Relations

Methods: Logic, Part 3a

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Topics covered

(i) tuples
(ii) Cartesian products
(iii) relations
(iv) properties of relations

Tuples

Order-sensitive collections

Sets

- order of elements is irrelevant
  $\{a, b\} = \{b, a\}$
- elements cannot reoccur
  $\{a, a\} = \{a\}$

Tuples

- order of elements is relevant
  $\langle a, b\rangle \neq \langle b, a\rangle$
- elements can reoccur
  $\langle a, a\rangle \neq \langle a\rangle$

Tuples

Terminology & notation

An *n-tuple* is a tuple with n elements (in order).
For $n = 1$, we conventionally define: $\langle x \rangle = x$.
For small $n \geq 1$, there are special words:

$n = 2$ ordered pair

$n = 3$ triple

$n = 4$ quadruple

$n = 5$ quintuple

$\ldots$

Cartesian products

The Cartesian product of two sets X and Y is a set of pairs:

$$X \times Y = \{\langle x, y \rangle \mid x \in X \text{ and } y \in Y\}$$

The Cartesian product of n sets is a set of n-tuples:

$$X_1 \times X_2, \ldots, \times X_n = \{\langle x_a, x_2, \ldots, x_n \rangle \mid x_i \in X_i \text{ for all } 1 \leq i \leq n\}$$

Examples

$$X = \{a, b\}$$

$$Y = \{c, d\}$$

$$X \times X = \{\langle a, a \rangle, \langle a, b \rangle, \langle b, a \rangle, \langle b, b \rangle\}$$

$$X \times Y = \{\langle a, c \rangle, \langle a, d \rangle, \langle b, c \rangle, \langle b, d \rangle\}$$

$$Y \times X = \{\langle c, a \rangle, \langle d, a \rangle, \langle c, b \rangle, \langle d, b \rangle\}$$

Cartesian products

Terminology & Notation

- the n-place Cartesian product with the same set X can also be written as X^n:

$$\underbrace{X \times X \times \cdots \times X}_{n \text{ times}} = X^n$$

Relations

Definition

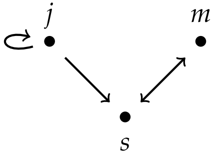
Any subset of an n-place Cartesian product is called an n-ary relation.

Example

$P = \{j, m, s\}$ is a set of people

$L \subseteq P \times P$ is binary relation encoding who loves whom:

$$\begin{aligned} L &= \{\langle x, y\rangle \in P \times P \mid x \text{ loves } y\} \\ &= \{\langle j, j\rangle, \langle j, s\rangle, \langle m, s\rangle, \langle s, m\rangle\} \end{aligned}$$



Relations

Terminology & notation

- if $\langle x, y \rangle \in R$, we can also use
  - prefix notation: Rxy [used in this course; except for math stuff like $1 \leq 2$]
  - infix notation: Rxy
  - postfix notation: xyR
- domain and range of binary relation $R \subseteq X \times X$:

$$\mathrm{dom}(R) = \{x \in X \mid \text{there is some } y \in Y \text{ with } Rxy\}$$

$$\mathrm{range}(R) = \{y \in Y \mid \text{there is some } x \in X \text{ with } Rxy\}$$

Properties of binary relations

Binary relation $R \subseteq X \times X$ is

reflexive iff Rxx for all $x \in X$

irreflexive iff Rxx for no $x \in X$

symmetric iff for all $x, y \in X$ if Rxy then also Ryx

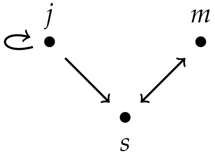
asymmetric iff for no $x, y \in X$ both Rxy and Ryx

anti-symmetric iff for all $x, y \in X$ if Rxy and Ryx, then $x = y$

transitive iff for all $x, y \in X$ if Rxy and Ryz, then also Rxz

intransitive iff for all $x, y \in X$ if Rxy and Ryz, then not Rxz

connected iff for all $x, y \in X$ either Rxy or Ryx or $x = y$



Orders and equivalence relations

Binary relation $R \subseteq X \times X$ is said to be:

- a *partial weak order* iff R is reflexive, anti-symmetric and transitive

  [example: relation "$\subseteq$" on $\mathcal{P}(Y)$]
- a *partial strict order* iff R is irreflexive, asymmetric and transitive

  [example: relation "$\subset$" on $\mathcal{P}(Y)$]
- a *linear weak order* iff R is a partial weak order and connected

  [example: relation "$\leq$" on $\mathbb{N}$]
- a *linear strict order* iff R is a partial strict order and connected

  [example: relation "$<$" on $\mathbb{N}$]
- an *equivalence relation* iff R is reflexive, symmetric and transitive

  [example: relation "has equal cardinality" on $\mathcal{P}(Y)$]