ANALYSIS-2 , HW-5

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Let V and W be finite dimensional vector spaces, and let $\{v_1, ..., v_n\}$ and $\{w_1, ..., w_m\}$ be bases of V and W respectively. Let $R : V \to K^n$ be the isomorphism given by $Rv_i = e_i$ and let $L : W \to K^m$ be the isomorphism given by $Lw_j = e'_j$, where $\{e_1, ..., e_n\}$ and $\{e'_1, ..., e'_m\}$ are the standard orthonormal bases of K^n and K^n respectively. Suppose $T : V \to W$ and $S : K^n \to K^m$ be linear maps such that

$$S = L \circ T \circ R^{-1}$$

1. We will show that the matrix of T with respect to the given bases is equal to the matrix of S with respect to the standard bases of K^n and K^m.

To show this, we will show the following: if

$$Tv_i = a_1w_1 + ... + a_mw_m$$

then

$$Se_i = a_1e'_1 + ... + a_me'_m$$

and since $(a_1, ..., a_m)$ is the i^{th} column of the matrix of T and S, it would follow that their matrices are equal.

So, suppose $Tv_i = a_1w_1 + ... + a_mw_m$. We have

$$\begin{aligned} Se_i &= LTR^{-1}e_i \\ &= LTv_i \\ &= L(a_1w_1 + ... + a_mw_m) \\ &= a_1L(w_1) + ... + a_mL(w_m) \\ &= a_1e'_1 + ... + a_me'_m \end{aligned}$$

and hence the claim follows.

Suppose $v_1, ..., v_n$ is a basis of $\mathbb{R}^n$ and $w_1, ..., w_m$ is a basis of $\mathbb{R}^m$. Let L and R be the automorphisms as defined. Suppose U is an open subset of $\mathbb{R}^n$, and let $U' = R(U)$. Let $f : U \to \mathbb{R}^m$ be a map.

2. We will show that U' is also open in $\mathbb{R}^n$. To show this, observe that R is an automorphism, and consider the inverse automorphism $\phi = R^{-1}$. Since ϕ is injective and surjective, it follows that $U' = \phi^{-1}(U)$. Now, ϕ being a linear operator on $\mathbb{R}^n$ is continuous, and since U is open on $\mathbb{R}^n$, it follows that $U' = \phi^{-1}(U)$ is also open in $\mathbb{R}^n$.

Date: March 2020.

3. Suppose $j \in \{1, ..., m\}$ is fixed. For $x_1v_1 + ... + x_nv_n \in U$, define $g_j(x_1, ..., x_n)$ to be the coefficient of w_j, when $f(x_1v_1 + ...x_nv_n)$ is written as a linear combination of $w_1, ..., w_m$.

Let's show that for $(x_1, ..., x_n) \in U'$, $g_j(x_1, ..., x_n)$ makes sense, i.e it is uniquely defined. Observe that if $(x_1, ..., x_n) \in U'$, then we can write $(x_1, ..., x_n) = x_1e_1 + ... + x_ne_n$, and hence $R^{-1}(x_1, ..., x_n) = x_1v_1 + ... + x_nv_n \in U$, which follows by the definition of R. Now, the vector $f(x_1v_1 + ... + x_nv_n)$, which is a vector in $\mathbb{R}^m$, has a *unique* representation as a linear combination of $w_1, ..., w_m$, and hence the coefficient of w_j, which is $g_j(x_1, ..., x_n)$, is unique. So, the map $(x_1, ..., x_n) \mapsto g_j(x_1, ..., x_n)$ is a well defined mapping from U' to $\mathbb{R}$.

In the following two problems, R will always refer to the restriction on the set U, i.e R will mean $R|_U$.

4. To show that the diagram commutes, it is enough to show that

$$g = L \circ f \circ R^{-1}$$

So, suppose $(x_1, ..., x_n) \in U'$. Then,

$$g(x_1, ..., x_n) = (g_1(x_1, ..., x_n), ..., g_m(x_1, ..., x_n))$$

Suppose $1 \leq j \leq m$. By the definition of g_j, $g_j(x_1, ..., x_n)$ is the coefficient of w_j when $f(x_1v_1 + ... + x_nv_n)$ is written as a linear combination of $\{w_1, ..., w_m\}$. Observe that

$$x_1v_1 + ... + x_nv_n = R^{-1}(x_1, ..., x_n)$$

via the isomorphism R. So, we see that

$$f(R^{-1}(x_1, ..., x_n)) = g_1(x_1, ..., x_n)w_1 + ... + g_n(x_1, ..., x_n)w_m$$

Finally, we see that

$$L(f(R^{-1}(x_1, ..., x_n))) = (g_1(x_1, ..., x_n), ..., g_m(x_1, ..., x_n))$$

and hence we have

$$g(x_1, ..., x_n) = L(f(R^{-1}(x_1, ..., x_n)))$$

Since this was true for any $(x_1, ..., x_n) \in U'$, we get that

$$g = L \circ f \circ R^{-1}$$

and hence the given diagram commutes.

5. Suppose $p \in U$ and let $q = R(p)$. Let g be as in problem **4.**. Suppose f is differentiable at p.

(a) We will show that g is differentiable at q. From problem **4.**, we have

$$g(q) = (L \circ f \circ R^{-1})(q)$$

Now, observe that R^{-1} is differentiable at q (being a linear map), f is differentiable at $R^{-1}(q) = p$ (by assumption), and L is differentiable at $f \circ R^{-1}(q)$ (being a linear map), and hence by the chain rule g is differentiable at q. Also, since linear maps are their own derivatives, we see that

$$Dg(q) = L \circ Df(p) \circ R^{-1}$$

where the last equation represents composition of linear maps, which follows from the chain rule.

(b) Consider the matrix of $f'(p)$ with respect to the basis $\{v_1, ..., v_n\}$ of $\mathbb{R}^n$ and $\{w_1, ..., w_m\}$ of $\mathbb{R}^m$.

Now, $(Jg)(q)$ is an $m \times n$ matrix, and the i^{th} column of $(Jg)(q)$ is $Dg(q)(e_i)$, where $1 \le i \le n$. By part **(a)** of this problem, we have

$$Dg(q)(e_i) = L(Df(p)(R^{-1}(e_i))) = L(Df(p)(v_i))$$

Now, consider the matrix of $Df(p)$ with respect to the basis $\{v_1, ..., v_n\}$ and $\{w_1, ..., w_m\}$. The i^{th} column of this matrix is the column vector $\begin{bmatrix} a_1 \\ ... \\ a_m \end{bmatrix}$, where

$$Df(p)(v_i) = a_1 w_1 + ... + a_m w_m$$

Also, we have

$$Dg(q)(e_i) = L(Df(p)(v_i)) = L(a_1 w_1 + ... + a_m w_m) = \begin{bmatrix} a_1 \\ ... \\ a_m \end{bmatrix}$$

and hence we see that the i^{th} columns of the matrix of $Dg(q)$ is equal to the i^{th} column of the matrix of $Df(p)$ with respect to the given basis. So, it follows that the matrix of $Df(p)$ with respect to the given basis is $(Jg)(q)$. **Maxima and Minima:** Let U be an open subset of $\mathbb{R}^n$ and let $h : U \to \mathbb{R}$, $f : U \to \mathbb{R}$ be two $\mathscr{C}^1$ functions on U. Let $c \in h(U)$ and let M be the level set $M = \{x \in U : h(x) = c\}$. We say $f|_M$ has a *local maximum* at $v \in M$ if there is an open neighborhood W of v in U such that $f(x) \le f(v)$ for all $x \in W \cap M$. Similarly, a *local minimum* is defined. $f|_M$ is said to have a *local extremum* at v if it either has a local maximum or local minimum.

6. Suppose h' does not vanish at any point of M.

(a) First, we will show that M has a tangent space at each of its points. Observe that the matrix of $h'(x)$ at any $x \in M$ is a $1 \times n$ matrix, and since h' does not vanish at any point of M, we see that the matrix of $h'(x)$ has rank 1 (i.e it is a full rank matrix). So, we see that the tangent space at any $x \in M$ makes sense.

(b) Suppose $f|_M$ has a local extremum at some point $v = (v_1, ..., v_n) \in M$. We will show that $\nabla f(v)$ is orthogonal to the tangent space of M at v.

By HW-4 problem **3.**, we know that T_v is spanned by velocity vectors of $\mathscr{C}^1$ paths in M passing through v. So, it is enough to show that $\nabla f(v)$ is orthogonal to any velocity vector such a $\mathscr{C}^1$ path.

So, let $\gamma : (a, b) \to M$ be a $\mathscr{C}^1$ path such that there is some $\theta \in (a, b)$ for which $\gamma(\theta) = v$ (the path passes through v). Consider the function $g : (a, b) \to \mathbb{R}$ defined by

$$g(x) = f(\gamma(x))$$

for all $x \in (a, b)$. Since f is $\mathscr{C}^1$ on U (and hence on M) and since γ is $\mathscr{C}^1$ on (a, b), it follows that g is $\mathscr{C}^1$ on (a, b). Moreover, we know that f has a local extremum at the point $v = \gamma(\theta)$. So, there is an open neighborhood W of $\gamma(\theta)$ in U such that $f(x) \le f(\gamma(\theta))$ for all $x \in W \cap M$. Since W is open and contains $\gamma(\theta)$, there is an open ball $B(\gamma(\theta), r)$ (for some $r > 0$) such that $B(\gamma(\theta), r) \subset W$. Also, since γ is continuous on (a, b), there is some interval I such that $\theta \in I \subset (a, b)$, and $\gamma(I) \subset B(\gamma(\theta), r)$. Since $\gamma(I) \subset M$ by definition, it follows that $\gamma(I) \subset W \cap M$, and hence $g(x) \le g(\theta)$

at all points $x \in I$, implying that $g'(\theta) = 0$ (which follows from one-variable calculus). Also, by problem (1) of HW-4, we know that

$$g'(\theta) = \langle \gamma'(\theta), \nabla f(v) \rangle$$

and hence it follows that $\langle \gamma'(\theta), \nabla f(v) \rangle = 0$ implying that $\nabla f(v)$ is orthogonal to the velocity vector. So, it follows that $\nabla f(v)$ is orthogonal to T_v, the tangent space of M at v.

7. Suppose h' does not vanish at any point of M and suppose $f|_M$ has a local extremum at $v \in M$.

As before, the matrix of $h'(x)$ for any $x \in M$ is a $1 \times n$ matrix, and since $h'(x)$ is non-zero (by assumption), the rank of this matrix is 1. By the rank-nullity theorem, we see that

$$n = \dim T_x + 1$$

where T_x is the tangent space at the point $x \in M$, and hence the dimension of normal space at x, which is $N_x = T_x^{\perp}$ satisfies

$$\dim N_x = 1$$

In particular, we know that $\dim N_v = 1$. First, observe that $\nabla h(v)$ is the row vector of the $1 \times n$ matrix $h'(v)$. By HW-5 problem **5.**, we know that the row vector $\nabla h(v)$ forms a basis of N_v.

Finally, since by the local extremum condition, we see that $\nabla f(v)$ lies in N_v by problem **6.** part **(b)**, and hence we see that

$$\nabla f(v) = \lambda \nabla h(v)$$

for some unique scalar $\lambda \in \mathbb{R}$.