ANALYSIS-2, HW-8

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1. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a polynomial of degree at most m (where the degree of the zero polynomial is $-\infty$). Suppose

$$\lim_{x\to 0} \frac{|f(x)|}{||x||^m} = 0$$

Then we will show that $f(x) = 0$ for all $x \in \mathbb{R}^n$. First, we will show this for one variable, i.e when $n = 1$.

So, suppose $f : \mathbb{R} \to \mathbb{R}$ is a polynomial of degree k, where $0 \le k \le m$ (if f is the zero polynomial then there is nothing to prove), and let

$$f(x) = a_k x^k + ... + a_0$$

We have that

$$\lim_{x\to 0} \frac{|a_k x^k + ... + a_0|}{|x|^m} = 0$$

Let $\epsilon > 0$ be fixed, and the above limit implies that there is some $\delta > 0$ for which

$$|a_k x^k + ... + a_0| \le \epsilon |x|^m$$

for all $0 < |x| < \delta$. Letting $x \to 0$, this shows that $a_0 = 0$.

So, we see that

$$\frac{|f(x)|}{|x|^m} = \frac{|a_k x^k + ... a_1 x|}{|x|^m} = \frac{|a_k x^{k-1} + ... + a_1|}{|x|^{m-1}}$$

Repeating the same procedure again, we will obtain that $a_1 = ... = a_k = 0$ (because $k \le m$), and hence it follows that $f(x) = 0$ for all $x \in \mathbb{R}$.

Now, consider the general case, i.e the case over n variables. Let f be a polynomial in n variables, and without loss of generality suppose the degree of f is m (if the degree of f is less than m) then the solution is still the same. So, we can write f as

$$f(x_1, ..., x_n) = \sum_{k=0}^{m} \sum_{i_1+i_2+...+i_n=k} a_{i_1...i_n} x_1^{i_1} ... x_n^{i_n}$$

where the inner sum ranges over all tuples $(i_1, ..., i_n)$ of non-negative integers with sum k. Also, suppose that

$$\lim_{x\to 0} \frac{|f(x)|}{||x||^m} = 0$$

Observe that the constant term of the polynomial is $f(0) = a_{000...0}$. By the above limit, we have that given any $\epsilon > 0$, there is some $\delta > 0$ such that

$$|f(x)| \le \epsilon ||x||^m$$

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for all $||x|| < \delta$. Letting $x \to 0$, we see that $f(0) = a_{000...0} = 0$ (because f is continuous) and hence the constant term is 0.

Now, the idea is to write f as a sum of homogeneous polynomials, and show that each homogeneous term is zero. So, for $1 \leq k \leq m$, define

$$P_k(x_1, ..., x_n) = \sum_{i_1+...+i_n=k} a_{i_1...i_n} x_1^{i_1}...x_n^{i_n}$$

and because the constant term is 0 we see that

$$f(x_1, ..., x_n) = P_1(x_1, ..., x_n) + ... + P_m(x_1, ..., x_n) = \sum_{k=1}^{m} P_k(x_1, ..., x_n)$$

Now, for any non-zero vector $v = (v_1, ..., v_n)$, define a polynomial $g_v : \mathbb{R} \to \mathbb{R}$ given by

$$g_v(t) = f(tv)$$

Now, observe that the degree of g is atmost m, because the degree of f is m.

$$\lim_{t\to 0} \frac{|g_v(t)|}{|t|^m} = ||v||^m \lim_{t\to 0} \frac{|f(tv)|}{|t^m|||v||^m} = ||v|| \lim_{t\to 0} \frac{|f(tv)|}{||tv||^m} = 0$$

and hence by the one-variable case it follows that $g_v(t) = 0$ for all $t \in \mathbb{R}$.

Moreover, observe that

$$g_v(t) = f(tv) = \sum_{k=1}^{m} P_k(tv_1, ..., tv_n) = \sum_{k=1}^{m} P_k(v)t^k$$

because each P_i is homogeneous. So, this shows that each coefficient $P_1(v) = ... = P_m(v) = 0$, and hence $f(v) = 0$. This shows that $f(x) = 0$ for all $x \in \mathbb{R}^n$.

2. Before doing this problem, we will prove the following lemma:

Lemma: Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a degree-two polynomial. Then, for $1 \leq i < j \leq n$, the functions $D_i i f : \mathbb{R}^n \to \mathbb{R}$ and $D_{ij} f : \mathbb{R}^n \to \mathbb{R}$ are constant functions. *Proof:* f can be written as

$$f(x_1, ..., x_n) = \sum_{i=1}^{n} a_i x_i^2 + \sum_{1\leq i\neq j\leq n} b_{ij}x_i x_j + \sum_{i=1}^{n} c_i x_i + d_0$$

where a_i, b_{ij}, c_i and d_0 are constants, and atleast one of the a_is or one of the b_{ij}s is non-zero. So, for any $1 \leq i < j \leq n$, we have

$$D_{ii} f(x_1, ..., x_n) = 2a_i$$
$$D_{ij} f(x_1, ..., x_n) = b_{ij} + b_{j_i}$$

and hence both of these are constant functions. This proves the **Lemma.**

We now get back to the problem. Suppose f is a degree-two polynomial in n variables, and let $a = (a_1, ..., a_n)$ be a critical point of f. We will show that there is a homogeneous degree two polynomial Q such that

$$f(x) - f(a) = Q(x - a)$$

for all $x \in \mathbb{R}^n$.

Since f is a polynomial, it is $\mathscr{C}^\infty$, and hence Taylor's theorem can be applied to any two points in $\mathbb{R}^n$. In particular, for any $x \in \mathbb{R}^n$, we have that

$$f(x) = \sum_{k=0}^{1} \sum_{i_1+...+i_n=k} \frac{D_1^{i_1}...D_n^{i_n} f(a)(x_1-a_1)^{i_1}...(x_n-a_n)^{i_n}}{i_1!...i_n!} + r(x-a)$$

where the remainder r is given by

$$r(x-a) = \sum_{i_1+...+i_n=2} \frac{D_1^{i_1}...D_n^{i_n} f(a+\theta(x-a))(x_1-a_1)^{i_1}...(x_n-a_n)^{i_n}}{i_1!...i_n!}$$

where $\theta \in [0,1]$. Now, also observe that

$$\sum_{k=0}^{1} \sum_{i_1+...+i_n=k} \frac{D_1^{i_1}...D_n^{i_n} f(a)(x_1-a_1)^{i_1}...(x_n-a_n)^{i_n}}{i_1!...i_n!} = f(a) + \langle \nabla f(a), x-a \rangle = f(a)$$

because $\nabla f(a) = 0$ (a is a critical point). and so we see that

$$f(x) - f(a) = r(x-a)$$

Now, we will show that r is a degree-two homogeneous polynomial, and by setting $Q = r$, the required Q will be found.

Observe that we have

$$\begin{aligned} r(x-a) &= \sum_{i_1+...+i_n=2} \frac{D_1^{i_1}...D_n^{i_n} f(a+\theta(x-a))(x_1-a_1)^{i_1}...(x_n-a_n)^{i_n}}{i_1!...i_n!} \\ &= \sum_{i=1}^{n} \frac{D_{ii} f(a+\theta(x-a))(x_i-a_i)^2}{2!} + \sum_{1\le i<j\le n} D_{ij} f(a+\theta(x-a))(x_i-a_i)(x_j-a_j) \end{aligned}$$

and by the **Lemma**, we know that each $D_{ii}f(a+\theta(x-a))$ and $D_{ij}f(a+\theta(x-a))$ is a constant. So, this shows that $r(x-a)$ is a two-degree homogeneous polynomial in $(x-a)$, which completes the proof.

3. Let f be a polynomial of degree 2 in three variables over $\mathbb{R}$ such that the Hessian H is non-singular. Let c be a constant, and suppose that the equation $f(x,y,z) = c$ has more than one solution. We will show that the surface S defined by the polynomial equation has a unique center.

We write f as

$$f(x,y,z) = Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fzx + Gx + Hy + Iz + J$$

where $A, B, C, D, E, F, G, H, I, J \in \mathbb{R}$ are constants. Now, we see that for any $(x,y,z) \in \mathbb{R}^3$,

$$Jf(x,y,z) = \begin{bmatrix} 2Ax+Dy+Fz+G & 2By+Dx+Ez+H & 2Cz+Ey+Fx+I \end{bmatrix}$$

and

$$Hf(x,y,z) = \begin{bmatrix} 2A & D & F \\ D & 2B & E \\ F & E & 2C \end{bmatrix}$$

Since the Hessian H is non-singular, it follows that there is a unique solution $a = (a_1, a_2, a_3)$ to the system of equations

$$\begin{aligned} 2Ax + Dy + Fz + G &= 0 \\ 2By + Dx + Ez + H &= 0 \\ 2Cz + Ey + Fx + I &= 0 \end{aligned}$$

and hence it follows that f has a unique critical point, which is a.

By problem **2.**, there is a homogeneous polynomial of degree 2 such that

$$f(x) - f(a) = Q(x - a)$$

for all $x \in \mathbb{R}^n$. In particular, if $(x, y, z) \in S$, then we have

$$f(a_1, a_2, a_3) + Q(x_1 - a_1, x_2 - a_2, x_3 - a_3) = c$$

which implies that

$$Q(x_1 - a_1, x_2 - a_2, x_3 - a_3) = c - f(a_1, a_2, a_3)$$

and note that the right hand side is a constant. So, it follows that $a = (a_1, a_2, a_3)$ is a center for S.

Now, suppose there was another center given by $b = (b_1, b_2, b_3)$. So, the equation $f(x, y, z) = c$ can be re-written as

$$P(x-b_1)^2+Q(y-b_2)^2+R(z-b_3)^2+S(x-b_1)(y-b_2)+T(y-b_2)(z-b_3)+U(z-b_3)(x-a_1) = \rho$$

Let the above expression be denoted by $g(x, y, z)$, and hence we see that

$$g(x, y, z) - \rho = f(x, y, z) - c$$

for all $(x, y, z) \in \mathbb{R}^3$ (because we are just rewriting f).

Now, it is easy to calculate that $Jg(b_1, b_2, b_3) = 0$, implying that $Jf(b_1, b_2, b_3) = 0$ (because f and g differ by a constant, their Jacobians are the same), and hence b is a critical point of f. Since f has a unique critical point, it follows that $b = a$, and hence the centre is unique.

4. This problem is solved after problem **6.**

In the following two problems, let $\mathrm{char}(A) = \det(A - tI)$ denote the characteristic polynomial of the matrix A.

5. In this problem, we will find the critical points of the given functions, and we will see if they correspond to points of local maximam, minima or saddle points.

(a) $f(x, y) = x^4 + y^4 - 4xy + 1$

First, we see that f is $\mathscr{C}^\infty$. For any $(x, y) \in \mathbb{R}^2$, we have

$$Jf(x, y) = \begin{bmatrix} 4x^3 - 4y & 4y^3 - 4x \end{bmatrix}$$

So, the critical points of f satisfy

$$4x^3 - 4y = 4y^3 - 4x = 0$$

and hence the critical points are: $(0, 0), (1, 1), (-1, 1)$.

Now, at any point $(x, y \in \mathbb{R}^2)$, the Hessian $Hf(x,y)$ is given by

$$Hf(x,y) = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x \partial y} \\ \dfrac{\partial^2 f}{\partial y \partial x} & \dfrac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{bmatrix}$$

So, it follows that $\det Hf(x,y) = 144x^2y^2 - 16$ and hence all critical points are non-degenerate.

Now, we have that

$$\begin{aligned} \operatorname{char}(Hf(0,0)) &= (t-4)(t+4) \\ \operatorname{char}(Hf(1,1)) &= (t-8)(t-16) \\ \operatorname{char}(Hf(-1,-1)) &= (t-8)(t-16) \end{aligned}$$

and hence we observe that $Hf(0,0)$ is indefinite, and hence 0 is a saddle point. Also, both $Hf(1,1)$ and $Hf(-1,-1)$ are positive definite, and hence both the points $(1,1)$ and $(-1,-1)$ are points of local minima.

(b) $f(x,y) = x^2 + y^2 - 2x - 6y + 14$.

First, we observe that f is $\mathscr{C}^\infty$. For any $(x,y) \in \mathbb{R}^2$, we have

$$Jf(x,y) = \begin{bmatrix} 2x-2 & 2y-6 \end{bmatrix}$$

and hence the only critical point of f is $(1,3)$. Also, for any $(x,y) \in \mathbb{R}^2$, the Hessian is given by

$$Hf(x,y) = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x \partial y} \\ \dfrac{\partial^2 f}{\partial y \partial x} & \dfrac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

and hence the point $(1,3)$ is a non-degenerate critical point. Also one can easily see that $Hf(1,3)$ is positive definite. So, the point $(1,3)$ is a point of local minima for this function.

(c) $f(x,y) = e^{4y-x^2-y^2}$

Again, we observe that f is $\mathscr{C}^\infty$. For any $(x,y) \in \mathbb{R}^2$, we have

$$Jf(x,y) = \begin{bmatrix} -2xe^{4y-x^2-y^2} & (4-2y)e^{4y-x^2-y^2} \end{bmatrix}$$

So, it follows that the only critical point of f is $(0,2)$.

Now, at any point $(x, y \in \mathbb{R}^2)$, the Hessian $Hf(x,y)$ is given by

$$Hf(x,y) = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x \partial y} \\ \dfrac{\partial^2 f}{\partial y \partial x} & \dfrac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 4x^2e^{4y-x^2-y^2} - 2e^{4y-x^2-y^2} & -2x(4-2y)e^{4y-x^2-y^2} \\ -2x(4-2y)e^{4y-x^2-y^2} & (4-2y)^2e^{4y-x^2-y^2} - 2e^{4y-x^2-y^2} \end{bmatrix}$$

So, the Hessian at the point $(0,2)$ is given by

$$Hf(0,2) = \begin{bmatrix} -2e^4 & 0 \\ 0 & -2e^4 \end{bmatrix}$$

and hence $(0,2)$ is a non-degenerate critical point. Also, it is easy to see that $Hf(0,2)$ is negative-definite, and hence $(0,2)$ is a point of local maxima for f.

6. In this exercise we will do the same thing as in exercise **5.**

(a) $f(x,y,z) = 3x^2 + 5y^2 + 3z^2 + 2yz + 2zx + 2xy - 4x - 8z + 5$

First, we observe that f is $\mathscr{C}^\infty$. For any $(x,y,z) \in \mathbb{R}^3$, we have

$$Jf(x,y,z) = \begin{bmatrix} 6x + 2z + 2y - 4 & 2x + 10y + 2z & 6z + 2y + 2x - 8 \end{bmatrix}$$

and hence the critical points of f satisfy the equation:

$$\begin{aligned} 6x + 2z + 2y - 4 &= 0 \\ 2x + 10y + 2z &= 0 \\ 6z + 2y + 2x - 8 &= 0 \end{aligned}$$

and hence the only critical point of f is $x_0 = \frac{1}{3}(1,-1,4)$.

Now, at any point $(x,y,z) \in \mathbb{R}^3$, the Hessian is given by

$$Hf(x,y,z) = \begin{bmatrix} 6 & 2 & 2 \\ 2 & 10 & 2 \\ 2 & 2 & 6 \end{bmatrix}$$

and hence we have that $\det Hf(x_0) = 288$, and hence x_0 is a non-degenerate critical point.

Finally, observe that

$$\text{char} Hf(x_0) = t^3 - 22t^2 + 144t - 288 = (t-4)(t-6)(t-12)$$

and hence $Hf(x_0)$ is positive definite. So, it follows that x_0 is a point of local minima.

(b) $f(x,y,z) = 3x^2 + 7y^2 + 3z^2 + 10yz - 2zx + 10xy + 4x - 12y + 4z + 1$

First, we observe that f is $\mathscr{C}^\infty$. For any $(x,y,z) \in \mathbb{R}^3$, we have

$$Jf(x,y,z) = \begin{bmatrix} 6x - 2z + 10y + 4 & 14y + 10z + 10x - 12 & 6z + 10y - 2x + 4 \end{bmatrix}$$

and hence the critical points of f satisfy the equation:

$$\begin{aligned} 6x - 2z + 10y + 4 &= 0 \\ 10x + 14y + 10z - 12 &= 0 \\ 6z + 10y - 2x + 4 &= 0 \end{aligned}$$

and hence the only critical point of f is $x_0 = \frac{1}{9}(11,-8,11)$.

Now, at any point $(x,y,z) \in \mathbb{R}^3$, the Hessian is given by

$$Hf(x,y,z) = \begin{bmatrix} 6 & 10 & -2 \\ 10 & 14 & 10 \\ -2 & 10 & 6 \end{bmatrix}$$

and hence we have that $\det Hf(x_0) = -1152$, and hence x_0 is a non-degenerate critical point.

Finally, observe that

$$\text{char} Hf(x_0) = t^3 - 26t^2 + 1152 = (t+6)(t-8)(t-24)$$

and hence $Hf(x_0)$ is indefinite. So, x_0 is a saddle point.

(c) $f(x,y,z) = 11y^2 + 14yz + 8zx + 14xy - 6x - 16y + 2z - 2$

Again, we observe that f is $\mathscr{C}^\infty$. For any $(x,y,z) \in \mathbb{R}^3$, we have

$$Jf(x,y,z) = \begin{bmatrix} 8z + 14y - 6 & 22y + 14z + 14x - 16 & 14y + 8x + 2 \end{bmatrix}$$

and hence the critical points of f satisfy the equation:

$$\begin{aligned} 8z + 14y - 6 &= 0 \\ 22y + 14z + 14x - 16 &= 0 \\ 14y + 8x + 2 &= 0 \end{aligned}$$

and hence the only critical point of f is $x_0 = \dfrac{1}{3}(1, -1, 4)$.

Now, at any point $(x,y,z) \in \mathbb{R}^3$, the Hessian is given by

$$Hf(x,y,z) = \begin{bmatrix} 0 & 14 & 8 \\ 14 & 22 & 14 \\ 8 & 14 & 0 \end{bmatrix}$$

and hence we have that $\det Hf(x_0) = 1728$, and hence x_0 is a non-degenerate critical point.

Finally, observe that

$$\text{char} Hf(x_0) = t^3 - 22t^2 - 456t - 1728 = (t+6)(t+8)(t-36)$$

and hence it follows that H is indefinite, implying that x_0 is a saddle point.

4. In this problem, we will find the centres and principles axes of the given conicoids.

Before continuing, I will highlight the method to find the principle directions of the conicoid. Suppose f is a degree two polynomial in three variabes. Let $x = (x_1, x_2, x_3)$. Then, we can write

$$f(x) = x^t Q x + P(x) + r$$

where $Q \in M_n(\mathbb{R})$ is a symmetric matrix (so x^tQx is a quadratic form), P is a linear homogeneous polynomial, and r is a constant. Now, we know that Q is diagonalisable by an orthogonal matrix Γ, i.e

$$\Gamma^t Q \Gamma = D$$

is a diagonal matrix. So, we can write

$$f(x) = (x^*)^t D(x^*) + P(\Gamma x^*) + r$$

where $x^* = \Gamma^{-1}x = \Gamma^t x$. If we let $P^*(x^*) = P(\Gamma x^*)$, then P^* will be a linear homogeneous polynomial. So, we can write

$$\begin{aligned} f(x) &= (x^*)^t D(x^*) + P^*(x^*) + r \\ &= \sum_{i=1}^{3} \lambda_i (\gamma_{i1}x_1 + \gamma_{i2}x_2 + \gamma_{i3}x_3)^2 + \sum_{i=1} \mu_i(\gamma_{i1}x_1 + \gamma_{i2}x_2 + \gamma_{i3}x_3) + r \end{aligned}$$

where γ_{ij} are the entries of Γ^{-1}. So, the principal directions will be the rows of Γ^{-1}, i.e the vectors $(\gamma_{j1}, \gamma_{j2}, \gamma_{j3})$. Also, it is easy to see that these row vectors are eigenvectors of Q. So, the principal directions will just be the set of eigenvectors of Q. The principle axes will be obtained from these.

(a) $f(x) = 3x^2 + 5y^2 + 3z^2 + 2yz + 2zx + 2xy - 4x - 8z + 5 = 0$

The jacobian is given by

$$Jf(x,y,z) = \begin{bmatrix} 6x + 2z + 2y - 4 & 2x + 10y + 2z & 6z + 2y + 2x - 8 \end{bmatrix}$$

The Hessian in this case is given by

$$Hf(x,y,z) = \begin{bmatrix} 6 & 2 & 2 \\ 2 & 10 & 2 \\ 2 & 2 & 6 \end{bmatrix}$$

which is non-singular because the determinant is 288. So, by problem 3., the unique critical point of f is a centre of the surface. This gives us that the point $x_0 = \frac{1}{3}(1,-1,4)$ is a centre of the surface.

Now, if $x \in \mathbb{R}^n$, then we can write

$$f(x) = x^t Q x + P(x) + 5$$

where

$$Q = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

and $P(x) = -4x_1 - 8x_3$, where $x = (x_1, x_2, x_3)$. By the above discussion, the principal directions are the eigenvectors of Q. Observe that $2Q = H$, and hence the eigenvalues of Q are $\lambda_1 = 2$,$\lambda_2 = 3$ and $\lambda_3 = 6$. The corresponding eigenvectors and hence the principal directions are $(1,0,-1)$, $(1,-1,1)$ and $(1,2,1)$. So, the principal axes are the following three lines

$$\begin{aligned} l_1 &:= x_0 + t(1,0,-1) \\ l_2 &:= x_0 + t(1,-1,1) \\ l_3 &:= x_0 + t(1,2,1) \end{aligned}$$

As in the discussion, we let $x^* = (x_1', x_2', x_3') = \Gamma^{-1}x$, where $x = (x_1, x_2, x_3)$. It is easy to see that the matrix Γ is

$$\Gamma = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{-1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

and we know that

$$\Gamma^{-1} Q \Gamma = \operatorname{diag}(2,3,6) = D$$

So, we have

$$\begin{aligned} f(x) &= (x^*)^t D x^* + P^*(x^*) + 5 \\ &= 2(x_1')^2 + 3(x_2')^2 + 6(x_3')^2 + P^*(x^*) + 5 \end{aligned}$$

Now, Γx^* can be easily calculated, and hence

$$\begin{aligned}
P^*(x^*) &= P(\Gamma x^*) \\
&= -4\left(\frac{x_1'}{\sqrt{2}} + \frac{x_2'}{\sqrt{3}} + \frac{x_3'}{\sqrt{6}}\right) - 8\left(\frac{-x_1'}{\sqrt{2}} + \frac{x_2'}{\sqrt{3}} + \frac{x_3'}{\sqrt{6}}\right) \\
&= 2\sqrt{2}x_1' - 4\sqrt{3}x_2' - 2\sqrt{6}x_3'
\end{aligned}$$

and hence in the new coordinate system, the equation of the conicoid is

$$2(x_1')^2 + 3(x_2')^2 + 6(x_3')^2 + 2\sqrt{2}x_1' - 4\sqrt{3}x_2' - 2\sqrt{6}x_3' + 5 = 0$$

Completing squares, we get

$$2\left(x_1' + \frac{1}{\sqrt{2}}\right)^2 + 3\left(x_2' - \frac{2}{\sqrt{3}}\right)^2 + 6\left(x_3 - \frac{1}{\sqrt{6}}\right)^2 = 1$$

Shifting the origin to $\left(\frac{-1}{\sqrt{2}}, \frac{2}{\sqrt{3}}, \frac{1}{\sqrt{6}}\right)$, the above equation becomes

$$2x^2 + 3y^2 + 6z^2 = 1$$

which is a standard ellipsoid.

(b) $f(x) = 3x^2 + 7y^2 + 3z^2 + 10yz - 2zx + 10xy + 4x - 12y + 4z + 1 = 0$

The jacobian is given by

$$Jf(x,y,z) = \begin{bmatrix} 6x - 2z + 10y + 4 & 14y + 10z + 10x - 12 & 6z + 10y - 2x + 4 \end{bmatrix}$$

The Hessian in this case is given by

$$Hf(x,y,z) = \begin{bmatrix} 6 & 10 & -2 \\ 10 & 14 & 10 \\ -2 & 10 & 6 \end{bmatrix}$$

which is non-singular because the determinant is -1152. So, by problem 3., the unique critical point of f is a centre of the surface. This gives us that the point $x_0 = \frac{1}{9}(11, -8, 11)$ is a centre of the surface.

Now, if $x \in \mathbb{R}^n$, then we can write

$$f(x) = x^t Q x + P(x) + 1$$

where

$$Q = \begin{bmatrix} 3 & 5 & -1 \\ 5 & 7 & 5 \\ -1 & 5 & 3 \end{bmatrix}$$

and $P(x) = 4x_1 - 12x_2 + 4x_3$, where $x = (x_1, x_2, x_3)$. By the above discussion, the principal directions are the eigenvectors of Q. Observe that $2Q = H$, and hence the eigenvalues of Q are $\lambda_1 = -3, \lambda_2 = 4$ and $\lambda_3 = 12$. The corresponding eigenvectors and hence the principal directions are $(1,-1,1)$, $(1,0,-1)$ and $(1,2,1)$. So, the principal axes are the following three lines

$$\begin{aligned}
l_1 &:= x_0 + t(1,-1,1) \\
l_2 &:= x_0 + t(1,0,-1) \\
l_3 &:= x_0 + t(1,2,1)
\end{aligned}$$

As in the discussion, we let $x^* = (x_1', x_2', x_3') = \Gamma^{-1}x$, where $x = (x_1, x_2, x_3)$. It is easy to see that the matrix Γ is

$$\Gamma = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

and we know that

$$\Gamma^{-1}Q\Gamma = \operatorname{diag}(-3, 4, 12) = D$$

So, we have

$$\begin{aligned} f(x) &= (x^*)^t Dx^* + P^*(x^*) + 1 \\ &= -3(x_1')^2 + 4(x_2')^2 + 12(x_3')^2 + P^*(x^*) + 1 \end{aligned}$$

Now, Γx^* can be easily calculated, and hence

$$\begin{aligned} P^*(x^*) &= P(\Gamma x^*) \\ &= 4\left(\frac{x_1'}{\sqrt{3}} + \frac{x_2'}{\sqrt{2}} + \frac{x_3'}{\sqrt{6}}\right) - 12\left(\frac{-x_1'}{\sqrt{3}} + \frac{2x_3'}{\sqrt{6}}\right) + 4\left(\frac{x_1'}{\sqrt{3}} - \frac{x_2'}{\sqrt{2}} + \frac{x_3'}{\sqrt{6}}\right) \\ &= \frac{20x_1'}{\sqrt{3}} - \frac{16x_3'}{\sqrt{6}} \end{aligned}$$

and hence in the new coordinate system, the equation of the conicoid is

$$-3(x_1')^2 + 4(x_2')^2 + 12(x_3')^2 + \frac{20x_1'}{\sqrt{3}} - \frac{16x_3'}{\sqrt{6}} + 1 = 0$$

Completing squares, we get

$$3\left(x_1' - \frac{10}{3\sqrt{3}}\right)^2 - 4x_2'^2 - 12\left(x_3' - \frac{2}{3\sqrt{6}}\right)^2 = \frac{101}{9}$$

Shifting the origin to $\left(\frac{10}{3\sqrt{3}}, 0, \frac{2}{3\sqrt{6}}\right)$, the above equation becomes

$$3x^2 - 4y^2 - 12z^2 = \frac{101}{9}$$

which after normalising is a standard hyperboloid of two sheets.

(c) $f(x) = 11y^2 + 14yz + 8zx + 14xy - 6x - 16y + 2z - 2$

The jacobian is given by

$$Jf(x,y,z) = \begin{bmatrix} 8z + 14y - 6 & 22y + 14z + 14x - 16 & 14y + 8x + 2 \end{bmatrix}$$

The Hessian in this case is given by

$$Hf(x,y,z) = \begin{bmatrix} 0 & 14 & 8 \\ 14 & 22 & 14 \\ 8 & 14 & 0 \end{bmatrix}$$

which is non-singular because the determinant is 1728. So, by problem 3., the unique critical point of f is a centre of the surface. This gives us that the point $x_0 = \frac{1}{3}(1, -1, 4)$ is a centre of the surface.

Now, if $x \in \mathbb{R}^n$, then we can write

$$f(x) = x^t Q x + P(x) - 2$$

where

$$Q = \begin{bmatrix} 0 & 7 & 4 \\ 7 & 11 & 7 \\ 4 & 7 & 0 \end{bmatrix}$$

and $P(x) = -6x_1 - 16x_2 + 2x_3$, where $x = (x_1, x_2, x_3)$. By the above discussion, the principal directions are the eigenvectors of Q. Observe that $2Q = H$, and hence the eigenvalues of Q are $\lambda_1 = -3$,$\lambda_2 = -4$ and $\lambda_3 = 18$. The corresponding eigenvectors and hence the principal directions are $(1, -1, 1)$, $(1, 0, -1)$ and $(1, 2, 1)$. So, the principal axes are the following three lines

$$\begin{aligned} l_1 &:= x_0 + t(1, -1, 1) \\ l_2 &:= x_0 + t(1, 0, -1) \\ l_3 &:= x_0 + t(1, 2, 1) \end{aligned}$$

As in the discussion, we let $x^* = (x_1', x_2', x_3') = \Gamma^{-1} x$, where $x = (x_1, x_2, x_3)$. It is easy to see that the matrix Γ is

$$\Gamma = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

and we know that

$$\Gamma^{-1} Q \Gamma = \operatorname{diag}(-3, -4, 18) = D$$

So, we have

$$\begin{aligned} f(x) &= (x^*)^t D x^* + P^*(x^*) - 2 \\ &= -3(x_1')^2 - 4(x_2')^2 + 18(x_3')^2 + P^*(x^*) - 2 \end{aligned}$$

Now, Γx^* can be easily calculated, and hence

$$\begin{aligned} P^*(x^*) &= P(\Gamma x^*) \\ &= -6\left(\frac{x_1'}{\sqrt{3}} + \frac{x_2'}{\sqrt{2}} + \frac{x_3'}{\sqrt{6}}\right) - 16\left(\frac{-x_1'}{\sqrt{3}} + \frac{2x_3'}{\sqrt{6}}\right) + 2\left(\frac{x_1'}{\sqrt{3}} - \frac{x_2'}{\sqrt{2}} + \frac{x_3'}{\sqrt{6}}\right) \\ &= 4\sqrt{3}x_1' - 4\sqrt{2}x_2' - 6\sqrt{6}x_3' \end{aligned}$$

and hence in the new coordinate system, the equation of the conicoid is

$$-3(x_1')^2 - 4(x_2')^2 + 18(x_3')^2 + 4\sqrt{3}x_1' - 4\sqrt{2}x_2' - 6\sqrt{6}x_3' - 2 = 0$$

Completing squares, we get

$$-3\left(x_1' - \frac{2\sqrt{3}}{3}\right)^2 - 4\left(x_2' + \frac{1}{\sqrt{2}}\right)^2 + 18\left(x_3 - \frac{1}{\sqrt{6}}\right)^2 = -1$$

Shifting the origin to $\left(\frac{2\sqrt{3}}{3}, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{6}}\right)$, the above equation becomes

$$-3x^2 - 4y^2 + 18z^2 = -1$$

and normalising both sides, this is an equation of a standard hyperboloid of one sheet.