Lecture 6 Scribe – Probability & Random Variables

(Exam Preparation Notes)

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1. Recap: Random Variables and Independent Events

Random Variables

A random variable (RV) is a numerical description of the outcome of a random experiment.

- Instead of studying outcomes directly, we assign numbers to them.
- A discrete random variable takes countable values.

If (X) is a random variable, then for each value (x):

$$P(X = x)$$

represents the probability that the random variable takes that value.

This probability assignment must satisfy:

1. ($P(X = x) \geq 0$)
2. ($\sum_x P(X = x) = 1$)

These conditions ensure that the probabilities are valid and account for all possible outcomes.

Independent Events

Two events (A) and (B) are independent if:

$$P(A \cap B) = P(A)P(B)$$

Reasoning

- If knowing that (A) occurred does not change the probability of (B), then the events are independent.
- The multiplication rule follows from the idea that both events occur without influencing each other.

For random variables, independence means:

$$P(X = x, Y = y) = P(X = x), P(Y = y)$$

2. Discrete Random Variable Types

(A) Bernoulli Random Variable

Definition

A Bernoulli RV represents one experiment with only two outcomes:

- Success (1)
- Failure (0)

Let:

- $(P(\text{success}) = p)$
- $(P(\text{failure}) = 1 - p)$

Probability Distribution

$$P(X = 1) = p, \quad P(X = 0) = 1 - p$$

Logical Basis

Since there are only two outcomes:

$$p + (1 - p) = 1$$

This satisfies the total probability rule.

(B) Binomial Random Variable

Definition

Counts the number of successes in (n) independent Bernoulli trials.
Each trial:

- Same success probability (p)
- Independent of others

Derivation of the Formula

We want the probability of exactly (k) successes.
Step 1: Select which (k) trials are successes
Number of ways:

$$\binom{n}{k}$$

Step 2: Probability of one specific arrangement

- (k) successes $\to$ (p^k)
- ($n - k$) failures $\to$ ($(1 - p)^{n-k}$)

Step 3: Multiply by number of arrangements

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

This accounts for all ways to obtain (k) successes.

(C) Geometric Random Variable

Definition

Represents the number of trials required until the first success occurs.

Derivation

For the first success to occur on trial (k):

- Trials 1 to ($k-1$): all failures
  Probability = ($(1-p)^{k-1}$)

- Trial (k): success
  Probability = (p)

Multiply:

$$P(X=k) = (1-p)^{k-1}p$$

Interpretation

The probability decreases as (k) increases because more failures must occur before the first success.

(D) Poisson Random Variable

Definition

Models the number of occurrences of an event in a fixed interval when:

- Events occur independently

- The average rate is constant

Let average number of events = (λ)

Probability Formula

$$P(X=k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

Derivation Idea

Poisson arises as a limit of the binomial distribution when:

- (n) is large

- (p) is small

- ($np = \lambda$) remains constant

As ($n \to \infty$), the binomial expression simplifies to the Poisson form.

3. Expectation

Definition

The expectation (mean) of a discrete random variable is:

$$E[X] = \sum x, P(X = x)$$

Reasoning

Each value is weighted by how likely it is to occur.
This gives the long-run average value.

Expectation of a Function

If $(Y = g(X))$:

$$E[g(X)] = \sum g(x), P(X = x)$$

Logic

1. Transform values using $(g(x))$
2. Then take the weighted average.

Linear Operations

Property 1

$$E[aX + b] = aE[X] + b$$

Property 2

$$E[X + Y] = E[X] + E[Y]$$

Reasoning

The expectation operator distributes across sums and constants because summation is linear.

4. Moments

Moments describe the shape and spread of a distribution.

nth Moment (About Origin)

$$E[X^n]$$

This measures the average of the nth power of the random variable.

Central Moments

Central moments measure variation around the mean.

Let:

$$\mu = E[X]$$

The nth central moment:

$$E[(X - \mu)^n]$$

Variance (2nd Central Moment)

$$\text{Var}(X) = E[(X - \mu)^2]$$

Alternative form:

$$\text{Var}(X) = E[X^2] - (E[X])^2$$

Meaning

Variance measures how spread out the values are from the mean.

Skewness (3rd Central Moment)

Describes asymmetry:

- Positive skew $\rightarrow$ long right tail
- Negative skew $\rightarrow$ long left tail

Kurtosis (4th Central Moment)

Measures peakedness:

- High kurtosis $\rightarrow$ sharp peak
- Low kurtosis $\rightarrow$ flatter distribution

5. CDF and PDF

CDF (Cumulative Distribution Function)

Definition:

$$F(x) = P(X \leq x)$$

Properties

1. Non-decreasing function
2. $(0 \leq F(x) \leq 1)$
3. Approaches 1 as (x) becomes large

Interpretation

Gives total probability accumulated up to value (x).

PMF (Discrete Case)

For a discrete RV:

$$P(X = x)$$

Relation to CDF:

$$F(x) = \sum_{t \leq x} P(X = t)$$

So the CDF is the cumulative sum of probabilities.

6. Applications of Poisson Distribution

Example 1: Misprints in a Book

If:

- Average number of misprints per page = (λ)

Then the probability of exactly (k) misprints on a page is:

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

Why Poisson fits

- Misprints occur randomly
- Independent occurrences
- Constant average rate

Example 2: Phone Calls / Phone Numbers

If:

- Calls arrive randomly
- Average rate is known

Then the number of calls in a fixed time interval follows a Poisson distribution.

Reasoning

- Calls occur independently
- Rate remains constant
- Count of events in time interval $\rightarrow$ Poisson model