Unit - II

2. Probability

Introduction

We are familiar with the word 'experiment'. we perform experiment in Physics, Chemistry or Biology.

e.g. In chemistry, you estimate the exact amount of alkali required to neutralize acid using titration method or In Biological experiments, a type of diet is fed to animals and increase in their weights are recorded.

But in statistics, the word 'experiment' is used in a wider sense. It is not necessarily restricted to laboratory experiments.

Experiment:- An experiment is virtually any operation that results in one or more outcomes.

e.g. ① Appearing for F.Y. BSc examination is an experiment with possible outcomes as PASS or FAIL

② Releasing a stone from hand is an experiment with the outcome that 'it will fall on the ground'

③ Tossing a coin is an experiment with two possible outcomes. 'Head up' or 'Tail up'

④ Rolling a six faced die: outcomes are 1, 2, 3, 4, 5, 6.

There are two types of experiments are follows.

① Deterministic / Non-random / Predictable experiment

② Non-Deterministic / random / Probabilistic experiment.

① **Deterministic experiment:-**

If an experiment is performed repeatedly under essential homogeneous and similar conditions and

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its result or outcome is not subject to chance i.e. the outcome is unique or certain, then such kind of experiment or model will be called deterministic.

e.g. ① If a ball is thrown in the air it is sure that it will fall down.

② If temperature of water is increased to 100°c then it is sure that it starts boiling.

② Non-deterministic experiment:

A non-deterministic experiment or a random experiment is an experiment, for which there are more than one possible outcomes and the result of the experiment cannot be predicted in advance.

e.g. ① Sex of a new born baby is recorded
outcomes: Male or Female

② Rolling a die experiment
outcomes: 1, 2, 3, 4, 5, 6

③ Tossing a coin experiment
outcomes: Head or Tail

* Sample Space :-

Defⁿ:- "The set of all possible distinct outcomes of an experiment is called as sample space."

Sample space is denoted by Ω or S

Thus, a sample space is nothing but the universal set concerned with the experiment.

e.g. Consider the experiment of tossing a coin. The corresponding sample space is $\Omega = \{\text{Head}, \text{Tail}\}$

The elements of sample space are different outcomes which are called as sample points.

Thus, H and T are sample points of the above sample space Ω.

Depending upon the number of sample points, the sample spaces are categorized into two types

i) Discrete

ii) Continuous

i) Discrete Sample Space :-

A sample space containing a finite number of points or countably infinite points is called a discrete sample space.

In other words a discrete sample space is either a finite sample space or a countably infinite sample space.

a) Finite Sample Space :

A sample space Ω is called a finite sample space if the number of elements contained in Ω is finite. Such a sample space can be denoted as

$$\Omega = \{ w_1, w_2, \ldots w_n \}$$

where, n = numbers of elements

i.e. number of possible outcomes of the experiment

$w_1, w_2, \ldots, w_n$ are the outcomes.

e.g ① Suppose two coins are tossed. Then

$$\Omega = \{ HH, HT, TH, TT \}$$

$$n = 4$$

② A die is rolled. Then
$\Omega = \{1,2,3,4,5,6\}$
$n = 6$

b) Countably Infinite Sample Space:
A sample space Ω is called a countably infinite sample space if the number of elements in Ω are countably infinite.
It may be represented as
$\Omega = \{\omega_1, \omega_2, \omega_3, --- \omega_n, ---\}$

e.g. ① A student appears for an examination till he passes. Then
$\Omega = \{P, FP, FFP, FFFP\}$

② Number of accidents on the Mumbai-Pune road in a month.
$\Omega = \{0,1,2,3, ----\}$

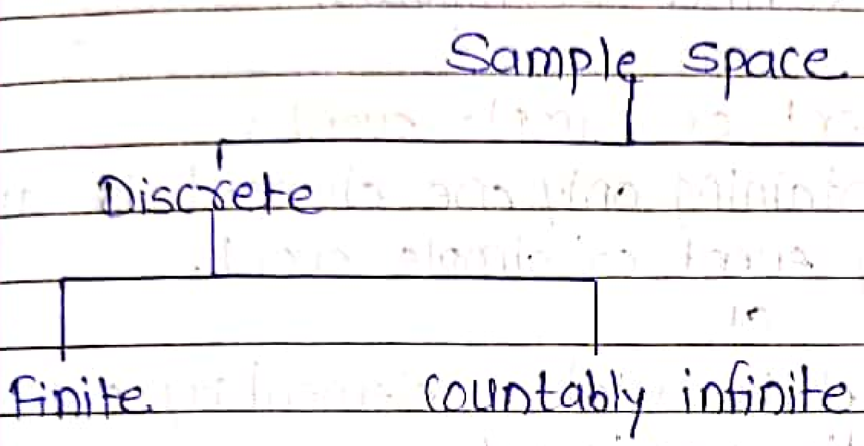
ii) Continuous Sample Space:
A sample space Ω is called a continuous sample space, if the number of elements in Ω are uncountably infinite.

e.g. consider the experiment of measuring height of a person. We may take $\Omega = (0, \infty)$. If we know that the lowest height is say 130 cm and highest is say 190 cm, we may as well consider $\Omega = [130, 190]$

Note: For both these sample spaces, the elements in the interval cannot be arrenged in a sequence. Thus, there are uncountably infinite elements in Ω.

In other words, whenever the observations on a characteristic can take any values in an interval, the concerned sample space is continuous.

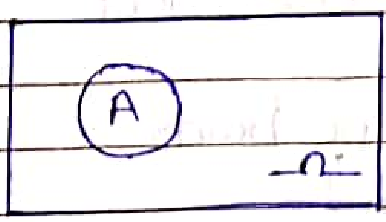
Different categories of sample space



* Events:

Consider the experiment of tossing two coins. Then the sample space is $\Omega = \{HH, HT, TH, TT\}$. We might be interested in getting a single head i.e. in the set $\{HT, TH\}$. Thus, we can associate the event of 'getting a single head' with the set $\{HT, TH\}$ which is the subset of Ω.

Defn: **Event:** An event is a subset of sample space. It consists of some or all points of the sample space.



Events are denoted by capital letters. A, B, C. . . .

Remark: For a sample space containing 'n' elements, there are '2^n' events (including ϕ and Ω)

* Types of Events:

According to the nature of the set, types of events are defined as follows.

1) Elementary event or simple event:

An event containing only one element is called as elementary event or simple event.

OR

A singleton set is called as elementary event.

e.g. Throwing a die experiment

$\Omega = \{1, 2, 3, 4, 5, 6\}$

event A = Getting a multiple of 5 on a die

$A = \{5\}$

2) Impossible event:

An event corresponding to empty set is called as an impossible event.

OR

An event which does not contain any sample point is called as an impossible event.

e.g. Tossing a coin experiment

$\Omega = \{H, T\}$

A = Getting a two heads

$A = \{\ \}$ or $\{\phi\}$

event A is impossible event

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3) Sure event or Certain event:

An event containing all the points of Ω is called a sure event or certain event.

OR

An event corresponding to the entire sample space is called a sure event.

e.g. Throwing a die experiment

$\Omega = \{1, 2, 3, 4, 5, 6\}$

A = Getting a number less than 7

$A = \{1, 2, 3, 4, 5, 6\}$

$\therefore$ event A is sure event

* Set Identities:

i) $(A')' = A$ where A' denotes the complement of A

$\Omega' = \phi$ or $\phi' = \Omega$

ii) $A \cup B = B \cup A$

$A \cup \phi = A$, $A \cup \Omega = \Omega$, $A \cup A' = \Omega$, $A \cup A = A$

iii) $A \cap B = B \cap A$

$A \cap \phi = \phi$, $A \cap \Omega = A$

$A \cap A' = \phi$, $A \cap A = A$

iv) De Morgan's laws:

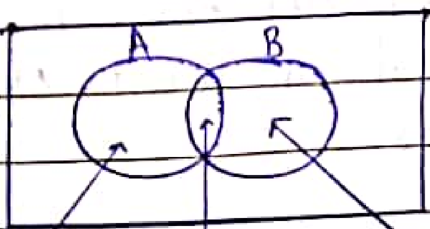
$(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$

v) If $A \subset B$ then $A \cup B = B$, $A \cap B = A$

vi) $A = (A \cap B) \cup (A \cap B')$

$B = (A \cap B) \cup (A' \cap B)$

vii) $(A \cup B) = (A \cap B') \cup (A \cap B) \cup (A' \cap B)$



$(A \cap B')$ $(A \cap B)$ $(A' \cap B)$

4) Mutually exclusive events: (Disjoint events)

Events A and B are said to be mutually exclusive if there is no common element in A and B. That is

$$A \cap B = \phi$$

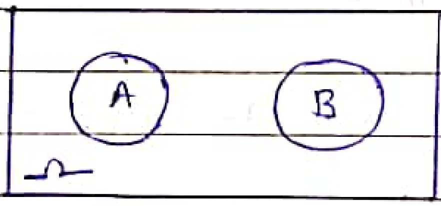
Events A and B are said to be mutually exclusive if there is no common element in A and B. That is

$$A \cap B = \phi$$

e.g. In an experiment of drawing a card from a well-shuffled pack of playing cards, if A - occurrence of red card and B - occurrence of a spade card, then

$$A \cap B = \phi$$

Hence, A and B are mutually exclusive events.



5) Complement of an event:

If A is an event on Ω, then complement of A is the event corresponding to the set of A'.

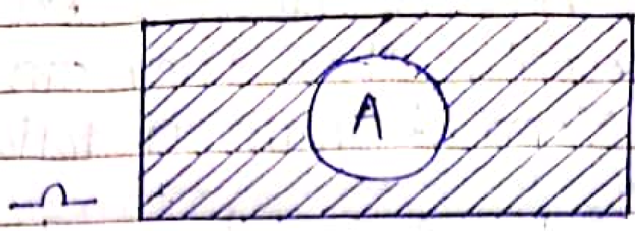
OR

A' is the event containing all points in Ω which are not in A.

e.g. If in the experiment of rolling a die,

A = occurrence of an even number

Then complement of event A is A' = occurrence of an odd number.



Note: A and A' are mutually exclusive events.

6) Exhaustive events:

Events A and B on Ω are said to be exhaustive events, if $A \cup B = \Omega$.

In general, events $A_1, A_2, \ldots, A_n$ are said to be exhaustive if $A_1 \cup A_2 \cup A_3 \ldots \cup A_n = \Omega$

e.g. $\Omega = \{1, 2, 3, 4, 5, 6\}$

$A_1 = \{1, 3\}$

$A_2 = \{2, 3, 5\}$

$A_3 = \{4, 5, 6\}$

Then

$A_1 \cup A_2 \cup A_3 = \Omega$

Therefore, A_1, A_2, A_3 are exhaustive events.

Remark:

1) A and A' are exhaustive
2) By mutually exclusive and exhaustive events we mean a partition of Ω.

eg. $\Omega = \{1, 2, 3, 4, 5, 6\}$

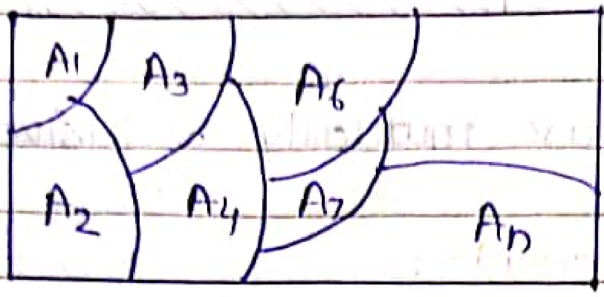
$A_1 = \{1, 3\}$, $A_2 = \{2, 5\}$, $A_3 = \{4, 6\}$

then

$A_1 \cap A_2 = \phi$, $A_1 \cap A_3 = \phi$, $A_2 \cap A_3 = \phi$

$A_1 \cup A_2 \cup A_3 = \Omega$

If $A_1, A_2, \ldots A_n$ are mutually exclusive and exhaustive events, then they can be represented as in the Venn diagram as follows,



7) Favourable Events:

The number of outcomes of a random experiment, which entail or result in happening of an event are known as favourable events.

e.g. In drawing a card from a pack of playing cards; the events favourable to "getting diamond" are the 13 diamond cards.

8) Equally likely Events:

The events A and B are said to be equally likely if none of them is expected to occur in preference to other i.e. chances of happening them are the same.

e.g. In tossing an unbiased coin, the events 'getting head' and 'getting tail' are equally likely events.

* Operations on Events:

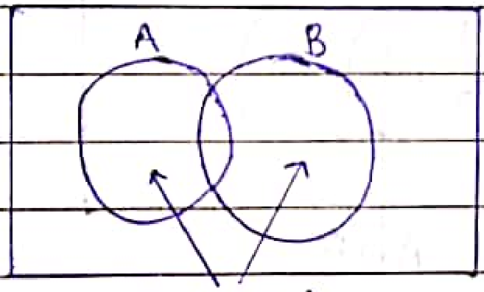
1) Union of two or more events:

let A and B be two events defined on a sample space Ω. The union of A and B denoted by $A \cup B$ is the event which contains all points

that belongs to either A or B or both
i.e. $A \cup B = \{x \mid x \in A \text{ and / or } x \in B\}$
Thus union of two events A and B corresponds to the occurrence of at least one of the two events.

e.g. Let $\Omega = \{1, 2, 3, 4, 5, 6, 7, 8\}$.
A = Occurrence of an even number
$= \{2, 4, 6, 8\}$
B = Occurrence of a multiple of 3
$= \{3, 6\}$
Here occurrence of number either even or multiple of 3 is given by.

$$A \cup B = \{2, 3, 4, 6, 8\}$$



Generalisation:
If $A_1, A_2, \ldots, A_n$ are n events defined on Ω. then union of $A_1, A_2, \ldots A_n$ denoted by $A_1 \cup A_2 \cup A_3 \ldots \cup A_n$ or $\bigcup_{i=1}^{n} A_i$ is the event which contains all points that belongs to at least one of the $A_1, A_2, \ldots A_n$
i.e. $\bigcup_{i=1}^{n} A = \{x \mid x \in A_i \text{, for at least one } i, i = 1, 2, \ldots n\}$

Remark:
i) $A \cup B = B \cup A$
ii) $A \cup A' = \Omega$
iii) $A \cup \phi = A$
iv) $A \cup \Omega = \Omega$

2) Intersection of two or more events:

Let A and B be two events defined on sample space Ω. The intersection of A and B denoted by $A \cap B$ is the event which contains all points which are in A and B both

i.e. $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

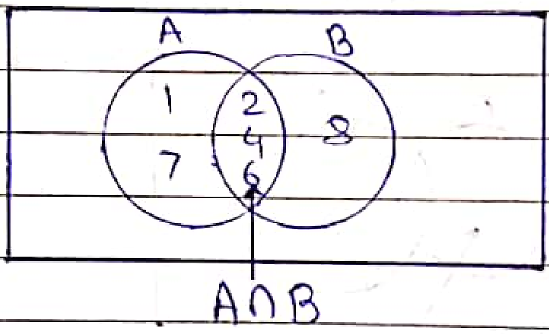
e.g. Let $\Omega = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

Let

$$A = \{1, 2, 4, 6, 7\}$$
$$B = \{2, 4, 6, 8\}$$

Then

$$A \cap B = \{2, 4, 6\}$$



Remark:

i) $A \cap B = B \cap A$ ii) $A \cap \phi = \phi$

iii) $A \cap A' = \phi$ iv) $A \cap \Omega = A$

Generalisation:

If $A_1, A_2, ---, A_n$ are n events defined on Ω then intersection of $A_1, A_2, ---, A_n$ denoted by $A_1 \cap A_2 \cap A_3 \cdots \cap A_n$ or by $\bigcap_{i=1}^{n} A_i$ is the event which consists of all points that belongs to every one of $A_1, A_2, ---, A_n$ i.e. $\bigcap_{i=1}^{n} A_i = \{x \mid x \in A_i, \forall i, i = 1, 2, -- n\}$

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3) Complement of an event:

If A is an event defined on Ω, then complement of A is the event which contains all points of Ω, except those which are in A. and this event is denoted by A' or A^c

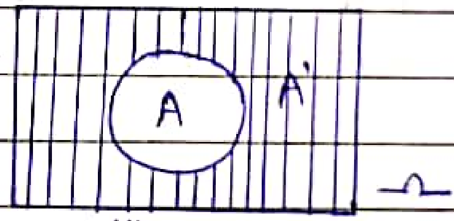
e.g. In tossing of 3 coins simultaneously the event A = getting at least one head

$\Omega = \{HHH, HHT, HTH, THH, THT, TTH, HTT, TTT\}$

$A = \{HHH, HHT, HTH, THH, THT, TTH, HTT\}$

The event A' is getting all three tails

$A' = \{TTT\}$

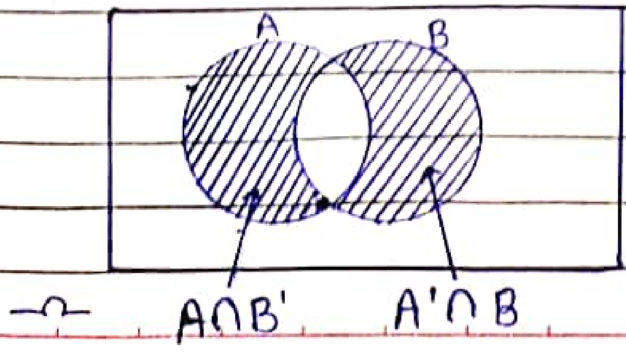


Remark:

i) $(A')' = A$ ii) $\Omega' = \phi$ iii) $\phi' = \Omega$

4) Relative Complementation:

Let A and B be two events on Ω. The relative complement of A with respect to B is given by $A' \cap B$ That is, it is the set of all points which are not in A but in B. Similarly, the relative complement of B with respect to A is given by $A \cap B'$. That is, it is the set of all points which are not in B but in A.



* **Set Identities:**

The events under the operations of union, intersection and complement satisfy the following important identities

1) Commutative Laws:

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

2) Associative Laws:

$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = (A \cap B) \cap C$$

3) Distributive Laws:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

4) De-Morgan's laws:

$$(A \cup B)' = A' \cap B'$$
$$(A \cap B)' = A' \cup A'$$

* Probability :

The probability theory tries to measure the possibility of an outcome in numeric terms. Thus probability of an outcome is a numeric measure of the possibility or chance of the occurrence of that outcome.

The classical or priori approach to probability is the oldest and simplest. This definition of probability is given by Laplace.

Classical definition of probability :

Defn:- If a random experiment results in 'n' mutually exclusive and equally likely outcomes ; out of which 'm' are favourable to the event A, then the probability of occurrence of the event A is denoted by P(A) and is given by,

$$P(A) = \frac{m}{n} \quad ; \quad 0 \leq m \leq n$$

In other words,

$$P(A) = \frac{\text{Number of elements belonging to A}}{\text{Total number of elements in the samplespace}}$$

Remark:

1) For any event A

$$0 \leq P(A) \leq 1 \quad \text{since} \quad 0 \leq \frac{m}{n} \leq \frac{n}{n} \Rightarrow 0 \leq P(A) \leq 1$$

Thus, probability of any event always lies between 0 and 1.

2) The classical definition of probability does not require the ~~logical~~ performance of the experiment.

Probability is obtained using logical reasoning without conducting the experiment.

3) 'n' is total number of mutually exclusive, equally likely and exhaustive outcomes in Ω.

4) If m out of n outcomes are in favour of an event A then (n-m) outcomes are against or not in favour of event A. Hence probability of non-occurrence of A i.e. probability of A' is defined by

$$P(A') = \frac{n-m}{n}$$

$$= \frac{n}{n} - \frac{m}{n}$$

$$= 1 - \frac{m}{n}$$

$$\therefore P(A') = 1 - P(A)$$

$$\therefore P(A') + P(A) = 1$$

Ex: ① If a pair of unbiased coins is tossed, obtain probability of occurrence of i) both heads ii) single head iii) at least one head

Soⁿ:- Here, $\Omega = \{HH, HT, TH, TT\}$

$\therefore n = 4$

i) suppose A = occurence of both heads

$A = \{HH\}$

$\therefore m = 1$

$\therefore P(A) = \frac{m}{n} = \frac{1}{4}$

ii) - B = Occurrence of single head

$B = \{HT, TH\}$

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$\therefore m = 2$

$$P(B) = \frac{m}{n} = \frac{2}{4} = \frac{1}{2}$$

iii) C = Occurrence of at least one head

$C = \{HT, TH, HH\}$

$\therefore m = 3$

$$P(C) = \frac{m}{n} = \frac{3}{4}$$

EX:② A card is selected at random from a well-shuffled ordinary deck of 52 playing cards. Find the probability of getting

i) a spade card   ii) a face card

Soln:- Since card is selected at random from the well-shuffled pack, here the outcomes are equally likely. Also the outcomes are mutually exclusive. So we use classical definition for finding the probability

i) let A - getting a spade card

The number of outcomes favourable to A is the number of spade cards = 13

i.e. $m = 13$ and $n = 52$ = total outcomes

$$P(A) = \frac{m}{n} = \frac{13}{52} = \frac{1}{4}$$

ii) Let B = getting a face card

A face card is Jack, Queen or King. In a suit there are 3 face cards. Therefore in a deck there are 12 face cards i.e number of face cards = m = 12 and total number of cards = n = 52

$$P(B) = \frac{m}{n} = \frac{12}{52} = \frac{3}{13}$$

* Limitions of classical definition of Probability:

i) In classical definition, it is assumed that all outcomes of the experiment under consideration are equally likely. This is not the case always e.g. probability of passing the examination and failing in the examination are not same. In such cases, probabilities of events can not be calculated ~~the~~ using the classical definition.

ii) Sometimes, n = the total number of possible outcomes is infinite.

e.g. For the experiment of tossing coin until head appears, the sample space Ω is $\Omega = \{H, TH, TTH, TTTH, \ldots\}$

Here, also classical definition fails.

iii) If the actual value of 'n' is not known, then also probabilities cannot be computed using the classical definition.

e.g. In the experiment of capturing fish from a pond, the total number of fish in the pond is not known. Hence, we cannot find probability of concerned events. Due to these drawbacks in the classical definition of probability, it is used only in limited situations. A Russian mathematicians A.N. Kolmogorov in 1933 formulated the axiomatic approach to the modern probability theory. This approach begin with certain notations and axioms, based on which the further theory is developed using logical reasoning. For finite sample spaces, the axiomatic approach reduces to the probability assignment approach which we consider first.

* Probability Models

Before we describe axiomatic approach to probability, firstly we discuss on probability model.
We know that non deterministic models can be used to describe the random phenomena. Such a model has three essential components; the set of all possible outcomes, events of interest and a measure of uncertainty.

Defn: Let Ω be a finite sample space containing the points $\omega_1, \omega_2, ---, \omega_n$ i.e. $\Omega = \{\omega_1, \omega_2, ----, \omega_n\}$
Assign a real number $P\{\omega_i\}$ to each $\omega_i \in \Omega$ such that

i) $0 \leq P\{\omega_i\} \leq 1$, for $i = 1, 2, --n$ and

ii) $P\{\omega_1\} + P\{\omega_2\} + ----+ P\{\omega_n\} = 1$

i.e. $\sum_{i=1}^{n} P\{\omega_i\} = 1$

$P\{\omega_i\}$ is called the probability of the elementary event $\{\omega_i\}$. In other words, $P\{\omega_i\}$ is the probability of occurrence of ω_i in a single trial of the experiment.

Probability Model :-

Suppose $\Omega = \{\omega_1, \omega_2, --- \}$ is a sample, then $\{\omega_i, P\{\omega_i\}\}$, $i = 1, 2, ---$ is a probability model if $P\{\omega_i\} \geq 0$ and $\sum P\{\omega_i\} = 1$

Ex: ① suppose $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\}$
which of the following probability models are valid?

	Sample Points					
Model	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
i)	0	0	0	0	$1/2$	$1/2$
ii)	$1/4$	$1/4$	$1/4$	$1/4$	$1/4$	$1/4$
iii)	1	0	0	0	0	0
iv)	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$	0	$\frac{1}{8}$
v)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	0	$\frac{1}{6}$
vi)	$\frac{-1}{3}$	$\frac{2}{3}$	0	$\frac{1}{3}$	0	0
vii)	π	$1-\pi$	0	0	0	0
viii)	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{3}{4}$	$\frac{1}{20}$

Solⁿ:- Only (i), (iii) and (viii) are valid probability assignment

ii) is not valid as $\Sigma P(\omega_i) = \frac{3}{2} > 1$

iv) is not valid as $\Sigma P(\omega_i) = \frac{3}{4} < 1$

v) is not valid as $\Sigma P(\omega_i) = \frac{5}{6} < 1$

vi) is not valid as $P(\omega_i) = \frac{-1}{3} < 0$

vii) is not valid as $P(\omega_1) = \pi > 1$ and $P(\omega_2) = 1-\pi < 0$

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Ex: ② A loaded die has following probability assignment to the six faces.

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

with $P(1) = 0.1$, $P(2) = 0.2$, $P(3) = 0.3$, $P(4) = 0.25$, $P(5) = 0.1$, $P(6) = 0.05$

what is the probability of i) an even number appears ii) a number less than four appears.

Soⁿ:- i) let event A = even number appears.

$$A = \{2, 4, 6\}$$

$$\therefore P(A) = \sum_{\omega_i \in A} P(\omega_i) = P(2) + P(4) + P(6)$$

$$= 0.2 + 0.25 + 0.05$$

$$P(A) = 0.5$$

ii) Let, event B = a number less than 4 appears

$$\therefore B = \{1, 2, 3\}$$

$$\therefore P(B) = P(1) + P(2) + P(3)$$

$$= 0.1 + 0.2 + 0.3$$

$$\therefore P(B) = 0.6$$

Ex: ③ Three girls and four boys take part in an antakshari competition. Those of same sex have equal probabilities of winning, but each girl is twice as likely to win as any boy. Find the probability that a boy wins in the competition.

Soⁿ:- let B denote a boy and G a girl

$$\therefore \Omega = \{G_1, G_2, G_3, B_1, B_2, B_3, B_4\}$$

let.

$$P(B_1) = P(B_2) = P(B_3) = P(B_4) = p$$

$$\& \ P(G_1) = P(G_2) = P(G_3) = 2p$$

since, sum of probabilities of all points in Ω must be 1.

$$4p + 6p = 1$$
$$10p = 1$$
$$p = \frac{1}{10}$$
$$\therefore p = 0.1$$

Now, let A be the event that a boy wins.

$$\therefore A = \{B_1, B_2, B_3, B_4\}$$
$$\therefore P(A) = \sum_{i=1}^{4} P(B_i) = 0.4$$

* Axiomatic Approach to Probability: (Modern Approach)

In probability models, probabilistic are commonly based on intuition, experience or experimentation, but if the assignments are in accurate the prediction of the model will be misleading. So to overcome this, we need a mathematical construction of probability measure, independent of intended application. This lead to a axiomatic approach to probability due to Kolmogorow in 1933.

* Axioms of Probability:

Let Ω be a sample space of a random experiment. Let A be any event defined on Ω. P is called the probability function or probability measure, which satisfies the following axioms.

Axiom 1: P(A) is a real number such that

$$P(A) \geq 0 \quad \text{for any event } A \text{ on } \Omega.$$

Axiom 2: $P(\Omega) = 1$

Axiom 3: If $A_1, A_2, \ldots, A_n$ are any mutually exclusive events of Ω then,

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

In particular, if A and B are two mutually exclusive (disjoint) events, then

$$P(A \cup B) = P(A) + P(B)$$

Remark: The above definition applies to countably infinite as well as uncountably infinite sample spaces.

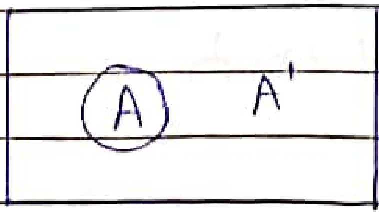
* Important Theorems on Probability:

We now learn some fundamental theorems on probability. Proofs of these theorems are based on the above axioms. Applications of these results makes the computations of probabilities of complex events very easy.

Thm. ① $P(A') = 1 - P(A)$, where A' is the complement of A.

Proof: Note that, for any event A, A and A' are mutually exclusive events

$$\therefore A \cap A' = \phi$$
$$A \cup A' = \Omega$$



Therefore, using axiom 3.

$$P(A \cup A') = P(A) + P(A')$$

$$\therefore \quad P(\Omega) = P(A) + P(A')$$

$$\therefore \quad 1 = P(A) + P(A') \qquad \because \text{by axiom 2}$$

$$\therefore \quad 1 - P(A) = P(A')$$

$$\therefore \quad P(A') = 1 - P(A)$$

Hence, the proof.

Th^m: 2 $P(\phi) = 0$, That is probability of impossible event is zero.

Proof: we know that

$$\phi' = \Omega$$

$$\therefore \quad P(\phi') = 1 - P(\Omega) \qquad \text{by theorem } 1$$

$$= 1 - 1 \qquad \text{by axiom 2}$$

$$\therefore P(\phi') = 0$$

Hence, the proof

Th^m: 3 For any event A of Ω

$$0 \leq P(A) \leq 1$$

Proof: by axiom 1 $\quad P(A) \geq 0 \quad$ —— (I)

Also, if A' is the complement of A, then

$$P(A') \geq 0 \qquad \text{by axiom 1}$$

$$1 - P(A) \geq 0 \qquad \text{by theorem 1}$$

$$-P(A) \geq -1$$

$$P(A) \leq 1 \quad \text{—— (II)}$$

From (I) and (II)

$$0 \leq P(A) \leq 1$$

Hence, the proof.

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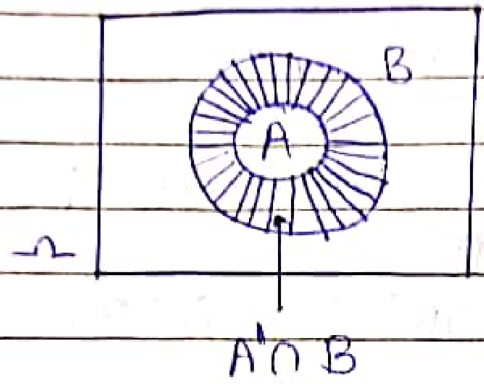
Thm ④ If $A \subset B$, then $P(A) \leq P(B)$

Proof: The event B is composed of two disjoint events A and $A' \cap B$.

i.e. $B = A \cup (A' \cap B)$



∴ Using axiom 3.

$$P(B) = P(A) + P(A' \cap B)$$

Now $P(A' \cap B) \geq 0$ — by axiom 1

$$\therefore P(B) \geq P(A)$$

$$\therefore P(A) \leq P(B)$$

Hence, the proof.

Thm ⑤ Addition theorem of probability (Theorem of total probability):

If A and B are any two events defined on Ω, then. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

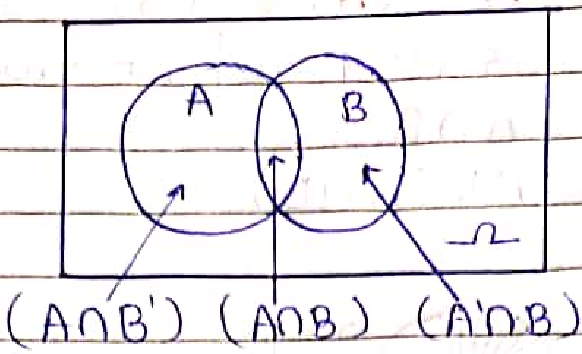
Proof: Since, A and B are any two events, we assumed that they are not disjoint.

Note that the event $A \cup B$ is composed of three mutually exclusive events $A \cap B'$, $A \cap B$ and $A' \cap B$.

Therefore, using axiom 3. we write

$$P(A \cup B) = P(A \cap B') + P(A \cap B) + P(A' \cap B) \quad \text{—— (I)}$$

the venn diagram will be as follows.



Similarly, A is the union of the disjoint event $A \cap B'$ and $A \cap B$

$\therefore P(A) = P(A \cap B') + P(A \cap B)$ —— (II)

Similarly,

$P(B) = P(A' \cap B) + P(A \cap B)$ —— (III)

Adding the equation (II) and (III) is.

$P(A) + P(B) = P(A \cap B') + P(A \cap B) + P(A' \cap B) + P(A \cap B)$
$= P(A \cap B') + P(A' \cap B) + 2P(A \cap B)$

$P(A) + P(B) = P(A \cup B) + P(A \cap B)$ ... from (I)

$P(A) + P(B) - P(A \cap B) = P(A \cup B)$

$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Hence, the proof.

Remark:

$P(A \cup B)$ is the probability of occurrence of at least one of the events A and B.

Ex: (1) Consider an experiment of rolling a fair die. Let A be the event than an even number appears. B = a number bigger than 3 occurs. Find the probability that the number appearing on the uppermost face is either even or bigger than 3.

Sol^n:- Here, $\Omega = \{1,2,3,4,5,6\}$

Since, die is fair, $P(i) = \frac{1}{6}$ for $i = 1,2,\ldots 6$.

$$A = \{2,4,6\}$$
$$B = \{4,5,6\}$$
$$A \cup B = \{2,4,5,6\}$$
$$A \cap B = \{4,6\}$$

$$P(A) = \frac{3}{6} = \frac{1}{2}$$

$$P(B) = \frac{3}{2} = \frac{1}{2}$$

$$P(A \cap B) = \frac{2}{6} = \frac{1}{3}$$

by using addition theorem.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{1}{2} + \frac{1}{2} - \frac{1}{3}$$

$$= 1 - \frac{1}{3}$$

$$\therefore P(A \cup B) = \frac{2}{3}$$

* **Conditional Probability:-**

Consider a family having two children. Suppose we wish to find the probability of the event A: that both the children are males.

$$\therefore \Omega = \{MM, MF, FM, FF\}$$
$$A = \{MM\}$$

Assuming the sample space to be equiprobable;
$P(A) = 1/4$.

Now, suppose it is already known that at least one of the children is a boy. Then the sample space will not contain FF and reduce to a new sample space, $B = \{MM, MF, FM\}$

As there are only 3 elements in the sample space, probability of A with respect to the reduced sample space B will be 1/3. Thus, the knowledge about the occurrence of some other event has altered the probability of the event under consideration. The above probability is called as conditional probability of A given B and is denoted by $P(A|B)$

observe that initially

$$P(A) = \frac{\text{No. of elements in } A}{\text{No. of elements in } \Omega} = \frac{n(A)}{n(\Omega)}$$

then,

$$P(A|B) = \frac{\text{Number of elements in } A \cap B}{\text{Number of elements in } B} = \frac{n(A \cap B)}{n(B)} = \frac{n(A \cap B)/n(\Omega)}{n(B)/n(\Omega)}$$

$$\therefore P(A|B) = \frac{P(A \cap B)}{P(B)} \quad ; \quad P(B) \neq 0$$

Defn:- Conditional Probability:

Let A and B be two events defined on sample space Ω of a random experiment then the conditional probability of A given B (or with respect to B), denoted by $P(A|B)$, is given by,

$$P(A|B) = \frac{P(A\cap B)}{P(B)} \quad ; \quad P(B) \neq 0$$

lly,

$$P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{P(B\cap A)}{P(A)} \quad ; \quad P(A) \neq 0$$

Remarks:

1) If A and B are mutually exclusive events defined on Ω then

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(\phi)}{P(B)} = 0$$

2) $$P(A|A') = \frac{P(A\cap A')}{P(A')} = \frac{P(\phi)}{P(A')} = 0$$

3) If $A \subset B$ then $P(B|A) = \frac{P(B\cap A)}{P(A)} = \frac{P(A)}{P(A)} = 1$

Note:

The probabilities calculated with respect to original sample space are called unconditional probabilities and those calculated with respect to reduced sample space due to occurrence of some event are called conditional probabities.

Ex: ① A pair ~~die~~ of fair dice is rolled. If the sum of 8 has appeared, find the probability that one of the dice shows 3.

Soⁿ:- The sample contains 36 elements.
Suppose B is the event that sum of the 8 appears. Then

$$B = \{(2,6), (3,5), (4,4), (5,3), (6,2)\}$$

Now, A = one of the dice show 3

$$\therefore A \cap B = \{(3,5), (5,3)\}$$

By definition, the conditional probability of A given B will be

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{2/36}{5/36} = \frac{2}{5}$$

Remark: $P(A|B) \neq P(B|A)$

* **Multiplication Theorem:**
(Theorem of Compound Probability)

Statement: Let A and B be any two events defined on a sample space Ω then,

$$P(A \cap B) = P(A|B) \cdot P(B)$$
$$= P(B|A) \cdot P(A)$$

Proof:- by using definition of conditional probability,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$\therefore P(A \cap B) = P(A|B) \cdot P(B)$$

lly,

$$P(A \cap B) = P(B|A) \cdot P(A)$$

Remark:

1) The theorem holds even if $P(B)=0$, provided we interpret $P(A|B)=0$ if $P(B)=0$ since. This is so because $A\cap B \subset B \Rightarrow P(A\cap B)=0$

2) The multiplication theorem is used for obtaining probability of simultaneous occurrence of two events, whenever conditional probability is given.

EX: ① A lot contains 12 items of which 4 are defective. Two items are drawn at random from the lot one after other (without replacement). Find the probability that both items are non-defective.

Sol^n:- Let

A = Event that the item drawn first is non-defective

B = Event that the item drawn at the second draw is non-defective.

12 items

4 Defective   8 Non-defective

$$\therefore P(A) = \frac{8}{12} = \frac{2}{3}$$

Now, probability that second item is non-defective given that first item is non-defective is

$$\therefore P(B|A) = \frac{7}{11}$$

Since, the item drawn at first draw is kept aside.

$$\therefore P(\text{both items non-defective}) = P(A\cap B)$$

$$= P(B|A)\cdot P(A) \quad \text{by th}^m$$

$$= \frac{7}{11}\times\frac{2}{3}$$

$$\therefore P(\text{both items non-defective}) = \frac{14}{33}$$

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* Independence of two events:

Event A and B defined on sample space Ω are said to be independent if the occurrences or non-occurrences of one of them does not influence the occurrence or non-occurrence of the other. In other words probability of occurrence of B is same as the conditional probability of B given that A has occured i.e. $P(B) = P(B|A)$ and similarly probability of occurrence of A is same as the conditional probability i.e. $P(A) = P(A|B)$. Using the definition of conditional probability both these conditions yield $P(A \cap B) = P(A) \cdot P(B)$

Def:- Two events are said to be independent ~~if the~~ i.e. A and B defined on a sample space Ω iff.

$$P(A \cap B) = P(A) \cdot P(B)$$

OR

The events A and B defined on sample space Ω are said to independent if and only if,

$P(A|B) = P(A)$ and $P(B|A) = P(B)$

or equivalently $P(A \cap B) = P(A) \cdot P(B)$

Ex: ① Consider the experiment of rolling a fair die

Soln:- $\Omega = \{1, 2, 3, 4, 5, 6\}$

Let, A = Occurrence of an even number

$\therefore A = \{2, 4, 6\}$

B = Occurrence of a number greater than 4.

$\therefore B = \{5, 6\}$

$P(A) = \frac{3}{6} = \frac{1}{2}$ and $P(B) = \frac{2}{6} = \frac{1}{3}$

$$P(A \cap B) = P\{6\} = \frac{1}{6} = P(A) \cdot P(B)$$

$\therefore$ By definition, A and B are independent events.

Remark:-

1) $A \cap B \neq \phi$, therefore A and B are not mutually exclusive events, although A and B are independent.

Ex: ② Consider the experiment of tossing three fair coins simultaneously.

Sol^n: $\Omega = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$

Let A = Getting two heads

$\therefore A = \{HHT, HTH, THH\}$

B = Getting two tails

$\therefore B = \{TTH, THT, HTT\}$

$P(A) = \frac{3}{8}$, $P(B) = \frac{3}{8}$

$A \cap B = \{\phi\}$

$P(A \cap B) = 0$

$\therefore P(A \cap B) \neq P(A) \cdot P(B)$

$\therefore$ A and B are not independent.

Remark:

2) In the abow example the events A and B are mutually exclusive ($\because A \cap B = \phi$) but they are not independent.

Thus, from the above two examples it it clear that, i) independance $\not\Rightarrow$ mutual exclusiveness and ii) mutual exclusiveness $\nRightarrow$ independence

Result 1: If A and B are independent events with $P(A)$ and $P(B)$ both non-zero, then A and B cannot be mutually exclusive.

Proof: Since A and B are independent,

$$P(A \cap B) = P(A) \cdot P(B)$$

$$\neq 0 \qquad \because P(A) \neq 0,\ P(B) \neq 0$$

$\therefore$ $A \cap B \neq \phi$ or A and B cannot be mutually exclusive.

Result: 2 If A and B are mutually exclusive with $P(A)$ and $P(B)$ both non-zero, then A and B cannot be independent ; i.e. A and B are dependent.

Proof: Since A and B are mutually exclusive

$$A \cap B = \phi$$

$$\therefore P(A \cap B) = 0$$

However, $P(A) \cdot P(B) \neq 0 \qquad \because P(A) \neq 0,\ P(B) \neq 0$

$$\therefore P(A \cap B) \neq P(A) \cdot P(B)$$

$\Rightarrow$ A and B are dependent events

Remark:-

3) If one of the events of A and B is an impossible event (ϕ), then A and B are independent as well as mutually exclusive.

e.g. Suppose $A = \phi$, then $A \cap B$ being a subset of A,

$A \cap B = \phi$

$\therefore$ A and B are mutually exclusive.

Also, $P(A \cap B) = 0$

$\therefore P(A \cap B) = P(A) \cdot P(B) \qquad \because P(A) = 0$

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* Independence of Three events:

Def.① Mutual independence of three events A, B, C.
Let A, B, C be three events defined on Ω. The three events A, B, C are said to be mutually independent or completely independent if and only if the following conditions are satisfied.

i) $P(A \cap B) = P(A) \cdot P(B)$

ii) $P(B \cap C) = P(B) \cdot P(C)$

iii) $P(A \cap C) = P(A) \cdot P(C)$

iv) $P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$

Def^n ② Pairwise independence of three events. The three events A, B, C defined on Ω are said to be pairwise independent if and only if the following conditions are satisfied.

i) $P(A \cap B) = P(A) \cdot P(B)$

ii) $P(B \cap C) = P(B) \cdot P(C)$

iii) $P(A \cap C) = P(A) \cdot P(C)$

Remark: The definition of mutual independence needs the conditions $P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$ in addition to the pairwise independence of A, B and C. Therefore,

A, B, C are mutually independent
$\Rightarrow$ A, B, C are pairwise independent.
The other way implication is not true.

Ex. ① Consider the experiment of rolling two fair dice. Let A = odd number on the first die, B = odd number on the second die and C = sum of two points is odd. Show that A, B, C are pairwise independent but not mutually independent.

Sol^n:- The sample space is

Ω = { (1,1), (1,2), (1,3), (1,4), (1,5), (1,6),
(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),
(3,1), (3,2), (3,3), (3,4), (3,5), (3,6),
(4,1), (4,2), (4,3), (4,4), (4,5), (4,6),
(5,1), (5,2), (5,3), (5,4), (5,5), (5,6),
(6,1), (6,2), (6,3), (6,4), (6,5), (6,6) }

$n(S) = 36$

Let. A = Odd number on the first die

∴ A = { (1,1), (1,2), (1,3), (1,4), (1,5), (1,6),
(3,1), (3,2), (3,3), (3,4), (3,5), (3,6),
(5,1), (5,2), (5,3), (5,4), (5,5), (5,6). }

B = odd number on the second

B = { (1,1), (1,3), (1,5), (2,1), (2,3), (2,5),
(3,1), (3,3), (3,5), (4,1), (4,3), (4,5),
(5,1), (5,3), (5,5), (6,1), (6,3), (6,5) }

C = Sum of two points is odd

C = { (1,2), (1,4), (1,6), (2,1), (2,3), (2,5),
(3,2), (3,4), (3,6), (4,1), (4,3), (4,5),
(5,2), (5,4), (5,6), (6,1), (6,3), (6,5) }

$A \cap B$ = { (1,1), (1,3), (1,5), (3,1), (3,3), (3,5),
(5,1), (5,3), (5,5) }.

$A \cap C$ = { (1,2), (1,4), (1,6), (3,2), (3,4), (3,6),
(5,2), (5,4), (5,6) }

$B \cap C$ = { (2,1), (2,3), (2,5), (4,1), (4,3), (4,5),
(6,1), (6,3), (6,5) }

$$P(A) = \frac{18}{36} = \frac{9}{18} = \frac{1}{2}$$

$$P(B) = \frac{18}{36} = \frac{9}{18} = \frac{1}{2}$$

$$P(C) = \frac{18}{36} = \frac{9}{18} = \frac{1}{2}$$

$$P(A \cap B) = \frac{9}{36} = \frac{1}{4}$$

$$P(A \cap C) = \frac{9}{36} = \frac{1}{4}$$

$$P(B \cap C) = \frac{9}{36} = \frac{1}{4}$$

$$\therefore P(A \cap B) = P(A) \cdot P(B)$$

$$\frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2}$$

$$\frac{1}{4} = \frac{1}{4}$$

Similarly

$$P(B \cap C) = P(B) \cdot P(C)$$
$$P(A \cap C) = P(A) \cdot P(C)$$

Hence, A, B, C are pairwise independent.
However, $A \cap B \cap C = \phi$ since if both the dice show odd number, there sum cannot be odd.

$$\therefore P(A \cap B \cap C) = 0 \neq P(A) \cdot P(B) \cdot P(C)$$

$\therefore$ A, B, C are not mutually independent.

Ex: ② Consider the experiment of tossing an unbiased coin three times. Let A_i denote the event that a head turns up on i^{th} toss, $i = 1, 2, 3$. Are events A_1, A_2, A_3 mutually independent?

Soln:- $\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

$A_1 = \{HHH, HHT, HTH, HTT\}$

$P(A_1) = \frac{4}{8} = \frac{1}{2}$

$A_2 = \{HHH, HHT, THH, THT\}$

$P(A_2) = \frac{4}{8} = \frac{1}{2}$

$A_3 = \{HHH, HTH, THH, TTH\}$

$P(A_3) = \frac{4}{8} = \frac{1}{2}$

further,

$A_1 \cap A_2 = \{HHH, HHT\}$

$P(A_1 \cap A_2) = \frac{2}{8} = \frac{1}{4}$

$A_2 \cap A_3 = \{HHH, THH\}$

$P(A_2 \cap A_3) = \frac{2}{8} = \frac{1}{4}$

$A_1 \cap A_3 = \{HHH, HTH\}$

$P(A_1 \cap A_3) = \frac{2}{8} = \frac{1}{4}$

$A_1 \cap A_2 \cap A_3 = \{HHH\}$

$P(A_1 \cap A_2 \cap A_3) = \frac{1}{8}$

Thus, i) $P(A_1 \cap A_2) = P(A_1) \cdot P(A_2)$

$\frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2}$

similarly,

ii) $P(A_2 \cap A_3) = P(A_2) \cdot P(A_3)$

iii) $P(A_1 \cap A_3) = P(A_1) \cdot P(A_3)$

iv) $P(A_1 \cap A_2 \cap A_3) = P(A_1) \cdot P(A_2) \cdot P(A_3)$

are all satisfied.

$\therefore$ A_1, A_2, A_3 are mutually independent.

* Partition of a Sample Space:

A collection of mutually exclusive and exhaustive events is called a partition of sample space. More specifically, the events $A_1, A_2, \ldots A_n$ defined on a sample space Ω are said to form a partition of Ω iff

i) $A_i \cap A_j = \phi$ for all i and j ; $i \neq j$

ii) $\bigcup_{i=1}^{n} A_i = \Omega$

e.g. Let $\Omega = \{1, 2, 3, 4, 5, 6\}$

$A = \{2, 4, 6\}$ and $B = \{1, 3, 5\}$

Then

$A \cap B = \phi$ and $A \cup B = \Omega$

Thus A and B form a partition of Ω.

Here note that $B = A'$

* Baye's Theorem:

Statement:- Suppose events $A_1, A_2, \ldots A_n$ form a partition of a sample space Ω of a random experiment. Suppose B is any other event with $P(B) > 0$, defined on Ω. Then

$$P(A_i | B) = \frac{P(A_i) \cdot P(B|A_i)}{\sum_{j=1}^{n} P(A_j) \cdot P(B|A_j)} \quad ; \text{for all } i = 1, 2, \ldots n.$$