Lecture 6 (L6) – Probability Theory

Exam Preparation Scribe

1. Previous Lecture Recap: Random Variables (RVs)

Definition: Random Variable

A random variable (RV) is a real-valued function defined on the sample space of a random experiment.

It assigns a numerical value to each outcome.

Usually denoted by capital letters such as X, Y.

Types of Random Variables

Discrete Random Variable: Takes countable values.

Continuous Random Variable: Takes values over a continuous interval.

2. Independent Events

Definition

Two events A and B are said to be independent if

$$P(A \cap B) = P(A)P(B)$$

This means that the occurrence of one event does not affect the probability of the other.

Example

If a coin is tossed and a die is rolled:

Event A: Coin shows Head

Event B: Die shows 3

Since the coin toss does not affect the die roll,

$$P(A \cap B) = P(A)P(B)$$

3. Types of Discrete Random Variables

3.1 Bernoulli Random Variable

Definition

A Bernoulli random variable represents a single experiment with two outcomes:

Success (1) with probability p

Failure (0) with probability $1 - p$

Probability Mass Function (PMF)

$$P(X = x) = \begin{cases} p, & x = 1 \\ 1 - p, & x = 0 \end{cases}$$

3.2 Binomial Random Variable

Definition

A Binomial random variable counts the number of successes in n independent Bernoulli trials with success probability p.

PMF

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}, \quad k = 0, 1, \dots, n$$

3.3 Geometric Random Variable

Definition

A Geometric random variable represents the number of trials needed to get the first success.

PMF

$$P(X = k) = (1 - p)^{k-1} p, \quad k = 1, 2, 3, \dots$$

3.4 Poisson Random Variable

Definition

A Poisson random variable models the number of occurrences of an event in a fixed interval.

PMF

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, 2, \ldots$$

where $\lambda > 0$ is the mean rate.

4. Expectation of Random Variables

4.1 Definition of Expectation

For a discrete random variable X,

$$E[X] = \sum_x x P(X = x)$$

Example
If

$$P(X = 0) = \frac{1}{2}, \quad P(X = 1) = \frac{1}{2}$$

then

$$E[X] = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2}$$

4.2 Expectation of a Function of an RV

If $Y = g(X)$, then

$$E[g(X)] = \sum_x g(x) P(X = x)$$

4.3 Linear Operation with Expectation

For constants a, b,

$$E[aX + b] = aE[X] + b$$

5. Moments of Random Variables

5.1 nth Moment

The nth moment of X is

$$E[X^n]$$

5.2 Central Moments

Variance

$$\mathrm{Var}(X) = E[(X - E[X])^2] = E[X^2] - (E[X])^2$$

Skewness
The third central moment:

$$E[(X - E[X])^3]$$

Kurtosis
The fourth central moment:

$$E[(X - E[X])^4]$$

6. Cumulative Distribution Function (CDF)

Definition
The CDF of a random variable X is

$$F_X(x) = P(X \leq x)$$

Properties

$$0 \leq F_X(x) \leq 1$$

$F_X(x)$ is non-decreasing

$$\lim_{x \to -\infty} F_X(x) = 0$$

$$\lim_{x \to \infty} F_X(x) = 1$$

Example
For a discrete RV,

$$F_X(x) = \sum_{t \le x} P(X = t)$$

7. Probability Density Function (PDF)

Definition
For a continuous random variable X, the PDF $f_X(x)$ satisfies

$$P(a \le X \le b) = \int_a^b f_X(x)\,dx$$

Properties

$$f_X(x) \ge 0$$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$$

Example
If

$$f_X(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

then

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$