

# Relations

Methods: Logic, Part 3a

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# Topics covered

- (i) tuples
- (ii) Cartesian products
- (iii) relations
- (iv) properties of relations

# Tuples

Order-sensitive collections

## Sets

- order of elements is irrelevant  
 $\{a, b\} = \{b, a\}$
- elements cannot reoccur  
 $\{a, a\} = \{a\}$

## Tuples

- order of elements is relevant  
 $\langle a, b \rangle \neq \langle b, a \rangle$
- elements can reoccur  
 $\langle a, a \rangle \neq \langle a \rangle$

# Tuples

## Terminology & notation

An *n-tuple* is a tuple with  $n$  elements (in order).

For  $n = 1$ , we conventionally define:  $\langle x \rangle = x$ .

For small  $n \geq 1$ , there are special words:

$n = 2$  ordered pair

$n = 3$  triple

$n = 4$  quadruple

$n = 5$  quintuple

...

# Cartesian products

The Cartesian product of two sets  $X$  and  $Y$  is a set of pairs:

$$X \times Y = \{\langle x, y \rangle \mid x \in X \text{ and } y \in Y\}$$

The Cartesian product of  $n$  sets is a set of  $n$ -tuples:

$$X_1 \times X_2, \dots, \times X_n = \{\langle x_1, x_2, \dots, x_n \rangle \mid x_i \in X_i \text{ for all } 1 \leq i \leq n\}$$

## Examples

$$X = \{a, b\}$$

$$Y = \{c, d\}$$

$$X \times Y = \{\langle a, c \rangle, \langle a, d \rangle, \langle b, c \rangle, \langle b, d \rangle\}$$

$$Y \times X = \{\langle c, a \rangle, \langle d, a \rangle, \langle c, b \rangle, \langle d, b \rangle\}$$

$$X \times X = \{\langle a, a \rangle, \langle a, b \rangle, \langle b, a \rangle, \langle b, b \rangle\}$$

# Relations

## Definition

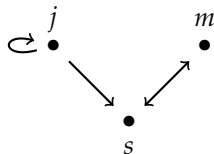
Any subset of an  $n$ -place Cartesian product is called an  $n$ -ary relation.

## Example

$P = \{j, m, s\}$  is a set of people

$L \subseteq P \times P$  is binary relation encoding who loves whom:

$$\begin{aligned} L &= \{\langle x, y \rangle \in P \times P \mid x \text{ loves } y\} \\ &= \{\langle j, j \rangle, \langle j, s \rangle, \langle m, s \rangle, \langle s, m \rangle\} \end{aligned}$$



# Relations

## Terminology & notation

- $R \subseteq X \times Y$  is also called a *binary* relation
- if  $\langle x, y \rangle \in R \subseteq X \times Y$ , we can also use
  - prefix notation:  $Rxy$  [used in this course; except for math stuff like  $1 \leq 2$ ]
  - infix notation:  $xRy$
  - postfix notation:  $xyR$
- domain and range of binary relation  $R \subseteq X \times Y$ :
  - $\text{dom}(R) = \{x \in X \mid \text{there is some } y \in Y \text{ with } Rxy\}$
  - $\text{range}(R) = \{y \in Y \mid \text{there is some } x \in X \text{ with } Rxy\}$

# Properties of binary relations

Binary relation  $R \subseteq X \times X$  is

*reflexive* iff  $Rxx$  for all  $x \in X$

*irreflexive* iff  $Rxx$  for no  $x \in X$

*symmetric* iff for all  $x, y \in X$  if  $Rxy$  then also  $Ryx$

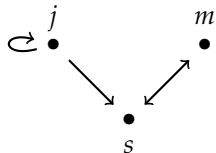
*asymmetric* iff for no  $x, y \in X$  both  $Rxy$  and  $Ryx$

*anti-symmetric* iff for all  $x, y \in X$  if  $Rxy$  and  $Ryx$ , then  $x = y$

*transitive* iff for all  $x, y \in X$  if  $Rxy$  and  $Ryz$ , then also  $Rxz$

*intransitive* iff for all  $x, y \in X$  if  $Rxy$  and  $Ryz$ , then not  $Rxz$

*connected* iff for all  $x, y \in X$  either  $Rxy$  or  $Ryx$  or  $x = y$





# Orders and equivalence relations

Binary relation  $R \subseteq X \times X$  is said to be:

- a *partial weak order* iff  $R$  is reflexive, anti-symmetric and transitive  
[example: relation " $\subseteq$ " on  $\mathcal{P}(Y)$ ]
- a *partial strict order* iff  $R$  is irreflexive, asymmetric and transitive  
[example: relation " $\subset$ " on  $\mathcal{P}(Y)$ ]
- a *linear weak order* iff  $R$  is a partial weak order and connected  
[example: relation " $\leq$ " on  $\mathbb{N}$ ]
- a *linear strict order* iff  $R$  is a partial strict order and connected  
[example: relation " $<$ " on  $\mathbb{N}$ ]
- an *equivalence relation* iff  $R$  is reflexive, symmetric and transitive  
[example: relation "has equal cardinality" on  $\mathcal{P}(Y)$ ]