ANALYSIS-2 , HW-7

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1. Suppose V is an inner-product space over $\mathbb{C}$. Let $T : V \to V$ be an operator. We will show that T is unitary if and only if T preserves norms.

First, suppose T is unitary. Then, we know that T preserves inner-products. So, if $v \in V$, then

$$\langle Tv, Tv\rangle = \langle v, v\rangle$$

and hence it follows that T preserves norms as well. This proves one direction.

For the other direction, suppose T preserves norms. Let $v, w \in V$. Then, we know that

$$\begin{aligned}\langle T(v-w), T(v-w)\rangle &= \langle v-w, v-w\rangle \\ &= \langle v-w, v\rangle - \langle v-w, w\rangle \\ &= \langle v, v\rangle - \langle w, v\rangle - \langle v, w\rangle + \langle w, w\rangle\end{aligned}$$

Also, by the linearity of T, we have

$$\begin{aligned}\langle T(v-w), T(v-w)\rangle &= \langle Tv - Tw, Tv - Tw\rangle \\ &= \langle Tv - Tw, Tv\rangle - \langle Tv - Tw, Tw\rangle \\ &= \langle Tv, Tv\rangle - \langle Tw, Tv\rangle - \langle Tv, Tw\rangle + \langle Tw, Tw\rangle \\ &= \langle v, v\rangle - \langle Tw, Tv\rangle - \langle Tv, Tw\rangle + \langle w, tw\rangle\end{aligned}$$

and hence we get

$$\langle v, w\rangle + \langle w, v\rangle = \langle Tv, Tw\rangle + \langle Tw, Tv\rangle$$

which implies that

$$\langle v, w\rangle + \overline{\langle v, w\rangle} = \langle Tv, Tw\rangle + \overline{\langle Tv, Tw\rangle}$$

and hence we see that

$$\mathrm{Re}\langle Tv, Tw\rangle = \mathrm{Re}\langle v, w\rangle$$

where $v, w \in V$ are arbitrary. To show that the imaginary parts are also equal, we have

$$\begin{aligned}\mathrm{Im}\langle Tv, Tw\rangle &= \mathrm{Re}[-i\langle Tv, Tw\rangle] \\ &= \mathrm{Re}\langle Tv, T(iw)\rangle \\ &= \mathrm{Re}\langle v, iw\rangle \\ &= \mathrm{Re}[-i\langle v, w\rangle] \\ &= \mathrm{Im}\langle v, w\rangle\end{aligned}$$

and hence the above equations imply that

$$\langle Tv, Tw\rangle = \langle v, w\rangle$$

which means that T is unitary. This completes the proof.

Date: March 2020.

2. Let U_n be the set of unitary matrices in $M_n(\mathbb{C})$. We will show that U_n is a compact subset of $M_n(\mathbb{C})$ by showing that it is closed and bounded (note that $M_n(\mathbb{C}) \cong \mathbb{R}^{2n^2}$ and hence the Heine-Borel theorem applies).

Consider the map $f : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ given by

$$f(A) = \overline{A^t}A$$

First, we claim that f is continuous (the default norm on $M_n(\mathbb{C})$ is the operator norm). To prove this, observe that the entries of $\overline{A}^t A$ are polynomials of the entries of A (and their conjugates), and hence each entry is a continuous function of the entries of A, and hence it follows that $\overline{A^t}A$ is a continuous function.

Now, observe that $U_n = f^{-1}(I_n)$. Now, $\{I_n\}$ being a singleton set in $M_n(\mathbb{C})$ is closed. Since f is continuous, it follows that U_n is also closed in $M_n(\mathbb{C})$.

Next, we will show that U_n is bounded. Suppose $A \in U_n$. We know that A preserves norms, i.e

$$||Ax|| = ||x||$$

for any $x \in \mathbb{C}^n$. So, if S is the unit sphere in $\mathbb{C}^n$, then we see that

$$\sup_{x \in S} ||Ax|| = 1$$

and hence $|A| = 1$. This means that $|A| < 2$ for all $A \in U_n$, and hence U_n is a bounded set. So, U_n is closed and bounded, and hence it is compact.

For the following problems, suppose $n = d + m$. Let $S = \{j_1, ..., j_m\}$ be a subset of $\{1, .., n\}$, where $j_1 < ... < j_m$. Let $\{i_1, ..., i_d\}$ be its complement, where $i_1 < ... < i_d$. Consider the automorphism $\Theta : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\Theta(x_1, ..., x_n) = (x_{i_1}, ..., x_{i_d}, x_{j_1}, ..., x_{j_m})$$

Let $\pi : \mathbb{R}^n \to \mathbb{R}^d$ and $\pi' : \mathbb{R}^n \to \mathbb{R}^m$ be the usual projections. Let U be an open subset of $\mathbb{R}^n$, and let $U_S = \Theta_S(U)$. Let $\Phi : U \to \mathbb{R}^m$ be a $\mathscr{C}^1$ map, and suppose $C_1, ..., C_n$ are the columns of $J\Phi$. Fix $p = (p_1, ..., p_n) \in U$, and let $c = \Phi(p)$, and let $M = \Phi^{-1}(c)$. Let $a_S = (p_{i_1}, ..., p_{i_d}) \in \mathbb{R}^d$ and $b_S = (p_{j_1}, ..., p_{j_m}) \in \mathbb{R}^m$.

3. Suppose $C_{j_1}(p), ..., C_{j_m}(p)$ are linearly independent. We will show that there is an open subset W_S in $\mathbb{R}^d$ of a_S and a unique $\mathscr{C}^1$ map $g_S : W_S \to \mathbb{R}^n$ *taking values in* M such that $g_S(a_S) = p$ and that $\pi \circ \Theta_S \circ g_S$ is the identity mapping on W_S.

First, consider the function $\Phi_S : U_S \to \mathbb{R}^m$ defined by $\Phi_S = \Phi \circ \Theta_S^{-1}$. Observe that Φ_S is a composition of two $\mathscr{C}^1$ maps (Θ_S^{-1} is $\mathscr{C}^1$ because it is a linear map on $\mathbb{R}^n$), and hence Φ_S is also $\mathscr{C}^1$. Also, at any point $x \in U_S$, we have that

$$(J\Phi_S)(x) = (J\Phi)(\Theta_S^{-1}(x))M(\Theta_S^{-1})$$

where in the above equation $M(\Theta_S^{-1})$ is the matrix of the map Θ_S^{-1}. Clearly, $M(\Theta_S^{-1})$ is a permutation matrix (since the underlying operator permutes the coordinates of a vector). Now, suppose $C_1(p), ..., C_n(p)$ are the columns of $(J\Phi)(p)$. Then, the columns of the matrix $(J\Phi)(p)M(\Theta_S^{-1})$ will be $C_{i_1}, ..., C_{i_d}, C_{j_1}, ..., C_{j_m}$ (because right multiplication by a permutation matrix permutes the columns according to the inverse permutation).

Now, consider the point $(a_S, b_S) \in U_S$. By definition, we know that $\Theta_S^{-1}(a_S, b_S) = p$. Also, by the above discussion, we see that

$$(J\Phi_S)(a_S, b_S) = (J\Phi)(p)M(\Theta_S^{-1})$$

and hence the columns of $(J\Phi_S)(a_S, b_S)$ are $C_{i_1}(p), ..., C_{i_d}(p), C_{j_1}(p), ..., C_{j_m}(p)$. Now, it is given that $C_{j_1}(p), ..., C_{j_m}(p)$ are linearly independent, and hence it follows that $\dfrac{\partial \Phi_S}{\partial y}(a_S, b_S)$ is invertible, and hence the implicit function theorem can be applied to Φ_S at the point (a_S, b_S).

Now, we have $\Phi_S(a_S, b_S) = \Phi(p) = c$. So, there is an open neighborhood W_S of a_S in $\mathbb{R}^d$, and a unique $\mathscr{C}^1$ map $f : W_S \to \mathbb{R}^m$ such that $f(a_S) = b_S$, $(x, f(x)) \in U_S$ for all $x \in W_S$, and $\Phi_S(x, f(x)) = c$ for all $x \in U_S$.

Now, define the map $g_S : W_S \to \mathbb{R}^n$ by

$$g_S(x) = \Theta_S^{-1}(x, f(x))$$

Observe that g_S can be written as a composition

$$g_S = \Theta_S^{-1} \circ q_S$$

where $q_S : W_S \to \mathbb{R}^n$ is given by $q_S(x) = (x, f(x))$. Since the components of q_S are $\mathscr{C}^1$, it follows that q_S is $\mathscr{C}^1$, and hence it follows that g_S is also $\mathscr{C}^1$.

First, if $x \in W_S$, then we have

$$\Phi(g_S(x)) = \Phi(\Theta_S^1(x, f(x))) = \Phi_S(x, f(x)) = c$$

by the definition of f, and hence $g_S(x) \in M$. So, g takes values in M. Also, we have

$$g_S(a_S) = \Theta_S^{-1}(a_S, b_S) = p$$

Finally, for any $x \in W_S$, we have

$$\pi \circ \Theta_S \circ g_S(x) = \pi \circ \Theta_S(\Theta_S^{-1}(x, f(x))) = \pi(x, f(x)) = x$$

and hence $\pi \circ \Theta_S \circ g_S$ is the identity mapping on W_S.

Finally, we prove the uniqueness of g_S. So, suppose $Q_S : W_S \to \mathbb{R}^n$ is another $\mathscr{C}^1$ function taking values in M such that $Q_S(a_S) = p$ and $\pi \circ \Theta_S \circ Q_S$ is the identity mapping on W_S. Let $T_S = \Theta_S \circ Q_S$, so that T_S is a map from W_S to $\mathbb{R}^n$. We write T_S as $T_S = (A_S, B_S)$, where $A_S : W_S \to \mathbb{R}^d$ and $B_S : W_S \to \mathbb{R}^m$ (we have just decomposed T_S to its component functions). Since Θ_S is an isomorphism, we have that

$$Q_S = \Theta_S^{-1} \circ T_S$$

Now, we know that $\pi \circ \Theta_S \circ Q_S$ is the identity map, which means that $\pi \circ \Theta_S \circ \Theta_S^{-1} \circ T_S$ is the identity mapping, which means that $\pi \circ T_S = A_S$ is the identity mapping. Hence, $T_S(x) = (x, B_S(x))$ for all $x \in W_S$. Also, since Q_S takes values in M, we see that for any $x \in W_S$,

$$\Phi(Q_S(x)) = \Phi(\Theta_S^{-1}(x, B_S(x))) = \Phi_S(x, B_S(x)) = c$$

and hence $B_S(x)$ is another implicit solution of $\Phi_S(x, y) = c$ on W_S, with $B_S(a_S) = b_S$. But from the implicit function theorem, we know that the implicit solution is unique, and hence it follows that $B_S(x) = f(x)$ on W_S, implying that $B_S = f$. So, this means that $Q_S(x) = \Theta_S^{-1}(x, f(x)) = g_S(x)$, and hence the uniqueness of g_S follows. This completes the proof.

4. It is easy to show that g_S is one-one. Suppose $g_S(x_1) = g_S(x_2)$, for some $x_1, x_2 \in W_S$. Then, we have

$$\Theta_S^{-1}(x_1, f(x_1)) = \Theta_S^{-1}(x_2, f(x_2))$$

and since Θ_S^{-1} is an isomorphism, we see that $(x_1, f(x_1)) = (x_2, f(x_2))$, implying that $x_1 = x_2$, so that g_S is one-one.

To prove the second part of the problem, we will follow a strategy very similar to the proof of the implicit function theorem from the inverse function theorem. Define the function $\Psi : U_S \to \mathbb{R}^n$ given by

$$\Psi(x, y) = (x, \Phi_S(x, y))$$

Then, at the point $(a_S, b_S) \in U_S$, the derivative of Ψ, which is nothing but

$$(J\Psi)(a_S, b_S) = \begin{bmatrix} I & 0 \\ \dfrac{\partial \Phi_S}{\partial x}(a_S, b_S) & \dfrac{\partial \Phi_S}{\partial y}(a_S, b_S) \end{bmatrix}$$

is invertible (because by hypothesis the matrix $\dfrac{\partial \Phi_S}{\partial y}(a_S, b_S))$ is invertible. So, by the inverse function theorem, there is an open set U_0 in $\mathbb{R}^n$ containing the point (a_S, b_S), and an open set W in $\mathbb{R}^n$ containing $\Psi(a_S, b_S) = (a_S, c)$, such that $\Psi : U_0 \to W$ is bijective, and the inverse map $\Psi^{-1} : W \to U_0$ is $\mathscr{C}^1$. Consider the decomposition $W = W_S \times W'_S$, where $W_S \subset \mathbb{R}^d$ and $W'_S \subset \mathbb{R}^m$ are open sets (this is the W_S which was guaranteed by the implicit function theorem in problem **3.**).

Then, consider the map Ψ restricted to the set $\Theta_S(M) \cap U_0$, i.e consider $\Psi|_{\Theta_S(M)\cap U_0}$ (this set is non-empty because $(a_S, b_S) \in \Theta_S(M) \cap U_0$). We will show that the image of this restriction map is contained in $W_S \times \{c\}$. To prove this, suppose $(x, y) \in \Theta_S(M) \cap U_0$. So, it follows that $\Theta_S^{-1}(x, y) \in M$, and hence $\Phi(\Theta_S^{-1}(x, y)) = c = \Phi_S(x, y)$, and hence

$$\Psi(x, y) = (x, \Phi_S(x, y)) = (x, c) \in W_S \times \{c\}$$

which proves that the image is contained in $W_S \times \{c\}$. Regard $W_S \times \{c\}$ as an open set in itself, and since Ψ is continuous, it follows that $V = \Psi^{-1}(W_S \times \{c\})$ is open in $\Theta_S(M) \cap U_0$. But, $\Theta_S(M) \cap U_0$ is open in $\Theta_S(M)$ (because U_0 is open in $\mathbb{R}^n$), and hence V is open in $\Theta_S(M)$.

Let $N = \{(x, f(x)) | x \in W_S\}$. We claim that $N = V$. To show this, suppose $(x, f(x)) \in N$, where $x \in \mathbb{R}^d$ and $f(x) \in \mathbb{R}^m$. Then, $\Psi(x, f(x)) = (x, \Phi_S(x, f(x))) = (x, c) \in W_S \times \{c\}$, and hence $(x, f(x)) \in V$. Conversely, suppose $(x, y) \in V$. Then, it follows that $\Psi(x, y) = (x, \Phi_S(x, y)) \in W_S \times \{c\}$, which implies that $x \in W_S$ and $\Phi_S(x, y) = c = \Phi_S(x, f(x))$. Since f was *unique*, it follows that $y = f(x)$, and hence $(x, y) \in N$. This proves the claim.

So, it follows that the set N is open in $\Theta_S(M)$. Since Θ_S is an isomorphism, it follows that $\Theta_S^{-1}(N)$ is open in M. But, observe that $\Theta_S^{-1}(N) = g_S(W_S)$, and hence $g_S(W_S)$ is open in M.

Now we are in position to prove that g_S maps open subsets of W_S to open subsets of M. Consider the map $g_S : W_S \to g(W_S)$ (we just made g_S a bijection, because we already showed that g_S was one-one). We know that for any $x \in W_S$, $g_S(x) = \Theta_S^{-1}(x, f(x))$, and hence the inverse mapping $g_S^{-1} = \chi : g_S(W_S) \to W_S$ is given by $\chi(x, y) = \pi(\Theta_S(x, y))$, so that $\chi = \pi \circ \Theta_S$, and hence χ is a continuous map.

Now, let K be an open subset of W_S. Then, we have that $g_S(K) = \chi^{-1}(K)$, and since χ is continuous and K is open in W_S, it follows that $g_S(K)$ is open in $g_S(W_S)$. But, $g_S(W_S)$ is open in M, and hence $g_S(K)$ is open in M. This completes the proof.

5. Suppose $C_{j_1}(p), ..., C_{j_m}(p)$ are linearly independent, and suppose $\{C_j(p) | j \in T\}$ are also linearly independent. Let g_S and g_T be as in problem **3.** So, we see that

$$g_S(x) = \Theta_S^{-1}(x, f_S(x))$$

for any $x \in W_S$, and

$$g_T(x) = \Theta_T^{-1}(x, f_T(x))$$

for any $x \in W_T$ (here f_S is the implicit solution for Φ_S as in problem **3.**, and similarly f_T is the implicit solution for Φ_T).

Let V be the open subset of M given by $V = g_S(W_S) \cap g_T(W_T)$. Let V_S and V_T be the open subsets of $\mathbb{R}^d$ given by $V_S = g_S^{-1}(V)$ and $V_T = g_T^{-1}(V)$. We will show that the map

$$(g_T|_{V_T})^{-1} \circ (g_S|_{V_S}) : V_S \to V_T$$

is $\mathscr{C}^1$. Observe that g_S was $\mathscr{C}^1$ on W_S, and hence it is $\mathscr{C}^1$ on $V_S \subset W_S$, and hence the restriction map $g_S|_{V_S}$ is $\mathscr{C}^1$.

Now, consider the restriction map $(g_T|_{V_T})$. In problem **3.**, we showed that g_S was one-one, and by a similar argument, g_T is also one-one on W_T, and hence it is one-one on $V_T \subset W_T$. In particular, $g_T|_{V_T} : V_T \to V$ is invertible, and consider the inverse map $(g_T|_{V_T})^{-1} : g_T(V_T) \to V_T$ (observe that $g_T(V_T)$ is a subset of V). First, we will show that the composition

$$(g_T|_{V_T})^{-1} \circ (g_S|_{V_S})$$

makes sense, i.e $g_S(V_S) \subset g_T(V_T)$. To show this, observe that $g_S : W_S \to g(W_S)$ is a bijection, and $g_T : W_T \to g(W_T)$ is a bijection (because both are one-one). By definition of V_S and V_T, it follows that $g_S(V_S) = V = g_T(V_T)$, and hence the composition makes sense.

Now, since $g_T(x) = \Theta_T^{-1}(x, f_T(x))$ for any $x \in V_T$, it follows that the inverse map $(g_T|_{V_T})^{-1}$ is given by

$$(g_T|_{V_T})^{-1} = \pi \circ \Theta_T$$

and hence it is $\mathscr{C}^1$, since π and Θ_T are $\mathbb{C}^1$. So, it follows that the composition

$$(g_T|_{V_T})^{-1} \circ (g_S|_{V_S})$$

is also $\mathscr{C}^1$, completing the proof of the claim.