# Proof strategies: Examples

## Methods: Logic, Part 2b

Michael Franke

# Proof strategies

(i) refutation by counterexample
(ii) direct proof
(iii) indirect proof
(iv) inductive proof

# Refutation by counterexample

Proposition

The following claim is false:
For any sets $X$ and $Y$, if $X \in Y$, then all the elements of $X$ are also elements of $Y$.

Proof.

A counterexample to the claim in question is given by the following two sets:

$$X = \{a, b\}$$

$$Y = \{c, d, X\} = \{c, d, \{a, b\}\}$$

Although $X \in Y$ and $a \in X$, it is not true that $a \in Y$. $\square$

# Direct proof

### Proposition

For any set $X$, $\emptyset \subseteq X$.

### Proof.

Consider an arbitrary set $X$.
For a set $Y$ be a subset of $X$, it is required that all elements of $Y$ are also in $X$.
I.o.w., there cannot be a single element $y \in Y$ for which $y \notin X$.
Since the empty set contains no elements at all, there cannot be any element in it, which is not also in $X$. $\square$

# Indirect proof

Proposition

For any set $X$, $\emptyset \subseteq X$.

Proof.

Assume that there is an $X$ for which $\emptyset \not\subseteq X$.
Then there must be an element in $\emptyset$ which is not in $X$.
But there are no elements in $\emptyset$. So, we have a contradiction. $\square$

# Inductive proof

### Definition

1. **anchor:** the symbol “*” is part of $\mathcal{F}$
2. **step:** if $f \in \mathcal{F}$, then so is “(x)”
3. **exhaustion:** nothing else is in $\mathcal{F}$

### Proposition

Each $f \in \mathcal{F}$ has an equal number of opening and closing parentheses.

### Proof.

The inductive proof is over the number $n$ of opening parentheses.
<u>Inductive base.</u> If $f \in \mathcal{F}$ has no opening parenthesis, it must be $f = *$, for which the number of opening and closing parentheses is equal.
<u>Inductive assumption.</u> Any $f \in \mathcal{F}$ with $n = k - 1$ opening parentheses has the same number of opening and closing parentheses.
<u>Inductive step.</u> If $f \in \mathcal{F}$ has $n = k$ opening parentheses, $f$ must be of the form $f =$ “(g)” where string $g \in \mathcal{F}$ has $k - 1$ opening parentheses.
By inductive assumption, $g$ has the same amount of opening and closing parentheses.
But since $f =$ “(g)”, and so exactly one parenthesis of each type is added to $g$, $f$ must have an equal number of parentheses, too. □