ANALYSIS-2 , HW-9(ADDITIONAL PROBLEMS)

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Throughout, let Γ be an orthogonal $n \times n$ matrix, U be an open set in $\mathbb{R}^n$ and $V = \Gamma^t(U)$. Since Γ is an isomorphism, this means $U = \Gamma(V)$. For a function $f : U \to \mathbb{R}$, let $f_\Gamma : V \to \mathbb{R}$ be the map $f_\Gamma := f \circ \Gamma$. Similarly, for $f = (f_1, ..., f_k) : U \to \mathbb{R}^k$, $f_\Gamma := f \circ \Gamma$.

6. Let f be $\mathscr{C}^1$. We show that $a \in V$ is a critical point of f_Γ if and only if Γa is a critical point of f.

First, observe that f_Γ, being a composite of $\mathscr{C}^1$ maps, is $\mathscr{C}^1$ on V, and by the chain rule

$$Df_\Gamma = Df \circ D\Gamma = Df \circ \Gamma$$

From this equation, the result clearly follows.

7. Let $m \leq n$ be non-negative integers, $h : U \to \mathbb{R}^m$ be a $\mathscr{C}^1$ map, $c \in h(U)$, M the level set $h^{-1}(c)$ satisfying the condition $\text{rank}[h'(x)] = m$ for all $x \in M$. Let $M_\Gamma = \Gamma^t(M)$.

(a) We first show that $M_\Gamma = h_\Gamma^{-1}(c)$. Suppose $x \in M_\Gamma$, implying that $\Gamma(x) \in M$, and hence $x \in V$ and $h(\Gamma(x)) = c$, implying that $x \in h_\Gamma^{-1}(c)$. Conversely, suppose $x \in h_\Gamma^{-1}(c)$, implying that $h(\Gamma(x)) = c$, and hence $\Gamma(x) \in M$, and hence $x \in \Gamma^{-1}(M) = M_\Gamma$. This shows that $M_\Gamma = h_\Gamma^{-1}(c)$.

Next, we show that $\text{rank}[h_\Gamma(x)] = m$ for all $x \in M_\Gamma$. Observe that by the chain rule, we have

$$Dh_\Gamma(x) = Dh(\Gamma(x)) \circ \Gamma$$

If $x \in M_\gamma$, then $\Gamma(x) \in M$, and hence $Dh(\Gamma(x))$ has rank m, and since Γ is an invertible matrix, we see that $\text{rank}[Dh(\Gamma(x)) \circ \Gamma] = \text{rank}[Dh(\Gamma(x))] = m$ (or in simple words, rank is invariant by multiplication with an invertible matrix). The converse also follows from the same equation, and this completes the proof.

(b) We show that $a \in M_\Gamma$ is a critical point of $f_\Gamma|_{M_\Gamma}$ if and only Γa is a critical point of $f|_M$. We assume that $a \in M_\Gamma$ in the following proof. Also, let Γ^i be the i^{th} column of Γ, for $1 \leq i \leq n$.

First, suppose a is a critical point of $f_\Gamma|_{M_\Gamma}$. Then, we know that

$$\nabla f_\Gamma(a) = \sum_{i=1}^{m} \lambda_i \nabla h_{\Gamma,i}(a)$$

where $\lambda_i \in \mathbb{R}$ are the Lagrange multipliers, and $h_{\Gamma,i}$ are the component functions of h_Γ. The above equation corresponds to the equations

$$D_i f_\Gamma(a) = \sum_{j=1}^{m} \lambda_j D_i h_{\Gamma,j}(a)$$

for each $1 \leq i \leq n$. Now, by the chain rule, we know that

$$D_i f_\Gamma(a) = \langle \nabla f(\Gamma a), \Gamma^i \rangle$$

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and also

$$D_i h_{\Gamma,j}(a) = \langle \nabla h_j(\Gamma a), \Gamma^i \rangle$$

and hence we see that

$$\langle \nabla f(\Gamma a), \Gamma^i \rangle = \left\langle \sum_{j=1}^{m} \lambda_j \nabla h_j(\Gamma a), \Gamma^i \right\rangle$$

and since this equation is true for every $1 \leq i \leq n$, we have that

$$\nabla f(\Gamma a) = \sum_{j=1}^{m} \lambda_j \nabla h_j(\Gamma a)$$

implying that Γa is a critical point of $f|_M$. The converse easily follows by reversing these arguments.

Remark 0.0.1. What we have shown in this proof is that if we change our coordinate system, there is no change on the Lagrange Multipliers and the behaviour of points of local maxima-minima.

8. Let $f(x, y, z) = \frac{1}{\sqrt{2}}(x - y)$ and let

$$h(x, y, z) = (3x^2 + 3y^2 + 2z^2 + 2\sqrt{2}zx + 2\sqrt{2}yz - 2xy, 2yz + 2zx - x + y)$$

and we wish to maximise f subject to the constraints $h = (16, 0)$.

We use the results of problem **7.** here. Note that the first component function of h is a quadratic form, which we can convert to a suitable diagonal form, i.e suppose Γ is the orthogonal matrix for which $\Gamma^t Q \Gamma$ is a diagonal matrix, and we will just work with the functions f_Γ and h_Γ. Clearly, both f and h are $\mathscr{C}^\infty$, so the Lagrange Mutliplier method is bound to work if we can ensure that the rank of h_Γ is 2.

In our case, we have

$$Q = \begin{bmatrix} 3 & -1 & \sqrt{2} \\ -1 & 3 & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & 2 \end{bmatrix}$$

so the eigenvalues of Q are $4, 4, 0$ and the corresponding unit eigenvectors are

$$\left(\frac{\sqrt{2}}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}}\right), \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(\frac{-1}{2}, \frac{-1}{2}, \frac{1}{\sqrt{2}}\right)$$

and hence the matrix Γ is

$$\Gamma = \begin{bmatrix} \frac{\sqrt{2}}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{-1}{2} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{2} \\ \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

So, we get that f_Γ and h_Γ are have the following formulas.

$$f_\Gamma(x, y, z) = \frac{x}{\sqrt{3}} - y$$

Might as well just to this problem using normal Lagrange Multipliers.

9. Let C be the ellipse given in problem **4.** We will compute its center. C is the intersection of the curves $x^2 + y^2 = z^2$ (a cone) and $x - 2z = -5$ (a plane).

First, observe that $(-5, 0, 0)$ lies on the plane. So, we shift the origin to $(-5, 0, 0)$, and observe that

$$(x', y', z') = (x + 5, y, z)$$

where the new coordinates are (x', y', z'). So, the new equations are

$$x' - 2z' = 0$$

$$(x' - 5)^2 + y'^2 = z'^2$$

and clearly the plane passes through the new origin. For a brief moment, we will replace the notation (x', y', z') by (x, y, z) and work with the new equations. Observe that $(1, 0, -2)$ is a normal to the plane, and the plane is spanned by the vectors $(2, 0, 1)$ and $(0, 1, 0)$. So, consider the unit vectors

$$\left(\frac{2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}}\right), (0, 1, 0), \left(\frac{1}{\sqrt{5}}, 0, \frac{-2}{\sqrt{5}}\right)$$

Consider the orthogonal matrix Γ given by

$$\Gamma = \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & \frac{-2}{\sqrt{5}} \end{bmatrix}$$

Now, view Γ as a change of basis matrix, where Γ changes coordinates from the axes given above to the standard axes. Let (x', y', z') be the coordinates w.r.t the above axes. Then, the equation of the plane is simply

$$z' = 0$$

If Γ_i is the i^{th} row of Γ, then the new equation of the cone becomes

$$(\Gamma_1 \cdot (x', y', z') - 5)^2 + (\Gamma_2 \cdot (x', y', z'))^2 = (\Gamma_3 \cdot (x', y', z'))^2$$

Putting $z' = 0$, we get the equation

$$3x'^2 + 5y'^2 - 20\sqrt{5}x' + 125 = 0$$

and the center of this ellipse in the new system is

$$\left(\frac{10\sqrt{5}}{3}, 0, 0\right)$$

So, the center with respect to the original axis is

$$\Gamma^{-1}\left(\frac{10\sqrt{5}}{3}, 0, 0\right) = \Gamma\left(\frac{10\sqrt{5}}{3}, 0, 0\right)$$

because Γ turns out to be symmetric. So, the center is the point

$$\left(\frac{20}{3}, 0, \frac{10}{3}\right)$$

So, shifting the origin back to $(0, 0, 0)$, we see that the center is

$$\left(\frac{20}{3} - 5, 0, \frac{10}{3}\right) = \frac{5}{3}(1, 0, 2)$$