Propositional logic

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Syntax & semantics of propositional logic; truth-tables; tautologies vs. contraditions vs. contingencies; translations from natural language into propositional logic; argument schemas & logical validity.

1 The language of propositional logic

Propositional logic (PropLog) studies how propositions are combined by logical operators, which closely correspond to certain sentential connectives in natural language (such as *and*, *or*, *if*, or *not*). A *proposition* in the sense of PropLog is a minimal unit of thought which can be evaluated as true or false independently of other propositions.[1] For example, the logical structure of the following sentence:

$$\underbrace{\text{The earth is round}}_{p} \; \underbrace{\text{and}}_{\wedge} \; \underbrace{\text{the moon is made of cheese.}}_{q}$$

could be analyzed as composed of two propositions, viz., the proposition denoted here with *proposition letter* p that the earth is round and the proposition denoted by q that the moon is made of cheese. These two propositions are connected by a logical operator "and", for which we write $\wedge$ in PropLog. The logical structure of the complex sentence above can therefore be written as $p \wedge q$ in PropLog.

[1] The notion of a proposition in this sense is not unproblematic. For example, a case like "*This pixel is red.*" seems like a minimal unit of truth-evaluable information about the color of a particular pixel, but it is not independent of another statement like "*This pixel is blue.*". (Historically, this problem related to color was a problem brought up famously by Frank Ramsey in response to the early logical work of Ludwig Wittgenstein.)

1.1 Proposition letters & sentential connectives

The language of PropLog is formed by:

(i) a set of *proposition letters* $\mathfrak{P} = \{p, q, r, s, p_1, q_{27}, \dots\}$, and

(ii) a set of *sentential connectives* $\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$[2]

The sentential connectives have names and are intended to correspond (approximately) to natural language paraphrases:[3]

name	paraphrase	symbol
negation	"not"	$\neg$
conjunction	"and"	$\wedge$
disjunction	"or"	$\vee$
implication	"if …, then …"	$\rightarrow$
equivalence	"if and only of"	$\leftrightarrow$

[2] You will also find terminology like *logical connectives* or *logical operators*. There may be additional connectives used by some logicians or textbooks, and you might find slightly different symbols for the same notions in some places

[3] The names and the (approximate) correspondence are earned by the *semantics* which we will give to these symbols later on.

1.2 Formulas

The language $\mathfrak{L}$ of PropLog is the set of all *formulas* which are recursively defined as follows:[4]

[4] We will make consistent use of Greek letters $\varphi, \psi, \chi, \dots$ as variables for formulas.

(i) Every proposition letter is a formula.

(ii) If φ is a formula, so is $\neg\varphi$.

(iii) If φ and ψ are formulas, so are:
a. $(\varphi \wedge \psi)$ b. $(\varphi \vee \psi)$ c. $(\varphi \rightarrow \psi)$ d. $(\varphi \leftrightarrow \psi)$

(iv) Anything that cannot be constructed by (i)–(iii) is not a formula.

Examples for formulas of PropLog are:[5]

[5] We conventionally omit the outermost parentheses of a formula.

$$pp \wedge p$$
$$p \rightarrow \neg q(p \vee q) \leftrightarrow r$$

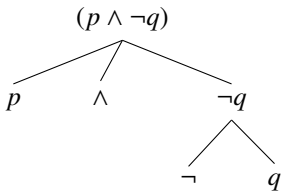
Examples of strings made of proposition letters and logical connectives which are *not* formulas of PropLog are:

$$(p) \qquad \neg pq$$
$$p\neg \rightarrow \neg q \qquad p \vee q \leftrightarrow r$$

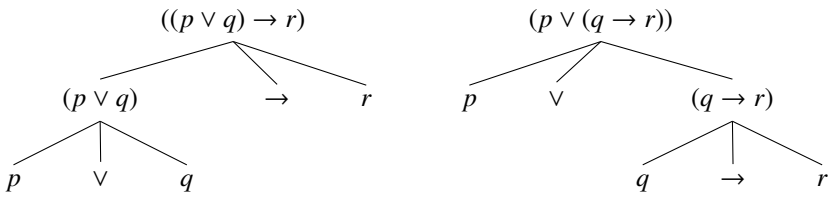
1.3 Syntactic trees

The recursive definition for formulas of PropLog gives an internal structure to each formula. Take the example $p \wedge \neg q$. There is only one way in which this formulas could have been generated by a constructive process that follows the recursive definition above. In the last step of that process, the two subformulas $\varphi = p$ and $\psi = \neg q$ have been combined to form an expression of the form $\varphi \wedge \psi$ using the rule (iii) part a. The first subformula $\varphi = p$ is constructed by rule (i). The second formula can only be constructed by first using rule (i) to introduce q and then using rule (ii) to introduce the negation sign.

A *syntactic tree* is a useful visual illustration of the internal structure of a formula. The syntactic tree of formula $p \wedge \neg q$ is this:



The construction of a complex formula, and therefore its syntactic tree, is always recoverable by following the introduction of the parentheses in step (iii). This is illustrated by the minimal pair in Figure 1.



(a) Tree for $((p \vee q) \to r)$

(b) Tree for $(p \vee (q \to r))$

Figure 1: Examples of syntactic trees

1.4 Terminology

A formula of PropLog which consist of a single proposition letter is also called *atomic formula.* Any formula of PropLog which is not atomic is also called a *complex formula*. Each complex formula has a *main connective*. The main connective is the last sentential connector introduced during the construction of the formula. A complex formula is also often called by the name of its main connective. For example, the formula $p \wedge \neg q$ has a conjunction as its main operator and could therefore be called a conjunction. The formula $(p \vee q) \to r$ from Figure 1(a) is an implication, while the formula $p \vee (q \to r)$ from Figure 1(b) is a disjunction.

For some two-place connectives, the subformulas can have special names as well. In the conjunction $\varphi \wedge \psi$, the subformulas φ and ψ are called *conjuncts*. In the disjunction $\varphi \vee \psi$, the subformulas φ and ψ are called *disjuncts*. In the implication $\varphi \to \psi$, the subformula φ is called *antecedent* and the subformula ψ is called *consequent*.

Exercise 1. Determine which of the following strings are formulas of propositional logic. For any formula, determine its main operator.

a. q_{12}

b. $p, q \wedge r$

c. $(p) \wedge q$

d. $p \to (p \wedge p)$

e. $(p \to)(p \wedge p)$

f. $(p \vee \neg q) \leftrightarrow (r \to (\neg(p \vee \neg p)))$

Exercise 2. Draw the syntactic tree for each of the following formulas.

a. $p \leftrightarrow q$

b. $\neg p \wedge p$

c. $p \to \neg(q \wedge r)$

d. $(\neg p \vee \neg q) \wedge (r \to p)$

2 *The semantics of propositional logic*

The language of PropLog defines rules of systematically combining proposition letters with sentential connectives. The *semantics* PropLog gives a meaning to each formula. Yet, PropLog is not concerned with the meaning of the proposition letters. Whether the proposition p that the earth is round or the proposition q that the moon is made of cheese is true or false in this world we live in, is of no concern to the logician. The (propositional) logician cares only about the meaning of the sentential connectives. More specifically, PropLog pays attention to a specific kind of meaning, namely *truth-conditional meaning*. Concretely, PropLog analyzes the meaning of a sentential connective $\odot \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$ in terms of how the truth or falsity of the complex sentence $\varphi \odot \psi$ depends on the truth or falsity of its components φ and ψ; and how the meaning of $\neg\varphi$ depends on the truth or falsity of φ.

2.1 *Truth tables*

One way of defining the semantics of the sentential connectives is by *truth tables*. Truth tables are also handy for systematically calculating the conditions under which complex formulas are true.

Take the complex formula $\neg\varphi$. We use φ here as a variable which could represent any formula, no matter if it is a proposition letter or extremely long and complex. For all we know, φ could be either true or false. There are just these two possibilities.[6] To give a truth-conditional meaning to the operator $\neg$ we need to define whether $\neg\varphi$ is true or false for each case: when φ is true, and when φ is false. The truth table that defines the meaning of negation is given in Table 1(a):[7] PropLog treats negation like a switch: take a formula φ as input, look at its truth value, and swap it for the other (since there are only two truth values).

Other sentential connectives than negation take two formulas as input, so to speak, and may therefore be conceived of as functions that take a pair of truth values as input. The semantics of the binary connectives is given in Table 1(b):

φ	$\neg\varphi$
1	0
0	1

(a) Semantics of negation

ϕ	ψ	$\phi \wedge \psi$	$\phi \vee \psi$	$\phi \rightarrow \psi$	$\phi \leftrightarrow \psi$
1	1	1	1	1	1
1	0	0	1	0	0
0	1	0	1	1	0
0	0	0	0	1	1

(b) Semantics of binary connectives

Table 1: Semantics of sentential connectives in propositional logic expressed in terms of truth tables

[6] That PropLog only considers two truth values is also referred to as the *principle of bivalence*. There are other logics, so-called many-valued logics, which allow for more than two truth values, e.g., to represent degrees of truth or uncertainty.

[7] We here write 1 for the truth value "true" and 0 for "false." It is also common to use letters T and F, or yet other symbols in truth tables.

2.2 *Working with truth tables*

Truth tables are useful for systematically working out the conditions under which complex formulas are true. Consider the complex formula $(\varphi \wedge \psi) \rightarrow \neg\chi$. We do not know whether the subformula φ, ψ and χ are atomic or themselves complex. But we want to know under which circumstances, i.e., truth value assignments to φ, ψ and χ, the formula $(\varphi \wedge \psi) \rightarrow \neg\chi$ is true or false. In order to find out, we can construct a truth table that starts by enumerating all logical possibilities of truth-value assignments for the formulas φ, ψ and χ (the first three columns in the table below). We then work towards the "goal formula" $(\varphi \wedge \psi) \rightarrow \neg\chi$ by following the recursive definition of formulas of PropLog. At each step we apply the definition of the semantics of the relevant sentential connective until we arrive at the "goal formula". Following this method, we construct the truth table in Table 2, which shows that the formula $(\varphi \wedge \psi) \rightarrow \neg\chi$ is always true, except when all three subformulas, φ, ψ and χ are all true at the same time.

φ	ψ	χ	$\varphi \wedge \psi$	$\neg\chi$	$(\varphi \wedge \psi) \rightarrow \neg\chi$
1	1	1	1	0	0
1	1	0	1	1	1
1	0	1	0	0	1
1	0	0	0	1	1
0	1	1	0	0	1
0	1	0	0	1	1
0	0	1	0	0	1
0	0	0	0	1	1

Table 2: Truth table for formula $(\varphi \wedge \psi) \rightarrow \neg\chi$

2.3 *Contingencies, tautologies & contradictions*

A common application of truth tables is to find out whether a given formula is always true, always false or whether it can sometimes be true and sometimes be false. The truth table in Table 2 shows that the formula $(\varphi \wedge \psi) \rightarrow \neg\chi$ belongs to the last category: there are cases, i.e., assignments of truth values to its components (= rows in the table), where it is true, and there are cases where it is false. A formula which can become true and false for different assignments of truth values to its components is called a *contingency*. A formula which is always true under any constellation of truth values for its components is called a *tautology*. A formula which is false for all ways of assigning truth to its subformulas is a *contradiction*. We say that a tautology is a formula that is *necessarily true*, or *true by logical necessity*. Similarly, a contradiction can be said to be *necessarily false*, or *false by logical necessity*.

To show that any formula of the form $\varphi \vee \neg\varphi$ is a tautology, we can construct the relevant truth table, shown in Table 3(a), and check whether the column for the "goal formula" contains only ones, no zeros. Similarly, the

truth table in Table 3(b) reveals that any formula of the logical form $\varphi \wedge \neg\varphi$ is a contradiction.

φ	$\neg\varphi$	$\varphi \vee \neg\varphi$
1	0	1
0	1	1

(a) Truth table for $\varphi \vee \neg\varphi$

φ	$\neg\varphi$	$\varphi \wedge \neg\varphi$
1	0	0
0	1	0

(b) Truth table for $\varphi \wedge \neg\varphi$

Table 3: Truth tables for a tautology and a contradiction

Exercise 3. For each of the following formulas, try to intuit whether it is a tautology, a contradiction or a contingency. Write down your best guess. Then use the truth-table method to find out for certain.

a. $\varphi \rightarrow \neg\varphi$

b. $\varphi \leftrightarrow (\varphi \wedge \psi)$

c. $(\varphi \wedge \neg\varphi) \rightarrow \psi$

d. $(\varphi \wedge \neg\psi) \leftrightarrow (\varphi \rightarrow \psi)$

2.4 *Excavating the (propositional) logical structure of natural language sentences*

The logical operators of PropLog bear a close correspondence to natural language expressions *and*, *or*, *if* and *if and only if*. But the correspondence is not a perfect match. Logical conjunction is order-insensitive: $\varphi \wedge \psi$ and $\psi \wedge \varphi$ are the same.[8] But natural language conjunctions are not necessarily. Saying that

They got married and had children.

might be understood differently from saying that

They had children and got married.

Logical disjunction is *inclusive*: $\varphi \vee \psi$ is true when φ and ψ are both true. Yet natural language disjunctions *can* be understood as exclusive disjunction according to which exactly one, but not both disjuncts are true:

She owns a Porsche or a Ferrari.

PropLog analyzes implication as so-called *material implication*, but it is controversial whether natural language conditional sentences might not have a much richer meaning, e.g., requiring a discernible causal-inferential relation between antecedent and consequent.

These superficial discrepancies notwithstanding, it is possible and highly informative to try to lay bare the logical structure of natural language sentences.[9] Take the following example sentence:

If drawing a red card means that I lost, I lost.

This sentences does have a propositional-logical structure, even though it is not at all apparent from the way it is realized in natural language. There are two atomic (truth-evaluable) propositions involved here:

- p: I drew a red card.
- q: I lost.

With this *translation key*, we can then determine the logical structure of the sentence above as:

$$(p \to q) \to q$$

Here are a few other examples.

(i) John is hungry.
Key: p: John is hungry. *Logical form:* p

(ii) Mary likes John.
Key: p: Mary likes John. *Logical form:* p

[8] In fact, they are *logically equivalent*, a notion we will define later.

[9] There are least two attitudes you can adopt to approach this endeavour. You can think that there *is* a true logical structure to be found, so that, if you struggle or find mismatches, it must mean that you are learning something about the nature of the true underlying logic. Or you can realize that trying to find the logical structure, even though you do not believe that there is single true one, is informative because it tells you something about the limits of logic and the aspects of natural language which cannot (easily) be captured by logical analysis (of the kind we use here).

(iii) John is hungry and Mary likes John.
Key: p: John is hungry. q: Mary likes John.
Logical form: $p \wedge q$

(iv) Berlin is east of Hamburg and Bremen.
Key: p: Berlin is east of Hamburg. q: Berlin is east of Bremen.
Logical form: $p \wedge q$

(v) Call me 'honey' or 'sweetie' once more and, no kidding, I'll step on your toe.
Key: p: I'm not kidding. q: I'll step on your toe.
r: You call me 'honey' or 'sweetie' once more.
Logical form: $p \wedge (r \rightarrow q)$

(vi) I will only play salsa, if you give me earplugs.[10]
Key: p: I will play salsa. q: You give me earplugs.
Logical form: $p \rightarrow q$

Exercise 4. For each of the following sentences give the translation key and the (propositional) logical form.

(a.) If John is hungry, then Mary likes John.

(b.) If John is hungry and Bill runs, then Mary doesn't likes John.

(c.) John is hungry or it's false that Bill runs.

(d.) Mary likes John or Mary doesn't like John.

(e.) Mary likes John but John is hungry.

(f.) Mary likes John provided Bill runs.

[10] There is a likely inclination to analyze this as $p \leftrightarrow q$, but it is not clear whether this is part of the semantic/logical content here. Suppose you know that the sentence is true. Now you learn that earplugs have been handed over. Does that *logically* entail that salsa was played? No, because the sentence only says that earplugs are a *necessary* but not necessarily *sufficient* condition for salsa.