ANALYSIS-2, HW-4

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In the following two problems, let U be an open subset of $\mathbb{R}^n$. Let $f : U \to \mathbb{R}$ be a $\mathscr{C}^1$ function. Let $\gamma : (a, b) \to U$ be a *smooth* path. Then, the velocity of γ at $\theta \in (a, b)$ is defined as

$$v_\gamma(\theta) = \gamma'(\theta) \in \mathbb{R}^n$$

(1). Let us show that

$$\frac{df(\gamma(t))}{dt} = \langle v_\gamma(t), \nabla f(\gamma(t)) \rangle$$

for all $t \in (a, b)$.

At any point $x \in U$, the derivative of f is a $1 \times n$ matrix, given by

$$Df(x) = \begin{bmatrix} D_1 f(x) & D_2 f(x) & ... & D_n f(x) \end{bmatrix}$$

Now, suppose $\gamma = (\gamma_1, ..., \gamma_n)$, where each $\gamma_i : (a, b) \to \mathbb{R}$. The derivative of γ at any point $t \in (a, b)$ is an $n \times 1$ matrix, given by

$$D\gamma(t) = v_\gamma(t) = \begin{bmatrix} D\gamma_1(t) \\ D\gamma_2(t) \\ ... \\ D\gamma_n(t) \end{bmatrix}$$

By the chain rule, for any $t \in (a, b)$ we have

$$\frac{df(\gamma(t))}{dt} = D(f \circ \gamma)(t) = Df(\gamma(t))D\gamma(t)$$

where the right hand side represents matrix multiplication. Now,

$$\begin{aligned} Df(\gamma(t))D\gamma(t) &= \begin{bmatrix} D_1 f(\gamma(t)) & D_2 f(\gamma(t)) & ... & D_n f(\gamma(t)) \end{bmatrix} \begin{bmatrix} D\gamma_1(t) \\ D\gamma_2(t) \\ ... \\ D\gamma_n(t) \end{bmatrix} \\ &= \sum_{i=1}^{n} D_i f(\gamma(t)) D\gamma_i(t) \\ &= \langle v_{\gamma(t)}, \nabla f(\gamma(t)) \rangle \end{aligned}$$

(2). Let S be a hypersurface in U given by the equation $f(x) = c$. We define a function $g : (a, b) \to \mathbb{R}$ as

$$g(t) = f(\gamma(t))$$

We claim that $g(t) = c$ for all $t \in (a, b)$. Since $\gamma(t) \in S$ for all $t \in (a, b)$, we have that

$$g(t) = f(\gamma(t)) = c$$

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for all $t \in (a, b)$. This means that g is a constant function on (a, b), and hence $g'(t) = 0$ for all $t \in (a, b)$. As proved in part **(1)**, this implies that

$$\frac{dg}{dt} = \frac{df(\gamma(t))}{dt} = 0$$

which means that

$$\langle v_\gamma(t), \nabla f(\gamma(t))\rangle = 0$$

for all $t \in (a, b)$. So, $v_\gamma(t)$ is orthogonal to $\nabla f(\gamma(t))$.

In the following two problems, l et U be open in $\mathbb{R}^n$, and let $g : U \to \mathbb{R}^m$ be a $\mathscr{C}^1$ map. Let $c \in \mathbb{R}^m$ be a point in the image of g and M be a subset of U given by the equation $g(x) = c$. Suppose $p \in M$ is a point such that the rank of $g'(p)$ is m. Let V_p be the null space of $g'(p)$. The space $T_p = V_p + p$ is called the tangent space to M at p.

(3). We will show that V_p is the space spanned by all velocity vectors $v_\gamma(p)$ for $\mathscr{C}^1$ paths taking values in M and passing through p.

First, suppose $\gamma : (a, b) \to M$ is a $\mathscr{C}^1$ path passing through p, so that there is some $\theta \in (a, b)$ for which $\gamma(\theta) = p$. Let us show that $\gamma'(\theta)$ (identified as a vector in $\mathbb{R}^n$) is in V_p. Observe that

$$g'(p)\gamma'(\theta) = \begin{bmatrix} g_1'(p) \\ g_2'(p) \\ \cdots \\ g_m'(p) \end{bmatrix} \gamma'(\theta) = \begin{bmatrix} \langle \gamma'(\theta), \nabla g_1(p)\rangle \\ \langle \gamma'(\theta), \nabla g_2(p)\rangle \\ \cdots \\ \langle \gamma'(\theta), \nabla g_m(p)\rangle \end{bmatrix} = 0$$

by problem (2).

Now, we will show that the space spanned by all velocity vectors is actually V_p. Observe that by the rank-nullity theorem, we have

$$\dim V_p = n - m = d$$

We will show that there are d linearly independent velocity vectors passing through p, proving the claim, which will show that V_p is spanned by these velocity vectors.

We are given that the rank of $g'(p)$ is m. Now, the matrix $g'(p)$ is given by

$$g'(p) = \begin{bmatrix} D_1 g(p) & D_2 g(p) & \dots & D_n g(p) \end{bmatrix}$$

So, there are some $\{i_1, ..., i_m\} \subset 1, 2, ..., n$ such that the set $\{D_{i_1} g(p), ..., D_{i_m} g(p)\}$ are linearly independent. Now, g is a function of n variables, denoted by $g(x_1, ..., x_n)$. We may also regard g as a function of n variables, where the variables are ordered as $g(x_{j_1}, ..., x_{j_d}, x_{i_1}, ..., x_{i_m})$. The benefit of this notation is that the partial derivatives with respect to the last m variables form a matrix of rank m, and hence it is invertible. So, without loss of generality, we may assume that the matrix of last m variables has rank m. Let $p = (a, b)$, where $a \in \mathbb{R}^d$ and $b \in \mathbb{R}^m$. Also, suppose $a = (a_1, ..., a_d)$ and $b = (b_1, ..., b_d)$.

So, it follows that the $m \times m$ matrix given by

$$\begin{bmatrix} D_{d+1} g(p) & D_{d+2} g(p) & \dots D_{d+m} g(p) \end{bmatrix}$$

has rank m, and hence it is invertible. So, we can apply the implicit function theorem to g at the point p. Applying the theorem, it follows that there is an open set W containing a in $\mathbb{R}^d$ and a $\mathscr{C}^1$ function $f : W \to \mathbb{R}^m$ such that $f(a) = b$, and that

$(q, f(q)) \in U$ for all $q \in W$, and $g(q, f(q)) = c$ for all $q \in W$, implying that $(q, f(q)) \in M$ for all $q \in W$.

Now, decompose this open subset W of $\mathbb{R}^d$ into the product

$$W = A_1 \times A_2 \times ... \times A_d$$

where each set A_i is open in $\mathbb{R}$ and A_i contains a_i, for $1 \leq i \leq d$. Since each A_i is open, there are open intervals $(p_i, q_i) \subset A_i$ for each $1 \leq i \leq d$ such that (p_i, q_i) contains a_i for each i.

We now define d $\mathscr{C}^1$ paths as follows: let $\gamma_i : (p_i, q_i) \to \mathbb{R}^m$ be such that

$$\gamma_i(t) = (a_1, ..., t, ..., a_d, f(a_1, ..., t, ..., a_d))$$

where t occurs at the i^{th} position above. Clearly, γ_i is a $\mathscr{C}^1$ path for every i, since the component functions are $\mathscr{C}^1$. Now, $(a_1, ..., t, ..., a_d) \in W$ for $t \in (p_i, q_i)$, and hence

$$g(a_1, ..., t, ..., a_d, f(a_1, ..., t, ..., a_d)) = c$$

showing that γ_i is a path taking values in M. Also, it is easy to see that each γ_i passes through p, because

$$\gamma_i(a_i) = (a_1, ..., a_i, ..., a_d, f(a_1, ..., a_i, ..., a_d)) = (a, b) = p$$

Finally, we see that for $t \in (p_i, q_i)$

$$v_{\gamma_i}(t) = \gamma_i'(t) = \left(0, ..., 1, ..., 0, \frac{\partial f}{\partial x_i}(t)\right)$$

and hence

$$v_{\gamma_i}(p) = \gamma_i'(a_i) = \left(0, ..., 1, ..., 0, \frac{\partial f}{\partial x_i}(a_i)\right)$$

Let us show that the vectors $\{v_{\gamma_1}(p), ..., v_{\gamma_d}(p)\}$ are linearly independent. If there are $c_1, ..., c_d \in \mathbb{R}$ such that

$$c_1 v_{\gamma_1}(p) + ... + c_d v_{\gamma_d}(p) = O$$

then it implies that

$$c_1 \left(1, 0, ..., 0, \frac{\partial f}{\partial x_1}(a_1)\right) + ... + c_d \left(0, 0, ..., 1, \frac{\partial f}{\partial x_d}(a_d)\right) = O$$

which implies that

$$c_1 = ... = c_d = 0$$

and hence these velocity vectors span V_p.

(4). Let W be an open set in $\mathbb{R}^n$ and let $f : W \to \mathbb{R}^m$ be a $\mathscr{C}^1$ function. Let

$$\Gamma := \{(x, f(x)) : x \in W\}$$

be the graph of f. Let $p \in W$, and let $A = f'(p)$ and $b = f(p)$.

We define a function $g : W \times R^m \to \mathbb{R}^m$ by the formula

$$g(x, y) = f(x) - y$$

Observe that $W \times \mathbb{R}^m$ is an open subset of R^{n+m}, since W is open in $\mathbb{R}^n$ and $\mathbb{R}^m$ is trivially open in $\mathbb{R}^m$.

We can see that g is $\mathscr{C}^1$, because it is a sum of two $\mathscr{C}^1$ functions. Also, the derivative of g at a point $(x, y) \in W \times \mathbb{R}^m$ is the block matrix

$$g'(x, y) = \begin{bmatrix} Df(x) & -I_m \end{bmatrix}$$

where I_m the the $m \times m$ identity matrix (this is easy to see).

Now, observe that Γ is precisely the set of all points $x \in W \times \mathbb{R}^m$ which satisfy

$$g(x) = 0$$

Now, let $p \in W$, so that $q = (p, f(p)) \in \Gamma$. Let $f'(p) = A$. We have shown that

$$g'(q) = \begin{bmatrix} Df(p) & -I_m \end{bmatrix} = \begin{bmatrix} A & -I_m \end{bmatrix}$$

Also, the rank of $g'(q)$ is clearly m, because it consists of I_m as one of its block matrices. So, the conditions given in problem **(3)** are satisfied by g at the point q.

Now, suppose $(x, y) \in \mathbb{R}^{m+n}$ ($x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$) is in the tangent space $T_q = V_q + q$, where V_q is the null space of $g'(q)$. So, it follows that $(x, y) - q$ is in the null space V_q, which means that $(x - p, y - b) \in V_q$, where $b = f(p)$. This means that $g'(q)(x - p, y - b) = 0$, where this equation represents matrix-multiplication. So,

$$\begin{bmatrix} A & -I_m \end{bmatrix} (x - p, y - b) = 0$$

and hence this implies that

$$A(x - p) - (y - b) = 0$$

and hence

$$y = A(x - p) + b$$