CHAPTER 18

Simplex-Based Sensitivity Analysis and Duality

CONTENTS

In Chapter 3 we defined sensitivity analysis as the study of how the changes in the coefficients of a linear program affect the optimal solution. In this chapter we discuss how sensitivity analysis information such as the ranges for the objective function coefficients, dual prices, and the ranges for the right-hand-side values can be obtained from the final simplex tableau. The topic of duality is also introduced. We will see that associated with every linear programming problem is a dual problem that has an interesting economic interpretation.

18.1 SENSITIVITY ANALYSIS WITH THE SIMPLEX TABLEAU

The usual sensitivity analysis for linear programs involves computing ranges for the objective function coefficients and the right-hand-side values, as well as the dual prices.

Objective Function Coefficients

Sensitivity analysis for an objective function coefficient involves placing a range on the coefficient's value. We call this range the **range of optimality.** As long as the actual value of the objective function coefficient is within the range of optimality, *the current basic feasible solution will remain optimal.* The range of optimality for a basic variable defines the objective function coefficient values for which that variable will remain part of the current optimal basic feasible solution. The range of optimality for a nonbasic variable defines the objective function coefficient values for which that variable will remain nonbasic.

In computing the range of optimality for an objective function coefficient, all other coefficients in the problem are assumed to remain at their original values; in other words, *only one coefficient is allowed to change at a time.* To illustrate the process of computing ranges for objective function coefficients, recall the HighTech Industries problem introduced in Chapter 17. The linear program for this problem is restated as follows:

$$
\begin{aligned}
&\text{Max} \quad 50x_1 + 40x_2 \\
&\text{s.t.} \\
&\qquad 3x_1 + 5x_2 \le 150 \quad \text{Assembly time} \\
&\qquad\qquad 1x_2 \le 20 \quad \text{Portable display} \\
&\qquad 8x_1 + 5x_2 \le 300 \quad \text{Warehouse capacity} \\
&\qquad x_1, x_2 \ge 0
\end{aligned}
$$

where

$$
\begin{aligned}
x_1 &= \text{number of units of the Deskpro} \\
x_2 &= \text{number of units of the Portable}
\end{aligned}
$$

The final simplex tableau for the HighTech problem is as follows.

		x_1	x_2	s_1	s_2	s_3	
Basis	c_B	50	40	0	0	0	
x_2	40	0	1	$8/25$	0	$-3/25$	**12**
s_2	0	0	0	$-8/25$	1	$3/25$	**8**
x_1	50	1	0	$-5/25$	0	$5/25$	**30**
z_j		50	40	$14/5$	0	$26/5$	**1980**
$c_j - z_j$		0	0	$-14/5$	0	$-26/5$	

Recall that when the simplex method is used to solve a linear program, an optimal solution is recognized when all entries in the net evaluation row ($c_j - z_j$) are ≤ 0. Because the preceding simplex tableau satisfies this criterion, the solution shown is optimal. However, if a change in one of the objective function coefficients were to cause one or more of the $c_j - z_j$ values to become positive, then the current solution would no longer be optimal; in such a case, one or more additional simplex iterations would be necessary to find the new optimal solution. *The range of optimality for an objective function coefficient is determined by those coefficient values that maintain*

$$c_j - z_j \leq 0 \tag{18.1}$$

for all values of j.

Let us illustrate this approach by computing the range of optimality for c_1, the profit contribution per unit of the Deskpro. Using c_1 (instead of 50) as the objective function coefficient of x_1, the final simplex tableau is as follows:

		x_1	x_2	s_1	s_2	s_3	
Basis	c_B	c_1	40	0	0	0	
x_2	40	0	1	$8/25$	0	$-3/25$	**12**
s_2	0	0	0	$-8/25$	1	$3/25$	**8**
x_1	c_1	1	0	$-5/25$	0	$5/25$	**30**
z_j		c_1	40	$\dfrac{64 - c_1}{5}$	0	$\dfrac{c_1 - 24}{5}$	$\mathbf{480 + 30c_1}$
$c_j - z_j$		0	0	$\dfrac{c_1 - 64}{5}$	0	$\dfrac{24 - c_1}{5}$	

Changing an objective function coefficient will result in changes in the z_j and $c_j - z_j$ rows, but not in the variable values.

Note that this tableau is the same as the previous optimal tableau except that c_1 replaces 50. Thus, we have a c_1 in the objective function coefficient row and the c_B column, and the z_j and $c_j - z_j$ rows have been recomputed using c_1 instead of 50. The current solution will remain optimal as long as the value of c_1 results in all $c_j - z_j \leq 0$. Hence, from the column for s_1 we must have

$$\frac{c_1 - 64}{5} \leq 0$$

and from the column for s_3, we must have

$$\frac{24 - c_1}{5} \leq 0$$

Using the first inequality, we obtain

$$c_1 - 64 \leq 0$$

or

$$c_1 \leq 64 \tag{18.2}$$

Similarly, from the second inequality, we obtain

$$24 - c_1 \leq 0$$

or

$$24 \leq c_1 \tag{18.3}$$

Because c_1 must satisfy both inequalities (18.2) and (18.3), the range of optimality for c_1 is given by

$$24 \leq c_1 \leq 64 \tag{18.4}$$

To see how management of HighTech can make use of this sensitivity analysis information, suppose an increase in material costs reduces the profit contribution per unit for the Deskpro to \$30. The range of optimality indicates that the current solution ($x_1 = 30$, $x_2 = 12$, $s_1 = 0$, $s_2 = 8$, $s_3 = 0$) is still optimal. To verify this solution, let us recompute the final simplex tableau after reducing the value of c_1 to 30.

We have simply set $c_1 = 30$ everywhere it appears in the previous tableau.

		x_1	x_2	s_1	s_2	s_3	
Basis	c_B	30	40	0	0	0	
x_2	40	0	1	$8/25$	0	$-3/25$	**12**
s_2	0	0	0	$-8/25$	1	$3/25$	**8**
x_1	30	1	0	$-5/25$	0	$5/25$	**30**
z_j		30	40	$34/5$	0	$6/5$	**1380**
$c_j - z_j$		0	0	$-34/5$	0	$-6/5$	

Because $c_j - z_j \leq 0$ for all variables, the solution with $x_1 = 30, x_2 = 12, s_1 = 0, s_2 = 8$, and $s_3 = 0$ is still optimal. That is, the optimal solution with $c_1 = 30$ is the same as the optimal solution with $c_1 = 50$. Note, however, that the decrease in profit contribution per unit of the Deskpro has caused a reduction in total profit from \$1980 to \$1380.

What if the profit contribution per unit were reduced even further—say, to \$20? Referring to the range of optimality for c_1 given by expression (18.4), we see that $c_1 = 20$ is outside the range; thus, we know that a change this large will cause a new basis to be optimal. To verify this new basis, we have modified the final simplex tableau by replacing c_1 by 20.

Basis	c_B	x_1 20	x_2 40	s_1 0	s_2 0	s_3 0	
x_2	40	0	1	$8/25$	0	$-3/25$	**12**
s_2	0	0	0	$-8/25$	1	$3/25$	**8**
x_1	20	1	0	$-5/25$	0	$5/25$	**30**
z_j		20	40	$44/5$	0	$-4/5$	**1080**
$c_j - z_j$		0	0	$-44/5$	0	$4/5$	

As expected, the current solution ($x_1 = 30$, $x_2 = 12$, $s_1 = 0$, $s_2 = 8$, and $s_3 = 0$) is no longer optimal because the entry in the s_3 column of the net evaluation row is greater than zero. This result implies that at least one more simplex iteration must be performed to reach the optimal solution. Continue to perform the simplex iterations in the previous tableau to verify that the new optimal solution will require the production of $16\frac{2}{3}$ units of the Deskpro and 20 units of the Portable.

At the endpoints of the range, the corresponding variable is a candidate for entering the basis if it is currently out or for leaving the basis if it is currently in.

The procedure we used to compute the range of optimality for c_1 can be used for any basic variable. The procedure for computing the range of optimality for nonbasic variables is even easier because a change in the objective function coefficient for a nonbasic variable causes only the corresponding $c_j - z_j$ entry to change in the final simplex tableau. To illustrate the approach, we show the following final simplex tableau for the original HighTech problem after replacing 0, the objective function coefficient for s_1, with the coefficient c_{s_1}:

Basis	c_B	x_1 50	x_2 40	s_1 c_{s_1}	s_2 0	s_3 0	
x_2	40	0	1	$8/25$	0	$-3/25$	**12**
s_2	0	0	0	$-8/25$	1	$3/25$	**8**
x_1	50	1	0	$-5/25$	0	$5/25$	**30**
z_j		50	40	$14/5$	0	$26/5$	**1980**
$c_j - z_j$		0	0	$c_{s_1} - 14/5$	0	$-26/5$	

Note that the only changes in the tableau are in the s_1 column. In applying inequality (18.1) to compute the range of optimality, we get

$$c_{s_1} - 14/5 \le 0$$

and hence

$$c_{s_1} \le 14/5$$

Therefore, as long as the objective function coefficient for s_1 is less than or equal to $14/5$, the current solution will be optimal. With no lower bound on how much the coefficient may be decreased, we write the range of optimality for c_{s_1} as

$$c_{s_1} \le 14/5$$

The same approach works for all nonbasic variables. In a maximization problem, the range of optimality has no lower limit, and the upper limit is given by z_j. Thus, the range of optimality for the objective function coefficient of any nonbasic variable is given by

$$c_j \leq z_j \tag{18.5}$$

Let us summarize the steps necessary to compute the range of optimality for objective function coefficients. In stating the following steps, we assume that computing the range of optimality for c_k, the coefficient of x_k, in a maximization problem is the desired goal. Keep in mind that x_k in this context may refer to one of the original decision variables, a slack variable, or a surplus variable.

Steps to Compute the Range of Optimality

Step 1. Replace the numerical value of the objective function coefficient for x_k with c_k everywhere it appears in the final simplex tableau.

Step 2. Recompute $c_j - z_j$ for each nonbasic variable (if x_k is a nonbasic variable, it is only necessary to recompute $c_k - z_k$).

Step 3. Requiring that $c_j - z_j \leq 0$, solve each inequality for any upper or lower bounds on c_k. If two or more upper bounds are found for c_k, the smallest of these is the upper bound on the range of optimality. If two or more lower bounds are found, the largest of these is the lower bound on the range of optimality.

Step 4. If the original problem is a minimization problem that was converted to a maximization problem in order to apply the simplex method, multiply the inequalities obtained in step 3 by -1, and change the direction of the inequalities to obtain the ranges of optimality for the original minimization problem.

Can you compute the range of optimality for objective function coefficients by working with the final simplex tableau? Try Problem 1.

By using the range of optimality to determine whether a change in an objective function coefficient is large enough to cause a change in the optimal solution, we can often avoid the process of formulating and solving a modified linear programming problem.

Right-Hand-Side Values

In many linear programming problems, we can interpret the right-hand-side values (the b_i's) as the resources available. For instance, in the HighTech Industries problem, the right-hand side of constraint 1 represents the available assembly time, the right-hand side of constraint 2 represents the available Portable displays, and the right-hand side of constraint 3 represents the available warehouse space. Dual prices provide information on the value of additional resources in these cases; the ranges over which these dual prices are valid are given by the ranges for the right-hand-side values.

Dual Prices In Chapter 3 we stated that the improvement in the value of the optimal solution per unit increase in a constraint's right-hand-side value is called a **dual price.**[1] When the simplex method is used to solve a linear programming problem, the values of the dual

[1]The closely related term *shadow price* is used by some authors. The shadow price is the same as the dual price for maximization problems; for minimization problems, the dual and shadow prices are equal in absolute value but have opposite signs. LINGO and The Management Scientist provide dual prices as part of the computer output. Some software packages, such as Premium Solver for Education, provide shadow prices.

prices are easy to obtain. They are found in the z_j row of the final simplex tableau. To illustrate this point, the final simplex tableau for the HighTech problem is again shown.

		x_1	x_2	s_1	s_2	s_3	
Basis	c_B	50	40	0	0	0	
x_2	40	0	1	$8/25$	0	$-3/25$	**12**
s_2	0	0	0	$-8/25$	1	$3/25$	**8**
x_1	50	1	0	$-5/25$	0	$5/25$	**30**
z_j		50	40	$14/5$	0	$26/5$	**1980**
$c_j - z_j$		0	0	$-14/5$	0	$-26/5$	

The z_j values for the three slack variables are $14/5$, 0, and $26/5$, respectively. Thus, the dual prices for the assembly time constraint, Portable display constraint, and warehouse capacity constraint are, respectively, $14/5 = \$2.80$, 0.00, and $26/5 = \$5.20$. The dual price of \$5.20 shows that more warehouse space will have the biggest positive impact on HighTech's profit.

To see why the z_j values for the slack variables in the final simplex tableau are the dual prices, let us first consider the case for slack variables that are part of the optimal basic feasible solution. Each of these slack variables will have a z_j value of zero, implying a dual price of zero for the corresponding constraint. For example, consider slack variable s_2, a basic variable in the HighTech problem. Because $s_2 = 8$ in the optimal solution, HighTech will have eight Portable display units unused. Consequently, how much would management of HighTech Industries be willing to pay to obtain additional Portable display units? Clearly the answer is nothing because at the optimal solution HighTech has an excess of this particular component. Additional amounts of this resource are of no value to the company, and, consequently, the dual price for this constraint is zero. In general, if a slack variable is a basic variable in the optimal solution, the value of z_j—and hence, the dual price of the corresponding resource—is zero.

Consider now the nonbasic slack variables—for example, s_1. In the previous subsection we determined that the current solution will remain optimal as long as the objective function coefficient for s_1 (denoted c_{s_1}) stays in the following range:

$$c_{s_1} \leq 14/5$$

It implies that the variable s_1 should not be increased from its current value of zero unless it is worth more than $14/5 = \$2.80$ to do so. We can conclude then that \$2.80 is the marginal value to HighTech of 1 hour of assembly time used in the production of Deskpro and Portable computers. Thus, if additional time can be obtained, HighTech should be willing to pay up to \$2.80 per hour for it. A similar interpretation can be given to the z_j value for each of the nonbasic slack variables.

With a greater-than-or-equal-to constraint, the value of the dual price will be less than or equal to zero because a one-unit increase in the value of the right-hand side cannot be helpful; a one-unit increase makes it more difficult to satisfy the constraint. For a maximization problem, then, the optimal value can be expected to decrease when the right-hand side of a greater-than-or-equal-to constraint is increased. The dual price gives the amount of the expected improvement—a negative number, because we expect a decrease. As a result, the dual price for a greater-than-or-equal-to constraint is given by the negative of the z_j entry for the corresponding surplus variable in the optimal simplex tableau.

TABLE 18.1 TABLEAU LOCATION OF DUAL PRICE BY CONSTRAINT TYPE

Constraint Type	Dual Price Given by
$\leq$	z_j value for the slack variable associated with the constraint
$\geq$	Negative of the z_j value for the surplus variable associated with the constraint
$=$	z_j value for the artificial variable associated with the constraint

Finally, it is possible to compute dual prices for equality constraints. They are given by the z_j values for the corresponding artificial variables. We will not develop this case in detail here because we have recommended dropping each artificial variable column from the simplex tableau as soon as the corresponding artificial variable leaves the basis.

To summarize, when the simplex method is used to solve a linear programming problem, the dual prices for the constraints are contained in the final simplex tableau. Table 18.1 summarizes the rules for determining the dual prices for the various constraint types in a maximization problem solved by the simplex method.

Try Problem 3, parts (a), (b), and (c), for practice in finding dual prices from the optimal simplex tableau.

Recall that we convert a minimization problem to a maximization problem by multiplying the objective function by -1 before using the simplex method. Nevertheless, the dual price is given by the same z_j values because improvement for a minimization problem is a decrease in the optimal value.

To illustrate the approach for computing dual prices for a minimization problem, recall the M&D Chemicals problem that we solved in Section 17.7 as an equivalent maximization problem by multiplying the objective function by -1. The linear programming model for this problem and the final simplex tableau are restated as follows, with x_1 and x_2 representing manufacturing quantities of products A and B, respectively.

$$\begin{array}{lll}
\text{Min} & 2x_1 + 3x_2 & \\
\text{s.t.} & & \\
& 1x_1 \qquad\quad \geq 125 & \text{Demand for product A} \\
& 1x_1 + 1x_2 \geq 350 & \text{Total production} \\
& 2x_1 + 1x_2 \leq 600 & \text{Processing time} \\
& x_1, x_2 \geq 0 &
\end{array}$$

		x_1	x_2	s_1	s_2	s_3	
Basis	c_B	-2	-3	0	0	0	
x_1	-2	1	0	0	1	1	**250**
x_2	-3	0	1	0	-2	-1	**100**
s_1	0	0	0	1	1	1	**125**
z_j		-2	-3	0	4	1	**-800**
$c_j - z_j$		0	0	0	-4	-1	

Following the rules in Table 18.1 for identifying the dual price for each constraint type, the dual prices for the constraints in the M&D Chemicals problem are given in Table 18.2.

TABLE 18.2 DUAL PRICES FOR M&D CHEMICALS PROBLEM

Constraint	Constraint Type	Dual Price
Demand for product A	$\geq$	0
Total production	$\geq$	-4
Processing time	$\leq$	1

Constraint 1 is not binding, and its dual price is zero. The dual price for constraint 2 shows that the marginal cost of increasing the total production requirement is \$4 per unit. Finally, the dual price of one for the third constraint shows that the per-unit value of additional processing time is \$1.

A change in b_i does not affect optimality ($c_j - z_j$ is unchanged), but it does affect feasibility. One of the current basic variables may become negative.

Range of Feasibility As we have just seen, the z_j row in the final simplex tableau can be used to determine the dual price and, as a result, predict the change in the value of the objective function corresponding to a unit change in a b_i. This interpretation is only valid, however, as long as the change in b_i is not large enough to make the current basic solution infeasible. Thus, we will be interested in calculating a range of values over which a particular b_i can vary without any of the current basic variables becoming infeasible (i.e., less than zero). This range of values will be referred to as the **range of feasibility.**

To demonstrate the effect of changing a b_i, consider increasing the amount of assembly time available in the HighTech problem from 150 to 160 hours. Will the current basis still yield a feasible solution? If so, given the dual price of \$2.80 for the assembly time constraint, we can expect an increase in the value of the solution of 10(2.80) = 28. The final simplex tableau corresponding to an increase in the assembly time of 10 hours is shown here.

		x_1	x_2	s_1	s_2	s_3	
Basis	c_B	50	40	0	0	0	
x_2	40	0	1	$8/25$	0	$-3/25$	**15.2**
s_2	0	0	0	$-8/25$	1	$3/25$	**4.8**
x_1	50	1	0	$-5/25$	0	$5/25$	**28.0**
z_j		50	40	$14/5$	0	$26/5$	**2008**
$c_j - z_j$		0	0	$-14/5$	0	$-26/5$	

The same basis, consisting of the basic variables x_2, s_2, and x_1, is feasible because all the basic variables are nonnegative. Note also that, just as we predicted using the dual price, the value of the optimal solution has increased by 10(\$2.80) = \$28, from \$1980 to \$2008.

You may wonder whether we had to re-solve the problem completely to find this new solution. The answer is no! The only changes in the final simplex tableau (as compared with the final simplex tableau with $b_1 = 150$) are the differences in the values of the basic variables and the value of the objective function. That is, only the last column of the simplex tableau changed. The entries in this new last column of the simplex tableau were obtained by adding 10 times the first four entries in the s_1 column to the last column in the previous tableau:

$$\text{New solution} = \overset{\substack{\text{Old}\\ \text{solution}\\ \downarrow}}{\left[\begin{array}{r} 12 \\ 8 \\ 30 \\ \hline 1980 \end{array}\right]} + \overset{\substack{\text{Change}\\ \text{in } b_1\\ \downarrow}}{10} \overset{\substack{s_1\\ \text{column}\\ \downarrow}}{\left[\begin{array}{r} 8/25 \\ -8/25 \\ -5/25 \\ \hline 14/5 \end{array}\right]} = \overset{\substack{\text{New}\\ \text{solution}\\ \downarrow}}{\left[\begin{array}{r} 15.2 \\ 4.8 \\ 28.0 \\ \hline 2008 \end{array}\right]}$$

Let us now consider why this procedure can be used to find the new solution. First, recall that each of the coefficients in the s_1 column indicates the amount of decrease in a basic variable that would result from increasing s_1 by one unit. In other words, these coefficients tell us how many units of each of the current basic variables will be driven out of solution if one unit of variable s_1 is brought into solution. Bringing one unit of s_1 into solution, however, is the same as reducing the availability of assembly time (decreasing b_1) by one unit; increasing b_1, the available assembly time, by one unit has just the opposite effect. Therefore, the entries in the s_1 column can also be interpreted as the changes in the values of the current basic variables corresponding to a one-unit increase in b_1.

To practice finding the new solution after a change in a right-hand side without re-solving the problem when the same basis remains feasible, try Problem 3, parts (d) and (e).

The change in the value of the objective function corresponding to a one-unit increase in b_1 is given by the value of z_j in that column (the dual price). In the foregoing case, the availability of assembly time increased by 10 units; thus, we multiplied the first four entries in the s_1 column by 10 to obtain the change in the value of the basic variables and the optimal value.

How do we know when a change in b_1 is so large that the current basis will become infeasible? We shall first answer this question specifically for the HighTech Industries problem and then state the general procedure for less-than-or-equal-to constraints. The approach taken with greater-than-or-equal-to and equality constraints will then be discussed.

We begin by showing how to compute upper and lower bounds for the maximum amount that b_1 can be changed before the current optimal basis becomes infeasible. We have seen how to find the new basic feasible solution values given a 10-unit increase in b_1. In general, given a change in b_1 of Δb_1, the new values for the basic variables in the HighTech problem are given by

$$\begin{bmatrix} x_2 \\ s_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 12 \\ 8 \\ 30 \end{bmatrix} + \Delta b_1 \begin{bmatrix} 8/25 \\ -8/25 \\ -5/25 \end{bmatrix} = \begin{bmatrix} 12 + 8/25\,\Delta b_1 \\ 8 - 8/25\,\Delta b_1 \\ 30 - 5/25\,\Delta b_1 \end{bmatrix} \tag{18.6}$$

As long as the new value of each basic variable remains nonnegative, the current basis will remain feasible and therefore optimal. We can keep the basic variables nonnegative by limiting the change in b_1 (i.e., Δb_1) so that we satisfy each of the following conditions:

$$12 + 8/25\,\Delta b_1 \geq 0 \tag{18.7}$$
$$8 - 8/25\,\Delta b_1 \geq 0 \tag{18.8}$$
$$30 - 5/25\,\Delta b_1 \geq 0 \tag{18.9}$$

The left-hand sides of these inequalities represent the new values of the basic variables after b_1 has been changed by Δb_1.

Solving for Δb_1 in inequalities (18.7), (18.8), and (18.9), we obtain

$$\begin{aligned}
\Delta b_1 &\geq (^{25}\!/_8)(-12) = -37.5 \\
\Delta b_1 &\leq (-^{25}\!/_8)(-8) = 25 \\
\Delta b_1 &\leq (-^{25}\!/_5)(-30) = 150
\end{aligned}$$

Because all three inequalities must be satisfied, the most restrictive limits on b_1 must be satisfied for all the current basic variables to remain nonnegative. Therefore, Δb_1 must satisfy

$$-37.5 \leq \Delta b_1 \leq 25 \tag{18.10}$$

The initial amount of assembly time available was 150 hours. Therefore, $b_1 = 150 + \Delta b_1$, where b_1 is the amount of assembly time available. We add 150 to each of the three terms in expression (18.10) to obtain

$$150 - 37.5 \leq 150 + \Delta b_1 \leq 150 + 25 \tag{18.11}$$

Replacing $150 + \Delta b_1$ with b_1, we obtain the range of feasibility for b_1:

$$112.5 \leq b_1 \leq 175$$

This range of feasibility for b_1 indicates that as long as the available assembly time is between 112.5 and 175 hours, the current optimal basis will remain feasible, which is why we call this range the range of feasibility.

Because the dual price for b_1 (assembly time) is $^{14}\!/_5$, we know profit can be increased by \$2.80 by obtaining an additional hour of assembly time. Suppose then that we increase b_1 by 25; that is, we increase b_1 to the upper limit of its range of feasibility, 175. The profit will increase to \$1980 + (\$2.80)25 = \$2050, and the values of the optimal basic variables become

$$\begin{aligned}
x_2 &= 12 + 25(^8\!/_{25}) = 20 \\
s_2 &= 8 + 25(-^8\!/_{25}) = 0 \\
x_1 &= 30 + 25(-^5\!/_{25}) = 25
\end{aligned}$$

What happened to the solution? The increased assembly time caused a revision in the optimal production plan. HighTech should produce more of the Portable and less of the Deskpro. Overall, the profit will be increased by (\$2.80)(25) = \$70. Note that although the optimal solution changed, the basic variables that were optimal before are still optimal.

The procedure for determining the range of feasibility has been illustrated with the assembly time constraint. The procedure for calculating the range of feasibility for the right-hand side of any less-than-or-equal-to constraint is the same. The first step for a general constraint i is to calculate the range of values for b_i that satisfies the following inequalities.

$$\begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \cdot \\ \cdot \\ \cdot \\ \bar{b}_m \end{bmatrix} + \Delta b_i \begin{bmatrix} \bar{a}_{1j} \\ \bar{a}_{2j} \\ \cdot \\ \cdot \\ \cdot \\ \bar{a}_{mj} \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \tag{18.12}$$

Current solution (last column of the final simplex tableau)

Column of the final simplex tableau corresponding to the slack variable associated with constraint i

The inequalities are used to identify lower and upper limits on Δb_i. The range of feasibility can then be established by the maximum of the lower limits and the minimum of the upper limits.

Similar arguments can be used to develop a procedure for determining the range of feasibility for the right-hand-side value of a greater-than-or-equal-to constraint. Essentially the procedure is the same, with the column corresponding to the surplus variable associated with the constraint playing the central role. For a general greater-than-or-equal-to constraint i, we first calculate the range of values for Δb_i that satisfy the inequalities shown in inequality (18.13).

$$\begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \cdot \\ \cdot \\ \cdot \\ \bar{b}_m \end{bmatrix} - \Delta b_i \begin{bmatrix} \bar{a}_{1j} \\ \bar{a}_{2j} \\ \cdot \\ \cdot \\ \cdot \\ \bar{a}_{mj} \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \tag{18.13}$$

Current solution (last column of the final simplex tableau)

Column of the final simplex tableau corresponding to the surplus variable associated with constraint i

Once again, these inequalities establish lower and upper limits on Δb_i. Given these limits, the range of feasibility is easily determined.

Try Problem 4 to make sure you can compute the range of feasibility by working with the final simplex tableau.

A range of feasibility for the right-hand side of an equality constraint can also be computed. To do so for equality constraint i, one could use the column of the final simplex tableau corresponding to the artificial variable associated with constraint i in equation (18.12). Because we have suggested dropping the artificial variable columns from the simplex tableau as soon as the artificial variable becomes nonbasic, these columns will not be available in the final tableau. Thus, more involved calculations are required to compute a range of feasibility for equality constraints. Details may be found in more advanced texts.

Changes that force b_i outside its range of feasibility are normally accompanied by changes in the dual prices.

As long as the change in a right-hand-side value is such that b_i stays within its range of feasibility, the same basis will remain feasible and optimal. Changes that force b_i outside its range of feasibility will force us to re-solve the problem to find the new optimal solution consisting of a different set of basic variables. (More advanced linear programming texts show how it can be done without completely re-solving the problem.) In any case, the calculation of the range of feasibility for each b_i is valuable management information and should be included as part of the management report on any linear programming project. The range of feasibility is typically made available as part of the computer solution to the problem.

Simultaneous Changes

In reviewing the procedures for developing the range of optimality and the range of feasibility, we note that only one coefficient at a time was permitted to vary. Our statements concerning changes within these ranges were made with the understanding that no other coefficients are permitted to change. However, sometimes we can make the same statements when either two or more objective function coefficients or two or more right-hand sides are varied simultaneously. When the simultaneous changes satisfy the 100 percent rule, the same statements are applicable. The 100 percent rule was explained in Chapter 3, but we will briefly review it here.

Let us define allowable increase as the amount a coefficient can be increased before reaching the upper limit of its range, and allowable decrease as the amount a coefficient can be decreased before reaching the lower limit of its range. Now suppose simultaneous changes are made in two or more objective function coefficients. For each coefficient changed, we compute the percentage of the allowable increase, or allowable decrease, represented by the change. If the sum of the percentages for all changes does not exceed 100 percent, we say that the 100 percent rule is satisfied and that the simultaneous changes will not cause a change in the optimal solution. However, just as with a single objective function coefficient change, the value of the solution will change because of the change in the coefficients.

Similarly, if two or more changes in constraint right-hand-side values are made, we again compute the percentage of allowable increase or allowable decrease represented by each change. If the sum of the percentages for all changes does not exceed 100 percent, we say that the 100 percent rule is satisfied. The dual prices are then valid for determining the change in value of the objective function associated with the right-hand-side changes.

NOTES AND COMMENTS

1. Sometimes, interpreting dual prices and choosing the appropriate sign can be confusing. It often helps to think of this process as follows. Relaxing a $\geq$ constraint means decreasing its right-hand side, and relaxing a $\leq$ constraint means increasing its right-hand side. Relaxing a constraint permits improvement in value; restricting a constraint (decreasing the right-hand side of a $\leq$ constraint or increasing the right-hand side of a $\geq$ constraint) has the opposite effect. In every case, the absolute value of the dual price gives the improvement in the optimal value associated with relaxing the constraint.
2. The Notes and Comments in Chapter 3 concerning sensitivity analysis are also applicable here. In particular, recall that the 100 percent rule cannot be applied to simultaneous changes in the objective function *and* the right-hand sides; it applies only to simultaneous changes in one or the other. Also note that this rule *does not* mean that simultaneous changes that do not satisfy the rule will necessarily cause a change in the solution. For instance, any proportional change in *all* the objective function coefficients will leave the optimal solution unchanged, and any proportional change in *all* the right-hand sides will leave the dual prices unchanged.

18.2 DUALITY

Every linear programming problem has an associated linear programming problem called the **dual problem.** Referring to the original formulation of the linear programming problem as the **primal problem,** we will see how the primal can be converted into its corresponding dual. Then we will solve the dual linear programming problem and interpret the results. A fundamental property of the primal-dual relationship is that the optimal solution to either the primal or the dual problem also provides the optimal solution to the other. In cases where the primal and the dual problems differ in terms of computational difficulty, we can choose the easier problem to solve.

Let us return to the HighTech Industries problem. The original formulation—the primal problem—is as follows:

$$
\begin{aligned}
&\text{Max} \quad 50x_1 + 40x_2 \\
&\text{s.t.} \\
&\qquad 3x_1 + 5x_2 \le 150 \quad \text{Assembly time} \\
&\qquad \phantom{3x_1 + {}} 1x_2 \le 20 \quad \text{Portable display} \\
&\qquad 8x_1 + 5x_2 \le 300 \quad \text{Warehouse space} \\
&\qquad x_1, x_2 \ge 0
\end{aligned}
$$

A maximization problem with all less-than-or-equal-to constraints and nonnegativity requirements for the variables is said to be in **canonical form.** For a maximization problem in canonical form, such as the HighTech Industries problem, the conversion to the associated dual linear program is relatively easy. Let us state the dual of the HighTech problem and then identify the steps taken to make the primal-dual conversion. The HighTech dual problem is as follows:

$$
\begin{aligned}
&\text{Min} \quad 150u_1 + 20u_2 + 300u_3 \\
&\text{s.t.} \\
&\qquad 3u_1 \phantom{{}+ 1u_2} + 8u_3 \ge 50 \\
&\qquad 5u_1 + 1u_2 + 5u_3 \ge 40 \\
&\qquad u_1, u_2, u_3 \ge 0
\end{aligned}
$$

This **canonical form for a minimization problem** is a minimization problem with all greater-than-or-equal-to constraints and nonnegativity requirements for the variables. Thus, the dual of a maximization problem in canonical form is a minimization problem in canonical form. The variables u_1, u_2, and u_3 are referred to as **dual variables.**

With the preceding example in mind, we make the following general statements about the *dual of a maximization problem in canonical form.*

1. The dual is a minimization problem in canonical form.
2. When the primal has n decision variables ($n = 2$ in the HighTech problem), the dual will have n constraints. The first constraint of the dual is associated with variable x_1 in the primal, the second constraint in the dual is associated with variable x_2 in the primal, and so on.
3. When the primal has m constraints ($m = 3$ in the HighTech problem), the dual will have m decision variables. Dual variable u_1 is associated with the first primal constraint, dual variable u_2 is associated with the second primal constraint, and so on.

4. The right-hand sides of the primal constraints become the objective function coefficients in the dual.
5. The objective function coefficients of the primal become the right-hand sides of the dual constraints.
6. The constraint coefficients of the ith primal variable become the coefficients in the ith constraint of the dual.

Try part (a) of Problem 17 for practice in finding the dual of a maximization problem in canonical form.

These six statements are the general requirements that must be satisfied when converting a maximization problem in canonical form to its associated dual: a minimization problem in canonical form. Even though these requirements may seem cumbersome at first, practice with a few simple problems will show that the primal-dual conversion process is relatively easy to implement.

We have formulated the HighTech dual linear programming problem, so let us now proceed to solve it. With three variables in the dual, we will use the simplex method. After subtracting surplus variables s_1 and s_2 to obtain the standard form, adding artificial variables a_1 and a_2 to obtain the tableau form, and multiplying the objective function by -1 to convert the dual problem to an equivalent maximization problem, we arrive at the following initial simplex tableau.

Basis	c_B	u_1	u_2	u_3	s_1	s_2	a_1	a_2	
		-150	-20	-300	0	0	$-M$	$-M$	
a_1	$-M$	3	0	8	-1	0	1	0	**50**
a_2	$-M$	5	1	5	0	-1	0	1	**40**
	z_j	$-8M$	$-M$	$-13M$	M	M	$-M$	$-M$	**$-90M$**
	$c_j - z_j$	$-150 + 8M$	$-20 + M$	$-300 + 13M$	$-M$	$-M$	0	0	

At the first iteration, u_3 is brought into the basis, and a_1 is removed. At the second iteration, u_1 is brought into the basis, and a_2 is removed. At this point, the simplex tableau appears as follows.

Basis	c_B	u_1	u_2	u_3	s_1	s_2	
		-150	-20	-300	0	0	
u_3	-300	0	$-3/25$	1	$-5/25$	$3/25$	**$26/5$**
u_1	-150	1	$8/25$	0	$5/25$	$-8/25$	**$14/5$**
	z_j	-150	-12	-300	30	12	**-1980**
	$c_j - z_j$	0	-8	0	-30	-12	

Because all the entries in the net evaluation row are less than or equal to zero, the optimal solution has been reached; it is $u_1 = 14/5$, $u_2 = 0$, $u_3 = 26/5$, $s_1 = 0$, and $s_2 = 0$. We have been maximizing the negative of the dual objective function; therefore, the value of the objective function for the optimal dual solution must be $-(-1980) = 1980$.

The final simplex tableau for the original HighTech Industries problem is shown here.

		x_1	x_2	s_1	s_2	s_3	
Basis	c_B	50	40	0	0	0	
x_2	40	0	1	8/25	0	−3/25	**12**
s_2	0	0	0	−8/25	1	3/25	**8**
x_1	50	1	0	−5/25	0	5/25	**30**
z_j		50	40	14/5	0	26/5	**1980**
$c_j - z_j$		0	0	−14/5	0	−26/5	

The optimal solution to the primal problem is $x_1 = 30$, $x_2 = 12$, $s_1 = 0$, $s_2 = 8$, and $s_3 = 0$. The optimal value of the objective function is 1980.

What observation can we make about the relationship between the optimal value of the objective function in the primal and the optimal value in the dual for the HighTech problem? The optimal value of the objective function is the same (1980) for both. This relationship is true for all primal and dual linear programming problems and is stated as property 1.

Property 1

If the dual problem has an optimal solution, the primal problem has an optimal solution, and vice versa. Furthermore, the values of the optimal solutions to the dual and primal problems are equal.

This property tells us that if we solved only the dual problem, we would know that HighTech could make a maximum of $1980.

Economic Interpretation of the Dual Variables

Before making further observations about the relationship between the primal and the dual solutions, let us consider the meaning or interpretation of the dual variables u_1, u_2, and u_3. Remember that in setting up the dual problem, each dual variable is associated with one of the constraints in the primal. Specifically, u_1 is associated with the assembly time constraint, u_2 with the Portable display constraint, and u_3 with the warehouse space constraint.

To understand and interpret these dual variables, let us return to property 1 of the primal-dual relationship, which stated that the objective function values for the primal and dual problems must be equal. At the optimal solution, the primal objective function results in

$$50x_1 + 40x_2 = 1980 \tag{18.14}$$

while the dual objective function is

$$150u_1 + 20u_2 + 300u_3 = 1980 \tag{18.15}$$

Using equation (18.14), let us restrict our interest to the interpretation of the primal objective function. With x_1 and x_2 as the number of units of the Deskpro and the Portable that are assembled respectively, we have

$$\begin{pmatrix}\text{Dollar value}\\\text{per unit of}\\\text{Deskpro}\end{pmatrix}\begin{pmatrix}\text{Number of}\\\text{units of}\\\text{Deskpro}\end{pmatrix}+\begin{pmatrix}\text{Dollar value}\\\text{per unit of}\\\text{Portable}\end{pmatrix}\begin{pmatrix}\text{Number of}\\\text{units of}\\\text{Portable}\end{pmatrix}=\begin{matrix}\text{Total dollar}\\\text{value of}\\\text{production}\end{matrix}$$

From equation (18.15), we see that the coefficients of the dual objective function (150, 20, and 300) can be interpreted as the number of units of resources available. Thus, because the primal and dual objective functions are equal at optimality, we have

$$\begin{pmatrix}\text{Units of}\\\text{resource}\\1\end{pmatrix}u_1+\begin{pmatrix}\text{Units of}\\\text{resource}\\2\end{pmatrix}u_2+\begin{pmatrix}\text{Units of}\\\text{resource}\\3\end{pmatrix}u_3=\begin{matrix}\text{Total dollar value}\\\text{of production}\end{matrix}$$

Thus, we see that the dual variables must carry the interpretations of being the value per unit of resource. For the HighTech problem,

$$\begin{aligned}u_1 &= \text{dollar value per hour of assembly time}\\u_2 &= \text{dollar value per unit of the Portable display}\\u_3 &= \text{dollar value per square foot of warehouse space}\end{aligned}$$

Have we attempted to identify the value of these resources previously? Recall that in Section 18.1, when we considered sensitivity analysis of the right-hand sides, we identified the value of an additional unit of each resource. These values were called dual prices and are helpful to the decision maker in determining whether additional units of the resources should be made available.

The analysis in Section 18.1 led to the following dual prices for the resources in the HighTech problem.

Resource	**Value per Additional Unit (dual price)**
Assembly time	\$2.80
Portable display	\$0.00
Warehouse space	\$5.20

The dual variables are the shadow prices, but in a maximization problem, they also equal the dual prices. For a minimization problem, the dual prices are the negative of the dual variables.

Let us now return to the optimal solution for the HighTech dual problem. The values of the dual variables at the optimal solution are $u_1 = 14/5 = 2.80$, $u_2 = 0$, and $u_3 = 26/5 = 5.20$. For this maximization problem, the values of the dual variables and the dual prices are the same. For a minimization problem, the dual prices and the dual variables are the same in absolute value but have opposite signs. Thus, the optimal values of the dual variables identify the dual prices of each additional resource or input unit at the optimal solution.

In light of the preceding discussion, the following interpretation of the primal and dual problems can be made when the primal is a product-mix problem.

Primal Problem Given a per-unit value of each product, determine how much of each should be produced to maximize the value of the total production. Constraints require the amount of each resource used to be less than or equal to the amount available.

Dual Problem Given the availability of each resource, determine the per-unit value such that the total value of the resources used is minimized. Constraints require the resource value per unit be greater than or equal to the value of each unit of output.

Using the Dual to Identify the Primal Solution

At the beginning of this section, we mentioned that an important feature of the primal-dual relationship is that when an optimal solution is reached, the value of the optimal solution for the primal problem is the same as the value of the optimal solution for the dual problem; see property 1. However, the question remains: If we solve only the dual problem, can we identify the optimal values for the primal variables?

Recall that in Section 18.1 we showed that when a primal problem is solved by the simplex method, the optimal values of the primal variables appear in the right-most column of the final tableau, and the dual prices (values of the dual variables) are found in the z_j row. The final simplex tableau of the dual problem provides the optimal values of the dual variables, and therefore the values of the primal variables should be found in the z_j row of the optimal dual tableau. This result is, in fact, the case and is formally stated as property 2.

Property 2

Given the simplex tableau corresponding to the optimal dual solution, the optimal values of the primal decision variables are given by the z_j entries for the surplus variables; furthermore, the optimal values of the primal slack variables are given by the negative of the $c_j - z_j$ entries for the u_j variables.

To test your ability to find the primal solution from the optimal simplex tableau for the dual and interpret the dual variables, try parts (b) and (c) of Problem 17.

This property enables us to use the final simplex tableau for the dual of the HighTech problem to determine the optimal primal solution of $x_1 = 30$ units of the Deskpro and $x_2 = 12$ units of the Portable. These optimal values of x_1 and x_2, as well as the values for all primal slack variables, are given in the z_j and $c_j - z_j$ rows of the final simplex tableau of the dual problem, which is shown again here.

		u_1	u_2	u_3	s_1	s_2	
Basis	c_B	−150	−20	−300	0	0	
u_3	−300	0	$-3/25$	1	$-5/25$	$3/25$	**26/5**
u_1	−150	1	$8/25$	0	$5/25$	$-8/25$	**14/5**
z_j		−150	−12	−300	30	12	**−1980**
$c_j - z_j$		0	−8	0	−30	−12	

Finding the Dual of Any Primal Problem

The HighTech Industries primal problem provided a good introduction to the concept of duality because it was formulated as a maximization problem in canonical form. For this form of primal problem, we demonstrated that conversion to the dual problem is rather easy. If the primal problem is a minimization problem in canonical form, then the dual is a maximization problem in canonical form. Therefore, finding the dual of a minimization problem in canonical form is also easy. Consider the following linear program in canonical form for a minimization problem:

$$\begin{aligned} \text{Min} \quad & 6x_1 + 2x_2 \\ \text{s.t.} \quad & \\ & 5x_1 - 1x_2 \geq 13 \\ & 3x_1 + 7x_2 \geq 9 \\ & x_1, x_2 \geq 0 \end{aligned}$$

The dual is the following maximization problem in canonical form:

$$\begin{aligned} \text{Max} \quad & 13u_1 + 9u_2 \\ \text{s.t.} \quad & \\ & 5u_1 + 3u_2 \leq 6 \\ & -1u_1 + 7u_2 \leq 2 \\ & u_1, u_2 \geq 0 \end{aligned}$$

Try Problem 18 for practice in finding the dual of a minimization problem in canonical form.

Although we could state a special set of rules for converting each type of primal problem into its associated dual, we believe it is easier to first convert any primal problem into an equivalent problem in canonical form. Then, we follow the procedures already established for finding the dual of a maximization or minimization problem in canonical form.

Let us illustrate the procedure for finding the dual of any linear programming problem by finding the dual of the following minimization problem:

$$\begin{aligned} \text{Min} \quad & 2x_1 - 3x_2 \\ \text{s.t.} \quad & \\ & 1x_1 + 2x_2 \leq 12 \\ & 4x_1 - 2x_2 \geq 3 \\ & 6x_1 - 1x_2 = 10 \\ & x_1, x_2 \geq 0 \end{aligned}$$

For this minimization problem, we obtain the canonical form by converting all constraints to greater-than-or-equal-to form. The necessary steps are as follows:

Step 1. Convert the first constraint to greater-than-or-equal-to form by multiplying both sides of the inequality by (-1). Doing so yields

$$-x_1 - 2x_2 \geq -12$$

Step 2. Constraint 3 is an equality constraint. For an equality constraint, we first create two inequalities: one with $\leq$ form, the other with $\geq$ form. Doing so yields

$$\begin{aligned} 6x_1 - 1x_2 &\geq 10 \\ 6x_1 - 1x_2 &\leq 10 \end{aligned}$$

Then, we multiply the $\leq$ constraint by (-1) to get two $\geq$ constraints.

$$\begin{aligned} 6x_1 - 1x_2 &\geq 10 \\ -6x_1 + 1x_2 &\geq -10 \end{aligned}$$

Now the original primal problem has been restated in the following equivalent form:

$$
\begin{aligned}
\text{Min} \quad & 2x_1 - 3x_2 \\
\text{s.t.} \quad & \\
& -1x_1 - 2x_2 \geq -12 \\
& 4x_1 - 2x_2 \geq 3 \\
& 6x_1 - 1x_2 \geq 10 \\
& -6x_1 + 1x_2 \geq -10 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

With the primal problem now in canonical form for a minimization problem, we can easily convert to the dual problem using the primal-dual procedure presented earlier in this section. The dual becomes[2]

$$
\begin{aligned}
\text{Max} \quad & -12u_1 + 3u_2 + 10u_3' - 10u_3'' \\
\text{s.t.} \quad & \\
& -1u_1 + 4u_2 + 6u_3' - 6u_3'' \leq 2 \\
& -2u_1 - 2u_2 - 1u_3' + 1u_3'' \leq -3 \\
& u_1, u_2, u_3', u_3'' \geq 0
\end{aligned}
$$

Can you write the dual of any linear programming problem? Try Problem 19.

The equality constraint required two $\geq$ constraints, so we denoted the dual variables associated with these constraints as u_3' and u_3''. This notation reminds us that u_3' and u_3'' both refer to the third constraint in the initial primal problem. Because two dual variables are associated with an equality constraint, the interpretation of the dual variable must be modified slightly. The dual variable for the equality constraint $6x_1 - 1x_2 = 10$ is given by the value of $u_3' - u_3''$ in the optimal solution to the dual. Hence, the dual variable for an equality constraint can be negative.

SUMMARY

In this chapter we showed how sensitivity analysis can be performed using the information in the final simplex tableau. This sensitivity analysis includes computing the range of optimality for objective function coefficients, dual prices, and the range of feasibility for the right-hand sides. Sensitivity information is routinely made available as part of the solution report provided by most linear programming computer packages.

We stress here that sensitivity analysis is based on the assumption that only one coefficient is allowed to change at a time; all other coefficients are assumed to remain at their original values. It is possible to do some limited sensitivity analysis on the effect of changing more than one coefficient at a time; the 100 percent rule was mentioned as being useful in this context.

In studying duality, we saw how the original linear programming problem, called the primal, can be converted into its associated dual linear programming problem. Solving either the primal or the dual provides the solution to the other. We learned that the value of the dual variable identifies the economic contribution or value of additional resources in the primal problem.

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[2]Note that the right-hand side of the second constraint is negative. Thus, we must multiply both sides of the constraint by -1 to obtain a positive value for the right-hand side before attempting to solve the problem with the simplex method.

GLOSSARY

Range of optimality The range of values over which an objective function coefficient may vary without causing any change in the optimal solution (i.e., the values of all the variables will remain the same, but the value of the objective function may change).

Dual price The improvement in value of the optimal solution per unit increase in a constraint's right-hand-side value.

Range of feasibility The range of values over which a b_i may vary without causing the current basic solution to become infeasible. The values of the variables in the solution will change, but the same variables will remain basic. The dual prices for constraints do not change within these ranges.

Dual problem A linear programming problem related to the primal problem. Solution of the dual also provides the solution to the primal.

Primal problem The original formulation of a linear programming problem.

Canonical form for a maximization problem A maximization problem with all less-than-or-equal-to constraints and nonnegativity requirements for the decision variables.

Canonical form for a minimization problem A minimization problem with all greater-than-or-equal-to constraints and nonnegativity requirements for the decision variables.

Dual variable The variable in a dual linear programming problem. Its optimal value provides the dual price for the associated primal resource.

PROBLEMS

SELF test

1. Consider the following linear programming problem.

$$
\begin{aligned}
\text{Max} \quad & 5x_1 + 6x_2 + 4x_3 \\
\text{s.t.} \quad & \\
& 3x_1 + 4x_2 + 2x_3 \le 120 \\
& x_1 + 2x_2 + x_3 \le 50 \\
& x_1 + 2x_2 + 3x_3 \ge 30 \\
& x_1, x_2, x_3 \ge 0
\end{aligned}
$$

The optimal simplex tableau is

		x_1	x_2	x_3	s_1	s_2	s_3	
Basic	c_B	5	6	4	0	0	0	
s_3	0	0	4	0	−2	7	1	**80**
x_3	4	0	2	1	−1	3	0	**30**
x_1	5	1	0	0	1	−2	0	**20**
z_j		5	8	4	1	2	0	**220**
$c_j - z_j$		0	−2	0	−1	−2	0	

a. Compute the range of optimality for c_1.
b. Compute the range of optimality for c_2.
c. Compute the range of optimality for c_{s_1}.

2. For the HighTech problem, we found the range of optimality for c_1, the profit contribution per unit of the Deskpro. The final simplex tableau is given in Section 18.1. Find the following:
a. The range of optimality for c_2.
b. The range of optimality for c_{s_2}.
c. The range of optimality for c_{s_3}.
d. Suppose the per-unit profit contribution of the Portable (c_2) dropped to \$35. How would the optimal solution change? What is the new value for total profit?

SELF test

3. Refer to the problem formulation and optimal simplex tableau given in Problem 1.
a. Find the dual price for the first constraint.
b. Find the dual price for the second constraint.
c. Find the dual price for the third constraint.
d. Suppose the right-hand side of the first constraint is increased from 120 to 125. Find the new optimal solution and its value.
e. Suppose the right-hand side of the first constraint is decreased from 120 to 110. Find the new optimal solution and its value.

SELF test

4. Refer again to the problem formulation and optimal simplex tableau given in Problem 1.
a. Find the range of feasibility for b_1.
b. Find the range of feasibility for b_2.
c. Find the range of feasibility for b_3.

5. For the HighTech problem, we found the range of feasibility for b_1, the assembly time available (see Section 18.1).
a. Find the range of feasibility for b_2.
b. Find the range of feasibility for b_3.
c. How much will HighTech's profit increase if there is a 20-square-foot increase in the amount of warehouse space available (b_3)?

6. Recall the Par, Inc., problem introduced in Chapter 2. The linear program for this problem is

$$\begin{array}{lrl}
\text{Max} & 10x_1 + 9x_2 & \\
\text{s.t.} & & \\
& \tfrac{7}{10}x_1 + 1x_2 \le 630 & \text{Cutting and dyeing time} \\
& \tfrac{1}{2}x_1 + \tfrac{5}{6}x_2 \le 600 & \text{Sewing time} \\
& 1x_1 + \tfrac{2}{3}x_2 \le 708 & \text{Finishing time} \\
& \tfrac{1}{10}x_1 + \tfrac{1}{4}x_2 \le 135 & \text{Inspection and packaging time} \\
& x_1, x_2 \ge 0 &
\end{array}$$

where

$$\begin{aligned}
x_1 &= \text{number of standard bags produced} \\
x_2 &= \text{number of deluxe bags produced}
\end{aligned}$$

The final simplex tableau is

		x_1	x_2	s_1	s_2	s_3	s_4	
Basis	c_B	10	9	0	0	0	0	
x_2	9	0	1	$30/16$	0	$-21/16$	0	**252**
s_2	0	0	0	$-15/16$	1	$5/32$	0	**120**
x_1	10	1	0	$-20/16$	0	$30/16$	0	**540**
s_4	0	0	0	$-11/32$	0	$9/64$	1	**18**
z_j		10	9	$70/16$	0	$111/16$	0	**7668**
$c_j - z_j$		0	0	$-70/16$	0	$-111/16$	0	

a. Calculate the range of optimality for the profit contribution of the standard bag.
b. Calculate the range of optimality for the profit contribution of the deluxe bag.
c. If the profit contribution per deluxe bag drops to \$7 per unit, how will the optimal solution be affected?
d. What unit profit contribution would be necessary for the deluxe bag before Par, Inc., would consider changing its current production plan?
e. If the profit contribution of the deluxe bags can be increased to \$15 per unit, what is the optimal production plan? State what you think will happen before you compute the new optimal solution.

7. For the Par, Inc., problem (Problem 6):
a. Calculate the range of feasibility for b_1 (cutting and dyeing capacity).
b. Calculate the range of feasibility for b_2 (sewing capacity).
c. Calculate the range of feasibility for b_3 (finishing capacity).
d. Calculate the range of feasibility for b_4 (inspection and packaging capacity).
e. Which of these four departments would you be interested in scheduling for overtime? Explain.

8. a. Calculate the final simplex tableau for the Par, Inc., problem (Problem 6) after increasing b_1 from 630 to $682\frac{4}{11}$.
b. Would the current basis be optimal if b_1 were increased further? If not, what would be the new optimal basis?

9. For the Par, Inc., problem (Problem 6):
a. How much would profit increase if an additional 30 hours became available in the cutting and dyeing department (i.e., if b_1 were increased from 630 to 660)?
b. How much would profit decrease if 40 hours were removed from the sewing department?
c. How much would profit decrease if, because of an employee accident, only 570 hours instead of 630 were available in the cutting and dyeing department?

10. The following are additional conditions encountered by Par, Inc. (Problem 6).
a. Suppose because of some new machinery Par, Inc., was able to make a small reduction in the amount of time it took to do the cutting and dyeing (constraint 1) for a standard bag. What effect would this reduction have on the objective function?
b. Management believes that by buying a new sewing machine, the sewing time for standard bags can be reduced from ½ to ⅓ hour. Do you think this machine would be a good investment? Why?

11. Recall the RMC problem (Chapter 17, Problem 9). The problem formulation is shown here:

$$
\begin{aligned}
\text{Max} \quad & 40x_1 + 30x_2 \\
\text{s.t.} \quad & \\
& \tfrac{2}{5}x_1 + \tfrac{1}{2}x_2 \le 20 \quad \text{Material 1} \\
& \qquad\quad \tfrac{1}{5}x_2 \le \ \ 5 \quad \text{Material 2} \\
& \tfrac{3}{5}x_1 + \tfrac{3}{10}x_2 \le 21 \quad \text{Material 3} \\
& x_1, x_2 \ge 0
\end{aligned}
$$

where

$$
\begin{aligned}
x_1 &= \text{tons of fuel additive produced} \\
x_2 &= \text{tons of solvent base produced}
\end{aligned}
$$

The final simplex tableau is

		x_1	x_2	s_1	s_2	s_3	
Basis	c_B	40	30	0	0	0	
x_2	30	0	1	$10/3$	0	$-20/9$	**20**
s_2	0	0	0	$-2/3$	1	$4/9$	**1**
x_1	40	1	0	$-5/3$	0	$25/9$	**25**
z_j		40	30	$100/3$	0	$400/9$	**1600**
$c_j - z_j$		0	0	$-100/3$	0	$-400/9$	

a. Compute the ranges of optimality for c_1 and c_2.
b. Suppose that because of an increase in production costs, the profit per ton on the fuel additive is reduced to $30 per ton. What effect will this change have on the optimal solution?
c. What is the dual price for the material 1 constraint? What is the interpretation?
d. If RMC had an opportunity to purchase additional materials, which material would be the most valuable? How much should the company be willing to pay for this material?

12. Refer to Problem 11.
a. Compute the range of feasibility for b_1 (material 1 availability).
b. Compute the range of feasibility for b_2 (material 2 availability).
c. Compute the range of feasibility for b_3 (material 3 availability).
d. What is the dual price for material 3? Over what range of values for b_3 is this dual price valid?

13. Consider the following linear program:

$$
\begin{aligned}
\text{Max} \quad & 3x_1 + 1x_2 + 5x_3 + 3x_4 \\
\text{s.t.} \quad & \\
& 3x_1 + 1x_2 + 2x_3 \qquad\quad = 30 \\
& 2x_1 + 1x_2 + 3x_3 + 1x_4 \ge 15 \\
& \qquad\quad 2x_2 \qquad\quad + 3x_4 \le 25 \\
& x_1, x_2, x_3, x_4 \ge 0
\end{aligned}
$$

a. Find the optimal solution.
b. Calculate the range of optimality for c_3.
c. What would be the effect of a four-unit decrease in c_3 (from 5 to 1) on the optimal solution and the value of that solution?
d. Calculate the range of optimality for c_2.
e. What would be the effect of a three-unit increase in c_2 (from 1 to 4) on the optimal solution and the value of that solution?

14. Consider the final simplex tableau shown here.

		x_1	x_2	x_3	x_4	s_1	s_2	s_3	
Basis	c_B	4	6	3	1	0	0	0	
x_3	3	$\frac{3}{60}$	0	1	$\frac{1}{2}$	$\frac{3}{10}$	0	$-\frac{6}{30}$	**125**
s_2	0	$\frac{195}{60}$	0	0	$-\frac{1}{2}$	$-\frac{5}{10}$	1	-1	**425**
x_2	6	$\frac{39}{60}$	1	0	$\frac{1}{2}$	$-\frac{1}{10}$	0	$\frac{12}{30}$	**25**
z_j		$\frac{81}{20}$	6	3	$\frac{9}{2}$	$\frac{3}{10}$	0	$\frac{54}{30}$	**525**
$c_j - z_j$		$-\frac{1}{20}$	0	0	$-\frac{7}{2}$	$-\frac{3}{10}$	0	$-\frac{54}{30}$	

The original right-hand-side values were $b_1 = 550$, $b_2 = 700$, and $b_3 = 200$.
a. Calculate the range of feasibility for b_1.
b. Calculate the range of feasibility for b_2.
c. Calculate the range of feasibility for b_3.

15. Consider the following linear program:

$$\begin{aligned}
\text{Max} \quad & 15x_1 + 30x_2 + 20x_3 \\
\text{s.t.} \quad & \\
& 1x_1 + 1x_3 \leq 4 \\
& 0.5x_1 + 2x_2 + 1x_3 \leq 3 \\
& 1x_1 + 1x_2 + 2x_3 \leq 6 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}$$

Solve using the simplex method, and answer the following questions:
a. What is the optimal solution?
b. What is the value of the objective function?
c. Which constraints are binding?
d. How much slack is available in the nonbinding constraints?
e. What are the dual prices associated with the three constraints? Which right-hand-side value would have the greatest effect on the value of the objective function if it could be changed?
f. Develop the appropriate ranges for the coefficients of the objective function. What is your interpretation of these ranges?
g. Develop and interpret the ranges of feasibility for the right-hand-side values.

16. Innis Investments manages funds for a number of companies and wealthy clients. The investment strategy is tailored to each client's needs. For a new client, Innis has been authorized to invest up to \$1.2 million in two investment funds: a stock fund and a money market fund. Each unit of the stock fund costs \$50 and provides an annual rate of return of 10%; each unit of the money market fund costs \$100 and provides an annual rate of return of 4%.

The client wants to minimize risk subject to the requirement that the annual income from the investment be at least \$60,000. According to Innis's risk measurement system, each unit invested in the stock fund has a risk index of 8, and each unit invested in the money market fund has a risk index of 3; the higher risk index associated with the stock fund simply indicates that it is the riskier investment. Innis's client also specified that at least \$300,000 be invested in the money market fund. Innis needs to determine how many units of each fund to purchase for the client to minimize the total risk index for the portfolio. Letting

$$
\begin{aligned}
x_1 &= \text{units purchased in the stock fund} \\
x_2 &= \text{units purchased in the money market fund}
\end{aligned}
$$

leads to the following formulation:

$$
\begin{array}{lrcrcrl}
\text{Min} & 8x_1 & + & 3x_2 & & & \text{Total risk} \\
\text{s.t.} & & & & & & \\
& 50x_1 & + & 100x_2 & \leq & 1{,}200{,}000 & \text{Funds available} \\
& 5x_1 & + & 4x_2 & \geq & 60{,}000 & \text{Annual income} \\
& & & 1x_2 & \geq & 3{,}000 & \text{Minimum units in money market} \\
& x_1, x_2 \geq 0 & & & & &
\end{array}
$$

a. Solve this problem using the simplex method.
b. The value of the optimal solution is a measure of the riskiness of the portfolio. What effect will increasing the annual income requirement have on the riskiness of the portfolio?
c. Find the range of feasibility for b_2.
d. How will the optimal solution and its value change if the annual income requirement is increased from \$60,000 to \$65,000?
e. How will the optimal solution and its value change if the risk measure for the stock fund is increased from 8 to 9?

SELF test

17. Suppose that in a product-mix problem x_1, x_2, x_3, and x_4 indicate the units of products 1, 2, 3, and 4, respectively, and we have

$$
\begin{array}{lrcrcrcrcrl}
\text{Max} & 4x_1 & + & 6x_2 & + & 3x_3 & + & 1x_4 & & & \\
\text{s.t.} & & & & & & & & & & \\
& 1.5x_1 & + & 2x_2 & + & 4x_3 & + & 3x_4 & \leq & 550 & \text{Machine A hours} \\
& 4x_1 & + & 1x_2 & + & 2x_3 & + & 1x_4 & \leq & 700 & \text{Machine B hours} \\
& 2x_1 & + & 3x_2 & + & 1x_3 & + & 2x_4 & \leq & 200 & \text{Machine C hours} \\
& x_1, x_2, x_3, x_4 \geq 0 & & & & & & & & &
\end{array}
$$

a. Formulate the dual to this problem.
b. Solve the dual. Use the dual solution to show that the profit-maximizing product mix is $x_1 = 0$, $x_2 = 25$, $x_3 = 125$, and $x_4 = 0$.
c. Use the dual variables to identify the machine or machines that are producing at maximum capacity. If the manager can select one machine for additional production capacity, which machine should have priority? Why?

SELF test

18. Find the dual for the following linear program:

$$\begin{aligned}
\text{Min} \quad & 2800x_1 + 6000x_2 + 1200x_3 \\
\text{s.t.} \quad & \\
& 15x_1 + 15x_2 + 1x_3 \geq 5 \\
& 4x_1 + 8x_2 \geq 5 \\
& 12x_1 + 8x_3 \geq 24 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}$$

SELF test

19. Write the following primal problem in canonical form, and find its dual.

$$\begin{aligned}
\text{Max} \quad & 3x_1 + 1x_2 + 5x_3 + 3x_4 \\
\text{s.t.} \quad & \\
& 3x_1 + 1x_2 + 2x_3 = 30 \\
& 2x_1 + 1x_2 + 3x_3 + 1x_4 \geq 15 \\
& 2x_2 + 3x_4 \leq 25 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{aligned}$$

20. Photo Chemicals produces two types of photograph-developing fluids at a cost of \$1.00 per gallon. Let

$$\begin{aligned}
x_1 &= \text{gallons of product 1} \\
x_2 &= \text{gallons of product 2}
\end{aligned}$$

Photo Chemicals management requires that at least 30 gallons of product 1 and at least 20 gallons of product 2 be produced. They also require that at least 80 pounds of a perishable raw material be used in production. A linear programming formulation of the problem is as follows:

$$\begin{aligned}
\text{Min} \quad & 1x_1 + 1x_2 \\
\text{s.t.} \quad & \\
& 1x_1 \geq 30 \quad \text{Minimum product 1} \\
& 1x_2 \geq 20 \quad \text{Minimum product 2} \\
& 1x_1 + 2x_2 \geq 80 \quad \text{Minimum raw material} \\
& x_1, x_2 \geq 0
\end{aligned}$$

a. Write the dual problem.

b. Solve the dual problem. Use the dual solution to show that the optimal production plan is $x_1 = 30$ and $x_2 = 25$.

c. The third constraint involves a management request that the current 80 pounds of a perishable raw material be used. However, after learning that the optimal solution calls for an excess production of five units of product 2, management is reconsidering the raw material requirement. Specifically, you have been asked to identify the cost effect if this constraint is relaxed. Use the dual variable to indicate the change in the cost if only 79 pounds of raw material have to be used.

21. Consider the following linear programming problem:

$$\begin{aligned}
\text{Min} \quad & 4x_1 + 3x_2 + 6x_3 \\
\text{s.t.} \quad & \\
& 1x_1 + 0.5x_2 + 1x_3 \geq 15 \\
& 2x_2 + 1x_3 \geq 30 \\
& 1x_1 + 1x_2 + 2x_3 \geq 20 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}$$

a. Write the dual problem.
b. Solve the dual.
c. Use the dual solution to identify the optimal solution to the original primal problem.
d. Verify that the optimal values for the primal and dual problems are equal.

22. A sales representative who sells two products is trying to determine the number of sales calls that should be made during the next month to promote each product. Based on past experience, representatives earn an average $10 commission for every call on product 1 and a $5 commission for every call on product 2. The company requires at least 20 calls per month for each product and not more than 100 calls per month on any one product. In addition, the sales representative spends about 3 hours on each call for product 1 and 1 hour on each call for product 2. If 175 selling hours are available next month, how many calls should be made for each of the two products to maximize the commission?
a. Formulate a linear program for this problem.
b. Formulate and solve the dual problem.
c. Use the final simplex tableau for the dual problem to determine the optimal number of calls for the products. What is the maximum commission?
d. Interpret the values of the dual variables.

23. Consider the linear program

$$
\begin{aligned}
\text{Max} \quad & 3x_1 + 2x_2 \\
\text{s.t.} \quad & \\
& 1x_1 + 2x_2 \le 8 \\
& 2x_1 + 1x_2 \le 10 \\
& x_1, x_2 \ge 0
\end{aligned}
$$

a. Solve this problem using the simplex method. Keep a record of the value of the objective function at each extreme point.
b. Formulate and solve the dual of this problem using the graphical procedure.
c. Compute the value of the dual objective function for each extreme-point solution of the dual problem.
d. Compare the values of the objective function for each primal and dual extreme-point solution.
e. Can a dual feasible solution yield a value less than a primal feasible solution? Can you state a result concerning bounds on the value of the primal solution provided by any feasible solution to the dual problem?

24. Suppose the optimal solution to a three-variable linear programming problem has $x_1 = 10$, $x_2 = 30$, and $x_3 = 15$. It is later discovered that the following two constraints were inadvertently omitted when formulating the problem.

$$
\begin{aligned}
6x_1 + 4x_2 - 1x_3 &\le 170 \\
\tfrac{1}{4}x_1 + 1x_2 \qquad &\ge 25
\end{aligned}
$$

Find the new optimal solution if possible. If it is not possible, state why it is not possible.

Self-Test Solutions and Answers to Even-Numbered Problems

Chapter 18

1. a. Recomputing the $c_j - z_j$ values for the nonbasic variables with c_1 as the coefficient of x_1 leads to the following inequalities that must be satisfied:

For x_2, we get no inequality because of the zero in the x_2 column for the row in which x_1 is a basic variable

For s_1, we get

$$\begin{aligned} 0 + 4 - c_1 &\leq 0 \\ c_1 &\geq 4 \end{aligned}$$

For s_2, we get

$$\begin{aligned} 0 - 12 + 2c_1 &\leq 0 \\ 2c_1 &\leq 12 \\ c_1 &\leq 6 \\ \text{Range: } 4 \leq c_1 &\leq 6 \end{aligned}$$

b. Because x_2 is nonbasic, we have

$$c_2 \leq 8$$

c. Because s_1 is nonbasic, we have

$$c_{s_1} \leq 1$$

2. a. $31.25 \leq c_2 \leq 83.33$

b. $-43.33 \leq c_{s_2} \leq 8.75$

c. $c_{s_3} \leq 26/5$

d. Variables do not change; Value = \$1920

3. a. It is the z_j value for s_1; dual price = 1

b. It is the z_j value for s_2; dual price = 2

c. It is the z_j value for s_3; dual price = 0

d.

$$\begin{aligned} s_3 &= 80 + 5(-2) = 70 \\ x_3 &= 30 + 5(-1) = 25 \\ x_1 &= 20 + 5(1) = 25 \\ \text{Value} &= 220 + 5(1) = 225 \end{aligned}$$

e.

$$\begin{aligned} s_3 &= 80 - 10(-2) = 100 \\ x_3 &= 30 - 10(-1) = 40 \\ x_1 &= 20 - 10(1) = 10 \\ \text{Value} &= 220 - 10(1) = 210 \end{aligned}$$

4. a.

$$\begin{aligned} 80 + \Delta b_1(-2) \geq 0 &\qquad \Delta b_1 \leq 40 \\ 30 + \Delta b_1(-1) \geq 0 &\qquad \Delta b_1 \leq 30 \\ 20 + \Delta b_1(1) \geq 0 &\qquad \Delta b_1 \geq -20 \\ -20 \leq \Delta b_1 &\leq 30 \\ 100 \leq b_1 &\leq 150 \end{aligned}$$

b.

$$\begin{aligned} 80 + \Delta b_2(7) \geq 0 &\qquad \Delta b_2 \geq -80/7 \\ 30 + \Delta b_2(3) \geq 0 &\qquad \Delta b_2 \geq -10 \\ 20 + \Delta b_2(-2) \geq 0 &\qquad \Delta b_2 \leq 10 \\ -10 \leq \Delta b_2 &\leq 10 \\ 40 \leq b_2 &\leq 60 \end{aligned}$$

c.

$$\begin{aligned} 80 - \Delta b_3(1) &\geq 0 \rightarrow \Delta b_3 \leq 80 \\ 30 - \Delta b_3(0) &\geq 0 \\ 20 - \Delta b_3(0) &\geq 0 \\ \Delta b_3 &\leq 80 \\ b_3 &\leq 110 \end{aligned}$$

6. a. $6.3 \leq c_1 \leq 13.5$

b. $6\frac{2}{3} \leq c_2 \leq 14\frac{2}{7}$

c. Variables do not change; Value = \$7164

d. Below $6\frac{2}{3}$ or above $14\frac{2}{7}$

e. $x_1 = 300$, $x_2 = 420$; Value = \$9300

8. a. $x_1 = 5220/11$, $x_2 = 3852/11$; Value = 86,868/11

b. No, s_1 would enter the basis

10. a. Increase in profit

b. No

12. a. $14 \leq b_1 \leq 21\frac{1}{2}$

b. $4 \leq b_2$

c. $18\frac{3}{4} \leq b_3 \leq 30$

d. Dual price = 400/9; Range: $18\frac{3}{4} \leq b_3 \leq 30$

14. a. $400/3 \leq b_1 \leq 800$

b. $275 \leq b_2$

c. $275/2 \leq b_3 \leq 625$

16. a. $x_1 = 4000$, $x_2 = 10{,}000$; Total risk = 62,000

b. Increase it by 2.167 per unit

c. $48{,}000 \leq b_2 \leq 102{,}000$

d. $x_1 = 5667$, $x_2 = 9167$; Total risk = 72,833

e. Variables do not change; Total risk = 66,000

17. a. The dual is given by:

$$\begin{aligned} \text{Min} \quad & 550u_1 + 700u_2 + 200u_3 \\ \text{s.t.} \quad & \\ & 1.5u_1 + 4u_2 + 2u_3 \geq 4 \\ & 2u_1 + 1u_2 + 3u_3 \geq 6 \\ & 4u_1 + 2u_2 + 1u_3 \geq 3 \\ & 3u_1 + 1u_2 + 2u_3 \geq 1 \\ & u_1, u_2, u_3 \geq 0 \end{aligned}$$

b. Optimal solution: $u_1 = 3/10$; $u_2 = 0$, $u_3 = 54/30$
The z_j values for the four surplus variables of the dual show $x_1 = 0$, $x_2 = 25$, $x_3 = 125$, and $x_4 = 0$

c. Because $u_1 = 3/10$, $u_2 = 0$, and $u_3 = 54/30$, machines A and C ($u_j > 0$) are operating at capacity; machine C is the priority machine since each hour is worth 54/30

18. The dual is given by

$$\begin{aligned} \text{Max} \quad & 5u_1 + 5u_2 + 24u_3 \\ \text{s.t.} \quad & \\ & 15u_1 + 4u_2 + 12u_3 \leq 2800 \\ & 15u_1 + 8u_2 \leq 6000 \\ & u_1 + 8u_3 \leq 1200 \\ & u_1, u_2, u_3 \geq 0 \end{aligned}$$

19. The canonical form is

$$\begin{array}{ll}
\text{Max} & 3x_1 + x_2 + 5x_3 + 3x_4 \\
\text{s.t.} & \\
& \begin{aligned}
3x_1 + 1x_2 + 2x_3 \phantom{{}- x_4} &\leq 30 \\
-3x_1 - 1x_2 - 2x_3 \phantom{{}- x_4} &\leq -30 \\
-2x_1 - 1x_2 - 3x_3 - x_4 &\leq -15 \\
2x_2 + 3x_4 &\leq 25 \\
x_1, x_2, x_3, x_4 &\geq 0
\end{aligned}
\end{array}$$

The dual is

$$\begin{array}{ll}
\text{Min} & 30u_1' - 30u_1'' - 15u_2 + 25u_3 \\
\text{s.t.} & \\
& \begin{aligned}
3u_1' - 3u_1'' - 2u_2 \phantom{{}+ 2u_3} &\geq 3 \\
u_1' - u_1'' - u_2 + 2u_3 &\geq 1 \\
2u_1' - 2u_1'' - 3u_2 \phantom{{}+ 2u_3} &\geq 5 \\
-u_2 + 3u_3 &\geq 3 \\
u_1', u_1'', u_2, u_3 &\geq 0
\end{aligned}
\end{array}$$

20. a. Max $30u_1 + 20u_2 + 80u_3$

s.t.

$$\begin{aligned}
u_1 + u_3 &\leq 1 \\
u_2 + 2u_3 &\leq 1 \\
u_1, u_2, u_3 &\geq 0
\end{aligned}$$

b. $x_1 = 30, x_2 = 25$

c. Reduce cost by \$0.50

22. a. Max $10x_1 + 5x_2$

s.t.

$$\begin{aligned}
x_1 &\geq 20 \\
x_2 &\geq 20 \\
x_1 &\leq 100 \\
x_2 &\leq 100 \\
3x_1 + x_2 &\leq 175 \\
x_1, x_2 &\geq 0
\end{aligned}$$

b. Min $-20u_1 - 20u_2 + 100u_3 + 100u_4 + 175u_5$

s.t.

$$\begin{aligned}
-u_1 + u_3 + 3u_5 &\geq 10 \\
-u_2 + u_4 + u_5 &\geq 5 \\
u_1, u_2, u_3, u_4, u_5 &\geq 0
\end{aligned}$$

Solution: $u_4 = 5/3, u_5 = 10/3$

c. $x_1 = 25, x_2 = 100$; commission = \$750

24. Check both constraints with $x_1 = 10, x_2 = 30, x_3 = 15$
Both constraints are satisfied; solution remains optimal