CSE400 – Fundamentals of Probability in Computing

Lecture 6: Discrete Random Variables, Expectation and Problem Solving
(Dhaval Patel, PhD — January 22, 2025)

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1. Random Variables (RV): Motivation and Concept

A **random variable** provides a numerical description of outcomes of a random experiment.
The distribution of a random variable can be visualized using a **bar diagram**:

- The **x-axis** represents the values the random variable can take.
- The **height of the bar** at value (a_i) represents the probability ($\Pr[X = a_i]$).

Each probability is computed by evaluating the probability of the corresponding event in the **sample space**.

2. Types of Random Variables

2.1 Discrete Random Variable

A random variable is **discrete** if it can take **at most a countable number of possible values**.

Properties (as listed in slides):

- Countable support
- Probability Mass Function (PMF)
- Probabilities assigned to **single values**
- Each possible value has **strictly positive probability**

Examples of discrete RVs mentioned:

- Bernoulli Random Variable
- Binomial Random Variable
- Geometric Random Variable
- Poisson Random Variable

2.2 Continuous Random Variable

A random variable is **continuous** if it has an **uncountable support**.

Properties (as listed in slides):

- Probability Density Function (PDF)
- Probabilities assigned to **intervals of values**
- Each individual value has **zero probability**

3. Example: Discrete Random Variable (Three Coin Tosses)

Experiment: Tossing 3 fair coins.
Let (Y) denote the number of heads obtained.

Step 1: Identify possible values

$$Y \in \{0, 1, 2, 3\}$$

Step 2: Compute probabilities

- $P(Y = 0) = P(t, t, t) = \frac{1}{8}$
- $P(Y = 1) = P(t, t, h) + P(t, h, t) + P(h, t, t) = \frac{3}{8}$
- $P(Y = 2) = P(h, h, t) + P(h, t, h) + P(t, h, h) = \frac{3}{8}$
- $P(Y = 3) = P(h, h, h) = \frac{1}{8}$

Step 3: Verify total probability

Since (Y) must take one of the values $(0, 1, 2, 3)$:

$$1 = P\left(\bigcup_{i=0}^{3}\{Y = i\}\right) = \sum_{i=0}^{3} P(Y = i)$$

4. Probability Mass Function (PMF)

4.1 Definition

A random variable that can take on **at most a countable number of possible values** is said to be **discrete**.

Let (X) be a discrete random variable with range:

$$R_X = x_1, x_2, x_3, \ldots \quad \text{(finite or countably infinite)}$$

The function:

$$P_X(x_k) = P(X = x_k), \quad k = 1, 2, 3, \ldots$$

is called the **Probability Mass Function (PMF)** of (X).

4.2 Fundamental Property of PMF

Since (X) must take on **one of the values** (x_k),

$$\sum_{k=1}^{\infty} P_X(x_k) = 1$$

5. Example: PMF Defined Using a Constant

The PMF of a random variable (X) is given by:

$$p(i) = c\frac{\lambda^i}{i!}, \quad i = 0, 1, 2, \ldots$$

where $(\lambda > 0)$.

Step 1: Use normalization condition

$$\sum_{i=0}^{\infty} p(i) = 1$$

Substitute the PMF:

$$\sum_{i=0}^{\infty} c\frac{\lambda^i}{i!} = c\sum_{i=0}^{\infty} \frac{\lambda^i}{i!}$$

Step 2: Use exponential series identity

$$e^{\lambda} = \sum_{i=0}^{\infty} \frac{\lambda^i}{i!}$$

Hence:

$$ce^{\lambda} = 1 \Rightarrow c = e^{-\lambda}$$

Step 3: Compute required probabilities

(a) Find $(P(X = 0))$**:**

$$P(X = 0) = e^{-\lambda}\frac{\lambda^0}{0!} = e^{-\lambda}$$

(b) Find $(P(X > 2))$**:**

$$P(X > 2) = 1 - P(X \leq 2)$$

$$= 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$$

$$= 1 - \left(e^{-\lambda} + \lambda e^{-\lambda} + \frac{\lambda^2}{2}e^{-\lambda}\right)$$

6. Cumulative Distribution Function (CDF)

The **Cumulative Distribution Function (CDF)** is defined for **both discrete and continuous random variables**.

For a discrete random variable, the CDF is obtained by **summing the PMF** up to a given value.

7. Probability Density Function (PDF)

The **Probability Density Function (PDF)** is defined for **continuous random variables only**.

Key relationship (as stated in slides):

$$\frac{d}{dx}[\text{CDF}] = \text{PDF}$$

8. Bayes' Theorem (Recap)

Using conditional probability:

$$\Pr(A|B_i) = \frac{\Pr(B_i|A)\Pr(A)}{\sum_{j=1}^{n} \Pr(A|B_j)\Pr(B_j)}$$

This is known as the **Bayes Formula (Proposition 3.1)**.

Terminology (from slides):

- ($\Pr(B_i)$): **a priori probability**
- ($\Pr(B_i|A)$): **posterior probability**

9. Bayes' Theorem – Example 1: Auditorium with 30 Rows

- Auditorium has **30 rows**.
- Row 1 has 11 seats, Row 2 has 12 seats, ... , Row 30 has 40 seats.
- A prize is given by **randomly selecting a row** (each row equally likely), then **randomly selecting a seat** within that row.

Tasks:

1. Compute the probability that **Seat 15** was selected given that **Row 20** was selected.
2. Compute the probability that **Row 20** was selected given that **Seat 15** was selected.

(Probabilities are computed exactly as shown using Bayes' formula in the slides.)

10. Bayes' Theorem – Example 2: Communication System

- Binary data (0 or 1) is sent and detected at the receiver.
- The receiver may make mistakes:
  - A 0 may be detected as 1
  - A 1 may be detected as 0

The system is described using **conditional probabilities**, and Bayes' theorem is applied to compute the required posterior probabilities.

This lecture scribe strictly follows the order, definitions, assumptions, derivations, and examples exactly as presented in the lecture slides, without adding any external material.