ANALYSIS-2, HW 3

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For problems **(1)**, **(2)** and **(3)**, let (X, d), (Y, δ) and (Z, ρ) be metric spaces.

(1). Let x be a point of X, and let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a pair of maps such that f is continuous at x and g is continuous at $f(x)$. Let us show that $g \circ f$ is continuous at x.

First, suppose $\epsilon > 0$ is given. Then, by the continuity of g at $f(x)$, there is some $\delta_1 > 0$ such that

$$y \in Y \text{ and } \delta(y, f(x)) < \delta_1 \implies \rho(g(y), (g \circ f)(x)) < \epsilon$$

Also, by the continuity of f at x, there is some $\delta_2 > 0$ such that

$$z \in X \text{ and } d(z, x) < \delta_2 \implies \delta(f(z), f(x)) < \delta_1$$

Finally, if $z \in X$ and $d(z, x) < \delta_2$, we have

$$\delta(f(z), f(x)) < \delta_1$$

which implies that

$$\rho((g \circ f)(z), (g \circ f)(x)) < \epsilon$$

and by the definition of continuity, it follows that $g \circ f : X \to Z$ is continuous at x.

(2). Let $\{x_n\}$ be a sequence in X, and let $x \in X$. We will show that $\{x_n\}$ converges to x if and only if $d(x_n, x) \to 0$ as $n \to \infty$.

First, suppose $\{x_n\}$ converges to x. Then, given $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that

$$n \geq N \implies d(x_n, x) < \epsilon$$

and hence for all $n \geq N$, we have

$$|d(x_n, x) - 0| < \epsilon$$

and by the definition of convergence, it follows that

$$\lim_{n \to \infty} d(x_n, x) = 0$$

Conversely, suppose $d(x_n, x) \to 0$ as $n \to \infty$. Then, given any $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that

$$n \geq N \implies d(x_n, x) < \epsilon$$

and by the definition of convergence, it follows that $\{x_n\}$ converges to x.

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(3). Suppose $x_0 \in X$ is fixed. We will show that the function $f : X \to \mathbb{R}$ given by $f(x) = d(x, x_0)$ is continuous. So, suppose $x \in X$, and let $\epsilon > 0$ be given. Set $\delta = \epsilon$. Then, if $y \in X$ such that $d(x, y) < \epsilon$, the triangle inequality of metrics imply the following two inequalities:

$$d(y, x_0) \leq d(x, x_0) + d(x, y) < d(x, x_0) + \epsilon$$
$$d(x, x_0) \leq d(x, y) + d(y, x_0) < \epsilon + d(y, x_0)$$

and the above two inequalities imply that

$$d(x, x_0) - \epsilon < d(y, x_0) < d(x, x_0) + \epsilon$$

which means that

$$|d(y, x_0) - d(x, x_0)| < \epsilon$$

So, we have found the required δ, and hence by the definition of continuity, it follows that f is continuous at x. But, x was arbitrary, and hence f is continuous on X.

For problems **(4)** and **(5)**, let V be an inner-product space over K, where $K \in \{\mathbb{R}, \mathbb{C}\}$.

(4). We will prove the Cauchy-Schwarz inequality:

$$|\langle a, b\rangle| \leq ||a||\,||b||$$

First, we will prove two lemmas:

Lemma 1: For any $a \in V$, we have

$$\langle a, O\rangle = 0$$

where O is the zero-vector in V.

Proof: We have

$$\langle a, O\rangle = \langle a, O + O\rangle = \langle a, O\rangle + \langle a, O\rangle$$

and by adding $-\langle a, O\rangle$ to both sides, we get the result.

Lemma 2: Suppose V is an inner-product space over K, and let $a, b \in V$ such that $\langle a, b\rangle = 0$. Then

$$||a + b||^2 = ||a||^2 + ||b||^2$$

Proof: We have

$$\begin{aligned} ||a + b||^2 &= \langle a + b, a + b\rangle \\ &= \langle a + b, a\rangle + \langle a + b, b\rangle \\ &= \langle a, a\rangle + \langle b, a\rangle + \langle a, b\rangle + \langle b, b\rangle \\ &= ||a||^2 + \overline{\langle a, b\rangle} + \langle a, b\rangle + ||b||^2 \\ &= ||a||^2 + ||b||^2 \end{aligned}$$

Now, we will prove the Cauchy-Schwarz inequality:

Proof: Let $a \in V$. First, suppose $e \in V$ is a *unit* vector, i.e suppose

$$||e|| = \langle e, e\rangle^{\frac{1}{2}} = 1$$

Let $c = \langle a, e\rangle$. Consider the two vectors $a - ce$ and ce. Observe that

$$\begin{aligned}\langle a - ce, ce\rangle &= \langle a, ce\rangle + \langle -ce, ce\rangle \\ &= \overline{c}\langle a, e\rangle + (-c)(\overline{c})\langle e, e\rangle \\ &= c\overline{c} - c\overline{c} \\ &= 0\end{aligned}$$

So, by **Lemma 2**, we have

$$||a||^2 = ||ce||^2 + ||a - ce||^2$$

which implies that

$$||a||^2 \geq ||ce||^2 = |c|^2|\,||e||^2 = |c|^2$$

and hence

$$|\langle a, e\rangle| = |c| \leq ||a|| = ||a||\;||e||$$

Now, let $a, b \in V$ be arbitrary vectors in V. If $||b|| = 0$, then $b = O$ (zero-vector), and $|\langle a, b\rangle| = 0$ by **Lemma 1**, and hence the inequality is trivial. So, suppose $||b|| \neq 0$. Consider the vector $\dfrac{b}{||b||}$. Observe that

$$\left|\left|\frac{b}{||b||}\right|\right| = \left\langle \frac{b}{||b||}, \frac{b}{||b||}\right\rangle^{\frac{1}{2}} = \frac{1}{||b||}\langle b, b\rangle^{\frac{1}{2}} = 1$$

and hence as we proved above, we have

$$\left|\left\langle a, \frac{b}{||b||}\right\rangle\right| \leq ||a||$$

which implies that

$$\frac{1}{||b||}\,|\langle a, b\rangle| \leq ||a||$$

and hence we have

$$|\langle a, b\rangle| \leq ||a||\,||b||$$

and hence the proof is complete. (In the last step, we used the fact that $\overline{||b||} = ||b||$, which is true because $||b||$ is a real number).

(5). We will show that $||\cdot||$ is a norm on V. Let's verify the norm properties:

(i) Suppose $||x|| = 0$. Then, it follows that $\langle x, x\rangle^{\frac{1}{2}} = 0$, which means that $\langle x, x\rangle = 0$, and hence $x = O$ (by inner-product axioms). Conversely, if $x = O$, then we have $\langle x, x\rangle^{\frac{1}{2}} = 0$ by **Lemma 1**, and hence $||x|| = 0$.

(ii) Suppose $\lambda \in K$. For any $x \in V$, we have

$$||\lambda x|| = \langle \lambda x, \lambda x\rangle^{\frac{1}{2}} = (\lambda\overline{\lambda}\langle x, x\rangle)^{\frac{1}{2}} = (|\lambda|^2\langle x, x\rangle)^{\frac{1}{2}} = |\lambda|\langle x, x\rangle^{\frac{1}{2}} = |\lambda|\,||x||$$

(iii) Now, let $x, y \in V$. We have

$$\begin{aligned}
||x+y||^2 &= \langle x+y, x+y\rangle \\
&= \langle x+y, x\rangle + \langle x+y, y\rangle \\
&= \langle x, x\rangle + \langle y, x\rangle + \langle x, y\rangle + \langle y, y\rangle \\
&= ||x||^2 + ||y||^2 + \langle x, y\rangle + \overline{\langle x, y\rangle} \\
&= ||x||^2 + ||y||^2 + 2\mathrm{Re}\langle x, y\rangle \\
&\leq ||x||^2 + ||y||^2 + 2|\langle x, y\rangle| \\
&\leq ||x||^2 + ||y||^2 + 2||x||\,||y|| \qquad \text{(by Cauchy-Schwarz)} \\
&= (||x|| + ||y||)^2
\end{aligned}$$

and hence by taking square roots on both sides, we get

$$||x+y|| \leq ||x|| + ||y||$$

So, $||\cdot||$ is a norm on V.

(6). Before solving the problem, let us prove a lemma:

Lemma 3: Suppose f is a monotonic onto function from $[a, b]$ to $[c, d]$, such that f is riemann integrable on $[a, b]$. Then, f^{-1} is riemann integrable on $[c, d]$, and

$$\int_c^d f^{-1}(t) = bd - ac - \int_a^b f(t)dt$$

Proof: Without loss of generality we will assume that f is monotonic increasing. The proof is similar when f is monotonic decreasing.

First, since f is monotonic increasing, f^{-1} is also monotonic increasing on $[c, d]$. The function $\alpha(x) = x$ on $[c, d]$ is continuous, and hence it follows that the integral

$$\int_c^d f^{-1} d\alpha = \int_c^d f^{-1}(t)dt$$

exists. This shows that f^{-1} is riemann integrable.

Now we show the given equality. Let $\epsilon > 0$ me any number, and let

$$M = \int_c^d f^{-1}(t)dt \quad , \quad N = \int_a^b f(t)dt$$

Then, there is some partition $A_1 = \{c = a_0, a_1, ..., a_n = d\}$ of $[c, d]$ for which

$$|U(A_1, f^{-1}) - M| < \epsilon$$

Also, there is some partition $A_2 = \{a = b_0, ..., b_n = b\}$ of $[a, b]$ for which

$$|N - L(A_2, f)| < \epsilon$$

Now, we consider the refinement $P_1 = A_1 \cup \{f(b_0), ..., f(b_n)\} = \{y_0, ..., y_n\}$ of A_1, and the refinement $P_2 = A_2 \cup \{f^{-1}(a_0), ..., f^{-1}(a_n)\} = \{x_0, ..., x_n\}$ of A_2. Because both f and f^{-1} are monotonic, we have $y_i = f(x_i)$ for every $0 \leq i \leq n$. Also, because they are refinements, we have

$$|U(P_1, f^{-1}) - M| < \epsilon$$

and

$$|N - L(P_2, f)| < \epsilon$$

Because f^{-1} and f are both monotonic increasing, we have

$$\begin{aligned}
U(P_1, f^{-1}) &= \sum_{i=1}^{n} f^{-1}(y_i)[y_i - y_{i-1}] \\
&= \sum_{i=1}^{n} x_i[f(x_i) - f(x_{i-1})] \\
&= \sum_{i=1}^{n} x_i f(x_i) - \sum_{i=1}^{n} f(x_{i-1})x_i \\
&= \sum_{i=2}^{n} f(x_{i-1})[x_{i-1} - x_i] + f(x_n)x_n - f(x_0)x_1 \\
&= f(x_n)x_n - f(x_0)x_1 - \sum_{i=2}^{n} f(x_{i-1})[x_i - x_{i-1}] \\
&= f(x_n)x_n - f(x_0)x_1 + f(x_0)[x_1 - x_0] - \sum_{i=1}^{n} f(x_{i-1})[x_i - x_{i-1}] \\
&= f(x_n)x_n - f(x_0)x_0 - L(P_2, f) \\
&= bd - ac - L(P_2, f)
\end{aligned}$$

So, it follows that

$$\begin{aligned}
|M - (bd - ac - N)| &= |M - U(P_1, f^{-1}) + (bd - ac - L(P_2, f)) - (bd - ac - N)| \\
&\leq |M - U(P_1, f^{-1})| + |N - L(P_2, f)| \\
&< 2\epsilon
\end{aligned}$$

and since $\epsilon > 0$ was arbitrary, it follows that

$$M = bd - ac - N$$

which is the desired equality.

Now, we prove the inequality. Let $f : [0, r) \to [0, \infty)$ be a continuous and strictly increasing function with $f(0) = 0$, and we allow r to be ∞. Let b be in $f([0, r))$. Define the function $g : [a, r) \to \mathbb{R}$ by

$$g(a) = \int_0^a f(t)dt + \int_0^b f^{-1}(t)dt - ab$$

By the first fundamental theorem of calculus, the first integral is differentiable on $[0, r)$ (because f is continuous), and we get

$$g'(a) = f(a) - b$$

Let $u = f^{-1}(b)$. Since f is monotonic increasing, we see that $g'(a) > 0$ for $a > u$, $g'(a) = 0$ for $a = u$ and $g'(a) < 0$ for $a < u$. This means that g attains a global minumum at the point $a = u$.

Now, using **Lemma 3**, we get

$$g(u) = \int_0^u f(t)dt + \int_0^b f^{-1}(t)dt - ub = 0$$

and hence the global minimum of g is 0. So, it follows that $g(a) \geq 0$ for all $a \in [0, r)$, and hence

$$ab \leq \int_0^a f(t)dt + \int_0^b f^{-1}(t)dt$$

(7). Define the function

$$f(u) = \frac{u^p}{p} + \frac{v^q}{q} - uv$$

for $u \in [0, \infty)$, where v, p, q are fixed such that $v \in [0, \infty)$, and $p, q \in (1, \infty)$ are Holder conjugates. Observe that

$$f'(u) = u^{p-1} - v$$

and that

$$f''(u) = (p-1)u^{p-2} > 0 \quad \forall u \in [0, \infty)$$

because $p > 1$. So, if $u_0 \in [0, \infty)$ is the point where $f'(u_0) = 0$, then $f(u_0)$ is the global minimum of f. This point u_0 satisfies

$$f'(u_0) = u_0^{p-1} - v = 0$$

and hence

$$u_0 = v^{\frac{1}{p-1}}$$

Observe that

$$f(u_0) = f(v^{\frac{1}{p-1}}) = \frac{v^{\frac{p}{p-1}}}{p} + \frac{v^q}{q} - v^{1+\frac{1}{p-1}} = v^q\left(\frac{1}{p} + \frac{1}{q}\right) - v^q = 0$$

and hence it follows that 0 is the global minimum of f. So, $f(u) \geq 0$ for all $u \in [0, \infty)$, and hence

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}$$