Natural deduction for propositional logic Michael Franke

natural deduction; soundness & completeness

- derivation (proof): chain of legitimate rewrite steps
- legitimate steps are: introduction and elimination of connectives

A derivation is a finite set of formulas

0.1 Introduction rule for conjunction I_{\wedge}

We may introduce the conjunction $\varphi \wedge \psi$ whenever both the conjuncts φ and ψ are available at previous lines m_1 and m_2 . It does not matter whether m_1 occurs before m_2 or the other way around.¹

```
m_1
Conjunction Intro I_{\wedge}
                                                             m_2
                                                              n.
                                                                                                 I_{\wedge}, m<sub>1</sub>, m<sub>2</sub>
                                                                         \varphi \wedge \psi
```

¹We adopt the same convention of omitting the outermost parentheses. Strictly speaking, we should write $(\varphi \land \psi)$ in line n. of this derivation.

We can use this rule to show that $p, q, r \vdash (r \land p) \land q$ like so:

1. pass. 2. ass. 3. ass. 4. $r \wedge p$ I_{\wedge} , 3, 1 5. $(r \wedge p) \wedge q$ I_{\wedge} , 4, 2

0.2 Elimination rule for conjunction E_{\wedge}

If we have the conjunction $\varphi \wedge \psi$, we are allowed to also derive each conjunct.²

```
\varphi \wedge \psi
                                                 m
Conjunction Elim E_{\wedge}
                                                                            E_{\wedge}, m
                                                n_1
                                                                            E_{\wedge}, m
                                                n_2
```

We can use this new rule to show that $p \land q \vdash q \land p$ like so:

1. $p \wedge q$ 2. E_{\wedge} , 1 p E_{\wedge} , 1 3. q I_{\wedge} , 3, 2 $q \wedge p$

0.3 Elimination rule for implication E_{\rightarrow}

If we have $\varphi \to \psi$ and φ somewhere in our derivation (no matter which one comes first), we can derive ψ .

²It is not necessary to derive both, we can also only derive one of the disjuncts.

$$\begin{array}{cccc} \vdots & \vdots & & & & \\ m_1 & \varphi \to \psi & & & \\ \vdots & \vdots & & & \\ m_2 & \varphi & & & \\ \vdots & \vdots & & & \\ n & \psi & E_\to, \, m_1, \, m_2 \end{array}$$

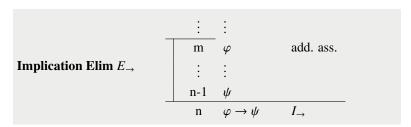
Using this rule, we can show that $p \wedge r, r \rightarrow q \vdash p \wedge q$:

ass.

- $p \wedge r$
- 2. $r \rightarrow q$ ass.
- 3. E_{\wedge} , 1
- E_{\wedge} , 1
- E_{\rightarrow} , 2, 4 5. q
- 6. $p \wedge q$ I_{\wedge} , 3, 5

0.4 Introduction rule for implication I_{\rightarrow}

The introduction rule for implication is slightly more complex. The idea is this. We can introduce $\varphi \to \psi$ if it is possible to derive ψ from the additional assumption that φ . We therefore allow *additional*, *temporary assumptions* to be introduced in order to make "thought experiments" like imagining that some formula was given as well. We use special notation to note where such an additional assumption was made and where this assumption is dropped again.3



We can use this rule to show that $\vdash (p \land q) \rightarrow q$:

1.	$p \wedge q$	ass.
2.	q	E_{\wedge} , 1
3.	$\vdash (p \land q) \rightarrow q$	I_{\rightharpoonup}

Another example, with explicit assumptions given is the following derivation showing that $(p \land q) \rightarrow r \vdash (q \land p) \rightarrow r$:

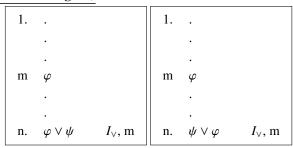
1.	$(p \land q) \to r$	ass.
2.	$q \wedge p$	ass.
3.	q	E_{\wedge} , 2
4.	p	E_{\wedge} , 3
5.	$p \wedge q$	I_{\wedge} , 4, 3
6.	r	E_{\rightarrow} , 1, 5
7.	$(q \land p) \rightarrow r$	$I_{ ightarrow}$

³Notice that we do not need to write down which previous lines this rule operates on as this is implicit in the notation used for marking the "thought experiment" or better put: the scope of the additional assumption.

Introductieregel I_{\rightarrow} :

 $\overline{I_{\rightarrow}}$ mag alleen met de laatste assumptie worden gebruikt.

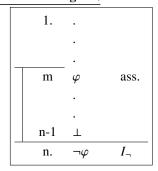
Introductieregel I_{\lor} :



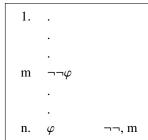
Eliminatieregel E_{\lor} :

Eliminatieregel E_{\neg} :

Introductieregel I_{\neg} :



$dubbelnegatie\ regel\ \neg\neg:$



Introductieregel I_{\exists} :

1. . m $[a/x]\varphi$ *I*∃, m n. $\exists x \varphi$

Eliminatieregel E_{\forall} :

1. . m $\forall x \varphi$ n. $[a/x]\varphi$ E_{\forall} , m