# CHAPTER 20

## Minimal Spanning Tree

In network terminology, the minimal spanning tree problem involves using the arcs of the network to reach *all* nodes of the network in such a fashion that the total length of all the arcs used is minimized. To better understand this problem, let us consider the communications system design problem encountered by a regional computer center.

The Southwestern Regional Computer Center must have special computer communications lines installed to connect five satellite users with a new central computer. The telephone company will install the new communications network. However, the installation is an expensive operation. To reduce costs, the center's management group wants the total length of the new communications lines to be as short as possible. Although the central computer could be connected directly to each user, it appears to be more economical to install a direct line to some users and let other users tap into the system by linking them with users already connected to the system. The determination of this minimal length communications system design is an example of the **minimal spanning tree** problem. The network for this problem with possible connection alternatives and distances is shown in Figure 20.1. An algorithm that can be used to solve this network model is explained in the following subsection.

## A Minimal Spanning Tree Algorithm

*For a network consisting of N nodes, a spanning tree will consist of N − 1 arcs.*

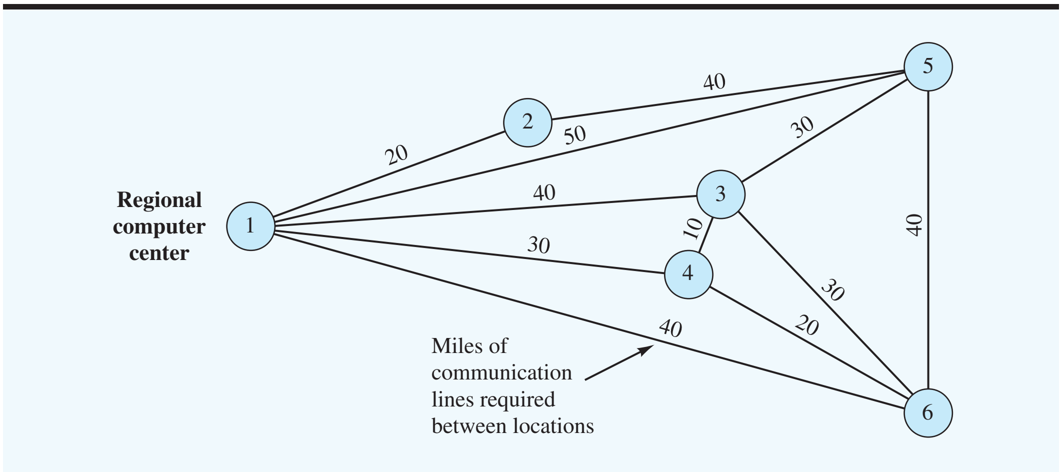
A **spanning tree** for an $N$-node network is a set of $N - 1$ arcs that connects every node to every other node. A minimal spanning tree provides this set of arcs at minimal total arc cost, distance, or some other measure. The network algorithm that can be used to solve the minimal spanning tree problem is simple. The steps of the algorithm are as follows:

Step 1. Arbitrarily begin at any node and connect it to the closest node in terms of the criterion being used (e.g., time, cost, or distance). The two nodes are referred to as *connected* nodes, and the remaining nodes are referred to as *unconnected* nodes.

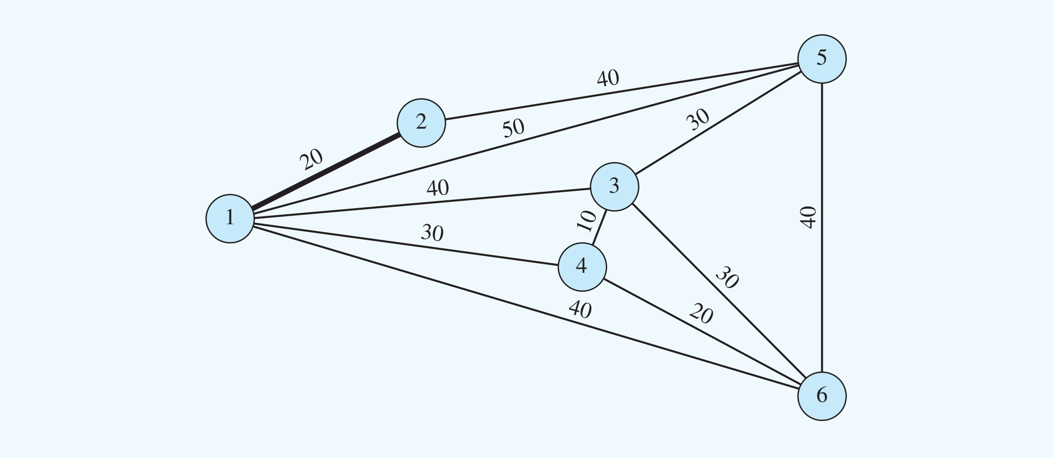
Step 2. Identify the unconnected node that is closest to one of the connected nodes. Break ties arbitrarily if two or more nodes qualify as the closest node. Add this new node to the set of connected nodes. Repeat this step until all nodes have been connected.

This network algorithm is easily implemented by making the connection decisions directly on the network.

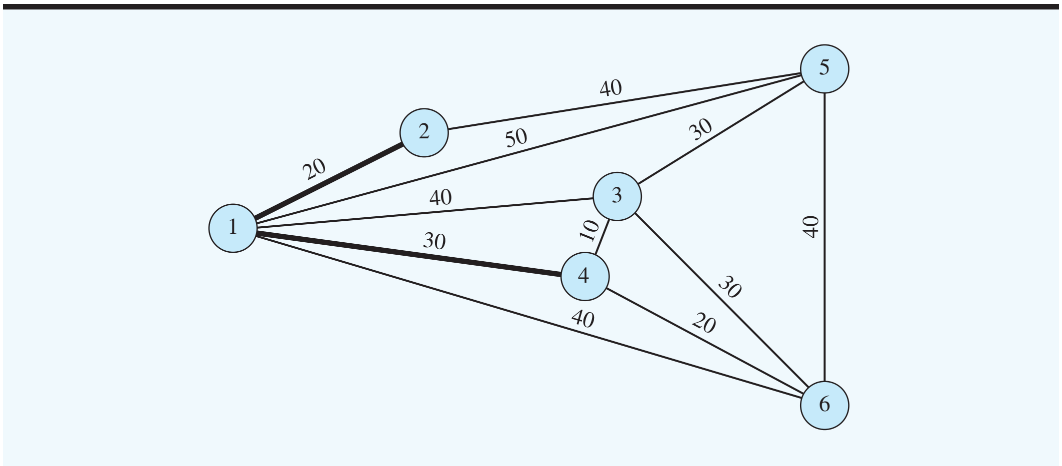
FIGURE 20.1 COMMUNICATIONS NETWORK FOR THE REGIONAL COMPUTER SYSTEM



Referring to the communications network for the regional computer center and arbitrarily beginning at node 1, we find the closest node is node 2 with a distance of 20. Using a bold line to connect nodes 1 and 2, step 1 of the algorithm provides the following result:




In step 2 of the algorithm, we find that the unconnected node closest to one of the connected nodes is node 4, with a distance of 30 miles from node 1. Adding node 4 to the set of connected nodes provides the following result:




Repeating the step of always adding the closest unconnected node to the connected segment of the network provides the minimal spanning tree solution shown in Figure 20.2. Follow the steps of the algorithm, and see whether you obtain this solution. The minimal length of the spanning tree is given by the sum of the distances on the arcs forming the spanning tree. In this case, the total distance is 110 miles for the computer center's communications network. Note that while the computer center's network arcs were measured in distance, other network models may measure the arcs in terms of other criteria such as cost, time, and so on. In such cases, the minimal spanning tree algorithm will identify the optimal solution (minimal cost, minimal time, etc.) for the criterion being considered.

*Can you now find a minimal spanning tree for a network? Try Problem 2.*

The computer solution to the regional computer center's problem is shown in Figure 20.3. The Management Scientist was used to obtain the minimal spanning tree solution of 110 miles.

**FIGURE 20.2** MINIMAL SPANNING TREE COMMUNICATIONS NETWORK FOR THE REGIONAL COMPUTER CENTER

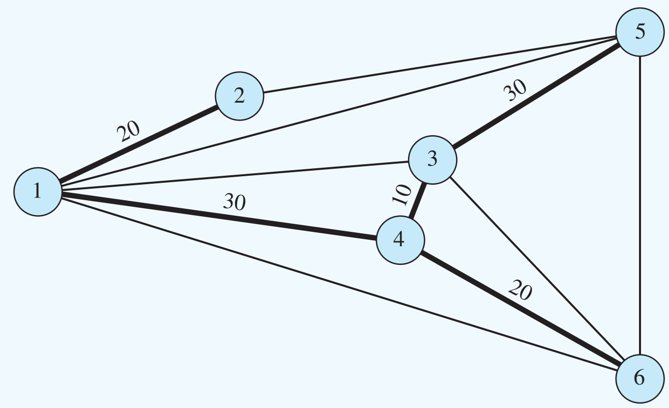



**FIGURE 20.3** THE MANAGEMENT SCIENTIST SOLUTION FOR THE REGIONAL COMPUTER CENTER MINIMAL SPANNING TREE PROBLEM

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****  NETWORK DESCRIPTION  ****

6 NODES AND 11 ARCS
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| ARC | START NODE | END NODE | DISTANCE |
|---|---|---|---|
| 1 | 1 | 2 | 20 |
| 2 | 1 | 3 | 40 |
| 3 | 1 | 4 | 30 |
| 4 | 1 | 5 | 50 |
| 5 | 1 | 6 | 40 |
| 6 | 2 | 5 | 40 |
| 7 | 3 | 4 | 10 |
| 8 | 3 | 5 | 30 |
| 9 | 3 | 6 | 30 |
| 10 | 4 | 6 | 20 |
| 11 | 5 | 6 | 40 |

```
MINIMAL SPANNING TREE
*********************
```

| START NODE | END NODE | DISTANCE |
|---|---|---|
| 1 | 2 | 20 |
| 1 | 4 | 30 |
| 4 | 3 | 10 |
| 4 | 6 | 20 |
| 3 | 5 | 30 |

TOTAL LENGTH 110

## NOTES AND COMMENTS

1. The Management Science in Action, EDS Designs a Communication Network, describes an interesting application of the minimal spanning tree algorithm.
2. The minimal spanning tree algorithm is considered a *greedy algorithm* because at each stage we can be "greedy" and take the best action available at that stage. Following this strategy at each successive stage will provide the overall optimal solution. Cases in which a greedy algorithm provides the optimal solution are rare. For many problems, however, greedy algorithms are excellent heuristics.

## MANAGEMENT SCIENCE IN ACTION

### EDS DESIGNS A COMMUNICATION NETWORK*

EDS, headquartered in Plano, Texas, is a global leader in information technology services. The company provides hardware, software, communications, and process solutions to many companies and governments around the world.

EDS designs communication systems and information networks for many of its customers. In one application, an EDS customer wanted to link together 64 locations for information flow and communications. Interactive transmission involving voice, video, and digital data had to be accommodated in the information flow between the various sites. The customer's locations included approximately 50 offices and information centers in the continental United States; they ranged from Connecticut to Florida to Michigan to Texas to California. Additional locations existed in Canada, Mexico, Hawaii, and Puerto Rico. A total of 64 locations formed the nodes of the information network.

EDS's task was to span the network by finding the most cost-effective way to link the 64 customer locations with each other and with existing EDS data centers. The arcs of the network represented communication links between pairs of nodes in the network. In cases where land communication lines were available, the arcs consisted of fiber-optic telephone lines. In other cases, the arcs represented satellite communication connections.

Using cost as the criterion, EDS developed the information network for the customer by solving a minimal spanning tree problem. The minimum cost network design made it possible for all customer locations to communicate with each other and with the existing EDS data centers.

*The authors are indebted to Greg A. Dennis of EDS for providing this application.

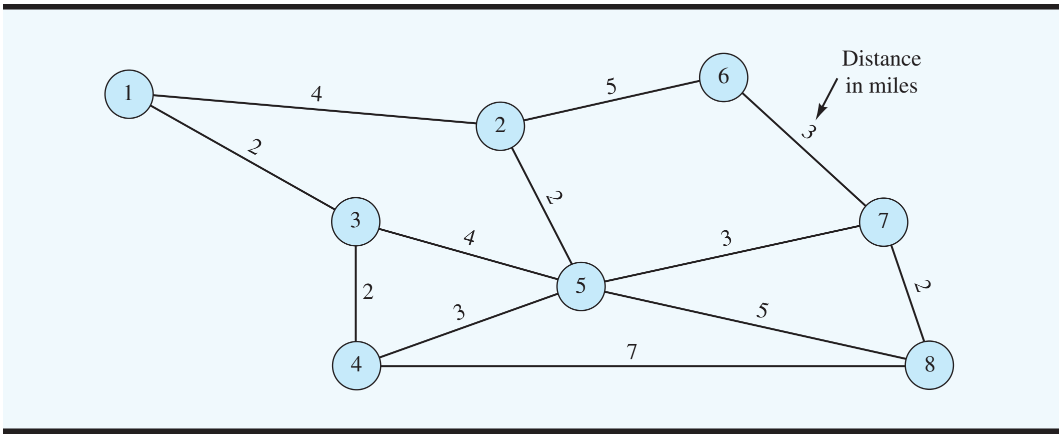
## GLOSSARY

**Minimal spanning tree** The spanning tree with the minimum length.

**Spanning tree** $N-1$ arcs that connect every node in the network with all other nodes where $N$ is the number of nodes.

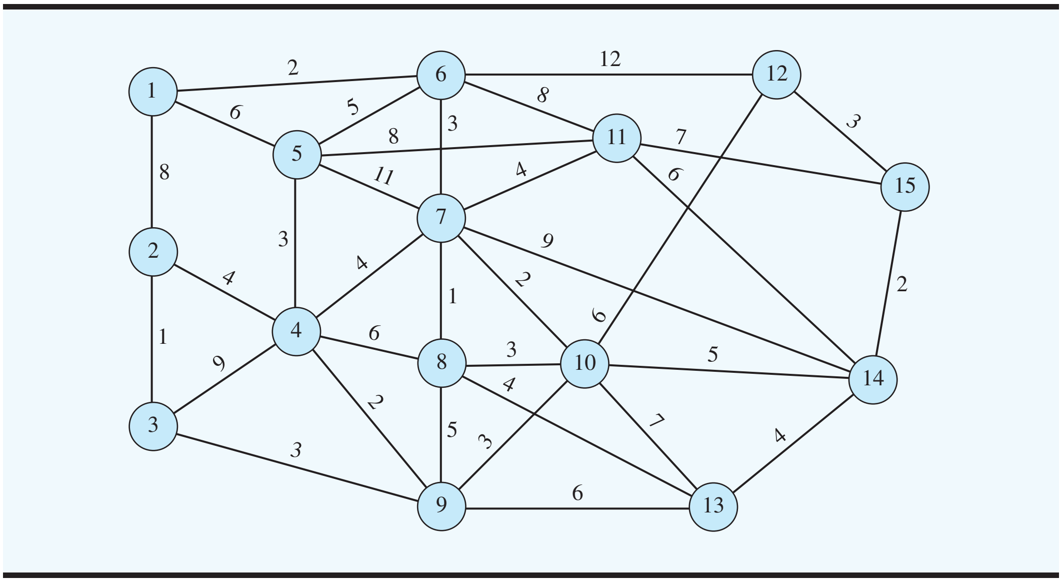
## PROBLEMS

1. Develop the minimal spanning tree solution for the following emergency communication network.

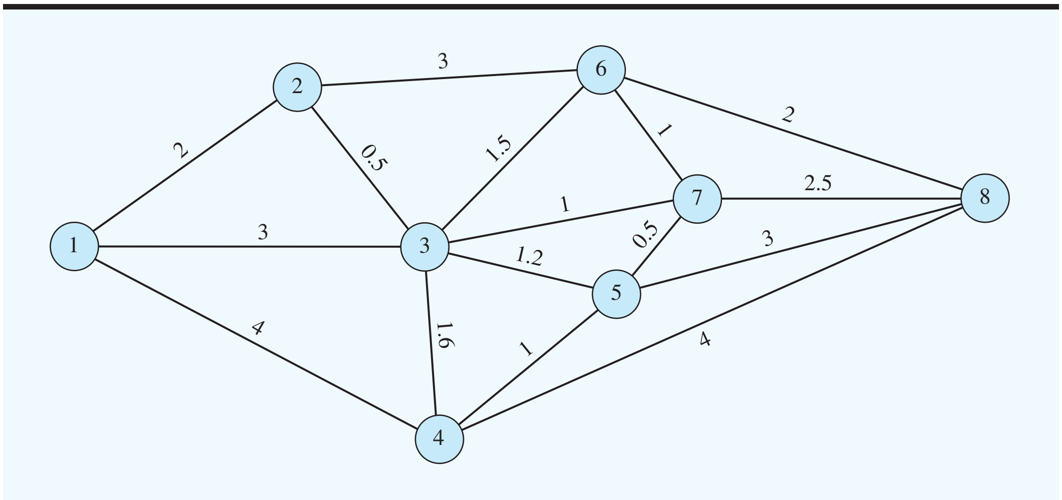



SELF test

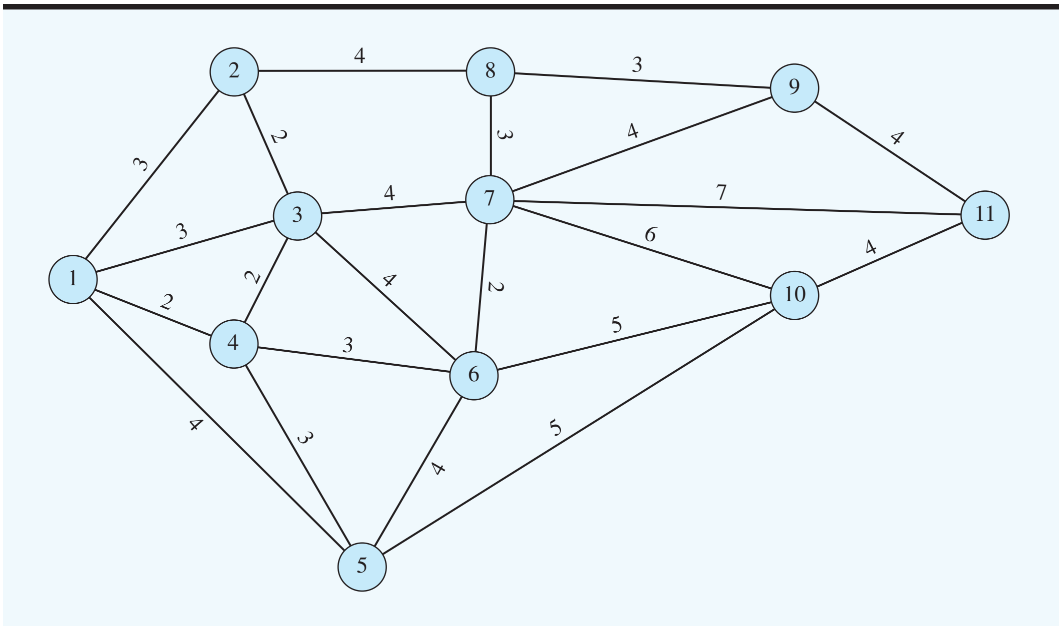
2. The State of Ohio recently purchased land for a new state park, and park planners identified the ideal locations for the lodge, cabins, picnic groves, boat dock, and scenic points of interest. These locations are represented by the nodes of the following network. The arcs of the network represent possible road connections in the park. If the state park designers want to minimize the total road miles that must be constructed in the park and still permit access to all facilities (nodes), which road connections should be constructed?



3. Midwest University is installing an electronic mail system. The following network shows the possible electronic connections among the offices. Distances between offices are shown in thousands of feet. Develop a design for the office communication system that will enable all offices to have access to the electronic mail service. Provide the design that minimizes the total length of connection among the eight offices.




4. The Metrovision Cable Company just received approval to begin providing cable television service to a suburb of Memphis, Tennessee. The nodes of the following network show the distribution points that must be reached by the company's primary cable lines. The arcs of the network show the number of miles between the distribution points. Determine the solution that will enable the company to reach all distribution points with the minimum length of primary cable line.



# Self-Test Solutions and Answers to Even-Numbered Problems

## Chapter 20

**2.**

| Connect | Distance |
|---|---|
| 1–6 | 2 |
| 6–7 | 3 |
| 7–8 | 1 |
| 7–10 | 2 |
| 10–9 | 3 |
| 9–4 | 2 |
| 9–3 | 3 |
| 3–2 | 1 |
| 4–5 | 3 |
| 7–11 | 4 |
| 8–13 | 4 |
| 14–15 | 2 |
| 15–12 | 3 |
| 14–13 | 4 |
| Total | 37 |

**4.** 1–4, 2–3, 3–4, 4–5, 4–6, 6–7, 7–8, 8–9, 9–11, 11–10
Minimum length = 28 miles