ANALYSIS-2, HW-9

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First, we will define the notion of critical points along a level set. Let U be open in $\mathbb{R}^n$, and let $f : U \to \mathbb{R}$ and let $h : U \to \mathbb{R}^m$ be two $\mathscr{C}^1$ functions (here $n = d + m$, and h represents m constraints). Let $c \in h(U)$, and let M be the level set $M = h^{-1}(c)$. Also, we will assume that the rank of $h'(x)$ for every $x \in M$ is m, so that the tangent space at every $x \in M$ makes sense, and that the tangent space is spanned by velocity vectors.

$a \in M$ is said to be a *critical* or *stationary* point of $f|_M$ if for every $\mathscr{C}^1$ path γ taking values in M and passing through a, the derivative of $f \circ \gamma$ at the point θ is 0, where $\gamma(\theta) = a$. Now, if a is a critical point of $f|_M$, then it is clear that $\nabla f(a)$ lies in the normal space N_a, i.e it is spanned by the vectors $\{\nabla h_i(a)\}_{1 \leq i \leq m}$. Conversely, if $\nabla f(a)$ lies in N_a, then it must be a critical point.

In these problems, we will find critical points over level sets.

1. $f(x, y, z) = 8x - 4z$, where the constraint is $x^2 + 10y^2 + z^2 = 5$.

So, define $h(x, y, z) = x^2 + 10y^2 + z^2$, and let M be the level set $M = h^{-1}(5)$. Observe that

$$h'(x, y, z) = \begin{bmatrix} 2x & 20y & 2z \end{bmatrix}$$

and hence $h'(x, y, z) = 0$ if and only if $(x, y, z) = 0$. In particular, the rank of $h'(p)$ for every $p \in M$ is 1, and hence the tangent space at every $p \in M$ exists.

Now, if $a \in M$ is a critical point of $f|_M$, then we have that

$$\nabla f(a) = \lambda h(a)$$

for some λ. Hence, we have

$$(8, 0, -4) = \lambda(2x, 20y, 2z)$$

which corresponds to the equations:

$$\begin{aligned} \lambda x &= 4 \\ \lambda y &= 0 \\ \lambda z &= -2 \end{aligned}$$

so we have that $y = 0$ (because λ cannot be 0 by the first equation) and also

$$\frac{4}{x} = \frac{-2}{z}$$

which implies that

$$x = -2z$$

Since (x, y, z) lies in M, this means that

$$(-2z)^2 + z^2 = 5$$

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and hence $z = \pm 1$. So, the two critical points are $(-2, 0, 1)$ and $(2, 0, -1)$.

2. $f(x, y) = e^{xy}$ where the constraint is $x^3 + y^3 = 16$.

Define $h(x, y) = x^3 + y^3$ and let M be the level set $M = h^{-1}(16)$.

Now, we have

$$h'(x, y) = \begin{bmatrix} 3x^2 & 3y^2 \end{bmatrix}$$

and hence $h'(x, y) = 0$ if and only if $(x, y) = 0$. In particular, the rank of $h'(p)$ for $p \in M$ is 1, and hence the tangent space at every $p \in M$ exists.

Now, if $a \in M$ is a critical point of $f|_M$, then we have that

$$\nabla f(a) = \lambda h(a)$$

for some λ. So if $a = (x, y)$ we have

$$(ye^{xy}, xe^{xy}) = \lambda(3x^2, 3y^2)$$

which corresponds to the equations

$$ye^{xy} = 3\lambda x^2$$
$$xe^{xy} = 3\lambda y^2$$

Clearly, none of x or y can be zero, because otherwise that would imply that $a = (0, 0)$, which is not in M. So, we see that

$$\frac{ye^{xy}}{x^2} = \frac{xe^{xy}}{y^2}$$

and since e^{xy} is never zero, we have

$$x^3 = y^3$$

Since (x, y, z) lies in M, this implies that

$$2x^3 = 16$$

and hence $x^3 = 8$. So, the only critical point is $(2, 2)$.

3. $f(x, y, z) = x + y + z$ where the constraints are $x^2 - y^2 = 1$ and $2x + z = 1$.

Define $h : \mathbb{R}^3 \to \mathbb{R}^2$ by $h(x, y, z) = (x^2 - y^2, 2x + z)$, and let M be the level set $M = h^{-1}(1, 1)$. Observe that we have

$$h'(x, y, z) = \begin{bmatrix} 2x & -2y & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

Now, observe that $(2x, -2y) = 0$ if and only if $(x, y) = 0$, and hence it follows that $h'(p)$ has rank 2 for all $p \in M$, and so the tangent space at every $p \in M$ exists.

Now, if $a \in M$ is a critical point of $f|_M$, then

$$\nabla f(a) = \lambda_1 \nabla h_1(a) + \lambda_2 \nabla h_2(a)$$

for some $\lambda_1, \lambda_2 \in \mathbb{R}$, where $h = (h_1, h_2)$. So if $a = (x, y, z)$, this means that

$$(1, 1, 1) = \lambda_1(2x, -2y, 0) + \lambda_2(2, 0, 1)$$

which corresponds to the equations

$$\begin{aligned} 2\lambda_1 x + 2\lambda_2 &= 1 \\ -2\lambda_1 y &= 1 \\ \lambda_2 &= 1 \end{aligned}$$

which implies that

$$\begin{aligned} 2\lambda_1 x &= -1 \\ -2\lambda_1 y &= 1 \end{aligned}$$

Clearly, we see that $(x, y) \neq 0$, and hence we have

$$\frac{-1}{2x} = \frac{-1}{2y}$$

which implies that $x = y$. However, because $a \in M$, we have $x^2 - y^2 = 1$, and hence no critical point exists on this level set.

4. Let C be the curve given by the intersection of $x^2 + y^2 = z^2$ with the plane $x - 2z = -5$.

(a) First, we find Δ, where $\Delta = \inf_{p \in C} ||p||$.

Define the function $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$. Also, define the map $h = (h_1, h_2)$, where

$$\begin{aligned} h_1(x, y, z) &= x^2 + y^2 - z^2 \\ h_2(x, y, z) &= x - 2z \end{aligned}$$

so that h is a $\mathscr{C}^\infty$ function. Clearly, we see that $C = h^{-1}(0, -5)$, and hence C is closed. Also, C is bounded, and hence it is compact. Since f is a continuous function on C, f attains a minimum on C. So, to find Δ, we need to minimise f over C.

First, observe that

$$h'(x, y, z) = \begin{bmatrix} 2x & 2y & -2z \\ 1 & 0 & -2 \end{bmatrix}$$

and hence $h'(p)$ has rank 2 for $p \in C$.

Now, if $a \in C$ is a point of local minimum of $f|_C$, then we must have

$$\nabla f(a) = \lambda_1 \nabla h_1(a) + \lambda_2 \nabla h_2(a)$$

for some $\lambda_1, \lambda_2 \in \mathbb{R}$. If $a = (x, y, z)$, this corresponds to the equations

$$\begin{aligned} \frac{x}{||a||} &= 2\lambda_1 x + \lambda_2 \\ \frac{y}{||a||} &= 2\lambda_1 y \\ \frac{z}{||a||} &= -2\lambda_1 z - 2\lambda_2 \end{aligned}$$

Now, there are two possibilities. If $y \neq 0$, then we will get that $\lambda_1 = \dfrac{1}{2||a||}$, which will imply that $\lambda_2 = 0$ and $z = 0$ which would imply that $x^2 + y^2 = 0$ (because $a \in C$) which is not possible. So, we see that $y = 0$. This leaves us with the possibilities

$x = \pm z$, and hence $a = (5, 0, 5)$ or $a = \frac{1}{3}(-5, 0, 5)$. So, the minimum of f over C occurs in the second case, and hence

$$\Delta = \frac{5\sqrt{2}}{3}$$

The only point on C which is at a distance of Δ from the origin is $\frac{1}{3}(-5, 0, 5)$.

(b) Suppose q is the centre of C (which is an ellipse). Consider the function $f(p) = ||p - q||$. Since C is compact, $f|_C$ attains a maximum and minimum on C. Let $\Delta = \max_{p \in C} f(p)$ and let $\delta = \min_{p \in C} f(p)$. Then, it follows that 2Δ is the length of the major axis of the ellipse, and 2δ is the length of the minor axis of the ellipse.

5. Let Q be a homogeneous degree two polynomial in n variables and M be the quadratic hypersurface $Q = c$ where $c \neq 0$ is not a constant. Suppose M is non-empty and let the hessian of Q be non-singular. Let $f : \mathbb{R}^n \to \mathbb{R}$ be $f(x) = ||x||^2$.

(a) Let A be the $n \times n$ symmetric matrix associated to Q. We will show that all eigenvalues of A are non-zero.

For $x \in \mathbb{R}^n$, we know that

$$Q(x) = x^t A x$$

From the lecture notes, we know that if H is the Hessian of this quadratic form, then

$$H = 2A$$

and since H is non-singular, A is also non-singular. So, it follows that all eigenvalues of A are non-zero.

(b) Suppose $a \in M$ is a stationary point of $f|_M$.

First, consider $\nabla Q(a)$. Observe that since Q is degree two homogeneous, we have

$$Ha = \nabla Q(a)$$

where we interpret $\nabla Q(a)$ as a column vector. Now, since $a \in M$, we see that $a \neq 0$, and since H is singular, we see that Ha is non-zero. So, it follows that $JQ(a) = (\nabla Q(a))^t$ has rank 1. So, Q has a tangent space to M at the point a.

Finally, since $a \in M$ is stationary, we see that

$$\nabla f(a) = \lambda \nabla Q(a)$$

for some λ. Also, observe that $\nabla f(a) = 2a$, and hence

$$2a = \lambda \nabla Q(a)$$

which means that

$$\nabla Q(a) = \frac{2}{\lambda} a$$

because $\lambda \neq 0$ as $a \neq 0$. So,

$$Ha = \frac{2}{\lambda} a$$

and hence $Aa = \frac{1}{\lambda} a$ which means that a is an eigenvector of A. So, a lies on a principal axis of M.

(c) First, suppose every principal axis of M has a stationary point of $f|_M$ on it. Let λ be an eigenvalue of A, and let v be an eigenvector on a corresponding principal axis of M (i.e, v is an eigenvector with eigenvalue λ) such that v is a stationary point of $f|_M$. This means that $v \in M$, and hence $v^t A v = c$. But, we know that

$$v^t A v = \lambda v^t v = c$$

and hence λ and c have the same sign (because $v^t v > 0$). So, it follows that all eigenvalues of A have the same sign.

Conversely, suppose all eigenvalues of A have the same sign, and hence A is either positive definite or negative definite. Without loss of generality, suppose A is positive definite (i.e each eigenvalue is positive), and hence $c > 0$. Now, let λ be an eigenvalue, and let v be an eigenvector (so v is along a principal axis of M). Choose $t \in \mathbb{R}$ such that

$$||tv||^2 = (tv)^t(tv) = \frac{c}{\lambda}$$

(which is possible because the right hand side is positive), and hence we have

$$\lambda(tv)^t(tv) = c$$

which implies that

$$w^t A w = c$$

where $w = tv$. So, $w \in M$. Also, observe that

$$\nabla Q(w) = Hw = 2Aw = 2\lambda w = \lambda 2w = \lambda \nabla f(w)$$

and hence

$$f(w) = \frac{1}{\lambda}\nabla Q(w)$$

which implies that w is a stationary point of $f|_M$ (this follows because $JQ(w) = (Hw)^t$ has rank 1, and hence the tangent space exists). So, this proves that every principal axis has a stationary point of $f|_M$ on it. This completes the proof.