Natural deduction for propositional logic Michael Franke

natural deduction; soundness & completeness

- derivation (proof): chain of legitimate rewrite steps
- legitimate steps are: introduction and elimination of connectives

A derivation is a finite set of formulas

0.1 Introduction rule for conjunction I_{\wedge}

We may introduce the conjunction $\varphi \wedge \psi$ whenever both the conjuncts φ and ψ are available at previous lines m_1 and m_2 . It does not matter whether m_1 occurs before m_2 or the other way around.¹

```
m_1
Conjunction Intro I_{\wedge}
                                                             m_2
                                                              n.
                                                                                                 I_{\wedge}, m<sub>1</sub>, m<sub>2</sub>
                                                                         \varphi \wedge \psi
```

¹We adopt the same convention of omitting the outermost parentheses. Strictly speaking, we should write $(\varphi \land \psi)$ in line n. of this derivation.

We can use this rule to show that $p, q, r \vdash (r \land p) \land q$ like so:

1. pass. 2. ass. 3. ass. 4. $r \wedge p$ I_{\wedge} , 3, 1 5. $(r \wedge p) \wedge q$ I_{\wedge} , 4, 2

0.2 Elimination rule for conjunction E_{\wedge}

If we have the conjunction $\varphi \wedge \psi$, we are allowed to also derive each conjunct.²

```
\varphi \wedge \psi
                                                 m
Conjunction Elim E_{\wedge}
                                                                            E_{\wedge}, m
                                                n_1
                                                                            E_{\wedge}, m
                                                n_2
```

We can use this new rule to show that $p \land q \vdash q \land p$ like so:

1. $p \wedge q$ 2. E_{\wedge} , 1 p E_{\wedge} , 1 3. q I_{\wedge} , 3, 2 $q \wedge p$

0.3 Elimination rule for implication E_{\rightarrow}

If we have $\varphi \to \psi$ and φ somewhere in our derivation (no matter which one comes first), we can derive ψ .

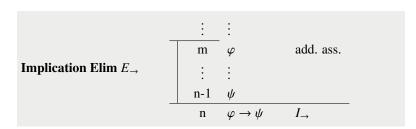
²It is not necessary to derive both, we can also only derive one of the disjuncts.

Using this rule, we can show that $p \wedge r, r \rightarrow q \vdash p \wedge q$:

- $p \wedge r$
- ass.
- $r \rightarrow q$ 2.
- ass.
- 4.
- E_{\wedge} , 1 E_{\wedge} , 1
- 5.
- E_{\rightarrow} , 2, 4
- $p \wedge q$
- I_{\wedge} , 3, 5

0.4 Introduction rule for implication I_{\rightarrow}

The introduction rule for implication is slightly more complex. The idea is this. We can introduce $\varphi \to \psi$ if it is possible to derive ψ from the additional assumption that φ . We therefore allow *additional*, *temporary assumptions* to be introduced in order to make "thought experiments" like imagining that some formula was given as well. We use special notation to note where such an additional assumption was made and where this assumption is dropped again.³ We are *not* allowed to use any of the formulas derived in lines m to n-1 after dismissing the additional assumption in line n.



We can use this rule to show that $\vdash (p \land q) \rightarrow q$:

$$\begin{array}{c|cccc}
1. & p \wedge q & \text{ass.} \\
2. & q & E_{\wedge}, 1 \\
\hline
3. & \vdash (p \wedge q) \rightarrow q & I_{\rightarrow}
\end{array}$$

Another example, with explicit assumptions given is the following derivation showing that $(p \land q) \rightarrow r \vdash (q \land p) \rightarrow r$:

³Notice that we do not need to write down which previous lines this rule operates on as this is implicit in the notation used for marking the "thought experiment" or better put: the scope of the additional assumption.

1.	$(p \land q) \rightarrow r$	ass.
2.	$q \wedge p$	ass.
3.	q	E_{\wedge} , 2
4.	p	E_{\wedge} , 3
5.	$p \wedge q$	I_{\wedge} , 4, 3
6.	r	E_{\rightarrow} , 1, 5
7.	$(q \land p) \rightarrow r$	$I_{ ightarrow}$

0.5 Introduction rule for disjunction I_{\vee}

A disjunction $\varphi \lor \psi$ can be introduced whenever at least one disjunct is available in the derivation.

$$\begin{array}{cccc} \vdots & \vdots & & & \\ m & \varphi & & & \\ \vdots & \vdots & & & \\ n_1 & \varphi \lor \psi & & I_{\lor}, m \\ n_2 & \psi \lor \varphi & & I_{\lor}, m \end{array}$$

insert example

0.6 Eliminiation rule for disjunction E_{\vee}

Intuitively, we can conclude χ from a disjunction $\varphi \lor \psi$ when χ follows from φ and also follows from ψ .

$$\begin{array}{cccc} \vdots & \vdots & & & & \\ m_1 & \varphi \lor \psi & & & & \\ \vdots & \vdots & & & & \\ m_2 & \varphi \to \chi & & & \\ \vdots & \vdots & & & & \\ m_3 & \psi \to \chi & & & \\ \vdots & \vdots & & & & \\ m_3 & \psi \to \chi & & & \\ \vdots & \vdots & & & & \\ n & \chi & E_{\lor}, \, m_1, \, m_2, \, m_3 \end{array}$$

insert example

0.7 Elimination rule for negation E_{\neg}

Negation is tricky in natural deduction. Though we will speak of an elimination rule for negation, strictly speaking we cannot just eliminate negation if we just have a formula $\neg \varphi$. But we can draw inferences from a negation like $\neg \varphi$ which are "reductive" in a sense: if we have derived both $\neg \varphi$ and φ we have derived a contradiction, which can be very informative. So, the elimi-

⁴Remember that the strategy of an indirect proof is to make a certain assumption in order to show that this will result in a contradiction.

nation rule for negation can best be thought of as an introduction rule for the sign \perp which we use as a special symbol for a contradiction.⁵

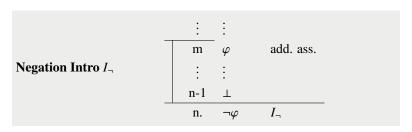
```
m_1
Negation Elim E_{\neg}
                                                      m_2
                                                       n
                                                                 \perp
                                                                                 E_{\neg}, m<sub>1</sub>, m<sub>2</sub>
```

⁵Strictly speaking, we should introduce ⊥ into the language of PropLog as a special formula, which can be used exactly like a proposition letter. We should the say that for all V it's always the case that $V(\bot) = 0$.

insert example

0.8 Introduction rule for negation I_{\neg}

The rule for introducing a negation follows the idea of an indirect proof. We make an additional assumption that φ . If we manage to derive from this assumption a contradiction (denoted as \perp), we have derived $\neg \varphi$.



⁶Negation introduction is essentially a derivation of $\varphi \rightarrow \bot$ which is logically equivalent to $\neg \varphi$.

insert example

0.9 Repetition rule R

We allow to just repeat any previously derived formula. This is really just for readability of a derivation.

```
m
                                             \varphi
Repetition
                                                        R, m
                                             \perp
```

0.10 Ex Falso Sequitor Quodlibet Rule EFSQ

The derivation rules introduced so far are still not enough to produce a derivation for every valid argument schema. One example is the logically valid argument schema $p \vee q$, $\neg p / q$. In order to obtain a system in which $p \lor q, \neg p \vdash q$, we need to introduce another rule, such as the (in)famous ex falso sequitur quodlibet (EFSQ) rule. ⁷ The EFSQ rule allows for the derivation of any formula if a contradiction has been derived. This is useful, of course, particularly when the contradiction is derived from an additional assumption, as in the introduction of implication (see example below).

⁷It is rather difficult to prove that there cannot be a derivation without this (or an equivalent rule). For our purposes, let's just accept that this is so.

Here is a derivation showing that $p \lor q, \neg p \vdash q$:

$p \lor q$	ass.
$\neg p$	ass.
p	add. ass.
\perp	E_{\neg} , 2,3
q	EFSQ, 4
$p \rightarrow q$	$I_{ ightarrow}$
q	add. ass.
q	R, 7
$q \rightarrow q$	$I_{ ightarrow}$
q	$E_{\lor}, 1, 6, 9$
	p \perp q $p \to q$ q q $q \to q$

0.11 Double-negation elimination rule $E_{\neg \neg}$

While it may seem innocuous to conclude φ from a doubly negated statement like $\neg\neg\varphi$, from the point of view of derivations (think: proofs), this is not so. In fact, the rules introduced so far do not allow for the elimination of double negation. We have to introduce a separate rule for this.

derive excluded middle

dubbelnegatie regel $\neg\neg$:

