Notes on Hyperbolic Geometry

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1 Introduction

For people who have taken real calculus, you know that the arc length of a curve in $\mathbb{R}^2$ $\gamma : [a, b] \to \mathbb{R}^2$, where $\gamma(t) = (x(t), y(t))$, is defined as

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

The reason behind this formula is that locally we have

$$(\Delta s)^2 \sim (\Delta x)^2 + (\Delta y)^2$$

by the Pythagorean Theorem. More precisely, when "$\Delta t \to 0''$", we get

$$(ds)^2 = (dx)^2 + (dy)^2 .$$

The differential ds is called the *arc length element*, or the *Riemannian metric*, of $\mathbb{R}^2$. The Riemannian metric introduced above is called the *standard Euclidean metric.*

One might wonder whether there exists other kinds of metric on $\mathbb{R}^2$. In fact, given a positive definite form

$$(ds)^2 = a(x, y)(dx)^2 + b(x, y) dxdy + c(x, y)(dy)^2$$

(positive definite means that $b^2 - 4ac > 0$ for all x, y), ds allows us to measure arc lengths along curves, and thus allows us to measure the "distance" between two points, i.e. a *metric*, and to determine the **distance minimizing curves**, called *geodesics.*

Remark 1.1. In the standard Euclidean metric, the geodesics are straight lines. (You can prove this using the triangle inequality.)

Definition 1.2. If (M, d) is a metric space, an isometry is a function $f : M \to M$ such that $d(f(x), f(y)) = d(x, y)$ for all $x, y \in M$.

Proposition 1.3. *Isometries send geodesics to geodesics.*

Proof. This is immediate from the definition of isometries. □

For the following two exercises, $\mathbb{R}^2$ will be equipped with the standard Euclidean metric.

Exercise 1.4. (This is a linear algebra exercise.) Show that every **linear** isometry on $\mathbb{R}^2$ is a composition of a rotation and possibly a reflection.

Exercise 1.5. (This is still a linear algebra exercise.) Show that every isometry on $\mathbb{R}^2$ is a composition of a translation, a rotation, and possibly a reflection. Moreover, a stronger result holds. See the following exercises.

Exercise 1.6. Show that every translation is a composition of two reflections, and every rotation is a composition of two reflections. Hence every isometry on $\mathbb{R}^2$ is a composition of reflections.

Theorem 1.7. *(Change of variables formula.) Given a Riemannian metric on $\mathbb{R}^2$, ds. A function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is an isometry if and only if ds is invariant under f.*

Example 1.8. Assume $\mathbb{R}^2$ has the standard Euclidean metric, and let $f : \mathbb{R} \to \mathbb{R}$ be the liner transformation represented by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, i.e., $f(x, y) = (ax + by, cx + dy)$. Then the new Riemannian metric under f would be

$$\begin{aligned} (ds')^2 &= (d(ax+by))^2 + (d(cx+dy))^2 \\ &= (adx + bdy)^2 + (cdx + ddy)^2 \\ &= \cdots \\ &= \left(a^2 + c^2\right)(dx)^2 + 2(ab + cd)(dx)(dy) + \left(b^2 + d^2\right)(dy)^2 . \end{aligned}$$

In order for f to be an isometry, we need $a^2 + c^2 = b^2 + d^2 = 1$ and $ab + cd = 0$. The readers can check that the only two possibilities are

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\theta & \mp\sin\theta \\ \sin\theta & \pm\cos\theta \end{pmatrix}$$

which correspond to a rotation by θ, and a rotation by θ followed by a reflection.

2 Hyperbolic Geometry

Definition 2.1. The Hyperbolic plane (without boundary) is defined as

$$\mathbb{H}^2 = \left\{(x, y) \in \mathbb{R}^2 \mid y > 0\right\}$$

with the Riemannian metric defined as

$$(ds)^2 = \frac{(dx)^2 + (dy)^2}{y^2}$$

Example 2.2. 1. Consider a horizontal line segment from (a, y) to (b, y). Note that $dy = 0$ since the curve is horizontal. The length of this line segment is

$$\begin{aligned}
\int_a^b ds &= \int_a^b \sqrt{\frac{(dx)^2 + (dy)^2}{y^2}} \\
&= \int_a^b \frac{dx}{y} \\
&= \frac{b-a}{y}
\end{aligned}$$

2. Consider a vertical line segment from (x, a) to (x, b). Note that $dx = 0$ since the curve is vertical. The length of this line segment is

$$\begin{aligned}
\int_a^b ds &= \int_a^b \sqrt{\frac{(dx)^2 + (dy)^2}{y^2}} \\
&= \int_a^b \frac{dy}{y} \\
&= \log(b) - \log(a)
\end{aligned}$$

Note that it does not depend on x. Also, since $\log(0) = -\infty$, the x-axis can never be reached within finite distance!

3. Consider a arc of a circle centered at the origin $\gamma(\theta) = (r\cos\theta, r\sin\theta), \theta \in [\alpha, \beta]$. We get that

$$\begin{aligned}
dx &= -r\sin\theta d\theta \\
dy &= r\cos\theta d\theta,
\end{aligned}$$

so $(dx)^2 + (dy)^2 = r^2 (d\theta)^2$. Hence the arc length is

$$\begin{aligned}
\int_\alpha^\beta ds &= \int_\alpha^\beta \sqrt{\frac{(dx)^2 + (dy)^2}{y^2}} \\
&= \int_\alpha^\beta \sqrt{\frac{r^2 (d\theta)^2}{r^2\cos^2\theta}} \\
&= \int_\alpha^\beta \sec\theta d\theta \\
&= \log|\sec\beta + \tan\beta| - \log|\sec\alpha + \tan\alpha|
\end{aligned}$$

Note that it does not depend on r. Also, the arc length goes to infinity when $\alpha \to 0$ or $\beta \to \pi$.

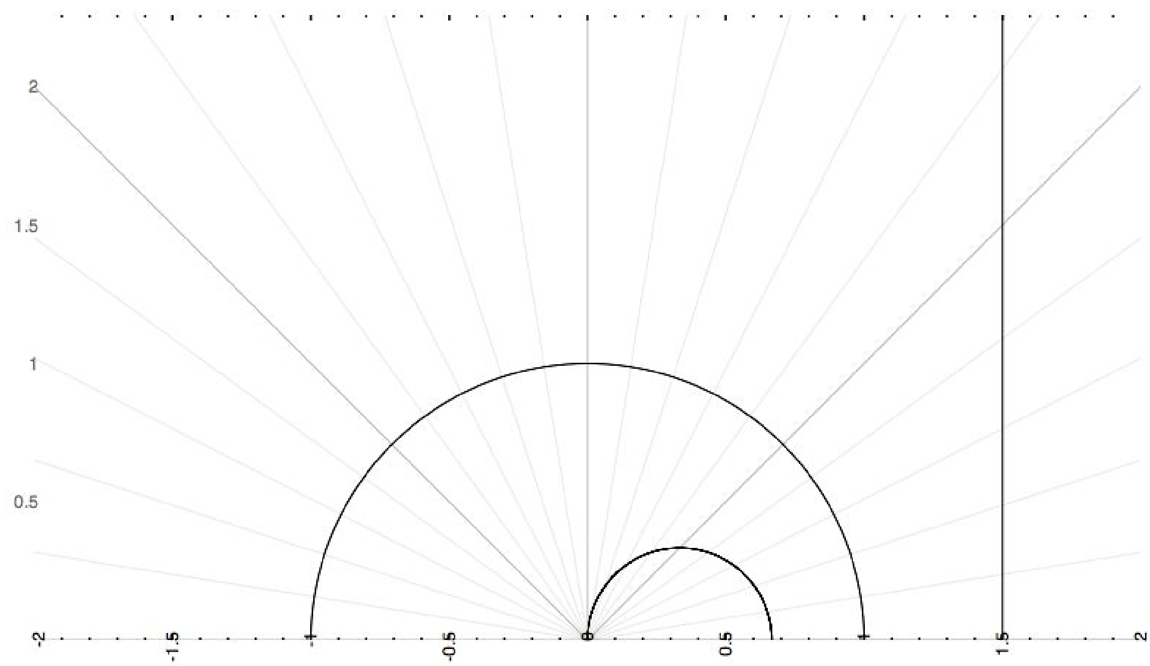

Figure 1: The inversion via the unit circle sends the vertical line to the smaller half-circle.

Exercise 2.3. Show that the following four operations are isometries on $\mathbb{H}^2$.

1. Horizontal Translation: $(x, y) \mapsto (x + t, y)$.
2. Dilation with respect to a point on the x-axis (say $((a, 0))$: $(x, y) \mapsto (r\,(x - a) + a, ry)$, where $r > 0$.
3. Horizontal Reflection (say via the line $x = a$): $(x, y) \mapsto (2a - x, y)$.
4. Inversion via a circle with center on the x-axis (say centered at $(a, 0)$ with radius r): $(x, y) \mapsto \left(\frac{r^2(x-a)}{(x-a)^2+y^2} + a, \frac{r^2 y}{(x-a)^2+y^2}\right)$.

In fact, one can show that all isometries on $\mathbb{H}^2$ are compositions of the four operations above. Moreover, a stronger result holds. See the following exercise.

Exercise 2.4. Show that a horizontal translation is a composition of two reflections, a dilation is a composition of two inversions, and a reflection is a composition of three inversions. Hence all isometries on $\mathbb{H}^2$ are compositions of inversions.

The following is a more advanced result in differential geometry.

Theorem 2.5. *The geodesics (length minimizing curves) in $\mathbb{H}^2$ are either parts of vertical lines or parts of semicircles whose centers are on the x-axis.*

3 Hyperbolic plane with boundary

Definition 3.1. The hyperbolic plane with boundary, denoted $\overline{\mathbb{H}^2}$, is defined as

$$\overline{\mathbb{H}^2} = \mathbb{H}^2 \cup \{(x, 0)\} \cup \{\infty\}$$

where ∞ represents all infinity points on the upper half plane, i.e., (∞, y), (x, ∞), and $(-\infty, y)$.

The part $\{(x, 0)\} \cup \{\infty\}$ is called the boundary of $\overline{\mathbb{H}^2}$. We can still define a metric on $\overline{\mathbb{H}^2}$ by defining the distance from a point on the boundary to any other point to be ∞.

Proposition 3.2. *The isometries introduced in Exercise 2.3 are still isometries on* $\overline{\mathbb{H}^2}$*.*

Proof. It suffices to show that these operations send points on the boundary to points on the boundary.

1. Horizontal Translation: $\infty \mapsto \infty, (x, 0) \mapsto (x + t, 0)$.
2. Dilation: $\infty \mapsto \infty, (x, 0) \mapsto (r(x - a) + a, 0)$.
3. Horizontal Reflection: $\infty \mapsto \infty, (x, 0) \mapsto (2a - x, 0)$.
4. Inversion: $\infty \mapsto (a, 0), (a, 0) \mapsto \infty$, and $(x, 0) \mapsto \left(\frac{r^2}{(x-a)} + a, 0\right)$, for $x \neq a$.

$\square$

4 Triangles on the Hyperbolic Plane

Definition 4.1. A *triangle* in $\overline{\mathbb{H}^2}$ consists of three points in $\overline{\mathbb{H}^2}$ with geodesics connecting the points. Two triangles are *congruent* if there exists an isometry sending one to the other. The *angle* between two edges is the angle between the tangent lines of the edges at their intersection.

Triangles in the hyperbolic plane behave differently from in the Euclidean plane. The following theorem is an example.

Theorem 4.2. *(Gauss-Bonnet) A triangle in* $\overline{\mathbb{H}^2}$ *with interior angles* $\theta_1, \theta_2, \theta_3$ *and area* A *satisfies the following relation.*

$$\theta_1 + \theta_2 + \theta_3 + A = \pi$$

In particular, every triangle with all three vertices on the boundary has the same area π, since its interior angles are all zero. In fact, all these triangles are congruent. For example, the following two triangles are congruent via the inversion via the unit circle.

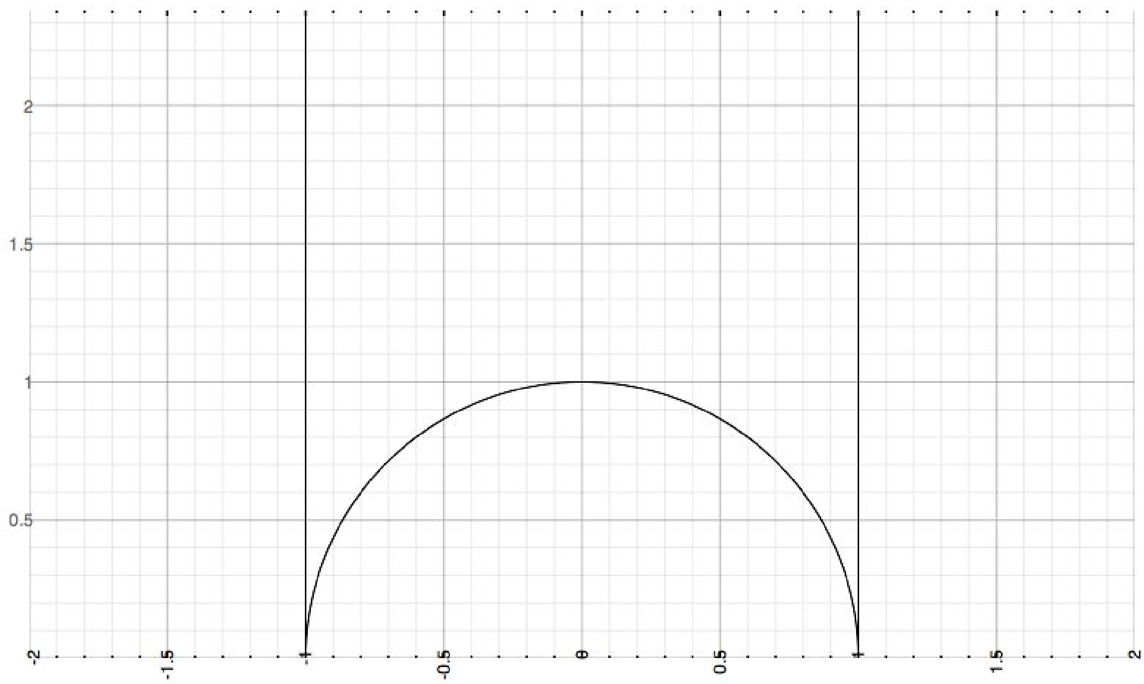

Figure 2: A triangle with vertices ∞, $(1, 0)$, and $(-1, 0)$.

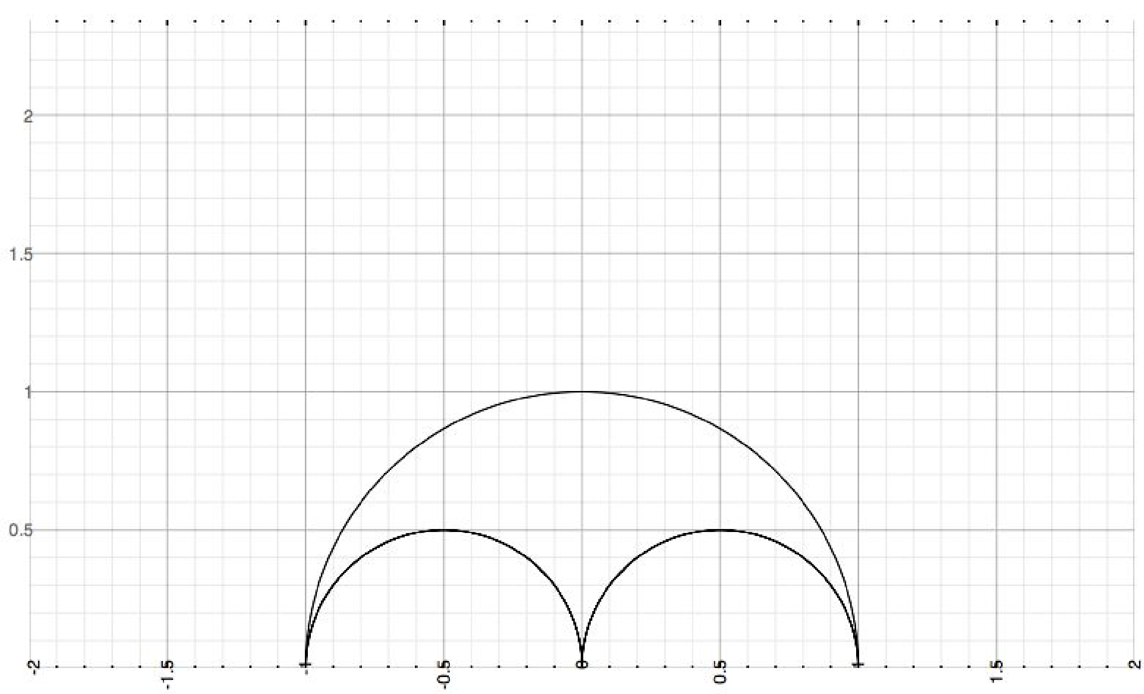

Figure 3: A triangle with vertices $(-1, 0)$, $(0, 0)$, and $(1, 0)$.