Predicate logic: semantics

Methods: Logic, Part 5

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Propositional logic

$$V(\varphi) \in \{0, 1\}$$

Predicate logic

$$V_M(\varphi) \in \{0, 1\}$$

Model of predicate logic

A model $M = \langle D, I \rangle$ for $\mathfrak{L}$ consists of:

domain $D \neq \emptyset$: set of entities

interpretation function I:

if c is a constant of $\mathfrak{L}$, then $I(c) \in D$, and
if P is an n-ary predicate letter of $\mathfrak{L}$, then $I(P) \subseteq D^n$.

D^n is the set of all n-tuples with elements from D

Assignments and term interpretations

Assignment function

An **assignment function** g for model $M = \langle D, I \rangle$ and language $\mathfrak{L}$ maps all variables in $\mathfrak{L}$ to elements in D.

Term interpretation

If t is a term of L, then $[\![t]\!]_{M,g}$ is the **term interpretation** relative to $M = \langle D, I \rangle$ and g:

$[\![t]\!]_{M,g} = I(t)$ if t is a constant, and $\qquad$ $[\![t]\!]_{M,g} = g(t)$ otherwise.

Valuation functions for predicate logic (1)

Simple formulas

$$V_{M,g}(At_1 \dots t_n) = 1 \quad \text{iff} \quad \left\langle [\![t_1]\!]_{M,g}, \dots, [\![t_n]\!]_{M,g} \right\rangle \in I(A)$$

Sentential connectives

$$
\begin{array}{lll}
V_{M,g}(\neg\phi) = 1 & \text{iff} & V_{M,g}(\phi) = 0 \\
V_{M,g}(\phi \wedge \psi) = 1 & \text{iff} & V_{M,g}(\phi) = 1 \text{ and } V_{M,g}(\psi) = 1 \\
V_{M,g}(\phi \vee \psi) = 1 & \text{iff} & V_{M,g}(\phi) = 1 \text{ or } V_{M,g}(\psi) = 1 \\
V_{M,g}(\phi \rightarrow \psi) = 0 & \text{iff} & V_{M,g}(\phi) = 1 \text{ and } V_{M,g}(\psi) = 0 \\
V_{M,g}(\phi \leftrightarrow \psi) = 1 & \text{iff} & V_{M,g}(\phi) = V_{M,g}(\psi)
\end{array}
$$

Valuation functions for predicate logic (2)

Quantifiers

$V_{M,g}(\forall x\,\phi) = 1 \quad \text{iff} \quad V_{M,g_{[x/d]}}(\phi) = 1$ for all $d \in D$

$V_{M,g}(\exists x\,\phi) = 1 \quad \text{iff} \quad V_{M,g_{[x/d]}}(\phi) = 1$ for at least one $d \in D$

Notation

We write $g_{[x/d]}$ and read "g with x mapped to d," for the assignment function which is like g except that x is mapped to $d \in D$.

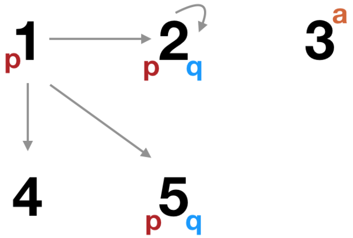
Truth in a model

If $V_{M,g}(\phi) = 1$ for all g, we write $V_M(\phi) = 1$ and say "ϕ is true in M".
Similarly for *false*.
Any sentence (no free variables) is either true or false in any given model M.

Example

Model

$$D = \{1, 2, 3, 4, 5\}$$
$$I(a) = 3$$
$$I(P) = \{1, 2, 5\}$$
$$I(Q) = \{2, 5\}$$
$$I(R) = \{\langle 1, 2\rangle, \langle 2, 2\rangle, \langle 1, 4\rangle, \langle 1, 5\rangle\}$$



Truth of formulas in the model

(i) $V_M(\exists x\,(Px \wedge Qx)) = 1$

(ii) $V_M(Pa) = 0$

(iii) $V_M(\forall x\,(Px \rightarrow Qx)) = 0$

(iv) $V_M(\forall x\,(Rax \rightarrow Qx)) = 1$