Natural deduction for propositional logic

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natural deduction; soundness & completeness

- derivation (proof): chain of legitimate rewrite steps
- legitimate steps are: introduction and elimination of connectives

A derivation is a finite set of formulas

0.1 *Introduction rule for conjunction $I_\wedge$*

We may introduce the conjunction $\varphi \wedge \psi$ whenever both the conjuncts φ and ψ are available at previous lines m_1 and m_2. It does not matter whether m_1 occurs before m_2 or the other way around.[1]

Conjunction Intro $I_\wedge$	$\vdots$	$\vdots$	
	m_1	φ	
	$\vdots$	$\vdots$	
	m_2	ψ	
	$\vdots$	$\vdots$	
	n.	$\varphi \wedge \psi$	$I_\wedge$, m_1, m_2

We can use this rule to show that $p, q, r \vdash (r \wedge p) \wedge q$ like so:

1.	p	ass.
2.	q	ass.
3.	r	ass.
4.	$r \wedge p$	$I_\wedge$, 3, 1
5.	$(r \wedge p) \wedge q$	$I_\wedge$, 4, 2

[1] We adopt the same convention of omitting the outermost parentheses. Strictly speaking, we should write $(\varphi \wedge \psi)$ in line n. of this derivation.

0.2 *Elimination rule for conjunction $E_\wedge$*

If we have the conjunction $\varphi \wedge \psi$, we are allowed to also derive each conjunct.[2]

Conjunction Elim $E_\wedge$	$\vdots$	$\vdots$	
	m	$\varphi \wedge \psi$	
	$\vdots$	$\vdots$	
	n_1	φ	$E_\wedge$, m
	n_2	ψ	$E_\wedge$, m

We can use this new rule to show that $p \wedge q \vdash q \wedge p$ like so:

1.	$p \wedge q$	ass.
2.	p	$E_\wedge$, 1
3.	q	$E_\wedge$, 1
4.	$q \wedge p$	$I_\wedge$, 3, 2

[2] It is not necessary to derive both, we can also only derive one of the disjuncts.

0.3 *Elimination rule for implication $E_\rightarrow$*

If we have $\varphi \rightarrow \psi$ and φ somewhere in our derivation (no matter which one comes first), we can derive ψ.

Implication Elim $E_{\to}$

$\vdots$	$\vdots$	
m_1	$\varphi \to \psi$	
$\vdots$	$\vdots$	
m_2	φ	
$\vdots$	$\vdots$	
n	ψ	$E_{\to}$, m_1, m_2

Using this rule, we can show that $p \wedge r, r \to q \vdash p \wedge q$:

1.	$p \wedge r$	ass.
2.	$r \to q$	ass.
3.	p	$E_{\wedge}$, 1
4.	r	$E_{\wedge}$, 1
5.	q	$E_{\to}$, 2, 4
6.	$p \wedge q$	$I_{\wedge}$, 3, 5

0.4 *Introduction rule for implication $I_{\to}$*

The introduction rule for implication is slightly more complex. The idea is this. We can introduce $\varphi \to \psi$ if it is possible to derive ψ from the additional assumption that φ. We therefore allow *additional, temporary assumptions* to be introduced in order to make "thought experiments" like imagining that some formula was given as well. We use special notation to note where such an additional assumption was made and where this assumption is dropped again.[3]

[3] Notice that we do not need to write down which previous lines this rule operates on as this is implicit in the notation used for marking the "thought experiment" or better put: the scope of the additional assumption.

Implication Elim $E_{\to}$

$\vdots$	$\vdots$	
m	φ	add. ass.
$\vdots$	$\vdots$	
n-1	ψ	
n	$\varphi \to \psi$	$I_{\to}$

We can use this rule to show that $\vdash (p \wedge q) \to q$:

1.	$p \wedge q$	ass.
2.	q	$E_{\wedge}$, 1
3.	$\vdash (p \wedge q) \to q$	$I_{\to}$

Another example, with explicit assumptions given is the following derivation showing that $(p \wedge q) \to r \vdash (q \wedge p) \to r$:

1.	$(p \wedge q) \to r$	ass.
2.	$q \wedge p$	ass.
3.	q	$E_{\wedge}$, 2
4.	p	$E_{\wedge}$, 3
5.	$p \wedge q$	$I_{\wedge}$, 4, 3
6.	r	$E_{\to}$, 1, 5
7.	$(q \wedge p) \to r$	$I_{\to}$

Introductieregel $I_{\rightarrow}$:

$I_{\rightarrow}$ mag alleen met de laatste assumptie worden gebruikt.

Introductieregel $I_{\vee}$:

1.	.	
	.	
	.	
m	φ	
	.	
	.	
n.	$\varphi \vee \psi$	$I_{\vee}$, m

1.	.	
	.	
	.	
m	φ	
	.	
	.	
n.	$\psi \vee \varphi$	$I_{\vee}$, m

Eliminatieregel $E_{\vee}$:

1.	.	
	.	
	.	
m_1	$\varphi \vee \psi$	
	.	
	.	
m_2	$\varphi \to \chi$	
	.	
	.	
m_3	$\psi \to \chi$	
	.	
	.	
n.	χ	$E_{\vee}$, m_1, m_2, m_3

Eliminatieregel $E_{\neg}$:

1.	.	
	.	
	.	
m_1	$\neg\varphi$	
	.	
	.	
m_2	φ	
	.	
	.	
n.	$\perp$	$E_{\neg}$, m_1, m_2

Introductieregel $I_{\neg}$:

1.	.	
	.	
	.	
m	φ	ass.
	.	
	.	
n-1	$\perp$	
n.	$\neg\varphi$	$I_{\neg}$

dubbelnegatie regel $\neg\neg$:

1.	.	
	.	
	.	
m	$\neg\neg\varphi$	
	.	
	.	
n.	φ	$\neg\neg$, m

Introductieregel $I_\exists$:

1.	.	
	.	
	.	
m	$[a/x]\varphi$	
	.	
	.	
n.	$\exists x\varphi$	$I_\exists$, m

Eliminatieregel $E_\forall$:

1.	.	
	.	
	.	
m	$\forall x\varphi$	
	.	
	.	
n.	$[a/x]\varphi$	$E_\forall$, m