VECTOR DIFFERENTIAL CALCULUS

A study of vector calculus is necessary in

— physics and engineering
— Engineering mechanics

Ex: 1. A boy is riding a bike with a velocity of 30 km/hr in a north-east direction. Defining a velocity, we need two things i.e the magnitude of the velocity and it's direction.

Scalar: A physical quantity which requires only magnitude for it's presentation is called scalar quantity.

Ex: Length, time, mass, temperature etc.

Vector: A physical quantity which requires both magnitude and direction for it's presentation is called vector quantity.

Ex: Velocity, acceleration, displacement etc.

Geometrical Representation:

$O \longrightarrow P$

Direction of arrow represents direction and length of line segment gives magnitude of the vector.

Notation : Vector is denoted by alphabetical letter with bar over it such as $\bar{a}, \bar{b}, \bar{x}, \bar{y}$ etc.

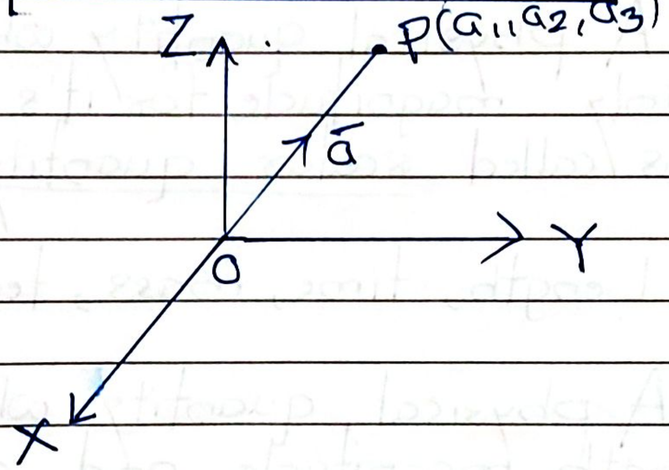
Magnitude : Magnitude of vector $\bar{a}$ is denoted by $|\bar{a}|$ or simply by a.

Analytically, vector in three dimension is given by,

$$\overline{OP} = \bar{a} = a_1 \bar{i} + a_2 \bar{j} + a_3 \bar{k}$$

(a_1, a_2, a_3) are co-ordinate of point P representing vector $\bar{a}$.

$$\boxed{|\bar{a}| = a = \sqrt{a_1^2 + a_2^2 + a_3^2}}$$



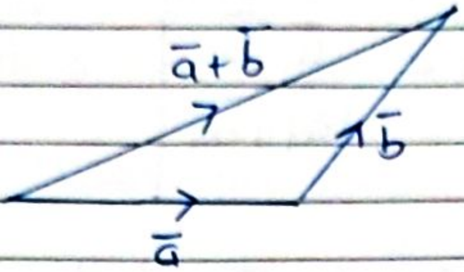
Algebra of vectors :

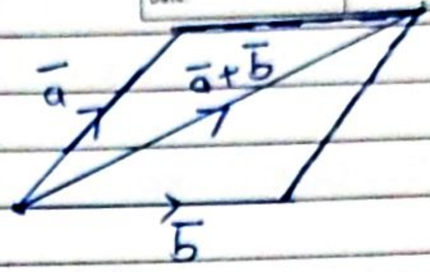
Let $\bar{a} = a_1 \bar{i} + a_2 \bar{j} + a_3 \bar{k}$ and

$\bar{b} = b_1 \bar{i} + b_2 \bar{j} + b_3 \bar{k}$ be any vectors.

1. **Addition** :

$$\bar{a} + \bar{b} = (a_1 + b_1)\bar{i} + (a_2 + b_2)\bar{j} + (a_3 + b_3)\bar{k}$$



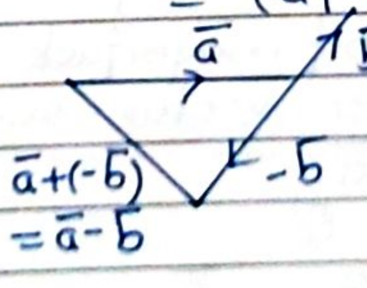


2. <u>Negative Vector</u> : Negative of vector say $\bar{a}$ is a vector denoted by $-\bar{a}$.

<u>Note</u> : $\bar{a}$ and $-\bar{a}$ have same magnitude but ~~of~~ they are in opposite direction.

3. <u>Subtraction of vectors</u> :

$$\bar{a}-\bar{b} = \bar{a}+(-\bar{b}).$$
$$= (a_1-b_1)i + (a_2-b_2)j + (a_3-b_3)k$$



<u>Product of vectors</u> :

(1). <u>Dot Product</u> (<u>Scalar Product</u>)

Dot product of two vectors say $\bar{a}$ and $\bar{b}$ is denoted by $\bar{a}\cdot\bar{b}$ and is defined as

$$\bar{a}\cdot\bar{b} = |\bar{a}||\bar{b}|\cos\theta = ab\cos\theta$$

where θ is angle between $\bar{a}$ and $\bar{b}$

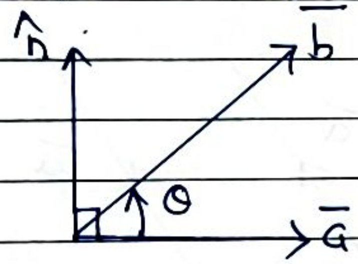
$$\therefore \boxed{\cos\theta = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}|\,|\bar{b}|}}$$

(2) Cross Product (Vector Product)

Cross product of $\bar{a}$ and $\bar{b}$ is denoted by $\bar{a}\times\bar{b}$ and is defined as,

$$\boxed{\bar{a}\times\bar{b} = |\bar{a}|\,|\bar{b}|\sin\theta\,\hat{n} = ab\sin\theta\,\hat{n}}$$

$$\bar{a}\times\bar{b} = (ab\sin\theta)\,\hat{n}$$



where $\hat{n}$ is unit vector normal to plane of $\bar{a}$ and $\bar{b}$.
Direction of $\hat{n}$ is given by right hand screw rule.

Note: A vector of unit magnitude along the direction of given vector say $\bar{a}$ is called unit vector along $\bar{a}$.
It is denoted by $\hat{a}$.
It is given by $\hat{a} = \frac{\bar{a}}{|\bar{a}|}$

Properties of Product of vectors:

Dot Product	Cross Product
(i) Dot product is commutative $\bar{a}\cdot\bar{b} = \bar{b}\cdot\bar{a}$	(i) Cross product is not commutative $\bar{a}\times\bar{b} \neq \bar{b}\times\bar{a}$; $\bar{a}\times\bar{b} = -(\bar{b}\times\bar{a})$
(ii) Dot product is scalar quantity	(ii) Cross product is vector quantity
(iii) If m is any constant $\bar{a}\cdot m\bar{b} = m\bar{a}\cdot b = m(\bar{a}\cdot b) = (\bar{a}\cdot\bar{b})m$	(iii) $\bar{a}\times m b = m\bar{a}\times b = m(\bar{a}\times\bar{b}) = (\bar{a}\times\bar{b})m$
(iv) If $\bar{a}\cdot\bar{b} = 0$, then $\bar{a}$ and $\bar{b}$ are perpendicular to each other	(iv) If $\bar{a}\times\bar{b} = 0$, then $\bar{a}$ and $\bar{b}$ are parallel to each other.

(3) Scalar triple product:

The product of $\bar{a}, \bar{b}, \bar{c}$ of form $\bar{a}\cdot\bar{b}\times\bar{c}$ is scalar quantity called as scalar triple product.

Notation: $\bar{a}\cdot\bar{b}\times\bar{c} \equiv [\bar{a}\ \bar{b}\ \bar{c}]$

(4) Vector triple product:

The product of $\bar{a}, \bar{b}, \bar{c}$ of the form

$\bar{a} \times (\bar{b} \times \bar{c})$ is a vector quantity called as vector triple product.
It is defined as,

$$\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \cdot \bar{c})\bar{b} - (\bar{b} \cdot \bar{a})\bar{c}$$

Note: $1 \times (2 \times 3) = (1 \cdot 3)2 - (2 \cdot 1)3$

Note: $\bar{i}, \bar{j}, \bar{k}$ are unit vectors along positive x, y, z direction respectively.

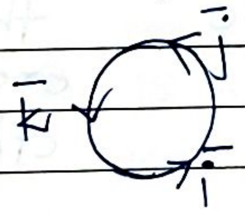
We have, $\bar{i} \cdot \bar{i} = 1$, $\bar{j} \cdot \bar{j} = 1$, $\bar{k} \cdot \bar{k} = 1$

$$\bar{i} \cdot \bar{j} = \bar{j} \cdot \bar{k} = \bar{i} \cdot \bar{k} = 0$$

$$\bar{i} \times \bar{i} = \bar{j} \times \bar{j} = \bar{k} \times \bar{k} = 0$$

$$\bar{i} \times \bar{j} = \bar{k}, \quad \bar{j} \times \bar{k} = \bar{i}, \quad \bar{k} \times \bar{i} = \bar{j}$$

& $\bar{j} \times \bar{i} = -\bar{k}$, $\bar{k} \times \bar{j} = -\bar{i}$, $\bar{i} \times \bar{k} = -\bar{j}$



Results: Let $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$

$\bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$

$\bar{c} = c_1\bar{i} + c_2\bar{j} + c_3\bar{k}$

be any vectors then

(1) $\bar{a} \cdot \bar{b} = (a_1\bar{i} + a_2\bar{j} + a_3\bar{k}) \cdot (b_1\bar{i} + b_2\bar{j} + b_3\bar{k})$

$$= a_1 b_1 + a_2 b_2 + a_3 b_3$$

(ii) $\bar{a} \times \bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

(iii) $\bar{a} \cdot \bar{b} \times \bar{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Note Properties of scalar triple product:

(1) ~~Vector~~ Value of scalar triple product remains unchanged if we change vectors $\bar{a}, \bar{b}, \bar{c}$ cyclically.

i.e $[\bar{a}\ \bar{b}\ \bar{c}] = [\bar{b}\ \bar{c}\ \bar{a}] = [\bar{c}\ \bar{a}\ \bar{b}]$

i.e $\bar{a} \cdot \bar{b} \times \bar{c} = \bar{b} \cdot \bar{c} \times \bar{a} = \bar{c} \cdot \bar{a} \times \bar{b}$

Thus,

$\bar{a} \cdot \bar{b} \times \bar{c} = \bar{c} \cdot \bar{a} \times \bar{b}$

$\Rightarrow \boxed{\bar{a} \cdot \bar{b} \times \bar{c} = \bar{a} \times \bar{b} \cdot \bar{c}}$ —— (*)

(As dot product is commutative)

(*) shows dot and cross product can be interchanged.

(2) In a scalar triple product, if two vectors are same or parallel, then value of scalar triple product is zero.

Ex: 1 : If $\bar{a} = 5\bar{i}+\bar{j}+7\bar{k}$, $\bar{b} = 2\bar{i}-3\bar{j}+4\bar{k}$
$\bar{c} = -\bar{i}+5\bar{j}-2\bar{k}$ then find

(i) $\bar{a}\cdot\bar{b}$ (ii) $\bar{a}\times\bar{c}$ (iii) $\bar{b}\times\bar{c}$
(iv) $|\bar{b}\times\bar{a}|$ (v) $\hat{a}$ (vi) $\hat{c}$ (vii) $\bar{a}\cdot\bar{c}$

Soln : (i) $\bar{a}\cdot\bar{b} = (5\bar{i}+\bar{j}+7\bar{k})\cdot(2\bar{i}-3\bar{j}+4\bar{k})$

$= 10-3+28$
$= 38-3 = 35$

(ii) $\bar{a}\times\bar{c} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 5 & 1 & 7 \\ -1 & 5 & -2 \end{vmatrix}$

$= \bar{i}(-2-35)-\bar{j}(-10+7)+\bar{k}(25+1)$

$= -37\bar{i}+3\bar{j}+26\bar{k}$

(iii) $\bar{b}\times\bar{c} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & -3 & 4 \\ -1 & 5 & -2 \end{vmatrix}$

$= \bar{i}(6-20)-\bar{j}(-4+4)+\bar{k}(10-3)$

$= -14\bar{i}+7\bar{k}$

(iv) $\bar{b}\times\bar{a} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & -3 & 4 \\ 5 & 1 & 7 \end{vmatrix} = \bar{i}(-21-4)-\bar{j}(14-20)+\bar{k}(2+15)$
$= -25\bar{i}+6\bar{j}+17\bar{k}$

Date:

$$|\bar{b}\times\bar{a}| = \sqrt{(-25)^2+6^2+(17)^2}$$
$$= \sqrt{950}$$

(v) $\hat{a} = \dfrac{\bar{a}}{|\bar{a}|} = \dfrac{5\bar{i}+\bar{j}+7\bar{k}}{\sqrt{25+1+49}} = \dfrac{5\bar{i}+\bar{j}+7\bar{k}}{\sqrt{75}}$

(vi) $\hat{c} = \dfrac{\bar{c}}{|\bar{c}|} = \dfrac{-\bar{i}+5\bar{j}-2\bar{k}}{\sqrt{80}}$

(vii) $\bar{a}\cdot\bar{c} = \underline{\underline{-14}}$

Ex: 2. If $\bar{a} = -10\bar{i}-6\bar{j}+5\bar{k}$, $\bar{b} = 2\bar{j}+15\bar{k}$,
$\bar{c} = 3\bar{i}-4\bar{j}+4\bar{k}$,
then find
(i) $\bar{a}\cdot\bar{c}$ (ii) $\bar{c}\times\bar{a}$ (iii) $(\bar{b}\times\bar{c})$ (iv) $\bar{a}\times\bar{c}$
(v) $\bar{c}\cdot\bar{b}$

Vector Function :

If for each scalar variable t, there corresponds a vector $\bar{r}$ by some definite rule, then $\bar{r}$ is called as vector function of scalar t.

It is written as $\bar{r} = \bar{F}(t)$

e.g (i) Velocity of a moving body
(ii) Force applied at certain points

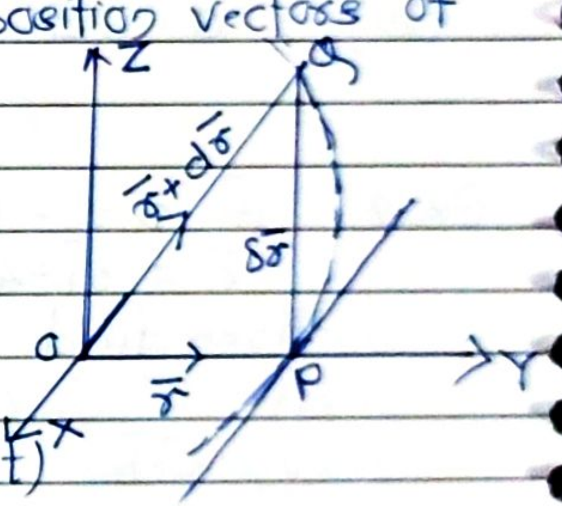
Derivative of a certain function

Let $\bar{r} = f(t)$ be any vector function.

Derivative of $\bar{r}$ w.r.t 't' is denoted by $\frac{d\bar{r}}{dt}$ and it is defined as -

$$\frac{d\bar{r}}{dt} = \lim_{\delta t \to 0} \frac{\bar{f}(t+\delta t) - \bar{f}(t)}{\delta t}$$

Let $\bar{r}$ and $\bar{r}+\delta\bar{r}$ be the position vectors of points P and Q respectively.



Where,

$\bar{r} = \bar{f}(t)$

$\bar{r} + \delta\bar{r} = \bar{f}(t+\delta t)$

From dig. $\overline{PQ} = \delta\bar{r} = \bar{f}(t+\delta t) - \bar{f}(t)$

$$\therefore \frac{d\bar{r}}{dt} = \lim_{\delta t \to 0} \frac{\delta\bar{r}}{\delta t} = \lim_{\delta t \to 0} \frac{\overline{PQ}}{\delta t}$$

As $\delta t \to 0$, $Q \longrightarrow P$

In this case, vector $\overline{PQ}$ attains the position of tangent to curve $\bar{r}$ at point P

$\therefore \frac{d\bar{r}}{dt}$ is vector along the direction of tangent to curve $\bar{r}$.

Note (i) If $x = x(t)$, $y = y(t)$, $z = z(t)$ be any arbitrary point on given curve then vector equation of curve c is given by —

$$\bar{r} = x(t)\,\bar{i} + y(t)\,\bar{j} + z(t)\,\bar{k}$$

(ii) In practice,

$$\frac{d\bar{r}}{dt} = \frac{dx}{dt}\bar{i} + \frac{dy}{dt}\bar{j} + \frac{dz}{dt}\bar{k}$$

(iii) In particular, if $\bar{r}$ is displacement vector of time 't' then $\frac{d\bar{r}}{dt}$ is velocity vector.

It is denoted by $\bar{v}$.

$$\therefore \bar{v} = \frac{d\bar{r}}{dt}$$

acceleration $\bar{a} = \frac{d\bar{v}}{dt} = \frac{d^2\bar{r}}{dt^2}$

(iv) Division of vectors say $\bar{u}$ and $\bar{v}$ i.e $\frac{\bar{u}}{\bar{v}}$ is not defined.

Rules of derivative:

Let $\bar{u}, \bar{v}, \bar{w}$ are vector functions of t, s is a scalar function of t, c is any constant.

then (i) $\frac{d}{dt} c = 0$

(ii) $\frac{d}{dt}(c\bar{u}) = c\frac{d\bar{u}}{dt}$

(iii) $\frac{d}{dt}(\bar{u} \pm \bar{v}) = \frac{d\bar{u}}{dt} \pm \frac{d\bar{v}}{dt}$

(iv) $\frac{d}{dt}(\bar{u}\cdot\bar{v}) = \bar{u}\cdot\frac{d\bar{v}}{dt} + \bar{v}\cdot\frac{d\bar{u}}{dt}$

(v) $\frac{d}{dt}(\bar{u}\times\bar{v}) = \bar{u}\times\frac{d\bar{v}}{dt} + \frac{d\bar{u}}{dt}\times\bar{v}$

(vi) $\frac{d}{dt}(\bar{u}\cdot\bar{v}\times\bar{w}) = \frac{d\bar{u}}{dt}\cdot\bar{v}\times\bar{w} + \bar{u}\cdot\frac{d\bar{v}}{dt}\times\bar{w}$
$+ \bar{u}\cdot\bar{v}\times\frac{d\bar{w}}{dt}$

(vii) $\frac{d}{dt}(s\bar{u}) = \frac{d\bar{u}}{dt}s + \frac{ds}{dt}\bar{u}$

Ex: If $\bar{r} = \bar{A}e^{nt} + \bar{B}e^{-nt}$ then show that $\frac{d^2\bar{r}}{dt^2} - n^2\bar{r} = 0$ where $\bar{A}, \bar{B}$ are constant vector and n is any constant.

Soln: Consider,
$$\bar{r} = \bar{A}e^{nt} + \bar{B}e^{-nt}$$

Diff. w.r.t 't'
$$\frac{d\bar{r}}{dt} = n\bar{A}e^{nt} + (-n)\bar{B}e^{-nt}$$

Differentiate again w.r.t t

$$\frac{d^2\bar{r}}{dt^2} = \bar{A}e^{nt}(n^2) + \bar{B}e^{-nt}(-n)^2$$

$$= n^2[\bar{A}e^{nt} + \bar{B}e^{-nt}]$$

$$= n^2\bar{r}$$

$$\therefore \frac{d^2\bar{r}}{dt^2} - n^2\bar{r} = 0$$

Ex: 2. If $\bar{r} = \bar{a}e^{2t} + \bar{b}e^{3t}$, where $\bar{a}, \bar{b}$ are constant vectors then show that

$$\frac{d^2\bar{r}}{dt^2} - 5\frac{d\bar{r}}{dt} + 6\bar{r} = 0$$

Ex: 3. If $\bar{A} = 5t^2\bar{i} + t\bar{j} - t^3\bar{k}$ and $\bar{B} = t\bar{i} + 2t\bar{j}$

then find value of

(i) $\frac{d}{dt}(\bar{A}\cdot\bar{B})$ (ii) $\frac{d}{dt}(\bar{A}\times\bar{B})$ (iii) $\frac{d}{dt}(\bar{A}\cdot\bar{A})$

Ans: (i) $\frac{d}{dt}(\bar{A}\cdot\bar{B}) = 15t^2 + 4t$

(ii) $\frac{d}{dt}(\bar{A}\times\bar{B}) = 8t^3\bar{i} - 4t^3\bar{j} + (30t^2 - 2t)\bar{k}$

(iii) $\frac{d}{dt}(\bar{A}\cdot\bar{A}) = 100t^3 + 2t + 6t^5$

Ex: 4. Find $\frac{d\bar{f}}{dt}$ if $\bar{f} = r^2\bar{r} + (\bar{a}\cdot\bar{r})\bar{b}$
where $\bar{r}$ is function of t, $r = |\bar{r}|$
and $\bar{a}, \bar{b}$ are constant vectors.

Soln: Consider $\bar{f} = r^2\bar{r} + (\bar{a}\cdot\bar{r})\bar{b}$

Step-1: Differentiate w.r.t 't'

$$\frac{d\bar{f}}{dt} = \frac{d}{dt}(r^2\bar{r}) + \frac{d}{dt}(\bar{a}\cdot\bar{r})\bar{b}$$

$$= r^2\frac{d\bar{r}}{dt} + \bar{r}\frac{d}{dt}(r^2) + \bar{b}\frac{d}{dt}(\bar{a}\cdot\bar{r})$$

Since $\bar{b}$ is constant vector

$$= r^2\frac{d\bar{r}}{dt} + 2r\bar{r}\frac{dr}{dt} + \bar{b}\left(\bar{a}\cdot\frac{d\bar{r}}{dt}\right)$$

Ex: 5. Find $\frac{d\bar{f}}{dt}$ if $\bar{f} = r^3\bar{r} + \bar{a} \times \frac{d\bar{r}}{dt}$ where $\bar{r}$ is function of t, $|\bar{r}| = r$ and $\bar{a}$ is a constant vector.

Soln : Consider, $\bar{f} = r^3\bar{r} + \bar{a} \times \frac{d\bar{r}}{dt}$ — (1)

Differentiating Eqn (1) w.r.t t

$$\frac{d\bar{f}}{dt} = \frac{d}{dt}\left[r^3\bar{r} + \bar{a} \times \frac{d\bar{r}}{dt}\right]$$

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$$= \frac{d}{dt}(r^3 \bar{r}) + \frac{d}{dt}\left(\bar{a} \times \frac{d\bar{r}}{dt}\right)$$

$$= r^3 \frac{d\bar{r}}{dt} + \bar{r}\frac{d}{dt}(r^3) + \bar{a} \times \frac{d^2\bar{r}}{dt^2}$$

$$= r^3 \frac{d\bar{r}}{dt} + \bar{r}\, 3r^2 \frac{dr}{dt} + \bar{a} \times \frac{d^2\bar{r}}{dt^2}$$

Ex. 6 If $\bar{r} = \bar{a}\sinh t + \bar{b}\cosh t$, where $\bar{a}, \bar{b}$ are constant vectors then prove that

(i) $\dfrac{d^2\bar{r}}{dt^2} = \bar{r}$

(ii) $\dfrac{d\bar{r}}{dt} \times \dfrac{d^2\bar{r}}{dt^2} = \bar{a} \times \bar{b}$

$$LHS = \frac{d\bar{r}}{dt} \times \frac{d^2\bar{r}}{dt^2} = (\bar{a}\cosh t + \bar{b}\sinh t) \times (\bar{a}\sinh t + \bar{b}\cosh t)$$

$$= \cosh t \sinh t(\bar{a} \times \bar{a}) + \cosh^2 t(\bar{a} \times \bar{b}) + \sinh^2 t(\bar{b} \times \bar{a}) + \sinh t \cosh t\,(\bar{b} \times \bar{b})$$

$$= \cosh^2 t\,(\bar{a} \times \bar{b}) - \sinh^2 t\,(\bar{a} \times \bar{b})$$

As $\bar{a} \times \bar{b} = -(\bar{b} \times \bar{a})$

$$= (\bar{a} \times \bar{b})\left[\cosh^2 t - \sinh^2 t\right]$$

$$= (\bar{a} \times \bar{b})(1)$$

$$= \bar{a} \times \bar{b}$$

(iii) $\bar{r} \cdot \frac{d\bar{r}}{dt} \times \frac{d^2\bar{r}}{dt^2} = \bar{r} \cdot \frac{d\bar{r}}{dt} \times \bar{r}$

$= 0$

As $(\bar{a} \cdot \bar{b} \times \bar{a}) = 0$.

Ex. 7. The position vector of particle at a time t is $\bar{r} = \cos(t-1)\bar{i} + \sinh(t-1)\bar{j} + mt^3\bar{k}$. Find condition imposed on m by r: at time t = 1, acceleration is normal to position vector.

Soln: Consider,

$$\bar{r} = \cos(t-1)\bar{i} + \sinh(t-1)\bar{j} + mt^3\bar{k} \quad \text{--- (1)}$$

$$\Rightarrow \frac{d\bar{r}}{dt} = -\sin(t-1)\bar{i} + \cosh(t-1)\bar{j} + 3mt^2\bar{k} \quad \text{--- (2)}$$

$$\Rightarrow \frac{d^2\bar{r}}{dt^2} = -\cos(t-1)\bar{i} + \sinh(t-1)\bar{j} + 6mt\bar{k} \quad \text{--- (3)}$$

Now at time t = 1,

$$(\bar{r})_{t=1} = -\cos 0\,\bar{i} + \sinh 0\,\bar{j} + m\bar{k}$$
$$= -\bar{i} + m\bar{k}$$

As $\cos 0 = 1$, $\sinh 0 = 0$

$$\bar{a} = \left(\frac{d^2\bar{r}}{dt^2}\right)_{t=1} = -\cos 0\,\bar{i} + \sinh 0\,\bar{j} + 6m\bar{k}$$
$$= -\bar{i} + 6m\bar{k}$$

Given: Since acceleration is normal to position vector.

$$\bar{a}\cdot\bar{r} = 0$$

$$(-\bar{i} + 6m\bar{k})\cdot(\bar{i} + m\bar{k}) = 0$$

$$\Rightarrow -1 + 6m^2 = 0$$

$$m^2 = 1/6 \quad \Rightarrow m = \pm\frac{1}{\sqrt{6}}$$

Ex: 8. Find magnitude of velocity and acceleration of particle which moves along the curve $x = 2\sin 3t$, $y = 2\cos 3t$, $z = 8t$ at any time $t > 0$. Find unit tangent vector to curve.

soln: The displacement vector of moving particle is given by –

$$\bar{r} = (2\sin 3t)\,i + (2\cos 3t)\,j + (8t)\,\bar{k}$$

$$\Rightarrow \bar{v} = \frac{d\bar{r}}{dt} = 6\cos 3t\, i - 6\sin 3t\, j + 8\bar{k}$$

$$\Rightarrow \bar{a} = \frac{d^2\bar{r}}{dt^2} = -18\sin 3t\, i - 18\cos 3t\, j + 0$$

$$|\bar{v}| = \sqrt{(6\cos 3t)^2 + (-6\sin 3t)^2 + 8^2}$$

$$= \sqrt{36(\cos^2 3t + \sin^3 3t) + 64}$$

$$= \sqrt{36 + 64}$$

$$= \sqrt{100} = 10$$

$$|\bar{a}| = \sqrt{(-18\sin 3t)^2 + (-18\cos 3t)^2}$$

$$= \sqrt{18^2} = 18$$

tangent

$\therefore$ Unit vector is given by

$$\hat{T} = \frac{\bar{T}}{|\bar{T}|} = \frac{\bar{V}}{|\bar{V}|} = \frac{1}{10}\left(6\cos 3t\,\bar{i} - 6\sin 3t\,\bar{j} + 8\bar{k}\right)$$

Ex:9 A particle moves along curve $\bar{r} = e^{-t}\bar{i} + e^{2t}\bar{j} + e^{t}\bar{k}$

Find velocity and acceleration at $t=0$

Ex:10 A curve is given by the equation

$x = (t^2+1),\ y = (4t-3),\ z = (2t^2-6t)$.

Find the angle between tangent at $t=2$ and $t=3$

Soln. Given curve is

$$\bar{r} = (t^2+1)\bar{i} + (4t-3)\bar{j} + (2t^2-6t)\bar{k} \quad \text{--- (1)}$$

Tangent to curve $\bar{r}$ at $t=t_0$ is given by

$$\left(\frac{d\bar{r}}{dt}\right)_{t=t_0}$$

Differentiating (1) w.r.t t

$$\frac{d\bar{r}}{dt} = 2t\bar{i} + 4\bar{j} + (4t-6)\bar{k} \quad \text{--- (2)}$$

Let $\bar{T}_1$ and $\bar{T}_2$ be tangents to given curve at $t=2$ and $t=3$ respectively

$$\bar{T}_1 = \left(\frac{d\bar{r}}{dt}\right)_{t=2}, \quad \bar{T}_2 = \left(\frac{d\bar{r}}{dt}\right)_{t=3}$$

From eqn (2)

$$\bar{T}_1 = 4\bar{i} + 4\bar{j} + 2\bar{k}, \quad \bar{T}_2 = 6\bar{i} + 4\bar{j} + 6\bar{k}$$

Let θ be the angle between $\bar{T_1}$ and $\bar{T_2}$

$$\therefore \cos\theta = \frac{\bar{T_1}\cdot\bar{T_2}}{|\bar{T_1}||\bar{T_2}|} = \frac{(4\bar{i}+4\bar{j}+2\bar{k})\cdot(6\bar{i}+4\bar{j}+6\bar{k})}{\sqrt{16+16+4}\sqrt{36+16+36}}$$

$$= \frac{24+16+12}{\sqrt{36}\sqrt{88}}$$

$$= \frac{52}{6\sqrt{4\times22}} = \frac{13}{3\sqrt{22}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{13}{3\sqrt{22}}\right)$$

Vector Differential Operator :

Def^n || A variable quantity whose value at any point in a region of space depends upon the position of the point, is called a point function.

(i) **Scalar Point function :**

Let R be region of space at each point of which a scalar $\phi = \phi(x, y, z)$ is given, then ϕ is called a scalar function and R is called a scalar field.

e.g (i) density of rigid body
(ii) temperature distribution in a medium.

Vector Point function :

Let R be a region of space at each point of which a vector $\bar{F} = \bar{F}(x, y, z)$ is given,

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then $\bar{F}$ is called a vector point function and R is called a vector field.

e.g. (i) Velocity of a moving fluid
(ii) gravitational force.

Level Surface:

A surface passing through those points where value of function is same is called a level surface.

It is denoted by $\phi(x, y, z) = C$

e.g density of uniform body.

A vector differential operator is denoted by ∇ called as Del or nabla and is given by

$$\nabla = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$$

Gradient of scalar point function:

Note: When ∇ is operated on scalar function say $\phi(x, y, z)$, we get a vector quantity.

This vector quantity is called gradient of ϕ.

Notation: grad ϕ or $\nabla\phi$

$$\therefore \nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

Directional Derivative:

The rate of change of ϕ at point say P along given direction is called directional derivative of ϕ at P.

Note: (i) Let $\frac{\partial \phi}{\partial r}$ rate of change of ϕ along any arbitrary direction $\overline{PQ}$ and $\frac{\partial \phi}{\partial n}$ be rate of change of ϕ along normal direction.

$$\overline{PQ} = \overline{\delta r}$$

(ii) $\frac{\partial \phi}{\partial r} \leq \frac{\partial \phi}{\partial n}$.

i.e Rate of change of ϕ along normal is maximum.

(iii) Grad ϕ i.e $\nabla \phi$ is the vector along direction of normal to surface ϕ.

~~(iv) Rate of change of ϕ is maximum~~

(iv) Magnitude of maximum rate of change of ϕ is given by $|\nabla \phi|$.

(v) The DD of scalar point function $\phi(x,y,z) = c$ at point $P(x,y,z)$ in the direction of given vector $\bar{a}$ is given by –

$$\text{Directional Derivative} = (\nabla \phi)_P \cdot \hat{a}$$

$$\text{where } \hat{a} = \frac{\bar{a}}{|\bar{a}|}$$

Ex: 1. Find unit normal to the surface $x^2+y^2+z^2-3$ at $(1,1,1)$

Soln: Let vector normal to the surface ϕ at the point P is given by $(\nabla\phi)_P$

$$\text{Let } \phi(x,y,z) = x^2+y^2+z^2-3$$

$\therefore (\nabla\phi)_P \equiv$ normal to surface ϕ at pt $(1,1,1)$

Consider,

$$\nabla\phi = \bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}(x^2+y^2+z^2-3)$$

$$= \bar{i}\frac{\partial}{\partial x}(x^2+y^2+z^2-3)+\bar{j}\frac{\partial}{\partial y}(x^2+y^2+z^2-3)$$
$$+\bar{k}\frac{\partial}{\partial z}(x^2+y^2+z^2-3)$$

$$= \bar{i}(2x)+\bar{j}(2y)+\bar{k}(2z)$$

Now $(\nabla\phi)_{(1,1,1)} = 2\bar{i}+2\bar{j}+2\bar{k} = \bar{N}$

Now, unit normal along $\bar{N}$ is given by,

$$\hat{N} = \frac{\bar{N}}{|\bar{N}|}$$

$$= \frac{2\bar{i}+2\bar{j}+2\bar{k}}{\sqrt{4+4+4}} = \frac{2i+2j+2k}{2\sqrt{3}}$$

$$\boxed{\hat{N} = \frac{i+j+k}{\sqrt{3}}}$$

Ex: 2. Find angle between normals to surfaces $x \log z = y^2 - 1$ and $x^2 y = 2 - z$ at the point $(1,1,1)$

Soln: Let $\phi = x \log z - y^2 + 1$

& $\psi = x^2 y - 2 + z$

~~Now,~~
Step-1: $\nabla\phi = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)(x \log z - y^2 + 1)$

$$= \log z\,\bar{i} - 2y\,\bar{j} + \frac{x}{z}\bar{k}$$

& $(\nabla\phi)_{(1,1,1)} = \left(\log z\,\bar{i} - 2y\,\bar{j} + \frac{x}{z}\bar{k}\right)_{(1,1,1)}$

$$\bar{N}_1 = -2\bar{j} + \bar{k}$$

where ~~say~~ $\bar{N}_1$ be the normal to the ~~plane~~ surface ϕ

~~Step~~
$$\nabla\psi = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)(x^2 y - 2 + z)$$

$$(\nabla\psi)_{(1,1,1)} = \bar{N}_2 = (2xy\,\bar{i} + x^2\,\bar{j} + \bar{k})_{(1,1,1)} = 2\bar{i} + \bar{j} + \bar{k}$$

where $\bar{N}_2$ be the normal to the surface ψ.

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Step-II : Let θ be the angle between $\bar{N_1}$ and $\bar{N_2}$.

$$\therefore \bar{N_1}\cdot\bar{N_2} = |\bar{N_1}||\bar{N_2}|\cos\theta$$

$$\therefore (-2\bar{j}+\bar{k})\cdot(2\bar{i}+\bar{j}+\bar{k}) = (\sqrt{4+1})(\sqrt{4+1+1})\cos\theta$$

$$\therefore -2+1 = \sqrt{5}\sqrt{6}\cos\theta$$

$$\cos\theta = \frac{-1}{\sqrt{30}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{-1}{\sqrt{30}}\right)$$

Note- 1. Two surfaces are said to be intersect orthogonally with each other if normal's to surfaces at point of contact intersect with each other at right angle.

2. If two vectors $\bar{a}$ and $\bar{b}$ are perpendicular then their dot product is zero.

i.e if $\bar{a} \perp \bar{b}$ then $\bar{a}\cdot\bar{b} = 0$

Ex: 1 : Find angle between the normals to the surface $xy = z^2$ at the points $(1, 2, 1)$ and $(3, 2, 3)$

Soln:

Let $\phi = xy - z^2$

Normal to surface ϕ at point (x, y, z) is given by $(\nabla\phi)(x, y, z)$

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<u>Step-1</u>: Consider, $\nabla\phi = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)(xy - z^2)$

$$\nabla\phi = \bar{i}(y) + \bar{j}(x) + \bar{k}(-2z)$$

$$\therefore \nabla\phi = y\bar{i} + x\bar{j} - 2z\bar{k}$$

<u>Step-II</u>: Let $\bar{N_1}$ and $\bar{N_2}$ be normals to surface ϕ at the points $(1,2,1)$ and $(3,2,3)$ resply.

$$\therefore \bar{N_1} = (\nabla\phi)_{(1,2,1)} = 2\bar{i} + \bar{j} - 2\bar{k}$$

$$\bar{N_2} = (\nabla\phi)_{(3,2,3)} = 2\bar{i} + 3\bar{j} - 6\bar{k}$$

<u>Step-III</u>: Let θ be the angle between $\bar{N_1}$ and $\bar{N_2}$

$$\therefore \bar{N_1}\cdot\bar{N_2} = |\bar{N_1}||\bar{N_2}|\cos\theta$$

$$\therefore \cos\theta = \frac{\bar{N_1}\cdot\bar{N_2}}{|\bar{N_1}||\bar{N_2}|}$$

$$= \frac{(2\bar{i}+\bar{j}-2\bar{k})\cdot(2\bar{i}+3\bar{j}-6\bar{k})}{\sqrt{4+1+4}\sqrt{4+9+36}}$$

$$= \frac{4+3+12}{(3)(7)}$$

$$\cos\theta = \frac{19}{21}$$

$$\theta = \cos^{-1}\left(\frac{19}{21}\right)$$

Ex: 2. Find constants a and b, so that the surface $ax^2 - byz = (a+2)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at the point $(1,-1,2)$

Soln: Two surfaces are said to be orthogonal to each other if normals to the two surfaces at the point of contact intersect each other at right angles.

Let $\phi = ax^2 - byz - (a+2)x$

$\psi = 4x^2y + z^3 - 4$

Let $\bar{N_1} = (\nabla\phi)_{(1,-1,2)}$, $(\bar{N_2} = (\nabla\psi)_{(1,-1,2)}$

Step-1: $\bar{N_1} = (\nabla\phi)_{(1,-1,2)} = \bar{i}(a-2) + \bar{j}(-2b) + \bar{k}b$

$\bar{N_2} = (\nabla\psi)_{(1,-1,2)} = -8\bar{i} + 4\bar{j} + 12\bar{k}$

Since $\bar{N_1}$ & $\bar{N_2}$ intersect each other at right angles, we have,

$$\bar{N_1} \cdot \bar{N_2} = 0.$$

$$\therefore (\bar{i}(a-2) + \bar{j}(-2b) + \bar{k}b) \cdot (-8\bar{i} + 4\bar{j} + 12\bar{k}) = 0$$

$$-8a + 4b + 16 = 0 \quad \text{——— (1)}$$

Since given point $(1,-1,2)$ lies on two surfaces ϕ & ψ,

Hence put $(1,-1,2)$ in the eqn of surface ϕ.

$$\therefore a + 2b = a + 2$$

$$\therefore b = 1.$$

put in (1) $\Rightarrow$ $a = 5/2$

Ex: 3. What is the greatest rate of increase of $\phi = x^2 + yz^2$ at point $(1,-1,3)$

Soln: Greatest rate of increase is along it's normal i.e along direction of $\nabla\phi$

it's value $= |\nabla\phi|$

Step-1: Consider,

$$(\nabla\phi)_{(1,-1,3)} = 2\bar{i} + 9\bar{j} - 6\bar{k}$$

$$|\nabla\phi|_{(1,-1,3)} = \sqrt{121} = 11$$

Working rule to find Directional Derivative:

Working rule to find directional derivative of a scalar function $\phi(x,y,z)$ at the point $P(x,y,z)$ in the direction of $\bar{a}$.

Step-1: Directional Derivative

$$DD = (\nabla\phi)_P \cdot \hat{a} \quad \text{——— (i)}$$

Step-2: Find $\nabla\phi$ as

$$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

Step-3: Find value of $\nabla\phi$ at the point P i.e find $(\nabla\phi)_{P(x,y,z)}$

Step-4: Find vector $\bar{a}$ by given information.

Step-5: Find unit vector $\hat{a}$ in the direction of vector $\bar{a}$ by using the formula.

$$\hat{a} = \frac{\bar{a}}{|\bar{a}|}$$

Step-6: Put the values in (i), then we get DD

Different types of Examples on Directional derivative:

Directional Derivative of $\phi(x,y,z)$ at the point $P(x,y,z)$ in direction of

(i) given vector $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$

$$D.D = (\nabla\phi)_P \cdot \hat{a}$$

(ii) Vector along line PQ where $Q(x_2,y_2,z_2)$

Here $\bar{a} = \overline{PQ} = (x_2-x)\bar{i} + (y_2-y)\bar{j} + (z_2-z)\bar{k}$

$$P \longrightarrow Q$$

(iii) Tangent to given curve

$x = x(t),\ y = y(t),\ z = z(t)$. at $t = t_0$.

Here $\bar{a} = \left(\frac{d\bar{r}}{dt}\right)_{t=t_0}$ where

$$\bar{r} = x(t)\bar{i} + y(t)\bar{j} + z(t)\bar{k}$$

(iv) Normal to surface say $\psi(x,y,z)$ at the point $Q(x_2,y_2,z_2)$

Ex: 1. Find directional derivative of $\phi = x^2yz + 4xz^2$ at the point $(1,-2,-1)$ in the direction of vector $2\bar{i} - \bar{j} - 2\bar{k}$

step-1:

Sol^n: We know that D.D at $P(x,y,z)$ in the direction of vector $\bar{a}$ is given by.

$$D.D = (\nabla\phi)_P \cdot \hat{a} \quad \text{——— (1)}$$

Here, $\phi = x^2yz + 4xz^2$, $P(x,y,z) = P(1,-2,-1)$

$\bar{a} = 2\bar{i} - \bar{j} - 2\bar{k}$

Consider,

$$\nabla\phi = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)(x^2yz + 4xz^2)$$

$$= \bar{i}\,(2xyz + 4z^2) + \bar{j}\,(x^2z) + \bar{k}\,(x^2y + 8xz)$$

Now

$$(\nabla\phi)_{(1,-2,-1)} = \bar{i}\left[2(1)(-2)(-1) + 4(-1)^2\right] + \bar{j}\left[(1)^2(-1)\right] + \bar{k}\left[(1)^2(-2) + 8(1)(-1)\right]$$

$$= \bar{i}\,[4+4] + \bar{j}\,[-1] + \bar{k}\,[-10]$$

$$= 8\bar{i} - \bar{j} - 10\bar{k}$$

$$= 8\bar{i} - \bar{j} - 10\bar{k}$$

Step-II : To find $\hat{a}$

$$\hat{a} = \frac{\bar{a}}{|\bar{a}|} = \frac{2\bar{i} - \bar{j} - 2\bar{k}}{\sqrt{4+1+4}} = \frac{2\bar{i} - \bar{j} - 2\bar{k}}{3}$$

Substitute in ①

$$D.D = (\nabla\phi)_P \cdot \hat{a}$$

$$= (8\bar{i} - \bar{j} - 10\bar{k}) \cdot \left(\frac{2\bar{i} - \bar{j} - 2\bar{k}}{3}\right)$$

$$D.D = \frac{16 + 1 + 20}{3} = \frac{37}{3}$$

Ex: 2. Find D.D of $\phi = xy^2 + yz^3$ at point $(2,-1,1)$ in the direction of vector $\bar{i} + 2\bar{j} + 2\bar{k}$

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Ans: $D.D = -11/3$

Ex: 3. Find magnitude of maximum value of directional derivative of $\phi = x^2y^2z^4$ at the point $(3, 1, -2)$.

Soln:

Magnitude of maximum value of

$$D.D = |(\nabla\phi)_p| = 418.45$$

Ex: 4. Find the Directional derivative of the function $\phi = xy^2 + yz^3$ at $(1, -1, 1)$ along the direction normal to the surface $x^2 + y^2 + z^2 = 9$ at $(1, 2, 1)$.

Soln: Step-1:

We know that D.D. of scalar point function at the point $P(x, y, z)$ along direction of $\bar{a}$ is given by -

$$\text{Directional Derivative} = (\nabla\phi)_{(1,-1,1)} \cdot \hat{a}$$

$$\nabla\phi = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)(xy^2 + yz^3)$$

$$= \bar{i}(y^2) + \bar{j}(2xy + z^3) + \bar{k}(3yz^2)$$

$$(\nabla\phi)_{(1,-1,1)} = \bar{i} + (-\bar{j}) - 3\bar{k})$$

$$= \bar{i} - \bar{j} - 3\bar{k}$$

Step-II: $\bar{a}$ = normal to given surface

$$\psi = x^2 + y^2 + z^2 - 9 \text{ at } (1, 2, 1)$$

$$= (\nabla\psi)_{(1,2,1)}$$

$$\bar{a} = (\nabla\psi)_{(1,2,1)} = 2\bar{i} + 4\bar{j} + 2\bar{k}$$

$$\hat{a} = \frac{2\bar{i} + 4\bar{j} + 2\bar{k}}{24}$$

Step-III : Directional Derivative $= -4/3$

Ex.5 : Find the directional derivative of $\phi = 4xz^3 - 3x^2y^2z$ at $(2,-1,2)$ in the direction of tangent to the curve $x = e^t \cos t$, $y = e^t \sin t$, $z = e^t$ at $t=0$

Soln: Step-1: $D.D = (\nabla\phi)_P \cdot \hat{a}$

$$(\nabla\phi)_{(2,-1,2)} = 8\bar{i} + 48\bar{j} + 84\bar{k}$$

Step-2: $\bar{a}$ = tangent to the curve $x = e^t \cos t$, $y = e^t \sin t$, $z = e^t$ at $t=0$

$$\bar{r} = (e^t \cos t)\bar{i} + (e^t \sin t)\bar{j} + e^t \bar{k}$$

$$\left(\frac{d\bar{r}}{dt}\right)_{t=0} = \bar{i} + \bar{j} + \bar{k}$$

Step-3: $\hat{a} = \frac{\bar{a}}{|a|} = \frac{\bar{i} + \bar{j} + \bar{k}}{\sqrt{1+1+1}}$

$$= \frac{\bar{i} + \bar{j} + \bar{k}}{\sqrt{3}}$$

Hence $D.D = 140/\sqrt{3}$

Ex: 6 : Find the directional derivative of $\operatorname{div}(x^5\bar{i}+y^5\bar{j}+z^5\bar{k})$ at $(1,2,3)$ in the direction of outward normal to the surface $x^2+y^2+z^2=9$ at $(1,2,3)$

Soln : We know that directional derivative ϕ at the point P in the direction of vector a is given by,

$$D \cdot D = (\nabla\phi)_P \cdot \hat{a} \quad \text{——— (1)}$$

Here. $\phi = \nabla \cdot (x^5\bar{i}+y^5\bar{j}+z^5\bar{k})$

$$\therefore \phi = \left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)\cdot\left(x^5\bar{i}+y^5\bar{j}+z^5\bar{k}\right)$$

$$= \frac{\partial}{\partial x}x^5+\frac{\partial}{\partial y}y^5+\frac{\partial}{\partial z}z^5$$

$$\phi = 5x^4+5y^4+5z^4$$

Now,

$$\nabla\phi = \left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)(5x^4+5y^4+5z^4)$$

$$= 20x^3\bar{i}+20y^3\bar{j}+20z^3\bar{k}$$

Hence $(\nabla\phi)_{(1,2,3)} = 20\bar{i}+20\times 8\bar{j}+20\cdot 27\bar{k}$

$$= 20\bar{i}+160\bar{j}+540\bar{k}$$

Step-II: Let $\psi = x^2+y^2+z^2-9$

A outward normal vector to surface ψ is given by $\nabla\psi$

Consider,

$$\nabla\psi = \left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2-9)$$

$$= (2x)\bar{i}+(2y)\bar{j}+(2z)\bar{k}$$

$$\therefore (\nabla\psi)_{(1,2,3)} = 2\bar{i}+4\bar{j}+6\bar{k} = \bar{a} \text{ (say)}$$

Step III : $\hat{a} = \dfrac{\bar{a}}{|\bar{a}|} = \dfrac{2\bar{i}+4\bar{j}+6\bar{k}}{\sqrt{4+16+36}}$

$$= \frac{2\bar{i}+4\bar{j}+6\bar{k}}{\sqrt{56}}$$

Hence eq^n 1 becomes -

$$D.D = (\nabla\phi)_p \cdot \hat{a}$$

$$= (20\bar{i}+160\bar{j}+540\bar{k})\cdot\frac{(2\bar{i}+4\bar{j}+6\bar{k})}{\sqrt{56}}$$

$$= \frac{3920}{\sqrt{56}}$$ // $40+640+3240$

Ex:7. Find D.D of $\phi = xy+yz^2$ at $(1,-1,1)$ towards the point $(2,1,2)$.

Ans: (i) $(\nabla\phi)_{(1,-1,1)} = -\bar{i}+2\bar{j}-2\bar{k}$

(ii) $\bar{a} = \overline{PQ} = \bar{Q}-\bar{P}$; $P=(1,-1,1)$, $Q=(2,1,2)$

$\therefore \bar{a} = \bar{i}+2\bar{j}+\bar{k}$

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(iii) $D \cdot D = 1/\sqrt{6}$

Ex: 8. Find directional derivative of $\phi = 2xz^4 - x^2y$ at $(2,-2,1)$ towards the point $(1,1,-1)$

Ans $D \cdot D = \frac{-54}{\sqrt{14}}$

Ex: 9. Find the directional derivative of the function $\phi = e^{2x-y-z}$ at $(1,1,1)$ in the direction of the tangent to the curve $x = e^{-t}$, $y = 2\sin t + 1$
$z = t - \cos t$ at $t=0$

Ans: $\bar{a} = \left(\frac{d\bar{r}}{dt}\right)_{t=0} = \bar{i} + 2\bar{j} + \bar{k}$

$D \cdot D = -5/\sqrt{6}$

Ex. 10. Find the D.D of $\phi = 3\log(x+y+z)$ at $(1,1,1)$ in the direction of tangent to the curve $x = b\sin t$, $y = b\cos t$, $z = bt$ at $t=0$

Ans. $D \cdot D = 2/\sqrt{2}$.

Ex: 11. Find the D.D. of $\phi = xy^2 + yz^3$ at $(1,-1,1)$ in the direction of the tangent to the curve
$x = \sin t$, $y = \cos t$, $z = t$, at $t = \pi/4$

$DD = 1 - 3/\sqrt{2}$

Ex: 12. Find the D.D of $\phi = 4xz^3 - 3x^2yz^2$ at $(2,-1,2)$ in the direction of normal to surface $x^3+y^3+3xyz=3$ at point $(1,2,-1)$

Ans: $D.D = \frac{192}{\sqrt{126}}$

Ex: 13 Find the directional derivative of xy^2+yz^3 at $(2,-1,1)$ along the line
$2(x-2) = y+1 = z-1$

Ans: $D.D = -11/3$

Hint: For $\bar{a}$:

$$x-2 = \frac{y+1}{2} = \frac{z-1}{2}$$

Vector say $\bar{a}$ along this line is given by,
$\bar{a} = \bar{i} + 2\bar{j} + 2\bar{k}$

Ex: 14. Find D.D of $\phi = 5x^2y - 5y^2z + 2z^2x$ at the point $(1,1,1)$ in the direction of the line $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$

Ans: $D.D = 11$

Ex: 15. Find D.D of $\phi = 4xz^3 - 3x^2y^2z$ at the point $(2,-1,2)$ along a line equally inclined with co-ordinate axes.

Hint: To find $\bar{a}$:
A vector along a line which is equally inclined with co-ordinate axes is given by
$\bar{a} = \bar{i} + \bar{j} + \bar{k}$

Ans: $D.D = \frac{140}{\sqrt{3}}$

Ex: 16 : If the D.D of $\phi = ax^2y + by^2z + cz^2x$ at (1,1,1) has maximum magnitude 15 in the direction parallel to
$$\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$$ then find values of a, b, c

Ans: (i) Magnitude of maximum D.D of ϕ is $|\nabla\phi|$

$$\nabla\phi = (2a+c)\bar{i} + (a+2b)\bar{j} + (b+2c)\bar{k}$$
$$|\nabla\phi| = \sqrt{(2a+c)^2 + (a+2b)^2 + (b+2c)^2}$$

(ii) From given line
$$\bar{a} = 2\bar{i} - 2\bar{j} + \bar{k}$$

Since $(\nabla\phi)_{(1,1,1)}$ is parallel to $2\bar{i} - 2\bar{j} + \bar{k}$,

hence unit vectors along $(\nabla\phi)_{(1,1,1)}$ and $2\bar{i} - 2\bar{j} + \bar{k}$ are equal.

$$\therefore \frac{(\nabla\phi)_{(1,1,1)}}{|\nabla\phi|_{(1,1,1)}} = \frac{\bar{a}}{|\bar{a}|}$$

Equating coefficients of $\bar{i}, \bar{j}, \bar{k}$ from LHS & RHS, & solving we get,
$$a = \frac{20}{9}, \quad b = -\frac{55}{9}, \quad c = \frac{50}{9}$$

Ex: 17 If the D.D of $\phi = axy + byz + cxz$ at (1,1,1) has maximum magnitude 4 in a direction parallel to x-axis then find values of a, b, c

Hint: $\frac{\nabla\phi}{|\nabla\phi|} = \bar{i}$

solving we get,

$a=2,\ b=-2,\ c=2.$

Ex:18 Find the values of constants a, b, c so that the D.D. of $\phi = axy^2 + byz + cz^2x^2$ at (2,1,1) has a maximum magnitude 12 in the direction parallel to X-axis

$a=4,\ b=-16,\ c=2.$

Divergence and curl of vector function:

Let $\bar{F} = f_1(x,y,z)\bar{i} + f_2(x,y,z)\bar{j} + f_3(x,y,z)k$ be given vector point function.

(1) When ∇ is operated on $\bar{F}$, we get scalar quantity called as divergence of $\bar{F}$.
It is denoted by $\mathrm{Div}\bar{F}$ or $\nabla\cdot\bar{F}$

$$\mathrm{Div}\bar{F} = \nabla\cdot\bar{F} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot(f_1 i + f_2 j + f_3 k)$$

$$= \frac{\partial}{\partial x}f_1 + \frac{\partial}{\partial y}f_2 + \frac{\partial}{\partial z}f_3$$

(2) When ∇ operated vectorially on $\bar{F}$, we get vector quantity called as curl of $\bar{F}$ and it is denoted by curl $\bar{F}$ or $\nabla\times\bar{F}$

$$\text{curl } \bar{F} = \nabla \times \bar{F} \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

Ex: 1. If $\phi = x^3 + y^3 + z^3 - 3xyz$ then find
(i) div(grad ϕ) (ii) curl (grad ϕ)

Soln: We have,

$$\nabla\phi = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)(x^3+y^3+z^3-3xyz)$$

$$= \bar{i}(3x^2-3yz) + \bar{j}(3y^2-3xz) + \bar{k}(3z^2-3xy)$$

$$\text{div}(\text{grad}\,\phi) = \nabla\cdot(\text{grad}\,\phi)$$
$$= \nabla\cdot(\nabla\phi)$$
$$= \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\Big(\bar{i}(3x^2-3yz)$$
$$+ \bar{j}(3y^2-3xz) + \bar{k}(3z^2-3xy)\Big]$$

$$= \frac{\partial}{\partial x}(3x^2-3yz) + \frac{\partial}{\partial y}(3y^2-3xz) + \frac{\partial}{\partial z}(3z^2-3xy)$$

$$= 6x + 6y + 6z$$

(ii) $\text{Curl}(\text{grad}\,\phi) = \nabla \times \nabla\phi$

$$\text{curl}(\text{grad}\,\phi) = \nabla \times \nabla\phi$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (3x^2-3yz) & (3y^2-3xz) & (3z^2-3xy) \end{vmatrix}$$

$$= \bar{i}\left[\frac{\partial}{\partial y}(3z^2-3xy) - \frac{\partial}{\partial z}(3y^2-3xz)\right]$$

$$-\bar{j}\left[\frac{\partial}{\partial x}(3z^2-3xy) - \frac{\partial}{\partial z}(3x^2-3yz)\right]$$

$$+\bar{k}\left[\frac{\partial}{\partial x}(3y^2-3xz) - \frac{\partial}{\partial y}(3x^2-3yz)\right]$$

$$= \bar{i}(-3x+3x) + \bar{j}(-3y+3y) + \bar{k}(-3z+3z)$$

$$= \bar{0}$$

Ex: 2. If $\bar{F} = 3xyz^2\bar{i} + 4x^3y\bar{j} - xy^2\bar{k}$ then find grad(Div $\bar{F}$) at $(-1,2,1)$

Ans: $\text{grad}(\text{Div}\bar{F})_{(-1,2,1)} = 12\bar{i} + 3\bar{j} + 12\bar{k}$

Ex: 3. Find curl(curl $\bar{F}$) at point $(1,1,1)$ if $\bar{F} = x^2y\,\bar{i} + xz\,\bar{j} + 2yz\,\bar{k}$

Ans: $(\text{curl}\,\text{curl}\,\bar{F})_{(1,1,1)} = 4\bar{j}$

Ex: 4. Find curl curl $\bar{F}$ at the point $(1,1,1)$ where $\bar{F} = x^2y\,\bar{i} + xyz\,\bar{j} + z^2y\,\bar{k}$

Ans: $\text{curl}\,\text{curl}\,\bar{F} = 2\bar{i} + 6\bar{j} + \bar{k}$

Solenoidal and Irrotational vector field :

1. A vector field $\bar{F}$ is said to be solenoidal if $\nabla \cdot \bar{F}$ i.e $Div \bar{F} = 0$

2. A vector field $\bar{F}$ is said to be irrotational or conservative if $\nabla \times \bar{F}$ i.e curl $\bar{F} = 0$

In this case, there exist a scalar potential ϕ such that $\bar{F} = \nabla \phi$

Method of finding ϕ:

If $\bar{F}$ is irrotational then $\bar{F} = \nabla \phi$

$\therefore \quad \bar{F} \cdot d\bar{r} = \nabla \phi \cdot d\bar{r}$

where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

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$\therefore \left[f_1(x,y,z)\bar{i} + f_2(x,y,z)\bar{j} + f_3(x,y,z)\bar{k}\right] \cdot \left[dx\bar{i} + dy\bar{j} + dz\bar{k}\right]$

$= \left[\bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}\right] \cdot \left[dx\bar{i} + dy\bar{j} + dz\bar{k}\right]$

$\therefore f_1(x,y,z)dx + f_2(x,y,z)dy + f_3(x,y,z)dz$

$= \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$

Hence,

$f_1(x,y,z)dx + f_2(x,y,z)dy + f_3(x,y,z)dz = d\phi$

If $\bar{F}$ is irrotational integral then LHS will be total differential of some function.

On integrating both sides.

$$\int_{y,z \text{ const}} F_1(x,y,z)\,dx + \int_{z \text{ const}} \left(\begin{matrix}\text{Terms of } F_2 \\ \text{free from } x\end{matrix}\right) dy + \int \left|\begin{matrix}\text{Terms of } F_3 \\ \text{free from} \\ x \,\&\, y\end{matrix}\right| dz + c = \phi$$

Ex: 1. If $\bar{F} = (x+3y)\bar{i} + (y-2z)\bar{j} + (az+x)\bar{k}$ is solenoidal. Find value of a

Soln: Since $\bar{F}$ is solenoidal we have,

$\nabla \cdot \bar{F} = 0$

$$\left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \cdot \left[(x+3y)\bar{i} + (y-2z)\bar{j} + (az+x)\bar{k}\right] = 0$$

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$$\frac{\partial}{\partial x}(x+3y)+\frac{\partial}{\partial y}(y-2z)+\frac{\partial}{\partial z}(az+x)=0$$

$$\Rightarrow 1+1+a=0$$

$$\Rightarrow \underline{a=-2}.$$

Ex: 2. Find the value of m if $\bar{F}=(x+2y)\bar{i}+(my+4z)\bar{j}+(5z+6x)\bar{k}$ is solenoidal.

<u>Soln</u>: Since $\bar{F}$ is solenoidal $\therefore \text{Div}\,\bar{F}=0$

i.e $\nabla\cdot\bar{F}=0$

$$\left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)\cdot\left[(x+2y)\bar{i}+(my+4z)\bar{j}+(5z+6x)\bar{k}\right]=0$$

$$\therefore \frac{\partial}{\partial x}(x+2y)+\frac{\partial}{\partial y}(my+4z)+\frac{\partial}{\partial z}(5z+6x)=0$$

$$\therefore 1+m+5=0$$

$$m+6=0$$

$$\underline{m=-6}$$

Ex: 3. If $\bar{F}=(ax+3y+4z)\bar{i}+(x-2y+3z)\bar{j}+(3x+2y-z)\bar{k}$ is solenoidal, then find value of a.

Ans: $a=3$

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Ex: 4 For what value of constant a, the vector field $\overline{F} = (axy - z^3)\overline{i} + (a-2)x^2\overline{j} + (1-a)xz^2\overline{k}$

Sol^n: Since $\overline{F}$ is irrotational

$$\Rightarrow \nabla \times \overline{F} = 0$$

$$\begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (axy - z^3) & (a-2)x^2 & (1-a)xz^2 \end{vmatrix} = 0$$

$$\Rightarrow \overline{i}\left(\frac{\partial}{\partial y}(1-a)xz^2 - \frac{\partial}{\partial z}(a-2)x^2\right)$$

$$-\overline{j}\left(\frac{\partial}{\partial x}(1-a)xz^2 - \frac{\partial}{\partial z}(axy - z^3)\right)$$

$$+\overline{k}\left(\frac{\partial}{\partial x}(a-2)x^2 - \frac{\partial}{\partial y}(axy - z^3)\right) = 0$$

$$\Rightarrow \overline{i}(0 - 0) - \overline{j}\left((1-a)z^2 - (-3z^2)\right) + \overline{k}\left(2x(a-2) - ax\right) = 0$$

$$-\overline{j}\left[(1-a)z^2 + 3z^2\right] + \overline{k}\left(2(a-2)x - ax\right) = 0.$$

$$-\overline{j}\left[(1-a)z^2 + 3z^2\right] + \overline{k}\left[2ax - 4x - ax\right] = 0\cdot\overline{j} + 0\overline{k}$$

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$\therefore\ z^2 - az^2 + 3z^2 = 0$ & $2ax - 4x - ax = 0$

$a\cancel{z^2} = 4\cancel{z^2}$ $\qquad$ $a\cancel{x} = 4\cancel{x}$

$a = 4$ $\qquad$ $a = 4$

$$\Rightarrow a = 4$$

Ex:5 Find constants a, b, c so that
$\bar{F} = (x + 2y + az)\,\bar{i} + (bx - 3y - z)\,\bar{j} + (4x + cy + 2z)\,\bar{k}$
is irrotational.

Sol^n: Since $\bar{F}$ is irrotational

$$\Rightarrow \nabla \times \bar{F} = 0$$

$$\therefore \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+2y+az) & (bx-3y-z) & (4x+cy+2z) \end{vmatrix} = 0$$

$$\therefore \bar{i}\left(\frac{\partial}{\partial y}(4x+cy+2z) - \frac{\partial}{\partial z}(bx-3y-z)\right)$$

$$-\bar{j}\left(\frac{\partial}{\partial x}(4x+cy+2z) - \frac{\partial}{\partial z}(x+2y+az)\right)$$

$$+\bar{k}\left(\frac{\partial}{\partial x}(bx-3y-z) - \frac{\partial}{\partial y}(x+2y+az)\right) = 0$$

$$\therefore \bar{i}\,\big(c - (-1)\big) - \bar{j}\,\big(4 - a\big) + \bar{k}\,\big(b - 2\big) = 0$$

$\therefore$ Hence, $(c+1)\,\bar{i} - (4-a)\,\bar{j} + (b-2)\,\bar{k} = 0\bar{i} + 0\bar{j} + 0\bar{k}$

$\Rightarrow c + 1 = 0$; $-(4-a) = 0$ $\Rightarrow b - 2 = 0$

$c = -1$ $\qquad$ $\Rightarrow a = 4$ $\qquad$ $b = 2$

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Ex: 6. If $\bar{F} = (y+z)\bar{i} + (z+x)\bar{j} + (x+y)k$
show that
$$\text{curl curl curl curl } \bar{F} = \nabla^4 F$$

Soln: $LHS = \text{curl curl curl curl } \bar{F} = \nabla \times \nabla \times \nabla \times \nabla \times \bar{F}$

$$= \nabla \times \nabla \times (\overset{1}{\nabla} \times \overset{2}{\nabla} \times \overset{3}{\bar{F}})$$
$$= \nabla \times \nabla \times [\nabla(\nabla \cdot \bar{F}) - (\nabla \cdot \nabla)\bar{F}] \quad \text{—— (1)}$$
$$(1 \cdot 3)\,2 \quad (2 \cdot 1)\,3$$

Consider,
$$\nabla \cdot \bar{F} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot [(y+z)\bar{i} + (z+x)\bar{j} + (x+y)\bar{k}]$$

$$= \frac{\partial}{\partial x}(y+z) + \frac{\partial}{\partial y}(z+x) + \frac{\partial}{\partial z}(x+y)$$

$$= 0$$

From eqn (1)

$$L.H.S = \nabla \times \nabla \times [0 - (\nabla \cdot \nabla)\bar{F}]$$
$$= \nabla \times \nabla \times [-\nabla^2 \bar{F}]$$
$$= -[\nabla \times \nabla \times \nabla^2 \bar{F}]$$
$$= -[\nabla(\nabla \cdot \nabla^2 \bar{F}) - (\nabla \cdot \nabla)\nabla^2 \bar{F}]$$
$$= -[\nabla(\nabla^2 \cdot \nabla \bar{F}) - (\nabla \cdot \nabla)\nabla^2 \bar{F}]$$

As dot product is commutative.

$$= -[-\nabla^2 \nabla^2 \bar{F}]$$
$$= \nabla^4 \bar{F} = RHS$$

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EX:7. If $\bar{u}$ and $\bar{v}$ are irrotational vectors then prove that $\bar{u} \times \bar{v}$ is solenoidal vector.

Soln: Since $\bar{u}$ & $\bar{v}$ are irrotational vectors

$\Rightarrow \nabla \times \bar{u} = \bar{0}$ and $\nabla \times \bar{v} = \bar{0}$

Now, consider,

$$\nabla \cdot (\bar{u} \times \bar{v}) = \bar{v} \cdot (\nabla \times \bar{u}) - \bar{u} \cdot (\nabla \times \bar{v})$$
$$= 0$$

$\therefore \bar{u} \times \bar{v}$ is solenoidal vector.

Vector Identities :

Properties of ∇ :

If $\bar{u}, \bar{v}$ are vector functions and ϕ and ψ are scalar functions then

(i) $\nabla \cdot (\bar{u} \pm \bar{v}) = \nabla \cdot \bar{u} \pm \nabla \cdot \bar{v}$

(ii) $\nabla \times (\bar{u} \pm \bar{v}) = \nabla \times \bar{u} \pm \nabla \times \bar{v}$

(iii) $\nabla \cdot (\phi \bar{u}) = \phi (\nabla \cdot \bar{u}) + (\nabla \phi) \cdot \bar{u}$

(iv) $\nabla \times (\phi \bar{u}) = \phi (\nabla \times \bar{u}) + (\nabla \phi) \times \bar{u}$

(v) $\nabla (\phi \psi) = \phi (\nabla \psi) + (\nabla \phi) \psi$

(vi) $\nabla \cdot (\bar{u} \times \bar{v}) = \bar{v} \cdot (\nabla \times \bar{u}) - \bar{u} \cdot (\nabla \times \bar{v})$

(vii) $\nabla \cdot (\nabla \phi) = \dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} + \dfrac{\partial^2 \phi}{\partial z^2} = \nabla^2 \phi$

(viii) $\nabla \times (\nabla \phi) = 0$

(ix) $\nabla \cdot (\nabla \times \bar{u}) = 0$

(x) $\overset{1}{\nabla} \times (\overset{2}{\nabla} \times \overset{3}{\bar{u}}) = \overset{2}{\nabla}\overset{(1\cdot 3)}{(\nabla \cdot \bar{u})} - (\overset{2}{\nabla} \cdot \overset{1}{\nabla}) \overset{3}{\bar{u}}$ $\quad ((1\cdot 3)2 - (2\cdot 1)3$

$= \nabla (\nabla \cdot \bar{u}) - \nabla^2 \bar{u}$

Result : Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$, $r = |\bar{r}|$

$\bar{a} = a_1 \bar{i} + a_2 \bar{j} + a_3 \bar{k}$

then,

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(i) $\nabla \cdot \bar{r} = 3$ (ii) $\nabla \times \bar{r} = 0$

(iii) $\nabla \cdot \bar{a} = 0$ (iv) $\nabla \times \bar{a} = 0$

(v) $\bar{r} \cdot \bar{r} = r^2$ (vi) $(\bar{a} \cdot \nabla)\bar{r} = \bar{a}$

(vii) $\nabla(\bar{a} \cdot \bar{r}) = \bar{a}$ (viii) $\nabla \cdot (\bar{a} \times \bar{r}) = 0$

(ix) $\nabla f(r) = \frac{f'(r)}{r}\bar{r}$ (x) $\nabla(\bar{a} \cdot \bar{b}) = 0$

Note: $\nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \nabla^2 \Leftarrow$ Laplace operator.

Ex. 1 Prove that $\frac{\bar{r}}{r^3}$ is solenoidal OR,

Prove that $\nabla \cdot \frac{\bar{r}}{r^3} = 0$.

Soln. We know ~~the~~ a vector function F is solenoidal if $\nabla \cdot \bar{F} = 0$

Hence ~~t~~ we have to prove $\nabla \cdot \frac{\bar{r}}{r^3} = 0$

Consider,

$$\nabla \cdot \frac{\bar{r}}{r^3} = \nabla \cdot \left(\underset{\phi}{\frac{1}{r^3}}\ \underset{\bar{u}}{\bar{r}}\right) = \left(\nabla \frac{1}{r^3}\right) \cdot \bar{r} + \frac{1}{r^3}(\nabla \cdot \bar{r})$$

$$= \left(\frac{-3r^{-4}}{r}\,\bar{r}\right)\cdot\bar{r} + \frac{3}{r^3}$$

Since $\nabla f(r) = \frac{f'(r)}{r}\,\bar{r}$

& $\nabla\cdot\bar{r} = 3$

$$= -\frac{3}{r^5}(\bar{r}\cdot\bar{r}) + \frac{3}{r^3}$$

$$= -\frac{3}{r^5}\,r^2 + \frac{3}{r^3} = -\frac{3}{r^3} + \frac{3}{r^3} = 0$$

Hence, $\frac{\bar{r}}{r^3}$ will be solenoidal

Ex: 2. If $\bar{a}$ is a constant vector and $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ then prove that:

$$\nabla\times(\bar{a}\times\bar{r}) = 2\bar{a}$$

Soln: LHS $= \overset{1}{\nabla}\times(\overset{2}{\bar{a}}\times\overset{3}{\bar{r}})$.

// $= (1\cdot 3)2 - (2\cdot 1)3$ //

$$\text{L.H.S} = (\nabla\cdot\bar{r})\bar{a} - (\bar{a}\cdot\nabla)\bar{r}$$

$$= 3\bar{a} - \bar{a}$$

$$= 2\bar{a}.$$

Ex: 3. Prove that $\frac{\bar{a}\times\bar{r}}{r^n}$ is solenoidal OR

Prove that $\nabla\cdot\left(\frac{\bar{a}\times\bar{r}}{r^n}\right) = 0$

Soln: Consider,

$$\nabla\cdot\left(\frac{\bar{a}\times\bar{r}}{r^n}\right) = \nabla\cdot\left(\underset{\phi}{\frac{1}{r^n}}\underset{\bar{u}}{(\bar{a}\times\bar{r})}\right)$$

$$= (\nabla\phi)\cdot\bar{u} + \phi(\nabla\cdot\bar{u})$$

$$= \left(\nabla\frac{1}{r^n}\right)\cdot(\bar{a}\times\bar{r}) + \frac{1}{r^n}\left(\nabla\cdot(\bar{a}\times\bar{r})\right)$$

$$= \left(\left(\frac{-n\,r^{-n-1}}{r}\bar{r}\right)\cdot(\bar{a}\times\bar{r}) + \frac{1}{r^n}(0)\right)$$

$$\nabla f(r) = \frac{f'(r)}{r}\bar{r}$$

$$\nabla\cdot(\bar{a}\times\bar{r}) = 0$$

$$= \frac{-n}{r^{n+2}}\,\bar{r}\cdot(\bar{a}\times\bar{r})$$

$$= 0$$

[Since in vector triple product, if two vectors are equal then scalar triple product is zero.] $\Rightarrow \bar{r}\cdot(\bar{a}\times\bar{r}) = 0$.

Ex: 3. Prove that $\nabla f(r) = \frac{f'(r)}{r}\bar{r}$

Soln: Let $\bar{r} = xi + yj + z\bar{k}$, $|\bar{r}| = \sqrt{x^2+y^2+z^2} = r$

Now,

$$\nabla f(r) = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) f(r)$$

$$\nabla f(r) = \bar{i}\frac{\partial}{\partial x} f(r) + \bar{j}\frac{\partial}{\partial y} f(r) + \bar{k}\frac{\partial}{\partial z} f(r)$$

Now, $\dfrac{\partial f}{\partial x} = \dfrac{df}{dr} \cdot \dfrac{\partial r}{\partial x}$

As $f(r) \longrightarrow r \left\langle \begin{matrix} x \\ y \\ z \end{matrix} \right.$

$\because \; r = \sqrt{x^2+y^2+z^2}$

$$\therefore \frac{\partial r}{\partial x} = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$$

Hence, $\dfrac{\partial f}{\partial x} = \dfrac{df}{dr} \cdot \dfrac{\partial r}{\partial x} = \dfrac{df}{dr} \cdot \dfrac{x}{r}$

$$\frac{\partial f}{\partial y} = \frac{df}{dr} \cdot \frac{\partial r}{\partial y} = \frac{df}{dr} \cdot \frac{y}{r}$$

$$\frac{\partial f}{\partial z} = \frac{df}{dr} \cdot \frac{\partial r}{\partial z} = \frac{df}{dr} \cdot \frac{z}{r}$$

Substituting in $\nabla f(r)$

$$\therefore \nabla f(r) = \bar{i}\frac{df}{dr} \cdot \frac{x}{r} + \bar{j}\frac{df}{dr} \cdot \frac{y}{r} + \bar{k}\frac{df}{dr} \cdot \frac{z}{r}$$

$$= \frac{1}{r}\frac{df}{dr}\left[x\bar{i} + y\bar{j} + z\bar{k}\right]$$

$$= \frac{df}{dr} \frac{\bar{r}}{r}$$

$$\nabla f(r) = f'(r) \cdot \frac{\bar{r}}{r} \qquad \because \frac{df}{dr} = f'(r)$$

Result

Ex: 4. Prove that $\boxed{\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)}$

Sol^n: We have, $\nabla^2 f(r) = \nabla \cdot \nabla f(r)$

$$= \nabla \cdot (\nabla f(r))$$

$$= \nabla \cdot \Big(\underbrace{\frac{f'(r)}{r}}_{\phi} \underbrace{\bar{r}}_{\bar{u}} \Big)$$

$$\nabla \cdot (\phi \bar{u}) = (\nabla \phi) \cdot \bar{u} + \phi (\nabla \cdot \bar{u})$$

$$\therefore \nabla^2 f(r) = \left(\nabla \frac{f'(r)}{r} \right) \cdot \bar{r} + \frac{f'(r)}{r} (\nabla \cdot \bar{r})$$

$$= \left[\frac{\left(\frac{d}{dr} \frac{f'(r)}{r} \right) \bar{r}}{r} \right] \cdot \bar{r} + \frac{f'(r)}{r} (3)$$

$$= \frac{1}{r} \left[\left(\frac{r f''(r) - f'(r) \cdot 1}{r^2} \right) \bar{r} \right] \cdot \bar{r} + \frac{3 f'(r)}{r}$$

$$= \frac{1}{r^3} [r f''(r) - f'(r)] (\bar{r} \cdot \bar{r}) + 3 \frac{f'(r)}{r}$$

$$= \frac{1}{r^{\cancel{3}}} [r f''(r) - f'(r)] \cancel{r^2} + 3 \frac{f'(r)}{r}$$

$$= \frac{\cancel{r} f''(r)}{\cancel{r}} - \frac{f'(r)}{r} + 3 \frac{f'(r)}{r}$$

$$= \frac{\cancel{r} f''(r)}{\cancel{r}} + \frac{2 f'(r)}{r}$$

$$\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$$

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Ex:5. Prove that: $\nabla\left(\frac{\bar{a}\cdot\bar{r}}{r^n}\right) = \nabla\left(\frac{1}{r^n}(\bar{a}\cdot\bar{r})\right)$

$= \frac{\bar{a}}{r^n} - \frac{n(\bar{a}\cdot\bar{r})\bar{r}}{r^{n+2}}$

(Here $\frac{1}{r^n}$ and $\bar{a}\cdot\bar{r}$ both are scalars)

Soln: Consider $\nabla\left(\frac{\bar{a}\cdot\bar{r}}{r^n}\right) = \nabla\left(\frac{1}{r^n}(\bar{a}\cdot\bar{r})\right)$

Since $\nabla(\phi\psi) = (\nabla\phi)\psi + \psi(\nabla\phi)$

Take $\phi = \frac{1}{r^n}$, $\psi = \bar{a}\cdot\bar{r}$

$$\nabla\left(\frac{\bar{a}\cdot\bar{r}}{r^n}\right) = \left(\nabla\frac{1}{r^n}\right)(\bar{a}\cdot\bar{r}) + \cancel{(\bar{a}\cdot\bar{r})}\left(\cancel{\nabla}\ \frac{1}{r^n}\left(\nabla(\bar{a}\cdot\bar{r})\right)\right)$$

$$= \frac{-n\,r^{-n-1}}{r}\bar{r}\,\underset{(\bar{a}\cdot\bar{r})}{+}\,\frac{1}{r^n}\bar{a}$$

As $\nabla f(r) = \frac{f'(r)}{r}\bar{r}$, $\nabla(\bar{a}\cdot\bar{r}) = \bar{a}$

$$= \frac{-n\cdot}{r^{n+2}}\bar{r}(\bar{a}\cdot\bar{r}) + \frac{1}{r^n}\bar{a}$$

$$= \frac{\bar{a}}{r^n} - \frac{n(\bar{a}\cdot\bar{r})\bar{r}}{r^{n+2}}.$$

Ex. 6. Prove that: $\nabla\left(\frac{\bar{a}\cdot\bar{r}}{r^3}\right) = \frac{\bar{a}}{r^3} - \frac{3(\bar{a}\cdot\bar{r})}{r^5}\,\bar{r}$

Soln: ~~Consider,~~ H.W. Refer Ex. 5

Ex. 7. Prove that: $\nabla\cdot\left[r\,\nabla\left(\frac{1}{r^n}\right)\right] = \frac{n(n-2)}{r^{n+1}}$

Soln:

$$L.H.S = \nabla\cdot\left[r\,\nabla r^{-n}\right]$$

$$= \nabla\cdot\left[\cancel{r}\left(\frac{(-n)\,r^{-n-1}}{\cancel{r}}\right)\bar{r}\right]$$

$$= (-n)\nabla\cdot\left[\underbrace{\left(r^{-n-1}\right)}_{\phi}\underbrace{\bar{r}}_{\bar{u}}\right]$$

$$= (-n)\left[(\nabla\phi)\cdot\bar{u} + \phi(\nabla\cdot\bar{u})\right]$$

$$= (-n)\left[\left(\frac{(-n-1)\,r^{-n-2}}{r}\,\bar{r}\right)\cdot\bar{r} + r^{-n-1}(\nabla\cdot\bar{r})\right]$$

$$= (-n)\left[\frac{(-n-1)}{r^{n+2+1}}(\bar{r}\cdot\bar{r}) + r^{-n-1}(3)\right]$$

$$= (-n)\left[\frac{-(n+1)}{r^{n+3}}\cdot r^2 + r^{-n-1}(3)\right]$$

$$\therefore L.H.S = -n\left[(-n-1)\, r^{-n-3} \cdot r^{2} + 3 r^{-n-1}\right]$$

$$= r^{-n-1}\left[(-n)(-n-1)\cancel{r^{-1}} - 3n\right]$$

$$= r^{-n-1}\left[n^{2} + n - 3n\right]$$

$$= r^{-n-1}\left[n^{2} - 2n\right]$$

$$= \frac{n^{2} - 2n}{r^{n+1}}$$

$$= \frac{n(n-2)}{r^{n+1}}.$$

Ex. 8 Prove that $\nabla^4 e^r = e^r + \frac{4}{r} e^r$

<u>Soln</u>: $\nabla^4 e^r = \nabla^2 (\nabla^2 e^r)$

Since, $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$

Here $f(r) = e^r \Rightarrow f'(r) = e^r \Rightarrow f''(r) = e^r$

Hence,

$$\nabla^4 e^r = \nabla^2 \left(e^r + \frac{2}{r} e^r \right)$$

$$= \nabla^2 \left(\left(1 + \frac{2}{r}\right) e^r \right) \quad \text{——— (1)}$$

$$\nabla^4 e^r = \frac{d^2}{dr^2}\left(\left(1+\frac{2}{r}\right)e^r\right) + \frac{2}{r}\frac{d}{dr}\left(\left(1+\frac{2}{r}\right)e^r\right) \quad \text{——— (1)}$$

Now, $\frac{d}{dr}\left(\left(1+\frac{2}{r}\right)e^r\right) = \left(1+\frac{2}{r}\right)e^r + e^r\left(-\frac{2}{r^2}\right)$

$$\therefore \frac{d}{dr}\left(\left(1+\frac{2}{r}\right)e^{r}\right) = \left(1+\frac{2}{r}-\frac{2}{r^2}\right)e^{r}$$

$$\therefore \frac{d^2}{dr^2}\left(\left(1+\frac{2}{r}\right)e^{r}\right) = \left(1+\frac{2}{r}-\frac{2}{r^2}\right)e^{r} + e^{r}\left(-\frac{2}{r^2}+\frac{4}{r^3}\right)$$

$$= \left(1+\frac{2}{r}-\frac{4}{r^2}+\frac{4}{r^3}\right)e^{r}$$

Substituting in (1)

$$\nabla^4 e^{r} = \left(1+\frac{2}{r}-\frac{4}{r^2}+\frac{4}{r^3}\right)e^{r} + \frac{2}{r}\left[\left(1+\frac{2}{r}-\frac{2}{r^2}\right)e^{r}\right]$$

$$= \left(1+\frac{2}{r}-\cancel{\frac{4}{r^2}}+\cancel{\frac{4}{r^3}}+\frac{2}{r}+\cancel{\frac{4}{r^2}}-\cancel{\frac{4}{r^3}}\right)e^{r}$$

$$= \left(1+\frac{4}{r}\right)e^{r}$$

$$= e^{r}+\frac{4}{r}e^{r}$$

Ex: 9 Prove that: $\nabla^4(\cancel{\nabla^2}\, r^2\log r) = \dfrac{6}{r^2}$

Soln: $\nabla^4(r^2\log r) = \nabla^2\nabla^2\left[(r^2\log r)\right]$ (1)

Here $f(r) = r^2\log r$

$$f'(r) = r^2\cdot\frac{1}{r} + \log r\cdot(2r) = r + 2r\log r$$

$$f''(r) = 1 + 2\left(r\cdot\frac{1}{r} + \log r\cdot 1\right)$$

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$\therefore f''(r) = 3 + 2\log r$

Hence, $\nabla^2 (r^2 \log r) = (3 + 2\log r) + \frac{2}{r}\left[(1 + 2\log r) r\right]$

$\nabla^2 (r^2 \log r) = 5 + 6\log r$ ——— (2)

$\therefore$ From eq^n (1)

$\nabla^4 r^2 \log r = \nabla^2 (5 + 6\log r)$

$= \frac{d^2}{dr^2}(5 + 6\log r) + \frac{2}{r}\frac{d}{dr}(5 + 6\log r)$

$= -6/r^2 + \frac{2}{r}(6/r)$

$= -6/r^2 + 12/r^2$

$\boxed{\nabla^4 r^2 \log r = 6/r^2}$

Ex: 10. Prove that: $\nabla^4 r^4 = 120$

Soln: $\nabla^4 r^4 = \nabla^2 (\nabla^2 r^4)$ ——— (1)

Since $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$

Here. $f(r) = r^4 \Rightarrow f'(r) = 4r^3 \Rightarrow f''(r) = 12r^2$

$\therefore \nabla^2 f(r) = \nabla^2 r^4 = 12r^2 + \frac{2}{r} \times 4r^3$

$= 12r^2 + 8r^2$

$= 20r^2$

Substitute in ①

$$\therefore \nabla^4 r^4 = \nabla^2(\nabla^2 r^4)$$

$$= \nabla^2(20 r^2)$$

$$= \frac{d^2}{dr^2}(20r^2) + \frac{2}{r}\frac{d}{dr}(20r^2)$$

$$= 40 + \frac{2}{\cancel{r}}(40\cdot\cancel{r})$$

$$= 40+80$$

$$\nabla^4 r^4 = 120$$

Ex: 11 Prove that: $\nabla\cdot\left[r\nabla\frac{1}{r^5}\right] = \frac{15}{r^6}$

Soln: $L.H.S = \nabla\cdot\left[r\nabla\left(\frac{1}{r^5}\right)\right]$

$$= \nabla\left[r\left(-5r^{-5-2}\bar{r}\right)\right]$$

$$\left(\because \nabla r^{-n} = (-n)\, r^{-n-2}\cdot\bar{r}\right)$$

$$= \nabla\left(-5r^{-6}\bar{r}\right)$$

$$= -5\,\nabla\left(\underbrace{\frac{1}{r^6}}_{\phi}\,\underbrace{\bar{r}}_{\bar{a}}\right)$$

$$= -5\left[\left(\nabla\frac{1}{r^6}\right)\cdot\bar{r} + \frac{1}{r^6}(\nabla\cdot\bar{r})\right]$$

$$= -5\left[\left((-6)\,r^{-6-2}\bar{r}\right)\cdot\bar{r} + \frac{1}{r^6}\cdot 3\right]$$

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$$= -5\left[-6\,r^{-6-2}(\bar{r}\cdot\bar{r}) + \frac{1}{r^6}(3)\right]$$

$$= -5\left(-6\,r^{-6} + 3/r^6\right)$$

$$= -5\left(-\frac{6}{r^6} + 3/r^6\right) = -5 \times 3/r^6$$

$$\boxed{\nabla\cdot\left(r\nabla\frac{1}{r^5}\right) = 15/r^6}$$