ANALYSIS-2 , HW-5

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1. In this problem, let V be an inner product space over K (where $K \in \{\mathbb{R}, \mathbb{C}\}$).

(a) Let us show that $H^\perp$ is a subspace of V. If $h \in H$, then we know that $\langle h, O\rangle = 0$, implying that $O \in H^\perp$. Next, if $u, v \in H^\perp$, then for any $c_1, c_2 \in K$ we have

$$\langle h, c_1u + c_2v\rangle = \langle h, c_1u\rangle + \langle h, c_2v\rangle = \overline{c_1}\langle h, u\rangle + \overline{c_2}\langle h, v\rangle = 0$$

and hence $c_1u + c_2v \in H^\perp$. So, $H^\perp$ is a subspace of V.

(b) First, suppose $H = V$. Let $u \in H^\perp$. Then, it follows that $\langle u, v\rangle = 0$ for all $v \in V$. In particular, $\langle u, u\rangle = 0$ and hence $u = O$. So, $H^\perp = 0$. Conversely, suppose $H^\perp = 0$. For the sake of contradiction, suppose $\dim H = k < n$. Let $\{w_1, ..., w_k\}$ be an orthonormal basis of H. Let $v \in V$ such that $v \notin H$. Define the vector v_0 by

$$v_0 = v - \langle v, w_1\rangle w_1 - ... - \langle v, w_k\rangle w_k$$

and it is clear that $v_0 \neq O$ because $v \notin H$. Also, observe that for any $1 \le i \le k$, we have

$$\langle v_0, w_i\rangle = \langle v - \langle v, w_1\rangle w_1 - ... - \langle v, w_k\rangle w_k, w_i\rangle = \langle v, w_i\rangle - \langle v, w_i\rangle = 0$$

and hence v_0 is orthogonal to every w_i, and hence $v_0 \in H^\perp$. But, this implies that $H^\perp \neq 0$, contradicting our hypothesis. Hence, $\dim H = n$, meaning that $H = V$.

(c) Here, we will show that $H^{\perp\perp} = H$. First, if $u \in H$, then by definition for all $h \in H^\perp$ we have

$$\langle u, h\rangle = 0$$

which implies that $u \in H^{\perp\perp}$, and hence $H \subset H^{\perp\perp}$. To prove the reverse containment, let $\{w_1, ..., w_k\}$ be an orthonormal basis of H. Suppose $v \in H^{\perp\perp}$. Then, it follows that for all $\overline{h} \in H^\perp$, we have

$$\langle v, \overline{h}\rangle = 0$$

As in part **(b)**, we define

$$v_0 = v - \langle v, w_1\rangle w_1 - ... - \langle v, w_k\rangle w_k$$

so that $v_0 \in H^\perp$ (because it is orthogonal to every w_i). Also, observe that for any $\overline{h} \in H^\perp$, we have

$$\langle v_0, \overline{h}\rangle = \langle v, \overline{h}\rangle - \langle v, w_1\rangle\langle w_1, \overline{h}\rangle - ... - \langle v, w_k\rangle\langle w_k, \overline{h}\rangle = 0 \tag{0.1}$$

By putting $\overline{h} = v_0$, equation (0.1), combined with the fact that $v_0 \in H^\perp$ implies that

$$\langle v_0, v_0\rangle = 0$$

which implies that $v_0 = O$. So, $v \in H$, and hence $H^{\perp\perp} \subset H$. This proves that $H = H^{\perp\perp}$.

Date: March 2020.

2. Here we will show that

$$V = H \oplus H^{\perp}$$

If $H = V$, then by problem **1.** part **(b)**, we know that $H^{\perp} = 0$, and hence we are done. If $H = 0$, then it follows that $H^{\perp}\perp = 0$ by part **(c)** of problem **1.**, and hence by part **(b)** again it follows that $H\perp = V$, and again the claim is true. So, we assume that $1 \leq k = \dim H < \dim V = n$.

Let $\{w_1, ..., w_k\}$ be an orthonormal basis of H. We will extend this to an orthonormal basis of V as follows: take $v_{k+1} \in V$ such that $v_{k+1} \notin H$. Put

$$w_{k+1} = v_{k+1} - \langle v_{k+1}, w_1\rangle w_1 - ... - \langle v_{k+1}, w_k\rangle w_k$$

Observe that $w_{k+1} \neq O$ because $v_{k+1} \notin H$. We will prove that $\{w_1, ..., w_{k+1}\}$ is a set of orthogonal linearly independent vectors. First, observe that for any $1 \leq i \leq k$, we have

$$\langle w_{k+1}, w_i\rangle = \langle\rangle = \langle v_{k+1}, w_i\rangle - \langle v_{k+1}, w_i\rangle = 0$$

and hence w_{k+1} is orthogonal to each w_i. To show linear independence, suppose

$$c_1 w_1 + ... + c_{k+1} w_{k+1} = O$$

for some $c_1, ..., c_k \in K$. Taking the inner product with w_i $(1 \leq i \leq k+1)$ on both sides, we get

$$\langle c_1 w_1 + ... + c_{k+1} w_{k+1}, w_i\rangle = \langle O, w_i\rangle = 0$$

and hence we get that

$$c_i = 0$$

for each $1 \leq i \leq k+1$, and hence we have shown that $\{w_1, ..., w_k, w_{k+1}\}$ is a linearly independent set of orthogonal vectors. If w_{k+1} is not a unit vector, we can replace it with $\dfrac{w_{k+1}}{||w_k||}$, and hence we can say that $\{w_1, ..., w_k, w_{k+1}\}$ is a set of orthonormal vectors.

Now, we continue this process again until we get n linearly independent vectors, and obtain a linearly independent set $\{w_1, ..., w_k, w_{k+1}, ..., w_n\}$ to get an orthonormal basis of V. Thus, we have extended an orthonormal basis of H to an orthonormal basis of V.

Now, let $W = \langle w_{k+1}, ..., w_n\rangle$. Clearly, we see that

$$V = H \oplus W$$

because $\{w_1, ..., w_k, w_{k+1}, ..., w_n\}$ is a basis of V. It remains to show that $W = H^{\perp}$.

It is easy to see that $W \subset H^{\perp}$, because if $u \in W$, then $\langle u, w_i\rangle = 0$ for each $1 \leq i \leq k$, because of the orthonormal basis. Next, suppose $u \in H^{\perp}$. Then,

$$u = c_1 w_1 + ... + c_k w_k + c_{k+1} w_{k+1} + ... + c_n w_n$$

Taking the inner product of u with w_i, for $1 \leq i \leq k$, we see that

$$0 = \langle u, w_i\rangle = \langle c_1 w_1 + ... + c_k w_k + c_{k+1} w_{k+1} + ... + c_n w_n, w_i\rangle = c_i$$

and hence $c_i = 0$. This implies that $u \in W$, and hence $H^{\perp} \subset W$. So, it follows that $H^{\perp} = W$, and

$$V = H \oplus H^{\perp}$$

3. Let $v \in V$, and let $f_v : V \to K$ be the map given by

$$f_v(w) = \langle w, v \rangle$$

Let us show that f_v is a functional. If there are constants $c_1, c_2 \in K$, and $v_1, v_2 \in V$, then we have

$$\begin{aligned} f_v(c_1v_1 + c_2v_2) &= \langle c_1v_1 + c_2v_2, v \rangle \\ &= \langle c_1v_1, v \rangle + \langle c_2v_2, v \rangle \\ &= c_1\langle v_1, v \rangle + c_2\langle v_2, v \rangle \\ &= c_1 f_v(v_1) + c_2 f_v(v_2) \end{aligned}$$

and hence f_v is a linear functional.

Now, let V^* be the dual of V. Consider the map $D : V \to V^*$ given by

$$D(v) = f_v$$

Let us show that D is an *anti-linear* isomorphism between V and V^*.

First, let us show that the map D is anti-linear. So, let $v_1, v_2 \in V$, and let $c_1, c_2 \in K$. Let $w \in V$ be arbitrary. Then, we have

$$\langle w, c_1v_1 + c_2v_2 \rangle = \overline{c_1}\langle w, v_1 \rangle + \overline{c_2}\langle w, v_2 \rangle$$

and this means that

$$f_{c_1v_1+c_2v_2}(w) = \overline{c_1} f_{v_1}(w) + \overline{c_2} f_{v_2}(w)$$

for all $w \in V$, and hence

$$f_{c_1v_1+c_2v_2} = \overline{c_1} f_{v_1} + \overline{c_2} f_{v_2}$$

and hence

$$D(c_1v_1 + c_2v_2) = \overline{c_1} D(v_1) + \overline{c_2} D(v_2)$$

showing that D is anti-linear.

Now, consider Ker$D$. If $v \in$ Ker$D$, then $f_v = O$ (zero-map), and hence

$$\langle w, v \rangle = 0$$

for all $w \in V$, implying that $v = O$. So, Ker$D = \{O\}$, showing that D is one-one.

Finally, since $\dim V = \dim V^*$ (because V is finite dimensional), by rank-nullity theorem we conclude that D is onto. So, D is an anti-linear isomorphism between V and V^*.

4. Let $T : V \to V$ be a K-linear map. We will show that there is a unique linear map T^* such that

$$\langle Tv, w \rangle = \langle v, T^*w \rangle$$

for all $v, w \in V$.

First, let $w \in V$ be fixed, and consider the map $f_w : V \to K$ given by

$$f_w(v) = \langle Tv, w \rangle$$

Let's show that f is a functional, i.e $f \in V^*$. Let $v_1, v_2 \in V$, and let $c_1, c_2 \in K$. Observe that

$$\begin{aligned} f_w(c_1v_1 + c_2v_2) &= \langle T(c_1v_1 + c_2v_2), w\rangle \\ &= \langle c_1Tv_1 + c_2Tv_2, w\rangle \\ &= c_1\langle Tv_1, w\rangle + c_2\langle Tv_2, w\rangle \\ &= c_1f_w(v_1) + c_2f_w(v_2) \end{aligned}$$

and hence $f \in V^*$. By problem (4) we see that there is some $w' \in V$ such for which

$$f_w(v) = \langle v, w'\rangle$$

for all $v \in V$. So, consider the map $T^* : V \to V$ that sends w to w'. The last equation can be written as

$$\langle Tv, w\rangle = \langle v, T^*w\rangle$$

Now we will show that T^* is a linear map, and then T^* will be the required linear map.

Now, let $w_1, w_2 \in V$, and let $c_1, c_2 \in K$. Now, for any $v \in V$, we have by definition of T^*

$$\langle Tv, c_1w_1 + c_2w_2\rangle = \langle v, T^*(c_1w_1 + c_2w_2)\rangle$$

and we also have

$$\begin{aligned} \langle Tv, c_1w_1 + c_2w_2\rangle &= \overline{c_1}\langle Tv, w_1\rangle + \overline{c_2}\langle Tv, w_2\rangle \\ &= \overline{c_1}\langle v, T^*w_1\rangle + \overline{c_2}\langle v, T^*w_2\rangle \\ &= \langle v, c_1T^*w_1\rangle + \langle v, c_2T^*w_2\rangle \\ &= \langle v, c_1T^*w_1 + c_2T^*w_2\rangle \end{aligned}$$

so it follows that for all $v \in V$, we have

$$\langle v, T^*(c_1w_1 + c_2w_2)\rangle = \langle v, c_1T^*w_1 + c_2T^*w_2\rangle$$

and by the non-degeneracy of the inner product, we see that

$$T^*(c_1w_1 + c_2w_2) = c_1T^*w_1 + c_2T^*w_2$$

proving that T^* is linear. This completes the proof.

For the following problems: suppose $n = d + m$, and identify $\mathbb{R}^n$ with $\mathbb{R}^d \times \mathbb{R}^m$. Suppose $\phi : U \to \mathbb{R}^m$ is a $\mathscr{C}^1$ map where U is an open subset of $\mathbb{R}^n$. Let $p = (a, b) \in U$ such that the rank of $\phi'(p)$ is m. Let $c = \phi(p)$ and M the subset of U consisting of points $(x, y) \in U$ such that $\phi(x, y) = c$. From now on, what was T_p in assignment 4 will be denoted T_p^{cl}, and V_p will be denoted by T_p. The normal space N_p is defined to be $T_p^\perp$, where the inner product is the standard inner product on $\mathbb{R}^n$.

5. We will show that the columns of $Q = ((J\phi)(p))^t$ form a basis of N_p. Suppose $\phi = (\phi_1, ..., \phi_m)$, where each component $\phi_m : U \to \mathbb{R}$. Observe that the i^{th} column of Q is nothing but $\nabla\phi_i(p)$.

First, we will show that each $\nabla\phi_i(p)$ for $1 \le i \le m$ lies in N_p.

Suppose $v \in T_p$. Then, we have that

$$(J\phi)(p)v = 0$$

where the above equation represents matrix multiplication. Now,

$$(J\phi)(p)v = \begin{bmatrix} \langle v, \nabla\phi_1(p)\rangle \\ \langle v, \nabla\phi_2(p)\rangle \\ \cdots \\ \langle v, \nabla\phi_m(p)\rangle \end{bmatrix}$$

which implies that $\langle v, \nabla\phi_i(p)\rangle = 0$ for each i, and hence since $v \in T_p$ was arbitrary, it follows that $\nabla\phi_i(p) \in N_p$ for each $1 \leq i \leq m$.

Now, it is given that the rank of $(J\phi)(p)$ is m, which means that $\dim T_p = d$ by rank-nullity. Also, this means that the row vectors of $(J\phi)(p)$, which are $\nabla\phi_i(p)$, which are also the column vectors of $((J\phi)(p))^t$ are linearly independent. Also, by problem **2.**, we know that

$$n = \dim T_p + \dim N_p$$

and hence $\dim N_p = m$. Since each $\nabla\phi_i(p)$ lies in N_p, it follows that $\{\nabla_1(p), ..., \nabla\phi_m(p)\}$ form a basis of N_p. Hence, the columns of $((J\phi)(p))^t$ form a basis of N_p.

6. Let $\pi : \mathbb{R}^n \to \mathbb{R}^d$ and $\pi' : \mathbb{R}^n \to \mathbb{R}^m$ be the usual projections.

(a) We will show that π' maps N_p surjectively on to $\mathbb{R}^m$ if and only if $\dfrac{\partial\phi}{\partial y}(p)$ is invertible.

First, suppose $\dfrac{\partial\phi}{\partial y}(p)$ is invertible, which means that the rank of $(J\phi)(p)$ is m, and hence the vectors $\{\nabla\phi_1(p), ..., \nabla\phi_m(p)\}$ are linearly independent, and they span N_p by problem **5.**. Observe that the vectors $\{\pi'(\nabla\phi_1(p)), ..., \pi'(\nabla\phi_m(p))\}$ are the rows of $\dfrac{\partial\phi}{\partial y}$, and by the given condition, these are linearly independent, and hence they form a basis of $\mathbb{R}^m$. So, π' maps a basis of N_p to a basis of $\mathbb{R}^m$, and since $\dim N_p = \dim \mathbb{R}^m = m$, it follows that π' is surjective.

Conversely, suppose π' is surjective. Now,

$$\dim N_p = \dim \mathrm{Ker}\pi' + \dim \mathbb{R}^m \geq m$$

and hence the dimension of N_p is atleast m. Moreover, observe that

$$n = \dim T_p + \dim N_p$$

and also

$$n = \dim T_p + \mathrm{rank}(J\phi)(p)$$

and hence

$$\dim N_p = \mathrm{rank}(J\phi)(p)$$

Clearly, the rank of $(J\phi)(p)$ is atmost m, and hence it follows that the rank of $(J\phi)(p) = m$. So, the row vectors of $(J\phi)(p)$, which are $\{\nabla\phi_1(p), ..., \nabla\phi_m(p)\}$ are linearly independent, and hence are a basis of N_p. Again, observe that the vectors $\{\pi'(\nabla\phi_1(p)), ..., \pi'(\nabla\phi_m(p))\}$ are the row vectors of $\dfrac{\partial\phi}{\partial y}(p)$, and since $\pi' : N_p \to \mathbb{R}^m$ is surjective, a basis of N_p is mapped to a basis of $\mathbb{R}^m$, and hence it follows that the rows of $\dfrac{\partial\phi}{\partial y}(p)$ are linearly independent, implying that $\dfrac{\partial\phi}{\partial y}(p)$ is invertible.

(b) The required criterion is the following: $\pi : T_p \to \mathbb{R}^d$ is surjective if and only if $\frac{\partial \phi}{\partial y}(p)$ is invertible.

Observe that $N_p = T_p^{\perp}$. We know that $\pi : T_p \to R^d$ is surjective if and only if $\pi' : T_p^{\perp} \to \mathbb{R}^m$ is surjective, which has been proven in class. This means that $\pi : T_p \to \mathbb{R}^d$ is surjective if and only if $\pi' : N_p \to \mathbb{R}^m$ is surjective. By part **(a)**, $\pi' : N_p \to \mathbb{R}^m$ is surjective if and only if $\frac{\partial \phi}{\partial y}(p)$ is invertible. So, it follows that $\pi : T_p \to \mathbb{R}^d$ is surjective if and only if $\frac{\partial \phi}{\partial y}(p)$ is invertible, proving the claim.

7. Suppose $\frac{\partial \phi}{\partial y}(p)$ is invertible and let $f : W \to \mathbb{R}^m$ be an implicit solution of $\phi(x, y) = c$ in an open neighborhood W of a in $\mathbb{R}^d$ with $f(a) = b$. We determine $(Jf)(a)$.

We have
$$\phi(x, f(x)) = c$$
for all $x \in W$. Define $h : W \to \mathbb{R}^m$ by
$$h(x) = \phi(x, f(x))$$
which implies that h is a constant function on W, so that $Dh(x) = 0$ for all $x \in W$. Also, observe that
$$h = \phi \circ q$$
where $q : W \to U$ given by $q(x) = (x, f(x))$. By the chain rule, we have
$$0 = Dh(a) = D\phi(q(a))Dq(a)$$
Observe that
$$Dq(a) = \begin{bmatrix} I_d \\ Df(a) \end{bmatrix}$$
so we get
$$\begin{aligned} 0 &= D\phi(a, b) \begin{bmatrix} I_d \\ Df(a) \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial \phi}{\partial x}(a, b) & \frac{\partial \phi}{\partial y}(a, b) \end{bmatrix} \begin{bmatrix} I_d \\ Df(a) \end{bmatrix} \\ &= \frac{\partial \phi}{\partial x}(a, b) + \frac{\partial \phi}{\partial y}(a, b) Df(a) \end{aligned}$$
and hence we get
$$Df(a) = -\left(\frac{\partial \phi}{\partial y}(a, b)\right)^{-1} \frac{\partial \phi}{\partial x}(a, b)$$
and hence the claim follow.s