CHAPTER 19

Solution Procedures for Transportation and Assignment Problems

CONTENTS

In Chapter 6, we introduced the **transportation** and **assignment problems** and showed how each could by solved using linear programming. In this chapter, we return to these two problems and describe special solution procedures that simplify the computations required to obtain an optimal solution.

19.1   TRANSPORTATION SIMPLEX METHOD: A SPECIAL-PURPOSE SOLUTION PROCEDURE

Solving transportation problems with a general-purpose linear programming code is fine for small to medium-sized problems. However, these problems often grow very large (a problem with 100 origins and 1000 destinations would have 100,000 variables), and more efficient solution procedures may be needed. The network structure of the transportation problem has enabled management scientists to develop special-purpose solution procedures that greatly simplify the computations.

In Section 6.1 we introduced the Foster Generators transportation problem and showed how to formulate and solve it as a linear program. The linear programming formulation involved 12 variables and 7 constraints. In this section we describe a special-purpose solution procedure, called the **transportation simplex method,** that takes advantage of the network structure of the transportation problem and makes possible the solution of large transportation problems efficiently on a computer and small transportation problems by hand.

The transportation simplex method, like the simplex method for linear programs, is a two-phase procedure; it involves first finding an initial feasible solution and then proceeding iteratively to make improvements in the solution until an optimal solution is reached. To summarize the data conveniently and to keep track of the calculations, we utilize a **transportation tableau.** The transportation tableau for the Foster Generators problem is presented in Table 19.1.

Note that the 12 *cells* in the tableau correspond to the 12 routes from one origin to one destination. Thus, each cell in the transportation tableau corresponds to a variable in the linear programming formulation. The entries in the right-hand margin of the tableau indicate the supply at each origin, and the entries in the bottom margin indicate the demand at each destination. Each row corresponds to a supply node, and each column corresponds to a demand node in the network model of the problem. The number of rows plus the number of columns equals the number of constraints in the linear programming formulation of the problem. The entries in the upper right-hand corner of each cell show the transportation cost per unit shipped over the corresponding route. Note also that for the Foster Generators problem total supply equals total demand. The transportation simplex method can be applied only to a balanced problem (total supply = total demand); if a problem is not balanced, a dummy origin or dummy destination must be added. The use of dummy origins and destinations will be discussed later in this section.

Phase I: Finding an Initial Feasible Solution

The first phase of the transportation simplex method involves finding an initial feasible solution. Such a solution provides arc flows that satisfy each demand constraint without shipping more from any origin node than the supply available. The procedures most often used to find an initial feasible solution to a transportation problem are called heuristics. A **heuristic** is a commonsense procedure for quickly finding a solution to a problem.

Several heuristics have been developed to find an initial feasible solution to a transportation problem. Although some heuristics can find an initial feasible solution quickly, often the solution they find is not especially good in terms of minimizing total cost. Other

TABLE 19.1 TRANSPORTATION TABLEAU FOR THE FOSTER GENERATORS TRANSPORTATION PROBLEM

Origin	Destination				Origin Supply
	Boston	Chicago	St. Louis	Lexington	
Cleveland	3	2	7	6	5000
Bedford	7	5	2	3	6000
York	2	5	4	5	2500
Destination Demand	6000	4000	2000	1500	13,500

Cell corresponding to shipments from Bedford to Boston

Total supply and total demand

heuristics may not find an initial feasible solution as quickly, but the solution they find is often good in terms of minimizing total cost. The heuristic we describe for finding an initial feasible solution to a transportation problem is called the **minimum cost method.** This heuristic strikes a compromise between finding a feasible solution quickly and finding a feasible solution that is close to the optimal solution.

We begin by allocating as much flow as possible to the minimum cost arc. In Table 19.1 we see that the Cleveland–Chicago, Bedford–St. Louis, and York–Boston routes each qualifies as the minimum cost arc because they each have a transportation cost of $2 per unit. When ties between arcs occur, we follow the convention of selecting the arc to which the most flow can be allocated. In this case it corresponds to shipping 4000 units from Cleveland to Chicago, so we write 4000 in the Cleveland–Chicago cell of the transportation tableau. This selection reduces the supply at Cleveland from 5000 to 1000; hence, we cross out the 5000-unit supply value and replace it with the reduced value of 1000. In addition, allocating 4000 units to this arc satisfies the demand at Chicago, so we reduce the Chicago demand to zero and eliminate the corresponding column from further consideration by drawing a line through it. The transportation tableau now appears as shown in Table 19.2.

Now we look at the reduced tableau consisting of all unlined cells to identify the next minimum cost arc. The Bedford–St. Louis and York–Boston routes tie with transportation cost of $2 per unit. More units of flow can be allocated to the York–Boston route, so we choose it for the next allocation. This step results in an allocation of 2500 units over the York–Boston route. To update the tableau, we reduce the Boston demand by 2500 units to

TABLE 19.2   TRANSPORTATION TABLEAU AFTER ONE ITERATION OF THE MINIMUM
COST METHOD

	Boston	Chicago	St. Louis	Lexington	Supply
Cleveland	3	2 4000	7	6	1000 ~~5000~~
Bedford	7	5	2	3	6000
York	2	5	4	5	2500
Demand	6000	~~4000~~ 0	2000	1500	

3500, reduce the York supply to zero, and eliminate this row from further consideration by lining through it. Continuing the process results in an allocation of 2000 units over the Bedford–St. Louis route and the elimination of the St. Louis column because its demand goes to zero. The transportation tableau obtained after carrying out the second and third iterations is shown in Table 19.3.

TABLE 19.3   TRANSPORTATION TABLEAU AFTER THREE ITERATIONS
OF THE MINIMUM COST METHOD

	Boston	Chicago	St. Louis	Lexington	Supply
Cleveland	3	2 4000	7	6	1000 ~~5000~~
Bedford	7	5	2 2000	3	4000 ~~6000~~
York	2 ~~2500~~	5	4	5	0 ~~2500~~
Demand	~~6000~~ 3500	~~4000~~ 0	~~2000~~ 0	1500	

TABLE 19.4   TRANSPORTATION TABLEAU AFTER FIVE ITERATIONS
OF THE MINIMUM COST METHOD

	Boston	Chicago	St. Louis	Lexington	**Supply**
Cleveland	3 1000	2 4000	7	6	0 1000 5000
Bedford	7	5	2 2000	3 1500	2500 4000 6000
York	2 2500	5	4	5	0 2500
Demand	6000 3500 2500	4000 0	2000 0	1500 0	

We now have two arcs that qualify for the minimum cost arc with a value of 3: Cleveland–Boston and Bedford–Lexington. We can allocate a flow of 1000 units to the Cleveland–Boston route and a flow of 1500 to the Bedford–Lexington route, so we allocate 1500 units to the Bedford–Lexington route. Doing so results in a demand of zero at Lexington and eliminates this column. The next minimum cost allocation is 1000 over the Cleveland–Boston route. After we make these two allocations, the transportation tableau appears as shown in Table 19.4.

The only remaining unlined cell is Bedford–Boston. Allocating 2500 units to the corresponding arc uses up the remaining supply at Bedford and satisfies all the demand at Boston. The resulting tableau is shown in Table 19.5.

This solution is feasible because all the demand is satisfied and all the supply is used. The total transportation cost resulting from this initial feasible solution is calculated in Table 19.6. Phase I of the transportation simplex method is now complete; we have an initial feasible solution. The total transportation cost associated with this solution is $42,000.

Summary of the Minimum Cost Method   Before applying phase II of the transportation simplex method, let us summarize the steps for obtaining an initial feasible solution using the minimum cost method.

Step 1. Identify the cell in the transportation tableau with the lowest cost, and allocate as much flow as possible to this cell. In case of a tie, choose the cell corresponding to the arc over which the most units can be shipped. If ties still exist, choose any of the tied cells.

Step 2. Reduce the row supply and the column demand by the amount of flow allocated to the cell identified in step 1.

TABLE 19.5   FINAL TABLEAU SHOWING THE INITIAL FEASIBLE SOLUTION
OBTAINED USING THE MINIMUM COST METHOD

	Boston	Chicago	St. Louis	Lexington	**Supply**
Cleveland	3 1000	2 4000	7	6	0 ~~1000~~ ~~5000~~
Bedford	7 2500	5	2 2000	3 1500	0 ~~2500~~ ~~4000~~ ~~6000~~
York	2 2500	5	4	5	0 ~~2500~~
Demand	~~6000~~ ~~3500~~ ~~2500~~ 0	~~4000~~ 0	~~2000~~ 0	~~1500~~ 0	

Step 3. If *all* row supplies and column demands have been exhausted, then stop; the allocations made will provide an initial feasible solution. Otherwise, continue with step 4.

Step 4. If the row supply is now zero, eliminate the row from further consideration by drawing a line through it. If the column demand is now zero, eliminate the column by drawing a line through it.

Step 5. Continue with step 1 for all unlined rows and columns.

TABLE 19.6   TOTAL COST OF THE INITIAL FEASIBLE SOLUTION OBTAINED USING
THE MINIMUM COST METHOD

Route		Units	Cost	
From	**To**	**Shipped**	**per Unit**	**Total Cost**
Cleveland	Boston	1000	$3	$ 3,000
Cleveland	Chicago	4000	$2	8,000
Bedford	Boston	2500	$7	17,500
Bedford	St. Louis	2000	$2	4,000
Bedford	Lexington	1500	$3	4,500
York	Boston	2500	$2	5,000
				$42,000

Phase II: Iterating to the Optimal Solution

Phase II of the transportation simplex method is a procedure for iterating from the initial feasible solution identified in phase I to the optimal solution. Recall that each cell in the transportation tableau corresponds to an arc (route) in the network model of the transportation problem. The first step at each iteration of phase II is to identify an incoming arc. The **incoming arc** is the currently unused route (unoccupied cell) where making a flow allocation will cause the largest per-unit reduction in total cost. Flow is then assigned to the incoming arc, and the amounts being shipped over all other arcs to which flow had previously been assigned (occupied cells) are adjusted as necessary to maintain a feasible solution. In the process of adjusting the flow assigned to the occupied cells, we identify and drop an **outgoing arc** from the solution. Thus, at each iteration in phase II, we bring a currently unused arc (unoccupied cell) into the solution, and remove an arc to which flow had previously been assigned (occupied cell) from the solution.

To show how phase II of the transportation simplex method works, we must explain how to identify the incoming arc (cell), how to make the adjustments to the other occupied cells when flow is allocated to the incoming arc, and how to identify the outgoing arc (cell). We first consider identifying the incoming arc.

As mentioned, the incoming arc is the one that will cause the largest reduction per unit in the total cost of the current solution. To identify this arc, we must compute for each unused arc the amount by which total cost will be reduced by shipping one unit over that arc. The *mo*dified *di*stribution or **MODI method** is a way to make this computation.

The MODI method requires that we define an index u_i for each row of the tableau and an index v_j for each column of the tableau. Computing these row and column indexes requires that the cost coefficient for each occupied cell equal $u_i + v_j$. Thus, when c_{ij} is the cost per unit from origin i to destination j, then $u_i + v_j = c_{ij}$ for each occupied cell. Let us return to the initial feasible solution for the Foster Generators problem, which we found using the minimum cost method (see Table 19.7), and use the MODI method to identify the incoming arc.

TABLE 19.7   INITIAL FEASIBLE SOLUTION FOR THE FOSTER GENERATORS PROBLEM

	Boston	Chicago	St. Louis	Lexington	**Supply**
Cleveland	3 1000	2 4000	7	6	5000
Bedford	7 2500	5	2 2000	3 1500	6000
York	2 2500	5	4	5	2500
Demand	6000	4000	2000	1500	

Requiring that $u_i + v_j = c_{ij}$ for all the occupied cells in the initial feasible solution leads to a system of six equations and seven indexes, or variables:

Occupied Cell	$u_i + v_j = c_{ij}$
Cleveland–Boston	$u_1 + v_1 = 3$
Cleveland–Chicago	$u_1 + v_2 = 2$
Bedford–Boston	$u_2 + v_1 = 7$
Bedford–St. Louis	$u_2 + v_3 = 2$
Bedford–Lexington	$u_2 + v_4 = 3$
York–Boston	$u_3 + v_1 = 2$

With one more index (variable) than equation in this system, we can freely pick a value for one of the indexes and then solve for the others. We will always choose $u_1 = 0$ and then solve for the values of the other indexes. Setting $u_1 = 0$, we obtain

$$0 + v_1 = 3$$
$$0 + v_2 = 2$$
$$u_2 + v_1 = 7$$
$$u_2 + v_3 = 2$$
$$u_2 + v_4 = 3$$
$$u_3 + v_1 = 2$$

Solving these equations leads to the following values for $u_1, u_2, u_3, v_1, v_2, v_3,$ and v_4:

$$u_1 = 0 \qquad v_1 = 3$$
$$u_2 = 4 \qquad v_2 = 2$$
$$u_3 = -1 \qquad v_3 = -2$$
$$ v_4 = -1$$

Management scientists have shown that for each *unoccupied* cell, $e_{ij} = c_{ij} - u_i - v_j$ provides the change in total cost per unit that will be obtained by allocating one unit of flow to the corresponding arc. Thus, we will call e_{ij} the **net evaluation index.** Because of the way u_i and v_j are computed, the net evaluation index for each occupied cell equals zero.

Rewriting the tableau containing the initial feasible solution for the Foster Generators problem and replacing the previous marginal information with the values of u_i and v_j, we obtain Table 19.8. We computed the net evaluation index (e_{ij}) for each unoccupied cell, which is the circled number in the cell. Thus, shipping one unit over the route from origin 1 to destination 3 (Cleveland–St. Louis) will increase total cost by $9; shipping one unit from origin 1 to destination 4 (Cleveland–Lexington) will increase total cost by $7; shipping one unit from origin 2 to destination 2 (Bedford–Chicago) will decrease total cost by $1; and so on.

On the basis of the net evaluation indexes, the best arc in terms of cost reduction (a net evaluation index of -1) is associated with the Bedford–Chicago route (origin 2–destination 2); thus, the cell in row 2 and column 2 is chosen as the incoming cell. Total cost decreases by $1 for every unit of flow assigned to this arc. The question now is: How much flow should we assign to this arc? Because the total cost decreases by $1 per unit assigned, we want to allocate the maximum possible flow. To find that maximum, we must recognize that, to maintain feasibility, each unit of flow assigned to this arc will require adjustments in the flow over the other currently used arcs. The **stepping-stone method** can be used to determine the adjustments necessary and to identify an outgoing arc.

TABLE 19.8   NET EVALUATION INDEXES FOR THE INITIAL FEASIBLE SOLUTION TO
THE FOSTER GENERATORS PROBLEM COMPUTED USING THE MODI
METHOD

u_i	v_j			
	3	2	−2	−1
0	3 1000	2 4000	7 (9)	6 (7)
4	7 2500	5 (−1)	2 2000	3 1500
−1	2 2500	5 (4)	4 (7)	5 (7)

The Stepping-Stone Method   Suppose that we allocate one unit of flow to the incoming
arc (the Bedford–Chicago route). To maintain feasibility—that is, not exceed the num-
ber of units to be shipped to Chicago—we would have to reduce the flow assigned to
the Cleveland–Chicago arc to 3999. But then we would have to increase the flow on the
Cleveland–Boston arc to 1001 so that the total Cleveland supply of 5000 units could be
shipped. Finally, we would have to reduce the flow on the Bedford–Boston arc by 1 to sat-
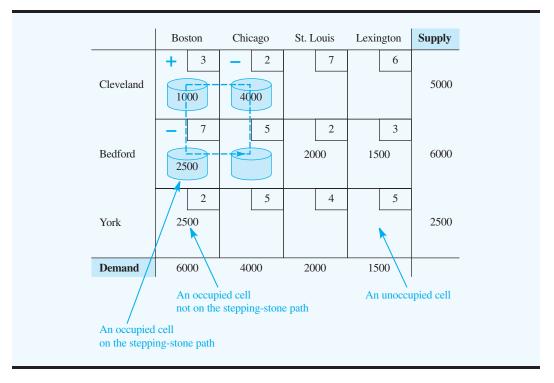isfy the Boston demand exactly. Table 19.9 summarizes this cycle of adjustments.

  The cycle of adjustments needed in making an allocation to the Bedford–Chicago cell
required changes in four cells: the incoming cell (Bedford–Chicago) and three currently oc-
cupied cells. We can view these four cells as forming a stepping-stone path in the tableau,
where the corners of the path are currently occupied cells. The idea behind the stepping-
stone name is to view the tableau as a pond with the occupied cells as stones sticking up in
it. To identify the stepping-stone path for an incoming cell, we start at the incoming cell and
move horizontally and vertically using occupied cells as the stones at the corners of the path;
the objective is to step from stone to stone and return to the incoming cell where we started.
To focus attention on which occupied cells are part of the stepping-stone path, we draw each
occupied cell in the stepping-stone path as a cylinder, which should reinforce the image of
these cells as stones sticking up in the pond. Table 19.10 depicts the stepping-stone path as-
sociated with the incoming arc of the Bedford–Chicago route.

  In Table 19.10 we placed a plus sign (+) or a minus sign (−) in each occupied cell on
the stepping-stone path. A plus sign indicates that the allocation to that cell will increase
by the same amount we allocate to the incoming cell. A minus sign indicates that the
allocation to that cell will decrease by the amount allocated to the incoming cell. Thus, to
determine the maximum amount that may be allocated to the incoming cell, we simply look
to the cells on the stepping-stone path identified with a minus sign. Because no arc can have
a negative flow, the minus-sign cell with the *smallest amount* allocated to it will determine
the maximum amount that can be allocated to the incoming cell. After allocating this

TABLE 19.9   CYCLE OF ADJUSTMENTS IN OCCUPIED CELLS NECESSARY TO MAINTAIN
FEASIBILITY WHEN SHIPPING ONE UNIT FROM BEDFORD TO CHICAGO

	Boston	Chicago	St. Louis	Lexington	**Supply**
Cleveland	3 1001 ~~1000~~	2 3999 ~~4000~~			5000
Bedford	7 2499 ~~2500~~	5 1	2 2000	3 1500	6000
York	2 2500	5	4	5	2500
Demand	6000	4000	2000	1500	

TABLE 19.10   STEPPING-STONE PATH WITH BEDFORD–CHICAGO
AS THE INCOMING ROUTE

	Boston	Chicago	St. Louis	Lexington	**Supply**
Cleveland	+ 3 1000	− 2 4000	7	6	5000
Bedford	− 7 2500	5	2 2000	3 1500	6000
York	2 2500	5	4	5	2500
Demand	6000	4000	2000	1500	

An occupied cell
not on the stepping-stone path

An unoccupied cell

An occupied cell
on the stepping-stone path

TABLE 19.11   NEW SOLUTION AFTER ONE ITERATION IN PHASE II
OF THE TRANSPORTATION SIMPLEX METHOD

	Boston	Chicago	St. Louis	Lexington	Supply
Cleveland	3 — 3500	2 — 1500	7	6	5000
Bedford	7	5 — 2500	2 — 2000	3 — 1500	6000
York	2 — 2500	5	4	5	2500
Demand	6000	4000	2000	1500	

maximum amount to the incoming cell, we then make all the adjustments necessary on the stepping-stone path to maintain feasibility. The incoming cell becomes an occupied cell, and the outgoing cell is dropped from the current solution.

In the Foster Generators problem, the Bedford–Boston and Cleveland–Chicago cells are the ones where the allocation will decrease (the ones with a minus sign) as flow is allocated to the incoming arc (Bedford–Chicago). The 2500 units currently assigned to Bedford–Boston is less than the 4000 units assigned to Cleveland–Chicago, so we identify Bedford–Boston as the outgoing arc. We then obtain the new solution by allocating 2500 units to the Bedford–Chicago arc, making the appropriate adjustments on the stepping-stone path and dropping Bedford–Boston from the solution (its allocation has been driven to zero). Table 19.11 shows the tableau associated with the new solution. Note that the only changes from the previous tableau are located on the stepping-stone path originating in the Bedford–Chicago cell.

We now try to improve on the current solution. Again, the first step is to apply the MODI method to find the best incoming arc, so we recompute the row and column indexes by requiring that $u_i + v_j = c_{ij}$ for all occupied cells. The values of u_i and v_j can easily be computed directly on the tableau. Recall that we begin the MODI method by setting $u_1 = 0$. Thus, for the two occupied cells in row 1 of the table, $v_j = c_{1j}$; as a result, $v_1 = 3$ and $v_2 = 2$. Moving down the column associated with each newly computed column index, we compute the row index associated with each occupied cell in that column by subtracting v_j from c_{ij}. Doing so for the newly found column indexes, v_1 and v_2, we find that $u_3 = 2 - 3 = -1$ and that $u_2 = 5 - 2 = 3$. Next, we use these row indexes to compute the column indexes for occupied cells in the associated rows, obtaining $v_3 = 2 - 3 = -1$ and $v_4 = 3 - 3 = 0$. Table 19.12 shows these new row and column indexes.

Also shown in Table 19.12 are the net changes (the circled numbers) in the value of the solution that will result from allocating one unit to each unoccupied cell. Recall that these are the net evaluation indexes given by $e_{ij} = c_{ij} - u_i - v_j$. Note that the net evaluation index

TABLE 19.12   MODI EVALUATION OF EACH CELL IN SOLUTION

u_i		v_j			
		3	2	−1	0
0	3	2	7	6	
		3500	1500	⑧	⑥
3	7	5	2	3	
		①	2500	2000	1500
−1	2	5	4	5	
		2500	④	⑥	⑥

for every unoccupied cell is now greater than or equal to zero. This condition shows that if current unoccupied cells are used, the cost will actually increase. Without an arc to which flow can be assigned to decrease the total cost, we have reached the optimal solution. Table 19.13 summarizes the optimal solution and shows its total cost.

Maintaining $m + n - 1$ Occupied Cells   Recall that m represents the number of origins and n represents the number of destinations. A solution to a transportation problem that has less than $m + n - 1$ cells with positive allocations is said to be **degenerate.** The solution to the Foster Generators problem is not degenerate; six cells are occupied and $m + n - 1 = 3 + 4 - 1 = 6$. The problem with degeneracy is that $m + n - 1$ occupied cells are required by the MODI method to compute all the row and column indexes. When degeneracy occurs, we must artificially create an occupied cell in order to compute the row and column indexes. Let us illustrate how degeneracy could occur and how to deal with it.

TABLE 19.13   OPTIMAL SOLUTION TO THE FOSTER GENERATORS
                 TRANSPORTATION PROBLEM

Route		Units	Cost	
From	**To**	**Shipped**	**per Unit**	**Total Cost**
Cleveland	Boston	3500	$3	$10,500
Cleveland	Chicago	1500	$2	3,000
Bedford	Chicago	2500	$5	12,500
Bedford	St. Louis	2000	$2	4,000
Bedford	Lexington	1500	$3	4,500
York	Boston	2500	$2	5,000
				$39,500

TABLE 19.14　TRANSPORTATION TABLEAU WITH A DEGENERATE INITIAL FEASIBLE SOLUTION

u_i	v_j = 3	6		Supply
0	3 35	6 25	7	60
−1	8	5 30	7	30
	4	9	11 30	30
Demand	35	55	30	

Table 19.14 shows the initial feasible solution obtained using the minimum cost method for a transportation problem involving $m = 3$ origins and $n = 3$ destinations. To use the MODI method for this problem, we must have $m + n − 1 = 3 + 3 − 1 = 5$ occupied cells. Since the initial feasible solution has only four occupied cells, the solution is degenerate.

Suppose that we try to use the MODI method to compute row and column indexes to begin phase II for this problem. Setting $u_1 = 0$ and computing the column indexes for each occupied cell in row 1, we obtain $v_1 = 3$ and $v_2 = 6$ (see Table 19.14). Continuing, we then compute the row indexes for all occupied cells in columns 1 and 2. Doing so yields $u_2 = 5 − 6 = −1$. At this point, we cannot compute any more row and column indexes because no cells in columns 1 and 2 of row 3 and no cells in rows 1 or 2 of column 3 are occupied.

To compute all the row and column indexes when fewer than $m + n − 1$ cells are occupied, we must create one or more "artificially" occupied cells with a flow of zero. In Table 19.14 we must create one artificially occupied cell to have five occupied cells. Any currently unoccupied cell can be made an artificially occupied cell if doing so makes it possible to compute the remaining row and column indexes. For instance, treating the cell in row 2 and column 3 of Table 19.14 as an artificially occupied cell will enable us to compute v_3 and u_3, but placing it in row 2 and column 1 will not.
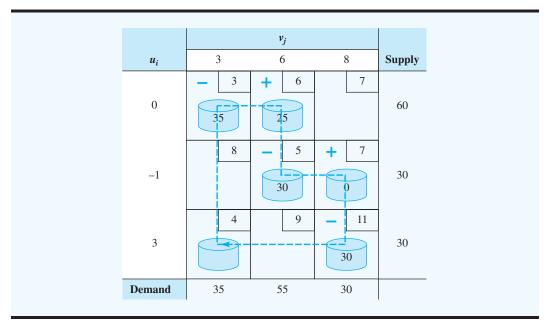
As we previously stated, whenever an artificially occupied cell is created, we assign a flow of zero to the corresponding arc. Table 19.15 shows the results of creating an artificially occupied cell in row 2 and column 3 of Table 19.14. Creation of the artificially occupied cell results in five occupied cells, so we can now compute the remaining row and column indexes. Using the row 2 index ($u_2 = −1$) and the artificially occupied cell in row 2, we compute the column index for column 3; thus, $v_3 = c_{23} − u_2 = 7 − (−1) = 8$. Then, using the column 3 index ($v_3 = 8$) and the occupied cell in row 3 and column 3 of the tableau, we compute the row 3 index: $u_3 = c_{33} − v_3 = 11 − 8 = 3$. Table 19.15 shows the complete set of row and column indexes and the net evaluation index for each unoccupied cell.

TABLE 19.15 TRANSPORTATION TABLEAU WITH AN ARTIFICIAL CELL IN ROW 2 AND COLUMN 3

u_i	v_j 3	6	8	Supply
0	3 35	6 25	7 (−1)	60
−1	8 (6)	5 30	7 0	30
3	4 (−2)	9 (0)	11 30	30
Demand	35	55	30	

Artificially occupied cell (pointing to the cell in row 2, column 3)

Reviewing the net evaluation indexes in Table 19.15, we identify the cell in row 3 and column 1 (net evaluation index = −2) as the incoming cell. The stepping-stone path and the adjustments necessary to maintain feasibility are shown in Table 19.16. Note that the stepping-stone path can be more complex than the simple one obtained for the incoming cell in the Foster Generators problem. The path in Table 19.16 requires adjustments in all

TABLE 19.16 STEPPING-STONE PATH FOR THE INCOMING CELL IN ROW 3 AND COLUMN 1

u_i	v_j 3	6	8	Supply
0	− 3 35	+ 6 25	7	60
−1	8	− 5 30	+ 7 0	30
3	4	9	− 11 30	30
Demand	35	55	30	

five occupied cells to maintain feasibility. Again, the plus- and minus-sign labels simply show where increases and decreases in the allocation will occur as units of flow are added to the incoming cell. The smallest flow in a decreasing cell is a tie between the cell in row 2 and column 2 and the cell in row 3 and column 3.

Because the smallest amount in a decreasing cell is 30, the allocation we make to the incoming cell is 30 units. However, when 30 units are allocated to the incoming cell and the appropriate adjustments are made to the occupied cells on the stepping-stone path, the allocation to two cells goes to zero (row 2, column 2 and row 3, column 3). We may choose either one as the outgoing cell, but not both. One will be treated as unoccupied; the other will become an artificially occupied cell with a flow of zero allocated to it. The reason we cannot let both become unoccupied cells is that doing so would lead to a degenerate solution, and as before, we could not use the MODI method to compute the row and column indexes for the next iteration. When ties occur in choosing the outgoing cell, we can choose any one of the tied cells as the artificially occupied cell and then use the MODI method to recompute the row and column indexes. As long as no more than one cell is dropped at each iteration, the MODI method will work.

The solution obtained after allocating 30 units to the incoming cell in row 3 and column 1 and making the appropriate adjustments on the stepping-stone path leads to the tableau shown in Table 19.17. Note that we treated the cell in row 2 and column 2 as the artificially occupied cell. After computing the new row and column indexes, we see that the cell in row 1 and column 3 will be the next incoming cell. Each unit allocated to this cell will further decrease the value of the solution by 1. The stepping-stone path associated with this incoming cell is shown in Table 19.18. The cell in row 2 and column 3 is the outgoing cell; the tableau after this iteration is shown in Table 19.19. Note that we have found the optimal solution and that, even though several earlier iterations were degenerate, the final solution is not degenerate.

TABLE 19.17   NEW ROW AND COLUMN INDEXES OBTAINED AFTER ALLOCATING 30 UNITS TO THE INCOMING CELL

u_i	v_j 3	6	8	Supply
0	[3] 5	[6] 55	[7] (−1)	60
−1	[8] (6)	[5] 0	[7] 30	30
1	[4] 30	[9] (2)	[11] (2)	30
Demand	35	55	30	

TABLE 19.18   STEPPING-STONE PATH ASSOCIATED WITH THE INCOMING CELL
IN ROW 1 AND COLUMN 3

u_i		v_j			
		3	6	8	**Supply**
0	3 — 5		6 55	7 (—)	60
−1	8 +		5 0	7 — 30	30
1	4 30		9	11	30
Demand	35		55	30	

TABLE 19.19   OPTIMAL SOLUTION TO A PROBLEM WITH A DEGENERATE INITIAL
FEASIBLE SOLUTION

u_i		v_j			
		3	6	7	**Supply**
0	3 5		6 25	7 30	60
−1	8 ⑥		5 30	7 ①	30
1	4 30		9 ②	11 ③	30
Demand	35		55	30	

Summary of the Transportation Simplex Method

The transportation simplex method is a special-purpose solution procedure applicable to any network model having the special structure of the transportation problem. It is actually a clever implementation of the general simplex method for linear programming that takes advantage of the special mathematical structure of the transportation problem; but because of the special structure, the transportation simplex method is hundreds of times faster than the general simplex method.

Try part (b) of Problem 2 for practice using the transportation simplex method.

To apply the transportation simplex method, you must have a transportation problem with total supply equal to total demand; thus, for some problems you may need to add a dummy origin or dummy destination to put the problem in this form. The transportation simplex method takes the problem in this form and applies a two-phase solution procedure. In phase I, apply the minimum cost method to find an initial feasible solution. In phase II, begin with the initial feasible solution and iterate until you reach an optimal solution. The steps of the transportation simplex method for a minimization problem are summarized as follows.

Phase I

Find an initial feasible solution using the minimum cost method.

Phase II

Step 1. If the initial feasible solution is degenerate with less than $m + n - 1$ occupied cells, add an artificially occupied cell or cells so that $m + n - 1$ occupied cells exist in locations that enable use of the MODI method.

Step 2. Use the MODI method to compute row indexes, u_i, and column indexes, v_j.

Step 3. Compute the net evaluation index $e_{ij} = c_{ij} - u_i - v_j$ for each unoccupied cell.

Step 4. If $e_{ij} \geq 0$ for all unoccupied cells, stop; you have reached the minimum cost solution. Otherwise, proceed to step 5.

Step 5. Identify the unoccupied cell with the smallest (most negative) net evaluation index and select it as the incoming cell.

Step 6. Find the stepping-stone path associated with the incoming cell. Label each cell on the stepping-stone path whose flow will increase with a plus sign and each cell whose flow will decrease with a minus sign.

Step 7. Choose as the outgoing cell the minus-sign cell on the stepping-stone path with the smallest flow. If there is a tie, choose any one of the tied cells. The tied cells that are not chosen will be artificially occupied with a flow of zero at the next iteration.

Step 8. Allocate to the incoming cell the amount of flow currently given to the outgoing cell; make the appropriate adjustments to all cells on the stepping-stone path, and continue with step 2.

Problem Variations

The following problem variations can be handled, with slight adaptations, by the transportation simplex method:

1. Total supply not equal to total demand
2. Maximization objective function
3. Unacceptable routes

The case where the total supply is not equal to the total demand can be handled easily by the transportation simplex method if we first introduce a dummy origin or a dummy

destination. If total supply is greater than total demand, we introduce a **dummy destination** with demand equal to the excess of supply over demand. Similarly, if total demand is greater than total supply, we introduce a **dummy origin** with supply equal to the excess of demand over supply. In either case, the use of a dummy destination or a dummy origin will equalize total supply and total demand so that we can use the transportation simplex method. When a dummy destination or origin is present, we assign cost coefficients of zero to every arc into a dummy destination and to every arc out of a dummy origin. The reason is that no shipments will actually be made from a dummy origin or to a dummy destination when the solution is implemented and thus a zero cost per unit is appropriate.

The transportation simplex method also can be used to solve maximization problems. The only modification necessary involves the selection of an incoming cell. Instead of picking the cell with the smallest or most negative e_{ij} value, we pick that cell for which e_{ij} is largest. That is, we pick the cell that will cause the largest increase per unit in the objective function. If $e_{ij} \leq 0$ for all unoccupied cells, we stop; the maximization solution has been reached.

To handle unacceptable routes in a minimization problem, infeasible arcs must carry an extremely high cost, denoted M, to keep them out of the solution. Thus, if we have a route (arc) from an origin to a destination that for some reason cannot be used, we simply assign this arc a cost per unit of M, and it will not enter the solution. Unacceptable arcs would be assigned a profit per unit of $-M$ in a maximization problem.

NOTES AND COMMENTS

1. Research devoted to developing efficient special-purpose solution procedures for network problems has shown that the transportation simplex method is one of the best. It is used in the transportation and assignment modules of The Management Scientist software package. A simple extension of this method also can be used to solve transshipment problems.

2. As we previously noted, each cell in the transportation tableau corresponds to an arc (route) in the network model of the problem and a variable in the linear programming formulation. Phase II of the transportation simplex method is thus the same as phase II of the simplex method for linear programming. At each iteration, one variable is brought into solution and another variable is dropped from solution. The reason the method works so much better for transportation problems is that the special mathematical structure of the constraint equations means that only addition and subtraction operations are necessary. We can implement the entire procedure in a transportation tableau that has one row for each origin and one column for each destination. A simplex tableau for such a problem would require a row for each origin, a row for each destination, and a column for each arc; thus, the simplex tableau would be much larger.

19.2   ASSIGNMENT PROBLEM: A SPECIAL-PURPOSE SOLUTION PROCEDURE

As mentioned previously, the assignment problem is a special case of the transportation problem. Thus, the transportation simplex method can be used to solve the assignment problem. However, the assignment problem has an even more special structure: All supplies and demands equal 1. Because of this additional special structure, special-purpose solution procedures have been specifically designed to solve the assignment problem; one such procedure is called the **Hungarian method.** In this section we will show how the Hungarian method can be used to solve the Fowle Marketing Research problem.

TABLE 19.20   ESTIMATED PROJECT COMPLETION TIMES (DAYS) FOR THE FOWLE
ASSIGNMENT PROBLEM

		Client	
Project Leader	**1**	**2**	**3**
Terry	10	15	9
Carle	9	18	5
McClymonds	6	14	3

Recall that the Fowle problem (see Section 6.2) involved assigning project leaders to clients; three project leaders were available and three research projects were to be completed for three clients. Fowle's assignment alternatives and estimated project completion times in days are restated in Table 19.20.

The Hungarian method involves what is called *matrix reduction.* Subtracting and adding appropriate values in the matrix yields an optimal solution to the assignment problem. Three major steps are associated with the procedure. Step 1 involves row and column reduction.

Step 1. Reduce the initial matrix by subtracting the smallest element in each row from every element in that row. Then, using the row-reduced matrix, subtract the smallest element in each column from every element in that column.

Thus, we first reduce the matrix in Table 19.20 by subtracting the minimum value in each row from each element in the row. With the minimum values of 9 for row 1, 5 for row 2, and 3 for row 3, the row-reduced matrix becomes

	1	**2**	**3**
Terry	1	6	0
Carle	4	13	0
McClymonds	3	11	0

The assignment problem represented by this reduced matrix is equivalent to the original assignment problem in the sense that the same solution will be optimal. To understand why, first note that the row 1 minimum element, 9, has been subtracted from every element in the first row. Terry must still be assigned to one of the clients, so the only change is that in this revised problem the time for any assignment will be 9 days less. Similarly, Carle and McClymonds are shown with completion times requiring 5 and 3 fewer days, respectively.

Continuing with step 1 in the matrix reduction process, we now subtract the minimum element in each column of the row-reduced matrix from every element in the column. This operation also leads to an equivalent assignment problem; that is, the same solution will still be optimal, but the times required to complete each project are reduced. With the minimum values of 1 for column 1, 6 for column 2, and 0 for column 3, the reduced matrix becomes

	1	2	3
Terry	0	0	0
Carle	3	7	0
McClymonds	2	5	0

The goal of the Hungarian method is to continue reducing the matrix until the value of one of the solutions is zero—that is, until an assignment of project leaders to clients can be made that, in terms of the reduced matrix, requires a total time expenditure of zero days. Then, as long as there are no negative elements in the matrix, the zero-valued solution will be optimal. The way in which we perform this further reduction and recognize when we have reached an optimal solution is described in the following two steps.

Step 2. Find the minimum number of straight lines that must be drawn through the rows and the columns of the current matrix so that all the zeros in the matrix will be covered. If the minimum number of straight lines is the same as the number of rows (or equivalently, columns), an optimal assignment with a value of zero can be made. If the minimum number of lines is less than the number of rows, go to step 3.

Applying step 2, we see that the minimum number of lines required to cover all the zeros is 2. Thus, we must continue to step 3.

	1	2	3	
Terry	0̶	0̶	0̶	Two straight lines will cover
Carle	3	7	0	all the zeros (step 2)
McClymonds	②	5	0	

Step 3. Subtract the value of the smallest unlined element from every unlined element, and add this same value to every element at the intersection of two lines. All other elements remain unchanged. Return to step 2, and continue until the minimum number of lines necessary to cover all the zeros in the matrix is equal to the number of rows.

The minimum unlined element is 2. In the preceding matrix we circled this element. Subtracting 2 from all unlined elements and adding 2 to the intersection element for Terry and client 3 produces the new matrix:

	1	2	3
Terry	0	0	2
Carle	1	5	0
McClymonds	0	3	0

Returning to step 2, we find that the minimum number of straight lines required to cover all the zeros in the current matrix is 3. The following matrix illustrates the step 2 calculations.

	1	2	3
Terry	0	0	2
Carle	1	5	0
McClymonds	0	3	0

Three lines must be drawn to cover all zeros; therefore, the optimal solution has been reached

According to step 2, then, it must be possible to find an assignment with a value of zero. To do so we first locate any row or column that contains only one zero. If all have more than one zero, we choose the row or column with the fewest zeros. We draw a square around a zero in the chosen row or column, indicating an assignment, and eliminate that row and column from further consideration. Row 2 has only one zero in the Fowle problem, so we assign Carle to client 3 and eliminate row 2 and column 3 from further consideration. McClymonds must then be assigned to client 1 (the only remaining zero in row 3) and, finally, Terry to client 2. The solution to the Fowle problem, in terms of the reduced matrix, requires a time expenditure of zero days, as follows:

	1	2	3
Terry	0	[0]	2
Carle	1	5	[0]
McClymonds	[0]	3	0

We obtain the value of the optimal assignment by referring to the original assignment problem and summing the solution times associated with the optimal assignment—in this case, 15 for Terry to client 2, 5 for Carle to client 3, and 6 for McClymonds to client 1. Thus, we obtain the solution time of $15 + 5 + 6 = 26$ days.

Finding the Minimum Number of Lines

Sometimes it is not obvious how the lines should be drawn through rows and columns of the matrix in order to cover all the zeros with the smallest number of lines. In these cases, the following heuristic works well. Choose any row or column with a single zero. If it is a row, draw a line through the column the zero is in; if it is a column, draw a line through the row the zero is in. Continue in this fashion until you cover all the zeros.

Can you solve an assignment problem using the Hungarian method? Try Problem 6.

If you make the mistake of drawing too many lines to cover the zeros in the reduced matrix and thus conclude incorrectly that you have reached an optimal solution, you will be unable to identify a zero-value assignment. Thus, if you think you have reached the optimal solution, but cannot find a set of zero-value assignments, go back to the preceding step and check to see whether you can cover all the zeros with fewer lines.

Problem Variations

We now discuss how to handle the following problem variations with the Hungarian method:

1. Number of agents not equal to number of tasks
2. Maximization objective function
3. Unacceptable assignments

TABLE 19.21 ESTIMATED PROJECT COMPLETION TIME (DAYS) FOR THE FOWLE
ASSIGNMENT PROBLEM WITH FOUR PROJECT LEADERS

	Client		
Project Leader	**1**	**2**	**3**
Terry	10	15	9
Carle	9	18	5
McClymonds	6	14	3
Higley	8	16	6

Number of Agents Not Equal to Number of Tasks   The Hungarian method requires that
the number of rows (agents) equal the number of columns (tasks). Suppose that in the Fowle
problem four project leaders (agents) had been available for assignment to the three new
clients (tasks). Fowle still faces the same basic problem, namely, which project leaders
should be assigned to which clients to minimize the total days required. Table 19.21 shows
the project completion time estimates with a fourth project leader.

   We know how to apply the Hungarian method when the number of rows and the num-
ber of columns are equal. We can apply the same procedure if we can add a new client. If
we do not have another client, we simply add a *dummy column*, or a dummy client. This
dummy client is nonexistent, so the project leader assigned to the dummy client in the op-
timal assignment solution, in effect, will be the unassigned project leader.

   What project completion time estimates should we show in this new dummy column?
The dummy client assignment will not actually take place, which means that a zero project
completion time for all project leaders seems logical. Table 19.22 shows the Fowle assign-
ment problem with a dummy client, labeled D. Problem 8 at the end of the chapter asks you
to use the Hungarian method to determine the optimal solution to this problem.

   Note that if we had considered the case of four new clients and only three project lead-
ers, we would have had to add a *dummy row* (dummy project leader) in order to apply the
Hungarian method. The client receiving the dummy leader would not actually be assigned
a project leader immediately and would have to wait until one becomes available. To obtain
a problem form compatible with the solution algorithm, adding several dummy rows or
dummy columns, but never both, may be necessary.

Maximization Objective   To illustrate how maximization assignment problems can be
handled, let us consider the problem facing management of Salisbury Discounts, Inc.

TABLE 19.22 ESTIMATED PROJECT COMPLETION TIME (DAYS) FOR THE FOWLE
ASSIGNMENT PROBLEM WITH A DUMMY CLIENT

	Client				Dummy client
Project Leader	**1**	**2**	**3**	**D**	
Terry	10	15	9	0	
Carle	9	18	5	0	
McClymonds	6	14	3	0	
Higley	8	16	6	0	

TABLE 19.23 ESTIMATED ANNUAL PROFIT ($1000s) FOR EACH DEPARTMENT-LOCATION COMBINATION

Department	Location			
	1	2	3	4
Shoe	10	6	12	8
Toy	15	18	5	11
Auto parts	17	10	13	16
Housewares	14	12	13	10
Video	14	16	6	12

Suppose that Salisbury Discounts has just leased a new store and is attempting to determine where various departments should be located within the store. The store manager has four locations that have not yet been assigned a department and is considering five departments that might occupy the four locations. The departments under consideration are shoes, toys, auto parts, housewares, and videos. After a careful study of the layout of the remainder of the store, the store manager has made estimates of the expected annual profit for each department in each location. These estimates are presented in Table 19.23.

This assignment problem requires a maximization objective. However, the problem also involves more rows than columns. Thus, we must first add a dummy column, corresponding to a dummy or fictitious location, in order to apply the Hungarian method. After adding a dummy column, we obtain the 5×5 Salisbury Discounts, Inc., assignment problem shown in Table 19.24.

We can obtain an equivalent minimization assignment problem by converting all the elements in the matrix to **opportunity losses.** We do so by subtracting every element in each column from the largest element in the column. Finding the assignment that minimizes opportunity loss leads to the same solution that maximizes the value of the assignment in the original problem. Thus, any maximization assignment problem can be converted to a minimization problem by converting the assignment matrix to one in which the elements represent opportunity losses. Hence, we begin the solution to this maximization assignment problem by developing an assignment matrix in which each element represents the opportunity loss for not making the "best" assignment. Table 19.25 presents the opportunity losses.

The opportunity loss from putting the shoe department in location 1 is $7000. That is, if we put the shoe department, instead of the best department (auto parts), in that location, we forgo the opportunity to make an additional $7000 in profit. The opportunity loss

TABLE 19.24 ESTIMATED ANNUAL PROFIT ($1000s) FOR EACH DEPARTMENT-LOCATION COMBINATION, INCLUDING A DUMMY LOCATION

Department	Location				Dummy location
	1	2	3	4	5
Shoe	10	6	12	8	0
Toy	15	18	5	11	0
Auto parts	17	10	13	16	0
Housewares	14	12	13	10	0
Video	14	16	6	12	0

TABLE 19.25 OPPORTUNITY LOSS ($1000s) FOR EACH DEPARTMENT-LOCATION COMBINATION

Department	1	2	3	4	5
Shoe	7	12	1	8	0
Toy	2	0	8	5	0
Auto parts	0	8	0	0	0
Housewares	3	6	0	6	0
Video	3	2	7	4	0

Location (columns 1–5). Column 5 is the Dummy location.

associated with putting the toy department in location 2 is zero because it yields the highest profit in that location. What about the opportunity losses associated with the dummy column? The assignment of a department to this dummy location means that the department will not be assigned a store location in the optimal solution. All departments earn the same amount from this dummy location, zero, making the opportunity loss for each department zero.

Try Problem 9 for practice in using the Hungarian method for a maximization problem.

Using steps 1, 2, and 3 of the Hungarian method on Table 19.25 will minimize opportunity loss and determine the maximum profit assignment.

Unacceptable Assignments   As an illustration of how we can handle unacceptable assignments, suppose that in the Salisbury Discounts, Inc., assignment problem the store manager believed that the toy department should not be considered for location 2 and that the auto parts department should not be considered for location 4. Essentially the store manager is saying that, based on other considerations, such as size of the area, adjacent departments, and so on, these two assignments are unacceptable alternatives.

Using the same approach for the assignment problem as we did for the transportation problem, we define a value of M for unacceptable minimization assignments and a value of $-M$ for unacceptable maximization assignments, where M is an arbitrarily large value. In fact, we assume M to be so large that M plus or minus any value is still extremely large. Thus, an M-valued cell in an assignment matrix retains its M value throughout the matrix reduction calculations. An M-valued cell can never be zero, so it can never be an assignment in the final solution.

Problem 10 at the end of this chapter asks you to solve this assignment problem.

The Salisbury Discounts, Inc., assignment problem with the two unacceptable assignments is shown in Table 19.26. When this assignment matrix is converted to an opportunity loss matrix, the $-M$ profit value will be changed to M.

TABLE 19.26 ESTIMATED PROFIT FOR THE SALISBURY DEPARTMENT-LOCATION COMBINATIONS

Department	1	2	3	4	5
Shoe	10	6	12	8	0
Toy	15	$-M$	5	11	0
Auto parts	17	10	13	$-M$	0
Housewares	14	12	13	10	0
Video	14	16	6	12	0

Location (columns 1–5).

GLOSSARY

Transportation problem   A network flow problem that often involves minimizing the cost of shipping goods from a set of origins to a set of destinations; it can be formulated and solved as a linear program by including a variable for each arc and a constraint for each node.

Assignment problem   A network flow problem that often involves the assignment of agents to tasks; it can be formulated as a linear program and is a special case of the transportation problem.

Transportation simplex method   A special-purpose solution procedure for the transportation problem.

Transportation tableau   A table representing a transportation problem in which each cell corresponds to a variable, or arc.

Heuristic   A commonsense procedure for quickly finding a solution to a problem. Heuristics are used to find initial feasible solutions for the transportation simplex method and in other applications.

Minimum cost method   A heuristic used to find an initial feasible solution to a transportation problem; it is easy to use and usually provides a good (but not optimal) solution.

Incoming arc   The unused arc (represented by an unoccupied cell in the transportation tableau) to which flow is assigned during an iteration of the transportation simplex method.

Outgoing arc   The arc corresponding to an occupied cell that is dropped from solution during an iteration of the transportation simplex method.

MODI method   A procedure in which a modified distribution method determines the incoming arc in the transportation simplex method.

Net evaluation index   The per-unit change in the objective function associated with assigning flow to an unused arc in the transportation simplex method.

Stepping-stone method   Using a sequence or path of occupied cells to identify flow adjustments necessary when flow is assigned to an unused arc in the transportation simplex method. This identifies the outgoing arc.

Degenerate solution   A solution to a transportation problem in which fewer than $m + n - 1$ arcs (cells) have positive flow; m is the number of origins and n is the number of destinations.

Dummy destination   A destination added to a transportation problem to make the total supply equal to the total demand. The demand assigned to the dummy destination is the difference between the total supply and the total demand.

Dummy origin   An origin added to a transportation problem in order to make the total supply equal to the total demand. The supply assigned to the dummy origin is the difference between the total demand and the total supply.

Hungarian method   A special-purpose solution procedure for solving an assignment problem.

Opportunity loss   For each cell in an assignment matrix, the difference between the largest value in the column and the value in the cell. The entries in the cells of an assignment matrix must be converted to opportunity losses to solve maximization problems using the Hungarian method.

PROBLEMS

1. Consider the following transportation tableau with four origins and four destinations.

Origin	Destination				Supply
	D_1	D_2	D_3	D_4	
O_1	5 / 25	7	10 / 50	5	75
O_2	6	5	8 / 100	2 / 75	175
O_3	6 / 100	6	12	7	100
O_4	8	5 / 100	14	4 / 50	150
Demand	125	100	150	125	

   a. Use the MODI method to determine whether this solution provides the minimum transportation cost. If it is not the minimum cost solution, find that solution. If it is the minimum cost solution, what is the total transportation cost?
   b. Does an alternative optimal solution exit? Explain. If so, find the alternative optimal solution. What is the total transportation cost associated with this solution?

2. Consider the following minimum cost transportation problem.

SELF test

Origin	Destination			Supply
	Los Angeles	San Francisco	San Diego	
San Jose	4	10	6	100
Las Vegas	8	16	6	300
Tucson	14	18	10	300
Demand	200	300	200	700

    a. Use the minimum cost method to find an initial feasible solution.

    b. Use the transportation simplex method to find an optimal solution.

    c. How would the optimal solution change if you must ship 100 units on the Tucson–San Diego route?

    d. Because of road construction, the Las Vegas–San Diego route is now unacceptable. Re-solve the initial problem.

3. Consider the following network representation of a transportation problem. The supplies, demands, and transportation costs per unit are shown on the network.

    a. Set up the transportation tableau for the problem.

    b. Use the minimum cost method to find an initial feasible solution.

4. A product is produced at three plants and shipped to three warehouses. The transportation costs per unit are shown in the following table.

Plant	Warehouse			Plant Capacity
	W_1	W_2	W_3	
P_1	20	16	24	300
P_2	10	10	8	500
P_3	12	18	10	100
Warehouse demand	200	400	300	

Use the transportation simplex method to find an optimal solution.

5.   Consider the following minimum cost transportation problem.

Origin	Destination			Supply
	D$_1$	D$_2$	D$_3$	
O$_1$	6	8	8	250
O$_2$	18	12	14	150
O$_3$	8	12	10	100
Demand	150	200	150	

    a.   Use the minimum cost method to find an initial feasible solution.
    b.   Use the transportation simplex method to find an optimal solution.
    c.   Using your solution to part (b), identify an alternative optimal solution.

SELF test

6.   Scott and Associates, Inc., is an accounting firm that has three new clients. Project leaders will be assigned to the three clients. Based on the different backgrounds and experiences of the leaders, the various leader-client assignments differ in terms of projected completion times. The possible assignments and the estimated completion times in days are

Project Leader	Client		
	1	2	3
Jackson	10	16	32
Ellis	14	22	40
Smith	22	24	34

Use the Hungarian method to obtain the optimal solution.

7.   CarpetPlus sells and installs floor covering for commercial buildings. Brad Sweeney, a CarpetPlus account executive, was just awarded the contract for five jobs. Brad must now assign a CarpetPlus installation crew to each of the five jobs. Because the commission Brad will earn depends on the profit CarpetPlus makes, Brad would like to determine an assignment that will minimize total installation costs. Currently, five installation crews are available for assignment. Each crew is identified by a color code, which aids in tracking of job progress on a large white board. The following table shows the costs (in hundreds of dollars) for each crew to complete each of the five jobs.

				Job		
		1	2	3	4	5
	Red	30	44	38	47	31
	White	25	32	45	44	25
Crew	Blue	23	40	37	39	29
	Green	26	38	37	45	28
	Brown	26	34	44	43	28

Use the Hungarian method to obtain the optimal solution.

8. Fowle Marketing Research has four project leaders available for assignment to three clients. Find the assignment of project leaders to clients that will minimize the total time to complete all projects. The estimated project completion times in days are as follows:

Project Leader		Client	
	1	2	3
Terry	10	15	9
Carle	9	18	5
McClymonds	6	14	3
Higley	8	16	6

Use the Hungarian method to obtain the optimal solution.

SELF test

9. Use the Hungarian method to solve the Salisbury Discount, Inc., problem using the profit data in Table 19.23.

10. Use the Hungarian method to solve the Salisbury Discount, Inc., problem using the profit data in Table 19.26.

Chapter 19

2. **a.** An initial solution is

	Los Angeles	San Francisco	San Diego
San Jose	4 / 100	10	6
Las Vegas	8 / 100	16	6 / 200
Tucson	14	18 / 300	10

Total cost = $7800

b. Note that the initial solution is degenerate because only 4 cells are occupied; a zero is assigned to the cell in row 3 and column 1 so that the row and column indexes can be computed

u_i	v_j 4	8	2
0	4 / 100	10 / ②	6 / ④
4	8 / 100	16 / ④	6 / 200
10	14 / 0	18 / 300	10 / ⊝2

Cell in row 3 and column 3 is identified as an incoming cell; however, 0 units can be added to this cell. Initial solution remains optimal

c.

San Jose–San Francisco:	100
Las Vegas–Los Angeles:	200
Las Vegas–San Diego:	100
Tucson–San Francisco:	200
Tucson–San Diego	100
Total Cost = $7800	

Note that this total cost is the same as for part (a); thus, we have alternative optimal solutions

d. The final transportation tableau is shown; the total transportation cost is $8000, an increase of $200 over the solution to part (a)

u_i	v_j 2	10	2	
0	4 / ②	10 / 100	6 / ④	100
6	8 / 200	16 / 100	M / ⓜ−8	300
8	14 / ④	18 / 100	10 / 200	300
	200	300	200	700

4. **b.** $x_{12} = 300$, $x_{21} = 100$, $x_{22} = 100$, $x_{23} = 300$, $x_{31} = 100$
Cost = 10,400

6. Subtract 10 from row 1, 14 from row 2, and 22 from row 3 to obtain:

	1	2	3
Jackson	0	6	22
Ellis	0	8	26
Smith	0	2	12

Subtract 0 from column 1, 2 from column 2, and 12 from column 3 to obtain:

	1	2	3
Jackson	0	④	10
Ellis	0	6	14
Smith	0	0	0

Two lines cover the zeros; the minimum unlined element is 4; step 3 yields:

	1	2	3
Jackson	0	0	6
Ellis	0	2	10
Smith	0	0	0

Optimal solution: Jackson–2
                Ellis–1
                Smith–3
Time requirement is 64 days

8. Terry 2; Carle 3; MacClymonds 1; Higley unassigned
Time = 26 days

9. We start with the opportunity loss matrix:

	1	2	3	4	D*
Shoe	4	11	0	5	0
Toy	0	0	8	3	1
Auto	0	10	2	0	3
Houseware	1	6	0	4	1
Video	0	1	6	1	0

*D = Dummy

	Optimal Solution	Profit
Toy	2	18
Auto	4	16
Housewares	3	13
Video	1	14
	Total	61

10. Toy: 2; Auto: 4; Housewares: 3; Video: 1

7	12	1	8	0		5	12	①	6	0
②	0	8	5	0		0	0	8	3	0
0	8	0	0	0	→	0	10	2	0	2
3	6	0	6	0		1	6	0	4	0
3	2	7	4	0		1	2	7	2	0