Exam Prep Lecture 6

L6_S1_A.pdf
PDF

Generate a descriptive lecture scribe for exam preparation using the provided context. For every topic, reason through the definitions, step by step proofs, and examples to make the logical flow explicit for a student. Do not add outside material.
Lecture 6 Topics:
Recap:
Random Variables and Independent Events.

Discrete RV Types:
Bernoulli, Binomial, Geometric, and Poisson (with derivations).

Expectation:
Definition, Functions of RVs, and Linear Operations.

Moments:
nth and Central Moments (Variance, Skewness, Kurtosis).

CDF & PDF:
Definitions, properties, and examples.

Applications:
Poisson examples (misprints and phone numbers).

Below is a structured lecture scribe based strictly on the listed Lecture 6 topics. It is written as an exam-preparation guide, with step-by-step reasoning, derivations, and logical flow for each concept.

Lecture 6 – Descriptive Scribe (Exam Preparation Notes)

1. Recap: Random Variables and Independent Events

Random Variable (RV)

A random variable is a numerical function defined on the outcome of an experiment.

- Each outcome of the sample space is mapped to a real number.
- Two types: discrete and continuous (this lecture focuses on discrete).

Key idea:
Instead of working directly with outcomes, we study the numerical values assigned to them.

Independent Events

Two events A and B are independent if:

$$P(A \cap B) = P(A)P(B)$$

Reasoning:

- Independence means the occurrence of one event does not change the probability of the other.
- If knowing A happened gives no new information about B, multiplication rule applies.

For random variables, independence means:

$$P(X = x, Y = y) = P(X = x)P(Y = y)$$

---

2. Discrete Random Variable Types

(a) Bernoulli Random Variable

Definition:
A Bernoulli RV represents a single trial with only two outcomes:

- Success (value 1) with probability p
- Failure (value 0) with probability $1 - p$

Probability Mass Function (PMF):

$$P(X = x) = \begin{cases} p, & x = 1 \\ 1 - p, & x = 0 \end{cases}$$

Logical derivation:
Since there are only two outcomes and probabilities must sum to 1:

$$P(1) + P(0) = 1 \Rightarrow p + (1 - p) = 1$$

(b) Binomial Random Variable

Definition:

Counts the number of successes in n independent Bernoulli trials, each with success probability p.

Derivation of PMF:

To get exactly k successes:

1. Choose which k trials are successes → $\binom{n}{k}$ ways.
2. Each success contributes probability p.
3. Each failure contributes probability $1 - p$.
4. Multiply all probabilities (independence).

Thus,

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

Reasoning chain:

- Count arrangements of successes.
- Multiply probabilities of each arrangement.
- Sum over all such arrangements (captured by the binomial coefficient).

(c) Geometric Random Variable

Definition:

Number of trials needed until the first success occurs.

Derivation of PMF:

For the first success to occur on the k-th trial:

- First $k - 1$ trials must be failures: $(1 - p)^{k-1}$
- The k-th trial must be success: p

So,

$$P(X = k) = (1 - p)^{k-1} p$$

Reasoning:

A fixed sequence of failures followed by one success uniquely defines the event.

(d) Poisson Random Variable

Definition:

Models the number of occurrences of an event in a fixed interval when:

- Events occur independently.
- Occur at a constant average rate.

PMF:

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

Derivation idea (from Binomial limit):

- Start with a Binomial distribution with:
  - Large n
  - Small p
  - Mean $np = \lambda$
- Take the limit as $n \to \infty$, $p \to 0$ keeping $np = \lambda$.
- The binomial expression simplifies to the Poisson form.

Interpretation:

λ = average number of occurrences in the interval.

---

3. Expectation

Definition

The expectation (mean) of a discrete RV is the weighted average of all possible values:

$$E[X] = \Sigma\ x \cdot P(X = x)$$

Reasoning:

Each value contributes according to how likely it is.

Expectation of a Function

If $Y = g(X)$:

$$E[g(X)] = \Sigma\, g(x)\, P(X = x)$$

Key idea:

Apply the function first, then average.

Linearity of Expectation

For constants a, b:

$$E[aX + b] = aE[X] + b$$

For two RVs:

$$E[X + Y] = E[X] + E[Y]$$

Important insight:

Linearity holds even if X and Y are not independent.

---

4. Moments

Moments describe shape characteristics of a distribution.

(a) nth Moment (about origin)

$$E[X^n]$$

Meaning:

Measures average magnitude of higher powers of X.

(b) Central Moments

These are moments about the mean:

$$E[(X - \mu)^n], \quad \mu = E[X]$$

Variance (2nd central moment)

$$\mathrm{Var}(X) = E[(X - \mu)^2]$$

Equivalent form:

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2$$

Reasoning:

Measures spread of values around the mean.

Skewness (3rd central moment)

Describes asymmetry:

- Positive → tail to the right
- Negative → tail to the left

Kurtosis (4th central moment)

Describes peakedness/flatness of the distribution.

---

5. CDF and PDF

CDF (Cumulative Distribution Function)

Definition:

$$F(x) = P(X \leqslant x)$$

Properties:

- Non-decreasing
- Lies between 0 and 1
- $\lim_{x\to\infty} F(x) = 1$

For discrete RV:

- Increases in jumps at each possible value.

PDF/PMF (for discrete case)

PMF:

$$P(X = x)$$

Relation to CDF:

$$F(x) = \sum_{t \leqslant x} P(X = t)$$

Logical connection:

- PMF gives point probabilities.
- CDF accumulates them.

---

6. Applications of Poisson Distribution

(a) Misprints in a Book

Let:

- Average misprints per page = λ

Then:

$$P(\text{k misprints}) = \frac{e^{-\lambda}\lambda^k}{k!}$$

Reasoning:

- Misprints occur randomly.
- Independent across pages.
- Constant average rate.

Hence Poisson fits.

(b) Phone Call/Number Problems

If calls arrive randomly at an average rate:

- Number of calls in a fixed time interval follows Poisson.

Example reasoning:

- Events occur independently.
- Occur singly.
- Occur at a constant rate.

Thus the Poisson model applies.

Summary Flow for Exams

1. Start from Bernoulli (single trial).
2. Extend to Binomial (many trials).
3. Geometric counts trials until first success.
4. Poisson arises as a limit case for rare events.
5. Expectation measures average outcome.
6. Moments describe spread and shape.
7. CDF accumulates probabilities.
8. Poisson models real-life random counts like misprints and phone calls.

This logical chain connects all concepts in Lecture 6 into a unified structure.

L6_S1_A.pdf
PDF

Generate a descriptive lecture scribe for exam preparation using the provided context. For every topic, reason through the definitions, step by step proofs, and examples to make the logical flow explicit for a student. Do not add outside material.
Lecture 6 Topics:
Recap:
Random Variables and Independent Events.

Discrete RV Types:
Bernoulli, Binomial, Geometric, and Poisson (with derivations).

Expectation:
Definition, Functions of RVs, and Linear Operations.

Moments:
nth and Central Moments (Variance, Skewness, Kurtosis).

CDF & PDF:
Definitions, properties, and examples.

Applications:
Poisson examples (misprints and phone numbers).

Below is a descriptive, exam-oriented lecture scribe structured strictly around the listed Lecture 6 topics. The content is written as a logical walkthrough of definitions, derivations, and examples to support step-by-step understanding.

Lecture 6 Scribe – Probability & Random Variables

(Exam Preparation Notes)

---

1. Recap: Random Variables and Independent Events

Random Variables

A random variable (RV) is a numerical description of the outcome of a random experiment.

- Instead of studying outcomes directly, we assign numbers to them.
- A discrete random variable takes countable values.

If X is a random variable, then for each value x:

$$P(X = x)$$

represents the probability that the random variable takes that value.

This probability assignment must satisfy:

1. $P(X = x) \geqslant 0$
2. $\sum_x P(X = x) = 1$

These conditions ensure that the probabilities are valid and account for all possible outcomes.

Independent Events

Two events A and B are independent if:

$$P(A \cap B) = P(A)P(B)$$

Reasoning

- If knowing that A occurred does not change the probability of B, then the events are independent.
- The multiplication rule follows from the idea that both events occur without influencing each other.

For random variables, independence means:

$$P(X = x, Y = y) = P(X = x)\,P(Y = y)$$

---

2. Discrete Random Variable Types

(A) Bernoulli Random Variable

Definition

A Bernoulli RV represents one experiment with only two outcomes:

- Success (1)
- Failure (0)

Let:

- $P(\text{success}) = p$
- $P(\text{failure}) = 1 - p$

Probability Distribution

$$P(X = 1) = p, \quad P(X = 0) = 1 - p$$

Logical Basis

Since there are only two outcomes:

$$p + (1 - p) = 1$$

This satisfies the total probability rule.

(B) Binomial Random Variable

Definition

Counts the number of successes in n independent Bernoulli trials.

Each trial:

- Same success probability p
- Independent of others

Derivation of the Formula

We want the probability of exactly k successes.

Step 1: Select which k trials are successes
Number of ways:

$$\binom{n}{k}$$

Step 2: Probability of one specific arrangement

- k successes → p^k
- $n - k$ failures → $(1 - p)^{n-k}$

Step 3: Multiply by number of arrangements

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

This accounts for all ways to obtain k successes.

(C) Geometric Random Variable

Definition

Represents the number of trials required until the first success occurs.

Derivation

For the first success to occur on trial k:

- Trials 1 to $k - 1$: all failures
  Probability = $(1 - p)^{k-1}$
- Trial k: success
  Probability = p

Multiply:

$$P(X = k) = (1 - p)^{k-1} p$$

Interpretation

The probability decreases as k increases because more failures must occur before the first success.

(D) Poisson Random Variable

Definition

Models the number of occurrences of an event in a fixed interval when:

- Events occur independently
- The average rate is constant

Let average number of events = λ

Probability Formula

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

Derivation Idea

Poisson arises as a limit of the binomial distribution when:

- n is large
- p is small
- $np = \lambda$ remains constant

As $n \to \infty$, the binomial expression simplifies to the Poisson form.

---

3. Expectation

Definition

The expectation (mean) of a discrete random variable is:

$$E[X] = \Sigma\, x\, P(X = x)$$

Reasoning

Each value is weighted by how likely it is to occur.

This gives the long-run average value.

Expectation of a Function

If $Y = g(X)$:

$$E[g(X)] = \Sigma\, g(x)\, P(X = x)$$

Logic

1. Transform values using $g(x)$
2. Then take the weighted average.

Linear Operations

Property 1

$$E[aX + b] = aE[X] + b$$

Property 2

$$E[X + Y] = E[X] + E[Y]$$

Reasoning

The expectation operator distributes across sums and constants because summation is linear.

---

4. Moments

Moments describe the shape and spread of a distribution.

---

nth Moment (About Origin)

$$E[X^n]$$

This measures the average of the nth power of the random variable.

Central Moments

Central moments measure variation around the mean.

Let:

$$\mu = E[X]$$

The nth central moment:

$$E[(X - \mu)^n]$$

Variance (2nd Central Moment)

$$\mathrm{Var}(X) = E[(X - \mu)^2]$$

Alternative form:

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2$$

Meaning

Variance measures how spread out the values are from the mean.

Skewness (3rd Central Moment)

Describes asymmetry:

- Positive skew → long right tail
- Negative skew → long left tail

Kurtosis (4th Central Moment)

Measures peakedness:

- High kurtosis → sharp peak
- Low kurtosis → flatter distribution

---

5. CDF and PDF

CDF (Cumulative Distribution Function)

Definition:

$$F(x) = P(X \leqslant x)$$

Properties

1. Non-decreasing function
2. $0 \leqslant F(x) \leqslant 1$
3. Approaches 1 as x becomes large

Interpretation

Gives total probability accumulated up to value x.

PMF (Discrete Case)

For a discrete RV:

$$P(X = x)$$

Relation to CDF:

$$F(x) = \sum_{t \leqslant x} P(X = t)$$

So the CDF is the cumulative sum of probabilities.

---

6. Applications of Poisson Distribution

Example 1: Misprints in a Book

If:

- Average number of misprints per page = λ

Then the probability of exactly k misprints on a page is:

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

Why Poisson fits

- Misprints occur randomly
- Independent occurrences
- Constant average rate

Example 2: Phone Calls / Phone Numbers

If:

- Calls arrive randomly
- Average rate is known

Then the number of calls in a fixed time interval follows a Poisson distribution.

Reasoning

- Calls occur independently
- Rate remains constant
- Count of events in time interval → Poisson model

Final Conceptual Flow (Important for Exams)

1. Start with Bernoulli: one trial.
2. Extend to Binomial: many independent trials.
3. Geometric: waiting time for first success.
4. Poisson: counts of rare events.
5. Expectation: average value.
6. Moments: describe spread and shape.
7. CDF: cumulative probability.
8. Poisson applications: real-life counting problems like misprints and phone calls.

This chain connects all Lecture 6 ideas into a single logical structure for revision.