{\mathscr{S}} \langle \hat{\mathbf{N}}, \mathbf{F} - \frac{\partial \mathbf{r}}{\partial t} \rangle \, \mathrm{d}S \end{aligned}$$

In order to find out the actual amount of fluid going through a moving surface $\mathscr{S}$, we integrate $\Phi(t)$ over a desired time range $[t_a, t_b]$:

$$\hat{\Phi} = \int_{t_a}^{t_b} \Phi(t) \, \mathrm{d}t.$$

Example

Suppose $\mathbf{F}(x, y, z, t) = (0, 0, t)$ and $\mathbf{r}(u, v, t) = (u, v, 2t)$, $A = [0, 1]^2$. Now we have

$$\begin{aligned}
\frac{\partial \mathbf{r}}{\partial u} &= (1, 0, 0), \\
\frac{\partial \mathbf{r}}{\partial v} &= (0, 1, 0), \\
\frac{\partial \mathbf{r}}{\partial t} &= (0, 0, 2).
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} \\
&= \mathbf{i} \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \\
&= \mathbf{k} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \\
&= \mathbf{k} \\
&= (0, 0, 1).
\end{aligned}$$

Now $\mathbf{F}(r(u, v, t), t) = (0, 0, t)$, $\frac{\partial \mathbf{r}}{\partial t} = (0, 0, 2)$ and $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = (0, 0, 1)$, so

$$\begin{aligned}
\Phi(t) &= \iint_{\mathscr{S}} \langle \hat{\mathbf{N}}, \mathbf{F} - \frac{\partial \mathbf{r}}{\partial t} \rangle \,\mathrm{d}S \\
&= \iint_A \langle \mathbf{F}(r(u, v, t), t) - \frac{\partial \mathbf{r}}{\partial t}, \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \rangle \,\mathrm{d}u \,\mathrm{d}v \\
&= \iint_A \langle (0, 0, t) - (0, 0, 2), (0, 0, 1) \rangle \,\mathrm{d}u \,\mathrm{d}v \\
&= \iint_A \langle (0, 0, t - 2), (0, 0, 1) \rangle \,\mathrm{d}u \,\mathrm{d}v \\
&= \iint_A t - 2 \,\mathrm{d}u \,\mathrm{d}v.
\end{aligned}$$

Since $A = [0, 1]^2$,

$$\begin{aligned}
\Phi(t) &= \iint_A t - 2 \,\mathrm{d}u \,\mathrm{d}v \\
&= t - 2.
\end{aligned}$$

Now the amount of fluid flowing across the surface $\mathscr{S}$ during time interval $[0, 10]$ is

$$\begin{aligned}
\hat{\Phi} &= \int_0^{10} t - 2 \, \mathrm{d}t \\
&= \left[\frac{1}{2} t^2 - 2t \right]_{t=0}^{t=10} \\
&= 30.
\end{aligned}$$