Precalculus Formulas

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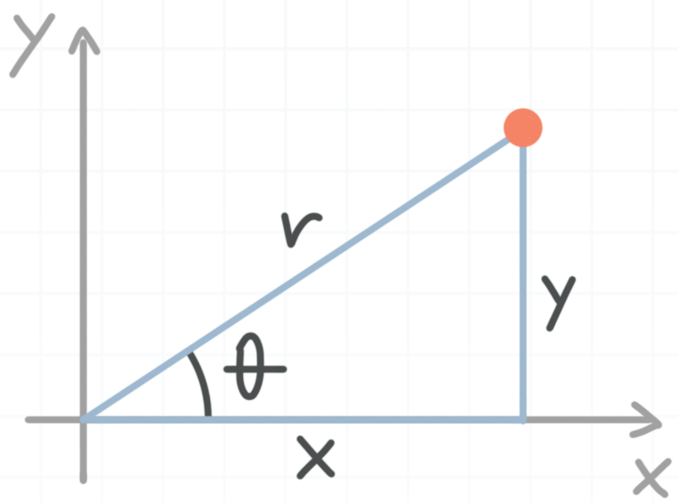
Polar curves

Polar coordinates



$$y = r\sin\theta \qquad x^2 + y^2 = r^2$$

$$x = r\cos\theta \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Multiple ways to express polar points

Coterminal angles: angles that differ by one full 2π rotation

Set of angles coterminal with θ

$\alpha = \theta + n(2\pi)$, where n is any integer

The pole: $(0,0)$, in polar coordinates

Expressing the same point in polar coordinates

1. Keep the value of r the same but add or subtract any multiple of 2π from θ.

2. Change the value of r to $-r$ while we add or subtract any odd integer multiple of π from θ.

Graphing polar curves in a rectangular system

Steps for sketching a polar curve in the rectangular plane

1. Set the argument of the trig function equal to $\pi/2$, and then solve this new equation for θ

2. Evaluate the polar curve at multiples of θ, starting with $\theta = 0$

3. Plot the resulting points in the xy-plane, treating the horizontal axis as the θ-axis (instead of the x-axis), and treating the vertical axis as the r-axis (instead of the y-axis).

4. Connect the plotted points with a smooth curve.

Graphing circles

Equations of circles

$r = a$	Circles centered at the pole
$r = c \sin\theta$	Circles centered on the vertical axis
$r = c \cos\theta$	Circles centered on the horizontal axis

Properties of circles

- Sine circles are symmetric around the vertical axis
- Cosine circles are symmetric around the horizontal axis
- When $c > 0$ the circle sits on the positive side of the axis
- When $c < 0$ the circle sits on the negative side of the axis

Graphing roses

Equations of roses

$$r = c\cos(n\theta)$$

$$r = c\sin(n\theta)$$

Properties of roses

- The rose has $|2n|$ petals when n is even
- Tho rose has $|n|$ petals when n is odd
- The petals extend to a distance of $r = |c|$
- If the rose has one petal on the horizontal axis and none on the vertical axis, it’s a cosine rose
- If the rose has one petal on the vertical axis and none on the horizontal axis, it’s a sine rose

Graphing cardioids

Equations of cardioids

$$r = c \pm c \cos \theta$$

$$r = c \pm c \sin \theta$$

Properties of cardioids

- $r = c + c \sin \theta$ is symmetric on the *vertical axis* and sits mostly *above* the horizontal axis
- $r = c - c \sin \theta$ is symmetric on the *vertical axis* and sits mostly *below* the horizontal axis
- $r = c + c \cos \theta$ is symmetric on the *horizontal axis* and sits mostly *above* the vertical axis
- $r = c - c \cos \theta$ is symmetric on the *horizontal axis* and sits mostly *below* the vertical axis
- Furthest distance from the pole is $2c$

Graphing limaçons

Equations of limaçons

$$r = a \pm b \cos \theta$$

$$r = a \pm b \sin \theta$$

Properties of limaçons

- $r = a + b\sin\theta$ is symmetric on the *vertical axis* and sits mostly *above* the horizontal axis
- $r = a - b\sin\theta$ is symmetric on the *vertical axis* and sits mostly *below* the horizontal axis
- $r = a + b\cos\theta$ is symmetric on the *horizontal axis* and sits mostly *above* the vertical axis
- $r = a - b\cos\theta$ is symmetric on the *horizontal axis* and sits mostly *below* the vertical axis

Ratio of a/b

$a/b = 1$	The limaçon is a cardioid
$a/b < 1$	The limaçon includes a small loop
$1 < a/b < 2$	The limaçon will include a small dip
$a/b \geq 2$	The limaçon will include the smallest dip yet

Graphing lemniscates

Equations of lemniscates

$$r^2 = \pm\, c^2 \sin(2\theta)$$

$$r^2 = \pm\, c^2 \cos(2\theta)$$

Properties of lemniscates

- Lemniscates are always symmetric around the pole
- The argument is always 2θ
- The lemniscate always has two loops
- The loops extend out a distance of c
- The positive sine lemniscate $r^2 = c^2 \sin(2\theta)$ lies in the first and third quadrants
- The negative sine lemniscate $r^2 = -c^2 \sin(2\theta)$ lies in the second and fourth quadrants
- The positive cosine lemniscate $r^2 = c^2 \cos(2\theta)$ lies along the horizontal axis
- The negative cosine lemniscate $r^2 = -c^2 \cos(2\theta)$ lies along the vertical axis

Complex numbers

Complex numbers

Imaginary number: $i^2 = -1$ or $i = \sqrt{-1}$

Powers of imaginary numbers

$$i^0 = 1$$

$i^1 = \sqrt{-1}$

$i^2 = -1$

$i^3 = -i$

Complex number: $z = a + bi$

- Real part a, or Re(z)
- Imaginary part b, or Im(z)
- If the complex number includes only the real part, it’s a real number; if it includes only the imaginary part, it’s a pure imaginary number

Complex number operations

Complex conjugate: The complex number created by switching the sign on the imaginary part of a complex number

Graphing complex numbers

Complex plane, Argand plane: A plane with a horizontal axis representing the real part of the complex number, and a vertical axis representing the imaginary part of the complex number

Distances and midpoints

Distance between two points

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Complex numbers in polar form

Absolute value, magnitude of a complex number: Distance from the origin of the complex point

$$r = |z| = \sqrt{a^2 + b^2}$$

Polar form of a complex number

$$z = r(\cos\theta + i\sin\theta), \text{ with } \theta = \arctan\left(\frac{b}{a}\right)$$

Exponential form of a complex number

$$z = re^{i\theta}$$

Multiplying and dividing polar forms

Product of complex numbers

$$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

Quotient of complex numbers

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

Powers of complex numbers and De Moivre's Theorem

De Moivre's Theorem

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$

Roots of complex numbers

nth roots of a complex number

$$\sqrt[n]{z} = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right]$$

$$\sqrt[n]{z} = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 360^\circ k}{n}\right) + i\sin\left(\frac{\theta + 360^\circ k}{n}\right)\right]$$

Matrices

Matrix dimensions and entries

Matrix: a rectangular array of values, where each value is an entry in both a row and a column. Matrix dimensions are always described as "rows × columns."

Representing systems with matrices

Augmented matrix: A matrix that represents a system of linear equations, where each row represents one equation in the system, and each column represents a different variable, or the constants

Simple row operations

Row operations that don't change the value of the matrix

- Switching any two rows
- Multiplying a row by a constant (multiplying by a scalar)
- Replacing a row with the sum of another row and itself

Gauss-Jordan elimination and reduced row-echelon form

Row-echelon form (ref)

1. All rows consisting of only 0s are at the bottom of the matrix, and
2. the first non-zero entry in each row sits in a column to the right of the first non-zero entries in all the rows above it. In other words, the non-zero entries sit in a staircase pattern.

Pivot, pivot entry: First non-zero entry in each row.

Pivot column: Any column that houses a pivot entry

Reduced row-echelon form (rref)

1. The matrix is in row-echelon form, and

2. all the pivot entries are equal to 1, and

3. all of the non-pivot entries in the matrix are equal to 0 (other than the constants in the far-right column).

Gauss-Jordan elimination: An algorithm for putting a matrix into reduced row-echelon form

1. Optional: Pull out any scalars from each row in the matrix.

2. If the first entry in the first row is 0, swap that first row with another row that has a non-zero entry in its first column. Otherwise, move to step 3.

3. Multiply through the first row by a scalar to make the leading entry equal to 1.

4. Add scaled multiples of the first row to every other row in the matrix until every entry in the first column, other than the leading 1 in the first row, is a 0.

5. Go back step 2 and repeat the process until the matrix is in reduced row-echelon form.

Matrix addition and subtraction

Properties of matrix addition: matrix addition is commutative and associative

$$A + B = B + A$$

$$(A + B) + C = A + (B + C)$$

Properties of matrix subtraction: matrix subtraction is not commutative and not associative

$$A - B \neq B - A$$

$$(A - B) - C \neq A - (B - C)$$

Scalar multiplication and zero matrices

Zero matrix: A matrix full of zero entries

Opposite matrices: Matrices K and $-K$. To get the opposite of a matrix, multiply it by a scalar of -1.

Matrix multiplication

Order matters: Matrix multiplication is not commutative so $A \cdot B$ won't give us the same result as $B \cdot A$.

Matching dimensions: Matrices can only be multiplied when we have the same number of columns in the first matrix as rows in the second matrix.

The dimensions of the product will be given by the number of rows from the first matrix and the number of columns from the second matrix.

Dot product: The tool we'll use to multiply an entire row of one matrix by an entire column of another matrix

Properties of matrix multiplication

Not commutative	$AB \neq BA$
Associative	$(AB)C = A(BC)$
Distributive	$A(B + C) = AB + AC$

Identity matrices

Identity matrix I: Always a square matrix in which the main diagonal is all entries of 1, and the rest of the matrix has entries of 0

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Properties of the identity matrix

$IA = A$	I has the same number of columns as A has rows
$AI = A$	I has the same number of rows as A has columns

Transformations

Transformation matrix: In a 2×2 transformation matrix, the first column tells us where the unit vector $(1,0)$ will land after the transformation, the second column tells us where the unit vector $(0,1)$ will land.

Matrix inverses, and invertible and singular matrices

Determinant of a matrix

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Inverse of a matrix

$$M^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Adjugate matrix

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Invertibility: Singular matrices don't have an inverse, invertible matrices have an inverse.

Solving systems with inverse matrices

Solution to the system of equations: $\vec{a} = M^{-1}\vec{b}$

Solving systems with Cramer's Rule

Cramer's Rule: The value of any variable in the system is give by D_v/D, where D_v is the determinant of the coefficient matrix with the answer column matrix substituted into the v-column, and where D is the determinant of the coefficient matrix.

Partial fractions

Fraction decomposition

Partial fractions decomposition: A tool we use to break down and rewrite rational functions

Rational functions: Fractions in which both the numerator and denominator are polynomial functions

Four types of factors

- Linear factors: First degree factors
- Quadratic factors: Second degree factors
- Distinct factors: Factors that are different
- Repeated factors: Factors that are the same

Expressions for the numerator: A for linear factors, $Ax+B$ for quadratic factors

Distinct linear factors

Solving for the constants in a distinct linear factor decomposition

1. Set each distinct linear factor equal to 0, and solve the equation for x.

2. If we're trying to solve for A, remove the distinct linear factor associated with A from the factored form of the original function.

3. Evaluate the new function at the value of x that we solved for in Step 1.

Repeated linear factors

Solving for the constants in a repeated linear factor decomposition

1. Combine the fractions on the right side of the equation by finding a common denominator equivalent to the function's original denominator.

2. Once the denominators on both sides are equivalent, set the numerators equal to each other.

3. Evaluate the equation at $x = k$, where k is the value that makes the repeated factor equal to 0. This will solve for the constant associated with the first-power factor.

4. Simplify the remaining equation and equate coefficients to build a system of equations that can be solved for the other constants.

Distinct quadratic factors

Solving for the constants in a distinct quadratic factor decomposition

1. Combine the fractions on the right side of the equation by finding a common denominator equivalent to the function's original denominator.

2. Once the denominators on both sides are equivalent, set the numerators equal to each other.

3. Simplify the equation and equate coefficients to build a system of equations that can be solved for the constants.

Repeated quadratic factors

Solving for the constants in a repeated quadratic factor decomposition

1. Combine the fractions on the right side of the equation by finding a common denominator equivalent to the function's original denominator.

2. Once the denominators on both sides are equivalent, set the numerators equal to each other.

3. Simplify the equation and equate coefficients to build a system of equations that can be solved for the constants.

Mixed factors

Solving for the constants in a mixed factor decomposition

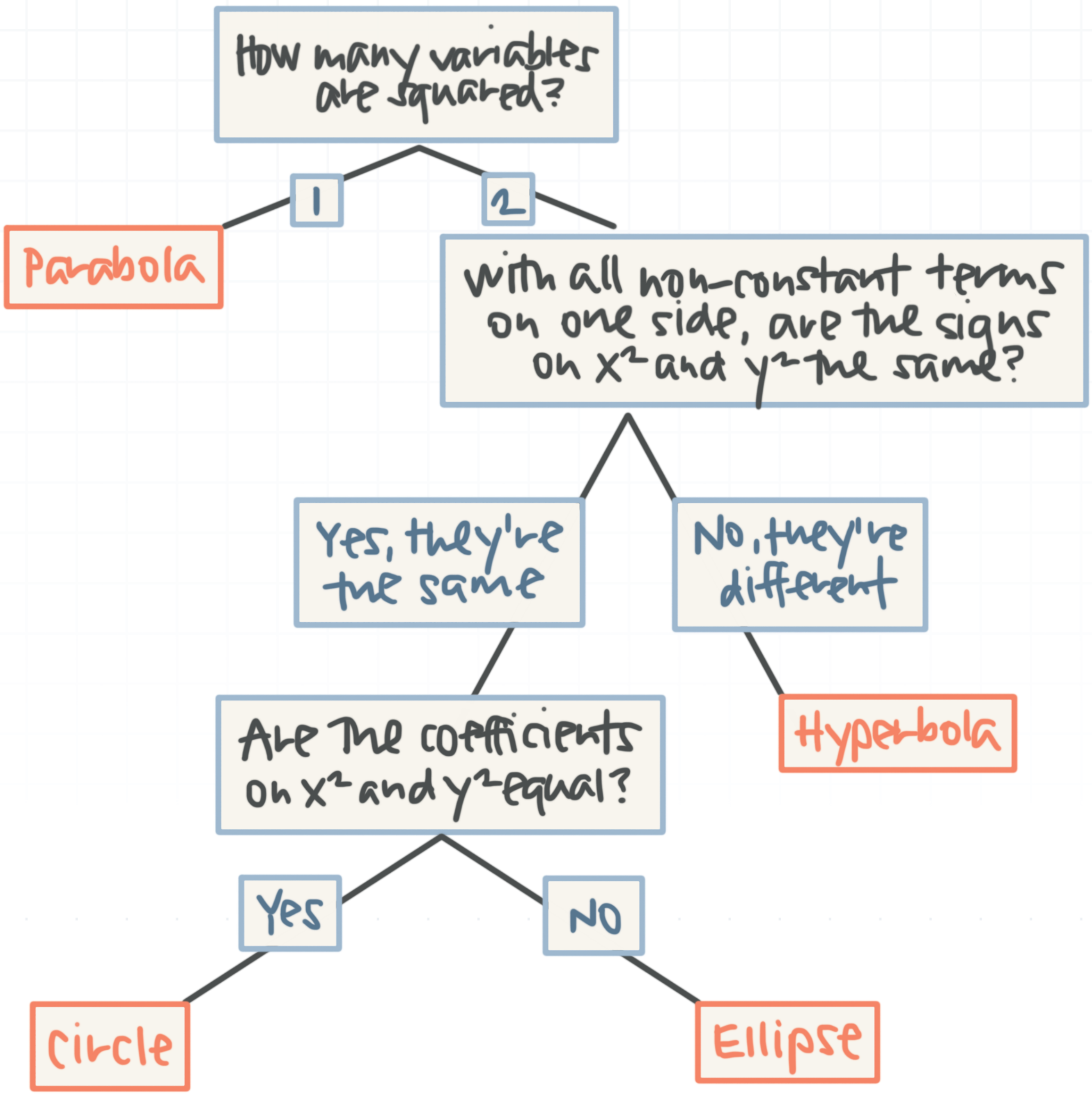
1. Combine the fractions on the right side of the equation by finding a common denominator equivalent to the function's original denominator.

2. Once the denominators on both sides are equivalent, set the numerators equal to each other.

3. Simplify the equation and equate coefficients to build a system of equations that can be solved for the constants.

Conic sections and analytic geometry

Identifying conic sections



Circles

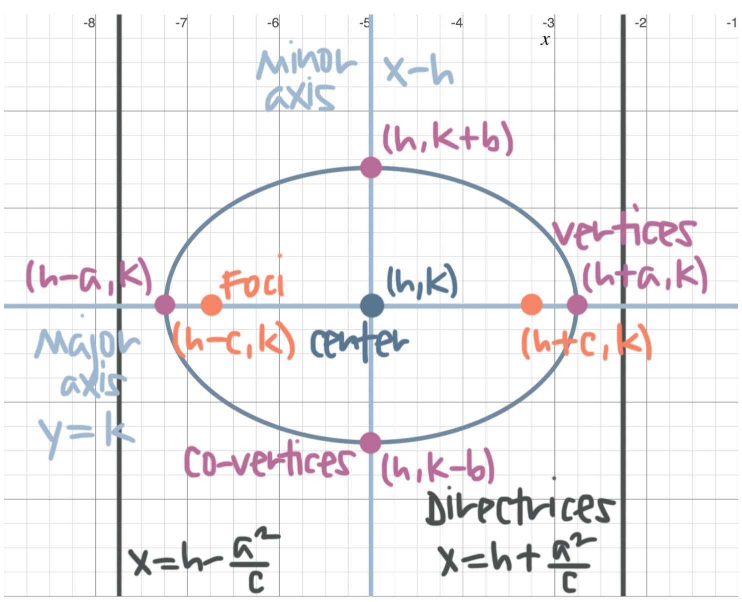
Standard form of the equation of a circle with center at (h, k) and radius r

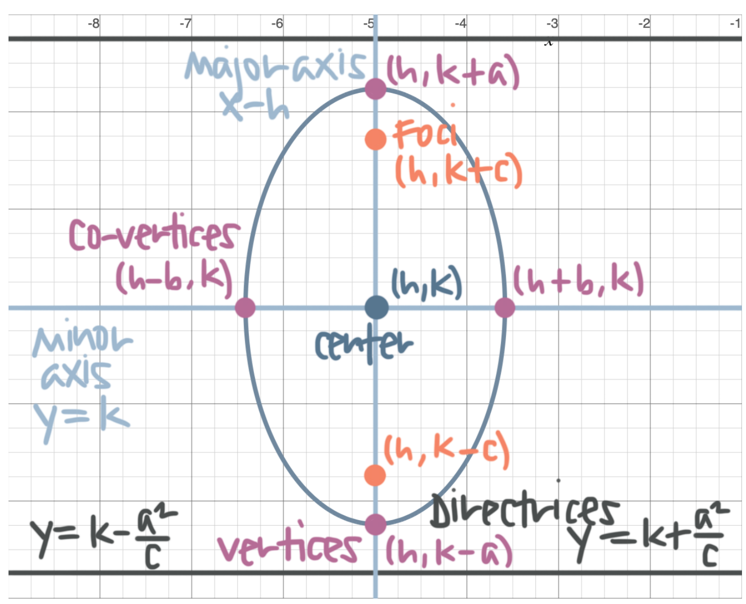
$$(x-h)^2+(y-k)^2=r^2$$

Ellipses

Geometry of ellipses centered at (h, k)

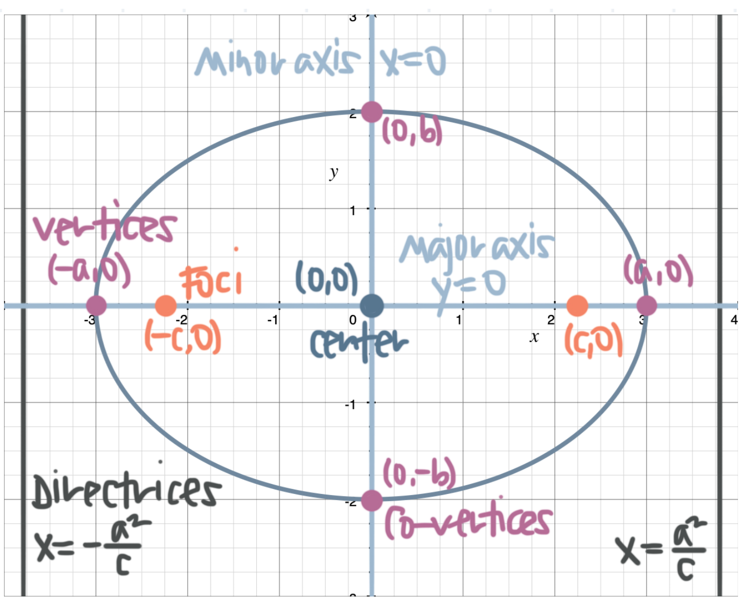
	Wide, $a \geq b > 0$	**Tall,** $a \geq b > 0$
Equation	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$	$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$
Center	(h, k)	(h, k)
Vertices	$(h \pm a, k)$	$(h, k \pm a)$
Co-vertices	$(h, k \pm b)$	$(h \pm b, k)$
Major axis	$y = k$	$x = h$
Minor axis	$x = h$	$y = k$
Foci	$(h \pm c, k)$ with $c^2 = a^2 - b^2$	$(h, k \pm c)$ with $c^2 = a^2 - b^2$
Directrices	$x = h \pm \frac{a^2}{c}$	$y = k \pm \frac{a^2}{c}$

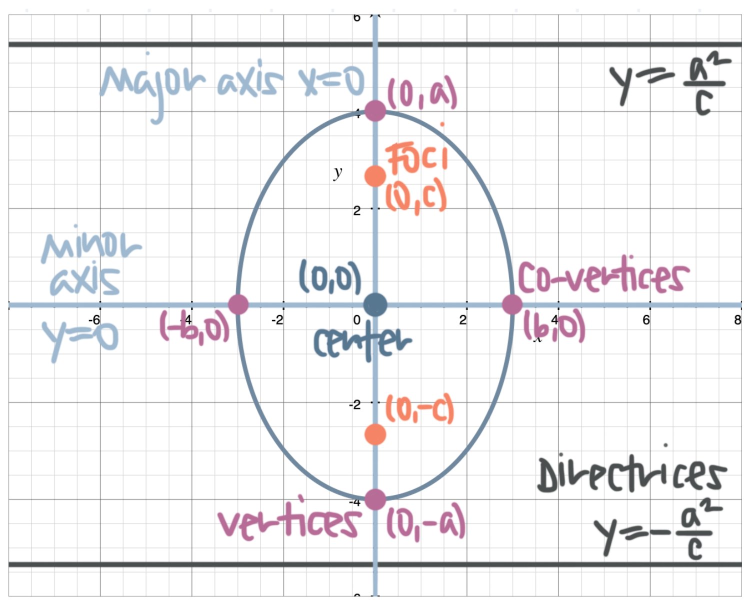




Geometry of ellipses centered at $(0,0)$

	Wide, $a \geq b > 0$	**Tall,** $a \geq b > 0$
Equation	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$
Center	$(0,0)$	$(0,0)$
Vertices	$(\pm a,0)$	$(0,\pm a)$
Co-vertices	$(0,\pm b)$	$(\pm b,0)$
Major axis	$y=0$	$x=0$
Minor axis	$x=0$	$y=0$
Foci	$(\pm c,0)$ with $c^2=a^2-b^2$	$(0,\pm c)$ with $c^2=a^2-b^2$
Directrices	$x=\pm\frac{a^2}{c}$	$y=\pm\frac{a^2}{c}$





Parabolas

Parabola: A parabola is the set of all points which are equidistant from its own focus and its directrix. The focus will be a point inside the parabola's "bowl", and the directrix will be a line outside the parabola that's perpendicular to the parabola's axis. The axis is the line of symmetry of the parabola, which runs through the parabola's vertex, the point at which the parabola intersects its axis.

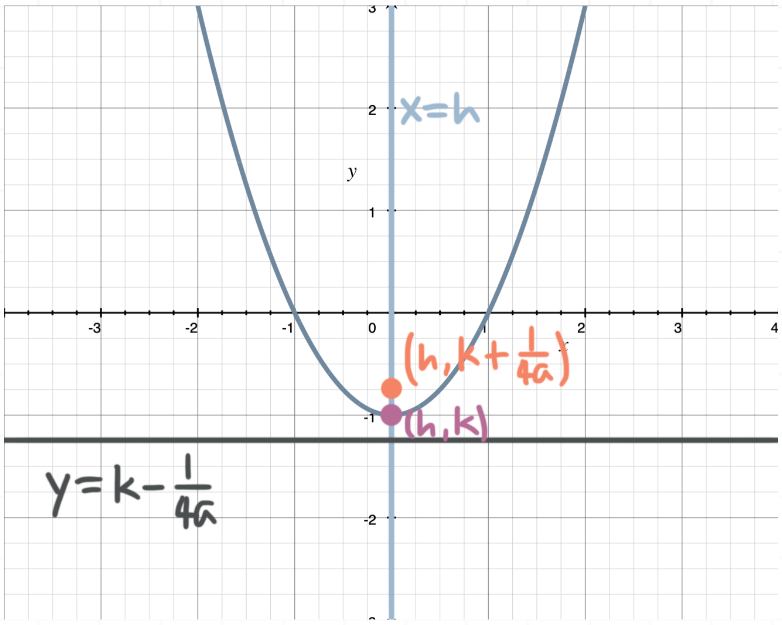
Three forms of the parabola's equation

Vertex form	$y = a(x-h)^2 + k$
Conics form	$4p(y-k) = (x-h)^2$
Standard form	$y = ax^2 + bx + c$

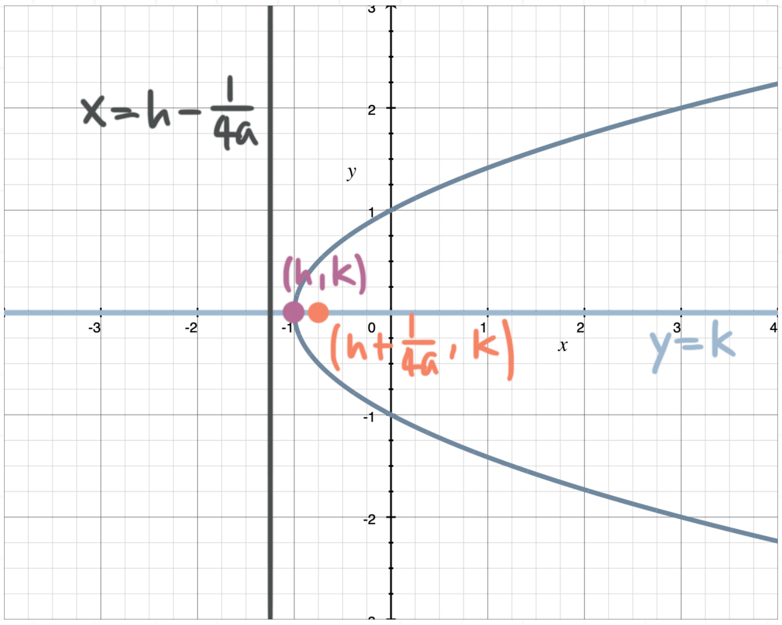
Geometry of vertex form

	Equation	**Vertex**	**Axis**	**Focus**	**Directrix**
Up/down shifted	$y = a(x-h)^2 + k$	(h, k)	$x = h$	$\left(h, k + \frac{1}{4a}\right)$	$y = k - \frac{1}{4a}$
Left/right shifted	$x = a(y-k)^2 + h$	(h, k)	$y = k$	$\left(h + \frac{1}{4a}, k\right)$	$x = h - \frac{1}{4a}$
Up/down origin	$y = ax^2$	$(0,0)$	$x = 0$	$\left(0, \frac{1}{4a}\right)$	$y = -\frac{1}{4a}$
Left/right origin	$x = ay^2$	$(0,0)$	$y = 0$	$\left(\frac{1}{4a}, 0\right)$	$x = -\frac{1}{4a}$

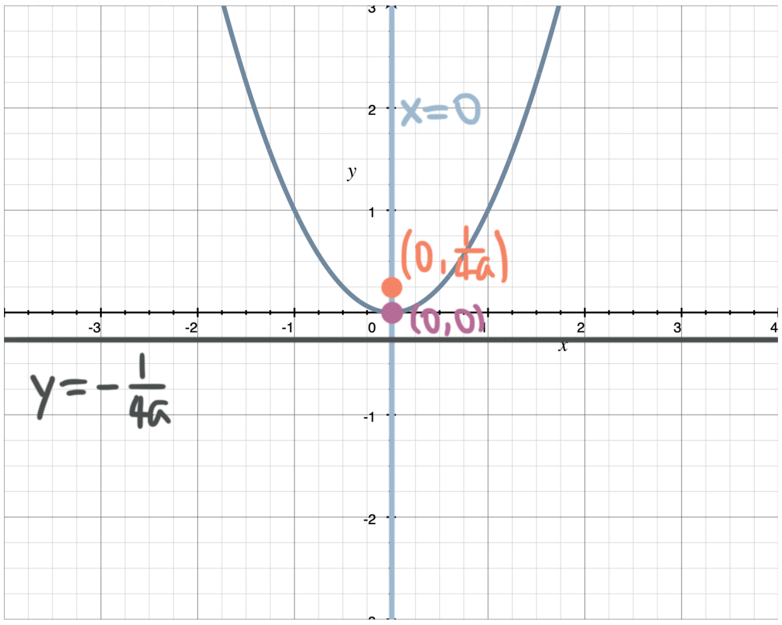
Up/down shifted:



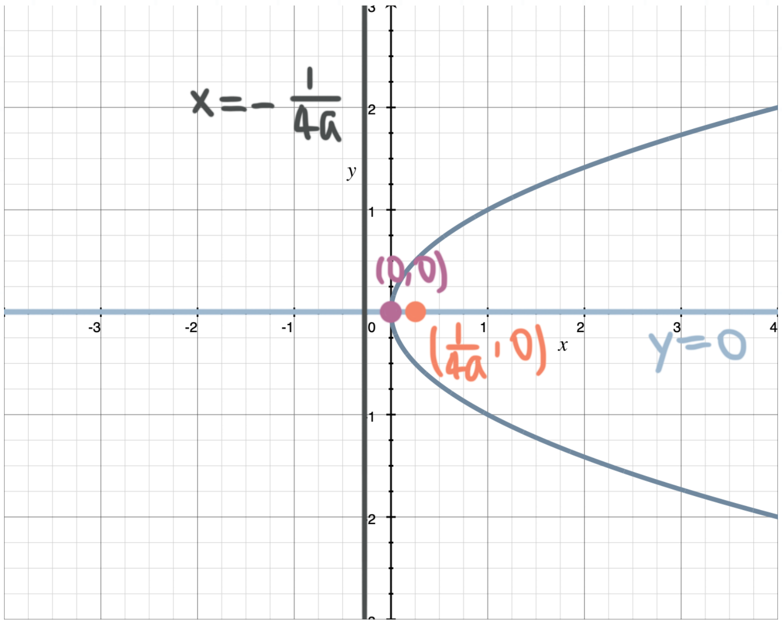
Left/right shifted:



Up/down origin:



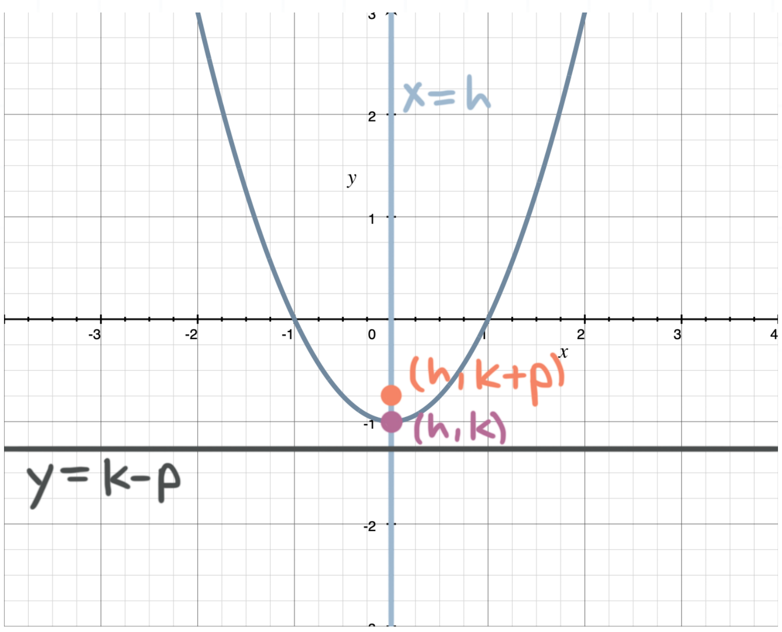
Left/right origin:



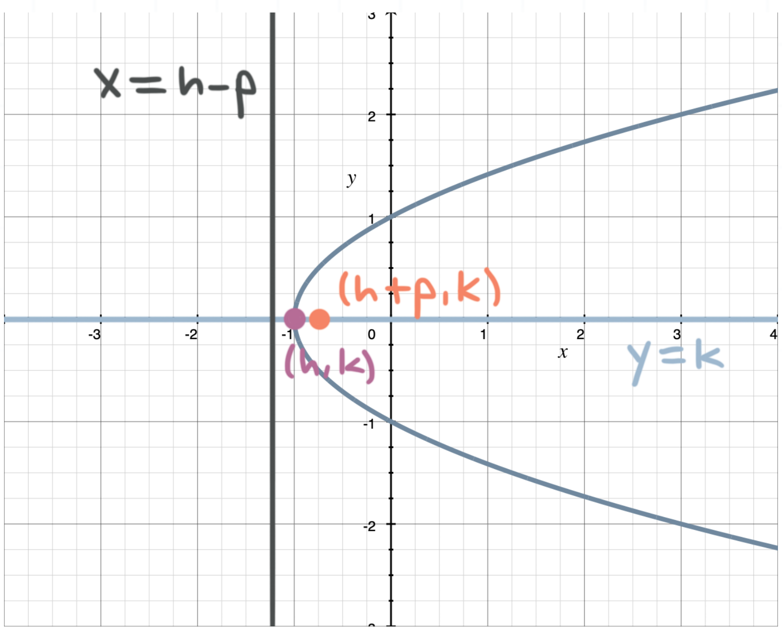
Geometry of conics form

	Equation	Vertex	Axis	Focus	Directrix
Up/down shifted	$4p(y-k)=(x-h)^2$	(h,k)	$x=h$	$(h,k+p)$	$y=k-p$
Left/right shifted	$4p(x-h)=(y-k)^2$	(h,k)	$y=k$	$(h+p,k)$	$x=h-p$
Up/down origin	$4py=x^2$	$(0,0)$	$x=0$	$(0,p)$	$y=-p$
Left/right origin	$4px=y^2$	$(0,0)$	$y=0$	$(p,0)$	$x=-p$

Up/down shifted:

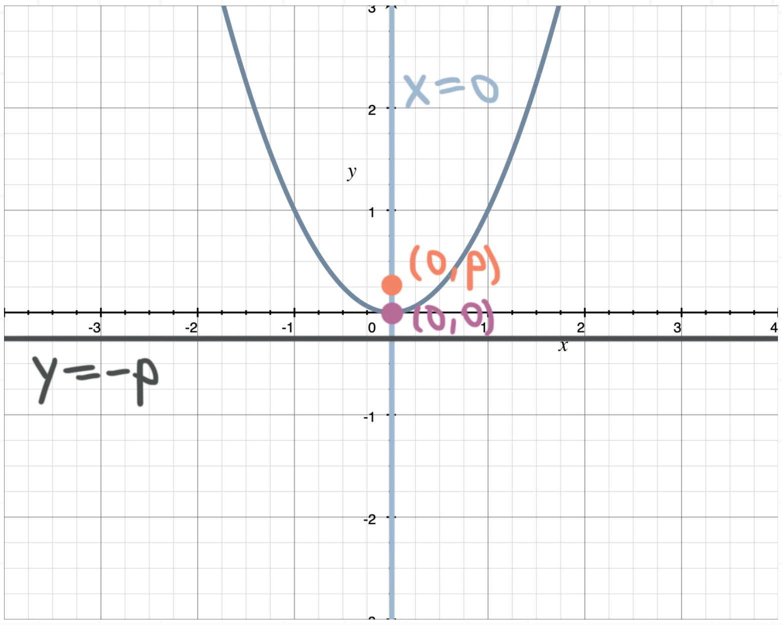


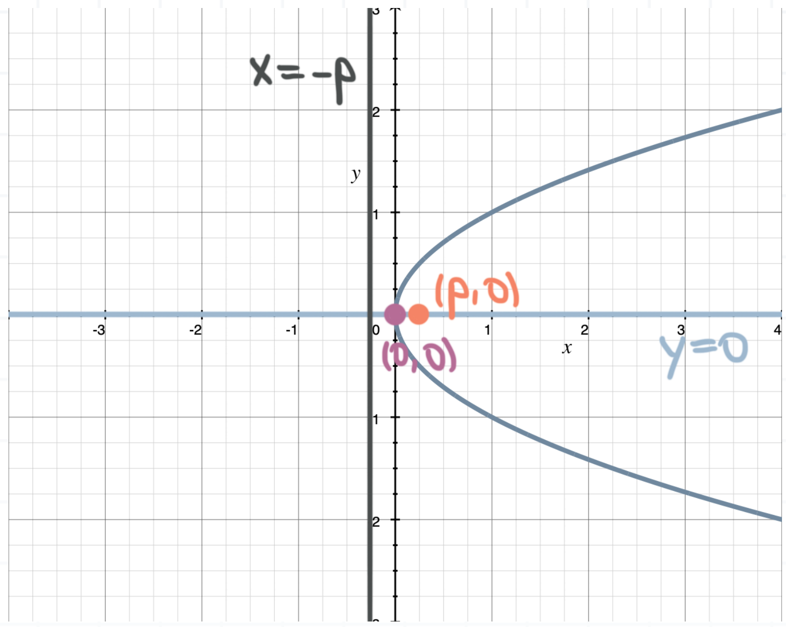
Left/right shifted:



Up/down origin:

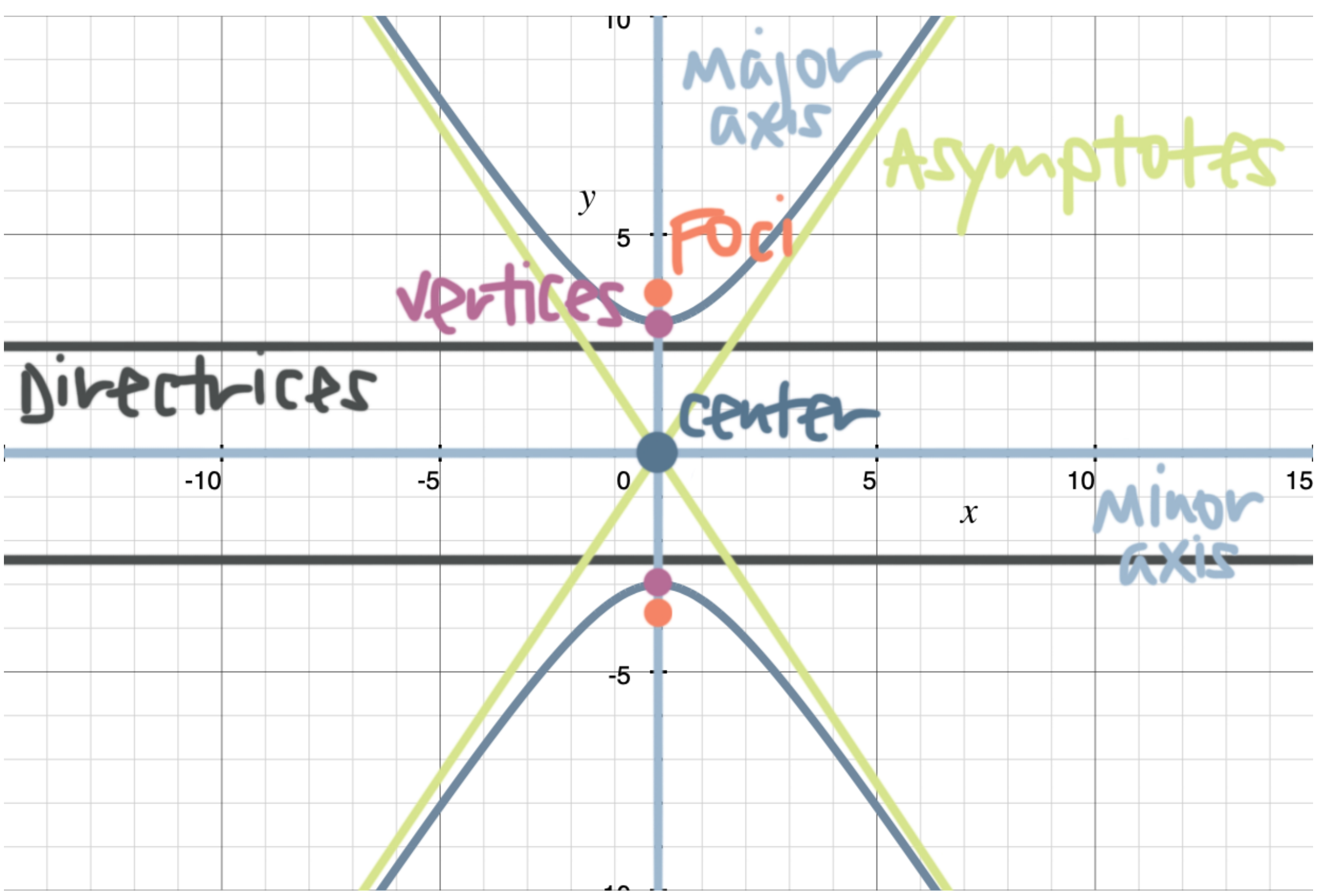
Left/right origin:





Hyperbolas

Hyperbola: A hyperbola is a lot like an ellipse, except that instead of the two halves opening toward each other, the two halves open away from each other. The hyperbola is the set of points for which the difference of distances from its two foci is constant.



Geometry of left/right hyperbolas

	Left/right shifted	Left/right origin
Equation	$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$
Center	(h,k)	$(0,0)$
Major, minor axis	$y=k$, $x=h$	$y=0$, $x=0$
Vertices	$(h\pm a,k)$	$(\pm a,0)$
Foci, $c=\sqrt{a^2+b^2}$	$(h\pm c,k)$	$(\pm c,0)$
Asymptotes	$y=\pm\frac{b}{a}x$	$y=\pm\frac{b}{a}x$
Directrices	$x=h\pm\frac{a^2}{c}$	$x=\pm\frac{a^2}{c}$

Geometry of up/down hyperbolas

	Up/down shifted	Up/down origin
Equation	$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$	$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$
Center	(h,k)	$(0,0)$
Major, minor axis	$x=h$, $y=k$	$x=0$, $y=0$
Vertices	$(h,k\pm a)$	$(0,\pm a)$
Foci, $c=\sqrt{a^2+b^2}$	$(h,k\pm c)$	$(0,\pm c)$

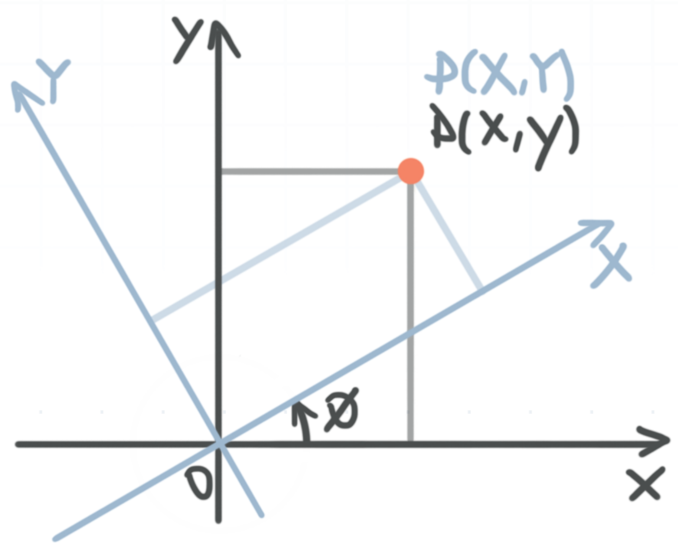
Asymptotes	$y = \pm \frac{a}{b}x$	$y = \pm \frac{a}{b}x$
Directrices	$y = k \pm \frac{a^2}{c}$	$y = \pm \frac{a^2}{c}$

Rotating axes

Standard form for the equation of a conic

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

Rotation of coordinate axes



Formulas for rotation

$$\cot(2\phi) = \frac{A - C}{B}$$

$$x = X\cos\phi - Y\sin\phi \qquad X = x\cos\phi + y\sin\phi$$

$$y = X\sin\phi + Y\cos\phi \qquad Y = -x\sin\phi + y\cos\phi$$

Half-angle identities

$$\cos\phi = \sqrt{\frac{1+\cos(2\phi)}{2}} \qquad \sin\phi = \sqrt{\frac{1-\cos(2\phi)}{2}}$$

Identifying the conic from the discriminant, $B^2 - 4AC$

- the conic section is a parabola when $B^2 - 4AC = 0$
- the conic section is an ellipse when $B^2 - 4AC < 0$
- the conic section is a hyperbola when $B^2 - 4AC > 0$

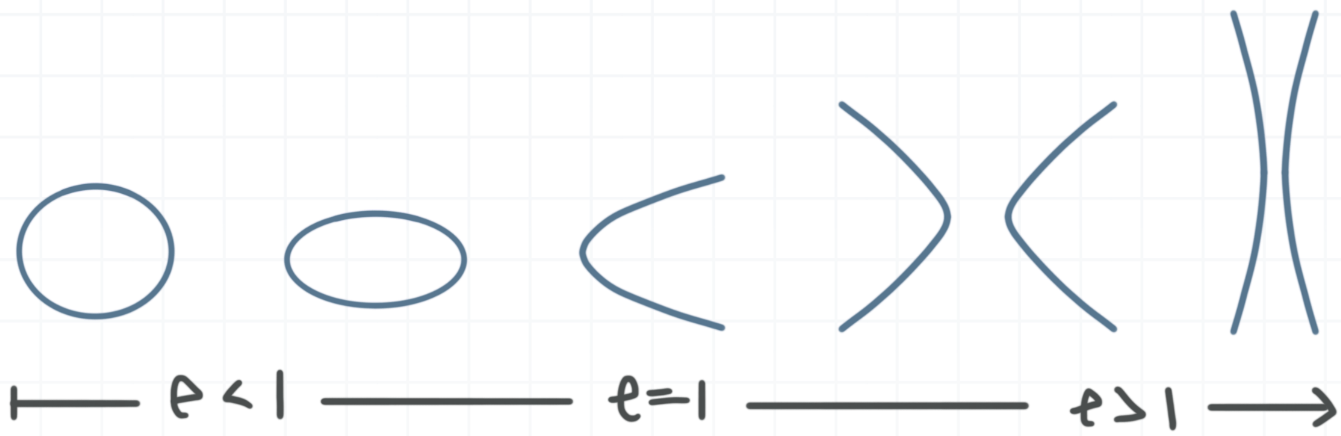
Polar equations of conics

Eccentricity: If we say that F is the focus, l is the directrix, and e is the eccentricity, a positive constant, then the distance of a point on the conic to the focus, divided by the distance of that same point to the directrix, is always equal to the eccentricity.

$$\frac{d(P,F)}{d(P,l)} = e$$

Identifying the conic from the eccentricity

- a parabola when $e = 1$
- an ellipse when $e < 1$
- a hyperbola when $e > 1$



Polar form of conics, in terms of eccentricity

$$r = \frac{ed}{1 + e\cos\theta} \qquad r = \frac{ed}{1 - e\cos\theta} \qquad r = \frac{ed}{1 + e\sin\theta} \qquad r = \frac{ed}{1 - e\sin\theta}$$

$$x = d \qquad x = -d \qquad y = d \qquad y = -d$$

Polar form of rotated conics, in terms of eccentricity

$$r = \frac{ed}{1 + e\cos(\theta - \alpha)} \qquad r = \frac{ed}{1 - e\cos(\theta - \alpha)} \qquad r = \frac{ed}{1 + e\sin(\theta - \alpha)} \qquad r = \frac{ed}{1 - e\sin(\theta - \alpha)}$$

$$x = d \qquad x = -d \qquad y = d \qquad y = -d$$

Parametric curves

Parametric curves and eliminating the parameter

Parametric equations: $x = f(t)$ and $y = g(t)$, which defined what happens to x and y individually as the parameter t changes

Eliminating the parameter: We can eliminate the parameter by substituting one parametric equation into the other, or by solving both equations for the parameter and then setting the curves equal to each other.

Direction of the parameter

Changing the value of t: Replacing t with $2t$ means the curve gets traced out twice as fast; replacing t with $-t$ means the curve gets traced out in the opposite direction.

W W W . K R I S T A K I N G M A T H . C O M