Relations

Methods: Logic, Part 3a

Michael Franke

Topics covered

(i) tuples
(ii) Cartesian products
(iii) relations
(iv) properties of relations

Tuples

Order-sensitive collections

Sets

- order of elements is irrelevant
  $\{a, b\} = \{b, a\}$
- elements cannot reoccur
  $\{a, a\} = \{a\}$

Tuples

- order of elements is relevant
  $\langle a, b\rangle \neq \langle b, a\rangle$
- elements can reoccur
  $\langle a, a\rangle \neq \langle a\rangle$

Tuples

Terminology & notation

An *n-tuple* is a tuple with n elements (in order).
For $n = 1$, we conventionally define: $\langle x \rangle = x$.
For small $n \geq 1$, there are special words:

$n = 2$	ordered pair
$n = 3$	triple
$n = 4$	quadruple
$n = 5$	quintuple
	...

Cartesian products

The Cartesian product of two sets X and Y is a set of pairs:

$$X \times Y = \{\langle x, y\rangle \mid x \in X \text{ and } y \in Y\}$$

Cartesian products

The Cartesian product of two sets X and Y is a set of pairs:

$$X \times Y = \{\langle x, y \rangle \mid x \in X \text{ and } y \in Y\}$$

The Cartesian product of n sets is a set of n-tuples:

$$X_1 \times X_2, \ldots, \times X_n = \{\langle x_1, x_2, \ldots, x_n \rangle \mid x_i \in X_i \text{ for all } 1 \leq i \leq n\}$$

Cartesian products

The Cartesian product of two sets X and Y is a set of pairs:

$$X \times Y = \{\langle x, y\rangle \mid x \in X \text{ and } y \in Y\}$$

The Cartesian product of n sets is a set of n-tuples:

$$X_1 \times X_2, \ldots, \times X_n = \{\langle x_1, x_2, \ldots, x_n\rangle \mid x_i \in X_i \text{ for all } 1 \leq i \leq n\}$$

Examples

$$\begin{aligned}
X &= \{a, b\} \\
Y &= \{c, d\} \\
X \times Y &= \{\langle a, c\rangle, \langle a, d\rangle, \langle b, c\rangle, \langle b, d\rangle\} \\
Y \times X &= \{\langle c, a\rangle, \langle d, a\rangle, \langle c, b\rangle, \langle d, b\rangle\} \\
X \times X &= \{\langle a, a\rangle, \langle a, b\rangle, \langle b, a\rangle, \langle b, b\rangle\}
\end{aligned}$$

Relations

Definition

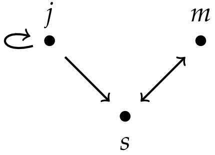
Any subset of an n-place Cartesian product is called an n-ary relation.

Example

$P = \{j, m, s\}$ is a set of people

$L \subseteq P \times P$ is binary relation encoding who loves whom:

$$\begin{aligned} L &= \{\langle x, y\rangle \in P \times P \mid x \text{ loves } y\} \\ &= \{\langle j, j\rangle, \langle j, s\rangle, \langle m, s\rangle, \langle s, m\rangle\} \end{aligned}$$



Relations

Terminology & notation

- $R \subseteq X \times Y$ is also called a *binary* relation
- if $\langle x, y \rangle \in R \subseteq X \times Y$, we can also use
  - prefix notation: Rxy [used in this course; except for math stuff like $1 \leq 2$]
  - infix notation: xRy
  - postfix notation: xyR
- domain and range of binary relation $R \subseteq X \times Y$:

$$\begin{aligned} \mathrm{dom}(R) &= \{x \in X \mid \text{there is some } y \in Y \text{ with } Rxy\} \\ \mathrm{range}(R) &= \{y \in Y \mid \text{there is some } x \in X \text{ with } Rxy\} \end{aligned}$$

Properties of binary relations

Binary relation $R \subseteq X \times X$ is

reflexive iff Rxx for all $x \in X$

irreflexive iff Rxx for no $x \in X$

symmetric iff for all $x, y \in X$ if Rxy then also Ryx

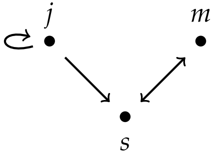
asymmetric iff for no $x, y \in X$ both Rxy and Ryx

anti-symmetric iff for all $x, y \in X$ if Rxy and Ryx, then $x = y$

transitive iff for all $x, y \in X$ if Rxy and Ryz, then also Rxz

intransitive iff for all $x, y \in X$ if Rxy and Ryz, then not Rxz

connected iff for all $x, y \in X$ either Rxy or Ryx or $x = y$



Orders and equivalence relations

Binary relation $R \subseteq X \times X$ is said to be:

- a *partial weak order* iff R is reflexive, anti-symmetric and transitive

  [example: relation "$\subseteq$" on $\mathcal{P}(Y)$]
- a *partial strict order* iff R is irreflexive, asymmetric and transitive

  [example: relation "$\subset$" on $\mathcal{P}(Y)$]
- a *linear weak order* iff R is a partial weak order and connected

  [example: relation "$\leq$" on $\mathbb{N}$]
- a *linear strict order* iff R is a partial strict order and connected

  [example: relation "$<$" on $\mathbb{N}$]
- an *equivalence relation* iff R is reflexive, symmetric and transitive

  [example: relation "has equal cardinality" on $\mathcal{P}(Y)$]