Lecture Scribe: Random Variables, Expectation, CDF and PDF

(1) Previous Lecture Recap: Random Variables (RVs)

Definition

A **Random Variable (RV)** is a function that assigns a real number to each outcome of a random experiment.

Let:

- S be the sample space
- X be a random variable

Then:

$$X : S \to \mathbb{R}$$

Each outcome $s \in S$ is mapped to a numerical value $X(s)$.

Classification of Random Variables

1. **Discrete Random Variable**
   Takes values from a **countable set**.
   - Example: Number of heads in coin tosses.
2. **Continuous Random Variable**
   Takes values from a **continuous interval**.
   - Example: Time, height, weight.

This lecture focuses primarily on **discrete random variables** and their properties.

(2) Independent Events

Definition

Two events A and B are said to be **independent** if the occurrence of one does not affect the probability of the other.

Mathematically:

$$P(A \cap B) = P(A)P(B)$$

Equivalent Form (Conditional Probability)

If $P(B) > 0$,

$$P(A \mid B) = P(A)$$

This shows that knowing B has occurred gives no additional information about A.

Example

Consider tossing two fair coins.

- Event A: First coin is Head
- Event B: Second coin is Head

$$P(A) = \frac{1}{2}, \quad P(B) = \frac{1}{2}$$

$$P(A \cap B) = \frac{1}{4}$$

Since:

$$P(A \cap B) = P(A)P(B)$$

The events are **independent**.

(3) Types of Discrete Random Variables

(a) Bernoulli Random Variable

Definition

A **Bernoulli RV** represents an experiment with exactly **two outcomes**:

- Success with probability p
- Failure with probability $1 - p$

The random variable X is defined as:

$$X = \begin{cases} 1, & \text{success} \\ 0, & \text{failure} \end{cases}$$

Probability Mass Function (PMF)

$$P(X = x) = p^x (1 - p)^{1-x}, \quad x \in \{0, 1\}$$

(b) Binomial Random Variable

Definition

A **Binomial RV** counts the number of successes in n independent Bernoulli trials, each having success probability p.

$$X \sim \text{Bin}(n, p)$$

Assumptions

1. Fixed number of trials n
2. Each trial is independent
3. Each trial has two outcomes
4. Probability of success is constant

PMF

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \ldots, n$$

(c) Geometric Random Variable

Definition

A **Geometric RV** represents the number of trials required until the **first success** occurs.

$$X \sim \text{Geom}(p)$$

PMF

$$P(X = k) = (1-p)^{k-1} p, \quad k = 1, 2, 3, \ldots$$

(d) Poisson Random Variable

Definition

A **Poisson RV** models the number of occurrences of an event in a fixed interval when events occur:

- Independently
- At a constant average rate λ

$$X \sim \text{Poisson}(\lambda)$$

PMF

$$P(X = k) = \frac{e^{-\lambda} \lambda^k}{k!}, \quad k = 0, 1, 2, \ldots$$

(4) Expectation of Random Variables

(a) Definition of Expectation

For a discrete RV X:

$$E[X] = \sum_x x \, P(X = x)$$

Example

Let X be a Bernoulli random variable with parameter p.

Possible values:

- $X = 1$ with probability p
- $X = 0$ with probability $1 - p$

Step-by-step:

$$E[X] = 1 \cdot p + 0 \cdot (1 - p) = p$$

(b) Expectation of a Function of an RV

If $Y = g(X)$, then:

$$E[g(X)] = \sum_x g(x)P(X = x)$$

This formula is used directly without finding the distribution of Y.

(c) Linear Operation with Expectation

For constants a and b:

$$E[aX + b] = aE[X] + b$$

More generally:

$$E[X + Y] = E[X] + E[Y]$$

This property holds **regardless of independence**.

(d) Moments and Central Moments

$\mathbf{n^{th}}$ Moment

$$E[X^n]$$

Central Moment

$$E[(X - \mu)^n], \quad \mu = E[X]$$

Variance

$$\mathrm{Var}(X) = E[(X - \mu)^2]$$

Expanding step-by-step:

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2$$

Skewness

$$\text{Skewness} = \frac{E[(X - \mu)^3]}{\sigma^3}$$

Kurtosis

$$\text{Kurtosis} = \frac{E[(X-\mu)^4]}{\sigma^4}$$

(6) Cumulative Distribution Function (CDF)

Definition

The **CDF** of a random variable X is defined as:

$$F_X(x) = P(X \le x)$$

Properties

1. $0 \le F_X(x) \le 1$
2. $F_X(x)$ is non-decreasing
3. $F_X(x)$ is right-continuous
4. $$\lim_{x\to-\infty} F_X(x) = 0, \quad \lim_{x\to\infty} F_X(x) = 1$$

Example

Let:

$$P(X=1) = 0.3, \quad P(X=2) = 0.5, \quad P(X=3) = 0.2$$

Step-by-step:

$$F_X(1) = 0.3$$
$$F_X(2) = 0.3 + 0.5 = 0.8$$
$$F_X(3) = 1$$

(7) Probability Density Function (PDF)

Definition

A function $f_X(x)$ is a **PDF** of a continuous random variable X if:

1. $f_X(x) \ge 0$
2. $$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

Relation between PDF and CDF

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$$

If $F_X(x)$ is differentiable:

$$f_X(x) = \frac{d}{dx}F_X(x)$$

Example

Given:

$$f_X(x) = \begin{cases} 2x, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Step-by-step:

- For $x < 0$: $F_X(x) = 0$
- For $0 \leq x \leq 1$:

$$F_X(x) = \int_0^x 2t\,dt = x^2$$

- For $x > 1$: $F_X(x) = 1$