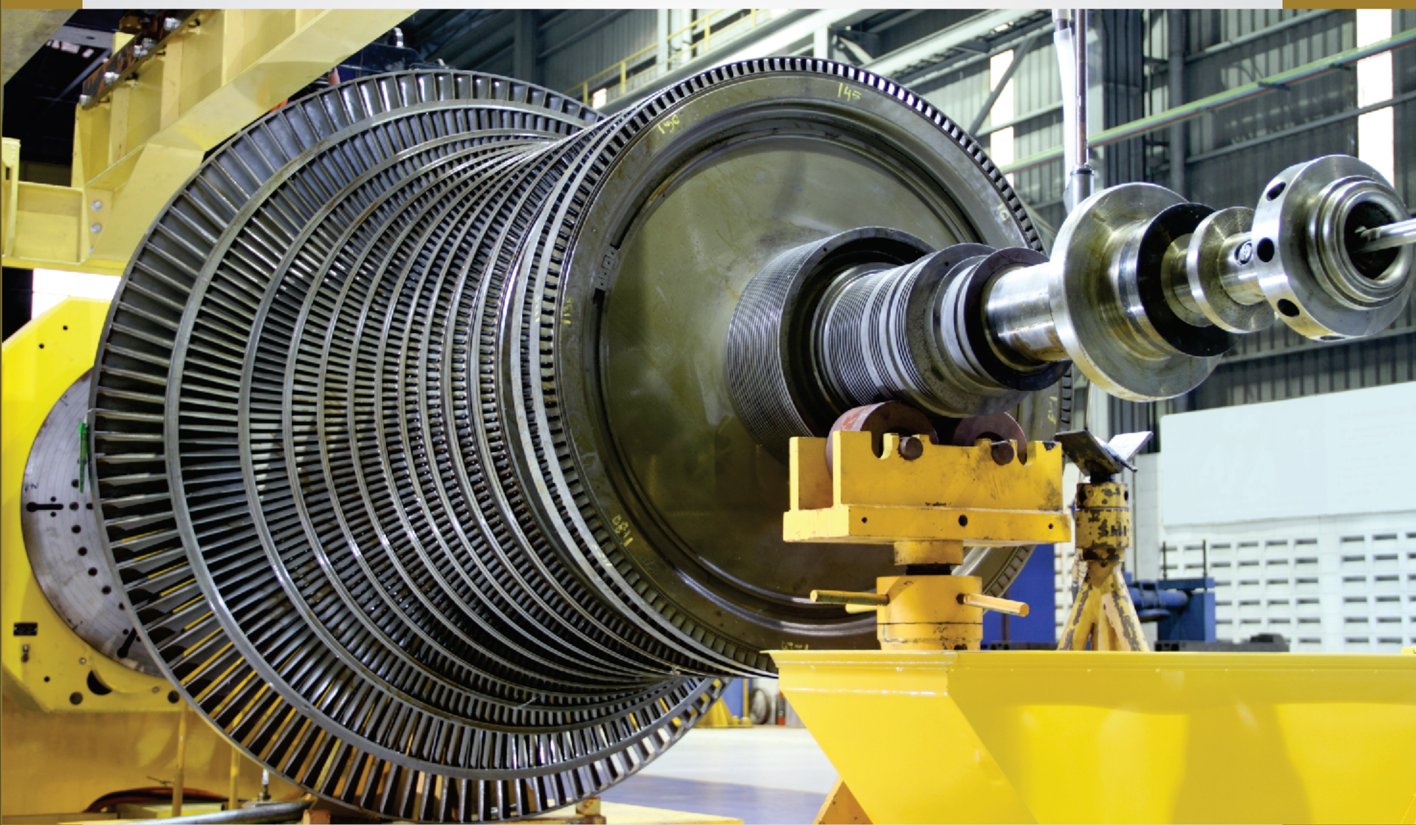
arihant

Understanding Physics

JEE Main & Advanced

MECHANICS

Volume 2



DC PANDEY

Understanding Physics

JEE Main & Advanced

MECHANICS

Volume 2

Understanding Physics

JEE Main & Advanced

MECHANICS

Volume 2

DC PANDEY
[B.Tech, M.Tech, Pantnagar, ID 15722]

arihant
ARIHANT PRAKASHAN (Series), MEERUT

arihant

ARIHANT PRAKASHAN (Series), MEERUT

卐 **ADMINISTRATIVE & PRODUCTION OFFICES**

Regd. Office
'Ramchhaya' 4577/15, Agarwal Road, Darya Ganj, New Delhi -110002
Tele: 011- 47630600, 43518550; Fax: 011- 23280316

Head Office
Kalindi, TP Nagar, Meerut (UP) - 250002
Tel: 0121-2401479, 2512970, 4004199; Fax: 0121-2401648

卐 **SALES & SUPPORT OFFICES**

Agra, Ahmedabad, Bengaluru, Bareilly, Chennai, Delhi, Guwahati,
Hyderabad, Jaipur, Jhansi, Kolkata, Lucknow, Meerut, Nagpur & Pune.

卐 **ISBN** 978-93-13190-56-1

卐

Published by
ARIHANT PUBLICATIONS (I) LTD.

For further information about the books published by Arihant, log on to
www.arihantbooks.com or e-mail at info@arihantbooks.com

/arihantpub /@arihantpub Arihant Publications /arihantpub

PREFACE

The overwhelming response to the previous editions of this book gives me an immense feeling of satisfaction and I take this an opportunity to thank all the teachers and the whole student community who have found this book really beneficial.

In the present scenario of ever-changing syllabus and the test pattern of JEE Main & Advanced.
The NEW EDITION of this book is an effort to cater all the difficulties being faced by the students during their preparation of JEE Main & Advanced. Almost all types and levels of questions are included in this book. My aim is to present the students a fully comprehensive textbook which will help and guide them for all types of examinations. An attempt has been made to remove all the printing errors that had crept in the previous editions.

I am very thankful to (Dr.) Mrs. Sarita Pandey, Mr. Anoop Dhyani and Mr. Nisar Ahmad

Comments and criticism from readers will be highly appreciated and incorporated in the subsequent editions.

DC Pandey

CONTENTS

SYLLABUS

JEE Main

Centre of Mass

Centre of mass of a two particle system, Centre of mass of a rigid body.

Rotational Motion

Basic concepts of rotational motion; Moment of a force, Torque, Angular momentum, conservation of angular momentum and its applications; Moment of inertia, Radius of gyration. Values of moments of inertia for simple geometrical objects, Parallel and perpendicular axes theorems and their applications. Rigid body rotation, Equations of rotational motion.

Gravitation

The universal law of gravitation. Acceleration due to gravity and its variation with altitude and depth. Kepler's laws of planetary motion. Gravitational potential energy; Gravitational potential, Escape velocity. Orbital, velocity of a satellite, Geo-stationary satellites.

Properties of Solids and Liquids

Elastic behaviour, Stress-strain relationship, Hooke's Law, Young's modulus, Bulk modulus, modulus of rigidity. Pressure due to a fluid column; Pascal's law and its applications. Viscosity, Stokes' law, Terminal velocity, Streamline and turbulent flow, Reynolds number. Bernoulli's principle and its applications. Surface energy and surface tension, Angle of contact, Application of surface tension – drops, bubbles and capillary rise.

Oscillations

Periodic motion – period, frequency, displacement as a function of time. Periodic functions. Simple harmonic motion (SHM) and its equation; Phase; Oscillations of a spring -restoring force and force constant; energy in SHM – Kinetic and potential energies; Simple pendulum – derivation of expression for its time period; Free, forced and damped oscillations, resonance.

JEE Advanced

General
Determination of g using simple pendulum. Young's modulus by Searle's method.

Centre of Mass and Collision
System of particles, Centre of mass and its motion, Impulse, Elastic and inelastic collisions.

Gravitation
Law of gravitation, Gravitational potential and field, Acceleration due to gravity, Motion of planets and satellites in circular orbits, Escape velocity.

Rotational Motion
Rigid body, moment of inertia, Parallel and perpendicular axes theorems, Moment of inertia of uniform bodies with simple geometrical shapes, Angular momentum, Torque, Conservation of angular momentum, Dynamics of rigid bodies with fixed axis of rotation, Rolling without slipping of rings, cylinders and spheres, Equilibrium of rigid bodies, Collision of point masses with rigid bodies.

Oscillations
Linear and angular simple harmonic motions.

Properties of Solids and Liquids
Hooke's law, Young's modulus. Pressure in a fluid, Pascal's law, Buoyancy, Surface energy and surface tension, capillary rise, Viscosity (Poiseuille's equation excluded), Stoke's law, Terminal velocity, Streamline flow, Equation of continuity, Bernoulli's theorem and its applications.

This book is dedicated to my honourable grandfather

(Late) Sh. Pitamber Pandey

a Kumaoni poet and a resident of Village
Dhaura (Almora), Uttarakhand

11

Centre of Mass Linear Momentum and Collision

Chapter Contents

11.1 Centre of Mass

When we consider the motion of a system of particles, there is one point in it which behaves as though the entire mass of the system (i.e. the sum of the masses of all the individual particles) is concentrated there and its motion is the same as would ensue if the resultant of all the forces acting on all the particles were applied directly to it. This point is called the centre of mass (COM) of the system. The concept of COM is very useful in solving many problems, in particular, those concerned with collision of particles.

Position of Centre of Mass

First of all we find the position of COM of a system of particles. Just to make the subject easy we classify a system of particles in three groups:

1. System of two particles.
2. System of a large number of particles and
3. Continuous bodies.

Now, let us take them separately.

Position of COM of Two Particles

Centre of mass of two particles of mass m_1 and m_2 separated by a distance of d lies in between the two particles. The distance of centre of mass from any of the particle (r) is inversely proportional to the mass of the particle (m).

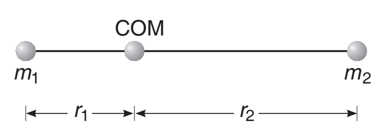


Fig. 11.1

i.e. $$r \propto \frac{1}{m}$$

or $$\frac{r_1}{r_2} = \frac{m_2}{m_1}$$

or $$m_1 r_1 = m_2 r_2$$

or $$\boxed{r_1 = \left(\frac{m_2}{m_2 + m_1}\right) d \quad \text{and} \quad r_2 = \left(\frac{m_1}{m_1 + m_2}\right) d}$$

Here, $r_1 =$ distance of COM from m_1

and $r_2 =$ distance of COM from m_2

From the above discussion, we see that

$r_1 = r_2 = \dfrac{d}{2}$ if $m_1 = m_2$, i.e. COM lies midway between the two particles of equal masses.

Similarly, $r_1 > r_2$ if $m_1 < m_2$ and $r_1 < r_2$ if $m_1 > m_2$, i.e. COM is nearer to the particle having larger mass.

Example 11.1 *Two particles of masses 1 kg and 2 kg are located at $x = 0$ and $x = 3$ m. Find the position of their centre of mass.*

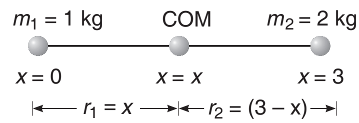


Fig. 11.2

Solution Since, both the particles lie on x-axis, the COM will also lie on x-axis. Let the COM is located at $x = x$, then

r_1 = distance of COM from the particle of mass 1 kg $= x$

and r_2 = distance of COM from the particle of mass 2 kg

$$= (3 - x)$$

Using $$\frac{r_1}{r_2} = \frac{m_2}{m_1}$$

or $$\frac{x}{3-x} = \frac{2}{1} \quad \text{or} \quad x = 2 \text{ m}$$

Thus, the COM of the two particles is located at $x = 2$ m.

Position of COM of a Large Number of Particles

If we have a system consisting of n particles, of mass $m_1, m_2, \ldots, m_n$ with $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$ as their position vectors at a given instant of time. The position vector $\mathbf{r}_{\text{COM}}$ of the COM of the system at that instant is given by:

$$\mathbf{r}_{\text{COM}} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2 + \ldots + m_n \mathbf{r}_n}{m_1 + m_2 + \ldots + m_n}$$

$$= \frac{\sum_{i=1}^{n} m_i \mathbf{r}_i}{\sum_{i=1}^{n} m_i} \quad \text{or} \quad \mathbf{r}_{\text{COM}} = \frac{\sum_{i=1}^{n} m_i \mathbf{r}_i}{M}$$

Here, $M = m_1 + m_2 + \ldots + m_n$ and $\Sigma m_i \mathbf{r}_i$ is called the first moment of the mass.

Further, $$\mathbf{r}_i = x_i \hat{\mathbf{i}} + y_i \hat{\mathbf{j}} + z_i \hat{\mathbf{k}}$$

and $$\mathbf{r}_{\text{COM}} = x_{\text{COM}} \hat{\mathbf{i}} + y_{\text{COM}} \hat{\mathbf{j}} + z_{\text{COM}} \hat{\mathbf{k}}$$

So, the cartesian co-ordinates of the COM will be

$$x_{\text{COM}} = \frac{m_1 x_1 + m_2 x_2 + \ldots + m_n x_n}{m_1 + m_2 + \ldots + m_n}$$

$$= \frac{\sum_{i=1}^{n} m_i x_i}{\Sigma m_i} \quad \text{or} \quad x_{\text{COM}} = \frac{\sum_{i=1}^{n} m_i x_i}{M}$$

Similarly, $$y_{\text{COM}} = \frac{\sum_{i=1}^{n} m_i y_i}{M}$$

and $$z_{\text{COM}} = \frac{\sum_{i=1}^{n} m_i z_i}{M}$$

Example 11.2 *The position vector of three particles of masses $m_1 = 1\ kg$, $m_2 = 2\ kg$ and $m_3 = 3\ kg$ are $\mathbf{r}_1 = (\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + \hat{\mathbf{k}})\ m$, $\mathbf{r}_2 = (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})\ m$ and $\mathbf{r}_3 = (2\hat{\mathbf{i}} - \hat{\mathbf{j}} - 2\hat{\mathbf{k}})\ m$ respectively. Find the position vector of their centre of mass.*

Solution The position vector of COM of the three particles is given by

$$\mathbf{r}_{COM} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3}{m_1 + m_2 + m_3}$$

Substituting the values, we get

$$\mathbf{r}_{COM} = \frac{(1)(\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + \hat{\mathbf{k}}) + (2)(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) + 3(2\hat{\mathbf{i}} - \hat{\mathbf{j}} - 2\hat{\mathbf{k}})}{1 + 2 + 3}$$

$$= \frac{9\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 3\hat{\mathbf{k}}}{6}$$

$$\mathbf{r}_{COM} = \frac{1}{2}(3\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}})\ \text{m}$$

Example 11.3 *Four particles of masses 1 kg, 2 kg, 3 kg and 4 kg are placed at the four vertices A, B, C and D of a square of side 1 m. Find the position of centre of mass of the particles.*

Solution Assuming D as the origin, DC as x-axis and DA as y-axis, we have

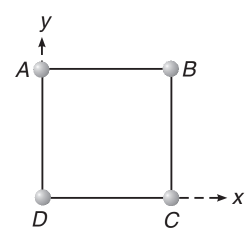


Fig. 11.3

$$m_1 = 1\ \text{kg}, (x_1, y_1) = (0, 1\ \text{m})$$
$$m_2 = 2\ \text{kg}, (x_2, y_2) = (1\ \text{m}, 1\ \text{m})$$
$$m_3 = 3\ \text{kg}, (x_3, y_3) = (1\ \text{m}, 0)$$

and
$$m_4 = 4\ \text{kg}, (x_4, y_4) = (0, 0)$$

Coordinates of their COM are

$$x_{COM} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4}{m_1 + m_2 + m_3 + m_4}$$

$$= \frac{(1)(0) + 2(1) + 3(1) + 4(0)}{1 + 2 + 3 + 4}$$

$$= \frac{5}{10} = \frac{1}{2}\ \text{m} = 0.5\ \text{m}$$

Similarly,

$$y_{COM} = \frac{m_1y_1 + m_2y_2 + m_3y_3 + m_4y_4}{m_1 + m_2 + m_3 + m_4}$$

$$= \frac{(1)(1) + 2(1) + 3(0) + 4(0)}{1 + 2 + 3 + 4}$$

$$= \frac{3}{10}\ \text{m} = 0.3\ \text{m}$$

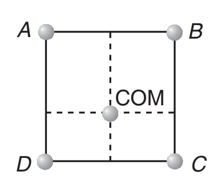


Fig. 11.4

$\therefore \quad (x_{COM}, y_{COM}) = (0.5\ \text{m}, 0.3\ \text{m})$

Thus, position of COM of the four particles is as shown in Fig. 11.4.

Position of COM of Continuous Bodies

If we consider the body to have continuous distribution of matter the summation in the formula of COM is replaced by integration. Suppose x, y and z are the co-ordinates of a small element of mass dm, we write the co-ordinates of COM as

$$x_{COM} = \frac{\int x\,dm}{\int dm} = \frac{\int x\,dm}{M}$$

$$y_{COM} = \frac{\int y\,dm}{\int dm} = \frac{\int y\,dm}{M}$$

and

$$z_{COM} = \frac{\int z\,dm}{\int dm} = \frac{\int z\,dm}{M}$$

Here, dm is the mass of small element and (x, y, z) co-ordinates of COM of this element.

Note *In most of the cases element is a particle. In this case, COM of this particle lies over the particle itself.*

Let us take an example.

Centre of Mass of a Uniform Rod

Suppose a rod of mass M and length L is lying along the x-axis with its one end at $x = 0$ and the other at $x = L$.

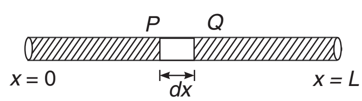


Fig. 11.5

Mass per unit length of the rod $= \frac{M}{L}$

Hence, the mass of the element PQ of length dx situated at $x = x$ is $dm = \frac{M}{L}\,dx$

The coordinates of the element PQ are $(x, 0, 0)$. Therefore, x-coordinate of COM of the rod will be

$$x_{COM} = \frac{\int_0^L x\,dm}{\int dm} = \frac{\int_0^L (x)\left(\frac{M}{L}\,dx\right)}{M} = \frac{1}{L}\int_0^L x\,dx = \frac{L}{2}$$

The y-coordinate of COM is

$$y_{COM} = \frac{\int y\,dm}{\int dm} = 0 \qquad \text{(as } y = 0\text{)}$$

Similarly, $z_{COM} = 0$

i.e. the coordinates of COM of the rod are $\left(\frac{L}{2}, 0, 0\right)$. Or it lies at the centre of the rod.

Proceeding in the similar manner, we can find the COM of certain rigid bodies. Centre of mass of some well known rigid bodies are given below :

1. Centre of mass of a uniform rectangular, square or circular plate lies at its centre.

Fig. 11.6

2. Centre of mass of a uniform **semicircular ring** lies at a distance of $h = \frac{2R}{\pi}$ from its centre, on the axis of symmetry where R is the radius of the ring.

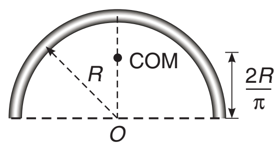


Fig. 11.7

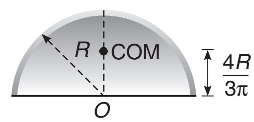


Fig. 11.8

3. Centre of mass of a uniform **semicircular disc** of radius R lies at a distance of $h = \frac{4R}{3\pi}$ from the centre on the axis of symmetry as shown in Fig. 11.8.

4. Centre of mass of a **hemispherical shell** of radius R lies at a distance of $h = \frac{R}{2}$ from its centre on the axis of symmetry as shown in figure 11.9.

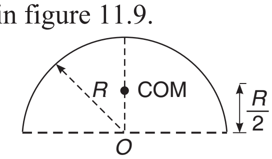


Fig. 11.9

5. Centre of mass of a **solid hemisphere** of radius R lies at a distance of $h = \frac{3R}{8}$ from its centre on the axis of symmetry.

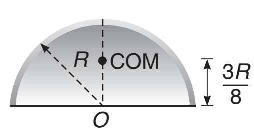


Fig. 11.10

Example 11.4 *A rod of length L is placed along the x-axis between $x = 0$ and $x = L$. The linear mass density (mass/length) ρ of the rod varies with the distance x from the origin as $\rho = a + bx$. Here, a and b are constants. Find the position of centre of mass of this rod.*

Solution Mass of element PQ of length dx situated at $x = x$ is

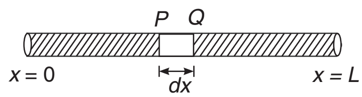


Fig. 11.11

$$dm = \rho\, dx = (a + bx)\, dx$$

The COM of the element has co-ordinates $(x, 0, 0)$. Therefore, x-coordinate of COM of the rod will be

$$x_{\text{COM}} = \frac{\int_0^L x\, dm}{\int_0^L dm} = \frac{\int_0^L (x)(a + bx)\, dx}{\int_0^L (a + bx)\, dx}$$

$$= \frac{\left[\frac{ax^2}{2} + \frac{bx^3}{3}\right]_0^L}{\left[ax + \frac{bx^2}{2}\right]_0^L} \quad \text{or} \quad x_{\text{COM}} = \frac{3aL + 2bL^2}{6a + 3bL}$$

The y-coordinate of COM of the rod is

$$y_{\text{COM}} = \frac{\int y\, dm}{\int dm} = 0 \qquad (\text{as } y = 0)$$

Similarly, $z_{\text{COM}} = 0$

Hence, the centre of mass of the rod lies at $\left[\frac{3aL + 2bL^2}{6a + 3bL}, 0, 0\right]$ **Ans.**

Extra Points to Remember

- For a laminar type (2-dimensional) body the formulae for finding the position of centre of mass are as under:

  (i) $$\mathbf{r}_{\text{COM}} = \frac{A_1\mathbf{r}_1 + A_2\mathbf{r}_2 + \ldots + A_n\mathbf{r}_n}{A_1 + A_2 + .. + A_n}$$

  (ii) $$x_{\text{COM}} = \frac{A_1x_1 + A_2x_2 + \ldots + A_nx_n}{A_1 + A_2 + \ldots + A_n}$$

  $$y_{\text{COM}} = \frac{A_1y_1 + A_2y_2 + \ldots + A_ny_n}{A_1 + A_2 + \ldots + A_n}$$

  and $$z_{\text{COM}} = \frac{A_1z_1 + A_2z_2 + \ldots + A_nz_n}{A_1 + A_2 + \ldots + A_n}$$

  Here, A stands for the area.

- If some mass or area is removed from a rigid body, then the position of centre of mass of the remaining portion is obtained from the following formulae:

  (i) $$\mathbf{r}_{\text{COM}} = \frac{m_1\mathbf{r}_1 - m_2\mathbf{r}_2}{m_1 - m_2} \quad \text{or} \quad \mathbf{r}_{\text{COM}} = \frac{A_1\mathbf{r}_1 - A_2\mathbf{r}_2}{A_1 - A_2}$$

(ii) $$x_{COM} = \frac{m_1x_1 - m_2x_2}{m_1 - m_2} \quad \text{or} \quad x_{COM} = \frac{A_1x_1 - A_2x_2}{A_1 - A_2}$$

$$y_{COM} = \frac{m_1y_1 - m_2y_2}{m_1 - m_2} \quad \text{or} \quad y_{COM} = \frac{A_1y_1 - A_2y_2}{A_1 - A_2}$$

and $$z_{COM} = \frac{m_1z_1 - m_2z_2}{m_1 - m_2} \quad \text{or} \quad z_{COM} = \frac{A_1z_1 - A_2z_2}{A_1 - A_2}$$

Here, m_1, A_1,$\mathbf{r}_1$ x_1, y_1 and z_1 are the values for the whole mass while m_2, A_2,$\mathbf{r}_2$, x_2, y_2 and z_2 are the values for the mass which has been removed.

Example 11.5 *Find the position of centre of mass of the uniform lamina shown in figure.*

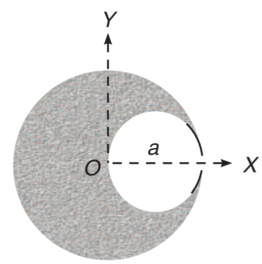


Fig. 11.12

Solution Here,

$$A_1 = \text{area of complete circle} = \pi a^2$$

$$A_2 = \text{area of small circle}$$

$$= \pi\left(\frac{a}{2}\right)^2 = \frac{\pi a^2}{4}$$

(x_1, y_1) = coordinates of centre of mass of large circle = $(0, 0)$

(x_2, y_2) = coordinates of centre of mass of small circle $= \left(\frac{a}{2}, 0\right)$

Using $$x_{COM} = \frac{A_1x_1 - A_2x_2}{A_1 - A_2}$$

we get $$x_{COM} = \frac{(\pi a^2)(0) - \left(\frac{\pi a^2}{4}\right)\left(\frac{a}{2}\right)}{\pi a^2 - \frac{\pi a^2}{4}}$$

$$= \frac{-\left(\frac{1}{8}\right)}{\left(\frac{3}{4}\right)} a = -\frac{a}{6}$$

and $y_{COM} = 0$ as y_1 and y_2 both are zero.

Therefore, coordinates of COM of the lamina shown in figure are $\left(-\frac{a}{6}, 0\right)$.

INTRODUCTORY EXERCISE 11.1

1. What is the difference between centre of mass and centre of gravity?
2. The centre of mass of a rigid body always lies inside the body. Is this statement true or false ?
3. The centre of mass always lies on the axis of symmetry if it exists. Is this statement true or false?
4. If all the particles of a system lie in y-z plane, the x-coordinate of the centre of mass will be zero. Is this statement true or false?
5. What can be said about the centre of mass of a solid hemisphere of radius r without making any calculation. Will its distance from the centre be more than $r/2$ or less than $r/2$?
6. All the particles of a body are situated at a distance R from the origin. The distance of the centre of mass of the body from the origin is also R. Is this statement true or false?
7. Three particles of masses 1 kg, 2 kg and 3 kg are placed at the corners A, B and C respectively of an equilateral triangle ABC of edge 1 m. Find the distance of their centre of mass from A.
8. Find the distance of centre of mass of a uniform plate having semicircular inner and outer boundaries of radii a and b from the centre O.

   Hint : Distance of COM of semicircular plate from centre is $\frac{4r}{3\pi}$.

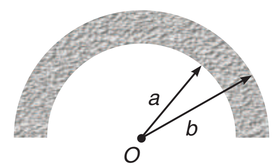


Fig. 11.13

9. Find the position of centre of mass of the section shown in figure 11.14.

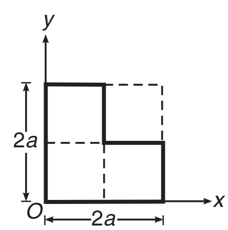


Fig. 11.14

Note *Solve the problem by using both the formulae:*

$$\text{(i) } x_{COM} = \frac{A_1x_1 + A_2x_2}{A_1 + A_2} \quad \text{and} \quad \text{(ii) } x_{COM} = \frac{A_1x_1 - A_2x_2}{A_1 - A_2}$$

10. Four particles of masses 1 kg, 2 kg, 3 kg and 4 kg are placed at the four vertices A, B, C and D of a square of side 1 m. Find square of distance of their centre of mass from A.
11. A square lamina of side a and a circular lamina of diameter a are placed touching each other as shown in Fig. 11.15. Find distance of their centre of mass from point O, the centre of square.

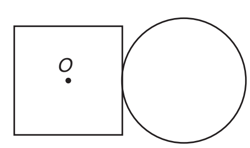


Fig. 11.15

12. The density of a thin rod of length l varies with the distance x from one end as $\rho = \rho_0 \frac{x^2}{l^2}$. Find the position of centre of mass of rod.
13. A straight rod of length L has one of its end at the origin and the other at $x = L$. If the mass per unit length of the rod is given by Ax where A is a constant, where is its centre of mass?

11.2 Motion of the Centre of Mass

Let us consider the motion of a system of n particles of individual masses $m_1, m_2, \ldots, m_n$ and total mass M. It is assumed that no mass enters or leaves the system during its motion, so that M remains constant. Then, as we have seen in the above article, we have the relation

$$\mathbf{r}_{\text{COM}} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \ldots + m_n\mathbf{r}_n}{m_1 + m_2 + \ldots + m_n}$$

$$= \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \ldots + m_n\mathbf{r}_n}{M}$$

or $$M\mathbf{r}_{\text{COM}} = m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \ldots + m_n\mathbf{r}_n$$

Differentiating this expression with respect to time t, we have

$$M\frac{d\mathbf{r}_{\text{COM}}}{dt} = m_1\frac{d\mathbf{r}_1}{dt} + m_2\frac{d\mathbf{r}_2}{dt} + \ldots + m_n\frac{d\mathbf{r}_n}{dt}$$

Since, $$\frac{d\mathbf{r}}{dt} = \text{velocity}$$

Therefore, $$M\mathbf{v}_{\text{COM}} = m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + \ldots + m_n\mathbf{v}_n \qquad \ldots\text{(i)}$$

or velocity of the COM is

$$\mathbf{v}_{\text{COM}} = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + \ldots + m_n\mathbf{v}_n}{M}$$

or $$\boxed{\mathbf{v}_{\text{COM}} = \frac{\sum_{i=1}^{n} m_i\mathbf{v}_i}{M}}$$

Further, $m\mathbf{v}$ = momentum of a particle $\mathbf{p}$. Therefore, Eq. (i) can be written as

$$\mathbf{p}_{\text{COM}} = \mathbf{p}_1 + \mathbf{p}_1 + \ldots + \mathbf{p}_n$$

or $$\boxed{\mathbf{p}_{\text{COM}} = \sum_{i=1}^{n} \mathbf{p}_i}$$

Differentiating Eq. (i) with respect to time t, we get

$$M\frac{d\mathbf{v}_{\text{COM}}}{dt} = m_1\frac{d\mathbf{v}_1}{dt} + m_2\frac{d\mathbf{v}_2}{dt} + \ldots + m_n\frac{d\mathbf{v}_n}{dt}$$

or $$M\mathbf{a}_{\text{COM}} = m_1\mathbf{a}_1 + m_2\mathbf{a}_2 + \ldots + m_n\mathbf{a}_n \qquad \ldots\text{(ii)}$$

or $$\mathbf{a}_{\text{COM}} = \frac{m_1\mathbf{a}_1 + m_2\mathbf{a}_2 + \ldots + m_n\mathbf{a}_n}{M}$$

or $$\boxed{\mathbf{a}_{\text{COM}} = \frac{\sum_{i=1}^{n} m_i\mathbf{a}_i}{M}}$$

Further, in accordance with Newton's second law of motion $\mathbf{F} = m\mathbf{a}$. Hence, Eq. (ii) can be written as

$$\mathbf{F}_{\text{COM}} = \mathbf{F}_1 + \mathbf{F}_2 + \ldots + \mathbf{F}_n$$

or

$$\boxed{\mathbf{F}_{\text{COM}} = \sum_{i=1}^{n} \mathbf{F}_i}$$

Thus, as pointed out earlier also, the centre of mass of a system of particles moves as though it were a particle of mass equal to that of the whole system with all the external forces acting directly on it.

Extra Points to Remember

- Students are often confused over the problems of centre of mass. They cannot answer even the basic problems of COM. For example, let us take a simple problem: two particles one of mass 1 kg and the other of 2 kg are projected simultaneously with the same speed from the roof of a tower, the one of mass 1 kg vertically upwards and the other vertically downwards. What is the acceleration of centre of mass of these two particles? When I ask this question in my first class of centre of mass, three answers normally come from the students g, $\frac{g}{3}$ and zero. The correct answer is g. Because

$$\mathbf{a}_{\text{COM}} = \frac{m_1\mathbf{a}_1 + m_2\mathbf{a}_2}{m_1 + m_2}$$

Here, $\mathbf{a}_1 = \mathbf{a}_2 = g$ (downwards)

$\therefore$ $\mathbf{a}_{\text{COM}} = \dfrac{(1)(g) + (2)(g)}{1+2} = g$ (downwards)

The idea behind this is that apply the basic equations when asked anything about centre of mass. Just as a revision I am writing below all the basic equations of COM at one place.

$$\mathbf{r}_{\text{COM}} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \ldots + m_n\mathbf{r}_n}{m_1 + m_2 + \ldots + m_n}$$

$$x_{\text{COM}} = \frac{m_1x_1 + m_2x_2 + \ldots + m_nx_n}{m_1 + m_2 + \ldots + m_n}$$

$$y_{\text{COM}} = \frac{m_1y_1 + m_2y_2 + \ldots + m_ny_n}{m_1 + m_2 + \ldots + m_n}$$

$$z_{\text{COM}} = \frac{m_1z_1 + m_2z_2 + \ldots + m_nz_n}{m_1 + m_2 + \ldots + m_n}$$

$$\mathbf{v}_{\text{COM}} = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + \ldots + m_n\mathbf{v}_n}{m_1 + m_2 + \ldots + m_n}$$

$$\mathbf{P}_{\text{COM}} = \mathbf{P}_1 + \mathbf{P}_2 + \ldots + \mathbf{P}_n$$

$$\mathbf{a}_{\text{COM}} = \frac{m_1\mathbf{a}_1 + m_2\mathbf{a}_2 + \ldots + m_n\mathbf{a}_n}{m_1 + m_2 + \ldots + m_n}$$

and $\mathbf{F}_{\text{COM}} = \mathbf{F}_1 + \mathbf{F}_2 + \ldots + \mathbf{F}_n$

Example 11.6 *Two particles A and B of masses 1 kg and 2 kg respectively are projected in the directions shown in figure with speeds $u_A = 200$ m/s and $u_B = 50$ m/s. Initially they were 90 m apart. They collide in mid air and stick with each other. Find the maximum height attained by the centre of mass of the particles. Assume acceleration due to gravity to be constant. ($g = 10$ m/s^2)*

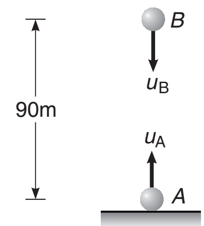


Fig. 11.16

Solution Using $m_A r_A = m_B r_B$

or $$(1)(r_A) = (2)(r_B)$$

or $$r_A = 2r_B \qquad \text{...(i)}$$

and $$r_A + r_B = 90 \text{ m} \qquad \text{...(ii)}$$

Solving these two equations, we get

$$r_A = 60 \text{ m} \quad \text{and} \quad r_B = 30 \text{ m}$$

i.e. COM is at height 60 m from the ground at time $t = 0$.

Further, $$\mathbf{a}_{\text{COM}} = \frac{m_A \mathbf{a}_A + m_B \mathbf{a}_B}{m_A + m_B}$$

$$= g = 10 \text{ m/s}^2 \qquad \text{(downwards)}$$

as $$\mathbf{a}_A = \mathbf{a}_g = g \qquad \text{(downwards)}$$

$$\mathbf{u}_{\text{COM}} = \frac{m_A \mathbf{u}_A + m_B \mathbf{u}_B}{m_A + m_B}$$

$$= \frac{(1)(200) - (2)(50)}{1 + 2} = \frac{100}{3} \text{ m/s} \qquad \text{(upwards)}$$

Let, h be the height attained by COM beyond 60 m. Using,

$$v_{\text{COM}}^2 = u_{\text{COM}}^2 + 2a_{\text{COM}} h$$

or $$0 = \left(\frac{100}{3}\right)^2 - (2)(10)h$$

or $$h = \frac{(100)^2}{180} = 55.55 \text{ m}$$

Therefore, maximum height attained by the centre of mass is

$$H = 60 + 55.55 = 115.55 \text{ m}$$ **Ans.**

Example 11.7 *In the arrangement shown in figure, $m_A = 2$ kg and $m_B = 1$ kg. String is light and inextensible. Find the acceleration of centre of mass of both the blocks. Neglect friction everywhere.*

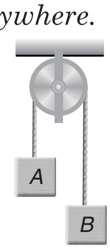


Fig. 11.17

Solution Net pulling force on the system is $(m_A - m_B)g$

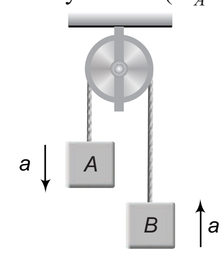


Fig. 11.18

or $$(2-1)g = g$$

Total mass being pulled is $m_A + m_B$ or 3 kg

$\therefore$ $$a = \frac{\text{Net pulling force}}{\text{Total mass}} = \frac{g}{3}$$

Now, $$\mathbf{a}_{\text{COM}} = \frac{m_A \mathbf{a}_A + m_B \mathbf{a}_B}{m_A + m_B} = \frac{(2)(a)-(1)(a)}{1+2} = \frac{a}{3} = \frac{g}{9} \quad \text{(downwards)}$$

Alternate Method

Free body diagram of block A is shown in Fig. 11.19.

$$2g - T = m_A(a)$$

or $$T = 2g - m_A a$$

$$= 2g - (2)\left(\frac{g}{3}\right) = \frac{4g}{3}$$

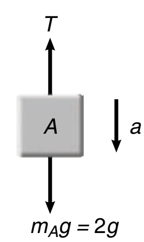


Fig. 11.19

Free body diagrams of A and B both are as shown in Fig. 11.20.

$$\mathbf{a}_{\text{COM}} = \frac{\text{Net force on both the blocks}}{m_A + m_B}$$

$$= \frac{(m_A + m_B)g - 2T}{2+1}$$

$$= \frac{3g - \frac{8g}{3}}{3} = \frac{g}{9} \quad \text{(downwards)}$$

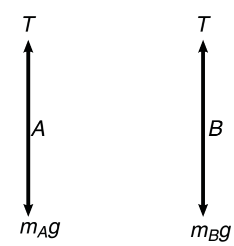


Fig. 11.20

Example 11.8 *Two blocks A and B of equal masses are released on two sides of a fixed wedge C as shown in figure. Find the acceleration of centre of mass of blocks A and B. Neglect friction.*

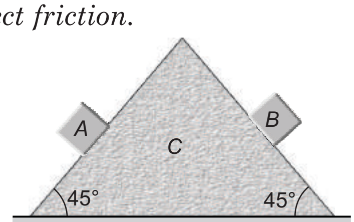


Fig. 11.21

Solution Acceleration of both the blocks will be $g \sin 45°$ or $\frac{g}{\sqrt{2}}$ at right angles to each other. Now,

$$\mathbf{a}_{COM} = \frac{m_A \mathbf{a}_A + m_B \mathbf{a}_B}{m_A + m_B}$$

Here, $m_A = m_B$

$\therefore$ $\mathbf{a}_{COM} = \frac{1}{2}(\mathbf{a}_A + \mathbf{a}_B) = \frac{1}{2} g$ (downwards)

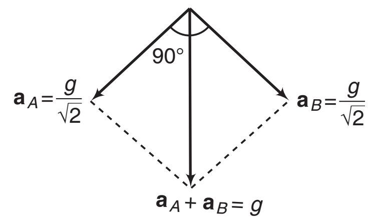


Fig. 11.22

INTRODUCTORY EXERCISE 11.2

1. A block of mass 1 kg is at $x = 10$ m and moving towards negative x-axis with velocity 6 m/s. Another block of mass 2 kg is at $x = 12$ m and moving towards positive x-axis with velocity 4 m/s at the same instant. Find position of their centre of mass after 2 s.
2. Two particles of masses 1 kg and 2 kg respectively are initially 10 m apart. At time $t = 0$, they start moving towards each other with uniform speeds 2 m/s and 1 m/s respectively. Find the displacement of their centre of mass at $t = 1$s.
3. There are two masses m_1 and m_2 placed at a distance l apart. Let the centre of mass of this system is at a point named C. If m_1 is displaced by l_1 towards C and m_2 is displaced by l_2 away from C. Find the distance, from C where new centre of mass will be located.
4. At one instant, the centre of mass of a system of two particles is located on the x-axis at $x = 3.0$ m and has a velocity of $(6.0 \text{ m/s})\hat{\mathbf{j}}$. One of the particles is at the origin, the other particle has a mass of 0.10 kg and is at rest on the x-axis at $x = 12.0$ m.
   (a) What is the mass of the particle at the origin ?
   (b) Calculate the total momentum of this system.
   (c) What is the velocity of the particle at the origin ?
5. A stone is dropped at $t = 0$. A second stone, with twice the mass of the first, is dropped from the same point at $t = 100$ ms.
   (a) How far below the release point is the centre of mass of the two stones at $t = 300$ ms ? (Neither stone has yet reached the ground).
   (b) How fast is the centre of mass of the two-stone system moving at that time ?
6. Two blocks A and B of equal masses are attached to a string passing over a smooth pulley fixed to a wedge as shown in figure. Find the magnitude of acceleration of centre of mass of the two blocks when they are released from rest. Neglect friction.

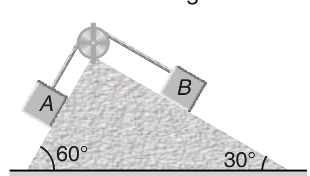


Fig. 11.23

11.3 Law of Conservation of Linear Momentum

The product of mass and the velocity of a particle is defined as its linear momentum ($\mathbf{p}$). So,

$$\boxed{\mathbf{p}=m\mathbf{v}}$$

The magnitude of linear momentum may be written as

$$p=mv \quad \text{or} \quad p^2=m^2v^2=2m\left(\frac{1}{2}mv^2\right)=2mK$$

Thus,
$$\boxed{p=\sqrt{2Km} \quad \text{or} \quad K=\frac{p^2}{2m}}$$

Here, K is the kinetic energy of the particle. In accordance with Newton's second law,

$$\mathbf{F}=m\mathbf{a}=m\frac{d\mathbf{v}}{dt}=\frac{d(m\mathbf{v})}{dt}=\frac{d\mathbf{p}}{dt}$$

Thus,
$$\mathbf{F}=\frac{d\mathbf{p}}{dt}$$

In case the external force applied to a particle (or a body) be zero, we have

$$\mathbf{F}=\frac{d\mathbf{p}}{dt}=0 \quad \text{or} \quad \mathbf{p}=\text{constant}$$

showing that in the absence of an external force, the linear momentum of a particle (or the body) remains constant. This is called the law of conservation of linear momentum. The law may be extended to a system of particles or to the centre of mass of a system of particles. For example, for a system of particles it takes the form :

If net force (or the vector sum of all the forces) on a system of particles is zero, the vector sum of linear momentum of all the particles remain conserved, or

If
$$\mathbf{F}=\mathbf{F}_1+\mathbf{F}_2+\mathbf{F}_3+\ldots+\mathbf{F}_n=0$$

Then,
$$\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3+\ldots+\mathbf{p}_n=\text{constant}$$

The same is the case for the centre of mass of a system of particles, i.e. if

$$\mathbf{F}_{\text{COM}}=0,\ \mathbf{p}_{\text{COM}}=\text{constant}.$$

Thus, the law of conservation of linear momentum can be applied to a single particle, to a system of particles or even to the centre of mass of the particles.

The law of conservation of linear momentum enables us to solve a number of problems which can not be solved by a straight application of the relation $\mathbf{F}=m\mathbf{a}$.

For example, suppose a particle of mass m initially at rest, suddenly explodes into two fragments of masses m_1 and m_2 which fly apart with velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively. Obviously, the forces resulting in the explosion of the particle must be internal forces, since no external force has been applied. In the absence of the external forces, therefore, the momentum must remain conserved and we should have

$$m\mathbf{v}_i=m_1\mathbf{v}_1+m_2\mathbf{v}_2$$

Since, the particle was initially at rest, $\mathbf{v}_i = 0$ and therefore,

$$m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2 = 0$$

$$\mathbf{v}_1 = -\frac{m_2}{m_1}\mathbf{v}_2 \quad \text{or} \quad \frac{|\mathbf{v}_1|}{|\mathbf{v}_2|} = \frac{m_2}{m_1}$$

Showing at once that the velocities of the two fragments must be inversely proportional to their masses and in opposite directions along the same line. This result could not possibly be arrived at from the relation $\mathbf{F} = m\mathbf{a}$, since we know nothing about the forces that were acting during the explosion. Nor, could we derive it from the law of conservation of energy.

Extra Points to Remember

- If law of conservation of linear momentum is applied to a single particle, then we can explain it like this.
  If net force or net external force (as there is no internal force on a single particle) acting on the particle is zero then its linear momentum remains constant. If mass of the particle is constant, then velocity of the particle is also constant. If the particle is at rest then it will remain at rest for ever. If it is moving then it will continue to be moving with constant velocity.
- If law of conservation of momentum is applied to a system of particles (may be a rigid body also) then the law is like this :
  It net force or net external force (as summation of net internal forces acting on a system of particles is already zero) on a system of particles (or on the centre of mass of the system of particles) is zero, then total momentum of the system of particles (or momentum of centre of mass of the system of particles) remains constant. If total mass of the system of particles is constant then velocity of centre of mass will also remain constant.
- If a projectile explodes in air in different parts, the path of the centre of mass remains unchanged. This is because during explosion no external force (except gravity) acts on the centre of mass. The situation is as shown in figure.

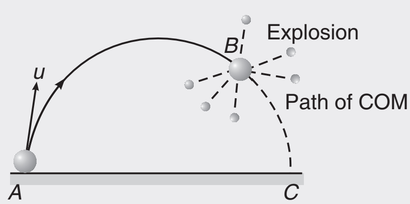


Fig. 11.24

Path of COM is parabola, even though the different parts travel in different directions after explosion.
This situation continues till the first particle strikes the ground. Because, after that force behaviour of system of particles will change.

Example 11.9 *Linear momentum of particle is increased by*
(a) 100 % *(b)* 1 %
without changing its mass. Find percentage increase in its kinetic energy in both cases.

Solution (a) Relation between kinetic energy K and momentum p is given by:

$$K = \frac{p^2}{2m} \qquad \text{......(i)}$$

Now, momentum is increased by 100%.
So, new momentum, $p' = 2p$

$\therefore \qquad K' = \dfrac{(p')^2}{2m} = \dfrac{(2p)^2}{2m} = 4\left[\dfrac{p^2}{2m}\right] = 4K$ [From Eq. (i)]

Now percentage change in kinetic energy,

$$= \left(\frac{\text{Final value} - \text{Initial value}}{\text{Initial value}}\right) \times 100$$

$$= \left(\frac{K' - K}{K}\right) \times 100 = \left(\frac{4K - K}{K}\right) \times 100$$

$$= +300\%$$ **Ans.**

Plus sign indicates that, with increase in linear momentum, kinetic energy will also increase.

(b) $K = \dfrac{p^2}{2m}$ or $K \propto p^2$ (as $m =$ constant)

Here, power of K is 1 and power of p is 2. For small changes, we can write it like this

(1) (% change in K) $= (2)$ (% change in p)

or % change in $K = (2)(1\%) = +2\%$ **Ans.**

Example 11.10 *Kinetic energy of a particle is increased by*

(a) 50 % *(b)* 1 %

Find percentage change in linear momentum.

Solution (a) $p = \sqrt{2Km}$ ...(i)

Kinetic energy is increased by 50 %. So, the new value of kinetic energy is

$$K' = 1.5K$$

$\therefore \qquad P' = \sqrt{2K'm}$

$$= \sqrt{2(1.5K)m} = \sqrt{1.5}\,(\sqrt{2Km})$$

$$= 1.22\sqrt{2Km} = 1.22\,p$$ [From Eq. (i)]

So, the percentage change in momentum is

$$= \left(\frac{p' - p}{p}\right) \times 100 = \left(\frac{1.22\,p - p}{p}\right) \times 100 = +22\%$$ **Ans.**

(b) $p = \sqrt{2Km}$ or $p \propto \sqrt{K}$ or $p \propto K^{\frac{1}{2}}$ (as $m =$ constant)

Here, power of p is 1 and the power of K is $\dfrac{1}{2}$.

For small percentage changes we can write as

$$(1)\ (\%\ \text{change in}\ P) = \left(\frac{1}{2}\right)(\%\ \text{change in}\ K)$$

or % change in $p = \dfrac{1}{2}(1\%) = +0.5\%$ **Ans.**

Example 11.11 *Two blocks A and B of masses 1 kg and 2 kg are connected together by means of a spring and are resting on a horizontal frictionless table. The blocks are then pulled apart so as to stretch the spring and then released. Find the ratio of their,*

(a) speed
(b) magnitude of momentum and
(c) kinetic energy at any instant.

Solution (a) Net force on the system of two blocks is zero. Therefore, linear momentum of the system will remain constant. Initially, they are at rest. So, at any instant the net momentum will be zero.

$$\therefore \quad \mathbf{p}_1 + \mathbf{p}_2 = 0$$

$$\Rightarrow \quad \mathbf{p}_1 = -\mathbf{p}_2 \qquad \ldots\text{(i)}$$

$$\text{or} \quad m_1 \mathbf{v}_1 = -m_2 \mathbf{v}_2$$

$$\text{or} \quad (1)\mathbf{v}_1 = -2\mathbf{v}_2$$

$$\text{or} \quad \frac{|\mathbf{v}_1|}{|\mathbf{v}_2|} = 2 \qquad \textbf{Ans.}$$

(b) From Eq. (i), $\dfrac{|\mathbf{p}_1|}{|\mathbf{p}_2|} = 1$

(c) $K = \dfrac{p^2}{2m}$ or $K \propto \dfrac{p^2}{m}$

$$\therefore \quad \frac{K_1}{K_2} = \left(\frac{p_1}{p_2}\right)^2 \left(\frac{m_2}{m_1}\right) = (1)^2 \left(\frac{2}{1}\right) = 2 \qquad \textbf{Ans.}$$

Example 11.12 *A gun (mass = M) fires a bullet (mass = m) with speed v_r relative to barrel of the gun which is inclined at an angle of 60° with horizontal. The gun is placed over a smooth horizontal surface. Find the recoil speed of gun.*

Solution Let the recoil speed of gun is v. Taking gun + bullet as the system. Net external force on the system in horizontal direction is zero. Initially the system was at rest. Therefore, applying the principle of conservation of linear momentum in horizontal direction, we get

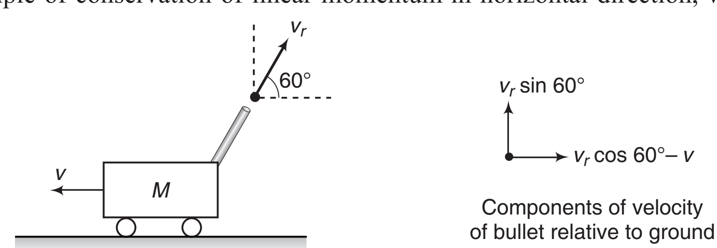


Fig. 11.25

$$Mv - m(v_r \cos 60° - v) = 0$$

$$\therefore \quad v = \frac{mv_r \cos 60°}{M + m} \quad \text{or} \quad v = \frac{mv_r}{2(M + m)} \qquad \textbf{Ans.}$$

Example 11.13 *A projectile of mass 3 m is projected from ground with velocity $20\sqrt{2}$ m/s at 45°. At highest point it explodes into two pieces. One of mass 2 m and the other of mass m. Both the pieces fly off horizontally in opposite directions. Mass 2 m falls at a distance of 100 m from point of projection. Find the distance of second mass from point of projection where it strikes the ground. ($g = 10\ m/s^2$)*

Solution Range of the projectile in the absence of explosion

$$R = \frac{u^2 \sin 2\theta}{g} = \frac{(20\sqrt{2})^2 \sin 90^\circ}{10} = 80\ \text{m}$$

The path of centre of mass of projectile will not change, i.e. x_{COM} is still 80 m. Now, from the definition of centre of mass

$$x_{COM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

or

$$80 = \frac{(m)(x_1) + (2m)(100)}{m + 2m}$$

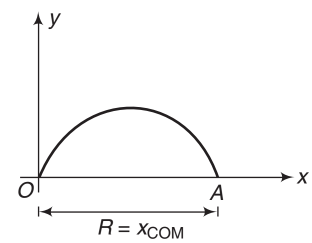


Fig. 11.26

Solving this equation, we get $x_1 = 40$ m

Therefore, the mass m will fall at a distance $x_1 = 40$ cm from point of projection. **Ans.**

INTRODUCTORY EXERCISE 11.3

1. Three particles of masses 20 g, 30 g and 40 g are initially moving along the positive direction of the three coordinate axes respectively with the same velocity of 20 cm/s. When due to their mutual interaction, the first particle comes to rest, the second acquires a velocity $(10\hat{\mathbf{i}} + 20\hat{\mathbf{k}})$ cm/s. What is then the velocity of the third particle?
2. A boy of mass 25 kg stands on a board of mass 10 kg which in turn is kept on a frictionless horizontal ice surface. The boy makes a jump with a velocity component 5 m/s in a horizontal direction with respect to the ice. With what velocity does the board recoil? With what rate are the boy and board separating from each other?
3. Find the ratio of the linear momenta of two particles of masses 1.0 kg and 4.0 kg if their kinetic energies are equal.
4. A uranium-238 nucleus, initially at rest, emits an alpha particle with a speed of 1.4×10^7 m/s. Calculate the recoil speed of the residual nucleus thorium-234. Assume that the mass of a nucleus is proportional to the mass number.
5. A man of mass 50 kg starts moving on the earth and acquires a speed of 1.8 m/s. With what speed does the earth recoil? Mass of earth $= 6 \times 10^{24}$ kg.
6. A man of mass 60 kg jumps from a trolley of mass 20 kg standing on smooth surface with absolute velocity 3 m/s. Find velocity of trolley and total energy produced by man.
7. A projectile is fired from a gun at an angle of 45° with the horizontal and with a speed of 20 m/s relative to ground. At the highest point in its flight the projectile explodes into two fragments of equal masses. One fragment, whose initial speed is zero falls vertically. How far from the gun does the other fragment land, assuming a level terrain? Take $g = 10$ m/s^2?

11.4 Variable Mass

In our discussion of the conservation of linear momentum, we have so far dealt with systems whose mass remains constant. We now consider those systems whose mass is variable, i.e. those in which mass enters or leaves the system. A typical case is that of the rocket from which hot gases keep on escaping, thereby continuously decreasing its mass.

In such problems you have nothing to do but apply a thrust force ($\mathbf{F}_t$) to the main mass in addition to the all other forces acting on it. This thrust force is given by,

$$\mathbf{F}_t = \mathbf{v}_{\text{rel}}\left(\pm\frac{dm}{dt}\right)$$

Here, $\mathbf{v}_{\text{rel}}$ is the velocity of the mass gained or mass ejected relative to the main mass. In case of rocket this is sometimes called the exhaust velocity of the gases. $\frac{dm}{dt}$ is the rate at which mass is increasing or decreasing.

The expression for the thrust force can be derived from the conservation of linear momentum in the absence of any external forces on a system as follows :

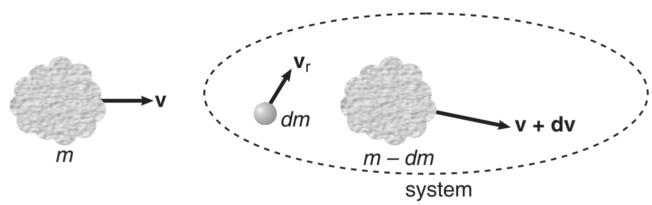


Fig. 11.27

Suppose at some moment mass of a body is m and its velocity is $\mathbf{v}$. After some time interval dt its mass becomes $(m - dm)$ and velocity becomes $\mathbf{v} + d\mathbf{v}$. The mass dm is ejected with relative velocity $\mathbf{v}_r$. Absolute velocity of mass 'dm' is therefore $(\mathbf{v}_r + \mathbf{v} + d\mathbf{v})$. If no external forces are acting on the system, the linear momentum of the system will remain conserved, or

$$\mathbf{p}_i = \mathbf{p}_f$$

or $$m\mathbf{v} = (m - dm)(\mathbf{v} + d\mathbf{v}) + dm(\mathbf{v}_r + \mathbf{v} + d\mathbf{v})$$

or $$m\mathbf{v} = m\mathbf{v} + md\mathbf{v} - dm\mathbf{v} - (dm)(d\mathbf{v}) + dm\mathbf{v} + \mathbf{v}_r dm + (dm)(d\mathbf{v})$$

$\therefore$ $$md\mathbf{v} = -\mathbf{v}_r dm \quad \text{or} \quad m\left(\frac{d\mathbf{v}}{dt}\right) = \mathbf{v}_r\left(-\frac{dm}{dt}\right)$$

Here, $$m\left(\frac{d\mathbf{v}}{dt}\right) = \text{thrust force } (\mathbf{F}_1)$$

and $$-\frac{dm}{dt} = \text{rate at which mass is ejecting}$$

Problems Related to Variable Mass can be Solved in Following Three Steps

1. Make a list of all the forces acting on the main mass and apply these forces on it.
2. Apply an additional thrust force $\mathbf{F}_t$ on the mass, the magnitude of which is $\left|\mathbf{v}_r\left(\pm\frac{dm}{dt}\right)\right|$ and direction is given by the direction of $\mathbf{v}_r$ in case the mass is increasing and in the direction of $-\mathbf{v}_r$ if it is decreasing.
3. Find net force on the mass and apply

$$\mathbf{F}_{\text{net}} = m\frac{d\mathbf{v}}{dt} \qquad (m = \text{mass at that particular instant})$$

Rocket Propulsion

Let m_0 be the mass of the rocket at time $t = 0$. m its mass at any time t and v its velocity at that moment. Initially let us assume that the velocity of the rocket is u.

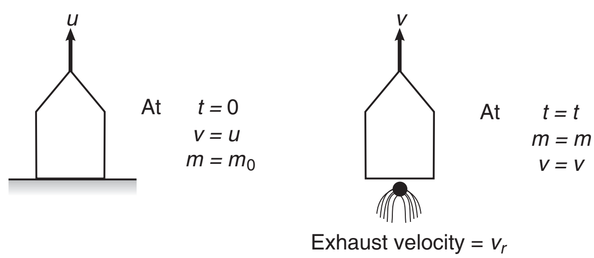


Fig. 11.28

Further, let $\left(\frac{-dm}{dt}\right)$ be the mass of the gas ejected per unit time and v_r the exhaust velocity of the gases. Usually $\left(\frac{-dm}{dt}\right)$ and v_r are kept constant throughout the journey of the rocket. Now, let us write few equations which can be used in the problems of rocket propulsion. At time $t = t$,

1. Thrust force on the rocket

$$F_t = v_r\left(-\frac{dm}{dt}\right) \qquad \text{(upwards)}$$

2. Weight of the rocket

$$w = mg \qquad \text{(downwards)}$$

3. Net force on the rocket

$$F_{\text{net}} = F_t - W \qquad \text{(upwards)}$$

or

$$F_{\text{net}} = v_r\left(\frac{-dm}{dt}\right) - mg$$

4. Net acceleration of the rocket $a = \frac{F}{m}$

or

$$\frac{dv}{dt} = \frac{v_r}{m}\left(\frac{-dm}{dt}\right) - g$$

or $$dv = v_r\left(\frac{-dm}{m}\right) - g\,dt$$

or $$\int_u^v dv = v_r \int_{m_0}^{m} \frac{-dm}{m} - g\int_0^t dt$$

or $$v - u = v_r \ln\left(\frac{m_0}{m}\right) - gt$$

Thus, $$\boxed{v = u - gt + v_r \ln\left(\frac{m_0}{m}\right)} \quad \text{...(i)}$$

Note (i) $F_t = v_r\left(-\frac{dm}{dt}\right)$ *is upwards, as* v_r *is downwards and* $\frac{dm}{dt}$ *is negative.*

(ii) If gravity is ignored and initial velocity of the rocket $u = 0$, *Eq. (i) reduces to* $v = v_r \ln\left(\frac{m_0}{m}\right)$.

Example 11.14 *(a) A rocket set for vertical firing weighs 50 kg and contains 450 kg of fuel. It can have a maximum exhaust velocity of 2 km/s. What should be its minimum rate of fuel consumption*

(i) to just lift it off the launching pad?

(ii) to give it an initial acceleration of 20 m/s²?

(b) What will be the speed of the rocket when the rate of consumption of fuel is 10 kg/s after whole of the fuel is consumed? (Take g = 9.8 m/s²)

Solution (a) (i) To just lift it off the launching pad

$$\text{weight} = \text{thrust force}$$

or $$mg = v_r\left(\frac{-dm}{dt}\right)$$

or $$\left(\frac{-dm}{dt}\right) = \frac{mg}{v_r}$$

Substituting the values, we get

$$\left(\frac{-dm}{dt}\right) = \frac{(450 + 50)(9.8)}{2 \times 10^3}$$

$$= 2.45 \text{ kg/s}$$ **Ans.**

(ii) Net acceleration $a = 20 \text{ m/s}^2$

$\therefore$ $$ma = F_t - mg$$

or $$a = \frac{F_t}{m} - g$$

or $$a = \frac{v_r}{m}\left(\frac{-dm}{dt}\right) - g$$

This gives $$\left(-\frac{dm}{dt}\right) = \frac{m(g + a)}{v_r}$$

Substituting the values, we get

$$\left(-\frac{dm}{dt}\right) = \frac{(450+50)(9.8+20)}{2\times 10^3}$$

$$= 7.45 \text{ kg/s}$$ **Ans.**

(b) The rate of fuel consumption is 10 kg/s.

So, the time for the consumption of entire fuel is

$$t = \frac{450}{10} = 45 \text{ s}$$

Using Eq. (i), i.e. $v = u - gt + v_r \ln\left(\frac{m_0}{m}\right)$

Here, $u = 0$, $v_r = 2\times 10^3$ m/s, $m_0 = 500$ kg and $m = 50$ kg

Substituting the values, we get

$$v = 0 - (9.8)(45) + (2\times 10^3)\ln\left(\frac{500}{50}\right)$$

or $$v = -441 + 4605.17$$

or $$v = 4164.17 \text{ m/s}$$

or $$v = 4.164 \text{ km/s}$$ **Ans.**

INTRODUCTORY EXERCISE 11.4

1. A rocket of mass 20 kg has 180 kg fuel. The exhaust velocity of the fuel is 1.6 km/s. Calculate the minimum rate of consumption of fuel so that the rocket may rise from the ground. Also, calculate the ultimate vertical speed gained by the rocket when the rate of consumption of fuel is $(g = 9.8 \text{ m/s}^2)$

   (i) 2 kg/s (ii) 20 kg/s

2. A rocket, with an initial mass of 1000 kg, is launched vertically upwards from rest under gravity. The rocket burns fuel at the rate of 10 kg per second. The burnt matter is ejected vertically downwards with a speed of 2000 ms^{-1} relative to the rocket. If burning ceases after one minute, find the maximum velocity of the rocket. (Take g as constant at 10 ms^{-2})

3. A rocket is moving vertically upward against gravity. Its mass at time t is $m = m_0 - \mu t$ and it expels burnt fuel at a speed u vertically downward relative to the rocket. Derive the equation of motion of the rocket but do not solve it. Here, μ is constant.

4. A rocket of initial mass m_0 has a mass $m_0(1 - t/3)$ at time t. The rocket is launched from rest vertically upwards under gravity and expels burnt fuel at a speed u relative to the rocket vertically downward. Find the speed of rocket at $t = 1$.

11.5 Linear Impulse

Consider a constant force $\mathbf{F}$ which acts for a time t on a body of mass m, thus, changing its velocity from $\mathbf{u}$ to $\mathbf{v}$. Because the force is constant, the body will travel with constant acceleration $\mathbf{a}$ where

$$\mathbf{F} = m\mathbf{a}$$

and $$\mathbf{a}t = \mathbf{v} - \mathbf{u}$$

hence, $$\frac{\mathbf{F}}{m} t = \mathbf{v} - \mathbf{u}$$

or $$\mathbf{F}t = m\mathbf{v} - m\mathbf{u}$$

The product of constant force $\mathbf{F}$ and the time t for which it acts is called the **impulse** ($\mathbf{J}$) of the force and this is equal to the change in linear momentum which it produces.

Thus, $$\boxed{\text{Impulse } (\mathbf{J}) = \mathbf{F}t = \Delta\mathbf{p} = \mathbf{p}_f - \mathbf{p}_i}$$

Instantaneous Impulse There are many occasions when a force acts for such a short time that the effect is instantaneous, e.g. a bat striking a ball. In such cases, although the magnitude of the force and the time for which it acts may each be unknown but the value of their product (*i.e.,* impulse) can be known by measuring the initial and final momenta. Thus, we can write

$$\boxed{\mathbf{J} = \int \mathbf{F}dt = \Delta\mathbf{p} = \mathbf{p}_f - \mathbf{p}_i}$$

Regarding the impulse it is important to note that impulse applied to an object in a given time interval can also be calculated from the area under force-time (F-t) graph in the same time interval.

In one dimensional motion, we can simply write as,

$$\boxed{J = \int F\, dt = \Delta p = p_f - p_i}$$

In this equation all vector quantities (J, F and p) are taken with proper signs.

Further, if mass is constant, then we can write,

$$p_f - p_i \text{ as } m(v_f - v_i)$$

Now depending on the nature of force there are following three cases :

Case 1 If force is constant, then linear impulse can be obtained by multiplying this constant force with the given time interval. Now, this impulse is change in linear momentum.

Case 2 If force is a function of time, then linear impulse can be obtained by integrating this linear function of time in the given time interval. Result of this integration (or linear impulse) is equal to the change in linear momentum.

Case 3 If force *versus* time graph is given, then linear impulse can be obtained by the area under F-t graph. This area (or the linear impulse) is equal to the change in linear momentum.

Note *Normally, the word impulse is used when a large force acts for a short interval of time but the equations discussed above can be used even if the time interval is large.*

Example 11.15 *A ball of mass 200 g is projected with a velocity of 30 m/s at 30° from horizontal. Using the concept of impulse, find change in velocity in 2 s. Take $g = 10\ m/s^2$.*

Solution In air, a constant force (= weight = $m\mathbf{g}$) will act on the ball. Therefore, the linear impulse can be obtained directly by multiplying this constant force with the given time interval. Further, this linear impulse is equal to the change in linear momentum. Thus,

$$\mathbf{J} = \mathbf{F} \times t = \Delta\mathbf{p} = m\Delta\mathbf{v}$$

$$\Rightarrow \qquad \Delta\mathbf{v} = \left(\frac{t}{m}\right)\mathbf{F} = \left(\frac{t}{m}\right)m\mathbf{g}$$

$$= \mathbf{g} \times t = (10\,\text{m/s}^2 \text{ downwards})(2\text{ s})$$

$$= 20\,\text{m/s} \qquad \text{(downwards)}$$ **Ans.**

Example 11.16 *A time varying force, $F = 2t$ is acting on a particle of mass 2 kg moving along x-axis. velocity of the particle is 4 m/s along negative x-axis at time $t = 0$. Find the velocity of the particle at the end of 4 s.*

Solution Given force is a function of time.

So, linear impulse can be obtained by integration. Further, this impulse is equal to change in linear momentum.

Thus, $$J = \int F\,dt = \Delta p = p_f - p_i = m(v_f - v_i)$$

or $$v_f = \frac{1}{m}\int F\,dt + v_i = \frac{1}{2}\int_0^4 (2t)\,dt - 4 \qquad (v_i = -4\text{ m/s})$$

$$= \frac{1}{2}[t^2]_0^4 - 4 = \frac{1}{2}[16 - 0] - 4$$

$$= +4\text{ m/s}$$ **Ans.**

Therefore, the final velocity is 4 m/s along positive x-direction.

Example 11.17 *A particle of mass 2 kg is initially at rest. A force starts acting on it in one direction whose magnitude changes with time. The force time graph is shown in figure. Find the velocity of the particle at the end of 10 s.*

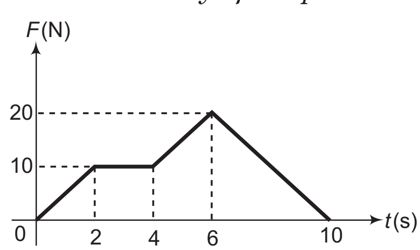


Fig. 11.29

Solution Using impulse = Change in linear momentum (or area under F-t graph)

We have, $$m(v_f - v_i) = \text{Area}$$

or $$2(v_f - 0) = \frac{1}{2} \times 2 \times 10 + 2 \times 10 + \frac{1}{2} \times 2 \times (10 + 20) + \frac{1}{2} \times 4 \times 20$$

$$= 10 + 20 + 30 + 40 \quad \text{or} \quad 2v_f = 100$$

$\therefore$ $$v_f = 50 \text{ m/s}$$ **Ans.**

Example 11.18 *A bullet of mass 10^{-3} kg strikes an obstacle and moves at 60° to its original direction. If its speed also changes from 20 m/s to 10 m/s. Find the magnitude of impulse acting on the bullet.*

Solution Mass of the bullet $m = 10^{-3}$ kg

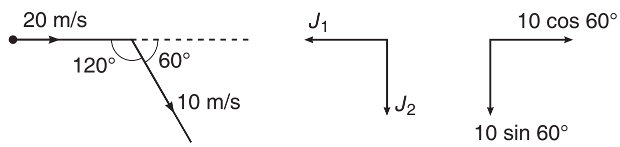


Fig. 11.30

Consider components parallel to J_1.

$$J_1 = 10^{-3}\,[-10\cos 60° - (-20)] \quad \text{or} \quad J_1 = 15 \times 10^{-3} \text{ N-s}$$

Similarly, parallel to J_2, we have

$$J_2 = 10^{-3}[10 \sin 60° - 0] = 5\sqrt{3} \times 10^{-3} \text{ N-s}$$

The magnitude of resultant impulse is given by

$$J = \sqrt{J_1^2 + J_2^2} = 10^{-3}\sqrt{(15)^2 + (5\sqrt{3})^2} \quad \text{or} \quad J = \sqrt{3} \times 10^{-2} \text{ N-s}$$ **Ans.**

INTRODUCTORY EXERCISE 11.5

1. A truck of mass 2×10^3 kg travelling at 4 m/s is brought to rest in 2 s when it strikes a wall. What force (assume constant) is exerted by the wall ?
2. A ball of mass *m*, travelling with velocity $2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$ receives an impulse $-3m\hat{\mathbf{i}}$. What is the velocity of the ball immediately afterwards ?
3. The net force *versus* time graph of a rocket is shown in figure. The mass of the rocket is 1200 kg. Calculate velocity of rocket, 16 seconds after starting from rest. Neglect gravity.

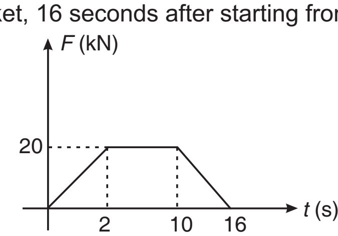


Fig. 11.31

4. A 5.0 g bullet moving at 100 m/s strikes a log. Assume that the bullet undergoes uniform deceleration and stops in 6.0 cm. Find (a) the time taken for the bullet to stop, (b) the impulse on the log and (c) the average force experienced by the log.

11.6 Collision

Contrary to the meaning of the term 'collision' in our everyday life, in physics it does not necessarily mean one particle 'striking' against other. Indeed two particles may not even touch each other and may still be said to collide. All that is implied is that as the particles approach each other,

(i) an impulse (a large force for a relatively short time) acts on each colliding particles.

(ii) the total momentum of the particles remain conserved.

The collision is in fact a redistribution of total momentum of the particles. Thus, law of conservation of linear momentum is indispensable in dealing with the phenomenon of collision between particles. Consider a situation shown in figure.

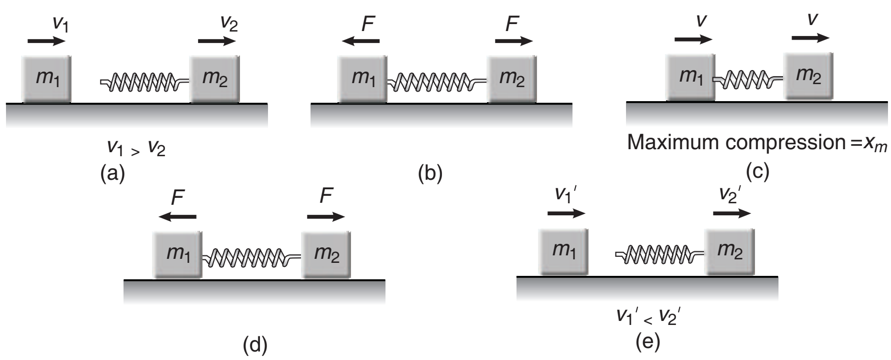


Fig. 11.32

Two blocks of masses m_1 and m_2 are moving with velocities v_1 and $v_2 (< v_1)$ along the same straight line in a smooth horizontal surface. A spring is attached to the block of mass m_2. Now, let us see what happens during the collision between two particles.

Figure (a) Block of mass m_1 is behind m_2. Since, $v_1 > v_2$, the blocks will collide after some time.

Figure (b) The spring is compressed. The spring force F $(= kx)$ acts on the two blocks in the directions shown in figure. This force decreases the velocity of m_1 and increases the velocity of m_2.

Figure (c) The spring will compress till velocity of both the blocks become equal. So, at maximum compression (say x_m) velocities of both the blocks are equal (say v).

Figure (d) Spring force is still in the directions shown in figure, i.e. velocity of block m_1 is further decreased and that of m_2 is increased. The spring now starts relaxing.

Figure (e) The two blocks are separated from one another. Velocity of block m_2 becomes more than the velocity of block m_1, i.e. $v_2' > v_1'$.

Equations Which can be Used in the Above Situation

Assuming spring to be perfectly elastic following two equations can be applied in the above situation.

(i) In the absence of any external force on the system the linear momentum of the system will remain conserved before, during and after collision, i.e.

$$m_1 v_1 + m_2 v_2 = (m_1 + m_2) v = m_1 v_1' + m_2 v_2' \qquad \ldots\text{(i)}$$

(ii) In the absence of any dissipative forces, the mechanical energy of the system will also remain conserved, i.e.

$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} (m_1 + m_2) v^2 + \frac{1}{2} k x_m^2$$

$$= \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2 \quad \ldots\text{(ii)}$$

Note *In the above situation we have assumed, spring to be perfectly elastic, i.e. it regains its original shape and size after the two blocks are separated. In actual practice there is no such spring between the two blocks. During collision, both the blocks (or bodies) are slightly deformed. This situation is similar to the compression of the spring. Due to deformation, two equal and opposite forces act on both the blocks. These two forces redistribute their linear momentum in such a manner that both the blocks are separated from one another. The collision is said to be elastic if both the blocks regain their original shape and size completely after they are separated. On the other hand if the blocks do not return to their original form the collision is said to be inelastic. If the deformation is permanent and the blocks move together with same velocity after the collision, the collision is said to be perfectly inelastic.*

Types of Collision

Collision between two bodies may be classified in two ways:

1. Elastic collision and inelastic collision.
2. Head on collision or oblique collision.

As discussed earlier also collision between two bodies is said to be **elastic** if both the bodies come to their original shape and size after the collision, i.e. no fraction of mechanical energy remains stored as deformation potential energy in the bodies. Thus in addition to the linear momentum, kinetic energy also remains conserved before and after collision. On the other hand, in an **inelastic** collision, the colliding bodies do not return to their original shape and size completely after collision and some part of the mechanical energy of the system goes to the deformation potential energy. Thus, only linear momentum remains conserved in case of an inelastic collision.

Further, a collision is said to be **head on (or direct)** if the directions of the velocity of colliding objects are along the line of action of the impulses, acting at the instant of collision. If just before collision, at least one of the colliding objects was moving in a direction different from the line of action of the impulses, the collision is called **oblique or indirect.**

Head on Elastic Collision

Let the two balls of masses m_1 and m_2 collide each other elastically with velocities v_1 and v_2 in the directions shown in Fig. 11.33(a). Their velocities become v_1' and v_2' after the collision along the same line. Applying conservation of linear momentum,

we get

(a) Before collision (b) After collision

Fig. 11.33

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2' \quad \ldots\text{(iii)}$$

In an elastic collision kinetic energy before and after collision is also conserved. Hence,

$$\frac{1}{2}m_1v_1^2+\frac{1}{2}m_2v_2^2=\frac{1}{2}m_1{v_1}'^2+\frac{1}{2}m_2{v_2}'^2 \quad \text{...(iv)}$$

Solving Eqs. (iii) and (iv) for v_1' and v_2', we get

$$v_1'=\left(\frac{m_1-m_2}{m_1+m_2}\right)v_1+\left(\frac{2m_2}{m_1+m_2}\right)v_2 \quad \text{...(v)}$$

and

$$v_2'=\left(\frac{m_2-m_1}{m_1+m_2}\right)v_2+\left(\frac{2m_1}{m_1+m_2}\right)v_1 \quad \text{...(vi)}$$

Special Cases

1. If $m_1=m_2$, then from Eqs. (v) and (vi), we can see that

$$v_1'=v_2 \text{ and } v_2'=v_1$$

i.e. when two particles of equal mass collide elastically and the collision is head on, they exchange their velocities., e.g.

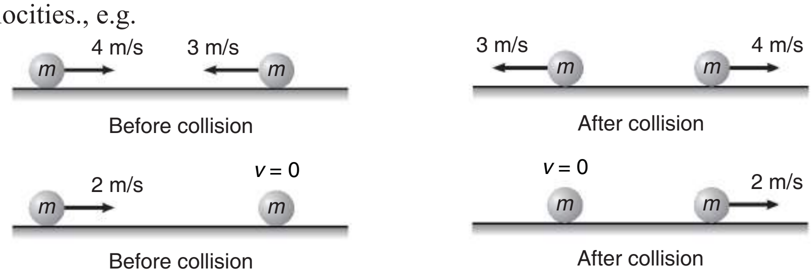


Fig. 11.34

2. If $m_1>>m_2$ and $v_1=0$. Then $\frac{m_2}{m_1}\approx 0$ with these two substitutions $\left(v_1=0 \text{ and } \frac{m_2}{m_1}=0\right)$

Fig. 11.35

we get the following two results

$$v_1'\approx 0 \quad \text{and} \quad v_2'\approx -v_2$$

i.e. the particle of mass m_1 remains at rest while the particle of mass m_2 bounces back with same speed v_2.

3. If $m_2>>m_1$ and $v_1=0$,

Fig. 11.36

with the substitution $\frac{m_1}{m_2} \approx 0$ and $v_1 = 0$, we get the results

$$v_1' \approx 2v_2 \quad \text{and} \quad v_2' \approx v_2$$

i.e. the mass m_1 moves with velocity $2v_2$ while the velocity of mass m_2 remains unchanged.

Note *It is important to note that Eqs. (v) and (vi) and their three special cases can be used only in case of a head on elastic collision between two particles. I have found that many students apply these two equations even if the collision is inelastic and do not apply these relations where clearly a head on elastic collision is given in the problem.*

Head on Inelastic Collision

As we have discussed earlier also, in an inelastic collision, the particles do not regain their shape and size completely after collision. Some fraction of mechanical energy is retained by the colliding particles in the form of deformation potential energy. Thus, the kinetic energy of the particles no longer remains conserved. However, in the absence of external forces, law of conservation of linear momentum still holds good.

Fig. 11.37

Suppose the velocities of two particles of mass m_1 and m_2 before collision be v_1 and v_2 in the directions shown in figure. Let v_1' and v_2' be their velocities after collision. The law of conservation of linear momentum gives

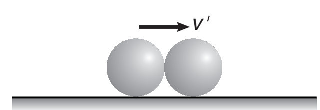


Fig. 11.38

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2' \qquad \text{...(vii)}$$

Collision is said to be **perfectly inelastic** if both the particles stick together after collision and move with same velocity, say v' as shown in Fig. 11.38. In this case, Eq. (vii) can be written as

$$m_1 v_1 + m_2 v_2 = (m_1 + m_2) v'$$

or

$$\boxed{v' = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}} \qquad \text{...(viii)}$$

Newton's Law of Restitution

When two objects are in direct (head on) impact, the speed with which they separate after impact is usually less than or equal to their speed of approach before impact.

Experimental evidence suggests that the ratio of these relative speeds is constant for two given set of objects. This property formulated by Newton, is known as the law of restitution and can be written in the form

$$\boxed{\frac{\text{separation speed}}{\text{approach speed}} = e} \qquad \text{...(ix)}$$

The ratio e is called the coefficient of restitution and is constant for two particular objects.

In general $\boxed{0 \le e \le 1}$

$e = 0$, for completely inelastic collision, as both the objects stick together. So, their separation speed is zero or $e = 0$ from Eq. (ix).

$e = 1$, for an elastic collision, as we can show from Eq. (v) and (vi), that

$$v_1' - v_2' = v_2 - v_1$$

or separation speed = approach speed

or $e = 1$

Fig. 11.39

Let us now find the velocities of two particles after collision if they collide directly and the coefficient of restitution between them is given as e.

Applying conservation of linear momentum

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2' \quad \text{...(x)}$$

Further, separation speed = e (approach speed)

or

$$v_1' - v_2' = e(v_2 - v_1) \quad \text{...(xi)}$$

Solving Eqs. (x) and (xi), we get

$$\boxed{v_1' = \left(\frac{m_1 - em_2}{m_1 + m_2}\right) v_1 + \left(\frac{m_2 + em_2}{m_1 + m_2}\right) v_2} \quad \text{...(xii)}$$

and

$$\boxed{v_2' = \left(\frac{m_2 - em_1}{m_1 + m_2}\right) v_2 + \left(\frac{m_1 + em_1}{m_1 + m_2}\right) v_1} \quad \text{...(xiii)}$$

Special Cases

1. If collision is elastic, i.e. $e = 1$, then

$$v_1' = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_1 + \left(\frac{2m_2}{m_1 + m_2}\right) v_2$$

and

$$v_2' = \left(\frac{m_2 - m_1}{m_1 + m_2}\right) v_2 + \left(\frac{2m_1}{m_1 + m_2}\right) v_1$$

which are same as Eqs. (v) and (vi).

2. If collision is perfectly inelastic, i.e. $e = 0$, then

$$v_1' = v_2' = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} = v' \text{ (say)}$$

which is same as Eq. (viii).

3. If $m_1 = m_2$ and $v_1 = 0$, then

$$v_1' = \left(\frac{1+e}{2}\right) v_2 \quad \text{and} \quad v_2' = \left(\frac{1-e}{2}\right) v_2 \qquad \ldots\text{(xiv)}$$

Fig. 11.40

Note *(i) If mass of one body is very-very greater than that of the other, then after collision velocity of heavy body does not change appreciably. (Whether the collision is elastic or inelastic).*

(ii) In the situation shown in figure if e is the coefficient of restitution between the ball and the ground, then after n^{th} collision with the floor the speed of ball will remain $e^n v_0$ and it will go upto a height $e^{2n} h$ or,

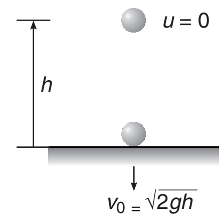


Fig. 11.41

$$v_n = e^n v_0 = e^n \sqrt{2gh}$$

and

$$h_n = e^{2n} h$$

EXERCISE *Derive the above two relations.*

Oblique Collision

During collision between two objects a pair of equal and opposite impulses act at the moment of impact. If just before impact at least one of the objects was moving in a direction different from the line of action of these impulses the collision is said to be oblique.

In the figure, two balls collide obliquely. During collision impulses act in the direction xx. Henceforth, we will call this direction as common normal direction and a direction perpendicular to it (i.e. yy) as common tangent.

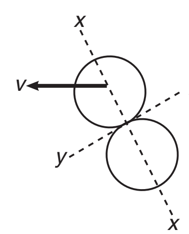


Fig. 11.42

Following four points are important regarding an oblique collision :

1. A pair of equal and opposite impulses act along common normal direction. Hence, linear momentum of individual bodies do change along common normal direction. If mass of the colliding bodies remain constant during collision, then we can say that linear velocity of the individual bodies change during collision in this direction.
2. No component of impulse act along common tangent direction. Hence, linear momentum or linear velocity of individual bodies (if mass is constant) remain unchanged along this direction.
3. Net impulse on both the bodies is zero during collision. Hence, net momentum of both the bodies remain conserved before and after collision in any direction.
4. Definition of coefficient of restitution can be applied along common normal direction, i.e. along common normal direction we can apply

   Relative speed of separation = e (relative speed of approach)

   Here, e is the coefficient of restitution between the particles.

Example 11.19 *Two blocks A and B of equal mass $m = 1.0$ kg are lying on a smooth horizontal surface as shown in figure. A spring of force constant $k = 200$ N/m is fixed at one end of block A. Block B collides with block A with velocity $v_0 = 2.0$ m/s. Find the maximum compression of the spring.*

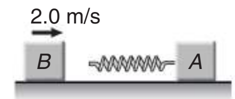


Fig. 11.43

Solution At maximum compression (x_m) velocity of both the blocks is same, say it is v. Applying conservation of linear momentum, we have

$$(m_A + m_B)v = m_B v_0$$

or $$(1.0 + 1.0)v = (1.0)\, v_0$$

or $$v = \frac{v_0}{2} = \frac{2.0}{2} = 1.0 \text{ m/s}$$

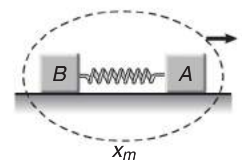


Fig. 11.44

Using conservation of mechanical energy, we have

$$\frac{1}{2} m_B v_0^2 = \frac{1}{2}(m_A + m_B)v^2 + \frac{1}{2} k x_m^2$$

Substituting the values, we get

$$\frac{1}{2} \times (1) \times (2.0)^2 = \frac{1}{2} \times (1.0 + 1.0) \times (1.0)^2 + \frac{1}{2} \times (200) \times x_m^2$$

or $$2 = 1.0 + 100 x_m^2$$

or $$x_m = 0.1 \text{ m} = 10.0 \text{ cm}$$ **Ans.**

Example 11.20 *Two balls of masses m and 2 m moving in opposite directions collide head on elastically with velocities v and 2v. Find their velocities after collision.*

Solution Here, $v_1 = -v$, $v_2 = 2v$, $m_1 = m$ and $m_2 = 2m$.

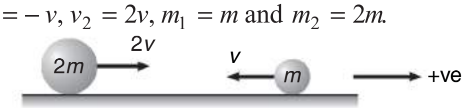


Fig. 11.45

Substituting these values in Eqs. (v) and (vi), we get

$$v_1' = \left(\frac{m - 2m}{m + 2m}\right)(-v) + \left(\frac{4m}{m + 2m}\right)(2v) \quad \text{or} \quad v_1' = \frac{v}{3} + \frac{8v}{3} = 3v$$

and $$v_2' = \left(\frac{2m - m}{m + 2m}\right)(2v) + \left(\frac{2m}{m + 2m}\right)(-v) \quad \text{or} \quad v_2' = \frac{2}{3}v - \frac{2}{3}v = 0$$

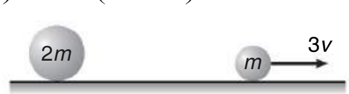


Fig. 11.46

i.e. the second ball (of mass $2m$) comes to a rest while the first (of mass m) moves with velocity $3v$ in the direction shown in Fig. 11.46.

Example 11.21 *Two pendulum bobs of masses m and 2m collide head on elastically at the lowest point in their motion. If both the balls are released from a height H above the lowest point, to what heights do they rise for the first time after collision?*

Solution Given, $m_1 = m, m_2 = 2m, v_1 = -\sqrt{2gH}$ and $v_2 = \sqrt{2gH}$

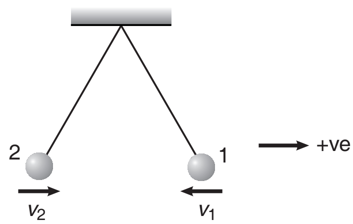


Fig. 11.47

Since, the collision is elastic. Using Eqs. (v) and (vi) discussed in the theory the velocities after collision are

$$v_1' = \left(\frac{m-2m}{m+2m}\right)(-\sqrt{2gH}) + \left(\frac{4m}{m+2m}\right)\sqrt{2gH}$$

$$= \frac{\sqrt{2gH}}{3} + \frac{4\sqrt{2gH}}{3} = \frac{5}{3}\sqrt{2gH}$$

and
$$v_2' = \left(\frac{2m-m}{m+2m}\right)(\sqrt{2gH}) + \left(\frac{2m}{m+2m}\right)(-\sqrt{2gH})$$

$$= \frac{\sqrt{2gH}}{3} - \frac{2\sqrt{2gH}}{3} = -\frac{\sqrt{2gH}}{3}$$

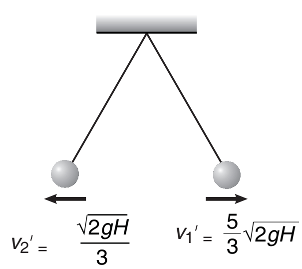


Fig. 11.48

i.e. the velocities of the balls after the collision are as shown in Fig. 11.48.

Therefore, the heights to which the balls rise after the collision are:

$$h_1 = \frac{(v_1')^2}{2g} \qquad \text{(using } v^2 = u^2 - 2gh\text{)}$$

or
$$h_1 = \frac{\left(\frac{5}{3}\sqrt{2gH}\right)^2}{2g} \quad \text{or} \quad h_1 = \frac{25}{9}H$$

and $$h_2 = \frac{(v_2')^2}{2g}$$

or $$h_2 = \frac{\left(\frac{\sqrt{2gH}}{3}\right)^2}{2g}$$

or $$h_2 = \frac{H}{9}$$

Note *Since, the collision is elastic, mechanical energy of both the balls will remain conserved, or*

$$E_i = E_f$$

$\Rightarrow$ $$(m + 2m)\, gH = mgh_1 + 2mgh_2$$

$\Rightarrow$ $$3mgH = (mg)\left(\frac{25}{9}H\right) + (2mg)\left(\frac{H}{9}\right)$$

$\Rightarrow$ $$3mgH = 3mgH$$

Example 11.22 *A ball of mass m moving at a speed v makes a head on inelastic collision with an identical ball at rest. The kinetic energy of the balls after the collision is $\frac{3}{4}$th of the original. Find the coefficient of restitution.*

Fig. 11.49

Solution For the given conditions, we can use Eq. (xiv) or

$$v_1' = \left(\frac{1+e}{2}\right)v \quad \text{and} \quad v_2' = \left(\frac{1-e}{2}\right)v$$

Given that $$K_f = \frac{3}{4}K_i$$

or $$\frac{1}{2}mv_1'^2 + \frac{1}{2}mv_2'^2 = \frac{3}{4}\left(\frac{1}{2}mv^2\right)$$

Substituting the value, we get

$$\left(\frac{1+e}{2}\right)^2 + \left(\frac{1-e}{2}\right)^2 = \frac{3}{4}$$

or $$(1+e)^2 + (1-e)^2 = 3$$

or $$2 + 2e^2 = 3$$

or $$e^2 = \frac{1}{2}$$

or $$e = \frac{1}{\sqrt{2}}$$ **Ans.**

Example 11.23 *A ball is moving with velocity 2 m/s towards a heavy wall moving towards the ball with speed 1 m/s as shown in figure. Assuming collision to be elastic, find the velocity of ball immediately after the collision.*

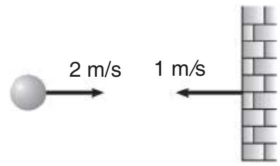


Fig. 11.50

Solution The speed of wall will not change after the collision. So, let v be the velocity of the ball after collision in the direction shown in figure. Since, collision is elastic ($e = 1$).

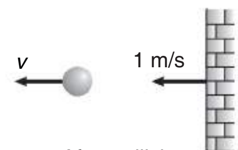


Fig. 11.51

$$\text{separation speed} = \text{approach speed}$$

or $$v - 1 = 2 + 1$$

or $$v = 4 \text{ m/s}$$ **Ans.**

Example 11.24 *A ball of mass m hits a floor with a speed v_0 making an angle of incidence α with the normal. The coefficient of restitution is e. Find the speed of the reflected ball and the angle of reflection of the ball.*

Solution The component of velocity v_0 along common tangent direction $v_0 \sin\alpha$ will remain unchanged. Let v be the component along common normal direction after collision. Applying

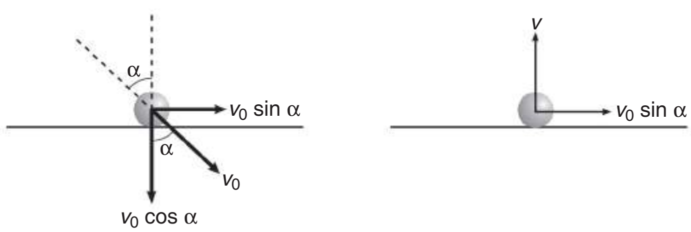


Fig. 11.52

Relative speed of separation = e (relative speed of approach)

along common normal direction, we get

$$v = ev_0 \cos\alpha$$

Thus, after collision components of velocity v' are $v_0 \sin\alpha$ and $ev_0 \cos\alpha$

$\therefore$ $$v' = \sqrt{(v_0 \sin\alpha)^2 + (ev_0 \cos\alpha)^2}$$

and $$\tan\beta = \frac{v_0 \sin\alpha}{ev_0 \cos\alpha}$$

or $$\tan\beta = \frac{\tan\alpha}{e}$$

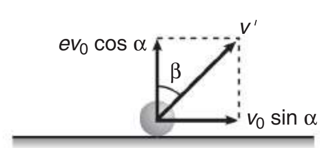


Fig. 11.53

Note *For elastic collision, $e = 1$*

$\therefore$ $$v' = v_0 \quad \textit{and} \quad \beta = \alpha.$$

Example 11.25 *After perfectly inelastic collision between two identical balls moving with same speed in different directions, the speed of the combined mass becomes half the initial speed. Find the angle between the two before collision.*

Solution Let θ be the desired angle. Linear momentum of the system will remain conserved.

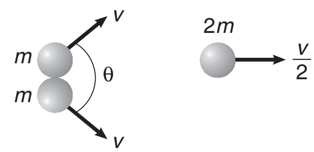


Fig. 11.54

Hence, $$p^2 = p_1^2 + p_2^2 + 2p_2 p_2 \cos\theta$$

or $$\left\{2m\left(\frac{v}{2}\right)\right\}^2 = (mv)^2 + (mv)^2 + 2(mv)(mv)\cos\theta$$

or $$1 = 1 + 1 + 2\cos\theta \quad \text{or} \quad \cos\theta = -\frac{1}{2}$$

∴ $$\theta = 120°$$ **Ans.**

Example 11.26 *The coefficient of restitution between a snooker ball and the side cushion is $\frac{1}{3}$. If the ball hits the cushion and then rebounds at right angles to its original direction, show that the angles made with the side cushion by the direction of motion before and after impact are 60° and 30° respectively.*

Solution Let the original speed be u, in a direction making an angle θ with the side cushion.

Using the law of restitution

Relative speed of separation along common normal direction

$$= e \text{ (relative speed of approach)}$$

or $$v = \frac{1}{3}(u\sin\theta) \Rightarrow \frac{u}{v} = \frac{3}{\sin\theta}$$

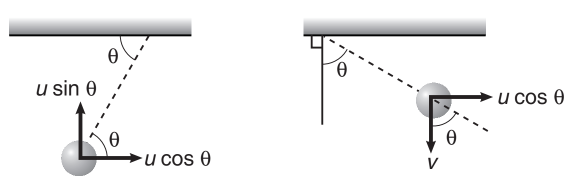


Fig. 11.55

After impact, $$\tan\theta = \frac{u\cos\theta}{v} = \frac{3\cos\theta}{\sin\theta}$$

$$\Rightarrow \tan^2\theta = 3$$

$$\Rightarrow \tan\theta = \sqrt{3}$$

$$\Rightarrow \theta = 60°$$

Therefore, the directions of motion before and after impact are at 60° and 30° to the cushion.

INTRODUCTORY EXERCISE 11.6

1. Two blocks of masses 3 kg and 6 kg respectively are placed on a smooth horizontal surface. They are connected by a light spring of force constant $k = 200$ N/m. Initially the spring is unstretched. The indicated velocities are imparted to the blocks. Find the maximum extension of the spring.

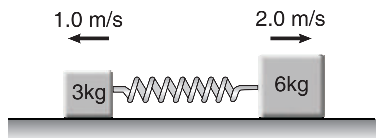


Fig. 11.56

2. A moving body of mass m makes a head on elastic collision with another body of mass $2m$ which is initially at rest. Find the fraction of kinetic energy lost by the colliding particle after collision.
3. What is the fractional decrease in kinetic energy of a body of mass m_1 when it makes a head on elastic collision with another body of mass m_2 kept at rest?
4. In one dimensional elastic collision of equal masses, the velocities are interchanged. Can velocities in a one dimensional collision be interchanged if the masses are not equal.
5. After an head on elastic collision between two balls of equal masses, one is observed to have a speed of 3 m/s along the positive x-axis and the other has a speed of 2 m/s along the negative x-axis. What were the original velocities of the balls ?
6. A ball of mass 1 kg moving with 4 ms^{-1} along $+x$-axis collides elastically with an another ball of mass 2 kg moving with 6 m/s is opposite direction. Find their velocities after collision.
7. Three balls A, B and C are placed on a smooth horizontal surface. Given that $m_A = m_C = 4m_B$. Ball B collides with ball C with an initial velocity v as shown in figure. Find the total number of collisions between the balls. All collisions are elastic.

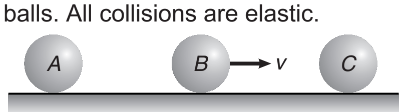
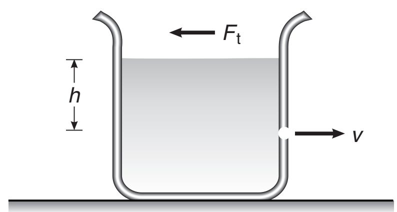


Fig. 11.57

8. Ball 1 collides directly with another identical ball 2 at rest. Velocity of second ball becomes two times that of 1 after collision. Find the coefficient of restitution between the two balls?
9. A sphere A of mass m, travelling with speed v, collides directly with a stationary sphere B. If A is brought to rest and B is given a speed V, find (a) the mass of B (b) the coefficient of restitution between A and B?
10. A smooth sphere is moving on a horizontal surface with velocity vector $2\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$ immediately before it hits a vertical wall. The wall is parallel to $\hat{\mathbf{j}}$ and the coefficient of restitution of the sphere and the wall is $e = \frac{1}{2}$. Find the velocity of the sphere after it hits the wall?
11. A ball falls vertically on an inclined plane of inclination α with speed v_0 and makes a perfectly elastic collision. What is angle of velocity vector with horizontal after collision.

Final Touch Points

1. **Centre of mass frame of reference or *C*-frame of reference or zero momentum frame** A frame of reference carried by the centre of mass of an isolated system of particles (i.e. a system not subjected to any external forces) is called the centre of mass or *C*-frame of reference. In this frame of reference.
   (i) Position vector of centre of mass is zero.
   (ii) Velocity and hence momentum of centre of mass is also zero.
2. A liquid of density ρ is filled in a container as shown in figure. The liquid comes out from the container through a orifice of area *'a'* at a depth *'h'* below the free surface of the liquid with a velocity *v*. This exerts a thrust force in the container in the backward direction. This thrust force is given by



$$F_t = v_r \left(-\frac{dm}{dt}\right)$$

Here, $v_r = v$ (in forward direction)

and $\left(-\frac{dm}{dt}\right) = \rho a v$

as $\left(\frac{dV}{dt}\right) =$ Volume of liquid flowing per second $= av$

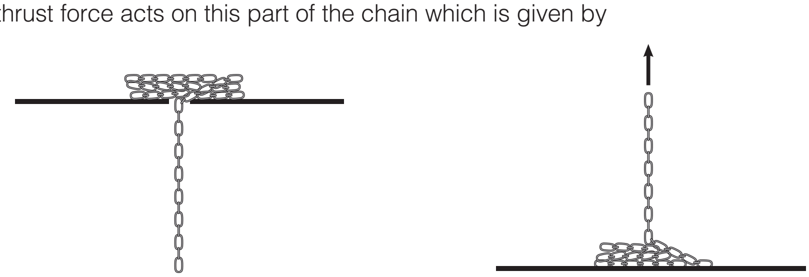
$\therefore$ $\left(-\frac{dm}{dt}\right) = \rho\left(\frac{dV}{dt}\right) = \rho a v$

$\therefore$ $F_t = v\,(\rho a v)$

or $F_t = \rho a v^2$ (in backward direction)

Further, we will see in the chapter of fluid mechanics that $v = \sqrt{2gh}$.

3. Suppose, a chain of mass per unit length λ begins to fall through a hole in the ceiling as shown in Fig. (a) or the end of the chain piled on the platform is lifted vertically as in Fig. (b). In both the cases, due to increase of mass in the portion of the chain which is moving with a velocity *v* at certain moment of time a thrust force acts on this part of the chain which is given by


(a) (b)

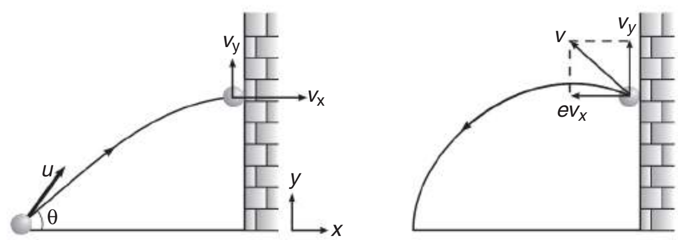
$$F_t = v_r\left(\frac{dm}{dt}\right). \quad \text{Here, } v_r = v \quad \text{and} \quad \frac{dm}{dt} = \lambda v$$

Here, v_r is upwards in case (a) and downwards in case (b). Thus,

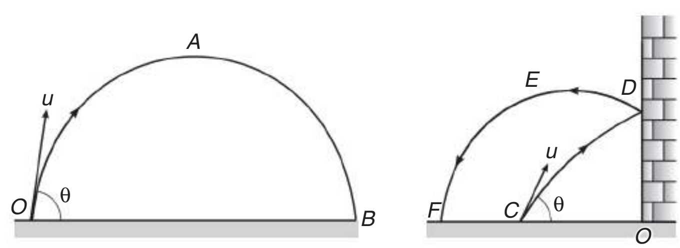
$$F_t = \lambda v^2$$

The direction of F_t is upwards in case (a) and downwards in case (b).

4. Suppose a ball is a projected with speed u at an angle θ with horizontal. It collides at some distance with a wall parallel to y-axis as shown in figure. Let v_x and v_y be the components of its velocity along x and y-directions at the time of impact with wall. Coefficient of restitution between the ball and the wall is e. Component of its velocity along y-direction (common tangent) v_y will remain unchanged while component of its velocity along x-direction (common normal) v_x will become ev_x is opposite direction.



Further, since v_y does not change due to collision, the time of flight (time taken by the ball to return to the same level) and maximum height attained by the ball will remain same as it would had been in the absence of collision with the wall. Thus,



$$t_{OAB} = t_{CD} + t_{DEF} = T = \frac{2u\sin\theta}{g}$$

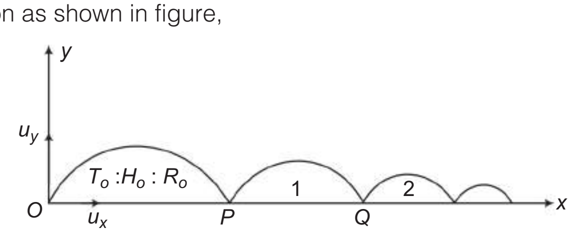
and $$h_A = h_E = \frac{u^2\sin^2\theta}{2g}$$

Further, $$CO + OF \leq \text{Range or } OB$$

It collision is elastic, then $$CO + OF = \text{Range} = \frac{u^2\sin 2\theta}{g}$$

and if it is inelastic, $$CO + OF < \text{Range}$$

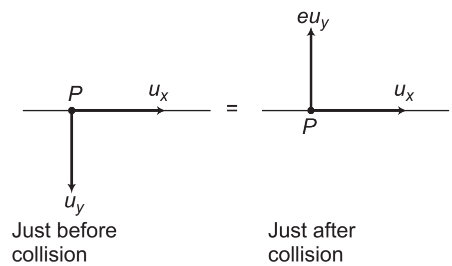
5. In the projectile motion as shown in figure,



$$T = \frac{2u_y}{g} \Rightarrow T \propto u_y$$

$$\Rightarrow \qquad H = \frac{u_y^2}{2g} \quad \Rightarrow \quad H \propto u_y^2$$

$$R = u_x T = u_x \left(\frac{2u_y}{g}\right) \Rightarrow R \propto u_x u_y$$



As shown in above figure, vertical component of velocity just after collision becomes eu_y or e times, while horizontal component remains unchanged. Hence, the next time, T will become e times (as $T \propto u_y$), H will become e^2 times (as $H \propto u_y^2$) and R will also becomes e times (as $R \propto u_x u_y$).

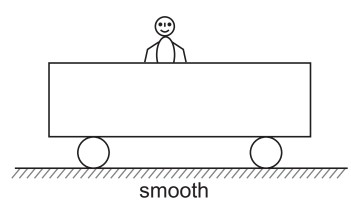
Thus, if T_0, H_0 and R_0 are the initial values then after first collision,

$$T_1 = eT_0, \; H_1 = e^2 H_0 \text{ and } R_1 = eR_0$$

Similarly after n-collisions,

$$T_n = e^n T_0, \; H_n = e^{2n} H_0 \text{ and } R_n = e^n R_0$$

6. Thrust force in variable mass system is nothing but a result of law of conservation of linear momentum. Let us take an example.



A boy is standing over a trolley kept over a smooth surface. If the boy throws a stone towards right, then mass of (trolley + boy) system is decreasing. Relative velocity of stone is towards right. So, a thrust force will act on (trolley + boy) system towards left (in a direction opposite to relative velocity, as mass is decreasing). Due to this thrust force (trolley + boy) system will move towards left.

And this is nothing but law of conservation of linear momentum. Initial momentum of system was zero. Therefore final momentum should also be zero. If momentum of stone is towards right, then momentum of (trolley + boy) system should be towards left (of same magnitude), to make total momentum equal to zero.

7. In perfectly inelastic collision, all bodies stick together and they have a common velocity given by:

$$\mathbf{v}_{\text{common}} = \frac{\text{Total momentum}}{\text{Total mass}} = \frac{\mathbf{p}_{\text{Total}}}{M_{\text{Total}}}$$

and this common velocity is also equal to the velocity of centre of mass of the system.

Solved Examples

TYPED PROBLEMS

Type 1. *Based on law of conversation of linear momentum.*

Concept

(i) If net force on a system is zero, then linear momentum of the system (or centre of mass remains constant).

(ii) Normally the ground is given smooth. So, the net force on a system in horizontal direction will be zero and momentum of the system in horizontal direction will remain constant.

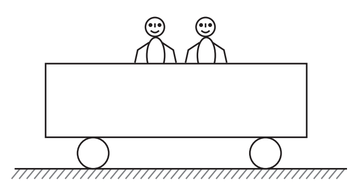
$\therefore$ Initial momentum = Final momentum

or $$p_i = p_f \qquad \text{(in horizontal direction)}$$

(iii) If all individual bodies/blocks of the system are initially at rest then, p_i will be zero. Therefore, total momentum of the system at any instant or p_f is also zero.

(iv) Since, we are using only one conservation law i.e. law of conservation of linear momentum, so we have only one equation and only one unknown.

Example 1 *A trolley of mass M is at rest over a smooth horizontal surface as shown in figure. Two boys each of mass 'm' are standing over the trolley. They jump from the trolley (towards right) with relative velocity v_r [relative to velocity of trolley just after jumping]*

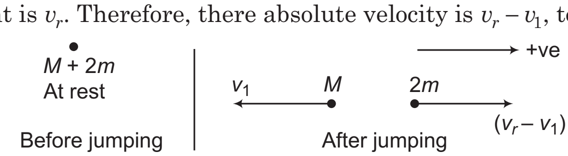


(a) together

(b) one after the other.

Find velocity of trolley in both cases.

Solution (a) Let velocity of trolley just after jumping is v_1 (towards left). Relative velocity of boys towards right is v_r. Therefore, there absolute velocity is $v_r - v_1$, towards right.



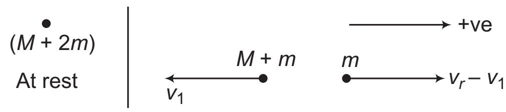
Net force on the system in horizontal direction is zero. Hence, linear momentum of the system in horizontal direction is zero.

or $$p_i = p_f$$

$\Rightarrow$ $$0 = 2m\,(v_r - v_1) - Mv_1$$

$\Rightarrow$ $$v_1 = \frac{2m\, v_r}{2m + M}$$ **Ans.**

(b) Let v_1 be the velocity of trolley after the first boy jumps. Then,

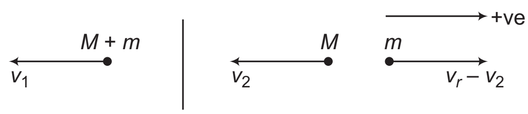


$$p_i = p_f$$

$\Rightarrow$ $$0 = m\,(v_r - v_1) - (M + m)\,v_1$$

$\Rightarrow$ $$v_1 = \frac{mv_r}{M + 2m} \qquad \text{...(i)}$$

Now, the second boy jumps from the moving trolley and let v_2 be the velocity of trolley after the second boy also jumps. Then,



$$p_i = p_f$$

$\Rightarrow$ $$-(M + m)v_1 = m\,(v_r - v_2) - Mv_2$$

$\therefore$ $$v_2 = \frac{mv_r + (M + m)\,v_1}{(M + m)}$$

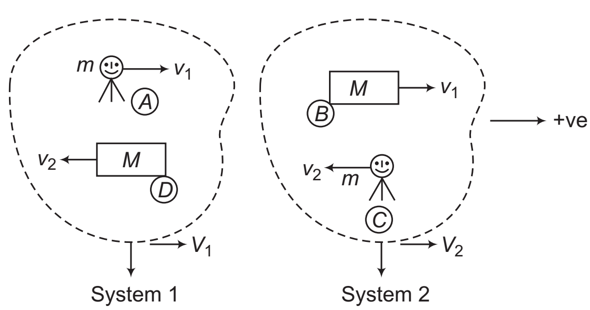
$$= \left(\frac{m}{M + m}\right) v_r + v_1$$

Substituting the value of v_1 from Eq. (i), we have

$$v_2 = mv_r\left[\frac{1}{M + m} + \frac{1}{M + 2m}\right]$$ **Ans.**

Example 2 *Two toy trains each of mass 'M' are moving in opposite directions with velocities v_1 and v_2 over two smooth rails. Two stuntmen of mass 'm' each are also moving with the trains (at rest w.r.t. trains). When trains are opposite to each other the stuntmen interchange their positions, then find the final velocities of the trains.*

Solution When the stuntmen are in air (after jumping) they also have horizontal velocities v_1 and v_2 in opposite directions. Initially (A) and (B) were together. Similarly (C) and (D) were together.



Now, (A) will fall over (D). So this is one system and suppose velocity of this system is V_1 (towards right or in positive direction) after jumping. Similarly, (C) will fall over (B) after jumping.

Let V_2 is the velocity of this system towards right after jumping.

Applying conservation of linear momentum.

System 1

$$p_i = p_f$$

$\Rightarrow$ $$m\,v_1 - M\,v_2 = (M+m)\,V_1$$

$\Rightarrow$ $$V_1 = \frac{m\,v_1 - M\,v_2}{M+m}$$ **Ans.**

System 2

$$p_i = p_f$$

$\Rightarrow$ $$M\,v_1 - m\,v_2 = (M+m)\,V_2$$

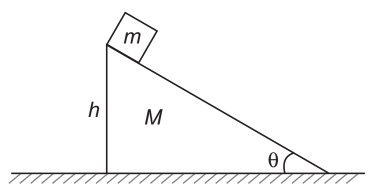
$\Rightarrow$ $$V_2 = \frac{Mv_1 - mv_2}{M+m}$$ **Ans.**

Type 2. *Based on conservation on linear momentum and mechanical energy.*

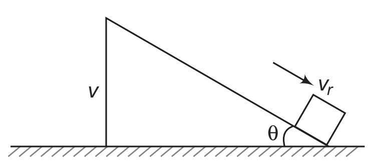
Concept

In these type of problems, all surfaces are given smooth. So, linear momentum in horizontal direction and mechanical energy of the system remains conserved. Since, we are applying two conservation laws. Therefore, number of equations are two. Hence, number of unknowns are also two.

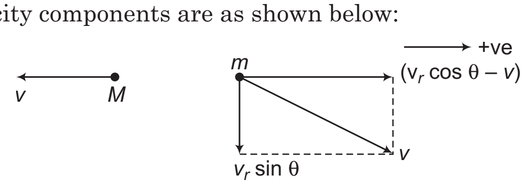
Example 3 *All surfaces shown in figure are smooth. Find velocity of wedge (of mass M) when the block (of mass m) reaches the bottom of the wedge.*



Solution When the block reaches to the bottom of the wedge, their velocities are as shown in figure. Here v is the absolute velocity (with respect to ground), but v_r is the relative velocity (relative to wedge).



Their absolute velocity components are as shown below:



$$v = \sqrt{(v_r \sin\theta)^2 + (v_r \cos\theta - v)^2}$$

Linear momentum in horizontal direction is conserved.

$\therefore$ $$p_i = p_f$$

or $$0 = m\,(v_r \cos\theta - v) - Mv \qquad \text{...(i)}$$

Mechanical energy is also conserved.

$\therefore$ $$E_i = E_f$$

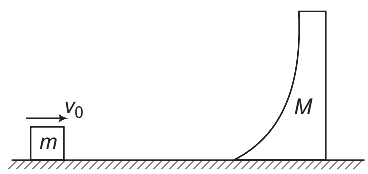
or $$mgh = \frac{1}{2}Mv^2 + \frac{1}{2}mv^2$$

or $$mgh = \frac{1}{2}Mv^2 + \frac{1}{2}m\,\left[(v_r \sin\theta)^2 + (v_r\cos\theta - v)^2\right] \qquad \text{...(ii)}$$

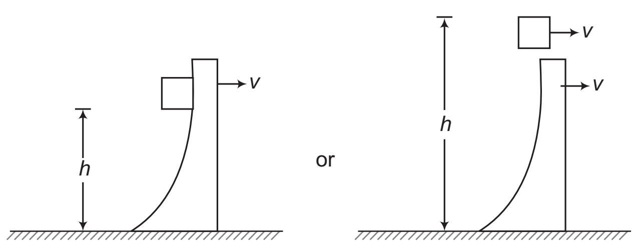
We have two equations and two unknowns v and v_r. Solving these equations, we can find the value of v.

Exercise: Solve these two equations to find the value of v.

Example 4 *All surfaces shown in figure are smooth. Wedge of mass 'M' is free to move. Block of mass 'm' is given a horizontal velocity v_0 as shown. Find the maximum height 'h' attained by 'm' (over the wedge or outside it).*



Solution At maximum height vertical component of velocity of 'm' will be zero. It will have only horizontal component of velocity and this is equal to the horizontal component of 'M' also (think why ?). So at the highest point figure is like this:



Applying law of conservation of linear momentum in horizontal direction,

$$p_i = p_f$$

$\Rightarrow$ $$mv_0 = (M + m)\,v \qquad \text{...(i)}$$

Now, applying law of conservation of mechanical energy,

$$E_i = E_f$$

$\therefore$ $$\frac{1}{2}mv_0^2 = \frac{1}{2}(M+m)\,v^2 + mgh \qquad \text{...(ii)}$$

We have two unknowns v and h.

So, we can find the value of 'h' by solving these two equations.

Exercise : Solve these two equations to find the value of 'h'.

Type 3. *Based on following four equations.*

$$\Sigma F_R = \Sigma F_L,\ \Sigma m_R a_R = \Sigma m_L a_L,\ \Sigma m_R v_R = \Sigma m_L v_L,\ \Sigma m_R x_R = \Sigma m_L x_L$$

In these equations, R stands for right hand side and L stands for left hand side. x is the absolute displacement with respect to ground.

Concept

If net force on a system in a particular direction is zero (normally in horizontal direction). And this can done by giving the ground smooth. Initially, the system is at rest. So, in this case individual bodies can move towards right or towards left, but centre of mass will remain stationary. Further, net force in horizontal direction is zero. Hence, total force towards right is equal to the total force towards left or,

$$\Sigma F_R = \Sigma F_L \quad \text{...(i)}$$

or
$$\Sigma m_R a_R = \Sigma m_L a_L \quad \text{...(ii)}$$

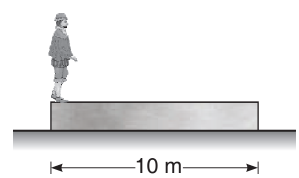
Now, integrating 'a' we will get 'v' and by further integrating 'v', we will get 'x'.

$\therefore$
$$\Sigma m_R v_R = \Sigma m_L v_L \quad \text{...(iii)}$$

and
$$\Sigma m_R x_R = \Sigma m_L x_L \quad \text{...(iv)}$$

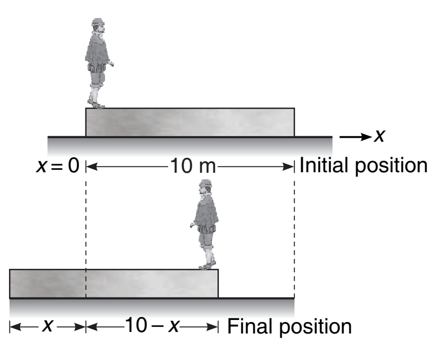
Note *If the system is initially not at rest but net force is zero, then Eqs. (i) and (ii) are still applicable but not the Eqs. (iii) and (iv). Think why ?*

Example 5 *A wooden plank of mass 20 kg is resting on a smooth horizontal floor. A man of mass 60 kg starts moving from one end of the plank to the other end. The length of the plank is 10 m. Find the displacement of the plank over the floor when the man reaches the other end of the plank.*



Solution Here, the system is man + plank. Net force on this system in horizontal direction is zero and initially the centre of mass of the system is at rest. Therefore, the centre of mass does not move in horizontal direction.

Let x be the displacement of the Plank. Assuming the origin, i.e. $x = 0$ at the position shown in figure.



As we said earlier also, the centre of mass will not move in horizontal direction (x-axis). Therefore, for centre of mass to remain stationary,

$$x_i = x_f$$

$$\frac{(60)(0)+20\left(\frac{10}{2}\right)}{60+20}=\frac{(60)(10-x)+20\left(\frac{10}{2}-x\right)}{60+20}$$

or $$\frac{5}{4}=\frac{6(10-x)+2\left(\frac{10}{2}-x\right)}{8}=\frac{60-6x+10-2x}{8}$$

or $$5=30-3x+5-x$$

or $$4x=30$$

or $$x=\frac{30}{4}\text{ m} \quad \text{or} \quad x=7.5\text{ m}$$

Note *The centre of mass of the plank lies at its centre.*

Alternate Method

x_L = displacement of plank towards left $= x$

m_L = mass of plank displaced towards left $= 20$ kg

x_R = displacement of man relative to ground towards right $= 10 - x$

and m_R = mass of man displaced towards right $= 60$ kg

Applying $x_R m_R = x_L m_L$, we get

$$(10-x)(60)=20x$$

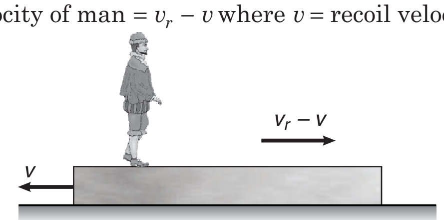
or $$x=30-3x$$

or $$4x=30$$

$\therefore$ $$x=\frac{30}{4}=7.5\text{ m}$$

Example 6 *A man of mass m_1 is standing on a platform of mass m_2 kept on a smooth horizontal surface. The man starts moving on the platform with a velocity v_r relative to the platform. Find the recoil velocity of platform.*

Solution Absolute velocity of man $= v_r - v$ where v = recoil velocity of platform.



Taking the platform and the man as a system, net external force on the system in horizontal direction is zero. The linear momentum of the system remains constant. Initially both the man and the platform were at rest.

Hence, $$0=m_1(v_r-v)-m_2v$$

$\therefore$ $$v=\frac{m_1v_r}{m_1+m_2}$$

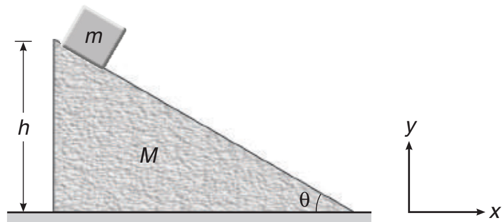
Alternate Method Using the equation,

$$\Sigma m_R v_R = \Sigma m_L v_L$$

we have, $$m_1(v_r-v)=m_2v$$

$\therefore$ $$v=\frac{m_1v_r}{m_1+m_2}$$

Example 7 *A block of mass m is released from the top of a wedge of mass M as shown in figure. Find the displacement of wedge on the horizontal ground when the block reaches the bottom of the wedge. Neglect friction everywhere.*



Solution Here, the system is wedge + block. Net force on the system in horizontal direction (x-direction) is zero, therefore, the centre of mass of the system will not move in x-direction so we can apply,

$$x_R m_R = x_L m_L \qquad \text{...(i)}$$

Let x be the displacement of wedge. Then,

$$x_L = \text{displacement of wedge towards left} = x$$

$$m_L = \text{mass of wedge moving towards left} = M$$

$$x_R = \text{displacement of block with respect to ground towards right} = h \cot \theta - x$$

and $$m_R = \text{mass of block moving towards right} = m$$

Substituting in Eq. (i), we get

$$m\,(h \cot \theta - x) = xM$$

$\therefore$ $$x = \frac{mh \cot \theta}{M + m}$$ **Ans.**

Type 4. *Explosion of a bomb or a projectile.*

Concept

If a bomb or a projectile explodes in two or more than two parts, then it explodes due to internal forces. Therefore, net force or net external force is zero. Hence, linear momentum of the system can be conserved just before and just after explosion. By this momentum conservation equation, we can find the velocity of some unknown part. If explosion takes place in air, then during the explosion, the external force due to gravity (= weight) can be neglected, as the time of explosion is very short. So, impulse of this force is negligible and impulse is change in linear momentum.Hence, change in linear momentum is also negligible.

Example 8 *A bomb of mass '5m' at rest explodes into three parts of masses 2m, 2m and m. After explosion, the equal parts move at right angles with speed v each. Find speed of the third part and total energy released during explosion.*

Solution Let the two equal parts move along positive x and positive y directions and suppose the velocity of third part is $\mathbf{V}$. From law of conservation of linear momentum,

we have, $$\mathbf{p}_i = \mathbf{p}_f$$

$\Rightarrow$ $$0 = 2m\,(v\hat{\mathbf{i}}) + 2\,m\,(v\,\hat{\mathbf{j}}) + m\mathbf{V}$$

Solving this equation we have, $\mathbf{V} = -2v\hat{\mathbf{i}} - 2v\hat{\mathbf{j}}$

$\therefore$ Speed of this particle

$$= |\mathbf{V}| \text{ or } V$$
$$= \sqrt{(-2v)^2 + (-2v)^2}$$
$$= 2\sqrt{2}v$$ **Ans.**

Energy released during explosion,

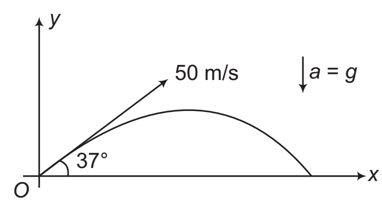
$$= \text{kinetic energy of all three parts}$$
$$= \frac{1}{2}(2m)v^2 + \frac{1}{2}(2m)v^2 + \frac{1}{2}(m)(2\sqrt{2}v)^2$$
$$= 6mv^2$$ **Ans.**

Example 9 *A projectile of mass 3 kg is projected with velocity 50 m/s at 37° from horizontal. After 2 s, explosion takes place and the projectile breaks into two parts of masses 1 kg and 2 kg. The first part comes to rest just after explosion. Find,*

(a) the velocity of second part just after explosion.

(b) maximum height attained by this part. Take $g = 10\ m/s^2$

Solution



(a)
$$\mathbf{u} = (50\cos 37°)\,\hat{\mathbf{i}} + (50\sin 37°)\,\hat{\mathbf{j}}$$
$$= (40\,\hat{\mathbf{i}} + 30\,\hat{\mathbf{j}})\text{ m/s}$$
$$\mathbf{a} = (-10\hat{\mathbf{j}})\text{ m/s}^2 = \text{constant}$$

After $t = 2$ s
$$\mathbf{v} = \mathbf{u} + \mathbf{a}t$$
$$= (40\hat{\mathbf{i}} + 30\hat{\mathbf{j}}) + (-10\hat{\mathbf{j}})(2) = (40\hat{\mathbf{i}} + 10\hat{\mathbf{j}})\text{ m/s}$$
$$\mathbf{S} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$
$$= (40\hat{\mathbf{i}} + 30\hat{\mathbf{j}})(2) + \frac{1}{2}(-10\hat{\mathbf{j}})(2)^2 = (80\hat{\mathbf{i}} + 40\hat{\mathbf{j}})\text{m}$$
$$s_y = 40\text{ m}$$

Hence, the explosion takes place at a height of 40 m and the velocity just before explosion is $(40\hat{\mathbf{i}} + 10\hat{\mathbf{j}})$ m/s.

Velocity of first part just after explosion is zero and let velocity of second part just after explosion is $\mathbf{V}$, then from conservation of linear momentum,

$$\mathbf{p}_i = \mathbf{p}_f$$
$$\Rightarrow \quad 3(40\hat{\mathbf{i}} + 10\hat{\mathbf{j}}) = (1)(0) + 2\mathbf{V}$$
$$\Rightarrow \quad \mathbf{V} = (60\,\hat{\mathbf{i}} + 15\hat{\mathbf{j}})\text{ m/s}$$ **Ans.**

(b) Vertical component of velocity of second part just after collision is 15 m/s and explosion has taken place at a height of 40 m. Therefore total height of this part from ground.

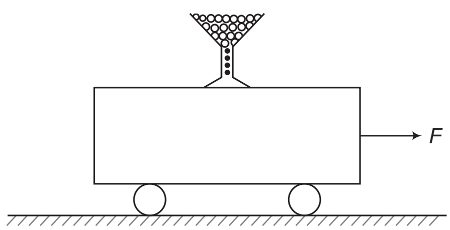
$$= 40 + \frac{v_y^2}{2g} = 40 + \frac{(15)^2}{2 \times 10} = 51.25\,\text{m}$$ **Ans.**

Type 5. *Based on variable mass system.*

Concept

In case of a variable mass system, a thrust force of magnitude $\left| v_r \times \frac{dm}{dt} \right|$ has to be applied on the system, whose mass is changing. Direction of this force is in the direction of $\mathbf{v}_r$ if mass is increasing and in the opposite direction of $\mathbf{v}_r$, if mass is decreasing, Further, $\mathbf{v}_r$ is zero if a mass is just dropped from a moving body. Because the dropped body has the same velocity as of the moving body at the time of dropping. So, no thrust force will act in this case.

Example 10 *A constant force F is applied on a trolley of initial mass m_0 kept over a smooth surface. Sand is poured gently over the trolley at a constant rate of (μ) kg/s. After time t, find*



(a) mass of the trolley (with sand)

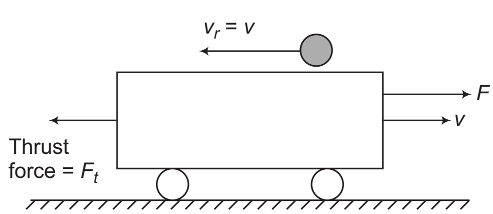
(b) net force on the trolley

(c) velocity of trolley

Solution (a) Mass of the trolley after time t is

$$m = \text{mass of trolley} + (\text{mass of sand poured per second})\,(\text{time})$$
$$= m_0 + \mu t$$

(b) Let v is the velocity of trolley at time t. Sand is poured gently. So, velocity of sand is zero or, relative velocity of sand is v in the opposite direction of velocity of trolley. Mass of trolley is increasing. So, thrust force on the trolley is in the direction of relative velocity or in the opposite direction of motion of trolley.



Net force on trolley $= F - F_t$

$$= F - v_r \frac{dm}{dt} = F - (v)(\mu)$$

or $$F_{\text{net}} = F - \mu v$$ **Ans.**

Note *In the above expression of F_{net}, velocity v is a function of time which has been asked in the next part.*

(c) $F_{\text{net}} = F - v\mu$

$\therefore$ $$ma = F - \mu v$$

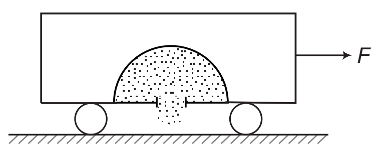
$\Rightarrow$ $$(m_0 + \mu t)\left(\frac{dv}{dt}\right) = F - \mu v$$

$\Rightarrow$ $$\int_0^v \frac{dv}{F - \mu v} = \int_0^t \frac{dt}{m_0 + \mu t}$$

Solving this equation we get

$$v = \frac{Ft}{m_0 + \mu t}$$ **Ans.**

Example 11 *A trolley of initial mass m_0 is kept over a smooth surface as shown in figure. A constant force F is applied on it. Sand kept inside the trolley drains out from its floor at a constant rate of (μ) kg/s. After time t find:*



(a) total mass of trolley and sand.
(b) net force on the trolley.
(c) velocity of trolley.

Solution (a) Total mass of trolley and sand

$$m = \text{initial mass} - \text{mass of sand drained out}$$

or $$m = m_0 - \mu t$$

(b) Let v is the velocity of trolley at time t. Then, velocity of sand drained out is also v or relative velocity is zero. Hence, no thrust force will act in this case. Therefore,

$$F_{\text{net}} = F$$

(c) $F_{\text{net}} = F$

or $$ma = F \quad \text{or} \quad m\frac{dv}{dt} = F$$

or $$(m_0 - \mu t)\frac{dv}{dt} = F$$

$\therefore$ $$\int_0^v dv = \int_0^t \frac{F dt}{m_0 - \mu t}$$

Solving this equation we get,

$$v = \frac{1}{\mu}\log_e\left(\frac{m_0}{m_0 - \mu t}\right)$$ **Ans.**

Type 6. *Problems based on linear impulse (J) and coefficient of restitution (e).*

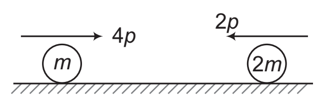
Concept

(i) During collision, equal and opposite impulses act on the two colliding bodies along common normal directions. This impulse changes the linear momentum of an individual body, but total momentum of the system remains constant, as the total linear impulse is zero. Further, this linear impulse is equal to change in linear momentum.

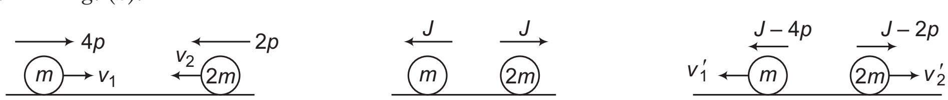
(ii) Coefficient of restitution between two colliding bodies is defined along common normal direction and this is given by:

$$e=\frac{\text{relative speed of separation}}{\text{relative speed of approach}}$$

Example 12 *Two balls of masses m and 2m and momenta 4p and 2p (in the directions shown) collide as shown in figure. During collision, the value of linear impulse between them is J. In terms of J and p find coefficient of restitution 'e'. Under what condition collision is elastic. Also find the condition of perfectly inelastic collision.*



Solution Directions of linear impulses on the two colliding bodies at the time of collision are shown in Fig. (b).



Before collision (a) | At the time to collision (b) | After collision (c)

Linear impulse is equal to the change in linear momentum. Hence, momenta of the balls after collision are shown in Fig. (c).

Let v_1 and v_2 are their velocities before collision [in the directions shown in figure (a)] and v_1' and v_2' are the velocities after collision as shown in figure (c).

$$v=\frac{\text{linear momentum}}{\text{mass}}$$

$$e=\frac{\text{relative speed of separation}}{\text{relative speed of approach}}$$

or

$$e=\frac{v_1'+v_2'}{v_1+v_2}=\frac{\dfrac{J-4p}{m}+\dfrac{J-2p}{2m}}{\dfrac{4p}{m}+\dfrac{2p}{2m}}$$

or

$$e=\frac{3J-10p}{10p}=\frac{3J}{10p}-1$$ **Ans.**

For elastic collision,

$$e=1 \quad \Rightarrow \quad \frac{3J}{10p}=2 \quad \text{or} \quad J=\frac{20}{3}p$$ **Ans.**

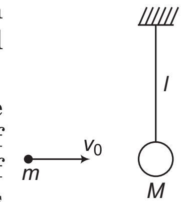
For perfectly inelastic collision,

$$e=0 \quad \Rightarrow \quad \frac{3J}{10p}=1 \quad \text{or} \quad J=\frac{10}{3}p$$ **Ans.**

Type 7. *Based on conservation of linear momentum and vertical circular motion.*

Concept

A ball of mass M is suspended from a massless string of length l as shown in figure. A bullet of mass m moving with velocity v_0 collides with the ball and sticks with the ball.



Now, velocity of the combined mass $(M+m)$ just after collision at the bottommost point (say it's u) can by obtained from law of conservation of linear momentum. After finding the value of u we can use the theory of vertical circular motion. For example, if $u \geq \sqrt{5gl}$, then the combined mass will complete the vertical circular motion.

If $\sqrt{2gl} < u < \sqrt{5gl}$, string will slack in upper half of the circle and if $0 < u \leq \sqrt{2gl}$, the combined mass will oscillate in lower half of the circle.

Example 13 *In the situation discussed above, find*

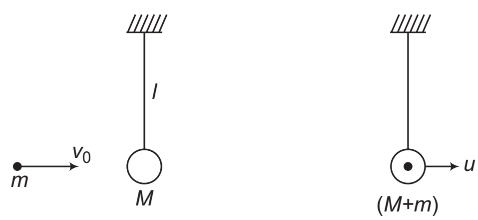
(a) velocity of combined mass just after collision at the bottommost point (or u).

(b) loss of mechanical energy during collision.

(c) minimum value of v_0 so that the combined mass completes the vertical circular motion.

Solution (a) Applying conservation of linear momentum, just before and just after collision.

$$p_i = p_f \quad \Rightarrow \quad mv_0 = (M+m)\,u \quad \Rightarrow \quad u = \frac{mv_0}{M+m}$$ **Ans.**



Just before collision   Just after collision

(b) Loss of mechanical energy during collision,

$$= E_i - E_f \qquad (E = \text{mechanical energy})$$

$$= \frac{1}{2} mv_0 - \frac{1}{2}(M+m)\,u^2$$

$$= \frac{1}{2} mv_0^2 - \frac{1}{2}(M+m)\left[\frac{mv_0}{M+m}\right]^2$$

$$= \frac{1}{2}\frac{Mm\,v_0^2}{(M+m)}$$ **Ans.**

(c) For completing the vertical circular motion, velocity at bottommost point, $u \geq \sqrt{5\,gl}$

$$\therefore \qquad \frac{mv_0}{M+m} \geq \sqrt{5\,gl} \quad \Rightarrow \quad \therefore \quad v_0 \geq \frac{M+m}{m}\sqrt{5\,gl}$$

or

$$(v_0)_{\min} = \frac{M+m}{m}\sqrt{5\,gl}$$ **Ans.**

Example 14 *A pendulum bob of mass 10^{-2} kg is raised to a height 5×10^{-2} m and then released. At the bottom of its swing, it picks up a mass 10^{-3} kg. To what height will the combined mass rise ?*

Solution Velocity of pendulum bob at bottom most point

$$v_1 = \sqrt{2gh} = \sqrt{2 \times 10 \times 5 \times 10^{-2}} = 1 \text{ m/s}$$

When the bob picks up a mass 10^{-3} kg at the bottom, then by conservation of linear momentum the velocity of combined mass is given by

$$m_1 v_1 + m_2 v_2 = (m_1 + m_2)\, v$$

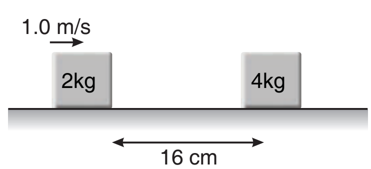
$$10^{-2} \times 1 + 10^{-3} \times 0 = (10^{-2} + 10^{-3})\, v$$

or $$v = \frac{10^{-2}}{1.1 \times 10^{-2}} = \frac{10}{11} \text{ m/s}$$

Now, $$h = \frac{v^2}{2g} = \frac{(10/11)^2}{2 \times 10} = 4.1 \times 10^{-2} \text{ m}$$ **Ans.**

Miscellaneous Examples

Example 15 *The friction coefficient between the horizontal surface and each of the block shown in the figure is 0.2. The collision between the blocks is perfectly elastic. Find the separation between them when they come to rest. (Take $g = 10 \text{ m/s}^2$).*



Solution Retardation, $a = \frac{\mu mg}{m} = \mu g = 0.2 \times 10 = 2 \text{ m/s}^2$

Velocity of first block before collision,

$$v_1^2 = 1^2 - 2\,(2) \times 0.16 = 1 - 0.64$$

$$v_1 = 0.6 \text{ m/s}$$

By conservation of momentum, $2 \times 0.6 = 2v_1' + 4v_2'$

also $v_2' - v_1' = v_1$ for elastic collision

It gives $$v_2' = 0.4 \text{ m/s}, \quad v_1' = -0.2 \text{ m/s}$$

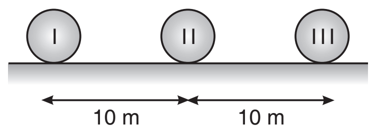
Now distance moved after collision

$$s_1 = \frac{(0.4)^2}{2 \times 2}$$

and $$s_2 = \frac{(0.2)^2}{2 \times 2}$$

$\therefore$ $$s = s_1 + s_2 = 0.05 \text{m} = 5 \text{ cm}$$ **Ans.**

Example 16 *Three identical balls, ball I, ball II and ball III are placed on a smooth floor on a straight line at the separation of* 10 *m between balls as shown in figure. Initially balls are stationary. Ball I is given velocity of* 10 *m/s towards ball II, collision between balls I and II is inelastic with coefficient of restitution* 0.5 *but collision between balls II and III is perfectly elastic. What is the time interval between two consecutive collisions between ball I and II?*



Solution Let velocity of I ball and II ball after collision be v_1 and v_2

$$v_2 - v_1 = 0.5 \times 10 \qquad \dots \text{(i)}$$

$$mv_2 + mv_1 = m \times 10 \qquad \dots \text{(ii)}$$

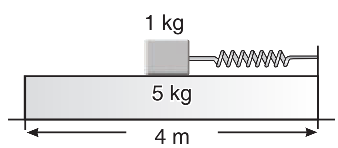
$\Rightarrow$
$$v_2 + v_1 = 10$$

Solving Eqs. (i) and (ii), we get $v_1 = 2.5$ m/s, $v_2 = 7.5$ m/s

Ball II after moving 10 m collides with ball III elastically and stops. But ball I moves towards ball II. Time taken between two consecutive collisions

$$\frac{10}{7.5} - \frac{10 - 10 \times \frac{2.5}{7.5}}{2.5} = 4\,\text{s}$$ **Ans.**

Example 17 *A plank of mass* 5 *kg is placed on a frictionless horizontal plane. Further a block of mass* 1 *kg is placed over the plank. A massless spring of natural length* 2 *m is fixed to the plank by its one end. The other end of spring is compressed by the block by half of spring's natural length. They system is now released from the rest. What is the velocity of the plank when block leaves the plank? (The stiffness constant of spring is* 100 *N/m)*



Solution Let the velocity of the block and the plank, when the block leaves the spring be u and v respectively.

By conservation of energy $\frac{1}{2}kx^2 = \frac{1}{2}mu^2 + \frac{1}{2}Mv^2$ [M = mass of the plank, m = mass of the block]

$\Rightarrow$
$$100 = u^2 + 5v^2 \qquad \dots \text{(i)}$$

By conservation of momentum

$$mu + Mv = 0 \quad \Rightarrow \quad u = -5v \qquad \dots \text{(ii)}$$

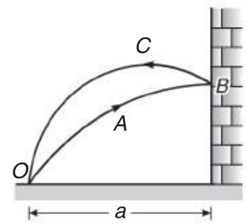
Solving Eqs. (i) and (ii) $\quad 30v^2 = 100 \quad \Rightarrow \quad v = \sqrt{\frac{10}{3}}$ m/s

From this moment until block falls, both plank and block keep their velocity constant.

Thus, when block falls, velocity of plank $= \sqrt{\frac{10}{3}}$ m/s. **Ans.**

Example 18 *A ball is projected from the ground with speed u at an angle α with horizontal. It collides with a wall at a distance a from the point of projection and returns to its original position. Find the coefficient of restitution between the ball and the wall.*

Solution As we have discussed in the theory, the horizontal component of the velocity of ball during the path OAB is $u \cos\alpha$ while in its return journey BCO it is $eu \cos\alpha$. The time of flight T also remains unchanged. Hence,



$$T = t_{OAB} + t_{BCO}$$

or $$\frac{2u \sin\alpha}{g} = \frac{a}{u\cos\alpha} + \frac{a}{eu\cos\alpha}$$

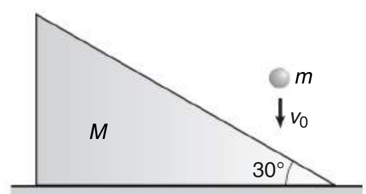
or $$\frac{a}{eu\cos\alpha} = \frac{2u\sin\alpha}{g} - \frac{a}{u\cos\alpha}$$

or $$\frac{a}{eu\cos\alpha} = \frac{2u^2 \sin\alpha\cos\alpha - ag}{gu\cos\alpha}$$

$$e = \frac{ag}{2u^2\sin\alpha\cos\alpha - ag}$$

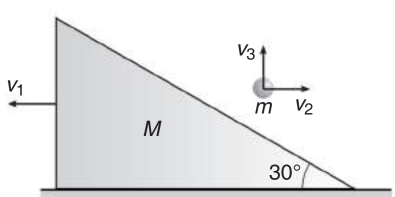
or $$e = \frac{1}{\left(\dfrac{u^2\sin 2\alpha}{ag} - 1\right)}$$

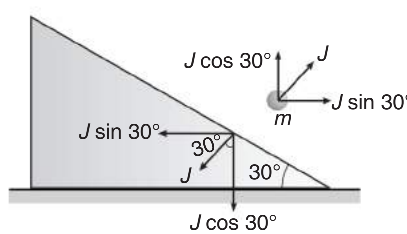
Example 19 *A ball of mass $m = 1$ kg falling vertically with a velocity $v_0 = 2$ m/s strikes a wedge of mass $M = 2$ kg kept on a smooth, horizontal surface as shown in figure. If impulse between ball and wedge during collision is J. Then make two equations which relate J with velocity components of wedge and ball. Also find impulse on wedge from ground during impact.*



Solution Given $M = 2$ kg and $m = 1$ kg

Let, J be the impulse between ball and wedge during collision and v_1, v_2 and v_3 be the components of velocity of the wedge and the ball in horizontal and vertical directions respectively.





Applying impulse = change in momentum

we get $$J \sin 30° = Mv_1 = mv_2$$

or $$\frac{J}{2} = 2v_1 = v_2 \qquad \text{...(i)}$$

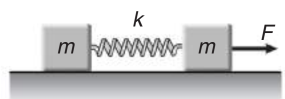
$$J \cos 30° = m\,(v_3 + v_0)$$

or $$\frac{\sqrt{3}}{2} J = (v_3 + 2) \qquad \ldots\text{(ii)}$$

So these are two equations relating J and velocity components of wedge and ball.

Further, net vertical impulse on wedge should be zero. Therefore, impulse on wedge from ground is $J \sin 30°$ or $\frac{J}{2}$ in upward direction.

Example 20 *Two blocks of equal mass m are connected by an unstretched spring and the system is kept at rest on a frictionless horizontal surface. A constant force F is applied on one of the blocks pulling it away from the other as shown in figure. (a) Find the displacement of the centre of mass at time t. (b) If the extension of the spring is x_0 at time t, find the displacement of the two blocks at this instant.*



Solution (a) The acceleration of the centre of mass is

$$a_{\text{COM}} = \frac{F}{2m}$$

The displacement of the centre of mass at time t will be

$$x = \frac{1}{2} a_{\text{COM}}\, t^2 = \frac{Ft^2}{4m} \qquad \textbf{Ans.}$$

(b) Suppose the displacement of the first block is x_1 and that of the second is x_2. Then,

$$x = \frac{mx_1 + mx_2}{2m} \quad \text{or} \quad \frac{Ft^2}{4m} = \frac{x_1 + x_2}{2}$$

or $$x_1 + x_2 = \frac{Ft^2}{2m} \qquad \ldots\text{(i)}$$

Further, the extension of the spring is $x_1 - x_2$. Therefore,

$$x_1 - x_2 = x_0$$

From Eqs. (i) and (ii),

$$x_1 = \frac{1}{2}\left(\frac{Ft^2}{2m} + x_0\right) \quad \text{and} \quad x_2 = \frac{1}{2}\left(\frac{Ft^2}{2m} - x_0\right).$$

Example 21 *A block of mass m is connected to another block of mass M by a massless spring of spring constant k. The blocks are kept on a smooth horizontal plane. Initially, the blocks are at rest and the spring is unstretched when a constant force F starts acting on the block of mass M to pull it. Find the maximum extension of the spring.*

Solution The centre of mass of the system (two blocks + spring) moves with an acceleration $a = \dfrac{F}{m + M}$. Let us solve the problem in a frame of reference fixed to the centre of mass of the system. As this frame is accelerated with respect to the ground, we have to apply a pseudo force ma towards left on the block of mass m and Ma towards left on the block of mass M. The net external force on m is

$$F_1 = ma = \frac{mF}{m + M} \qquad \text{(towards left)}$$

and the net external force on M is

$$F_2 = F - Ma = F - \frac{MF}{m + M} = \frac{mF}{m + M} \qquad \text{(towards right)}$$

As the centre of mass is at rest in this frame, the blocks move in opposite directions and come to instantaneous rest at some instant. The extension of the spring will be maximum at this instant. Suppose, the left block is displaced through a distance x_1 and the right block through a distance x_2 from the initial positions. The total work done by the external forces F_1 and F_2 in this period are

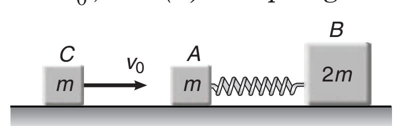
$$W = F_1 x_1 + F_2 x_2 = \frac{mF}{m + M}(x_1 + x_2)$$

This should be equal to the increase in the potential energy of the spring, as there is no change in the kinetic energy. Thus,

$$\frac{mF}{m + M}(x_1 + x_2) = \frac{1}{2}k(x_1 + x_2)^2 \quad \text{or} \quad x_1 + x_2 = \frac{2mF}{k(m + M)} \qquad \textbf{Ans.}$$

This is the maximum extension of the spring.

Example 22 *Two blocks A and B of masses m and 2m respectively are placed on a smooth floor. They are connected by a spring. A third block C of mass m moves with a velocity v_0 along the line joining A and B and collides elastically with A, as shown in figure. At a certain instant of time t_0 after collision, it is found that the instantaneous velocities of A and B are the same. Further, at this instant the compression of the spring is found to be x_0. Determine (i) the common velocity of A and B at time t_0, and (ii) the spring constant.*



Solution Initially, the blocks A and B are at rest and C is moving with velocity v_0 to the right. As masses of C and A are same and the collision is elastic the body C transfers its whole momentum mv_0 to body A and as a result the body C stops and A starts moving with velocity v_0 to the right. At this instant the spring is uncompressed and the body B is still at rest.

The momentum of the system at this instant $= mv_0$

Now, the spring is compressed and the body B comes in motion. After time t_0, the compression of the spring is x_0 and common velocity of A and B is v (say).

As external force on the system is zero, the law of conservation of linear momentum gives

$$mv_0 = mv + (2m)v \quad \text{or} \quad v = \frac{v_0}{3} \qquad \textbf{Ans.}$$

The law of conservation of energy gives

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + \frac{1}{2}(2m)v^2 + \frac{1}{2}kx_0^2$$

or
$$\frac{1}{2}mv_0^2 = \frac{3}{2}mv^2 + \frac{1}{2}kx_0^2$$

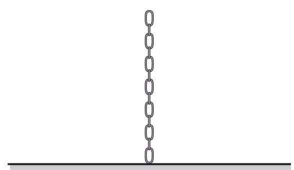
$$\frac{1}{2}mv_0^2 = \frac{3}{2}m\left(\frac{v_0}{3}\right)^2 + \frac{1}{2}kx_0^2$$

∴
$$\frac{1}{2}kx_0^2 = \frac{1}{2}mv_0^2 - \frac{1}{6}mv_0^2$$

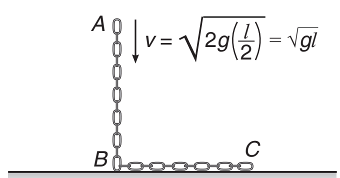
or
$$\frac{1}{2}kx_0^2 = \frac{1}{3}mv_0^2$$

$$k = \frac{2}{3}\frac{mv_0^2}{x_0^2}$$ **Ans.**

Example 23 *A uniform chain of mass m and length l hangs on a thread and touches the surface of a table by its lower end. Find the force exerted by the table on the chain when half of its length has fallen on the table. The fallen part does not form heap.*



Solution Force exerted by the chain on the table consists of two parts:



1. Weight of the portion BC of the chain lying on the table,

$$W = \frac{mg}{2} \quad \text{(downwards)}$$

2. Thrust force $F_t = \lambda v^2$

Here, λ = mass per unit length of chain $= \dfrac{m}{l}$

$$v^2 = (\sqrt{gl})^2 = gl$$

∴
$$F_t = \left(\frac{m}{l}\right)(gl) = mg \qquad \text{(downwards)}$$

∴ Net force exerted by the chain on the table is

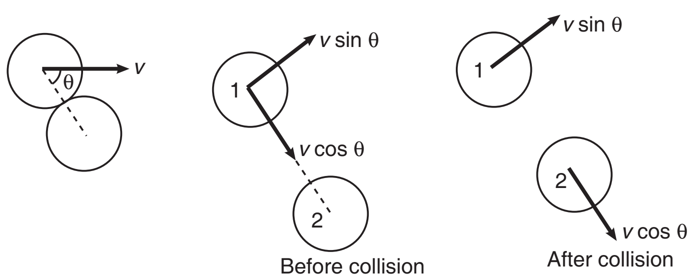
$$F = W + F_t = \frac{mg}{2} + mg = \frac{3}{2}mg \qquad \text{(downwards)}$$

So, from Newton's third law the force exerted by the table on the chain will be $\frac{3}{2}mg$ (vertically upwards).

Note *Here, the thrust force (F_t) applied by the chain on the table will be vertically downwards, as* $F_t = v_r\left(\frac{dm}{dt}\right)$ *and in this expression v_r is downwards plus $\frac{dm}{dt}$ is positive. So, F_t will be downwards.*

Example 24 *A ball of mass m makes an elastic collision with another identical ball at rest. Show that if the collision is oblique, the bodies go at right angles to each other after collision.*

Solution In head on elastic collision between two particles, they exchange their velocities. In this case, the component of ball 1 along common normal direction, $v \cos\theta$ becomes zero after collision, while that of 2 becomes $v \cos\theta$. While the components along common tangent direction of both the balls remain unchanged. Thus, the components along common tangent and common normal direction of both the balls in tabular form are given below.



Ball	Component along common tangent direction		Component along common normal direction	
	Before collision	After collision	Before collision	After collision
1	$v \sin\theta$	$v \sin\theta$	$v \cos\theta$	0
2	0	0	0	$v \cos\theta$

From the above table and figure, we see that both the balls move at right angles after collision with velocities $v \sin\theta$ and $v \cos\theta$.

Exercises

LEVEL 1

Assertion and Reason

Directions : *Choose the correct option.*

(*a*) *If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion.***

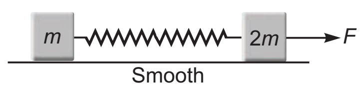
(*b*) *If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion.***

(*c*) *If **Assertion** is true, but the **Reason** is false.*

(*d*) *If **Assertion** is false but the **Reason** is true.*

1. Assertion : Centre of mass of a rigid body always lies inside the body.
Reason : Centre of mass and centre of gravity coincide if gravity is uniform.

2. Assertion : A constant force F is applied on two blocks and one spring system as shown in figure. Velocity of centre of mass increases linearly with time.



Reason : Acceleration of centre of mass is constant.

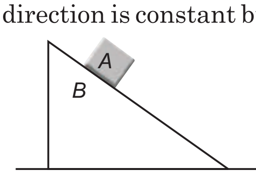
3. Assertion : To conserve linear momentum of a system, no force should act on the system.
Reason : If net force on a system is zero, its linear momentum should remain constant.

4. Assertion : A rocket moves forward by pushing the surrounding air backwards.
Reason : It derives the necessary thrust to move forward according to Newton's third law of motion.

5. Assertion : Internal forces cannot change linear momentum.
Reason : Internal forces can change the kinetic energy of a system.

6. Assertion : In case of bullet fired from gun, the ratio of kinetic energy of gun and bullet is equal to ratio of mass of bullet and gun.
Reason : Kinetic energy $\propto \frac{1}{\text{mass}}$; if momentum is constant.

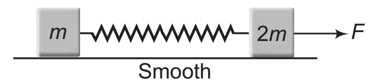
7. Assertion : All surfaces shown in figure are smooth. System is released from rest. Momentum of system in horizontal direction is constant but overall momentum is not constant.



Reason : A net vertically upward force is acting on the system.

8. Assertion : During head on collision between two bodies let Δp_1 is change in momentum of first body and Δp_2 the change in momentum of the other body, then $\Delta p_1 = \Delta p_2$.
Reason : Total momentum of the system should remain constant.

9. Assertion : In the system shown in figure spring is first stretched then left to oscillate. At some instant kinetic energy of mass m is K. At the same instant kinetic energy of mass $2m$ should be $\frac{K}{2}$.



Reason : Their linear momenta are equal and opposite and $K = \frac{p^2}{2m}$ or $K \propto \frac{1}{m}$.

10. Assertion : Energy can not be given to a system without giving it momentum.

Reason : If kinetic energy is given to a body it means it has acquired momentum.

11. Assertion : The centre mass of an electron and proton, when released moves faster towards proton.

Reason : Proton is heavier than electron.

12. Assertion : The relative velocity of the two particles in head-on elastic collision is unchanged both in magnitude and direction.

Reason : The relative velocity is unchanged in magnitude but gets reversed in direction.

13. Assertion : An object of mass m_1 and another of mass $m_2 (m_2 > m_1)$ are released from certain distance. The objects move towards each other under the gravitational force between them. In this motion, centre of mass of their system will continuously move towards the heavier mass m_2.

Reason : In a system of a heavier and a lighter mass, centre of mass lies closer to the heavier mass.

14. Assertion : A given force applied in turn to a number of different masses may cause the same rate of change in momentum in each but not the same acceleration to all.

Reason : $\mathbf{F} = \frac{d\mathbf{p}}{dt}$ and $\mathbf{a} = \frac{\mathbf{F}}{m}$

15. Assertion : In an elastic collision between two bodies, the relative speed of the bodies after collision is equal to the relative speed before the collision.

Reason : In an elastic collision, the linear momentum of the system is conserved.

Objective Questions

Single Correct Option

1. A ball is dropped from a height of 10 m. Ball is embedded in sand through 1 m and stops.
(a) only momentum remains conserved
(b) only kinetic energy remains conserved
(c) both momentum and kinetic energy are conserved
(d) neither kinetic energy nor momentum is conserved

2. If no external force acts on a system
(a) velocity of centre of mass remains constant
(b) position of centre of mass remains constant
(c) acceleration of centre of mass remains non-zero and constant
(d) All of the above

3. When two blocks connected by a spring move towards each other under mutual interaction
(a) their velocities are equal
(b) their accelerations are equal
(c) the force acting on them are equal and opposite
(d) All of the above

4. If two balls collide in air while moving vertically, then momentum of the system is conserved because
(a) gravity does not affect the momentum of the system
(b) force of gravity is very less compared to the impulsive force
(c) impulsive force is very less than the gravity
(d) gravity is not acting during collision

5. When a cannon shell explodes in mid air, then identify the **incorrect** statement
(a) the momentum of the system is conserved at the time of explosion
(b) the kinetic energy of the system always increases
(c) the trajectory of centre of mass remains unchanged
(d) None of the above

6. In an inelastic collision
(a) momentum of the system is always conserved
(b) velocity of separation is less than the velocity of approach
(c) the coefficient of restitution can be zero
(d) All of the above

7. The momentum of a system is defined
(a) as the product of mass of the system and the velocity of centre of mass
(b) as the vector sum of the momentum of individual particles
(c) for bodies undergoing translational, rotational and oscillatory motion
(d) All of the above

8. The momentum of a system with respect to centre of mass
(a) is zero only if the system is moving uniformly
(b) is zero only if no external force acts on the system
(c) is always zero
(d) can be zero in certain conditions

9. Three identical particles are located at the vertices of an equilateral triangle. Each particle moves along a meridian with equal speed towards the centroid and collides inelastically.
(a) all the three particles will bounce back along the meridians with lesser speed.
(b) all the three particles will become stationary
(c) all the particles will continue to move in their original directions but with lesser speed
(d) nothing can be said

10. The average resisting force that must act on a 5 kg mass to reduce its speed from 65 to 15 ms^{-1} in 2s is
(a) 12.5 N (b) 125 N (c) 1250 N (d) None of these

11. In a carbon monoxide molecule, the carbon and the oxygen atoms are separated by a distance 1.2×10^{-10} m. The distance of the centre of mass from the carbon atom is
(a) 0.48×10^{-10} m (b) 0.51×10^{-10} m
(c) 0.74×10^{-10} m (d) 0.68×10^{-10} m

12. A bomb of mass 9 kg explodes into two pieces of masses 3 kg and 6 kg. The velocity of mass 3 kg is 16 ms^{-1}. The kinetic energy of mass 6 kg is

(a) 96 J (b) 384 J (c) 192 J (d) 768 J

13. A heavy ball moving with speed v collides with a tiny ball. The collision is elastic, then immediately after the impact, the second ball will move with a speed approximately equal to

(a) v (b) $2v$ (c) $\frac{v}{2}$ (d) $\frac{v}{3}$

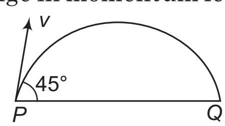
14. A loaded 20, 000 kg coal wagon is moving on a level track at 6 ms^{-1}. Suddenly 5000 kg of coal is dropped out of the wagon. The final speed of the wagon is

(a) 6 ms^{-1} (b) 8 ms^{-1} (c) 4.8 ms^{-1} (d) 4.5 ms^{-1}

15. A machine gun shoots a 40 g bullet at a speed of 1200 ms^{-1}. The man operating the gun can bear a maximum force of 144 N. The maximum number of bullets shot per second is

(a) 3 (b) 5 (c) 6 (d) 9

16. A projectile of mass m is fired with a velocity v from point P at an angle 45°. Neglecting air resistance, the magnitude of the change in momentum leaving the point P and arriving at Q is



(a) $\sqrt{2}\,mv$ (b) $2mv$ (c) $\frac{mv}{2}$ (d) $\frac{mv}{\sqrt{2}}$

17. A ball after freely falling from a height of 4.9 m strikes a horizontal plane. If the coefficient of restitution is 3/4, the ball will strike second time with the plane after

(a) $\frac{1}{2}$ s (b) 1 s (c) $\frac{3}{2}$ s (d) $\frac{3}{4}$ s

18. The centre of mass of a non uniform rod of length L, whose mass per unit length varies as $\rho = \frac{k \cdot x^2}{L}$ where k is a constant and x is the distance of any point from one end is (from the same end)

(a) $\left(\frac{3}{4}\right)L$ (b) $\left(\frac{1}{4}\right)L$ (c) $\left(\frac{1}{6}\right)L$ (d) $\left(\frac{2}{3}\right)L$

19. A boat of length 10 m and mass 450 kg is floating without motion in still water. A man of mass 50 kg standing at one end of it walks to the other end of it and stops. The magnitude of the displacement of the boat in metres relative to ground is

(a) zero (b) 1 m (c) 2 m (d) 5 m

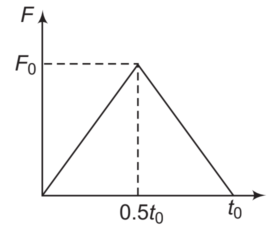
20. A man of mass M stands at one end of a stationary plank of length L, lying on a smooth surface. The man walks to the other end of the plank. If the mass of the plank is $M/3$, the distance that the man moves relative to the ground is

(a) $\frac{3L}{4}$ (b) $\frac{L}{4}$ (c) $\frac{4L}{5}$ (d) $\frac{L}{3}$

21. A ball of mass m moving at a speed v collides with another ball of mass $3m$ at rest. The lighter block comes to rest after collision. The coefficient of restitution is

(a) $\frac{1}{2}$ (b) $\frac{2}{3}$ (c) $\frac{1}{4}$ (d) None of these

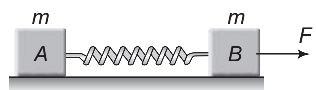
22. A particle of mass m moving with velocity u makes an elastic one-dimensional collision with a stationary particle of mass m. They come in contact for a very small time t_0. Their force of interaction increases from zero to F_0 linearly in time $0.5t_0$, and decreases linearly to zero in further time $0.5\,t_0$ as shown in figure. The magnitude of F_0 is



(a) $\dfrac{mu}{t_0}$ (b) $\dfrac{2mu}{t_0}$

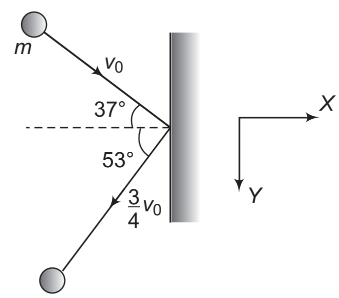
(c) $\dfrac{mu}{2t_0}$ (d) None of these

23. Two identical blocks A and B of mass m joined together with a massless spring as shown in figure are placed on a smooth surface. If the block A moves with an acceleration a_0, then the acceleration of the block B is



(a) a_0 (b) $-a_0$ (c) $\dfrac{F}{m} - a_0$ (d) $\dfrac{F}{m}$

24. A ball of mass m moving with velocity v_0 collides a wall as shown in figure. After impact it rebounds with a velocity $\dfrac{3}{4}v_0$. The impulse acting on ball during impact is



(a) $-\dfrac{m}{2}v_0\,\hat{\mathbf{j}}$

(b) $-\dfrac{3}{4}mv_0\,\hat{\mathbf{i}}$

(c) $\dfrac{-5}{4}mv_0\,\hat{\mathbf{i}}$

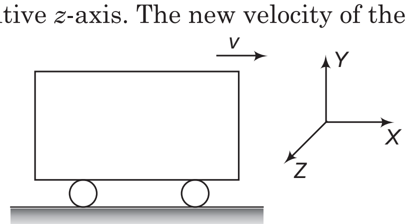
(d) None of the above

25. A steel ball is dropped on a hard surface from a height of 1 m and rebounds to a height of 64 cm. The maximum height attained by the ball after n^{th} bounce is (in m)

(a) $(0.64)^{2n}$ (b) $(0.8)^{2n}$

(c) $(0.5)^{2n}$ (d) $(0.8)^n$

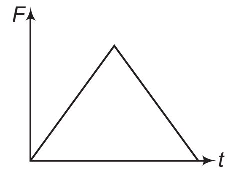
26. A car of mass 500 kg (including the mass of a block) is moving on a smooth road with velocity 1.0 ms^{-1} along positive x-axis. Now a block of mass 25 kg is thrown outside with absolute velocity of 20 ms^{-1} along positive z-axis. The new velocity of the car is (ms^{-1})

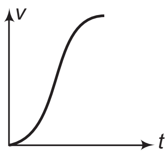


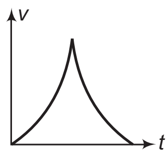
(a) $10\,\hat{\mathbf{i}} + 20\,\hat{\mathbf{k}}$ (b) $10\,\hat{\mathbf{i}} - 20\,\hat{\mathbf{k}}$

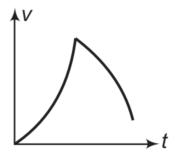
(c) $\dfrac{20}{19}\hat{\mathbf{i}} - \dfrac{20}{19}\hat{\mathbf{k}}$ (d) $10\,\hat{\mathbf{i}} - \dfrac{20}{19}\hat{\mathbf{k}}$

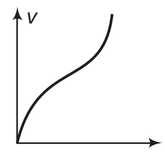
27. The net force acting on a particle moving along a straight line varies with time as shown in the diagram. Force is parallel to velocity. Which of the following graph is best representative of its speed with time ? (Initial velocity of the particle is zero)



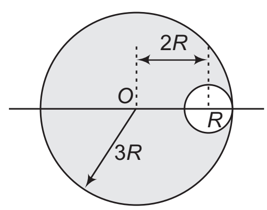
(a) 

(b) 

(c) 

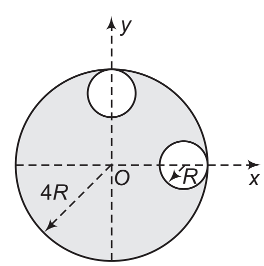
(d) 

28. In the figure shown, find out centre of mass of a system of a uniform circular plate of radius $3R$ from O in which a hole of radius R is cut whose centre is at $2R$ distance from the centre of large circular plate



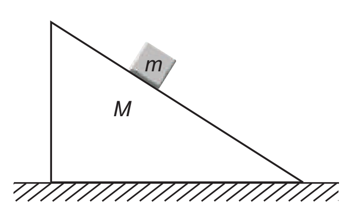
(a) $\frac{R}{2}$
(b) $\frac{R}{5}$
(c) $\frac{R}{4}$
(d) None of these

29. From the circular disc of radius $4R$ two small discs of radius R are cut off. The centre of mass of the new structure will be at



(a) $\hat{\mathbf{i}}\frac{R}{5} + \hat{\mathbf{j}}\frac{R}{5}$
(b) $-\hat{\mathbf{i}}\frac{R}{5} + \hat{\mathbf{j}}\frac{R}{5}$
(c) $-\hat{\mathbf{i}}\frac{R}{5} - \hat{\mathbf{j}}\frac{R}{5}$
(d) None of the above

30. A block of mass m rests on a stationary wedge of mass M. The wedge can slide freely on a smooth horizontal surface as shown in figure. If the block starts from rest



(a) the position of the centre of mass of the system will change
(b) the position of the centre of mass of the system will change along the vertical but not along the horizontal
(c) the total energy of the system will remain constant
(d) All of the above

31. A bullet of mass m hits a target of mass M hanging by a string and gets embedded in it. If the block rises to a height h as a result of this collision, the velocity of the bullet before collision is

(a) $v = \sqrt{2gh}$ (b) $v = \sqrt{2gh}\left[1 + \frac{m}{M}\right]$ (c) $v = \sqrt{2gh}\left[1 + \frac{M}{m}\right]$ (d) $v = \sqrt{2gh}\left[1 - \frac{m}{M}\right]$

32. A loaded spring gun of mass M fires a bullet of mass m with a velocity v at an angle of elevation θ. The gun is initially at rest on a horizontal smooth surface. After firing, the centre of mass of the gun and bullet system

(a) moves with velocity $\frac{v}{M}m$

(b) moves with velocity $\frac{vm}{M\cos\theta}$ in the horizontal direction

(c) does not move in horizontal direction

(d) moves with velocity $\frac{v(M-m)}{M+m}$ in the horizontal direction

33. Two bodies with masses m_1 and m_2 ($m_1 > m_2$) are joined by a string passing over fixed pulley. Assume masses of the pulley and thread negligible. Then the acceleration of the centre of mass of the system ($m_1 + m_2$) is

(a) $\left(\frac{m_1 - m_2}{m_1 + m_2}\right)g$ (b) $\left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2 g$

(c) $\frac{m_1 g}{m_1 + m_2}$ (d) $\frac{m_2 g}{m_1 + m_2}$

34. A rocket of mass m_0 has attained a speed equal to its exhaust speed and at that time the mass of the rocket is m. Then the ratio $\frac{m_0}{m}$ is (neglect gravity)

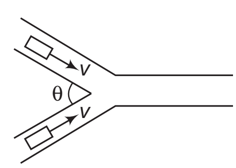
(a) 2.718 (b) 7.8 (c) 3.14 (d) 4

35. A jet of water hits a flat stationary plate perpendicular to its motion. The jet ejects 500g of water per second with a speed of 1 m/s. Assuming that after striking, the water flows parallel to the plate, then the force exerted on the plate is

(a) 5 N (b) 1.0 N

(c) 0.5 N (d) 10 N

36. Two identical vehicles are moving with same velocity v towards an intersection as shown in figure. If the collision is completely inelastic, then



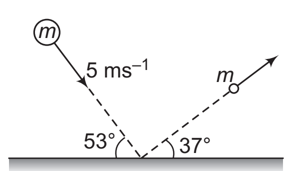
(a) the velocity of separation is zero

(b) the velocity of approach is $2v\sin\frac{\theta}{2}$

(c) the common velocity after collision is $v\cos\frac{\theta}{2}$

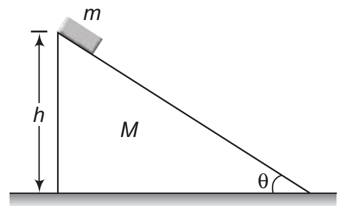
(d) All of the above

37. A ball of mass $m = 1$ kg strikes a smooth horizontal floor as shown in figure. The impulse exerted on the floor is



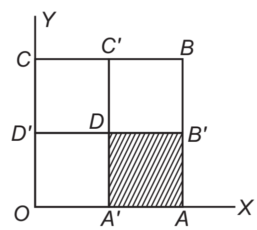
(a) 6.25 Ns (b) 1.76 Ns (c) 7.8 Ns (d) 2.2 Ns

38. A small block of mass m is placed at rest on the top of a smooth wedge of mass M, which in turn is placed at rest on a smooth horizontal surface as shown in figure. It h be the height of wedge and θ is the inclination, then the distance moved by the wedge as the block reaches the foot of the wedge is



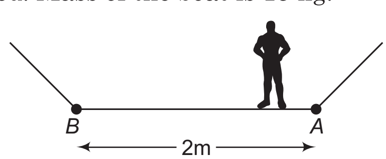
(a) $\frac{Mh\cot\theta}{M+m}$ (b) $\frac{mh\cot\theta}{M+m}$ (c) $\frac{Mh\,\text{cosec}\theta}{M+m}$ (d) $\frac{mh\,\text{cosec}\theta}{M+m}$

39. A square of side 4 cm and uniform thickness is divided into four squares. The square portion $A'AB'D$ is removed and the removed portion is placed over the portion $DB'BC'$. The new position of centre of mass is



(a) (2 cm, 2 cm) (b) (2 cm, 3 cm) (c) (2 cm, 2.5 cm) (d) (3 cm, 3 cm)

40. A boy having a mass of 40 kg stands at one end A of a boat of length 2 m at rest. The boy walks to the other end B of the boat and stops. What is the distance moved by the boat? Friction exists between the feet of the boy and the surface of the boat. But the friction between the boat and the water surface may be neglected. Mass of the boat is 15 kg.



(a) 0.49 m (b) 2.46 m (c) 1.46 m (d) 3.2 m

41. Three identical particles with velocities $v_0\hat{\mathbf{i}}, -3v_0\hat{\mathbf{j}}$ and $5v_0\hat{\mathbf{k}}$ collide successively with each other in such a way that they form a single particle. The velocity vector of resultant particle is

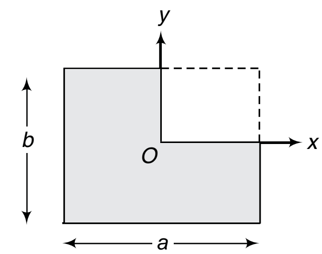
(a) $\frac{v_0}{3}(\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})$ (b) $\frac{v_0}{3}(\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}})$ (c) $\frac{v_0}{3}(\hat{\mathbf{i}}-3\hat{\mathbf{j}}+\hat{\mathbf{k}})$ (d) $\frac{v_0}{3}(\hat{\mathbf{i}}-3\hat{\mathbf{j}}+5\hat{\mathbf{k}})$

42. A mortar fires a shell of mass M which explodes into two pieces of mass $\frac{M}{5}$ and $\frac{4M}{5}$ at the top of the trajectory. The smaller mass falls very close to the mortar. In the same time the bigger piece lands a distance D from the mortar. The shell would have fallen at a distance R from the mortar if there was no explosion. The value of D is (neglect air resistance)

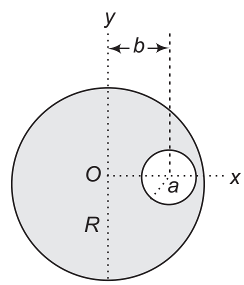
(a) $\frac{3R}{2}$ (b) $\frac{4R}{3}$ (c) $\frac{5R}{4}$ (d) None of these

Subjective Questions

1. Consider a rectangular plate of dimensions $a \times b$. If this plate is considered to be made up of four rectangles of dimensions $\frac{a}{2} \times \frac{b}{2}$ and we now remove one out of four rectangles. Find the position where the centre of mass of the remaining system will be zero.



2. The uniform solid sphere shown in the figure has a spherical hole in it. Find the position of its centre of mass.



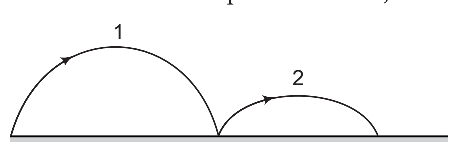
3. A gun fires a bullet. The barrel of the gun is inclined at an angle of 45° with horizontal. When the bullet leaves the barrel it will be travelling at an angle greater than 45° with the horizontal. Is this statement true or false?

4. Two blocks A and B of masses m_A and m_B are connected together by means of a spring and are resting on a horizontal frictionless table. The blocks are then pulled apart so as to stretch the spring and then released. Show that the kinetic energies of the blocks are, at any instant inversely proportional to their masses.

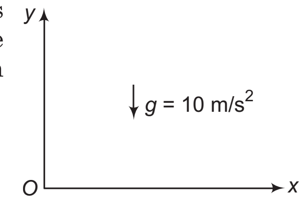
5. Show that in a head on elastic collision between two particles, the transference of energy is maximum when their mass ratio is unity.

6. A particle moving with kinetic energy K makes a head on elastic collision with an identical particle at rest. Find the maximum elastic potential energy of the system during collision.

7. A ball is projected from the ground at some angle with horizontal. Coefficient of restitution between the ball and the ground is e. Let a, b and c be the ratio of times of flight, horizontal range and maximum height in two successive paths. Find a, b and c in terms of e.



8. x-y is the vertical plane as shown in figure. A particle of mass 1 kg is at (10 m, 20 m) at time $t = 0$. It is released from rest. Another particle of mass 2 kg is at (20 m, 40 m) at the same instant. It is projected with velocity $(10\hat{\mathbf{i}} + 10\hat{\mathbf{j}})$ m/s. After 1 s. Find



(a) acceleration,
(b) velocity and
(c) position of their centre of mass.

9. A system consists of two particles. At $t = 0$, one particle is at the origin; the other, which has a mass of 0.60 kg, is on the y-axis at $y = 80$ m. At $t = 0$, the centre of mass of the system is on the y-axis at $y = 24$ m and has a velocity given by $(6.0 \text{ m/s}^3)\, t^2 \hat{\mathbf{j}}$.

(a) Find the total mass of the system.
(b) Find the acceleration of the centre of mass at any time t.
(c) Find the net external force acting on the system at $t = 3.0$ s.

10. A particle of mass 2 kg moving with a velocity $5\hat{\mathbf{i}}$ m/s collides head-on with another particle of mass 3 kg moving with a velocity $-2\hat{\mathbf{i}}$ m/s. After the collision the first particle has speed of 1.6 m/s in negative x direction. Find

(a) velocity of the centre of mass after the collision,
(b) velocity of the second particle after the collision,
(c) coefficient of restitution.

11. A rocket of mass 40 kg has 160 kg fuel. The exhaust velocity of the fuel is 2.0 km/s. The rate of consumption of fuel is 4 kg/s. Calculate the ultimate vertical speed gained by the rocket. $(g = 10 \text{ m/s}^2)$

12. A boy of mass 60 kg is standing over a platform of mass 40 kg placed over a smooth horizontal surface. He throws a stone of mass 1 kg with velocity $v = 10$ m/s at an angle of 45° with respect to the ground. Find the displacement of the platform (with boy) on the horizontal surface when the stone lands on the ground. Take $g = 10 \text{ m/s}^2$.

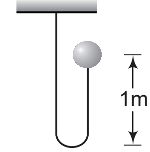
13. A man of mass m climbs to a rope ladder suspended below a balloon of mass M. The balloon is stationary with respect to the ground.

(a) If the man begins to climb the ladder at speed v (with respect to the ladder), in what direction and with what speed (with respect to the ground) will the balloon move ?
(b) What is the state of the motion after the man stops climbing ?

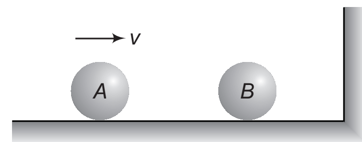
14. Find the mass of the rocket as a function of time, if it moves with a constant acceleration a, in absence of external forces. The gas escapes with a constant velocity u relative to the rocket and its mass initially was m_0.

15. A particle of mass $2m$ is projected at an angle of 45° with horizontal with a velocity of $20\sqrt{2}$ m/s. After 1 s explosion takes place and the particle is broken into two equal pieces. As a result of explosion one part comes to rest. Find the maximum height attained by the other part. Take $g = 10 \text{ m/s}^2$.

16. A ball of mass 1 kg is attached to an inextensible string. The ball is released from the position shown in figure. Find the impulse imparted by the string to the ball immediately after the string becomes taut. (Take $g = 10 \text{ m/s}^2$)



17. Two balls shown in figure are identical. Ball A is moving towards right with a speed v and the second ball is at rest. Assume all collisions to be elastic. Show that the speeds of the balls remain unchanged after all the collisions have taken place.



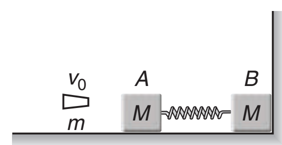
18. A particle of mass 0.1 kg moving at an initial speed v collides with another particle of same mass kept initially at rest. If the total energy becomes 0.2 J after the collision, what would be the minimum and maximum values of v ?

19. A particle of mass m moving with a speed v hits elastically another stationary particle of mass $2\,m$ on a smooth horizontal circular tube of radius r. Find the time when the next collision will take place?

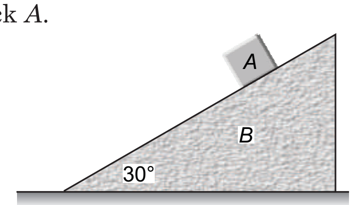
20. In a one-dimensional collision between two identical particles A and B, B is stationary and A has momentum p before impact. During impact, B gives an impulse J to A. Find the coefficient of restitution between A and B ?

21. Two billiard balls of same size and mass are in contact on a billiard table. A third ball of same mass and size strikes them symmetrically and remains at rest after the impact. Find the coefficient of restitution between the balls?

22. Two identical blocks each of mass $M = 9\,kg$ are placed on a rough horizontal surface of frictional coefficient $\mu = 0.1$. The two blocks are joined by a light spring and block B is in contact with a vertical fixed wall as shown in figure. A bullet of mass $m = 1$ kg and $v_0 = 10$ m/s hits block A and gets embedded in it. Find the maximum compression of spring. (Spring constant = 240 N/m, $g = 10$ m/s^2)



23. Block A has a mass of 5 kg and is placed on top of a smooth triangular block, B having a mass of 30 kg. If the system is released from rest, determine the distance, B moves when A reaches the bottom. Neglect the size of block A.



24. A trolley was moving horizontally on a smooth ground with velocity v with respect to the earth. Suddenly a man starts running from rear end of the trolley with a velocity (3/2) v with respect to the trolley. After reaching the other end, the man turns back and continues running with a velocity (3/2) v with respect to trolley in opposite direction. If the length of the trolley is L, find the displacement of the man with respect to earth when he reaches the starting point on the trolley. Mass of the trolley is equal to the mass of the man.

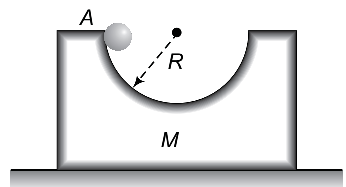
25. A 4.00 g bullet travelling horizontally with a velocity of magnitude 500 m/s is fired into a wooden block with a mass of 1.00 kg, initially at rest on a level surface. The bullet passes through the block and emerges with speed 100 m/s. The block slides a distance of 0.30 m along the surface from its initial position.

(a) What is the coefficient of kinetic friction between block and surface ?
(b) What is the decrease in kinetic energy of the bullet ?
(c) What is the kinetic energy of the block at the instant after the bullet has passed through it ? Neglect friction during collision of bullet with the block.

26. A bullet of mass 0.25 kg is fired with velocity 302 m/s into a block of wood of mass $m_1 = 37.5$ kg. It gets embedded into it. The block m_1 is resting on a long block m_2 and the horizontal surface on which it is placed is smooth. The coefficient of friction between m_1 and m_2 is 0.5. Find the displacement of m_1 on m_2 and the common velocity of m_1 and m_2. Mass $m_2 = 1.25$ kg.

27. A wagon of mass M can move without friction along horizontal rails. A simple pendulum consisting of a sphere of mass m is suspended from the ceiling of the wagon by a string of length l. At the initial moment the wagon and the pendulum are at rest and the string is deflected through an angle α from the vertical. Find the velocity of the wagon when the pendulum passes through its mean position.

28. A block of mass M with a semicircular track of radius R rests on a horizontal frictionless surface shown in figure. A uniform cylinder of radius r and mass m is released from rest at the point A. The cylinder slips on the semicircular frictionless track. How far has the block moved when the cylinder reaches the bottom of the track ? How fast is the block moving when the cylinder reaches the bottom of the track ?

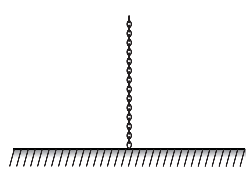


29. A ball of mass 50 g moving with a speed 2 m/s strikes a plane surface at an angle of incidence 45°. The ball is reflected by the plane at equal angle of reflection with the same speed. Calculate

(a) The magnitude of the change in momentum of the ball.

(b) The change in the magnitude of the momentum of the wall.

30. A uniform rope of mass m per unit length, hangs vertically from a support so that the lower end just touches the table top shown in figure. If it is released, show that at the time a length y of the rope has fallen, the force on the table is equivalent to the weight of a length $3y$ of the rope.



31. Sand drops from a stationary hopper at the rate of 5 kg/s on to a conveyor belt moving with a constant speed of 2 m/s. What is the force required to keep the belt moving and what is the power delivered by the motor, moving the belt ?

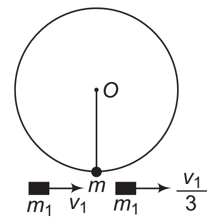
32. A 3.0 kg block slides on a frictionless horizontal surface, first moving to the left at 50 m/s. It collides with a spring as it moves left, compresses the spring and is brought to rest momentarily. The body continues to be accelerated to the right by the force of the compressed spring. Finally, the body moves to the right at 40 m/s. The block remains in contact with the spring for 0.020 s. What were the magnitude and direction of the impulse of the spring on the block? What was the spring's average force on the block?

33. Block A has a mass 3 kg and is sliding on a rough horizontal surface with a velocity $v_A = 2$ m/s when it makes a direct collision with block B, which has a mass of 2 kg and is originally at rest. The collision is perfectly elastic. Determine the velocity of each block just after collision and the distance between the blocks when they stop sliding. The coefficient of kinetic friction between the blocks and the plane is $\mu_k = 0.3$. (Take $g = 10\ \text{m/s}^2$)

LEVEL 2

Objective Questions

Single Correct Option



1. A pendulum consists of a wooden bob of mass m and length l. A bullet of mass m_1 is fired towards the pendulum with a speed v_1 and it emerges from the bob with speed $\frac{v_1}{3}$. The bob just completes motion along a vertical circle. Then v_1 is

(a) $\frac{m}{m_1}\sqrt{5gl}$ (b) $\frac{3m}{2m_1}\sqrt{5gl}$

(c) $\frac{2}{3}\left(\frac{m}{m_1}\right)\sqrt{5gl}$ (d) $\left(\frac{m_1}{m}\right)\sqrt{gl}$

2. A bob of mass m attached with a string of length l tied to a point on ceiling is released from a position when its string is horizontal. At the bottom most point of its motion, an identical mass m gently stuck to it. Find the maximum angle from the vertical to which it rotates.

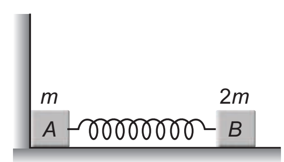
(a) $\cos^{-1}\left(\frac{2}{3}\right)$ (b) $\cos^{-1}\left(\frac{3}{4}\right)$

(c) $\cos^{-1}\left(\frac{1}{4}\right)$ (d) $60°$

3. A train of mass M is moving on a circular track of radius R with constant speed v. The length of the train is half of the perimeter of the track. The linear momentum of the train will be

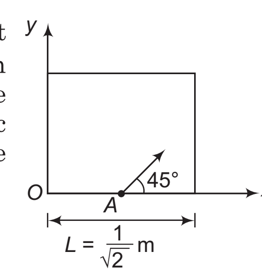
(a) zero (b) $\frac{2Mv}{\pi}$ (c) MvR (d) Mv

4. Two blocks A and B of mass m and $2\,m$ are connected together by a light spring of stiffness k. The system is lying on a smooth horizontal surface with the block A in contact with a fixed vertical wall as shown in the figure. The block B is pressed towards the wall by a distance x_0 and then released. There is no friction anywhere. If spring takes time Δt to acquire its natural length then average force on the block A by the wall is



(a) zero (b) $\frac{\sqrt{2mk}}{\Delta t}x_0$ (c) $\frac{\sqrt{mk}}{\Delta t}x_0$ (d) $\frac{\sqrt{3mk}}{\Delta t}x_0$

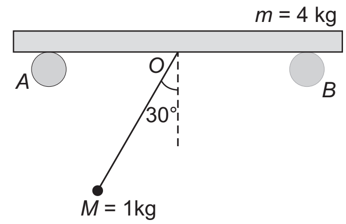
5. A striker is shot from a square carom board from a point A exactly at midpoint of one of the walls with a speed of 2 ms^{-1} at an angle of 45° with the x-axis as shown in the figure. The collisions of the striker with the walls of the fixed carom are perfectly elastic. The coefficient of kinetic friction between the striker and board is 0.2. The coordinate of the striker when it stops (taking point O to be the origin) is (in SI units)



(a) $\frac{1}{2\sqrt{2}}, \frac{1}{\sqrt{2}}$ (b) $0, \frac{1}{2\sqrt{2}}$

(c) $\frac{1}{2\sqrt{2}}, 0$ (d) $\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}$

6. A ball of mass 1 kg is suspended by an inextensible string 1 m long attached to a point O of a smooth horizontal bar resting on fixed smooth supports A and B. The ball is released from rest from the position when the string makes an angle 30° with the vertical. The mass of the bar is 4 kg. The displacement of bar when ball reaches the other extreme position (in m) is



(a) 0.4 (b) 0.2

(c) 0.25 (d) 0.5

7. A ball falls vertically onto a floor with momentum p and then bounces repeatedly. If coefficient of restitution is e, then the total momentum imparted by the ball to the floor is

(a) $p(1+e)$ (b) $\frac{p}{1-e}$

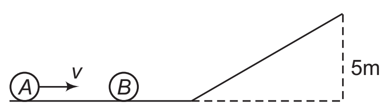
(c) $p\left(\frac{1-e}{1+e}\right)$ (d) $p\left(\frac{1+e}{1-e}\right)$

8. A bullet of mass m penetrates a thickness h of a fixed plate of mass M. If the plate was free to move, then the thickness penetrated will be

(a) $\frac{Mh}{M+m}$ (b) $\frac{2Mh}{M+m}$

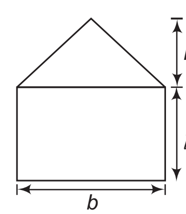
(c) $\frac{mh}{2(M+m)}$ (d) $\frac{Mh}{2(M+m)}$

9. Two identical balls of equal masses A and B, are lying on a smooth surface as shown in the figure. Ball A hits the ball B (which is at rest) with a velocity $v = 16\ \text{ms}^{-1}$. What should be the minimum value of coefficient of restitution e between A and B so that B just reaches the highest point of inclined plane? ($g = 10\ \text{ms}^{-2}$)



(a) $\frac{2}{3}$ (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) $\frac{1}{3}$

10. The figure shows a metallic plate of uniform thickness and density. The value of l in terms of L so that the centre of mass of the system lies at the interface of the triangular and rectangular portion is



(a) $l = \frac{L}{3}$ (b) $l = \frac{L}{2}$

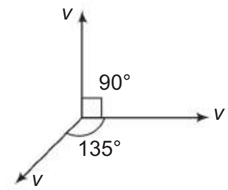
(c) $l = \frac{L}{\sqrt{3}}$ (d) $l = \sqrt{\frac{2}{3}}\,L$

11. Particle A makes a head on elastic collision with another stationary particle B. They fly apart in opposite directions with equal speeds. The mass ratio will be

(a) $\frac{1}{3}$ (b) $\frac{1}{2}$

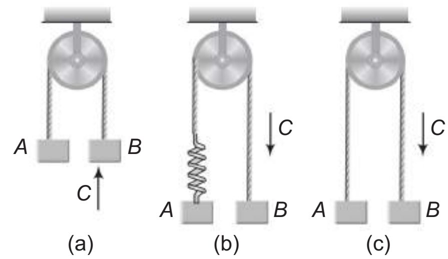
(c) $\frac{1}{4}$ (d) $\frac{2}{3}$

12. A particle of mass $4m$ which is at rest explodes into four equal fragments. All four fragments scattered in the same horizontal plane. Three fragments are found to move with velocity v as shown in the figure. The total energy released in the process is



(a) $mv^2(3-\sqrt{2})$
(b) $\frac{1}{2}mv^2(3-\sqrt{2})$
(c) $2mv^2$
(d) $\frac{1}{2}mv^2(1+\sqrt{2})$

13. In figures (a), (b) and (c) shown, the objects A, B and C are of same mass. String, spring and pulley are massless. C strikes B with velocity u in each case and sticks it. The ratio of velocity of B in case (a) to (b) to (c) is



(a) 1 : 1 : 1
(b) 3 : 3 : 2
(c) 3 : 2 : 2
(d) 1 : 2 : 3

14. A ladder of length L is slipping with its ends against a vertical wall and a horizontal floor. At a certain moment, the speed of the end in contact with the horizontal floor is v and the ladder makes an angle $\theta = 30°$ with horizontal. Then, the speed of the ladder's centre of mass must be

(a) $\frac{\sqrt{3}}{2}v$
(b) $\frac{v}{2}$
(c) v
(d) $2v$

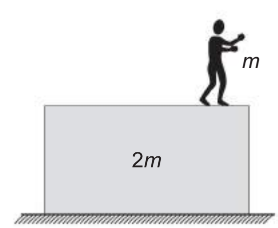
15. A body of mass 2 g, moving along the positive x-axis in gravity free space with velocity 20 cms^{-1} explodes at $x = 1$ m, $t = 0$ into two pieces of masses 2/3 g and 4/3 g. After 5s, the lighter piece is at the point (3m, 2m, − 4 m). Then the position of the heavier piece at this moment, in metres is

(a) (1.5, −1, −2)
(b) (1.5, −2, −2)
(c) (1.5, −1, −1)
(d) None of these

16. A body of mass m is dropped from a height of h. Simultaneously another body of mass $2m$ is thrown up vertically with such a velocity v that they collide at height $\frac{h}{2}$. If the collision is perfectly inelastic, the velocity of combined mass at the time of collision with the ground will be

(a) $\sqrt{\frac{5gh}{4}}$
(b) $\sqrt{gh}$
(c) $\sqrt{\frac{gh}{4}}$
(d) None of these

17. A man is standing on a cart of mass double the mass of man. Initially cart is at rest. Now, man jumps horizontally with velocity u relative to cart. Then work done by man during the process of jumping will be



(a) $\frac{mu^2}{2}$
(b) $\frac{3mu^2}{4}$
(c) mu^2
(d) None of these

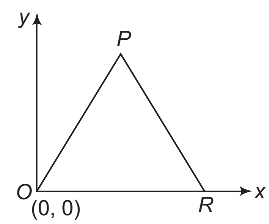
18. Two balls of equal mass are projected upwards simultaneously, one from the ground with initial velocity 50 ms^{-1} and the other from a 40m tower with initial velocity of 30 ms^{-1}. The maximum height attained by their COM will be

(a) 80 m (b) 60 m
(c) 100 m (d) 120 m

19. A particle of mass m and momentum $\mathbf{p}$ moves on a smooth horizontal table and collides directly and elastically with a similar particle (of mass m) having momentum $-2\,\mathbf{p}$. The loss (–) or gain (+) in the kinetic energy of the first particle in the collision is

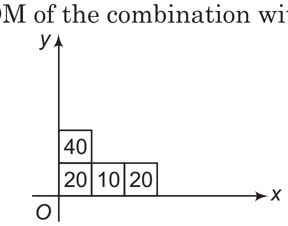
(a) $+\frac{p^2}{2m}$ (b) $-\frac{p^2}{4m}$ (c) $+\frac{3p^2}{2m}$ (d) zero

20. An equilateral triangular plate of mass $4m$ of side a is kept as shown. Consider two cases : (i) a point mass $4m$ is placed at the vertex P of the plate (ii) a point mass m is placed at the vertex R of the plate. In both cases the x-coordinate of centre of mass remains the same. Then x coordinate of centre of mass of the plate is



(a) $\frac{a}{3}$ (b) $\frac{a}{6}$
(c) $\frac{6a}{7}$ (d) $\frac{2a}{3}$

21. Four cubes of side a each of mass 40 g, 20 g, 10 g and 20 g are arranged in XY plane as shown in the figure. The coordinates of COM of the combination with respect to point O is

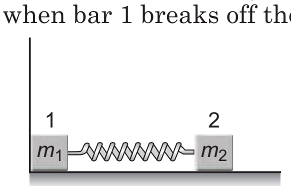


(a) $\frac{19a}{18}, \frac{17a}{18}$ (b) $\frac{17a}{18}, \frac{11a}{18}$ (c) $\frac{17a}{18}, \frac{13a}{18}$ (d) $\frac{13a}{18,}\ \frac{17a}{18}$

22. A particle of mass m_0, travelling at speed v_0, strikes a stationary particle of mass $2m_0$. As a result the particle of mass m_0 is deflected through 45° and has a final speed of $\frac{v_0}{\sqrt{2}}$. Then the speed of the particle of mass $2m_0$ after this collision is

(a) $\frac{v_0}{2}$ (b) $\frac{v_0}{2\sqrt{2}}$ (c) $\sqrt{2}v_0$ (d) $\frac{v_0}{\sqrt{2}}$

23. Two bars of masses m_1 and m_2, connected by a weightless spring of stiffness k, rest on a smooth horizontal plane. Bar 2 is shifted by a small distance x_0 to the left and released. The velocity of the centre of mass of the system when bar 1 breaks off the wall is



(a) $x_0\sqrt{\frac{km_2}{m_1+m_2}}$ (b) $\frac{x_0}{m_1+m_2}\sqrt{km_2}$ (c) $x_0k\frac{m_1+m_2}{m_2}$ (d) $x_0\frac{\sqrt{km_1}}{(m_1+m_2)}$

24. n elastic balls are placed at rest on a smooth horizontal plane which is circular at the ends with radius r as shown in the figure. The masses of the balls are $m, \frac{m}{2}, \frac{m}{2^2} \ldots\ldots\ldots\ldots \frac{m}{2^{n-1}}$ respectively. What is the minimum velocity which should be imparted to the first ball of mass m such that n^{th} ball completes the vertical circle

(a) $\left(\frac{3}{4}\right)^{n-1} \sqrt{5gr}$ (b) $\left(\frac{4}{3}\right)^{n-1} \sqrt{5gr}$

(c) $\left(\frac{3}{2}\right)^{n-1} \sqrt{5gr}$ (d) $\left(\frac{2}{3}\right)^{n-1} \sqrt{5gr}$

More than One Correct Options

1. A particle of mass m, moving with velocity v collides a stationary particle of mass $2m$. As a result of collision, the particle of mass m deviates by 45° and has final speed of $\frac{v}{2}$. For this situation mark out the correct statement (s).

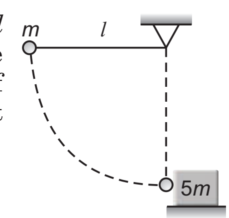
(a) The angle of divergence between particles after collision is $\frac{\pi}{2}$

(b) The angle of divergence between particles after collision is less than $\frac{\pi}{2}$

(c) Collision is elastic

(d) Collision is inelastic

2. A pendulum bob of mass m connected to the end of an ideal string of length l is released from rest from horizontal position as shown in the figure. At the lowest point the bob makes an elastic collision with a stationary block of mass $5m$, which is kept on a frictionless surface. Mark out the correct statement(s) for the instant just after the impact.



(a) Tension in the string is $\frac{17}{9} mg$

(b) Tension in the string is 3 mg.

(c) The velocity of the block is $\frac{\sqrt{2gl}}{3}$

(d) The maximum height attained by the pendulum bob after impact is (measured from the lowest position) $\frac{4l}{9}$

3. A particle of mass m strikes a horizontal smooth floor with a velocity u making an angle θ with the floor and rebound with velocity v making an angle ϕ with the floor. The coefficient of restitution between the particle and the floor is e. Then

(a) the impulse delivered by the floor to the body is $mu\,(1 + e)\sin\theta$

(b) $\tan\phi = e\tan\theta$

(c) $v = u\sqrt{1 - (1 - e^2)\sin^2\theta}$

(d) the ratio of the final kinetic energy to the initial kinetic energy is $\cos^2\theta + e^2\sin^2\theta$

4. A particle of mass m moving with a velocity $(3\hat{\mathbf{i}} + 2\hat{\mathbf{j}})\ \text{ms}^{-1}$ collides with another body of mass M and finally moves with velocity $(-2\hat{\mathbf{i}} + \hat{\mathbf{j}})\ \text{ms}^{-1}$. Then during the collision

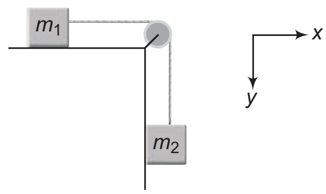
(a) impulse received by m is $m\ (5\hat{\mathbf{i}} + \hat{\mathbf{j}})$

(b) impulse received by m is $m\ (-5\hat{\mathbf{i}} - \hat{\mathbf{j}})$

(c) impulse received by M is $m\ (-5\hat{\mathbf{i}} - \hat{\mathbf{j}})$

(d) impulse received by M is $m\ (5\hat{\mathbf{i}} + \hat{\mathbf{j}})$

5. All surfaces shown in figure are smooth. System is released from rest. x and y components of acceleration of COM are



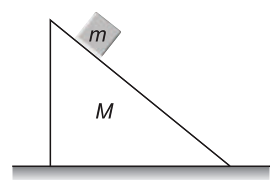
(a) $(a_{\text{cm}})_x = \dfrac{m_1 m_2 g}{m_1 + m_2}$

(b) $(a_{\text{cm}})_x = \dfrac{m_1 m_2 g}{(m_1 + m_2)^2}$

(c) $(a_{\text{cm}})_y = \left(\dfrac{m_2}{m_1 + m_2}\right)^2 g$

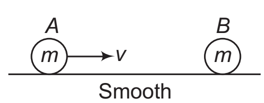
(d) $(a_{\text{cm}})_y = \left(\dfrac{m_2}{m_1 + m_2}\right) g$

6. A block of mass m is placed at rest on a smooth wedge of mass M placed at rest on a smooth horizontal surface. As the system is released



(a) the COM of the system remains stationary
(b) the COM of the system has an acceleration g vertically downward
(c) momentum of the system is conserved along the horizontal direction
(d) acceleration of COM is vertically downward $(a < g)$

7. In the figure shown, coefficient of restitution between A and B is $e = \dfrac{1}{2}$, then



(a) velocity of B after collision is $\dfrac{v}{2}$

(b) impulse between two during collision is $\dfrac{3}{4}mv$

(c) loss of kinetic energy during the collision is $\dfrac{3}{16}mv^2$

(d) loss of kinetic energy during the collision is $\dfrac{1}{4}mv^2$

8. In case of rocket propulsion, choose the correct options.
(a) Momentum of system always remains constant
(b) Newton's third law is applied
(c) If exhaust velocity and rate of burning of mass is kept constant, then acceleration of rocket will go on increasing
(d) Newton's second law can be applied

Comprehension Based Questions

Passage 1 (Q. Nos. 1 to 2)

A block of mass 2 kg is attached with a spring of spring constant 4000 Nm^{-1} and the system is kept on smooth horizontal table. The other end of the spring is attached with a wall. Initially spring is stretched by 5 cm from its natural position and the block is at rest. Now suddenly an impulse of 4 kg-ms^{-1} is given to the block towards the wall.

1. Find the velocity of the block when spring acquires its natural length

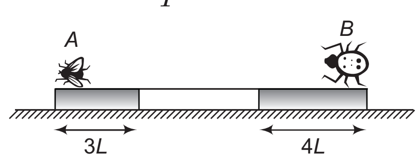
(a) 5 ms^{-1} (b) 3 ms^{-1}
(c) 6 ms^{-1} (d) None of these

2. Approximate distance travelled by the block when it comes to rest for a second time (not including the initial one) will be (Take $\sqrt{45} = 6.70$)

(a) 30 cm (b) 25 cm
(c) 40 cm (d) 20 cm

Passage 2 (Q. Nos. 3 to 7)

A uniform bar of length $12L$ and mass 48 m is supported horizontally on two fixed smooth tables as shown in figure. A small moth (an insect) of mass $8m$ is sitting on end A of the rod and a spider (an insect) of mass $16m$ is sitting on the other end B. Both the insects moving towards each other along the rod with moth moving at speed $2v$ and the spider at half this speed (absolute). They meet at a point P on the rod and the spider eats the moth. After this the spider moves with a velocity $\frac{v}{2}$ relative to the rod towards the end A. The spider takes negligible time in eating on the other insect. Also, let $v = \frac{L}{T}$ where T is a constant having value 4 s.



3. Displacement of the rod by the time the insect meet the moth is

(a) $\frac{L}{2}$ (b) L (c) $\frac{3L}{4}$ (d) zero

4. The point P is at

(a) the centre of the rod
(b) the edge of the table supporting the end B
(c) the edge of the table supporting end A
(d) None of the above

5. The speed of the rod after the spider eats up the moth and moves towards A is

(a) $\frac{v}{2}$ (b) v (c) $\frac{v}{6}$ (d) $2v$

6. After starting from end B of the rod the spider reaches the end A at a time

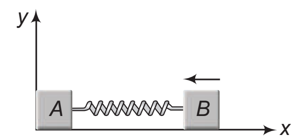
(a) 40 s (b) 30 s (c) 80 s (d) 10 s

7. By what distance the centre of mass of the rod shifts during this time?

(a) $\frac{8L}{3}$ (b) $\frac{4L}{3}$ (c) L (d) $\frac{L}{3}$

Match the Columns

1. Two identical blocks A and B are connected by a spring as shown in figure. Block A is not connected to the wall parallel to y-axis. B is compressed from the natural length of spring and then left. Neglect friction. Match the following two columns.



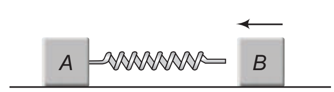
Column I	Column II
(a) Acceleration of centre of mass of two blocks	(p) remains constant
(b) Velocity of centre of mass of two blocks	(q) first increases then becomes constant
(c) x-coordinate of centre of mass of two blocks	(r) first decreases then becomes zero
(d) y-coordinate of centre of mass of two blocks	(s) continuously increases

2. One particle is projected from ground upwards with velocity 20 ms^{-1}. At the same time another identical particle is dropped from a height of 180 m but not along the same vertical line. Assume that collision of first particle with ground is perfectly inelastic. Match the following two columns for centre of mass of the two particles ($g = 10\ ms^{-2}$)

Column I	Column II
(a) Initial acceleration	(p) 5 SI units
(b) Initial velocity	(q) 10 SI units
(c) Acceleration at $t = 5$ s	(r) 20 SI units
(d) Velocity at $t = 5$ s	(s) 25 SI units

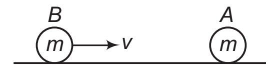
Note *Only magnitudes are given in column-II.*

3. Two identical blocks of mass 0.5 kg each are shown in figure. A massless elastic spring is connected with A. B is moving towards A with kinetic energy of 4 J. Match the following two columns. Neglect friction.



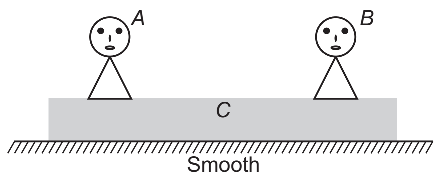
Column I	Column II
(a) Initial momentum of B	(p) zero
(b) Momentum of centre of mass of two blocks	(q) 1 kg-ms^{-1}
(c) Momentum of A at maximum compression	(r) 2 kg-ms^{-1}
(d) Momentum of B when spring is relaxed after compression	(s) 4 kg-ms^{-1}

4. Two identical balls A and B are kept on a smooth table as shown. B collides with A with speed v. For different conditions mentioned in Column I, match with speed of A after collision given in Column II.



Column I	Column II
(a) Elastic collision	(p) $\frac{3}{4}v$
(b) Perfectly inelastic collision	(q) $\frac{5}{8}v$
(c) Inelastic collision with $e = \frac{1}{2}$	(r) v
(d) Inelastic collision with $e = \frac{1}{4}$	(s) $\frac{v}{2}$

5. Two boys A and B of masses 30 kg and 60 kg are standing over a plank C of mass 30 kg as shown. Ground is smooth. Match the displacement of plank of Column II with the conditions given in Column I.

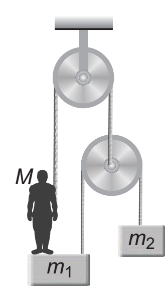


Column I	Column II
(a) A moves x towards right	(p) x towards right
(b) B moves x towards left	(q) $2x$ towards left
(c) A moves x towards right and B moves x towards left	(r) $\frac{x}{3}$ towards left
(d) A and B both move x towards right	(s) None

Note *All displacements mentioned in two columns are with respect to ground.*

6. A man of mass M is standing on a platform of mass m_1 and holding a string passing over a system of ideal pulleys. Another mass m_2 is hanging as shown.

$$(m_2 = 20\text{kg},\ m_1 = 10\text{kg},\ g = 10\ \text{ms}^{-2})$$

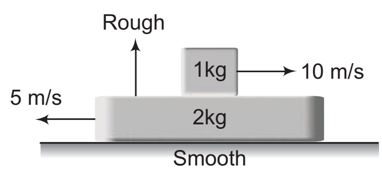


Column I	Column II
(a) Weight of man for equilibrium	(p) 100 N
(b) Force exerted by man on string to accelerate the COM of system upwards	(q) 150 N
(c) Force exerted by man on string to accelerate the COM of system downwards	(r) 500 N
(d) Normal reaction of platform on man in equilibrium	(s) 600 N

7. Two blocks of masses 3 kg and 6 kg are connected by an ideal spring and are placed on a frictionless horizontal surface. The 3 kg block is imparted a speed of 2 ms^{-1} towards left. (consider left as positive direction)

Column I	Column II
(a) When the velocity of 3 kg block is $\frac{2}{3}$ ms^{-1}	(p) velocity of centre of mass is $\frac{2}{3}$ ms^{-1}
(b) When the velocity of 6 kg block is $\frac{2}{3}$ ms^{-1}	(q) deformation of the spring is zero
(c) When the speed of 3 kg block is minimum	(r) deformation of the spring is maximum
(d) When the speed of 6 kg block is maximum	(s) both the blocks are at rest with respect to each other

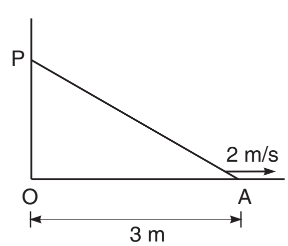
8. In a two block system shown in figure match the following



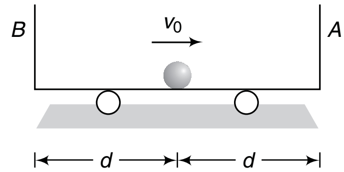
Column I	Column II
(a) Velocity of centre of mass	(p) Keep on changing all the time
(b) Momentum of centre of mass	(q) First decreases then become zero
(c) Momentum of 1 kg block	(r) Zero
(d) Kinetic energy of 2 kg block	(s) Constant

Subjective Questions

1. A ladder AP of length 5 m inclined to a vertical wall is slipping over a horizontal surface with velocity of 2 m/s, when A is at distance 3 m from ground. What is the velocity of COM at this moment?

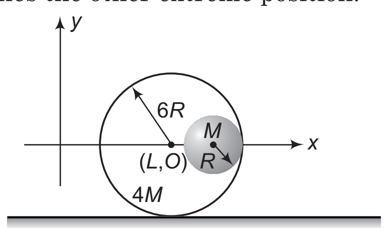


2. A ball of negligible size and mass m is given a velocity v_0 on the centre of the cart which has a mass M and is originally at rest. If the coefficient of restitution between the ball and walls A and B is e. Determine the velocity of the ball and the cart just after the ball strikes A. Also, determine the total time needed for the ball to strike A, rebound, then strike B, and rebound and then return to the centre of the cart. Neglect friction.



3. Two point masses m_1 and m_2 are connected by a spring of natural length l_0. The spring is compressed such that the two point masses touch each other and then they are fastened by a string. Then the system is moved with a velocity v_0 along positive x-axis. When the system reached the origin, the string breaks ($t = 0$). The position of the point mass m_1 is given by $x_1 = v_0 t - A(1 - \cos \omega t)$ where A and ω are constants. Find the position of the second block as a function of time. Also, find the relation between A and l_0.

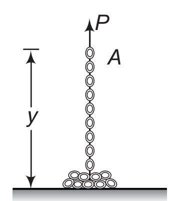
4. A small sphere of radius R is held against the inner surface of larger sphere of radius $6R$ (as shown in figure). The masses of large and small spheres are $4M$ and M respectively. This arrangement is placed on a horizontal table. There is no friction between any surfaces of contact. The small sphere is now released. Find the coordinates of the centre of the large sphere, when the smaller sphere reaches the other extreme position.



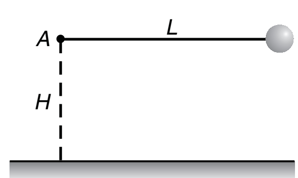
5. A chain of length l and mass m lies in a pile on the floor. If its end A is raised vertically at a constant speed v_0, express in terms of the length y of chain which is off the floor at any given instant.

(a) The magnitude of the force P applied to end A.

(b) Energy lost during the lifting of the chain.



6. A is a fixed point at a height H above a perfectly inelastic smooth horizontal plane. A light inextensible string of length $L(> H)$ has one end attached to A and other to a heavy particle. The particle is held at the level of A with string just taut and released from rest. Find the height of the particle above the plane when it is next instantaneously at rest.



7. A particle of mass $2\,m$ is projected at an angle of 45° with horizontal with a velocity of $20\sqrt{2}$ m/s. After 1 s, explosion takes place and the particle is broken into two equal pieces.

As a result of explosion one part comes to rest. Find the maximum height attained by the other part. (Take $g = 10\ \text{m/s}^2$)

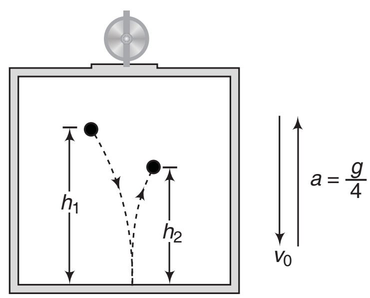
8. A sphere of mass m, impinges obliquely on a sphere, of mass M, which is at rest. Show that, if $m = eM$, the directions of motion of the spheres after impact are at right angles.

9. A gun of mass M (including the carriage) fires a shot of mass m. The gun along with the carriage is kept on a smooth horizontal surface. The muzzle speed of the bullet v_r is constant. Find

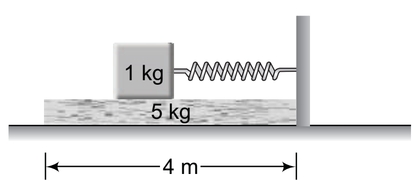
(a) The elevation of the gun with horizontal at which maximum range of bullet with respect to the ground is obtained.

(b) The maximum range of the bullet.

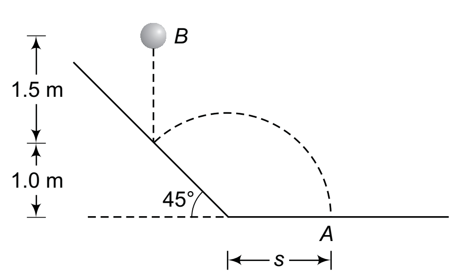
10. A ball is released from rest relative to the elevator at a distance h_1 above the floor. The speed of the elevator at the time of ball release is v_0. Determine the bounce height h_2 relative to elevator of the ball (a) if v_0 is constant and (b) if an upward elevator acceleration $a = \dfrac{g}{4}$ begins at the instant the ball is released. The coefficient of restitution for the impact is e.



11. A plank of mass 5 kg is placed on a frictionless horizontal plane. Further a block of mass 1 kg is placed over the plank. A massless spring of natural length 2 m is fixed to the plank by its one end. The other end of spring is compressed by the block by half of spring's natural length. The system is now released from the rest. What is the velocity of the plank when block leaves the plank? (The stiffness constant of spring is 100 N/m).

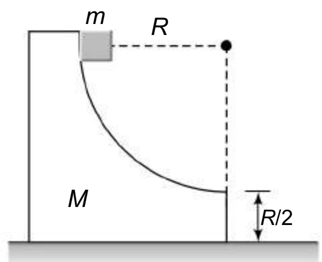


12. To test the manufactured properties of 10N steel balls, each ball is released from rest as shown and strikes a 45° inclined surface. If the coefficient of restitution is to be $e = 0.8$, determine the distance s, where the ball must strike the horizontal plane at A. At what speed does the ball strike at A?($g = 9.8$ m/s^2)

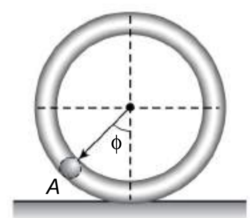


13. Two particles A and B of equal masses lie close together on a horizontal table and are connected by a light inextensible string of length l. A is projected vertically upwards with a velocity $\sqrt{10gl}$. Find the velocity with which it reaches the table again.

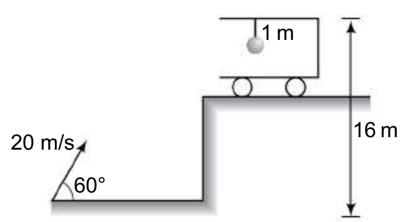
14. A small cube of mass m slides down a circular path of radius R cut into a large block of mass M, as shown in figure. M rests on a table, and both blocks move without friction. The blocks are initially at rest, and m starts from the top of the path. Find the horizontal distance from the bottom of block where cube hits the table.



15. A thin hoop of mass M and radius r is placed on a horizontal plane. At the initial instant, the hoop is at rest. A small washer of mass m with zero initial velocity slides from the upper point of the hoop along a smooth groove in the inner surface of the hoop. Determine the velocity u of the centre of the hoop at the moment when the washer is at a certain point A of the hoop, whose radius vector forms an angle ϕ with the vertical (figure). The friction between the hoop and the plane should be neglected.

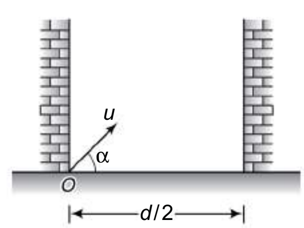


16. A shell of mass 1 kg is projected with velocity 20 m/s at an angle 60° with horizontal. It collides inelastically with a ball of mass 1 kg which is suspended through a thread of length 1 m. The other end of the thread is attached to the ceiling of a trolley of mass $\frac{4}{3}$ kg as shown in figure. Initially the trolley is stationary and it is free to move along horizontal rails without any friction. What is the maximum deflection of the thread with vertical? String does not slack. Take $g = 10\ \text{m/s}^2$.

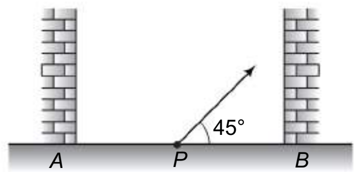


17. A small ball is projected at an angle α between two vertical walls such that in the absence of the wall its range would have been $5d$. Given that all the collisions are perfectly elastic, find

(a) maximum height attained by the ball.
(b) total number of collisions with the walls before the ball comes back to the ground, and
(c) point at which the ball finally falls. The walls are supposed to be very tall.

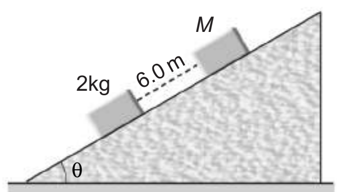


18. Two large rigid vertical walls A and B are parallel to each other and separated by 10 m. A particle of mass 10 g is projected with an initial velocity of 20 m/s at 45° to the horizontal from point P on the ground, such that $AP = 5$ m. The plane of motion of the particle is vertical and perpendicular to the walls. Assuming that all the collisions are perfectly elastic , find the maximum height attained by the particle and the total number of collisions suffered by the particle with the walls before it hits ground. Take $g = 10\ \text{m/s}^2$.



19. Two blocks of masses 2 kg and M are at rest on an inclined plane and are separated by a distance of 6.0 m as shown. The coefficient of friction between each block and the inclined plane is 0.25. The 2 kg block is given a velocity of 10.0 m/s up the inclined plane. It collides with M, comes back and has a velocity of 1.0 m/s when it reaches its initial position. The other block M after the collision moves 0.5 m up and comes to rest. Calculate the coefficient of restitution between the blocks and the mass of the block M.

[Take $\sin\theta \approx \tan\theta = 0.05$ and $g = 10\ \text{m/s}^2$]



20. A small block of mass m is placed on top of a smooth hemisphere also of mass m which is placed on a smooth horizontal surface. If the block begins to slide down due to a negligible small impulse, show that it will loose contact with the hemisphere when the radial line through vertical makes an angle θ given by the equation $\cos^3\theta - 6\cos\theta + 4 = 0$.

21. A ball is projected from a given point with velocity u at some angle with the horizontal and after hitting a vertical wall returns to the same point. Show that the distance of the point from the wall must be less than $\dfrac{eu^2}{(1+e)g}$, where e is the coefficient of restitution.

Answers

Introductory Exercise 11.1

1. $\mathbf{r}_{COM} = \dfrac{\sum_{i=1}^{n} m_i \mathbf{r}_i}{\sum_{i=1}^{n} m_i}$ while $\mathbf{r}_{CG} = \dfrac{\sum_{i=1}^{n} w_i \mathbf{r}_i}{\sum_{i=1}^{n} w_i}$ Here, w = weight (mg), $\mathbf{r}_{COM} = \mathbf{r}_{CG}$ when $\mathbf{g}$ = constant

2. False **3.** True **4.** True **5.** less than $\dfrac{r}{2}$

6. False **7.** $\dfrac{\sqrt{19}}{6}$ m **8.** $\dfrac{4}{3\pi}\left\{\dfrac{a^2 + ab + b^2}{a + b}\right\}$ **9.** $\left(\dfrac{5a}{6}, \dfrac{5a}{6}\right)$

10. 0.74 m^2 **11.** $\left(\dfrac{\pi}{\pi + 4}\right) a$ **12.** $x_{COM} = \dfrac{3l}{4}$ **13.** $\dfrac{2}{3} L$

Introductory Exercise 11.2

1. x_{CM} = 12.67 m **2.** zero **3.** $\dfrac{m_1 l_1 + m_2 l_2}{m_1 + m_2}$ **4.** (a) 0.30 kg (b) (2.4 kg-ms^{-1})$\hat{\mathbf{j}}$ (c) (8.0 ms^{-1}) $\hat{\mathbf{j}}$

5. (a) 28 cm (b) 2.3 ms^{-1} **6.** $\left(\dfrac{\sqrt{3} - 1}{4\sqrt{2}}\right) g$

Introductory Exercise 11.3

1. $(2.5\,\hat{\mathbf{i}} + 15\hat{\mathbf{j}} + 5\hat{\mathbf{k}})$ cm/s **2.** 12.5 m/s in opposite direction, 17.5 m/s **3.** 1 : 2

4. 2.4×10^5 m/s **5.** 1.5×10^{-23} m/s **6.** 9 ms^{-1}, 1.08 kJ **7.** 60 m

Introductory Exercise 11.4

1. 1.225 kgs^{-1}, (i) 2.8 kms^{-1}, (ii) 3.6 kms^{-1} **2.** 1232.6 ms^{-1}

3. $(m_0 - \mu t)\dfrac{d^2x}{dt^2} = \mu u - (m_0 - \mu t)\, g$ **4.** $u \ln\left(\dfrac{3}{2}\right) - g$

Introductory Exercise 11.5

1. 4×10^3 N **2.** $-\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$ **3.** 200 m/s **4.** (a) 1.2×10^{-3} s (b) 0.5 N-s (c) 417 N

Introductory Exercise 11.6

1. 30 cm **2.** $\dfrac{8}{9}$ **3.** $\dfrac{4m_1 m_2}{(m_1 + m_2)^2}$ **4.** No

5. 2 ms^{-1} in negative x-axis, 3 m/s in positive x-axis.

6. $v_1 = \dfrac{28}{3}$ ms^{-1} (in negative x-direction) and $v_2 = \dfrac{2}{3}$ ms^{-1} (in positive x-direction)

7. Two **8.** $\dfrac{1}{3}$ **9.** (a) $\dfrac{mv}{V}$ (b) $\dfrac{V}{v}$ **10.** $-\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$ **11.** $90° - 2\alpha$

Exercises

LEVEL 1

Assertion and Reason

1. (d) **2.** (a) **3.** (d) **4.** (d) **5.** (b) **6.** (a) **7.** (c) **8.** (d) **9.** (a) **10.** (d)
11. (d) **12.** (d) **13.** (d) **14.** (a) **15.** (d)

Single Correct Option

1. (a)	**2.** (a)	**3.** (c)	**4.** (b)	**5.** (b)	**6.** (d)	**7.** (d)	**8.** (c)	**9.** (d)	**10.** (b)
11. (d)	**12.** (c)	**13.** (b)	**14.** (a)	**15.** (a)	**16.** (a)	**17.** (c)	**18.** (a)	**19.** (b)	**20.** (b)
21. (d)	**22.** (b)	**23.** (c)	**24.** (c)	**25.** (b)	**26.** (c)	**27.** (a)	**28.** (c)	**29.** (d)	**30.** (d)
31. (c)	**32.** (c)	**33.** (b)	**34.** (a)	**35.** (c)	**36.** (d)	**37.** (a)	**38.** (b)	**39.** (c)	**40.** (c)
41. (d)	**42.** (c)								

Subjective Questions

1. $\left(-\frac{a}{12}, -\frac{b}{12}\right)$ **2.** $x_{COM} = -\frac{a^3 b}{R^3 - a^3}$ **3.** True **6.** $\frac{K}{2}$

7. $\frac{1}{e}, \frac{1}{e}, \frac{1}{e^2}$

8. (a) $(-10\hat{\mathbf{j}})\,\text{ms}^{-2}$ (b) $\frac{10}{3}(2\hat{\mathbf{i}} - \hat{\mathbf{j}})\,\text{ms}^{-1}$ (c) $\left(\frac{70}{3}\hat{\mathbf{i}} + 35\hat{\mathbf{j}}\right)$ m

9. (a) 2.0 kg (b) $(12.0\ \text{ms}^{-2})t\,\hat{\mathbf{j}}$ (c) $(72.0\ \text{N})\hat{\mathbf{j}}$ **10.** (a) $0.8\,\hat{\mathbf{i}}$ m/s (b) $2.4\,\hat{\mathbf{i}}$ m/s (c) $\frac{4}{7}$

11. 2.82 kms^{-1} **12.** 10 cm **13.** (a) $\frac{mv}{M+m}$ (b) balloon will also stop moving

14. $m = m_0 e^{-(a/u)t}$ **15.** 35 m **16.** $2\sqrt{10}$ N-s

18. 2 ms^{-1}, $2\sqrt{2}$ ms^{-1} **19.** $\frac{2\pi r}{v}$

20. $\frac{2J}{P} - 1$ **21.** $\frac{2}{3}$

22. $\frac{1}{6}$ m

23. 71.4 mm **24.** $\frac{4L}{3}$

25. (a) 0.43 (b) 480 J (c) 1.28 J

26. 0.011mm, 1.94 m/s **27.** $2m \sin\left(\frac{\alpha}{2}\right)\sqrt{\frac{gl}{M(M+m)}}$

28. $\frac{m(R-r)}{M+m}, m\sqrt{\frac{2g(R-r)}{M(M+m)}}$

29. (a) 0.14 kg-ms^{-1} (b) zero

31. 10 N, 20 W

32. 270 N-s (to the right), 13.5 kN (to the right)

33. 0.4 ms^{-1}, 2.4 ms^{-1}, 0.93 m

LEVEL 2

Single Correct Option

1. (b)	**2.** (b)	**3.** (b)	**4.** (b)	**5.** (a)	**6.** (b)	**7.** (d)	**8.** (a)	**9.** (b)	**10.** (c)
11. (a)	**12.** (a)	**13.** (b)	**14.** (c)	**15.** (d)	**16.** (d)	**17.** (d)	**18.** (c)	**19.** (c)	**20.** (b)
21. (a)	**22.** (b)	**23.** (b)	**24.** (a)						

More than One Correct Options

1. (b,d) **2.** (a,c,d) **3.** (all) **4.** (b,d) **5.** (b,c) **6.** (c,d) **7.** (b,c) **8.** (b,c,d)

Comprehension Based Questions

1. (b) **2.** (b) **3.** (d) **4.** (b) **5.** (c) **6.** (c) **7.** (a)

Match the Columns

1. (a) → (r) (b) → (q) (c) → (s) (d) → (p)
2. (a) → (q) (b) → (q) (c) → (p) (d) → (s)
3. (a) → (r) (b) → (r) (c) → (q) (d) → (p)
4. (a) → (r) (b) → (s) (c) → (p) (d) → (q)
5. (a) → (r) (b) → (p) (c) → (p) (d) → (s)
6. (a) → (r) (b) → (r,s) (c) → (p,q) (d) → (p)
7. (a) → (p,r,s) (b) → (p,r,s) (c) → (p) (d) → (p,q)
8. (a) → (r,s) (b) → (r,s) (c) → (q) (d) → (q)

Subjective Questions

1. 1.25 ms^{-1} **2.** $v_{ball} = \left(\frac{eM - m}{M + m}\right) v_0$ (leftwards), $v_{cart} = \left(\frac{e+1}{M+m}\right) mv_0$ (rightwards); $t = \frac{d}{v_0}\left(1 + \frac{2}{e} + \frac{1}{e^2}\right)$

3. $x_2 = v_0 t + \frac{m_1}{m_2} A(1 - \cos \omega t)$, $l_0 = \left(\frac{m_1}{m_2} + 1\right) A$ **4.** $(L + 2R, 0)$

5. (a) $\frac{m}{l}(gy + v_0^2)$ (b) $\frac{myv_0^2}{2l}$ **6.** $\frac{H^5}{L^4}$

7. 35 m **9.** (a) 45° (b) $\left(\frac{M}{M+m}\right)\frac{v_r^2}{g}$

10. (a) $e^2 h_1$ (b) $e^2 h_1$

11. $\sqrt{\frac{10}{3}}$ ms^{-1} **12.** 0.93 m, 6.6 ms^{-1}

13. $2\sqrt{gl}$ **14.** $R\sqrt{\frac{2(M+m)}{M}}$

15. $v = m \cos \phi \sqrt{\frac{2gr(1 + \cos \phi)}{(M+m)(M + m \sin^2 \phi)}}$ **16.** 60°

17. (a) $\frac{u^2 \sin^2 \alpha}{2g}$ (b) nine (c) point O

18. 10 m, Four **19.** 0.84, 15.12 kg

12

Rotational Mechanics

Chapter Contents

12.1 Introduction

A particle means mass with negligible volume. A rigid body is made up of too many particles but distance between any two particles is always constant. In any type of motion of a rigid body this distance always remains constant. A particle has only translational motion. Even if a particle is rotating in a circle it has only translational motion and it has only translational kinetic energy $\frac{1}{2}mv^2$.

A rigid body may have either of the following three types of motions :

(i) Translational motion
(ii) Rotational motion
(iii) Translational plus rotational motion

In translational motion of the rigid body all particles of the rigid body have same linear displacement, same linear velocity and same linear acceleration. In rest two motions, different particles have different linear displacement, different linear velocity and different linear acceleration.

As far as translational motion is concerned we do not differentiate between a particle and a rigid body. This motion is already discussed in the chapter of kinematics. This is the reason, in the chapter of kinematics, sometimes we write: a particle is moving and sometimes we write: a block (or a body) is moving. Rest two motions are only defined for a rigid body. In the present chapter, we shall discuss these two motions of a rigid body.

12.2 Moment of Inertia

Like the centre of mass, the moment of inertia is a property of an object that is related to its mass distribution. The moment of inertia (denoted by I) is an important quantity in the study of system of particles that are rotating . The role of the moment of inertia in the study of rotational motion is analogous to that of mass in the study of linear motion. Moment of inertia gives a measurement of the resistance of a body to a change in its rotational motion. If a body is at rest, the larger the moment of inertia of a body, the more difficult it is to put that body into rotational motion. Similarly, the larger the moment of inertia of a body, the more difficult it is to stop its rotational motion. The moment of inertia is calculated about some axis (usually the rotational axis) and it depends on the mass as well as its distribution about that axis.

Moment of Inertia of a Single Particle

For a very simple case the moment of inertia of a single particle about an axis is given by,

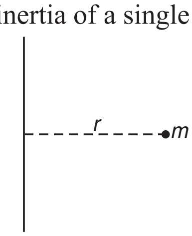


Fig. 12.1

$$\boxed{I = mr^2} \qquad \text{...(i)}$$

Here, m is the mass of the particle and r its distance from the axis under consideration.

Moment of Inertia of a System of Particles

The moment of inertia of a system of particles about an axis is given by,

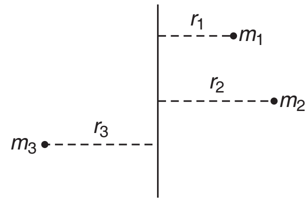


Fig. 12.2

$$\boxed{I = \sum_i m_i r_i^2} \qquad \text{...(ii)}$$

where, r_i is the perpendicular distance from the axis to the i^{th} particle, which has a mass m_i.

For example, in Fig. 12.2:

$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2$$

Moment of Inertia of Rigid Bodies

For a continuous mass distribution such as found in a rigid body, we replace the summation of Eq. (ii) by an integral. If the system is divided into infinitesimal elements of mass dm and if r is the distance from a mass element to the axis of rotation, the moment of inertia is,

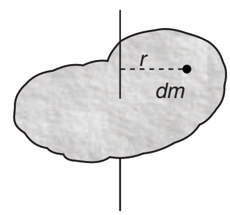


Fig. 12.3

$$\boxed{I = \int r^2 \, dm}$$

where the integral is taken over the system.

Moment of Inertia of a Uniform Cylinder

Let us find the moment of inertia of a uniform cylinder about an axis through its centre of mass and perpendicular to its base. Mass of the cylinder is M and radius is R.

We first divide the cylinder into annular shells of width dr and length l as shown in figure. The moment of inertia of one of these shells is

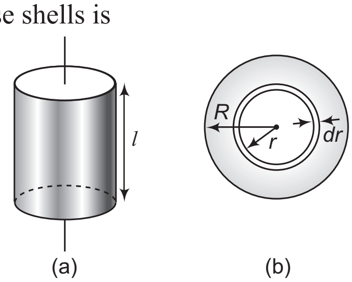


Fig. 12.4

$$dI = r^2 dm = r^2 \, (\rho \cdot dV)$$

Here, ρ = density of cylinder

and dV = volume of shell $= 2\pi r l \, dr$

$\therefore$ $dI = 2\pi\rho l \, r^3 \, dr$

The cylinder's moment of inertia is found by integrating this expression between 0 and R,

So, $$I = 2\pi\rho l \int_0^R r^3\,dr = \frac{\pi\rho l}{2} R^4 \qquad \ldots\text{(iii)}$$

The density ρ of the cylinder is the mass divided by the volume.

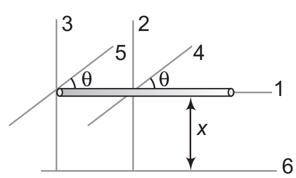
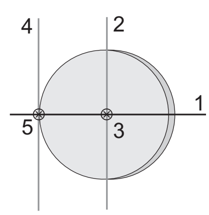
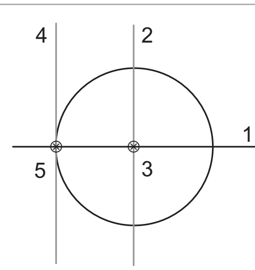
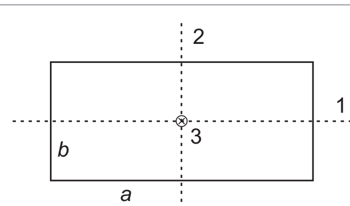
∴ $$\rho = \frac{M}{\pi R^2 l} \qquad \ldots\text{(iv)}$$

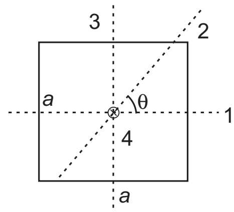
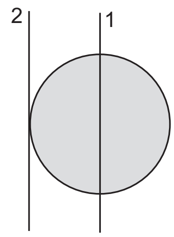
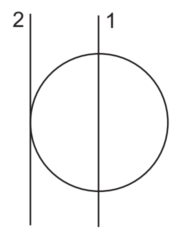
From Eqs. (iii) and (iv), we have

$$\boxed{I = \frac{1}{2} MR^2}$$

Proceeding in the similar manner we can find the moment of inertia of certain rigid bodies about some given axis. Moments of inertia of several rigid bodies with symmetry are listed in Table. 12.1.

Table 12.1

Thin rod		$I_1 = 0,\ I_2 = \frac{ml^2}{12}$ $I_3 = \frac{ml^2}{3},\ I_4 = \frac{ml^2}{12}\sin^2\theta$ $I_5 = \frac{ml^2}{3}\sin^2\theta,\ I_6 = mx^2$
Circular disc		$I_1 = I_2 = \frac{mR^2}{4}$ $I_3 = I_1 + I_2 = \frac{mR^2}{2}$ $I_4 = I_2 + mR^2 = \frac{5}{4}mR^2$ $I_5 = I_3 + mR^2 = \frac{3}{2}mR^2$
Circular ring		$I_1 = I_2 = \frac{mR^2}{2}$ $I_3 = I_1 + I_2 = mR^2$ $I_4 = I_2 + mR^2 = \frac{3}{2}mR^2$ $I_5 = I_3 + mR^2 = 2mR^2$
Rectangular slab		$I_1 = \frac{mb^2}{12}$ $I_2 = \frac{ma^2}{12}$ $I_3 = I_1 + I_2$ $= \frac{m}{12}(a^2 + b^2)$

Square slab		$I_1 = I_2 = I_3 = \frac{ma^2}{12}$ $I_4 = I_1 + I_3 = \frac{ma^2}{6}$
Solid sphere		$I_1 = \frac{2}{5} mR^2$ $I_2 = I_1 + mR^2$ $= \frac{7}{5} mR^2$ m = mass of sphere
Hollow sphere		$I_1 = \frac{2}{3} mR^2$ $I_2 = I_1 + mR^2$ $= \frac{5}{3} mR^2$

Theorems on Moment of Inertia

There are two important theorems on moment of inertia, which, in some cases, enable the moment of inertia of a body to be determined about any general axis, if its moment of inertia about some other axis is known. Let us now discuss both of them.

Theorem of Parallel Axes

A very useful theorem, called the parallel axes theorem relates the moment of inertia of a rigid body about two parallel axes, one of which passes through the centre of mass.

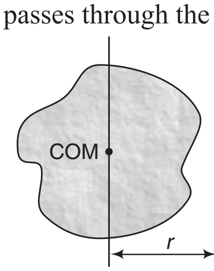


Fig. 12.5

Two such axes are shown in figure for a body of mass M. If r is the distance between the axes and I_{COM} and I are the respective moments of inertia about them then, these two are related by,

$$\boxed{I = I_{COM} + Mr^2}$$

We now present a proof of the above theorem.

Proof To prove this theorem, we consider two axes, both parallel to the z-axis, one through the center of mass and the other through a point P (Fig 12.6). First we take a very thin slice of the body, parallel to the xy-plane and perpendicular to the z-axis. We take the origin of our coordinate system to be at the centre of mass of the body; the coordinates of the centre of mass are then $x_{\text{cm}} = y_{\text{cm}} = z_{\text{cm}} = 0$. The axis through the centre of mass passes through this thin slice at point O and the parallel axis passes through point P, whose x and y coordinates are (a, b). The distance of this axis from the axis through the centre of mass is r, where $r^2 = a^2 + b^2$.

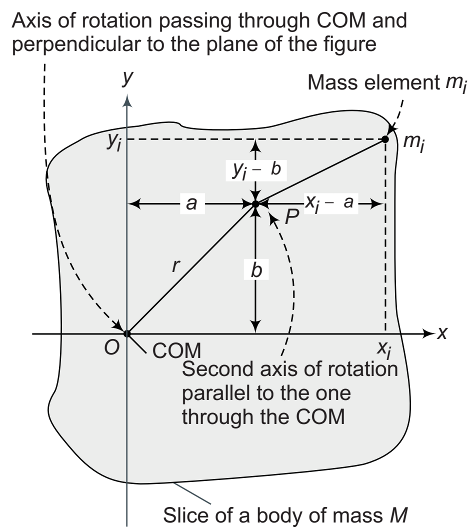


Fig. 12.6

We can write an expression for the moment of inertia I_P about the axis through point P. Let m_i be a mass element in our slice, with coordinates (x_i, y_i, z_i). Then, the moment of inertia I_{COM} of the slice about the axis through the centre of mass (at O) is

$$I_{\text{COM}} = \sum_i m_i (x_i^2 + y_i^2)$$

The moment of inertia of the slice about the axis through P is

$$I_P = \sum_i m_i [(x_i - a)^2 + (y_i - b)^2]$$

These expressions don't involve the coordinates z_i measured perpendicular to the slice, so we can extend the sums to include all particles in all slices. Then, I_P becomes the moment of inertia of the entire body for an axis through P. We then expand the squared terms, regroup and obtain

$$I_P = \sum_i m_i (x_i^2 + y_i^2) - 2a \sum_i m_i x_i - 2b \sum_i m_i y_i + (a^2 + y^2) \sum_i m_i$$

The first sum is I_{COM}. From the, definition of the centre of mass the second and third sums are proportional to x_{cm} and y_{cm}. These are zero because we have taken our origin to be the centre of mass. The final term is r^2 multiplied by the total mass or r^2. This completes our proof that

$$I_P = I_{\text{COM}} + Mr^2$$

Note *From the above theorem we can see that among several parallel axes, moment of inertia is least about an axis which passes through centre of mass. e. g.* I_2 *is least among* I_1, I_2 *and* I_3. *Similarly,* I_5 *is least among* I_4, I_5 *and* I_6.

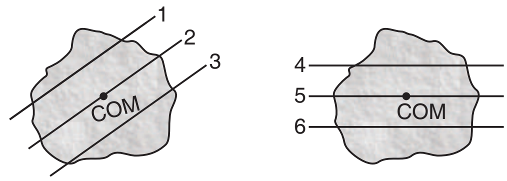


Fig. 12.7

Theorem of Perpendicular Axes

This theorem is applicable only to the plane bodies (two dimensional). The theorem states that the moment of inertia of a plane lamina about an axis perpendicular to the plane of the lamina is equal to the sum of the moments of inertia of the lamina about two axes perpendicular to each other, in its own plane and intersecting each other, at the point where the perpendicular axis passes through it. Let x and y axes be chosen in the plane of the body and z-axis perpendicular, to this plane, three axes being mutually perpendicular, then the theorem states that

$$\boxed{I_z = I_x + I_y}$$

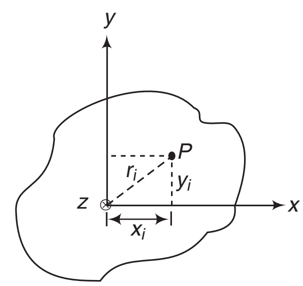


Fig. 12.8

Proof Consider an arbitrary particle P of mass m_i, distant r_i from O and x_i and y_i are the perpendicular distances of point P from the x and y-axes respectively, we have

$$I_z = \sum_i m_i r_i^2, I_x = \sum_i m_i y_i^2 \text{ and } I_y = \sum_i m_i x_i^2$$

So that,
$$I_x + I_y = \sum_i m_i y_i^2 + \sum_i m_i x_i^2$$

$$= \sum_i m_i (y_i^2 + x_i^2) = \sum_i m_i r_i^2 = I_z$$

i.e.
$$I_z = I_x + I_y$$
Hence Proved.

Radius of Gyration

Radius of gyration (K) of a body about an axis is the effective distance from this axis where the whole mass can be assumed to be concentrated so that the moment of inertia remains the same. Thus,

$$\boxed{I = MK^2 \quad \text{or} \quad K = \sqrt{\frac{I}{M}}}$$

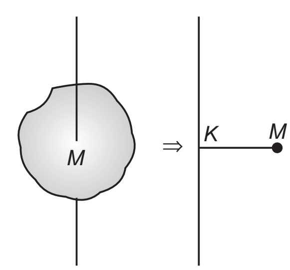


Fig. 12.9

e.g. radius of gyration of a disc about an axis perpendicular to its plane and passing through its centre of mass is

$$K = \sqrt{\frac{\frac{1}{2}MR^2}{M}} = \frac{R}{\sqrt{2}}$$

Extra Points to Remember

- Theorem of parallel axes is applicable for any type of rigid body whether it is a two dimensional or three dimensional, while the theorem of perpendicular axes is applicable for laminar type or two dimensional bodies only.
- In theorem of perpendicular axes, the point of intersection of the three axes (x, y and z) may be any point on the plane of body (it may even lie outside the body also). This point may or may not be the centre of mass of the body.
- If whole mass of the rigid body is kept at same distance x or R from the axis, then moment of intertia is mx^2 or mR^2, where m is the mass of whole body.

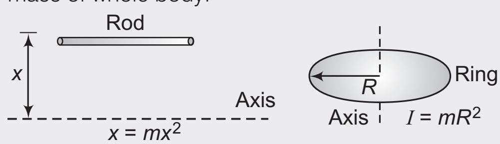


Fig. 12.10

- If a portion is symmetrically cut about an axis and mass of remaining portion is M. Then, moment of inertia of the remaining portion is same as the moment of inertia of the whole body of same mass M. e.g. in figure 12.11(a) moment of inertia of the section shown (a part of circular disc) about an axis perpendicular to its plane and passing through point O is $\frac{1}{2}MR^2$ as the moment of inertia of the complete disc is also $\frac{1}{2}MR^2$.

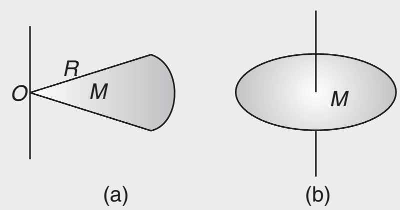


Fig. 12.11

Proof : Suppose the given section is $\frac{1}{n}$ th part of the disc, then mass of the disc will be nM.

$$I_{\text{disc}} = \frac{1}{2}(nM)R^2$$

$$\therefore \qquad I_{\text{section}} = \frac{1}{n}I_{\text{disc}} = \frac{1}{2}MR^2$$

- If whole mass of the rigid body is kept over the axis then, moment of inertia is zero. For example, moment of inertia of a thin rod about an axis passing through the rod is zero.

Fig. 12.12

Example 12.1 *Three particles of masses 1g, 2g and 3g are kept at points (2cm, 0), (0,6 cm), (4cm, 3cm). Find moment of inertia of all three particles (in gm-cm^2) about, (a) x-axis (b) y-axis (c) z-axis*

Solution (a) **About x-axis**

$$I_x = I_1 + I_2 + I_3$$

$$= m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2$$

Here r = perpendicular distance of the particle from x-axis

$$\therefore \qquad I_1 = (1)(0)^2 + (2)(6)^2 + (3)(3)^2$$

$$= 99 \text{g-cm}^2 \qquad \textbf{Ans.}$$

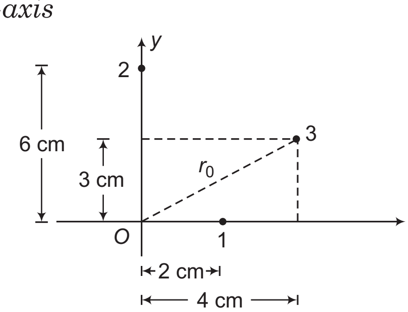


Fig. 12.13

(b) **About y-axis**

$$I_y = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2$$

Here, r = perpendicular distance of the particle from y-axis

$\therefore$ $$I_y = (1)(2)^2 + (2)(0)^2 + (3)(4)^2 = 52\,\text{g-cm}^2$$ **Ans.**

(c) **About z-axis** $$I_z = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2$$

Here, r = perpendicular distance of the particle from z-axis.

$$r_0 = \sqrt{(3)^2 + (4)^2} = 5\,\text{cm}$$

$\therefore$ $$I_z = (1)(2)^2 + (2)(6)^2 + 3(5)^2 = 151\,\text{g-cm}^2$$ **Ans.**

Note *In the above example, from theorem of perpendicular axes, we can see that*

$$I_z = I_x + I_y$$

Example 12.2 *Three rods each of mass m and length l are joined together to form an equilateral triangle as shown in figure. Find the moment of inertia of the system about an axis passing through its centre of mass and perpendicular to the plane of the triangle.*

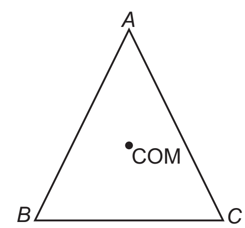


Fig. 12.14

Solution Moment of inertia of rod BC about an axis perpendicular to plane of triangle ABC and passing through the mid-point of rod BC (i.e. D) is

$$I_1 = \frac{ml^2}{12}$$

$$r = BD \tan 30°$$

or $$r = \left(\frac{l}{2}\right)\left(\frac{1}{\sqrt{3}}\right) = \frac{l}{2\sqrt{3}}$$

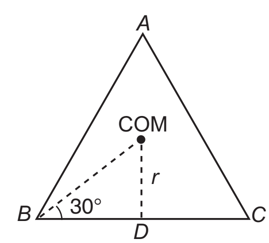


Fig. 12.15

From theorem of parallel axes, moment of inertia of this rod about the asked axis is

$$I_2 = I_1 + mr^2 = \frac{ml^2}{12} + m\left(\frac{l}{2\sqrt{3}}\right)^2 = \frac{ml^2}{6}$$ **Ans.**

$\therefore$ Moment of inertia of all the three rods is

$$I = 3I_2 = 3\left(\frac{ml^2}{6}\right) = \frac{ml^2}{2}$$ **Ans.**

Example 12.3 *Find the moment of inertia of a solid sphere of mass M and radius R about an axis XX shown in figure. Also find radius of gyration about the given axis.*

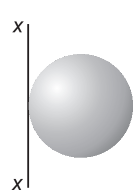


Fig. 12.16

Solution From theorem of parallel axis,

$$I_{XX} = I_{COM} + Mr^2 = \frac{2}{5}MR^2 + MR^2$$

$$= \frac{7}{5}MR^2 \qquad \textbf{Ans.}$$

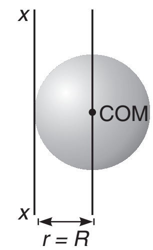


Fig. 12.17

Radius of gyration, $K = \sqrt{\frac{I}{M}} = \sqrt{\frac{\frac{7}{5}MR^2}{M}} = \sqrt{\frac{7}{5}}R$

Note *If whole mass M is kept at a distance* $K\left(=\sqrt{\frac{7}{5}}R\right)$ *as a particle, then moment of inertia is again* $\frac{7}{5}MR^2$.

Example 12.4 *Consider a uniform rod of mass m and length 2l with two particles of mass m each at its ends. Let AB be a line perpendicular to the length of the rod and passing through its centre. Find the moment of inertia of the system about AB.*

Solution

$$I_{AB} = I_{\text{rod}} + I_{\text{both particles}}$$

$$= \frac{m(2l)^2}{12} + 2(ml^2)$$

$$= \frac{7}{3}ml^2 \qquad \textbf{Ans.}$$

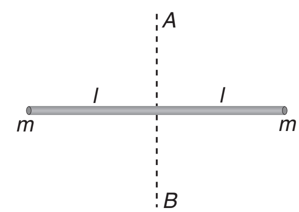


Fig. 12.18

Example 12.5 *Find the moment of inertia of the rod AB about an axis yy as shown in figure. Mass of the rod is m and length is l.*

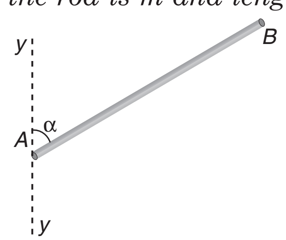


Fig. 12.19

Solution Mass per unit length of the rod $= \frac{m}{l}$

Mass of an element PQ of the rod is, $dm = \left(\frac{m}{l}\right)dx$

Perpendicular distance of this elemental mass about yy is, $r = x\sin\alpha$

∴ Moment of inertia of this small element of the rod (can be assumed as a point mass) about yy is,

$$dI = (dm)r^2 = \left(\frac{m}{l}dx\right)(x\sin\alpha)^2 = \left(\frac{m}{l}\sin^2\alpha\right)x^2\,dx$$

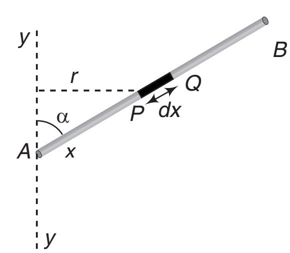


Fig. 12.20

$\therefore$ Moment of inertia of the complete rod,

$$I = \int_{x=0}^{x=l} dI = \frac{m}{l}\sin^2\alpha \int_0^l x^2\, dx = \frac{ml^2}{3}\sin^2\alpha \qquad \textbf{Ans.}$$

Note *(i)* $I = 0$ if $\alpha = 0$ *(ii)* $I = \dfrac{ml^2}{3}$ if $\alpha = \dfrac{\pi}{2}$ or $90°$

INTRODUCTORY EXERCISE 12.1

1. Find the radius of gyration of a rod of mass m and length $2l$ about an axis passing through one of its ends and perpendicular to its length.
2. A mass of 1 kg is placed at (1 m, 2 m, 0). Another mass of 2 kg is placed at (3 m, 4 m, 0). Find moment of inertia of both the masses about z-axis.
3. Four thin rods each of mass m and length l are joined to make a square. Find moment of inertia of all the four rods about any side of the square.
4. About what axis would a uniform cube have its minimum moment of inertia?
5. There are four solid balls with their centres at the four corners of a square of side a. The mass of each sphere is m and radius is r. Find the moment of inertia of the system about (i) one of the sides of the square (ii) one of the diagonals of the square.
6. A non-uniform rod AB has a mass M and length $2l$. The mass per unit length of the rod is mx at a point of the rod distant x from A. Find the moment of inertia of this rod about an axis perpendicular to the rod (a) through A (b) through the mid-point of AB.
7. The uniform disc shown in the figure has a moment of inertia of $0.6\ \text{kg-m}^2$ around the axis that passes through O and is perpendicular to the plane of the page. If a segment is cut out from the disc as shown, what is the moment of inertia of the remaining disc?

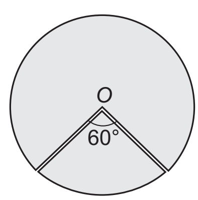


Fig. 12.21

8. If two circular disks of the same weight and thickness are made from metals having different densities. Which disk, if either will have the larger moment of inertia about its central axis.
9. Particles of masses $1\text{g}, 2\text{g}, 3\text{g}, \ldots 100\text{g}$ are kept at the marks $1\text{cm}, 2\text{cm}, 3\text{cm}, \ldots, 100\text{cm}$ respectively on a metre scale. Find the moment of inertia of the system of particles about a perpendicular bisector of the metre scale.
10. If I_1 is the moment of inertia of a thin rod about an axis perpendicular to its length and passing through its centre of mass and I_2 the moment of inertia of the ring formed by the same rod about an axis passing through the centre of mass of the ring and perpendicular to the plane of the ring. Then find the ratio $\dfrac{I_1}{I_2}$.

12.3 Angular Velocity

Angular velocity is a vector quantity. It is represented by ω. Its SI unit is radian per second. It can be defined for following three situations :

(i) Angular velocity of a particle (in motion) about a fixed point.

(ii) Angular velocity of a rigid body in pure rotational motion.

(iii) Angular velocity of a rigid body in rotational and translational motion.

Angular Velocity of a Particle (in motion) about a Fixed Point

At a given instant a particle P has velocity $\mathbf{v}$. It has position vector $\mathbf{r}$ with respect to a fixed point O as shown in figure.

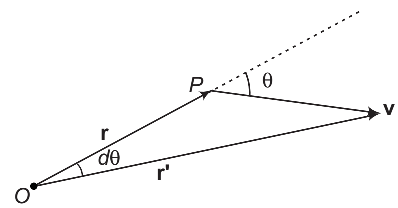


Fig. 12.22

After some time position vector has become $\mathbf{r}'$. We can see two changes in its position vector.

First, its magnitude $|\mathbf{r}|$ has changed, second its direction has changed or we can say, its position vector has been rotated. If we resolve $\mathbf{v}$ along $\mathbf{r}$ and perpendicular to $\mathbf{r}$ then its two components $v_{\parallel}$ and $v_{\perp}$ have the following meanings.

$$v_{\parallel} = v\cos\theta = \frac{\mathbf{v}\cdot\mathbf{r}}{r}$$

$$= \text{component of } \mathbf{v} \text{ along } \mathbf{r} = \frac{d|\mathbf{r}|}{dt}$$

$=$ rate by which magnitude of $\mathbf{r}$ changes

$=$ rate by which distance of P from O changes.

$\theta =$ angle between $\mathbf{r}$ and $\mathbf{v}$

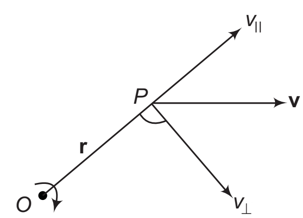


Fig. 12.23

$$\frac{v_{\perp}}{|\mathbf{r}|} = \frac{v_{\perp}}{r} = \frac{v_{\perp}}{OP} = \omega$$

$=$ angular velocity of particle P about point O at this instant

$=$ rate by which $\mathbf{r}$ rotates

If θ is acute, $\cos\theta$ or $v_{\parallel}$ is positive i.e. distance of P from O is increasing. If θ is obtuse, $\cos\theta$ or $v_{\parallel}$ is negative i.e. distance of P from O is decreasing. If θ is 90°, then $\cos\theta$ or $v_{\parallel}$ is zero or distance of P from O is constant. For example, when particle rotates in a circle then with respect to centre, θ is always 90°. This is the reason, why distance of particle from centre always remains constant. With respect to any other point θ is sometimes acute and sometimes obtuse. Therefore, distance sometimes increases and sometimes decreases.

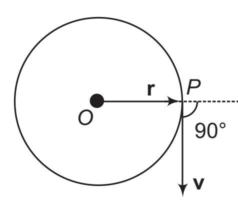


Fig. 12.24

$\mathbf{v}_{\perp} = \omega \times \mathbf{r}$ i.e. perpendicular component of velocity in vector form is the cross product of ω and $\mathbf{r}$.

Direction of ω is given by right hand screw law. For example: in Fig. 12.24 rotation is clockwise. So, ω is perpendicular to paper inwards. If a particle moves in a straight line, then about any point lying on this line angle between $\mathbf{r}$ and $\mathbf{v}$ is 0° or 180°. Hence, $v_{\perp} = 0$ or $\omega = 0$.

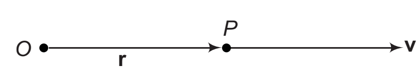


Fig. 12.25

Angular Velocity of a Rigid Body in Pure Rotational Motion

Consider a rigid body rotating about a fixed line AB. Consider a particle P. Draw a perpendicular PO to the axis of rotation. In time Δt, this particle moves to point Q.

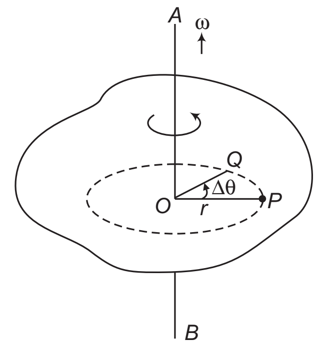


Fig. 12.26

Let $$\angle QOP = \Delta\theta$$

Then, we say that the particle has rotated through an angle $\Delta\theta$. In fact, all the particles of the rigid body have rotated the same angle $\Delta\theta$ or we can say that the whole body has rotated through an angle $\Delta\theta$.

The average angular velocity of the rigid body during the time interval Δt is

$$\omega_{av} = \frac{\Delta\theta}{\Delta t}$$

The instantaneous angular velocity of the rigid body is

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

Direction of angular velocity is given by right hand rule. The direction of angular velocity is defined to be the direction in which the thumb of your right hand points when you curl your fingers in the direction of rotation.

For example, direction of ω in the given figure is along the axis of rotation from B to A. The magnitude of angular velocity is called angular speed. However, we shall continue to use the word angular velocity.

Angular Velocity of a Rigid Body in Rotational and Translational Motion

Consider two particles A and B on a rigid body (in translational and rotational motion). In general, velocity of A is not equal to the velocity of B.

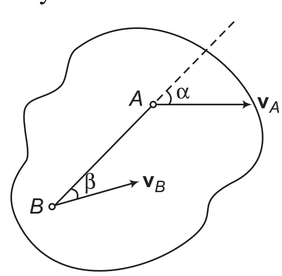


Fig. 12.27

or $$\mathbf{v}_A \neq \mathbf{v}_B$$

Find their components along AB and perpendicular to AB. Now,

(a) Along AB their components are always equal or,

$$v_A \cos\alpha = v_B \cos\beta$$

This is because, in a rigid body distance between two particles (here A and B) is always constant.

If the components are not equal then the distance AB will either increase (when $v_A \cos\alpha > v_B \cos\beta$) or decrease (when $v_B \cos\beta > v_A \cos\alpha$).

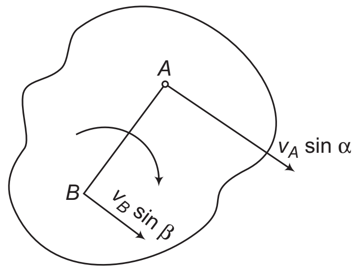


Fig. 12.28

(b) Find relative component perpendicular to AB and divide it by the distance AB to find angular velocity of the rigid body. In the given figure, the perpendicular components are $v_A \sin\alpha$ and $v_B \sin\beta$ (in the same direction). Suppose $v_A \sin\alpha > v_B \sin\beta$. Then, the relative component perpendicular to AB is

$$v_r = v_A \sin\alpha - v_B \sin\beta$$

$$\therefore \qquad \omega = \frac{v_r}{AB} = \frac{v_A \sin\alpha - v_B \sin\beta}{AB}$$

As, $v_A \sin\alpha > v_B \sin\beta$, so rotation of the body is clockwise and according to right hand rule, angular velocity vector is perpendicular to paper inwards. This direction is also shown like ⊗.

Extra Points to Remember

- If a particle P is moving in a circle, its angular velocity about centre of the circle (ω_C) is two times the angular velocity about any point on the circumference of the circle (ω_O) or $\omega_C = 2\omega_O$

  This is because $\angle P'CP = 2\angle P'OP$ (by property of a circle)

  $$\omega_C = \frac{\angle P'CP}{t_{pp'}}, \quad \omega_O = \frac{\angle P'OP}{t_{pp'}}$$

  From these relations we can see that $\omega_C = 2\omega_O$.

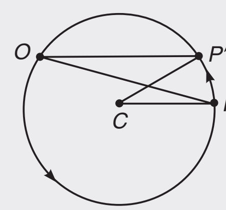


Fig. 12.29

- In pure translational motion of a rigid body its angular velocity will be zero.
- According to right hand rule direction of angular velocity in some cases (in pure rotational motion) have been shown below :

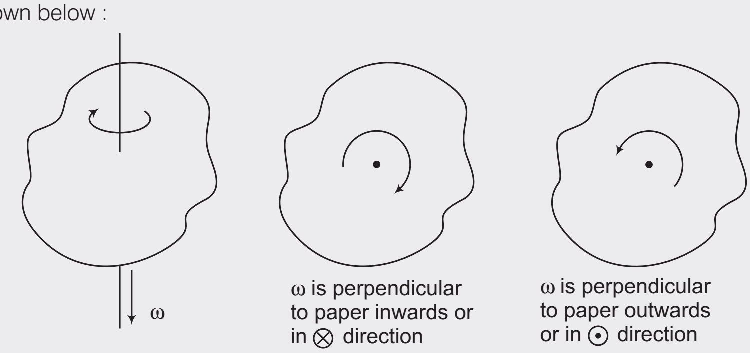


Fig. 12.30

Example 12.6 *Rod AB has a length L. Velocity of end A of the rod has velocity v_0 at the given instant.*

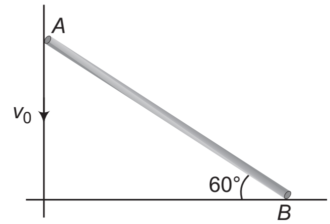


Fig. 12.31

(a) Which type of motion the rod has?
(b) Find velocity of end B at the given instant.
(c) Find the angular velocity of the rod.

Solution (a) The rod has rotational plus translational motion.

(b) Let velocity of end B is v_B in the direction shown in figure.

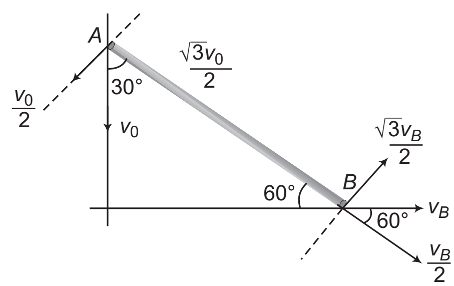


Fig. 12.32

Components of $\mathbf{v}_A$ and $\mathbf{v}_B$ along AB and perpendicular to AB are also shown in the same figure.

Rod is a rigid body. So, distance AB should remain constant or the components along AB should be same.

$\therefore$ $$\frac{\sqrt{3}v_0}{2} = \frac{v_B}{2} \quad \text{or} \quad v_B = \sqrt{3}v_0$$ **Ans.**

(c) Perpendicular to AB, components are in opposite directions. So, the relative component will be

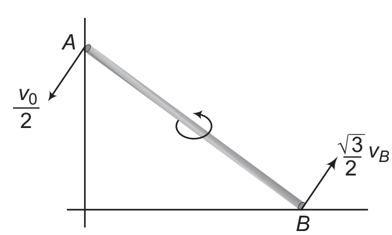


Fig. 12.33

$$v_r = \frac{v_0}{2} + \frac{\sqrt{3}v_B}{2}$$

Substituting $v_B = \sqrt{3}v_0$ we get, $v_r = 2v_0$

Now angular velocity of the rod is $\omega = \frac{v_r}{AB}$

or $\omega = \frac{2v_o}{L}$ **Ans.**

Rotation of the rod is anticlockwise. Therefore, from right hand rule ω is perpendicular to paper outwards or in $\odot$ direction.

INTRODUCTORY EXERCISE 12.2

1. Find angular speed of second's clock.
2. Two points P and Q, diametrically opposite on a disc of radius R have linear velocities v and $2v$ as shown in figure. Find the angular speed of the disc.

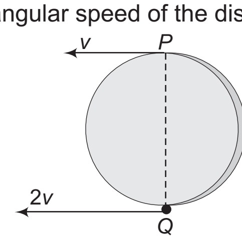


Fig. 12.34

3. A particle is located at (3 m, 4 m) and moving with $\mathbf{v} = (4\hat{\mathbf{i}} - 3\hat{\mathbf{j}})$ m/s. Find its angular velocity about origin at this instant.
4. In the figure shown, the instantaneous speed of end A of the rod is v to the left. Find angular velocity of the rod at given instant.

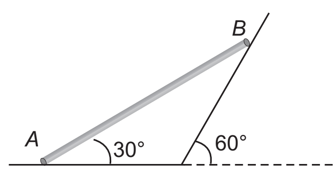


Fig. 12.35

12.4 Torque

Suppose a force **F** is acting on a particle P and let **r** be the position vector of this particle about some reference point O. The torque of this force **F**, about O is defined as,

$$\boxed{\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}}$$

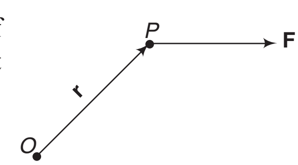


Fig. 12.36

This is a vector quantity having its direction perpendicular to both **r** and **F**, according to the rule of cross product.

Note *Here,* $\mathbf{r} = \mathbf{r}_P - \mathbf{r}_O$

$\mathbf{r}_P$ = *position vector of point, where force is acting and*

$\mathbf{r}_O$ = *position vector of point about which torque is required.*

Torque of a Force about a Line

Consider a rigid body rotating about a fixed axis AB. Let $\mathbf{F}$ be a force acting on the body at point P. Take the origin O somewhere on the axis of rotation. The torque of $\mathbf{F}$ about O is

$$\tau = \mathbf{r} \times \mathbf{F}$$

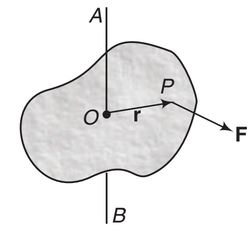


Fig. 12.37

Its component along AB is called the torque of $\mathbf{F}$ about AB. This is also equal to

$$\tau = F \times r_{\perp}$$

Here, $r_{\perp}$ = perpendicular distance of point of application of force from the line AB

Extra Points to Remember

- When a rigid body is rotating about a fixed axis and a force is applied on it at some point then we are concerned with the component of torque of this force about the axis of rotation not with the net torque.
- The component of torque about axis of rotation is independent of the choice of the origin O, so long as it is chosen on the axis of rotation, i.e. we may choose point O anywhere on the line AB.
- Component of torque along axis of rotation AB is zero if
  (a) $\mathbf{F}$ is parallel to AB (b) $\mathbf{F}$ intersects AB at some point
- If $\mathbf{F}$ is perpendicular to AB, but does not intersect it, then component of torque about line AB = magnitude of force $\mathbf{F}$ × perpendicular distance of $\mathbf{F}$ from the line AB (called the lever arm or moment arm) of this torque.
- If there are more than one force $\mathbf{F}_1, \mathbf{F}_2, \ldots$ acting on a body, the total torque will be

$$\tau = \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 + \ldots$$

But if the forces act on the same particle, one can add the forces and then take the torque of the resultant force, or

$$\tau = \mathbf{r} \times (\mathbf{F}_1 + \mathbf{F}_2 + \ldots)$$

Example 12.7 *Find the torque of a force $\mathbf{F} = (\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}})$ N about a point O. The position vector of point of application of force about O is $\mathbf{r} = (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}})$ m.*

Solution Torque $\tau = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 3 & -1 \\ 1 & 2 & -3 \end{vmatrix}$

$$= \hat{\mathbf{i}}(-9+2) + \hat{\mathbf{j}}(-1+6) + \hat{\mathbf{k}}(4-3)$$

or $\tau = (-7\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + \hat{\mathbf{k}})$ N-m **Ans.**

Example 12.8 *A small ball of mass 1.0 kg is attached to one end of a 1.0 m long massless string and the other end of the string is hung from a point O. When the resulting pendulum is making 30° from the vertical, what is the magnitude of net torque about the point of suspension?*
[Take $g = 10\ m/s^2$]

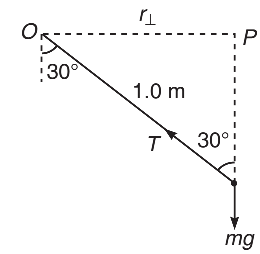


Fig. 12.38

Solution Two forces are acting on the ball

(*i*) tension (T) (*ii*) weight (mg)

Torque of tension about point O is zero, as it passes through O.

$$\tau_{mg} = F \times r_\perp$$

Here, $r_\perp = OP = 1.0 \sin 30° = 0.5\ \text{m}$

$\therefore$ $\tau_{mg} = (mg)(0.5)$

$= (1)(10)(0.5) = 5\ \text{N-m}$ **Ans.**

Example 12.9 *A force* $\mathbf{F} = (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 4\hat{\mathbf{k}})$ *N is acting at point* $P(2\ m, -3\ m, 6\ m)$. *Find torque of this force about a point O whose position vector is* $(2\hat{\mathbf{i}} - 5\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$ *m.*

Solution $\tau = \mathbf{r} \times \mathbf{F}$ Here, $\mathbf{r} = \mathbf{r}_P - \mathbf{r}_O = (2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}) - (2\hat{\mathbf{i}} - 5\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) = (2\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$ m

Now, $$\tau = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 2 & 3 \\ 2 & 3 & -4 \end{vmatrix} = (-17\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 4\hat{\mathbf{k}})\ \text{N-m}$$ **Ans.**

INTRODUCTORY EXERCISE 12.3

1. A force $\mathbf{F} = (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 2\hat{\mathbf{k}})$ N is acting on a body at point (2 m, 4 m, –2 m). Find torque of this force about origin.
2. A particle of mass $m = 1$kg is projected with speed $u = 20\sqrt{2}$ m/s at angle $\theta = 45°$ with horizontal. Find the torque of the weight of the particle about the point of projection when the particle is at the highest point.
3. Point C is the centre of mass of the rigid body shown in figure. Find the total torque acting on the body about point C.

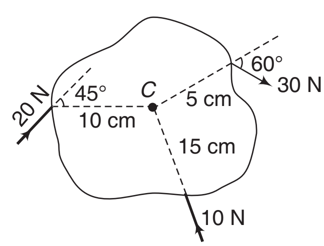


Fig. 12.39

4. Find the net torque on the wheel in figure about the point O if $a = 10$ cm and $b = 25$ cm.

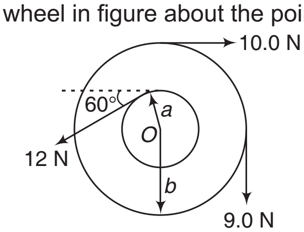


Fig. 12.40

12.5 Rotation of a Rigid Body about a Fixed Axis

When a body is rotating about a fixed axis, any point P located in the body travels along a circular path. Before, analyzing the circular motion of point P, we will first study the angular motion properties of a rigid body.

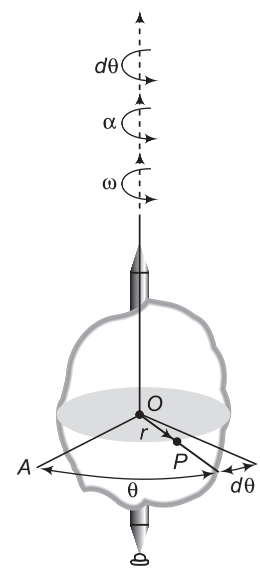


Fig. 12.41

Angular Motion

Since, a point is without dimension, it has no angular motion. Only lines or bodies undergo angular motion. Let us consider the angular motion of a radial line r located with the shaded plane.

Angular Position

The angular position of r is defined by the angle θ, measured between a fixed reference line OA and r.

Angular Displacement

The change in the angular position, often measured as a differential $d\theta$ is called the angular displacement. (Finite angular displacements are not vector quantities, although differential rotations $d\theta$ are vectors). This vector has a magnitude $d\theta$ and the direction of $d\theta$ is along the axis.

Specifically, the direction of $d\theta$ is determined by right hand rule; that is, the fingers of the right hand are curled with the sense of rotation, so that in this case the thumb or $d\theta$ points upward.

Angular Velocity

The time rate of change in the angular position is called the angular velocity ω. Thus,

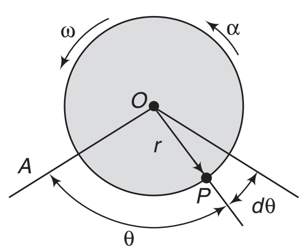


Fig. 12.42

$$\boxed{\omega = \frac{d\theta}{dt}} \qquad \text{...(i)}$$

It is expressed here in scalar form, since its direction is always along the axis of rotation, i.e. in the same direction as $d\theta$.

Angular Acceleration

The angular acceleration α measures the time rate of change of the angular velocity. Hence, the magnitude of this vector may be written as,

$$\alpha = \frac{d\omega}{dt} \qquad \text{...(ii)}$$

It is also possible to express α as,

$$\alpha = \frac{d^2\theta}{dt^2}$$

The line of action of α is the same as that for ω, however its sense of direction depends on whether ω is increasing or decreasing with time. In particular, if ω is decreasing, α is called an angular deceleration and therefore, has a sense of direction which is opposite to ω.

Torque and Angular Acceleration for a Rigid Body

The angular acceleration of a rigid body is directly proportional to the sum of the torque components along the axis of rotation. The proportionality constant is the inverse of the moment of inertia about that axis, or

$$\alpha = \frac{\Sigma\tau}{I}$$

Thus, for a rigid body we have the rotational analog of Newton's second law :

$$\Sigma\tau = I\alpha \qquad \text{...(iii)}$$

Following two points are important regarding the above equation :

(i) The above equation is valid only for rigid bodies. If the body is not rigid like a rotating tank of water, the angular acceleration α is different for different particles.

(ii) The sum $\Sigma\tau$ in the above equation includes only the torques of the external forces, because all the internal torques add to zero.

Rotation with Constant Angular Acceleration

If the angular acceleration of the body is constant then Eqs. (i) and (ii) when integrated yield a set of formulae which relate the body's angular velocity, angular position and time. These equations are similar to equations used for rectilinear motion. Table given ahead compares the linear and angular motion with constant acceleration.

Table 12.2

Straight line motion with constant linear acceleration	Fixed axis rotation with constant angular acceleration
$a =$ constant	$\alpha =$ constant
$v = u + at$	$\omega = \omega_0 + \alpha t$
$s = s_0 + ut + \frac{1}{2}at^2$	$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 = u^2 + 2a(s - s_0)$	$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$

Here, θ_0 and ω_0 are the initial values of the body's angular position and angular velocity respectively.

Note *If α is not constant then we will have to take help of either differentiation or integration (with limits). The equations involved are :*

Equations of Differentiation

$$\omega = \frac{d\theta}{dt} \text{ and } \alpha = \frac{d\omega}{dt} = \omega\frac{d\omega}{d\theta}$$

Equations of Integration

$$\int d\theta = \int \omega dt, \int d\omega = \int \alpha dt \text{ and } \int \omega d\omega = \int \alpha d\theta$$

Kinetic Energy of a Rigid Body Rotating about a Fixed Axis

Suppose a rigid body is rotating about a fixed axis with angular speed ω. Then, kinetic energy of the rigid body will be

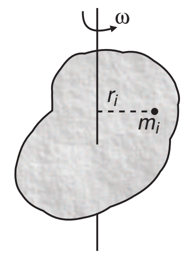


Fig. 12.43

$$K = \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i (\omega r_i)^2$$

$$= \frac{1}{2}\omega^2 \sum_i m_i r_i^2 = \frac{1}{2} I\omega^2 \quad (\text{as } \sum_i m_i r_i^2 = I)$$

Thus, $$\boxed{\text{KE} = \frac{1}{2} I\omega^2}$$

Sometimes it is called the rotational kinetic energy.

Extra Points to Remember

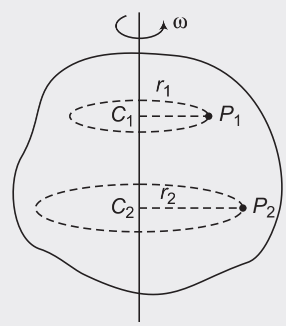


Fig. 12.44

- In pure rotational motion of a rigid body all particles rotate in circular paths except the particles lying on the axis.The particles lying on the axis are at rest.
- Planes of all circular paths are mutually parallel with their centres lying on the axis.
- Linear velocity of any particle is tangential to its own circle and its linear speed is,

$$v = r\omega$$

In this equation, ω is same for all particles (and that is also called ω of the rigid body).

$\therefore$ $$v \propto r$$

In the figure shown, $r_2 > r_1 \Rightarrow \therefore v_2 > v_1$

Further, r for the particles lying on the axis is zero. Therefore, their linear velocity is zero or they are at rest.

- Since every particle is rotating in a circle (except the particles lying on the axis). Hence acceleration of the particle has two components :

(i) Radial component or centripetal acceleration given by

$$a_r = r\omega^2 = \frac{v^2}{r}$$

This component is always towards the centre and this can't be zero.

(ii) Tangential component given by $a_t = \frac{dv}{dt} = r\alpha$ where, $\alpha = \frac{d\omega}{dt}$

This component is tangential and it may be zero, positive or negative. If v or ω is constant then a_t is zero. If v or ω is increasing then this component is positive and in the direction of linear velocity. If v or ω is decreasing then this component is negative and in the opposite direction of linear velocity.

(iii) Net acceleration of the particle is the resultant of these two perpendicular components.

Note *From the above discussions, we have seen that pure rotational motion of a rigid body is nothing but circular motion of its different particles. So, before solving the problems of this topic make yourself expert in circular motion.*

Example 12.10 *A solid sphere of mass 2 kg and radius 1 m is free to rotate about an axis passing through its centre. Find a constant tangential force F required to rotate the sphere with 10 rad/s in 2 s. Also find the number of rotations made by the sphere in that time interval.*

Solution Since, the force is constant, the torque produced by it and the angular acceleration α will be constant. Hence, we can apply

$$\omega = \omega_0 + \alpha t \quad \Rightarrow \quad 10 = 0 + (\alpha)(2) \quad \Rightarrow \quad \alpha = 5 \text{ rad/s}^2$$

Further, the force is tangential. Therefore, the perpendicular distance from the axis of rotation will be equal to the radius of the sphere.

$$\therefore \qquad \alpha = \frac{\tau}{I} = \frac{F \cdot R}{\frac{2}{5} mR^2} = \frac{5F}{2mR} \quad \text{or} \quad F = \frac{2mR\alpha}{5}$$

Substituting the value, we have $F = \dfrac{(2)(2)(1)(5)}{(5)} = 4 \text{ N}$ **Ans.**

Further, angle rotated $\theta = \dfrac{1}{2}\alpha t^2 = \dfrac{1}{2}(5)(2)^2 = 10 \text{ rad}$

$\therefore$ Number of rotations $n = \dfrac{\theta}{2\pi} = \dfrac{10}{2\pi} = \dfrac{5}{\pi}$ **Ans.**

Example 12.11 *The angular position of a point on the rim of a rotating wheel is given by $\theta = 4t - 3t^2 + t^3$, where θ is in radians and t is in seconds. What are the angular velocities at*

(a) $t = 2.0$ s and *(b) $t = 4.0$ s ?*

(c) What is the average angular acceleration for the time interval that begins at $t = 2.0$ s and ends at $t = 4.0$ s?

(d) What are the instantaneous angular acceleration at the beginning and the end of this time interval?

Solution Angular velocity $\omega = \dfrac{d\theta}{dt} = \dfrac{d}{dt}(4t - 3t^2 + t^3)$ or $\omega = 4 - 6t + 3t^2$

(a) At $t = 2.0$ s, $\omega = 4 - 6 \times 2 + 3(2)^2$ or $\omega = 4$ rad/s **Ans.**

(b) At $t = 4.0$ s, $\omega = 4 - 6 \times 4 + 3(4)^2$ or $\omega = 28$ rad/s **Ans.**

(c) Average angular acceleration $\alpha_{av} = \dfrac{\omega_f - \omega_i}{t_f - t_i} = \dfrac{28 - 4}{4 - 2}$ or $\alpha_{av} = 12 \text{ rad/s}^2$ **Ans.**

(d) Instantaneous angular acceleration is,

$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}(4 - 6t + 3t^2) \quad \text{or} \quad \alpha = -6 + 6t$$

At $t = 2.0$ s, $\alpha = -6 + 6 \times 2 = 6 \text{ rad/s}^2$ **Ans.**

At $t = 4.0$ s, $\alpha = -6 + 6 \times 4 = 18 \text{ rad/s}^2$ **Ans.**

Example 12.12 *A circular disc is rotating with an angular speed (in radian per sec)*

$$\omega = 2t^2$$

Given, $CP = 2\,m$

In terms of $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ *and* $\hat{\mathbf{k}}$, *at* $t = 1$ s *find,*

(a) ω

(b) α

(c) linear velocity of the particle lying at P

(d) linear acceleration of the particle lying at P

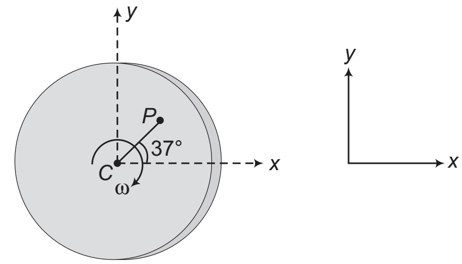


Fig. 12.45

Solution $\omega = 2t^2 \quad \Rightarrow \quad \alpha = \dfrac{d\omega}{dt} = 4t$

At $t = 1$s, $\qquad \omega = 2\,\text{rad/s}$ and $\alpha = 4\,\text{rad/s}^2$

For the particle at P, $\qquad r = CP = 2$m

(a) Rotation is clockwise. So, according to right hand rule, ω is perpendicular to paper inwards along negative z-direction.

$\therefore \qquad \boldsymbol{\omega} = (-2\hat{\mathbf{k}})\,\text{rad/s}$

(b) ω is increasing. So, α is also in the direction of ω.

$\therefore \qquad \boldsymbol{\alpha} = (-4\hat{\mathbf{k}})\,\text{rad/s}^2$

(c) $v = r\omega = (2)(2) = 4$ m/s

This velocity is tangential to the doted circle of P as shown in figure.

$\therefore \qquad \mathbf{v} = (4\cos 53°)\hat{\mathbf{i}} - (4\sin 53°)\hat{\mathbf{j}}$

or $\qquad \mathbf{v} = (2.4\hat{\mathbf{i}} - 3.2\hat{\mathbf{j}})$ m/s

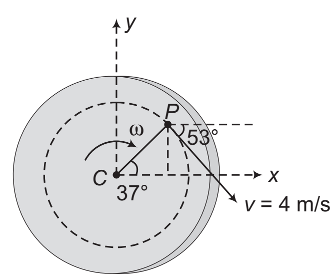


Fig. 12.46

(d) Acceleration of the particle has two components

(i) $a_r = r\omega^2$ (radial component)

$= (2)(2)^2 = 8\,\text{m/s}^2$

This components is towards centre C.

(ii) $a_t = r\alpha$ (tangential component)

$= (2)(4) = 8\,\text{m/s}^2$

This component is in the direction of linear velocity, as ω or v is increasing.

$\therefore \qquad \mathbf{a} = (8\cos 53° - 8\cos 37°)(\hat{\mathbf{i}}) + (8\sin 53° + 8\sin 37°)(-\hat{\mathbf{j}})$

or $\qquad \mathbf{a} = (-1.6\hat{\mathbf{i}} - 11.2\hat{\mathbf{j}})\,\text{m/s}^2$ **Ans.**

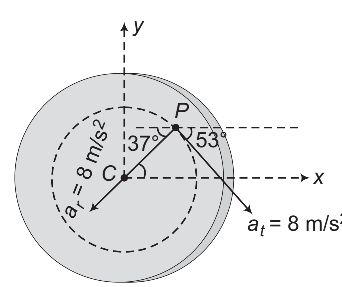


Fig. 12.47

INTRODUCTORY EXERCISE 12.4

1. A wheel rotating with uniform angular acceleration covers 50 rev in the first five seconds after the start. Find the angular acceleration and the angular velocity at the end of five seconds.
2. A body rotates about a fixed axis with an angular acceleration 1 rad/s^2. Through what angle does it rotates during the time in which its angular velocity increases from 5 rad/s to 15 rad/s?
3. A flywheel of moment of inertia 5.0 kg-m^2 is rotated at a speed of 10 rad/s. Because of the friction at the axis it comes to rest in 10 s. Find the average torque of the friction.
4. A wheel starting from rest is uniformly accelerated at 4 rad/s^2 for 10 s. It is allowed to rotate uniformly for the next 10 s and is finally brought to rest in the next 10 s. Find the total angle rotated by the wheel.
5. A wheel of mass 10 kg and radius 0.2 m is rotating at an angular speed of 100 rpm, when the motion is turned off. Neglecting the friction at the axis, calculate the force that must be applied tangentially to the wheel to bring it to rest in 10 rev. Assume wheel to be a disc.
6. A solid body rotates about a stationary axis according to the law $\theta = 6t - 2t^3$. Here, θ is in radian and t in seconds. Find
   (a) the mean values of the angular velocity and angular acceleration averaged over the time interval between $t = 0$ and the complete stop,
   (b) the angular acceleration at the moment when the body stops.

Hint If $y = y(t)$, then mean/average value of y between t_1 and t_2 is $\langle y \rangle = \dfrac{\int_{t_1}^{t_2} y(t)\,dt}{t_2 - t_1}$.

7. A body rotating at 20 rad/s is acted upon by a constant torque providing it a deceleration of 2 rad/s^2. At what time will the body have kinetic energy same as the initial value if the torque continues to act ?
8. A wheel whose moment of inertia is 0.03 kg m^2, is accelerated from rest to 20 rad/s in 5 s. When the external torque is removed, the wheel stops in 1 min. Find
   (a) the frictional torque, (b) the external torque.
9. A flywheel whose moment of inertia about its axis of rotation is 16 kg-m^2 is rotating freely in its own plane about a smooth axis through its centre. Its angular velocity is 9 rad s^{-1} when a torque is applied to bring it to rest in t_0 seconds. Find t_0 if
   (a) the torque is constant and of magnitude 4 N-m,
   (b) the magnitude of the torque after t seconds is given by kt.
10. A shaft is turning at 65 rad/s at time zero. Thereafter, angular acceleration is given by $\alpha = -10$ rad/s^2 $-5t$ rad/s^2
    where t is the elapsed time.
    (a) Find its angular speed at $t = 3.0$ s.
    (b) How much angle does it turn in these 3s ?
11. The angular velocity of a gear is controlled according to $\omega = 12 - 3t^2$ where ω, in radian per second, is positive in the clockwise sense and t is the time in seconds. Find the net angular displacement $\Delta\theta$ from the time $t = 0$ to $t = 3$ s. Also, find the number of revolutions N through which the gear turns during the 3 s.

12.6 Angular Momentum

A mass moving in a straight line has linear momentum $(\mathbf{P})$. When a mass rotates about some point/axis, there is momentum associated with rotational motion called the angular momentum $(\mathbf{L})$. Just as net external force is required to change the linear momentum of an object a net external torque is required to change the angular momentum of an object. Keeping in view the problems asked in JEE, the angular momentum is classified in following three types.

Angular Momentum of a Particle about a Fixed Point

Suppose a particle A of mass m is moving with linear momentum $\mathbf{P} = m\mathbf{v}$. Its angular momentum $\mathbf{L}$ about point O is defined as

$$\boxed{\mathbf{L} = \mathbf{r} \times \mathbf{P} = \mathbf{r} \times (m\mathbf{v}) = m(\mathbf{r} \times \mathbf{v})}$$

Here, $\mathbf{r}$ is the radius vector of particle A about O at that instant of time. The magnitude of $\mathbf{L}$ is

$$\boxed{L = mvr \sin\theta = mvr_{\perp}}$$

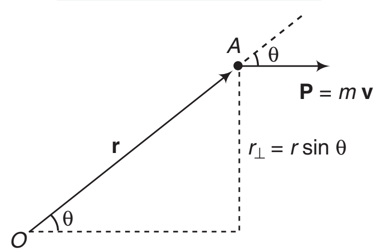


Fig. 12.48

Here, $r_{\perp} = r \sin\theta$ is the perpendicular distance of line of action of velocity $\mathbf{v}$ from point O. The direction of $\mathbf{L}$ is same as that of $\mathbf{r} \times \mathbf{v}$.

Direction of angular momentum can also be by right hand rule as shown below:

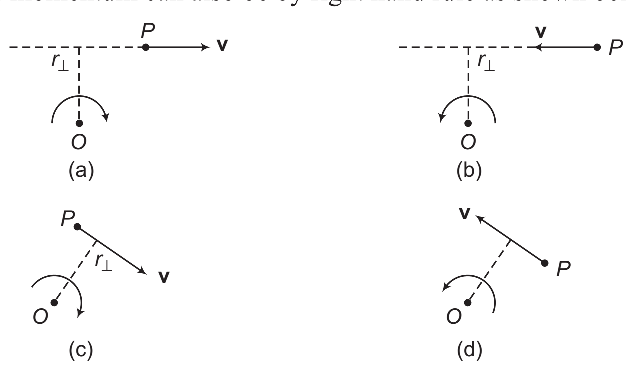


Fig. 12.49

In figures (a) and (c), rotation is clockwise. Hence, according to right hand rule angular momentum is perpendicular to paper inwards or in $\otimes$ direction.

In figures (b) and (d), rotation is anticlockwise. Hence, direction of angular momentum is perpendicular to paper outwards or in $\odot$ direction.

Extra Points to Remember

- If line of action of velocity passes through the point O, then $r_\perp = 0$. Therefore, angular momentum is zero.

Fig. 12.50

- If a particle rotates in a circle then $r_\perp$ from centre is always equal to R (= radius of circle). Or, θ between $\mathbf{r}$ and $\mathbf{v}$ (or $\mathbf{P}$) is always 90°. Therefore, angular momentum about centre is $L = mvR$

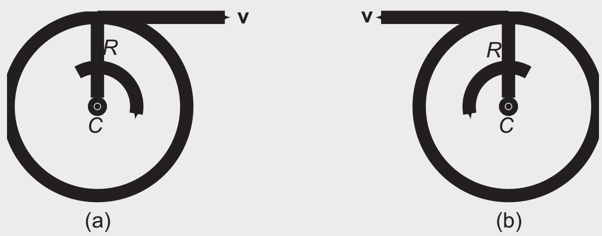


Fig. 12.51

In figure (a), direction of angular momentum is perpendicular to paper inwards ⊗ and in figure (b) outwards or ʘ.

- If a particle is moving with a constant velocity (speed and direction of velocity both are constant) then angular momentum about any point always remains constant. But this constant value will be different about different points. In the figure shown,

$L_{O_1} = 0$ as $r_\perp = 0$

$L_{O_2} = mvb$ (= constant), perpendicular to paper inwards

$L_{O_3} = mvc$ (= constant), perpendicular to paper outwards.

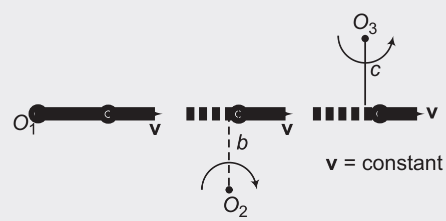


Fig. 12.52

Angular Momentum of a Rigid Body Rotating about a Fixed Axis

Suppose a particle P of mass m is going in a circle of radius r and at some instant the speed of the particle is v. For finding the angular momentum of the particle about the axis of rotation, the origin may be chosen anywhere on the axis. We choose it at the centre of the circle. In this case $\mathbf{r}$ and $\mathbf{P}$ are perpendicular to each other and $\mathbf{r} \times \mathbf{P}$ is along the axis. Thus, component of $\mathbf{r} \times \mathbf{P}$ along the axis is mvr itself. The angular momentum of the whole rigid body about AB is the sum of components of all particles, i.e.

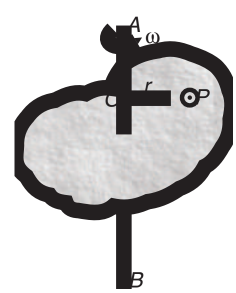


Fig. 12.53

$$L = \sum_i m_i r_i v_i$$

Here, $$v_i = r_i \omega$$

∴ $$L = \sum_i m_i r_i^2 \omega$$

or $$L = \omega \sum_i m_i r_i^2 \quad \text{or} \quad \boxed{L = I\omega} \qquad \left(\text{as } \sum_i m_i r_i^2 = I\right)$$

Here, I is the moment of inertia of the rigid body about AB.

Thus, $L = I\omega$ is the component of angular momentum of the whole rigid body about axis of rotation AB. Direction of this component is again given by right hand rule. For example, in the given figure $L = I\omega$ is upwards or along the axis of rotation from B to A.

Extra Points to Remember

- The vector relation $\mathbf{L} = I\omega$ is not correct in the above case because $\mathbf{L}$ and ω do not point in the same direction, but we could write $L_{AB} = I\omega$. If however the body is symmetric about the axis of rotation $\mathbf{L}$ and ω are parallel and we can write $(L = I\omega)$ in vector form as $\mathbf{L} = I\omega$.
- By symmetric we mean that for every mass element in the body there must be an identical mass element diametrically opposite the first element and at the same distance from the axis of rotation.
- Thus, remember that $\mathbf{L} = I\omega$ applies only to bodies that have symmetry about the (fixed) rotational axis. Here, $\mathbf{L}$ stands for total angular momentum. However the relation $L_{AB} = I\omega$ holds for any rigid body symmetrical or not that is rotating about a fixed axis.

Angular Momentum of a Rigid Body in Combined Rotation and Translation

Let O be the fixed point in an inertial frame of reference.

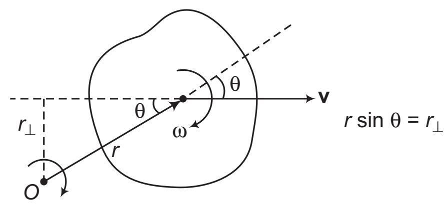


Fig. 12.54

Angular momentum of the rigid body about O is the vector sum of two terms (as discussed above).

(i) $mvr\sin\theta$ or $mvr_\perp$ (ii) $I\omega$

Here, $\mathbf{v}$ is the velocity of centre of mass. Moment of inertia I is about an axis passing through centre of mass. Directions of above two terms can be determined by right hand rule. If both the terms are in the same direction then these two terms are additive and if they are in opposite directions, then they are subtractive.

Example 12.13 *A particle of mass m is moving along the line $y = b, z = 0$ with constant speed v. State whether the angular momentum of particle about origin is increasing, decreasing or constant.*

Solution $|\mathbf{L}| = mvr\sin\theta = mvr_\perp = mvb$

$\therefore$ $|\mathbf{L}| =$ constant as m, v and b all are constants.

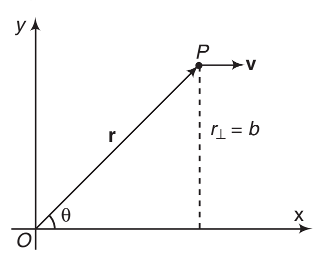


Fig. 12.55

Direction of $\mathbf{r} \times \mathbf{v}$ also remains the same. Therefore, angular momentum of particle about origin remains constant with due course of time.

Note *In this problem $|\mathbf{r}|$ is increasing, θ is decreasing but $r\sin\theta$, i.e. b remains constant. Hence, the angular momentum remains constant.*

Example 12.14 *A particle of mass m is projected from origin O with speed u at an angle θ with positive x-axis. Positive y-axis is in vertically upward direction. Find the angular momentum of particle at any time t about O before the particle strikes the ground again.*

Solution $$\mathbf{L} = m(\mathbf{r} \times \mathbf{v})$$

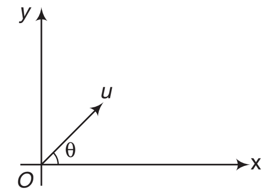


Fig. 12.56

Here, $$\mathbf{r}(t) = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} = (u\cos\theta)t\hat{\mathbf{i}} + (ut\sin\theta - \frac{1}{2}gt^2)\hat{\mathbf{j}}$$

and $$\mathbf{v}(t) = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} = (u\cos\theta)\hat{\mathbf{i}} + (u\sin\theta - gt)\hat{\mathbf{j}}$$

$\therefore$ $$\mathbf{L} = m(\mathbf{r} \times \mathbf{v}) = m\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ (u\cos\theta)t & (u\sin\theta)t - \frac{1}{2}gt^2 & 0 \\ u\cos\theta & u\sin\theta - gt & 0 \end{vmatrix}$$

$$= m\left[(u^2\sin\theta\cos\theta)t - (u\cos\theta)gt^2 - (u^2\sin\theta\cos\theta)t + \frac{1}{2}(u\cos\theta)gt^2\right]\hat{\mathbf{k}}$$

$$= -\frac{1}{2}m(u\cos\theta)gt^2\,\hat{\mathbf{k}}$$ **Ans.**

Example 12.15 *A rod of mass 2kg and length 2m is rotating about its one end O with an angular velocity $\omega = 4$ rad/s. Find angular momentum of the rod about the axis rotation.*

Fig. 12.57

Solution In pure rotational motion of a rigid body, component of total angular momentum about axis of rotation is given by

$$L = I\omega = \left(\frac{ml^2}{3}\right)\omega \qquad \left(I_0 = \frac{ml^2}{3}\right)$$

Substituting the values we have,

$$L = \frac{(2)(2)^2}{3}(4)$$

$$= \frac{32}{3}\text{ kg-m}^2/\text{s}$$ **Ans.**

Direction of this component is perpendicular to paper inwards (from right hand rule), as the rotation is clockwise.

Example 12.16 *A circular disc of mass m and radius R is set into motion on a horizontal floor with a linear speed v in the forward direction and an angular speed $\omega = \frac{v}{R}$ in clockwise direction as shown in figure. Find the magnitude of the total angular momentum of the disc about bottommost point O of the disc.*

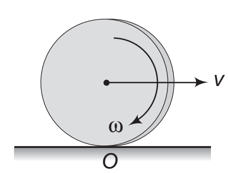


Fig. 12.58

Solution As we have discussed, angular momentum about O is the vector sum of two terms:

$$I\omega \text{ and } mvr_{\perp}$$

Here, $$I\omega = I_{cm}\omega = \left(\frac{1}{2}mR^2\right)\left(\frac{v}{R}\right) = \frac{1}{2}mvR \quad \text{(perpendicular to paper inwards)}$$

and $$mvr_{\perp} = mv_c r_{\perp} = mRv \quad \text{(perpendicular to paper inwards)}$$

Since, both the terms are in the same direction.

$$L_{\text{Total}} = \frac{1}{2}mvR + mvR$$

or $$L_{\text{Total}} = \frac{3}{2}mvR \quad \text{(perpendicular to paper inwards)}$$

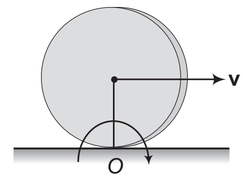


Fig. 12.59

INTRODUCTORY EXERCISE 12.5

1. A uniform rod of mass m is rotated about an axis passing through point O as shown. Find angular momentum of the rod about rotational axis.

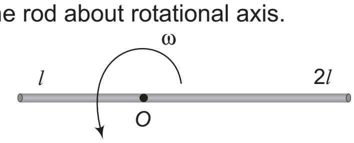


Fig. 12.60

2. A particle of mass 1 kg is moving along a straight line $y = x + 4$. Both x and y are in metres. Velocity of the particle is 2 m/s. Find magnitude of angular momentum of the particle about origin.
3. A particle of mass m is projected from the ground with an initial speed u at an angle α. Find the magnitude of its angular momentum at the highest point of its trajectory about the point of projection.
4. If the angular momentum of a body is zero about some point. Is it necessary that it will be zero about a different point?
5. A solid sphere of mass m and radius R is rolling without slipping as shown in figure. Find angular momentum of the sphere about z-axis.

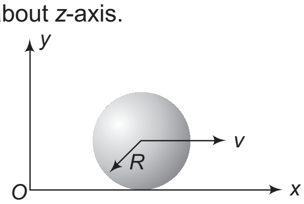


Fig. 12.61

6. In example number 12.16 suppose the disc starts rotating anticlockwise with the same angular velocity $\omega = \frac{v}{R}$, then what will be the angular momentum of the disc about bottommost point in this new situation?
7. Two particles each of mass m and speed v, travel in opposite directions along parallel lines separated by a distance d. Show that the vector angular momentum of this system of particles is the same about any point taken as origin.

12.7 Conservation of Angular Momentum

As we have seen in Article 12.6, the angular momentum of a particle about some reference point O is defined as,

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \qquad \ldots\text{(i)}$$

Here, **P** is the linear momentum of the particle and **r** its position vector with respect to the reference point O. Differentiating Eq. (i) with respect to time, we get

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \frac{d\mathbf{p}}{dt} + \frac{d\mathbf{r}}{dt} \times \mathbf{p} \qquad \ldots\text{(ii)}$$

Here, $\frac{d\mathbf{p}}{dt} = \mathbf{F}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{v}$ (velocity of particle)

Hence, Eq. (ii) can be rewritten as,

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \mathbf{F} + \mathbf{v} \times \mathbf{p}$$

Now, $\mathbf{v} \times \mathbf{p} =$ a null vector, because **v** and **p** are parallel to each other and the cross product of two parallel vectors is a null vector. Thus,

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \mathbf{F} = \tau \quad \text{or} \quad \boxed{\tau = \frac{d\mathbf{L}}{dt}} \qquad \ldots\text{(iii)}$$

Which states that the time rate of change of angular momentum of a particle about some reference point in an inertial frame of reference is equal to the net torques acting on it. This result is rotational analog of the equation $\mathbf{F} = \frac{d\mathbf{p}}{dt}$, which states that the time rate of change of the linear momentum of a particle is equal to the force acting on it. Eq. (iii), like all vector equations, is equivalent to three scalar equations, namely

$$\tau_x = \left(\frac{dL}{dt}\right)_x, \quad \tau_y = \left(\frac{dL}{dt}\right)_y \quad \text{and} \quad \tau_z = \left(\frac{dL}{dt}\right)_z$$

The same equation can be generalised for a system of particles as, $\tau_{\text{ext}} = \frac{d\mathbf{L}}{dt}$. According to which the time rate of change of the total angular momentum of a system of particles about some reference point in an inertial frame of reference is equal to the sum of all external torques (of course the vector sum) acting on the system about the same reference point.

Now, suppose that $\tau_{\text{ext}} = 0$, then $\frac{d\mathbf{L}}{dt} = 0$, so that $\mathbf{L} =$ constant.

When the resultant external torque acting on a system is zero, the total vector angular momentum of the system remains constant. This is the principle of the conservation of angular momentum.

For a rigid body rotating about an axis (the z-axis, say) that is fixed in an inertial reference frame, we have

$$L_z = I\omega$$

It is possible for the moment of inertia I of a rotating body to change by rearrangement of its parts. If no net external torque acts, then L_z must remain constant and if I does change, there must be a compensating change in ω. The principle of conservation of angular momentum in this case is expressed as

$$I\omega = \text{constant} \quad \text{or} \quad \boxed{I_1\omega_1 = I_2\omega_2} \qquad \text{...(iv)}$$

Note *According to above equation, if some mass moves away from the axis of rotation, its moment of inertia will increase. So, to conserve angular momentum or $I\omega$ its angular speed ω should decrease. Therefore, time period of rotation $\left(T = \frac{2\pi}{\omega}\right)$ should also increase.*

Example 12.17 *A wheel of moment of inertia I and radius R is rotating about its axis at an angular speed ω_0. It picks up a stationary particle of mass m at its edge. Find the new angular speed of the wheel.*

Solution Net external torque on the system is zero. Therefore, angular momentum will remain conserved. Thus,

$$I_1\omega_1 = I_2\omega_2 \quad \text{or} \quad \omega_2 = \frac{I_1\omega_1}{I_2}$$

Here, $I_1 = I, \quad \omega_1 = \omega_0, \ I_2 = I + mR^2$

$\therefore$ $$\omega_2 = \frac{I\omega_0}{I + mR^2}$$ **Ans.**

INTRODUCTORY EXERCISE 12.6

1. A thin circular ring of mass M and radius R is rotating about its axis with an angular speed ω_0. Two particles each of mass m are now attached at diametrically opposite points. Find the new angular speed of the ring.
2. If the ice at the poles melts and flows towards the equator, how will it affect the duration of day-night?
3. When tall buildings are constructed on earth, the duration of day night slightly increases. Is this statement true or false?
4. If radius of earth is increased, without change in its mass, will the length of day increase, decrease or remain same?

12.8 Combined Translational and Rotational Motion of a Rigid Body

This is the most complex motion of a rigid body. But this can be simplified by splitting this motion into following two parts:

(i) pure translational motion with the velocity $(\mathbf{v})$ and acceleration $(\mathbf{a})$ of centre of mass.

(ii) pure rotational motion about an axis passing through centre of mass with angular velocity ω and angular acceleration α.

As we have discussed earlier also, different particles of the rigid body in this type of motion have different linear velocity and linear acceleration. So, now the question is how to find linear velocity and linear acceleration of a general particle P on the rigid body?

Method of Finding Linear Velocity of a General Particle *P*

Suppose linear velocity v (of centre of mass) and angular velocity ω (of the rigid body) are known to us and we wish to find linear velocity of a general particle P at a distance r from C.

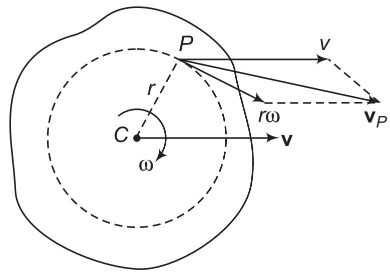


Fig. 12.62

As we know that

$$\mathbf{v}_{PC} = \mathbf{v}_P - \mathbf{v}_C$$

$$\therefore \quad \mathbf{v}_P = \mathbf{v}_C + \mathbf{v}_{PC}$$

or, absolute velocity of P is the vector sum of $\mathbf{v}_C$ and $\mathbf{v}_{PC}$. Here, $\mathbf{v}_C = \mathbf{v}$ and motion of P with respect to C is a circle (dotted circle shown in figure). In circular motion, velocity of a particle is $r\omega$, tangential to its circle (in the direction of rotation). So, $\mathbf{v}_{PC}$ is $r\omega$ in the tangential direction as shown in Fig. 12.62. Thus net velocity of P is the vector sum of following two terms:

(i) v and (ii) $r\omega$

For different particles, values of v and ω are same. But values of r and therefore $r\omega$ and direction of $r\omega$ are different . This is the reason why different particles have different linear velocities.

Method of Finding Acceleration of a General Particle *P*

Suppose linear acceleration a (of centre of mass) angular velocity ω and angular acceleration α (of the rigid body) are known to us and we wish to find linear acceleration of a general particle P at a distance r from C.

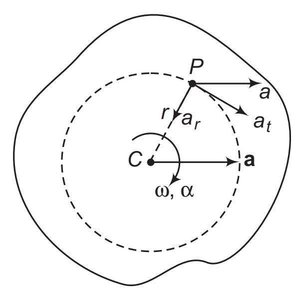


Fig. 12.63

As we know that

$$\mathbf{a}_{PC} = \mathbf{a}_P - \mathbf{a}_C$$

$$\therefore \quad \mathbf{a}_P = \mathbf{a}_C + \mathbf{a}_{PC}$$

Thus, absolute acceleration of P is the vector sum of $\mathbf{a}_C$ and $\mathbf{a}_{PC}$.

Here, $\mathbf{a}_C = \mathbf{a}$ and motion of P with respect to C is a circle (dotted circle shown in figure). In circular motion, acceleration of a particle has two components:

(i) tangential component, a_t (ii) radial component, a_r

Here, $$a_t = r\alpha \qquad \left(\text{where, } \alpha = \frac{d\omega}{dt}\right)$$

This component is tangential in the direction of linear velocity if ω is increasing and in the opposite direction of linear velocity if ω is decreasing. Further, this component is zero if ω is constant.

$$a_r = r\omega^2$$

and this component is always towards centre C.

Hence, net acceleration of the particle P is the vector sum of the following three terms:

(i) a (ii) $a_t = r\alpha$ (iii) $a_r = r\omega^2$

For different particles, values of a, ω and α are same. But value of r is different. Therefore $r\alpha$, $r\omega^2$ and their directions are different. This is the reason why different particles have different linear accelerations.

Kinetic Energy of Rigid Body in Combined Translational and Rotational Motion

Here, two energies are associated with the rigid body. One is translational $\left(=\frac{1}{2} m\, v_{\text{COM}}^2\right)$ and another is rotational $\left(=\frac{1}{2} I_{\text{COM}}\, \omega^2\right)$. Thus, total kinetic energy of the rigid body is

$$\boxed{K = \frac{1}{2} m v_{\text{COM}}^2 + \frac{1}{2} I_{\text{COM}}\, \omega^2.}$$

Example 12.18

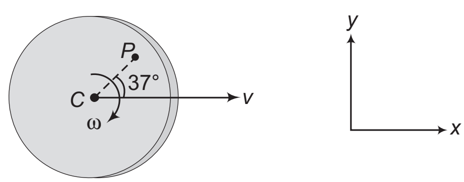


Fig. 12.64

In the figure shown $v = 2$ *m/s*, $\omega = 5$ *rad/s and* $CP = 1$ *m*

In terms of $\hat{\mathbf{i}}$ *and* $\hat{\mathbf{j}}$ *find linear velocity of particle P.*

Solution For particle P,

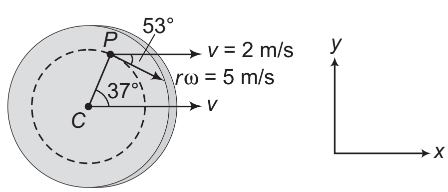


Fig. 12.65

$$r = CP = 1 \text{ m}$$

$$\Rightarrow \qquad r\omega = (1)(5) = 5 \text{ m/s}$$

Net velocity of P is the vector sum of v and $r\omega$ as shown in figure.

$$\therefore \qquad \mathbf{v}_P = 2\hat{\mathbf{i}} + (5\cos 53°\,\hat{\mathbf{i}} - 5\sin 53°\,\hat{\mathbf{j}})$$

$$= 2\hat{\mathbf{i}} + 3\hat{\mathbf{i}} - 4\hat{\mathbf{j}} = (5\hat{\mathbf{i}} - 4\hat{\mathbf{j}}) \text{ m/s}$$

Ans.

Example 12.19 *A disc of radius R has linear velocity v and angular velocity ω as shown in the figure. Given $v = r\omega$. Find velocity of points A, B, C and D on the disc.*

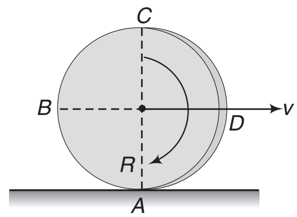


Fig. 12.66

Solution As stated in above article, velocity of any point of the rigid body in rotation plus translation is the vector sum of v (the velocity of centre of mass) and $r\omega$. Here, r is the distance of the point under consideration from the centre of mass of the body. Direction of this $r\omega$ is perpendicular to the line joining the point with centre of mass in the sense of rotation. Based on this, velocities of points A, B, C and D are as shown below

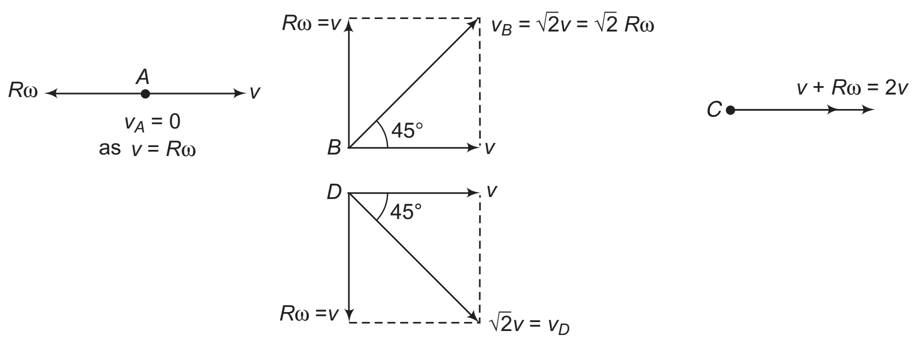


Fig. 12.67

Thus, v_A is zero, velocity of B and D is $\sqrt{2}v$ or $\sqrt{2}R\omega$ and velocity of C is $2v$ or $2R\omega$ in the directions shown in figure.

Example 12.20 *In the shown figure,*

$$a = 2\, m/s^2,\ \omega = (2t)\ rads^{-1}\ \text{and}\ CP = 1\ m.$$

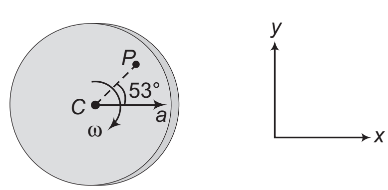


Fig. 12.68

In terms of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, find linear acceleration of the particle at P at $t = 1$ s.

Solution For particle at P

$$r = CP = 1\text{m}$$

$\Rightarrow$
$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}(2t) = 2\,\text{rad/s}^2$$

At $t = 1\text{s}$,

$$\omega = 2\,\text{rad/s}$$
$$\alpha = 2\,\text{rad/s}^2$$
$$a_t = r\alpha = 2\,\text{m/s}^2$$
$$a_r = r\omega^2 = 4\,\text{m/s}^2$$

and
$$a = 2\,\text{m/s}^2$$

Net acceleration of P is the vector sum of three terms a, a_r and a_t as shown in figure below.

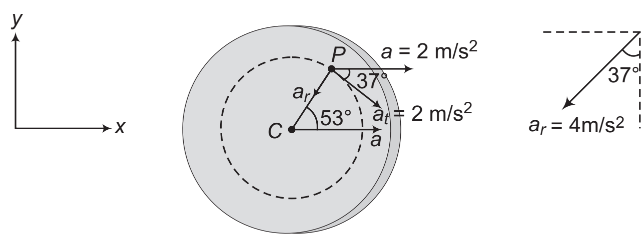


Fig. 12.69

$\therefore$
$$\mathbf{a}_P = 2\hat{\mathbf{i}} + (2\cos 37°\,\hat{\mathbf{i}} - 2\sin 37°\,\hat{\mathbf{j}}) + (-4\sin 37°\,\hat{\mathbf{i}} - 4\cos 37°\,\hat{\mathbf{j}})$$
$$= 2\hat{\mathbf{i}} + 1.6\hat{\mathbf{i}} - 1.2\hat{\mathbf{j}} - 2.4\hat{\mathbf{i}} - 3.2\hat{\mathbf{j}}$$
$$= (1.2\hat{\mathbf{i}} - 4.4\hat{\mathbf{j}})\,\text{m/s}^2$$ **Ans.**

INTRODUCTORY EXERCISE 12.7

1. In the figure shown, $\omega = \dfrac{v}{2R}$. In terms of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, find linear velocities of particles M, N, R and S.

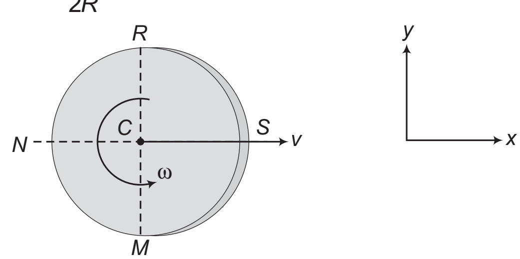


Fig. 12.70

2. In the same figure, if v and ω both are constant, then find linear accelerations of points M, N, R and S in terms of R, ω, $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, where R is the radius of disc.

12.9 Uniform Pure Rolling

Pure rolling means no relative motion (or no slipping) at point of contact between two bodies.

For example, consider a disc of radius R moving with linear velocity v and angular velocity ω on a horizontal ground. The disc is said to be moving without slipping if velocities of points P and Q (shown in figure 12.71 (b) are equal, i.e.

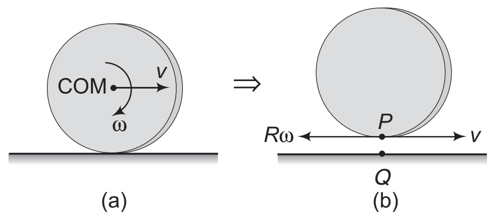


Fig. 12.71

$$v_P = v_Q$$

or $$v - R\omega = 0$$

or $$\boxed{v = R\omega}$$

If $v_P > v_Q$ or $v > R\omega$, the motion is said to be in **forward slipping** and if $v_P < v_Q$ or $v < R\omega$, the motion is said to be in **backward slipping** (or sometimes called forward English).

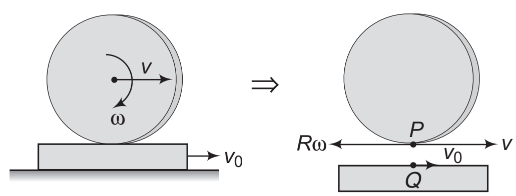


Fig. 12.72

In pure rolling ($v = R\omega$) over a stationary ground net velocity of bottommost point of the body is zero. In forward slip condition ($v > R\omega$) net velocity is in the direction of motion. In backward slip condition ($v < R\omega$) net velocity is in the opposite direction of motion.

Thus, $v = R\omega$ is the condition of pure rolling on a stationary ground. Sometimes it is simply said rolling. Suppose the base over which the disc in rolling, is also moving with some velocity (say v_0) then in that case condition of pure rolling is different.

For example, in the Fig. 12.72,

$$v_P = v_Q$$

or $$v - R\omega = v_0$$

Thus, in this case $v - R\omega \neq 0$, but $v - R\omega = v_0$. By uniform pure rolling we mean that v and ω are constant.

They are neither increasing nor decreasing.

Extra Points to Remember

In case of pure rolling on a stationary horizontal ground (when $v = R\omega$), following points are important to note:

- Distance moved by the centre of mass of the rigid body in one full rotation is $2\pi R$.

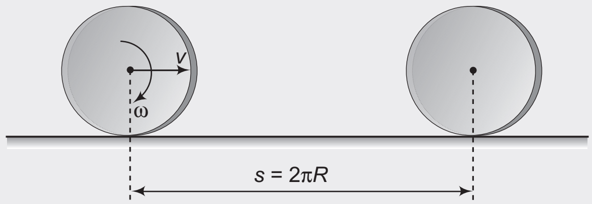


Fig. 12.73

This is because $\quad s = v \cdot T = (\omega R)\left(\dfrac{2\pi}{\omega}\right) = 2\pi R$

In forward slipping $\quad s > 2\pi R \quad$ (as $v > \omega R$)

and in backward slipping $\quad s < 2\pi R \quad$ (as $v < \omega R$)

- The speed of a point on the circumference of the body at the instant shown in figure is $2v \sin\dfrac{\theta}{2}$ or $2R\omega \sin\dfrac{\theta}{2}$. i.e.

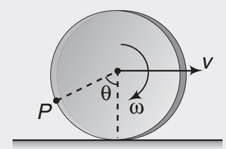


Fig. 12.74

$$|\mathbf{v}_P| = v_p = 2v \sin\frac{\theta}{2} = 2R\omega \sin\frac{\theta}{2}$$

This can be shown as :

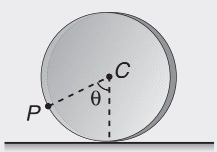


Fig. 12.75

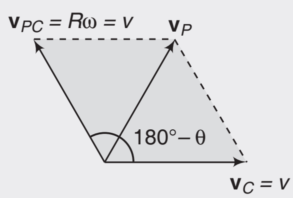


Fig. 12.76

$$\mathbf{v}_P = \mathbf{v}_C + \mathbf{v}_{PC}$$

$$\therefore \qquad |\mathbf{v}_P| = \sqrt{v^2 + v^2 + 2v \cdot v \cos(180° - \theta)} = 2v \sin\frac{\theta}{2}$$

- From the above expression we can see that :

$$v_A = 0 \quad \text{as} \quad \theta = 0°$$

$$v_B = \sqrt{2}v \quad \text{as} \quad \theta = 90°$$

and $\quad v_C = 2v \quad$ as $\quad \theta = 180°$

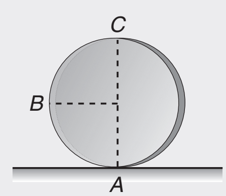


Fig. 12.77

- The path of a point on circumference is a cycloid and the distance moved by this point in one full rotation is $8R$.

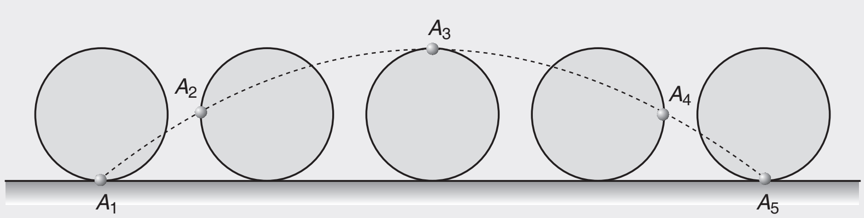


Fig. 12.78

In the figure, the dotted line is a cycloid and the distance $A_1 A_2 \ldots A_5$ is $8R$. This can be proved as under. In Fig. 12.79

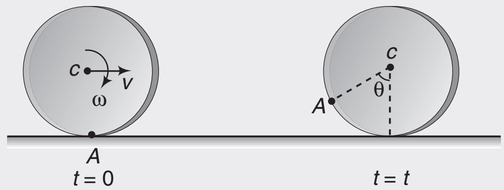


Fig. 12.79

$$\theta = \omega t$$

Speed of point A at this moment is, $\quad v_A = 2R\omega \sin\frac{\theta}{2} = 2R\omega \sin\left(\frac{\omega t}{2}\right)$

Distance moved by it in time dt is, $\quad ds = v_A\, dt = 2R\omega \sin\left(\frac{\omega t}{2}\right) dt$

Therefore, total distance moved in one full rotation is,

$$s = \int_0^{T = 2\pi/\omega} ds \quad \text{or} \quad s = \int_0^{T = 2\pi/\omega} 2R\omega \sin\left(\frac{\omega t}{2}\right) dt$$

On integration we get, $\quad s = 8R.$

- $$\frac{K_R}{K_T} = 1 \quad \text{for a ring}$$
$$= \frac{1}{2} \quad \text{for a disc}$$
$$= \frac{2}{5} \quad \text{for a solid sphere}$$
$$= \frac{2}{3} \quad \text{for a hollow sphere etc.}$$

Here, K_R stands for rotational kinetic energy $\left(= \frac{1}{2} I\omega^2\right)$ and K_T for translational kinetic energy $\left(= \frac{1}{2} mv^2\right)$.

For example, for a disc

$$K_R = \frac{1}{2} I\omega^2 = \frac{1}{2}\left(\frac{1}{2} mR^2\right)\left(\frac{v}{R}\right)^2$$
$$= \frac{1}{4} mv^2 \quad \text{and} \quad K_T = \frac{1}{2} mv^2$$

$\therefore$
$$\frac{K_R}{K_T} = \frac{1}{2}$$

Example 12.21 *A solid disc is rolling without slipping on a horizontal ground as shown in figure. Its total kinetic energy is 100 J. What is its translational and rotational kinetic energy.*

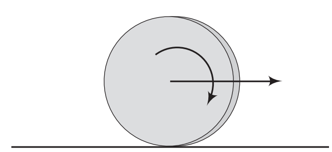


Fig. 12.80

Solution In case of pure rolling, ratio of rotational kinetic energy and translational kinetic energy is $\frac{1}{2}$.

or $\quad \frac{K_R}{K_T} = \frac{1}{2} \quad \Rightarrow \quad \therefore \quad K_R = \left(\frac{1}{1+2}\right)$(Total kinetic energy)

$$= \frac{1}{3}(100\,\text{J}) = \frac{100}{3}\,\text{J} \qquad \textbf{Ans.}$$

Similarly,
$$K_T = \left(\frac{2}{1+2}\right)(\text{Total kinetic energy})$$

$$= \frac{2}{3}(100\,\text{J}) = \frac{200}{3}\,\text{J} \qquad \textbf{Ans.}$$

Example 12.22 *A disc of radius R starts at time $t = 0$ moving along the positive x-axis with linear speed v and angular speed ω. Find the x and y coordinates of the bottommost point at any time t.*

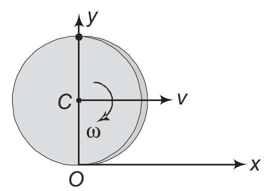


Fig. 12.81

Solution At time t the bottommost point will rotate an angle $\theta = \omega t$ with respect to the centre of the disc C. The centre C will travel a distance $s = vt$.

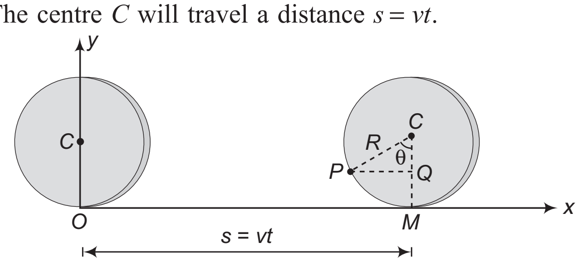


Fig. 12.82

In the figure, $PQ = R\sin\theta = R\sin\omega t$ and $CQ = R\cos\theta = R\cos\omega t$

Coordinates of point P at time t are,

$$x = OM - PQ = vt - R\sin\omega t$$

and
$$y = CM - CQ = R - R\cos\omega t$$

$\therefore$
$$(x, y) \equiv (vt - R\sin\omega t, R - R\cos\omega t)$$

INTRODUCTORY EXERCISE 12.8

1. A solid sphere is rolling without slipping on a horizontal ground. Its rotational kinetic energy is 10 J. Find its translational and total kinetic energy.
2. Under forward slip condition, translational kinetic energy of a ring is greater than its rotational kinetic energy. Is this statement true or false?
3. In backward slip condition, translational kinetic energy of a disc may be equal to its rotational kinetic energy. Is this statement true of false?

12.10 Instantaneous Axis of Rotation

As we have seen that combined rotational and translational motion of a rigid body is the most complex motion. But this motion can be simplified and may be assumed to be in pure rotational motion (with same ω) about an axis called instantaneous axis of rotation.

Further as we know that, in pure rotational motion, points lying on the axis of rotation are at rest. Therefore, we can say that, instantaneous axis of rotation passes through those points which are at rest.

For example, in pure rolling over ground instantaneous axis of rotation (*IAOR*) passes through the bottommost point, as it is a point of zero velocity. Thus, the combined motion of rotation and translation can be assumed to be pure rotational motion about bottommost point with same angular speed ω.

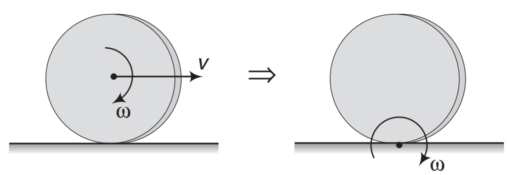


Fig. 12.83

Now, there are two uses of the concept of instantaneous axis of rotation.

(i) Velocity of any point P can obtained by a single term

$$v = r\omega$$

as in pure rotational motion this is the expression of velocity of any point P. Here r is the distance of P from instantaneous axis of rotation. With respect to O, point P is rotating in a circle with centre at O and radius as r. Velocity of P is tangential to this circle (or perpendicular to OP), in the direction of rotation.

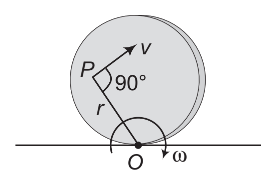


Fig. 12.84

(ii) We can find total kinetic energy of the body by a single term.

$$K = \frac{1}{2} I\omega^2$$

But here, I is the moment of inertia about instantaneous axis of rotation.

Example 12.23 *Using the concept of instantaneous axis of rotation. Find speed of particle P as shown in figure, under pure rolling condition.*

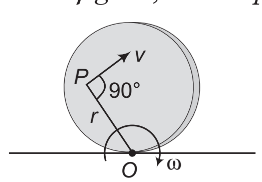


Fig. 12.85

Solution In pure rolling, combined rotation and translation motion may be assumed to be a pure rotational motion about an axis passing through bottommost point (with same ω) or instantaneous axis of rotation.

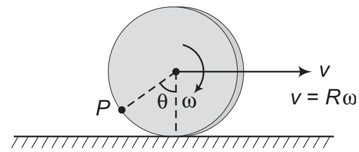


Fig. 12.86

$$|\mathbf{v}_P| = (OP)\omega \quad \text{Here, } OP = 2R\sin\frac{\theta}{2}$$

$$\therefore \qquad |\mathbf{v}_P| = \left(2R\sin\frac{\theta}{2}\right)\omega = 2R\omega\sin\left(\frac{\theta}{2}\right) = 2v\sin\frac{\theta}{2}$$

Example 12.24 *A disc is rolling (without slipping) on a horizontal surface. C is its centre and Q and P are two points equidistant from C. Let v_P, v_Q and v_C be the magnitude of velocities of points P, Q and C respectively, then* **(JEE 2004)**

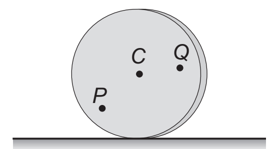


Fig. 12.87

(a) $v_Q > v_C > v_P$ (b) $v_Q < v_C < v_P$

(c) $v_Q = v_P, v_C = \frac{1}{2}v_P$ (d) $v_Q < v_C > v_P$

Solution In case of pure rolling bottom most point is the instantaneous centre of zero velocity.

Velocity of any point on the disc, $v = r\omega$, where r is the distance of point from O.

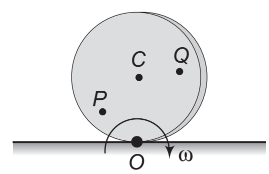


Fig. 12.88

$$r_Q > r_C > r_P \quad \Rightarrow \quad \therefore \quad v_Q > v_C > v_P$$

Therefore, the correct option is (a).

INTRODUCTORY EXERCISE 12.9

1. A disc is rolling without slipping with linear velocity v as shown in figure. With the concept of instantaneous axis of rotation, find velocities of points A, B, C and D.

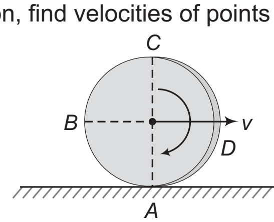


Fig. 12.89

2. A solid sphere is rolling without slipping as shown in figure. Prove that

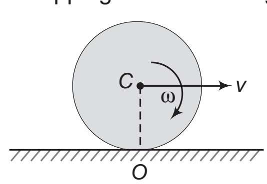


Fig. 12.90

$$\frac{1}{2}mv^2 + \frac{1}{2}I_C\omega^2 = \frac{1}{2}I_0\omega^2$$

12.11 Accelerated Pure Rolling

Till now we were discussing the uniform pure rolling in which v and ω were constants. Now, suppose an external force is applied to the rigid body, the motion will no longer remain uniform. The condition of pure rolling on a stationary ground is,

$$v = R\omega$$

Differentiating this equation with respect to time, we have

$$\frac{dv}{dt} = R \cdot \frac{d\omega}{dt}$$

or $$a = R\alpha$$

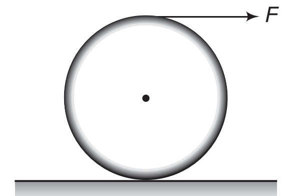


Fig. 12.91

Thus, in addition to $v = R\omega$ at every instant of time there is one additional condition, linear acceleration $= R \times$ angular acceleration or $a = R\alpha$ for pure rolling to take place. Here, friction plays an important role in maintaining the pure rolling. The friction may sometimes act in forward direction, sometimes in backward direction or under certain conditions it may be zero. Here, we should not forget the basic nature of friction, which is a self adjusting force (upto a certain maximum limit) and which has a tendency to stop the relative motion between two bodies in contact and here the relative motion stops when at every instant $v = R\omega$. To satisfy this equation all the time, $a = R\alpha$ equation should also be satisfied. Let us take an example illustrating the above theory.

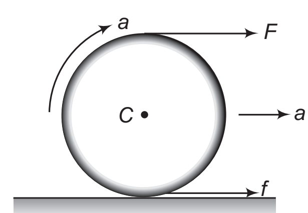


Fig. 12.92

Suppose a force F is applied at the topmost point of a rigid body of radius R, mass M and moment of inertia I about an axis passing through the centre of mass. Now, the applied force F can produce by itself:

(i) a linear acceleration a and

(ii) an angular acceleration α.

If $a = R\alpha$, then there is no need of friction and force of friction $f = 0$. If $a < R\alpha$, then to support the linear motion the force of friction f will act in forward direction. Similarly, if $a > R\alpha$, then to support the angular motion the force of friction will act in backward direction. So, in this case force of friction will be either backward, forward or even zero also. It all depends on M, I and R. For calculation purpose initially we can choose any direction of friction. Let we assume it in forward direction,

Let, a = linear acceleration, α = angular acceleration

then, $$a = \frac{F_{\text{net}}}{M} = \frac{F + f}{M} \qquad \text{...(i)}$$

$$\alpha = \frac{\tau_c}{I} = \frac{(F - f)R}{I} \qquad \text{...(ii)}$$

For pure rolling to take place, $$a = R\alpha \qquad \text{...(iii)}$$

Solving Eqs. (i), (ii) and (iii), we get $$f = \frac{(MR^2 - I)}{(MR^2 + I)} \cdot F \qquad \text{...(iv)}$$

From Eq. (iv) following conclusions can be drawn

(i) If $I = MR^2$ (e.g. in case of a ring) $f = 0$ i.e. if a force F is applied on the top of a ring, the force of friction will be zero and the ring will roll without slipping.

(ii) If $I < MR^2$, (e.g. in case of a solid sphere or a hollow sphere), f is positive, i.e. force of friction will be forward.

(iii) If $I > MR^2$, f is negative, i.e. force of friction will be backwards. Although under no condition $I > MR^2$. (Think why?). So, force of friction is either in forward direction or zero. Here, it should be noted that the force of friction f obtained in Eq. (iv) should be less than the limiting friction (μMg) for pure rolling to take place. Further, we have seen that for $I < MR^2$ force of friction acts in forward direction. This is because α is more if I is small $\left(\alpha = \dfrac{\tau}{I}\right)$ i.e. to support the linear motion force of friction is in forward direction.

Note *It is often said that rolling friction is less than the sliding friction. This is because the force of friction calculated by equation number (iv) is normally less than the sliding friction ($\mu_k N$) and sometimes it is in forward direction, i.e. it supports the motion.*

Extra Points to Remember

- In accelerated pure rolling, $v = R\omega$ and $a = R\alpha$ are not the only conditions to be satisfied, sense of rotation is also important as per the direction of linear acceleration. Sense of rotation (or direction of α) should be as shown below in following two figures:

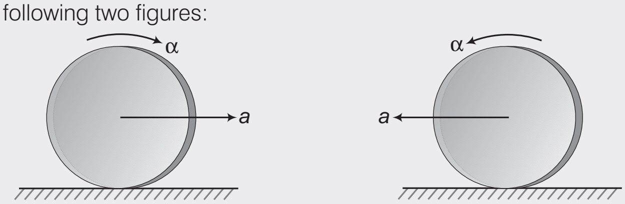


Fig. 12.93

- In following two figures accelerated pure rolling is not possible.

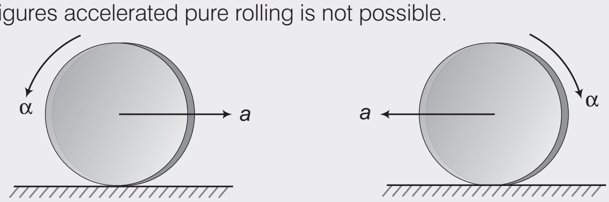


Fig. 12.94

- There are certain situations in which the direction of friction is fixed. For example in the following situations the force of friction is backwards. This is because linear acceleration 'a' due to the applied force (= $mg \sin\theta$ in first case) is in the direction shown in figure. So, direction of α should also be in the shown direction. We have only friction force which can provide 'α' in that direction. Hence, it should be in backward direction.

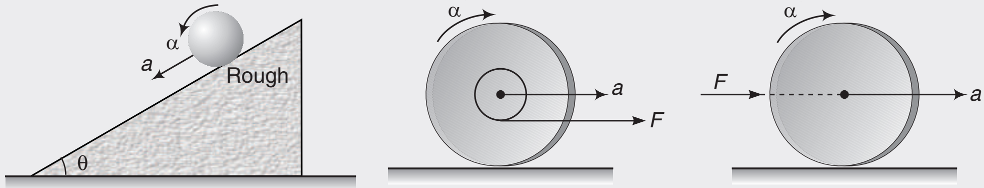


Fig. 12.95

Rolling on Rough Inclined Plane

A body of mass M, radius R and moment of inertia I is kept over a rough ground as shown in figure. In this case no external force is applied for accelerated pure rolling. But $mg\sin\theta$ is already acting at centre.

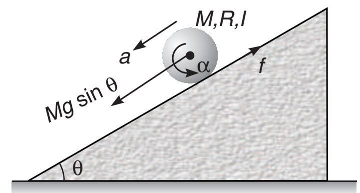


Fig. 12.96

As we said earlier also, force of friction in this case will be backward. Equations of motion are

$$a = \frac{Mg\sin\theta - f}{M} \qquad \text{...(i)}$$

$$\alpha = \frac{fR}{I} \qquad \text{...(ii)}$$

For pure rolling to take place,

$$a = R\alpha \qquad \text{...(iii)}$$

Solving Eqs. (i), (ii) and (iii), we get

$$f = \frac{Mg\sin\theta}{1 + \dfrac{MR^2}{I}} \qquad \text{...(iv)}$$

and

$$a = \frac{g\sin\theta}{1 + \dfrac{I}{MR^2}} \qquad \text{...(v)}$$

From Eq. (v), we can see that if a solid sphere and a hollow sphere of same mass and radius are released from a rough inclined plane and pure rolling is taking place, then the solid sphere reaches the bottom first because

$$I_{\text{solid}} < I_{\text{hollow}} \quad \text{or} \quad a_{\text{solid}} > a_{\text{hollow}} \Rightarrow \therefore \quad t_{\text{solid}} < t_{\text{hollow}}$$

Further, the force of friction calculated in Eq. (iv) for pure rolling to take place should be less than or equal to the maximum available friction $\mu Mg\cos\theta$.

or

$$\frac{Mg\sin\theta}{1 + \dfrac{MR^2}{I}} \le \mu\, Mg\cos\theta \quad \text{or} \quad \mu \ge \frac{\tan\theta}{1 + \dfrac{MR^2}{I}}$$

Thus, minimum value of friction required for pure rolling is

$$\mu_{\min} = \frac{\tan\theta}{1 + MR^2/I}$$

If given value of $\mu > \mu_{\min}$, then friction acting on the body is

$$f = \frac{Mg\sin\theta}{1 + MR^2/I}$$

and in this case linear acceleration of the body is

$$a = \frac{g\sin\theta}{1 + I/MR^2}$$

Example 12.25 *In the shown figure, accelerated pure rolling will takes place, if $a = R\alpha$. Find the case if*

(a) $a > R\alpha$ *(b) $a < R\alpha$*

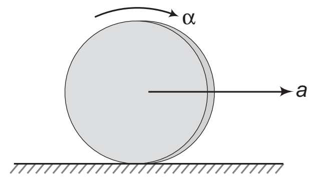


Fig. 12.97

Solution (a) If $a > R\alpha$, then at any instant $v > R\omega$. So, it is a case of forward slipping.

(b) If $a < R\alpha$, then at any instant $v < R\omega$. So, it is a case of backward slipping.

Example 12.26 *If accelerated pure rolling is taking place on a stationary ground, then work done by friction is always zero. Comment on this.*

Solution In pure rolling on stationary ground the bottommost point of the rigid body (where force of friction is acting) is at rest. Therefore, work done by friction is zero.

Example 12.27 *In the shown figure, M is mass of the body, R its radius and I the moment of inertia about an axis passing through centre. Find force of friction 'f' acting on the body (upwards), its linear acceleration 'a' (down the plane) and type of motion if:*

(a) $\mu = 0$ *(b) $\mu < \mu_{min}$* *(c) $\mu > \mu_{min}$*

where, μ_{min} is the minimum value of coefficient of friction required for pure rolling.

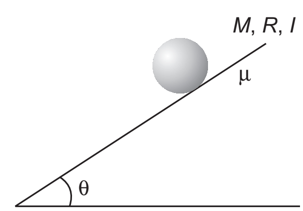


Fig. 12.98

Solution (a) If $\mu = 0$ then,

$$f = 0 \quad \text{and} \quad a = g\sin\theta = a_1 \text{ (say)}$$

and the motion is only translational.

(b) If $\mu < \mu_{min}$, then maximum value of friction will act, as friction is insufficient to provide accelerated pure rolling or to stop the relative motion.

$$\therefore \qquad f = f_{max} = \mu\, mg\cos\theta$$

$$\text{and} \qquad a = \frac{Mg\sin\theta - \mu\, Mg\cos\theta}{m}$$

$$= g\sin\theta - \mu g\cos\theta = a_2 \text{ (say)}$$

In this case, motion is rotation + translation with forward slip (as $a > R\alpha$).

(c) If $\mu > \mu_{min}$, Then we have discussed in the above article that

$$f = \frac{Mg\sin\theta}{1 + \dfrac{MR^2}{I}} \quad \text{and} \quad a = \frac{g\sin\theta}{1 + \dfrac{I}{MR^2}} = a_3 \text{ (say)}$$

Motion in this case is rotation + translation with accelerated pure rolling.

Note *In the above example, we can see that a_1 and a_2 are independent of moment of inertia I, but a_3 depends on it.*

Example 12.28 *A tangential force F acts at the top of a thin spherical shell of mass m and radius R. Find the acceleration of the shell if it rolls without slipping.*

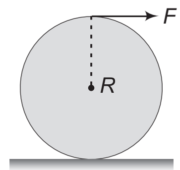


Fig. 12.99

Solution Let f be the force of friction between the shell and the horizontal surface.

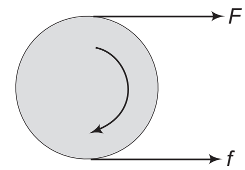


Fig. 12.100

For translational motion,

$$F + f = ma \qquad \ldots \text{(i)}$$

For rotational motion,

$$FR - fR = I\alpha = I\frac{a}{R}$$

[$\because a = R\alpha$ for pure rolling]

$$\Rightarrow \qquad F - f = I\frac{a}{R^2} \qquad \ldots \text{(ii)}$$

Adding Eqs. (i) and (ii), we get

$$2F = \left(m + \frac{I}{R^2}\right)a$$

$$= \left(m + \frac{2}{3}m\right)a = \frac{5}{3}ma$$

or

$$F = \frac{5}{6}ma \qquad \left[\because I_{\text{shell}} = \frac{2}{3}mR^2\right]$$

$$\Rightarrow \qquad a = \frac{6F}{5m}$$ **Ans.**

Example 12.29 *A horizontal force F acts on the sphere at its centre as shown. Coefficient of friction between ground and sphere is μ. What is maximum value of F, for which there is no slipping?*

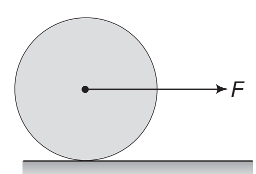


Fig. 12.101

Solution $$F - f = Ma \qquad \ldots \text{(i)}$$

$$\tau = I\alpha = I\left(\frac{a}{R}\right)$$

$$\Rightarrow \qquad f \cdot R = \frac{2}{5}MR^2\frac{a}{R}$$

$$\Rightarrow \qquad f = \frac{2}{5}Ma \qquad \ldots \text{(ii)}$$

Solving Eqs. (i) and (ii) we get,

$$f = \frac{2}{7}F$$

$$\frac{2}{7}F \le \mu mg$$

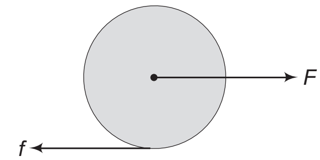


Fig. 12.102

$$\Rightarrow \qquad F \le \frac{7}{2}\mu mg \qquad \textbf{Ans.}$$

INTRODUCTORY EXERCISE 12.10

1. Work done by friction in pure rolling is always zero. Is this statement true or false?
2. In the figure shown, a force F is applied at the top of a disc of mass 4 kg and radius 0.25 m. Find maximum value of F for no slipping.

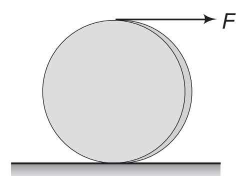


Fig. 12.103

3. In the figure shown a solid sphere of mass 4 kg and radius 0.25 m is placed on a rough surface. Find $(g = 10\ \text{ms}^2)$
   (a) minimum coefficient of friction for pure rolling to take place.
   (b) If $\mu > \mu_{min}$, find linear acceleration of sphere.
   (c) If $\mu = \frac{\mu_{min}}{2}$, find linear acceleration of cylinder.
   Here, μ_{min} is the value obtained in part (a).

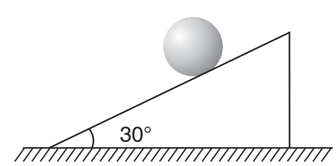


Fig. 12.104

4. A ball of mass M and radius R is released on a rough inclined plane of inclination θ. Friction is not sufficient to prevent slipping. The coefficient of friction between the ball and the plane is μ. Find:
   (a) the linear acceleration of the ball down the plane,
   (b) the angular acceleration of the ball about its centre of mass.
5. A spool is pulled by a force in vertical direction as shown in figure. What is the direction of friction in this case? The spool does not loose contact with the ground.

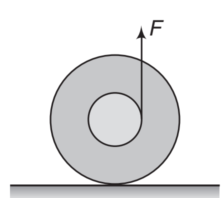


Fig. 12.105

12.12 Angular Impulse

In the previous chapter, we have learnt that linear impulse

$$= \mathbf{F} \cdot \Delta t = \mathbf{J} = \text{change in linear momentum } \Delta \mathbf{P}$$

or,
$$\mathbf{J} = \Delta \mathbf{P} = \mathbf{P}_f - \mathbf{P}_i = m(\mathbf{v}_f - \mathbf{v}_i)$$

In one dimension, we can simply write as:

$$J = \Delta P = P_f - P_i = m(v_f - v_i)$$

If $v_i = 0$ and $v_f = v$, then

$$J = mv \quad \text{or} \quad \boxed{v = \frac{J}{m}}$$

In the similar manner, angular impulse

$$= \tau \cdot \Delta t = A \cdot I = \text{change in angular momentum } \Delta L$$

or
$$A \cdot I = \tau \, \Delta t = \Delta L = L_f - L_i$$

But
$$\tau = F \times r_\perp$$

∴
$$A \cdot I = F \times r_\perp \times \Delta t$$
$$= J \times r_\perp \qquad \text{(as } F \times \Delta t = J\text{)}$$

Thus, $A \cdot I = J \times r_\perp = L_f - L_i$

If $L_i = 0$, then $L_f = L = I\,\omega$

∴
$$A \cdot I = J \times r_\perp = I\omega \quad \text{or} \quad \boxed{\omega = \frac{J \times r_\perp}{I}}$$

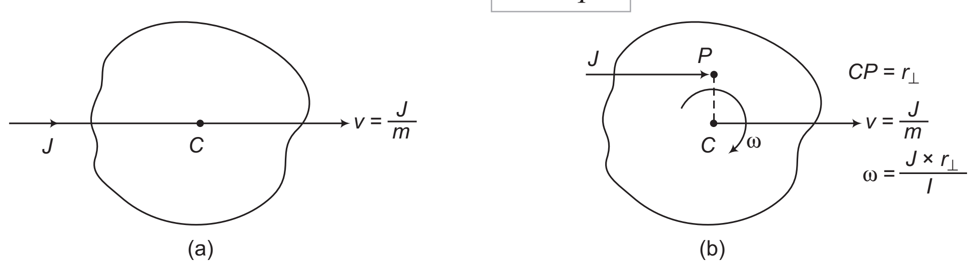


Fig. 12.106

In Fig. (a) A linear impulse J is applied at centre of mass C of the rigid body. Just after hitting, it will have only translational motion and its linear velocity will be given by

$$v = \frac{J}{m}$$

In Fig. (b) A linear impulse J is applied at point P, at a perpendicular distance $r_\perp = CP$. Just after hitting it will have both translational and rotational motion. Its linear velocity v and angular velocity ω will be given by

$$v = \frac{J}{m} \text{ and } \omega = \frac{J \times r_\perp}{I}$$

If $r_\perp$ is increased (keeping J to be constant) then v will remain same but ω will increase. So, the translational kinetic energy will have the same value but rotational kinetic energy will be more.

Extra Points to Remember

- Angular impulse $A{\cdot}I = \tau \times \Delta t = \Delta L$

  Now, there are following three cases:

  (i) If torque is constant, then angular impulse can be obtained by directly multiplying this constant torque with the given time interval.

  (ii) If torque is a function of time then angular impulse can be obtained by integration.

  $\therefore$ $$A{\cdot}I = \int_{t_i}^{t_f} \tau\, dt$$

  (iii) If torque *versus* time graph is given then angular impulse can be obtained by the area under that graph.

  In all three cases, angular impulse is equal to the change in angular momentum.

Example 12.30 *A solid sphere of mass M and radius R is hit by a cue at a height h above the centre C. For what value of h the sphere will roll without slipping ?*

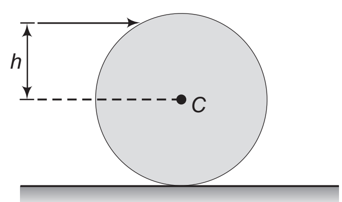


Fig. 12.107

Solution For rolling without slipping,

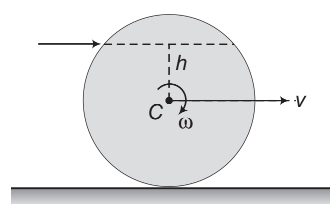


Fig. 12.108

$$v = R\omega$$

Here, v and ω are the values obtained just after hitting.

$$\therefore \qquad \frac{J}{M} = R\left(\frac{J \times r_\perp}{I}\right) \qquad \text{...(i)}$$

Here, $r_\perp = h$ and $I = I_C = \frac{2}{5}MR^2$

Substituting these values in Eq. (i), we have

$$h = \frac{2}{5}R$$ **Ans.**

Example 12.31 *A uniform sphere of mass m and radius R starts rolling without slipping down an inclined plane. Find the time dependence of the angular momentum of the sphere relative to the point of contact at the initial moment. How will the result be affected in the case of a perfectly smooth inclined plane? The angle of inclination of the plane is* θ.

Solution Applying the equation (about bottommost point)

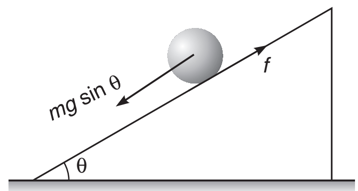


Fig. 12.109

Angular impulse = change in angular momentum about point of contact we have,

$$\tau = (mg\sin\theta)R = \text{constant}$$

$\therefore$ Angular impulse $= \tau \times t = (mg\sin\theta)Rt$

or $L = (mg\sin\theta)Rt$ **Ans.**

Note *There will be no change in the result even if body pure rolls or slides, as the torque of force of friction is zero about point of contact. So, it hardly matters whether the surface is rough or smooth.*

INTRODUCTORY EXERCISE 12.11

1. A cylinder is rolling down a rough inclined plane. Its angular momentum about the point of contact remains constant. Is this statement true or false?
2. A solid sphere and a hollow sphere both of same mass and same radius are hit by a cue at a height h above the centre C. In which case,

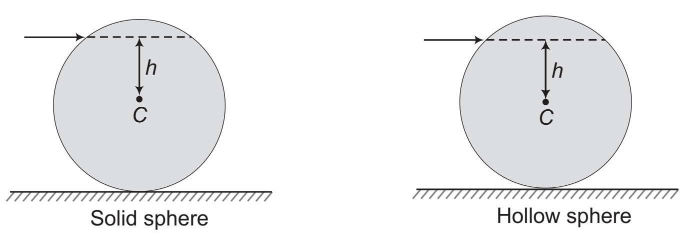


Fig. 12.110

(a) linear velocity will be more ?
(b) angular velocity will be more ?
(c) rotational kinetic energy will be more ?

Note *Linear impulse in both cases is same.*

12.13 Toppling

You might have seen in your practical life that if a force F is applied to a block A of smaller width and greater height it is more likely to topple down before sliding while if the same force F is applied to an another block B of broader base, chances of its sliding are more compared to its toppling. Have you ever thought why it happens so. To understand it in a better way let us take an example.

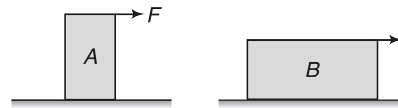


Fig. 12.111

Suppose a force F is applied at a height b above the base AE of the block. Further, suppose the friction f is sufficient to prevent sliding. In this case, if the normal reaction N also passes through C, then despite the fact that the block is in translational equilibrium ($F = f$ and $N = mg$), an unbalanced torque (due to the couple of forces F and f) is there.

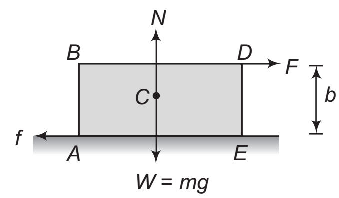


Fig. 12.112

This torque has a tendency to topple the block about point E. To cancel the effect of this unbalanced torque the normal reaction N is shifted towards right a distance 'a' such that, net anticlockwise torque is equal to the net clockwise torque or

$$Fb = (mg)a$$

or
$$a = \frac{Fb}{mg}$$

Now, if F or b (or both) is increased, distance a also increases. But it can not go beyond the right edge of the block. So, in extreme case (beyond which the block will topple down), the normal reaction passes through E as shown in Fig. 12.113 (b).

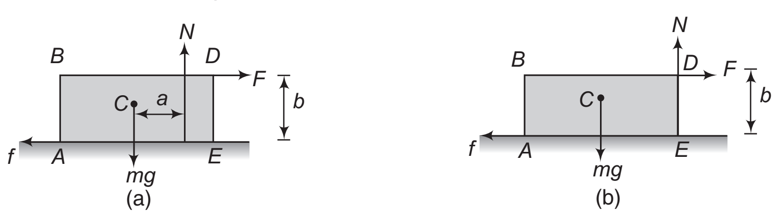


Fig. 12.113

Now, if F or b is further increased, the block will topple down. This is why the block having the broader base has less chances of toppling in comparison to a block of smaller base. Because the block of larger base has more margin for the normal reaction to shift. On the similar ground we can see why the rolling is so easy.

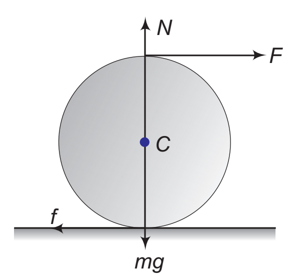


Fig. 12.114

Because in this case the normal reaction has zero margin to shift. So even if the body is in translational equilibrium ($F = f$, $N = mg$) an unbalanced torque is left behind and the body starts toppling and here the toppling means motion. Under ideal conditions, the body will start moving by a very small force F tending to zero also.

Example 12.32 *A uniform cube of side a and mass m rests on a rough horizontal table. A horizontal force F is applied normal to one of the faces at a point directly above the centre of the face, at a height $\frac{3a}{4}$ above the base. What is the minimum value of F for which the cube begins to tip about an edge?*

Solution In the limiting case normal reaction will pass through O. The cube will tip about O if torque of F exceeds the torque of mg.

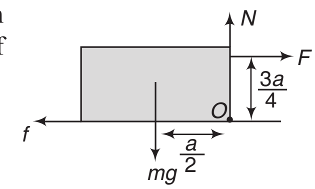


Fig. 12.115

Hence, $$F\left(\frac{3a}{4}\right) > mg\left(\frac{a}{2}\right)$$

or $$F > \frac{2}{3}mg$$

Therefore, minimum value of F is $\frac{2}{3}mg$.

Example 12.33 *A uniform cylinder of height h and radius r is placed with its circular face on a rough inclined plane and the inclination of the plane to the horizontal is gradually increased. If μ is the coefficient of friction, then under what condition the cylinder will (a) slide before toppling (b) topple before sliding.*

Solution (a) The cylinder will slide if

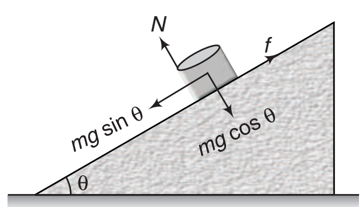


Fig. 12.116

$$mg\sin\theta > \mu mg\cos\theta$$

or $$\tan\theta > \mu \quad \ldots\text{(i)}$$

The cylinder will topple if $(mg\sin\theta)\frac{h}{2} > (mg\cos\theta)r$

or $$\tan\theta > \frac{2r}{h} \quad \ldots\text{(ii)}$$

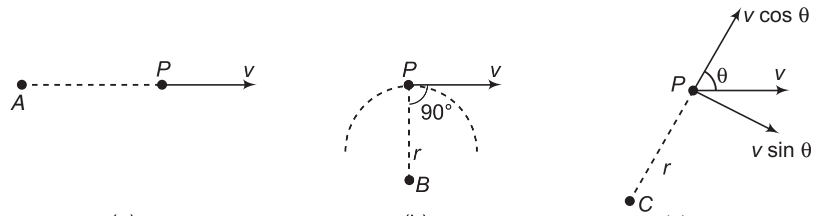
Thus, the condition of sliding is $\tan\theta > \mu$ and condition of toppling is $\tan\theta > \frac{2r}{h}$. Hence, the cylinder will slide before toppling if

$$\mu < \frac{2r}{h}$$

(b) The cylinder will topple before sliding if $\mu > \frac{2r}{h}$

Final Touch Points

1. Whether a particle is in translational motion, rotational motion or in both it merely depends on the reference point with respect to which the motion of the particle is described.



(a) (b) (c)

For example: a particle P of mass m is moving in a straight line as shown in figures (a), (b) and (c).

Refer Fig. (a) With respect to point A, the particle is in pure translational motion. Hence, kinetic energy of the particle can be written as

$$KE = \frac{1}{2} mv^2$$

Refer Fig. (b) With respect to point B, the particle is in pure rotational motion. Hence, the kinetic energy of the particle can be written as

$$KE = \frac{1}{2} I\omega^2 = \frac{1}{2}(mr^2)\left(\frac{v}{r}\right)^2 = \frac{1}{2} mv^2$$

Refer Fig. (c) With respect to point C, the particle can be assumed to be in rotational as well as translational motion. Hence, the kinetic energy of the particle can be written as

$$KE = \frac{1}{2} m(v\cos\theta)^2 + \frac{1}{2} I\omega^2$$

$$= \frac{1}{2} m(v\cos\theta)^2 + \frac{1}{2}(mr^2)\left(\frac{v\sin\theta}{r}\right)^2$$

$$= \frac{1}{2} mv^2$$

Thus, in all the three cases, the kinetic energy of the particle comes out to be the same.

2. In cases where pulley is having some mass and friction is sufficient enough to prevent slipping, the tension on two sides of the pulley will be different and rotational motion of the pulley is also to be considered.

3. Finite angular displacements are not vector quantities, the reason being that they do not obey the law of vector addition. This law asserts that the order in which vectors are added does not affect their sum.

$$A + B = B + A$$

It can be seen applying two successive 90° rotations-one about the x-axis, and the other about the z-axis to a six-sided dice. In the first case, the z-rotation is applied before the x-rotation and *vice versa* in the second case. It can be seen that the dice ends up in two completely different states. Clearly, the z-rotation plus the x-rotation does not equal the x-rotation plus the z-rotation. This non-commutative algebra cannot be represented by vectors. We conclude that, rotations are not, in general, vector quantities.

However infinitesimal angles do commute under addition, making it possible to treat them as vectors.

4. Rotation plus translation motion of a rigid body is simplified by splitting this motion into two parts.

(i) pure translation motion with the linear velocity and acceleration of the centre of mass.

(ii) pure rotational motion about an axis passing through centre of mass and perpendicular to the plane of motion of the particles.

But this motion may be considered as pure rotational motion about an axis called instantaneous axis of rotation (say *IAOR*). If this *IAOR* is non-inertial, then we cannot apply,

$$\tau_{ext} = I\alpha$$

about this axis. This is because in the derivation of this equation we use $F = ma$ for each particle. If *IAOR* has an acceleration $\mathbf{a}$, we have to apply a pseudo force $-m\mathbf{a}$ to each particle. These pseudo forces produce a pseudo torque about this axis.

But this equation can be applied about an axis passing through centre of mass even if this is non-inertial. Let us prove this:

Take the origin at the centre of mass. The total torque of the pseudo force is

$$\begin{aligned}\Sigma \mathbf{r}_i \times (-m_i \mathbf{a}_i) &= -(\Sigma m_i \mathbf{r}_i) \times \mathbf{a} \\ &= -M\left(\frac{\Sigma m_i \mathbf{r}_i}{M}\right) \times \mathbf{a}\end{aligned}$$

But $\dfrac{\Sigma m_i \mathbf{r}_i}{M}$ is the position vector of the centre of mass and that is zero as the centre of mass is at the origin. Hence, the torque of pseudo forces acting on all particles of the rigid body is zero.

Solved Examples

TYPED PROBLEMS

Type 1. *Based on rotational equilibrium about a fixed axis.*

Concept

Given that a rigid body can only rotate about a fixed axis, (i.e. hinged at some point) still it is not rotating or it is at rest or it is in equilibrium.

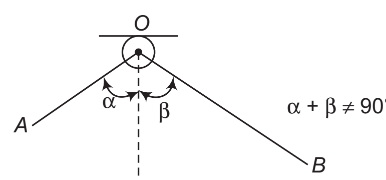
How to Solve?

- Net torque about hinge point should be zero. But torque of hinge force about the same point is already zero.
- Net force on the rigid body is also zero.

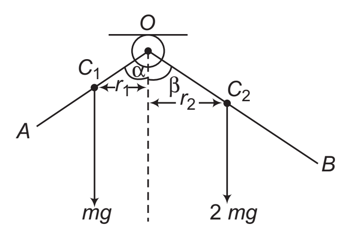
Example 1 *A uniform L shaped rod of mass 3m is hinged at point O. Length OB is two times the length OA. It is in equilibrium.*

Find

(a) relation between α *and* β *(b) net hinge force.*



Solution (a) Length OB is two times the length OA. Therefore, mass of OB is $2m$ and that of OA is m and their weight will act at their centres (as the rod is uniform).



If total length is $3l$ then,

$$OA = l \Rightarrow OC_1 = \frac{OA}{2} = \frac{l}{2}$$

and $$r_1 = OC_1 \sin\alpha = \frac{l}{2}\sin\alpha$$

$$OB = 2l \Rightarrow OC_2 = \frac{OB}{2} = l$$

and $$r_2 = OC_2 \sin\beta = l\sin\beta$$

Net torque about $O = 0$

$\Rightarrow$ anticlockwise torque of mg = clockwise torque of $2\,mg$

$\Rightarrow$ $$(mg)r_1 = (2mg)r_2$$

$\Rightarrow$ $$mg\left(\frac{l}{2}\sin\alpha\right) = (2mg)\,l\sin\beta$$

or $\dfrac{\sin\alpha}{\sin\beta} = 4$ is the required relation between α and β.

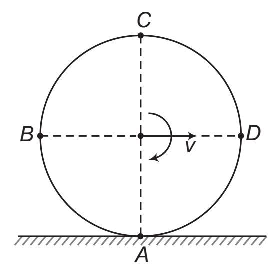
(b) Net force on the rod is also zero. Therefore, hinge force is $3\,mg (= mg + 2mg)$ in upward direction.

Type 2. *To find total kinetic energy of a system of rigid bodies in rotation plus translation.*

Concept

A particle has only translation kinetic energy $\frac{1}{2}mv^2$. But a rigid body may have $\frac{1}{2}mv^2$ and $\frac{1}{2}I\omega^2$. Here, v is velocity of centre of mass, ω is angular speed of the rigid body and I is the moment of inertia about an axis passing through centre of mass and perpendicular to the plane of motion of the body.

Example 2 *A ring of mass 'm' is rolling without slipping with linear speed v as shown in figure. Four particles each of mass 'm' are also attached at points A, B, C and D. Find total kinetic energy of the system.*



Solution Earlier we have learned that in case of pure rolling,

$$\omega = \frac{v}{R},\ v_A = 0,\ v_B = v_D = \sqrt{2}v \quad \text{and} \quad v_C = 2v$$

Now, total kinetic energy = [translational kinetic energy of four particles]+[translational kinetic energy of ring + rotational kinetic energy of ring]

$$\therefore \qquad (\text{KE})_{\text{total}} = \left[\frac{1}{2}m(0)^2 + \frac{1}{2}m(\sqrt{2}\,v)^2 + \frac{1}{2}m(\sqrt{2}\,v)^2 + \frac{1}{2}m(2v)^2\right] + \left[\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2\right]$$

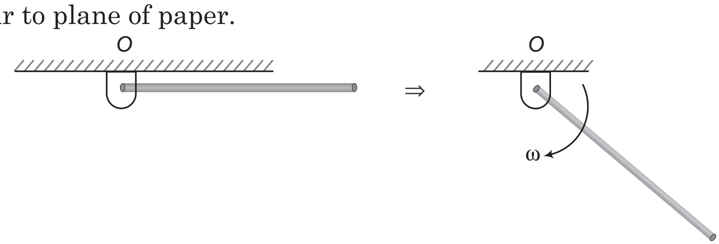
Substituting $I = mR^2$ and $\omega = \frac{v}{R}$

we get, $$(\text{KE})_{\text{total}} = 5mv^2$$ **Ans.**

Type 3. *Energy conservation in pure rotational motion.*

Concept

A rigid body (suppose a rod) is hinged at O as shown in figure. It can have only rotational motion in a vertical plane about a smooth horizontal axis passing through O and perpendicular to plane of paper.



It is released from the horizontal position. As the rod rotates downwards its gravitational potential energy decreases and rotational kinetic energy increases.

How to Solve?

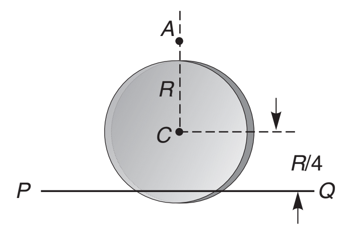
- Put decrease in gravitational potential energy (mgh) equal to increase in rotational kinetic energy $\left(\frac{1}{2}I\omega^2\right)$.

  Here, h is the fall of height of centre of mass of the rigid body. In case of particle, 'h' is decrease in height of the particle. From this equation, we can find the value of ω.
- In pure rotational motion velocity of any point is

$$v = r\omega$$

Here, r is the distance of that point from the axis of rotation.

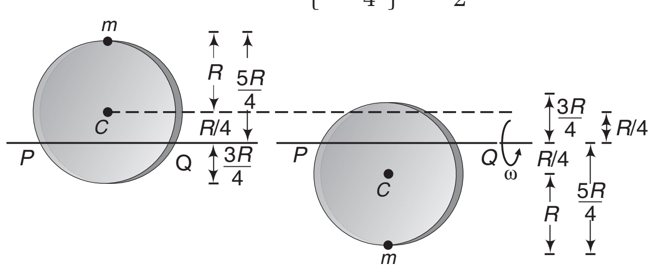
Example 3 *A uniform circular disc has radius R and mass m. A particle, also of mass m, is fixed at a point A on the edge of the disc as shown in the figure. The disc can rotate freely about a horizontal chord PQ that is at a distance R/4 from the centre C of the disc. The line AC is perpendicular to PQ. Initially the disc is held vertical with the point A at its highest position. It is then allowed to fall, so that it starts rotation about PQ. Find the linear speed of the particle as it reaches its lowest position.* **(JEE 1998)**



Solution Initial and final positions are shown below.

Decrease in potential energy of mass

$$= mg\left\{2 \times \frac{5R}{4}\right\} = \frac{5\,mgR}{2}$$



Decrease in potential energy of disc

$$= mg\left\{2 \times \frac{R}{4}\right\} = \frac{mgR}{2}$$

Therefore, total decrease in potential energy of system

$$= \frac{5\,mgR}{2} + \frac{mgR}{2} = 3mgR$$

Gain in kinetic energy of system $= \frac{1}{2} I\,\omega^2$

where, I = moment of inertia of system (disc + mass) about axis PQ

= moment of inertia of disc + moment of inertia of mass

$$= \left\{\frac{mR^2}{4} + m\left(\frac{R}{4}\right)^2\right\} + m\left(\frac{5R}{4}\right)^2$$

$$I = \frac{15\,mR^2}{8}$$

From conservation of mechanical energy,

Decrease in potential energy = Gain in kinetic energy

$\therefore$ $$3\,mgR = \frac{1}{2}\left(\frac{15\,mR^2}{8}\right)\omega^2$$

$\Rightarrow$ $$\omega = \sqrt{\frac{16g}{5R}}$$

Therefore, linear speed of particle at its lowest point

$$v = r\omega = \left(\frac{5R}{4}\right)$$

$$\omega = \frac{5R}{4}\sqrt{\frac{16g}{5R}}$$

$\Rightarrow$ $$v = \sqrt{5gR}$$ **Ans.**

Type 4. *Based on Angular Impulse.*

Concept

Angular impulse A.I $= \tau\,\Delta t$

If torque is a function of time, then

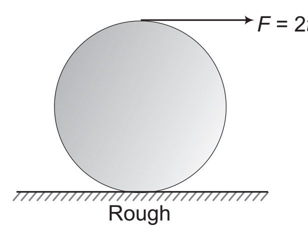
$$\text{A.I} = \int \tau\, dt$$

and this angular impulse is equal to the change in angular momentum.

Example 4 *A solid sphere of mass* 'm' and radius 'R' is kept over a rough ground. A time varying force $F = 2t$ is acting at the topmost point as shown in figure.

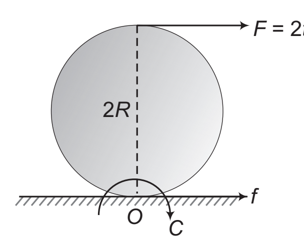
(a) Find angular momentum of the sphere about the bottommost point as a function of time 't'.

(b) Does this result depend on the fact whether the ground is rough or smooth?



Solution (a) Suppose 'f' is the force of friction acting on the sphere in forward direction as shown in figure.

Taking torque about bottommost point O. Torque of friction is already zero, as this force already passes through O. Torque of applied force is only there.



$\therefore$ $$\tau = F \times r_{\perp}$$
$$= (2t)\,(2R) = 4\,RT$$

As, this toque is a function of time.

$\therefore$ Angular impulse $= \int_0^t \tau\, dt = \int_0^t (4Rt)\, dt = 2\,Rt^2$

This angular impulse is equal to change in angular momentum. Hence, angular momentum at time 't' is $2Rt^2$.

(b) This result is independent of the nature of surface (smooth or rough), as the torque of friction about bottommost point is already zero. It does not contribute in angular impulse and therefore in angular momentum.

Type 5. *Role of friction in acceleration pure rolling.*

Concept

Friction has a tendency to stop the relative motion. In case of rolling (rotation + translation) over a stationary ground relative motion is stopped when $v = R\omega$ equation is satisfied at all instants. To satisfy this equation, a should be equal to '$R\alpha$'. Hence, friction has a tendency to satisfy the equation, $a = R\alpha$. Sometimes friction is sufficient and sometimes not.

If friction is sufficient, then accelerated pure rolling will takes place. Otherwise forward or backward slipping occurs.

How to Solve?

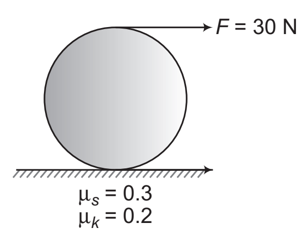
- Find N, $\mu_s N$ and $\mu_k N$ (or μN)
- Find requirement of friction to satisfy the equation $a = R\alpha$.

  For this, apply a general value of f in either forward or backward direction.
- Put $a = R\alpha$

  or
$$\frac{F_{\text{net}}}{m} = R\left(\frac{\tau_{\text{net}}}{I}\right)$$
  and find the required value of f for accelerated pure rolling.
- If $f \le \mu_s N$ i.e. requirement $\le$ availability then pure rolling will take place and $a = R\alpha$. In this case the obtained value of friction in step (iii) will act.
- If $f > \mu_s N$ then either forward or backward slip will take place and $a \ne R\alpha$. But,
$$a = \frac{F_{\text{net}}}{m} \quad \text{and} \quad \alpha = \frac{\tau_{\text{net}}}{I}$$
  are still applicable.

  In this case, $\mu_k N$ friction will act.
- If required value of friction in step (iii) comes out to be negative then just change the direction of friction which was initially assumed. Otherwise, magnitude wise all calculations are same.
- After calculation, if force of friction comes in forward direction, then if there is slip, it is backward slip and $a < R\alpha$.

Example 5 *A solid sphere of mass 5 kg and radius 1 m is kept over a rough surface as shown in figure. A force $F = 30\text{N}$ is acting at the topmost point.*



(a) Check whether the pure rolling will take place or not.

(b) Find direction and magnitude of friction actually acting on the sphere.

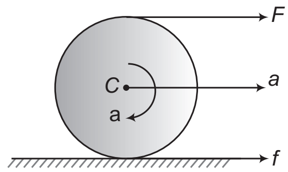
(c) Find linear acceleration 'a' and angular acceleration 'α'.

Take $g = 10\,m/s^2$

Solution (a) Availability of friction.

$$N = mg = 5 \times 10 = 50\text{ N}$$
$$\mu_s N = 0.3 \times 50 = 15\text{ N}$$
$$\mu_k N = 0.2 \times 50 = 10\text{ N}$$

Requirement of friction for accelerated pure rolling (or to satisfy $a = R\alpha$).



Let the friction f acts in forward direction.

$$a = R\alpha$$

$$\Rightarrow \qquad \frac{F_{\text{net}}}{m} = R\left(\frac{\tau_{\text{net}}}{I}\right)$$

$$\text{or} \qquad \frac{F+f}{m} = R\left[\frac{FR - fR}{\frac{2}{5}mR^2}\right]$$

Solving this equation, we get

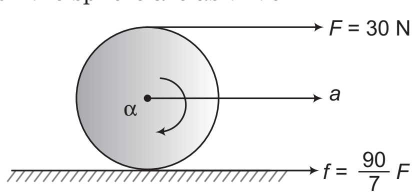
$$f = \frac{3}{7}F$$

$$= \frac{3}{7}(30) = \frac{90}{7}\text{ N}$$

Since, this value of f is less than $\mu_s N$. Therefore, friction is sufficient for accelerated pure rolling to take place. Hence, pure rolling is taking place.

(b) The above value of f is positive, hence direction of friction is forward and $\frac{90}{7}$ N friction will be acting.

(c) The actual forces acting on the sphere are as under



$$a = \frac{F_{\text{net}}}{m} = \frac{30 + \frac{90}{7}}{5} = 8.57\text{ m/s}^2 \qquad \textbf{Ans.}$$

Since, $a = R\alpha$

$$\therefore \qquad \alpha = \frac{a}{R} = \frac{8.57}{1} = 8.57\text{ rad/s}^2 \qquad \textbf{Ans.}$$

Example 6 *Repeat all parts of above problem for $F = 40N$.*

Solution (a) and (b): We have already calculated that, requirement of friction for pure rolling is

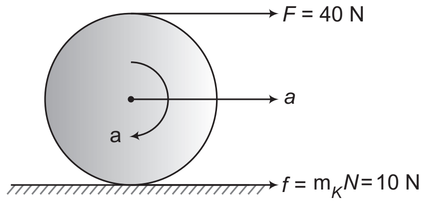
$$f = \frac{3}{7}F$$

For,

$$F = 40\text{ N} \Rightarrow f = \frac{120}{7}\text{ N}$$

Now, this value of f is more than $\mu_s N$, so backward slip ($a < R\alpha$) will take place because f is positive so kinetic friction or 10 N will act in forward direction.

(c) The actual forces acting on the sphere are as under :



$$a = \frac{F_{\text{net}}}{m} = \frac{40+10}{5} = 10 \text{ m/s}^2$$ **Ans.**

$$\alpha \neq \frac{a}{R}$$

But

$$\alpha = \frac{\tau_{\text{net}}}{I} = \frac{FR - fR}{\frac{2}{5}mR^2}$$

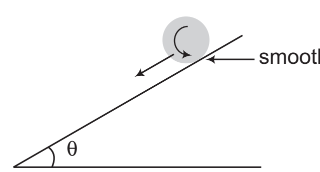
$$= \frac{2.5\,(F-f)}{mR} = \frac{2.5\,(40-10)}{(5)\,(1)}$$

$$= 15 \text{ rad/s}^2.$$ **Ans.**

Note *We can see that $a < R\alpha$.*

Type 6. *Based on conservation of mechanical energy and no change in rotational kinetic energy.*

Concept



If a body is kept in translational plus rotational motion condition over a smooth inclined plane then two forces acting on the body (weight and normal reaction) pass through centre of mass. So, they cannot provide the torque and therefore angular acceleration. Therefore, its angular speed and rotational kinetic energy remains constant. Same is the case, when the body moves freely under gravity. Here, the only force acting on the body is 'mg' which also passes through centre of mass. When the body moves up, its translational kinetic energy decreases and gravitational potential energy increases but rotational kinetic energy remains constant. Opposite is the case when the body moves down.

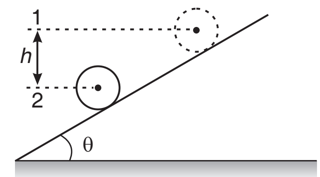
Further, in case of accelerated pure rolling over a stationary rough ground, work done by friction is zero. Hence, mechanical energy will remain constant. In this case,

$$\frac{K_R}{K_T} = 1 \text{ for ring } = \frac{1}{2} \text{ for disc}$$

$$= \frac{2}{5} \text{ for solid sphere etc.}$$

Thus, total mechanical energy ($K_R + K_T \pm mgh$) remains constant in case of accelerated pure rolling, over a smooth ground or under gravity.

Example 7 *A solid cylinder of mass m and radius r starts rolling down an inclined plane of inclination θ. Friction is enough to prevent slipping. Find the speed of its centre of mass when its centre of mass has fallen a height h.*



Solution Considering the two shown positions of the cylinder. As it does not slip hence total mechanical energy will be conserved.

Energy at position 1 is $E_1 = mgh$

Energy at position 2 is $E_2 = \frac{1}{2} mv_{COM}^2 + \frac{1}{2} I_{COM}\,\omega^2$

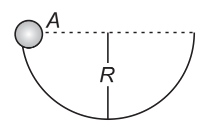
$$\frac{v_{COM}}{r} = \omega \quad \text{and} \quad I_{COM} = \frac{mr^2}{2}$$

$$\Rightarrow \qquad E_2 = \frac{3}{4} mv_{COM}^2$$

From conservation of energy, $E_1 = E_2$ or $mgh = \frac{3}{4} mv_{COM}^2$

$$\Rightarrow \qquad v_{COM} = \sqrt{\frac{4}{3} gh}$$ **Ans.**

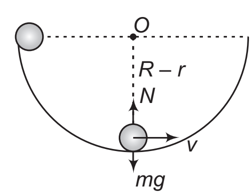
Example 8 *A small solid cylinder of radius r is released coaxially from point A inside the fixed large cylindrical bowl of radius R as shown in figure. If the friction between the small and the large cylinder is sufficient enough to prevent any slipping, then find :*



(a) What fractions of the total energy are translational and rotational, when the small cylinder reaches the bottom of the larger one?

(b) The normal force exerted by the small cylinder on the larger one when it is at the bottom.

Solution (a) $K_{trans} = \frac{1}{2} mv^2$



$$K_{rot} = \frac{1}{2} I\omega^2 = \frac{1}{2}\left(\frac{1}{2} mr^2\right)\left(\frac{v}{r}\right)^2 = \frac{1}{4} mv^2$$

$$K = K_{trans} + K_{rot} = \frac{3}{4} mv^2$$

$$\therefore \qquad \frac{K_{trans}}{K} = \frac{2}{3} \quad \Rightarrow \quad \frac{K_{rot}}{K} = \frac{1}{3}$$

(b) From conservation of energy,

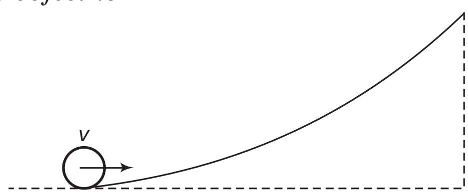
$$mg\,(R - r) = \frac{3}{4} mv^2$$

$$\therefore \qquad \frac{mv^2}{R - r} = \frac{4}{3} mg$$

Now, $$N - mg = \frac{mv^2}{R - r} = \text{centripetal force} = \frac{4}{3} mg$$

$$\therefore \qquad N = \frac{7}{3} mg$$ **Ans.**

Example 9 *A small object of uniform density rolls up a curved surface with an initial velocity v. It reaches up to a maximum height of $\frac{3v^2}{4g}$ with respect to the initial position. The object is* **(JEE 2007)**



(a) ring *(b) solid sphere* *(c) hollow sphere* *(d) disc*

Solution

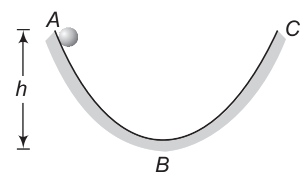
$$\frac{1}{2}mv^2 + \frac{1}{2}I\left(\frac{v}{R}\right)^2 = mg\left(\frac{3v^2}{4g}\right)$$

$\therefore$

$$I = \frac{1}{2}mR^2$$

$\therefore$ Body is disc. The correct option is (d).

Example 10 *A solid ball rolls down a parabolic path ABC from a height h as shown in figure. Portion AB of the path is rough while BC is smooth. How high will the ball climb in BC ?*



Solution At B, total kinetic energy $= mgh$

Here, $m =$ mass of ball

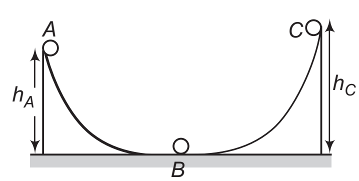
The ratio of rotational to translational kinetic energy would be,

$$\frac{K_R}{K_T} = \frac{2}{5} \quad \Rightarrow \quad \therefore \quad K_R = \frac{2}{7}mgh \quad \text{and} \quad K_T = \frac{5}{7}mgh$$

In portion BC, friction is absent. Therefore, rotational kinetic energy will remain constant and translational kinetic energy will convert into potential energy. Hence, if H be the height to which ball climbs in BC, then

$$mgH = K_T \quad \text{or} \quad mgH = \frac{5}{7}mgh \quad \text{or} \quad H = \frac{5}{7}h$$

Example 11 *A ball moves over a fixed track as shown in the figure. From A to B the ball rolls without slipping. If surface BC is frictionless and K_A, K_B and K_C are kinetic energies of the ball at A, B and C respectively, then* **(JEE 2006)**



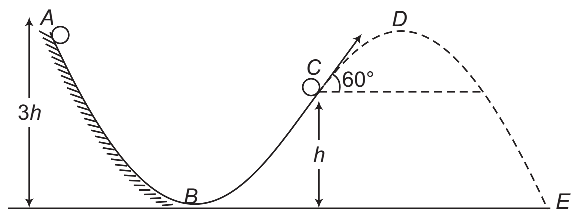
(a) $h_A > h_C; K_B > K_C$ *(b)* $h_A > h_C; K_C > K_A$

(c) $h_A = h_C; K_B = K_C$ *(d)* $h_A < h_C; K_B > K_C$

Solution On smooth part BC, due to zero torque, angular velocity and hence the rotational kinetic energy remains constant. While moving from B to C translational kinetic energy converts into gravitational potential energy.

$\therefore$ The correct option is (a)

Example 12 *A small solid sphere of mass 'm' is released from point A. Portion AB is sufficiently rough (to provide accelerated pure rolling), BC is smooth and after C, the ball moves freely under gravity. Find gravitational potential energy (U), rotational kinetic energy (K_R) and translational kinetic energy (K_T) at points A, B C, D and E.*



Solution AB is sufficiently rough BC is smooth and after C motion is under gravity. So, total mechanical energy ($U + K_R + K_T$) is always constant. On AB:

$$\frac{K_R}{K_T} = \frac{2}{5} \text{ (for solid sphere)}$$

as accelerated pure rolling is taking place.

After B, rotational kinetic energy will become constant. After C, centre of mass of the ball will follow a projectile motion.

$$\therefore \quad v_D = v_C \cos 60° = \frac{v_C}{2} = \text{half of } v_C$$

$$\therefore \quad (K_T)_D = \frac{1}{4}(K_T)_C$$

At point *A*

$$U = 3\,mgh$$

$$K_R = 0 \quad \Rightarrow \quad K_T = 0$$

$\therefore$ Total mechanical energy $E = 3\,mgh$ = constant.

At point *B*

$$U = 0 \quad \Rightarrow \quad E = 3\,mgh$$

$$\therefore \quad K = E = 3\,mgh \qquad (K = \text{total K.E})$$

But
$$\frac{K_R}{K_T} = \frac{2}{5}$$

$$\therefore \quad K_R = \frac{2}{7}K = \frac{6}{7}mgh$$

$$K_T = \frac{5}{7}K = \frac{15}{7}mgh$$

At point *C*

$$U = mgh$$

$$K_R = (K_R)_B = \frac{6}{7}mgh$$

$$\therefore \quad K_T = E - U - K_R$$

$$= 3\,mgh - mgh - \frac{6}{7}mgh = \frac{8}{7}mgh$$

At point *D*

$$K_R = (K_R)_B = \frac{6}{7}mgh$$

$$K_T = \frac{1}{4}(K_T)_C = \frac{2}{7}mgh$$

$\therefore$

$$U = E - K_R - K_T$$

$$= 3mgh - \frac{6}{7}mgh - \frac{2}{7}mgh$$

$$= \frac{13}{7}mgh$$

At point *E*

$$U = 0$$

$$K_R = (K_R)_B = \frac{6}{7}mgh$$

$\therefore$

$$K_T = E - K_R - U$$

$$= 3\,mgh - \frac{6}{7}mgh - 0 = \frac{15}{7}mgh$$

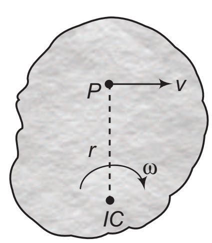
Type 7. *Based on instantaneous axis of rotation (IAOR).*

Concept

We have seen that in case of pure of rolling ($v = R\omega$) over a stationary ground *IAOR* passes through the bottommost point of the rigid body. Following are three more cases where we can locate the position of *IAOR*. In the following cases *IAOR* is written as *IC*.

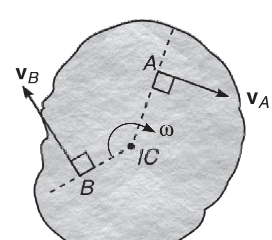
(i) **Given the velocity of a point (normally the centre of mass) on the body and the angular velocity of the body**

If v and ω are known, the *IC* is located along the line drawn perpendicular to $\mathbf{v}$ at P, such that the distance from P to *IC* is, $r = \dfrac{v}{\omega}$. Note that *IC* lie on that side of P which causes rotation about the *IC*, which is consistent with the direction of motion caused by ω and $\mathbf{v}$.



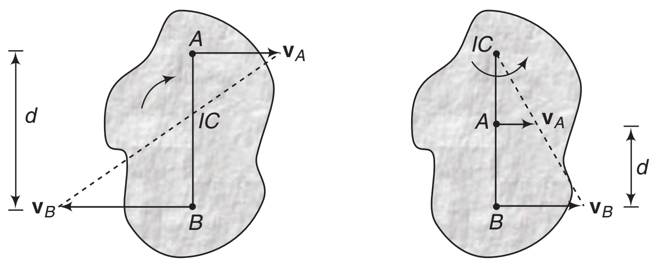
(ii) **Given the lines of action of two non-parallel velocities**

Consider the body shown in figure where the line of action of the velocities $\mathbf{v}_A$ and $\mathbf{v}_B$ are known. Draw perpendiculars at A and B to these lines of action. The point of intersection of these perpendiculars as shown locates the *IC* at the instant considered.



(iii) **Given the magnitude and direction of two parallel velocities**

When the velocities of points A and B are parallel and have known magnitudes v_A and v_B then the location of the IC is determined by proportional triangles as shown in figure.



(a) (b)

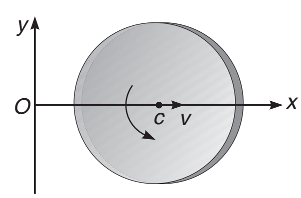
In both the cases, $$r_{A,IC} = \frac{v_A}{\omega} \quad \text{and} \quad r_{B,IC} = \frac{v_B}{\omega}$$

In Fig. (a) $$r_{A,IC} + r_{B,IC} = d$$

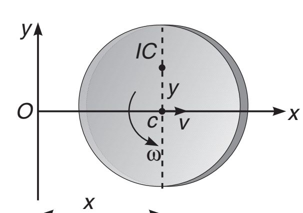
and in Fig. (b) $$r_{B,IC} - r_{A,IC} = d$$

As a special case, if the body is translating, $v_A = v_B$ and the IC would be located at infinity, in which case $\omega = 0$.

Example 13 *A rotating disc moves in the positive direction of the x-axis. Find the equation $y(x)$ describing the position of the instantaneous axis of rotation if at the initial moment the centre c of the disc was located at the point O after which it moved with constant velocity v while the disc started rotating counterclockwise with a constant angular acceleration α. The initial angular velocity is equal to zero.*



Solution $t = \frac{x}{v}$ and $\omega = \alpha t = \frac{\alpha x}{v}$



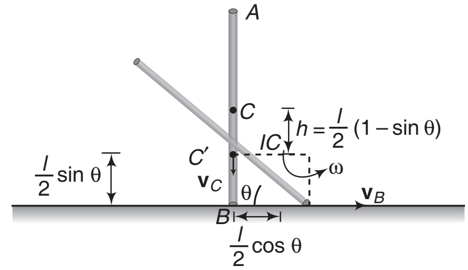
The position of IC will be at a distance

$$y = \frac{v}{\omega} \quad \text{or} \quad y = \frac{v}{\frac{\alpha x}{v}}$$

or $$y = \frac{v^2}{\alpha x} \quad \text{or} \quad xy = \frac{v^2}{\alpha} = \text{constant}$$

This is the desired x-y equation. This equation represents a rectangular hyperbola.

Example 14 *A uniform thin rod of mass m and length l is standing on a smooth horizontal surface. A slight disturbance causes the lower end to slip on the smooth surface and the rod starts falling. Find the velocity of centre of mass of the rod at the instant when it makes an angle θ with horizontal.*



Solution As the floor is smooth, mechanical energy of the rod will remain conserved. Further, no horizontal force acts on the rod, hence the centre of mass moves vertically downwards in a straight line. Thus velocities of *COM* and the lower end *B* are in the directions shown in figure. The location of *IC* at this instant can be found by drawing perpendiculars to $\mathbf{v}_C$ and $\mathbf{v}_B$ at respective points. Now, the rod may be assumed to be in pure rotational motion about *IAOR* passing through *IC* with angular speed ω.

Applying conservation of mechanical energy. Decrease in gravitational potential energy of the rod = increase in rotational kinetic energy about IC

$$\therefore \qquad mgh = \frac{1}{2} I_{IC}\, \omega^2$$

or
$$mg\frac{l}{2}(1 - \sin\theta) = \frac{1}{2}\left(\frac{ml^2}{12} + \frac{ml^2}{4}\cos^2\theta\right)\omega^2$$

Solving this equation, we get

$$\omega = \sqrt{\frac{12g(1 - \sin\theta)}{l(1 + 3\cos^2\theta)}}$$

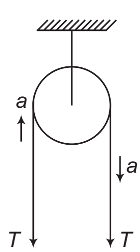
Now,
$$|\mathbf{v}_C| = \left(\frac{l}{2}\cos\theta\right)\omega$$

$$= \sqrt{\frac{3gl(1 - \sin\theta)\cos^2\theta}{(1 + 3\cos^2\theta)}}$$ **Ans.**

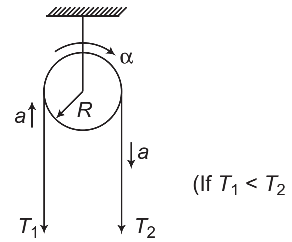
Type 8. *Based on rotational pulleys.*

Concept

If the pulley is not massless and is sufficiently rough then tension on a string passing over it on its both sides will be different. In this case, the string does not slip over the pulley but the pulley also rotates.

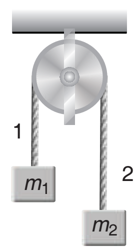


In the above figure, pulley is massless, smooth and stationary. String slips over the pulley.

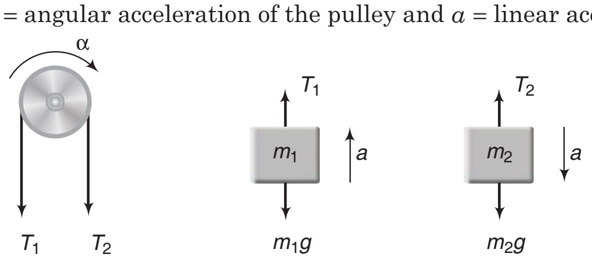


In this figure, pulley is neither massless nor smooth. It rotates with the string. If there is no slip then, $a = R\alpha$.

Example 15 *In the arrangement shown in figure the mass of the uniform solid cylindrical pulley of radius R is equal to m and the masses of two bodies are equal to m_1 and m_2. The thread slipping and the friction in the axle of the pulley are supposed to be absent. Find the angular acceleration of the cylinder and the ratio of tensions $\frac{T_1}{T_2}$ of the vertical sections of the thread in the process of motion.*



Solution Let α = angular acceleration of the pulley and a = linear acceleration of two bodies



Equations of motion are

For mass m_1, $$T_1 - m_1 g = m_1 a \quad \text{...(i)}$$

For mass m_2, $$m_2 g - T_2 = m_2 a \quad \text{...(ii)}$$

For pulley, $$\alpha = \frac{(T_2 - T_1)R}{\frac{1}{2}mR^2} \quad \text{...(iii)}$$

For no slipping condition $$a = R\alpha \quad \text{...(iv)}$$

Solving these equations, we get $$\alpha = \frac{2(m_2 - m_1)g}{(2m_1 + 2m_2 + m)R}$$ **Ans.**

and $$\frac{T_1}{T_2} = \frac{m_1(m + 4m_2)}{m_2(m + 4m_1)}$$ **Ans.**

Type 9. *When friction converts forward or backward slip into pure rolling.*

Concept

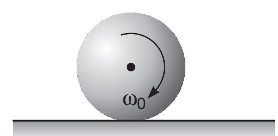
If a body is kept in forward or backward slip condition ($v \neq R\omega$) over a rough horizontal ground, then kinetic friction will act in the opposite direction of slip. This friction provides both linear acceleration 'a' and angular acceleration 'α'. They are in the directions so as the equation $v = R\omega$ (with proper sense of rotation) is satisfied so that pure rolling may start. For example, if initially $v > r\omega$, then 'a' is in the opposite direction of 'v' (to decrease it) and 'α' is in the direction of 'ω' (to increase it).

Once pure rolling starts, relative motion (or slipping) is stopped and friction becomes zero.

During slip, mechanical energy is not conserved. But, some part of mechanical energy is used up in doing work against friction. Once pure rolling starts, mechanical energy becomes constant.

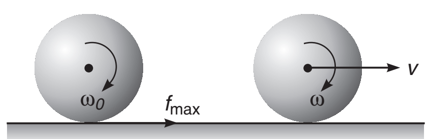
If only one coefficient of friction or μ is given in the question then, instead of kinetic friction, apply μN during the slip.

Example 16 *A solid sphere of radius r is gently placed on a rough horizontal ground with an initial angular speed ω_0 and no linear velocity. If the coefficient of friction is μ, find the time t when the slipping stops. In addition, state the linear velocity v and angular velocity ω at the end of slipping.*



Solution Let m be the mass of the sphere.

Since, it is a case of backward slipping, force of friction is in forward direction. Limiting friction will act in this case.



Linear acceleration $a = \dfrac{f}{m} = \dfrac{\mu mg}{m} = \mu g$

Angular retardation $\alpha = \dfrac{\tau}{I} = \dfrac{f \cdot r}{\frac{2}{5}mr^2} = \dfrac{5}{2}\dfrac{\mu g}{r}$

Slipping is ceased when $v = r\omega$

or $$(at) = r(\omega_0 - \alpha t)$$

or $$\mu g t = r\left(\omega_0 - \frac{5}{2}\frac{\mu g t}{r}\right) \quad \text{or} \quad \frac{7}{2}\mu g t = r\omega_0$$

$\therefore$ $$t = \frac{2}{7}\frac{r\omega_0}{\mu g}$$ **Ans.**

$$v = at = \mu g t = \frac{2}{7}r\omega_0$$ **Ans.**

and $$\omega = \frac{v}{r} = \frac{2}{7}\omega_0$$ **Ans.**

Alternate Solution

Net torque on the sphere about the bottommost point is zero as friction is passing through that point. Therefore, angular momentum of the sphere will remain conserved about the bottommost point.

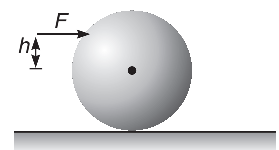
$$L_i = L_f$$

$\therefore$ $$I\omega_0 = I\omega + mrv$$

or $$\frac{2}{5}mr^2\omega_0 = \frac{2}{5}mr^2\omega + mr(\omega r)$$

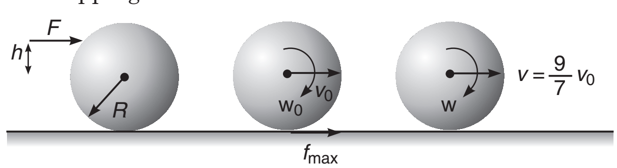
$\therefore$ $$\omega = \frac{2}{7}\omega_0 \quad \text{and} \quad v = r\omega = \frac{2}{7}r\omega_0$$ **Ans.**

Example 17 *A billiard ball, initially at rest, is given a sharp impulse by a cue. The cue is held horizontally a distance h above the centre line as shown in figure. The ball leaves the cue with a speed v_0 and because of its forward english (backward slipping) eventually acquires a final speed $\frac{9}{7}v_0$. Show that $h = \frac{4}{5}R$*



where R is the radius of the ball.

Solution Let ω_0 be the angular speed of the ball just after it leaves the cue. The maximum friction acts in forward direction till the slipping continues. Let v be the linear speed and ω the angular speed when slipping is ceased.



$\therefore$ $$v = R\omega \quad \text{or} \quad \omega = \frac{v}{R}$$

Given, $$v = \frac{9}{7}v_0 \qquad \text{...(i)}$$

$\therefore$ $$\omega = \frac{9}{7}\frac{v_0}{R} \qquad \text{...(ii)}$$

Applying, Linear impulse = change in linear momentum

$\therefore$ $$F\,dt = mv_0 \qquad \text{...(iii)}$$

Angular impulse = change in angular momentum

$\therefore$ $$\tau\,dt = I\omega_0 \quad \text{or} \quad (Fh)dt = \frac{2}{5}mR^2\omega_0 \qquad \text{...(iv)}$$

During the slip, angular momentum about bottommost point will remain conserved.

i.e. $$L_i = L_f$$

or $$I\omega_0 + mRv_0 = I\omega + mRv$$

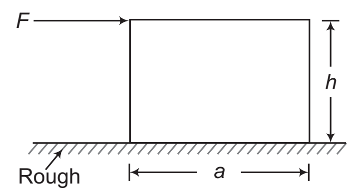
$\therefore$ $$\frac{2}{5}mR^2\omega_0 + mRv_0 = \frac{2}{5}mR^2\left(\frac{9}{7}\frac{v_0}{R}\right) + \frac{9}{7}mRv_0 \qquad \text{...(v)}$$

Solving Eqs. (iii), (iv) and (v), we get $$h = \frac{4}{5}R$$ **Proved.**

Type 10. *Based on critical value of* μ.

Concept

If a force is applied on a block as shown in figure and the force is increased then in some cases the block slides before toppling and in other cases it topples before sliding. It mainly depends on:



(i) base length a
(ii) height of point of application of force h
(iii) coefficient of friction μ.

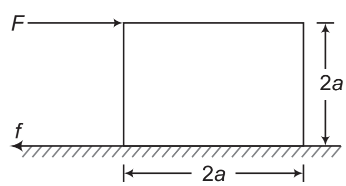
For example, if μ is small then chances of sliding are more. If a is small then chances of toppling are more.

For given values of a and h it only depends on the value of μ. In such problems, there is a critical value μ_{cr}. If given value of $\mu > \mu_{cr}$, then the block topples before sliding and if $\mu < \mu_{cr}$ then the block slides before toppling.

How to Solve?

- Make two conditions:
  Condition of sliding
  Condition of toppling, when the normal reaction (just before toppling) shifts to the right side edge).
  From these, two conditions we can find μ_{cr}.

Example 18 *For the given dimensions shown in figure, find critical value of coefficient of friction* μ.



Solution **Condition of sliding**

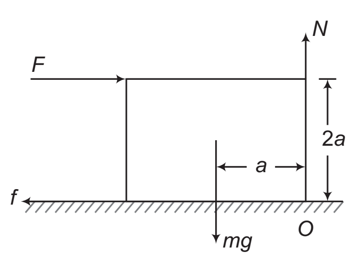
The block will slide if,

$$F > \mu N$$

but $\quad N = mg \quad$ (m = mass of the block)

$\therefore \quad F > \mu mg \quad$ ...(i)

Condition of toppling



Block will topple about an axis passing through O and perpendicular to plane of paper if: clockwise torque of $F >$ anticlockwise torque of mg

$\therefore$ $$F(2a) > (mg)(a)$$

or $$F > \frac{1}{2} mg \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we can see that,

$$\mu_{cr} = \frac{1}{2}$$ **Ans.**

If given value of μ is less than μ_{cr} (say it is $\frac{1}{4}$) then Eq. (i) is,

$$F > \frac{1}{4} mg$$

So, Eq. (i) is satisfied before Eq. (ii). Therefore, the block will slide before toppling.

If given value of μ is greater than μ_{cr}(say it is $\frac{3}{4}$) then Eq. (i) is,

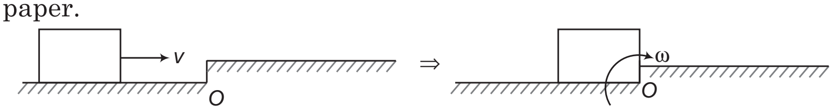
$$F > \frac{3}{4} mg$$

So, Eq. (ii) is satisfied before Eq. (i). Hence, the block will topple before sliding.

Type 11. *When pure translational motion of a rigid body converts into pure rotational motion by a jerk (or linear impulse).*

Concept

A block in pure translational motion (with linear velocity v) meets an obstacle at O. A linear impulse will act on the block at point O. Just after the impact, the block starts rotating about point O (with an angular speed say ω). The value of this 'ω' can be found by conservation of angular momentum about O because during the impact, the angular impulse of the linear impulse about point O will be zero (as $r_\perp = 0$). Just before impact motion is pure translational. So, angular momentum is $m\,vr\sin\theta$ or $mvr_\perp$. Here, v is the velocity of centre of mass. Just after impact motion is pure rotational. So, angular momentum is $I\omega$, where, I is the moment of inertia passing through O and perpendicular to plane of paper.



Furthermore, there will be loss of mechanical energy during impact $\left(= E_i - E_f = \frac{1}{2} mv^2 - \frac{1}{2} I\omega^2\right)$. But after impact, the mechanical energy remains constant. As the block moves up, its potential energy increases and rotational kinetic energy decreases.

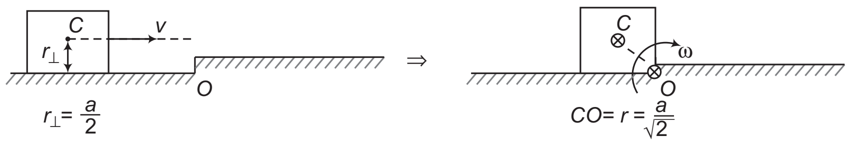
Example 19 *In the figure shown in the text, if the block is a cube of side 'a'. Find*

(a) ω just after impact

(b) loss of mechanical energy during impact

(c) minimum value of v so as the block overcomes the obstacle and does not turn back.

Solution



(a) From conservation of angular momentum about O,

$$L_i = L_f$$

$$mv_c r_\perp = I_o\omega = (I_c + mr^2)\omega$$

$$\Rightarrow \quad mv\left(\frac{a}{2}\right) = \left[\frac{ma^2}{6} + m\left(\frac{a}{\sqrt{2}}\right)^2\right]\omega$$

$$\Rightarrow \quad \omega = \frac{3}{4}\left(\frac{v}{a}\right)$$ **Ans.**

(b) Loss of mechanical energy,

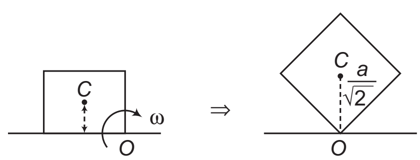
$$= E_i - E_f$$

$$= \frac{1}{2}mv^2 - \frac{1}{2}I_o\omega^2$$

$$= \frac{1}{2}mv^2 - \frac{1}{2}\left[\frac{ma^2}{6} + m\left(\frac{a}{\sqrt{2}}\right)^2\right]\left(\frac{3}{4}\frac{v}{a}\right)^2$$

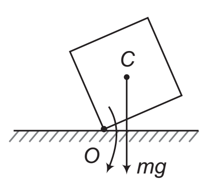
$$= \frac{5}{16}mv^2$$ **Ans.**

(c)



Block overcomes the obstacle at O if centre of mass rises upto a height $\frac{a}{\sqrt{2}}$ as shown in figure (from the initial height $\frac{a}{2}$).

Because after that torque of 'mg' about O will itself rotate the block on other side as shown in figure.



∴ Decrease in rotational kinetic energy = increase in gravitational potential energy

$$\therefore \quad \frac{1}{2}I_o\omega^2 = mg\left(\frac{a}{\sqrt{2}} - \frac{a}{2}\right)$$

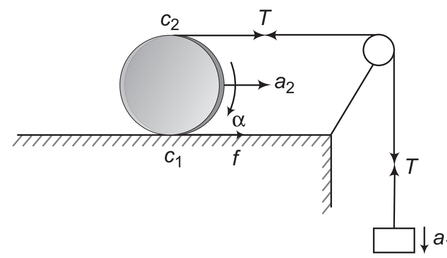
or $$\frac{1}{2}\left[\frac{ma^2}{6} + m\left(\frac{a}{\sqrt{2}}\right)^2\right]\left[\frac{3}{4}\frac{v}{a}\right]^2 = mg \left(\frac{a}{\sqrt{2}} - \frac{a}{2}\right)$$

$$\therefore \quad v = \sqrt{1.1\, g\, a}$$ **Ans.**

Type 12. *To identify number of unknowns and then make equations corresponding to that.*

Concept

A disc and block system are released from rest as shown in figure. Ground is sufficiently rough, so that there is no slip anywhere. We have to find accelerations of both, tension in the string and force of friction.



In such problems, first of all find the number of unknowns:

(i) Block can have only translational motion. So, it has some linear acceleration say a_1

(ii) Disc can have translational as well as rotational motion. So, it has linear acceleration a_2 and angular acceleration 'α'.

(iii) Ground is rough. So, there is one unknown friction 'f'

(iv) One more unknown is tension in the string T.

Therefore, there are total five unknowns in this problem, a_1, a_2, α, T and f.

How to Solve?

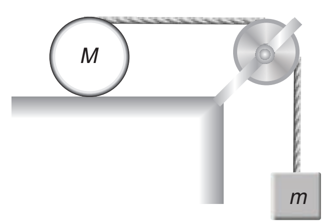
We will make three acceleration equations for a_1, a_2 and α by using the equations

$$a = \frac{F_{\text{net}}}{m} \quad \text{and} \quad \alpha = \frac{\tau_{\text{net}}}{I}$$

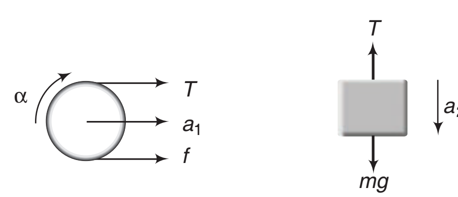
There are two contact equations at c_1 and c_2.

At c_1, disc is in contact with ground and at c_2 with string. At the other end string is connected to the block. So, there should not be any slip at c_1 between disc and ground and at c_2 between disc and string. Following two examples illustrate the method of making these equations.

Example 20 *Consider the arrangement shown in figure. The string is wrapped around a uniform cylinder which rolls without slipping. The other end of the string is passed over a massless, frictionless pulley to a falling weight. Determine the acceleration of the falling mass m in terms of only the mass of the cylinder M, the mass m and g.*



Solution Let T be the tension in the string and f the force of (static) friction, between the cylinder and the surface.



a_1 = acceleration of centre of mass of cylinder towards right

a_2 = downward acceleration of block m

α = angular acceleration of cylinder (clockwise)

Acceleration equations

For block, $\quad mg - T = ma_2 \quad$ ...(i)

For cylinder, $$T + f = Ma_1 \quad \text{...(ii)}$$

$$\alpha = \frac{(T-f)R}{\frac{1}{2}MR^2} \quad \text{...(iii)}$$

Contact equations

The string is attached to the mass m at the highest point of the cylinder, hence

$$v_m = v_{COM} + R\omega$$

Differentiating, we get $$a_2 = a_1 + R\alpha \quad \text{...(iv)}$$

We also have (for rolling without slipping)

$$a_1 = R\alpha \quad \text{...(v)}$$

Solving these equations, we get $$a_2 = \frac{8mg}{3M + 8m}$$ **Ans.**

Alternate Solution (Energy Method)

Since, there is no slipping at all contacts mechanical energy of the system will remain conserved.

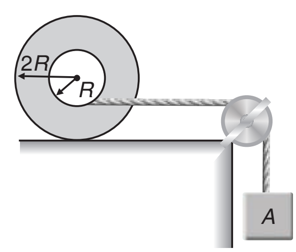
∴ Decrease in gravitational potential energy of block m in time t = increase in translational kinetic energy of block + increase in rotational as well as translational kinetic energy of cylinder.

∴ $$mgh = \frac{1}{2}mv_2^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}Mv_1^2$$

or $$mg\left(\frac{1}{2}a_2t^2\right) = \frac{1}{2}m(a_2t)^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)(\alpha t)^2 + \frac{1}{2}M(a_1t)^2 \quad \text{...(vi)}$$

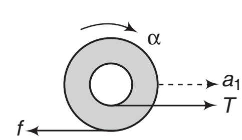
Solving Eqs. (iv), (v) and (vi), we get the same result.

Example 21 *A thin massless thread is wound on a reel of mass 3 kg and moment of inertia 0.6 kg-m². The hub radius is $R = 10$ cm and peripheral radius is $2R = 20$ cm. The reel is placed on a rough table and the friction is enough to prevent slipping. Find the acceleration of the centre of reel and of hanging mass of 1 kg.*



Solution Here, number of unknowns are five:

a_1 = acceleration of centre of mass of reel
a_2 = acceleration of 1 kg block
α = angular acceleration of reel (clockwise)
T = tension in the string
and f = force of friction



Acceleration equations :

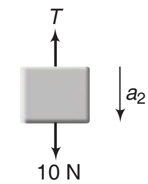
Free body diagram of reel is as shown in figure: (only horizontal forces are shown).

Equations of motion are

$$T - f = 3a_1 \quad \text{...(i)}$$

$$\alpha = \frac{\tau}{I} = \frac{f(2R) - T.R}{I} = \frac{0.2f - 0.1T}{0.6} = \frac{f}{3} - \frac{T}{6} \quad \text{...(ii)}$$

Free body diagram of mass is,



Equation of motion is,

$$10 - T = a_2 \qquad \ldots\text{(iii)}$$

Contact equations :

For no slipping condition,

$$a_1 = 2R\alpha \quad \text{or} \quad a_1 = 0.2\,\alpha \qquad \ldots\text{(iv)}$$

and $$a_2 = a_1 - R\alpha \quad \text{or} \quad a_2 = a_1 - 0.1\,\alpha \qquad \ldots\text{(v)}$$

Solving the above five equations, we get

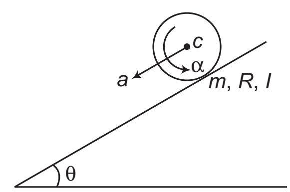
$$a_1 = 0.27 \text{ m/s}^2 \quad \text{and} \quad a_2 = 0.135 \text{ m/s}^2$$ **Ans.**

Type 13. *Energy method of solving problems of accelerated pure rolling.*

Concept

In accelerated pure rolling over a stationary ground work done by friction is zero. So, mechanical energy remains constant. Therefore, some problems of accelerated pure rolling can also be solved by using energy conservation principle.

Example 22 *A body of mass m, radius R and moment of inertia I (about an axis passing through the centre of mass and perpendicular to plane of motion) is released from rest over a sufficiently rough ground (to provide accelerated pure rolling). Find linear acceleration of the body.*



Solution Let linear acceleration is 'a' and angular acceleration 'α'.

For accelerated pure rolling,

$$\alpha = \frac{a}{R}$$

After time t, displacement of centre of mass along the plane, $s = \frac{1}{2}at^2$

$\therefore$ Height fallen by centre of mass

$$h = (s)(\sin\theta) = \frac{1}{2}at^2\sin\theta$$

linear velocity $v = at$

angular velocity $\omega = \alpha\, t = \frac{at}{R}$

From energy conservation principle, decrease in potential energy = increase in translational and rotational kinetic energy.

or $$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

Substituting the value we have,

$$mg\left(\frac{1}{2}at^2\sin\theta\right) = \frac{1}{2}m\,(at)^2 + \frac{1}{2}I\left(\frac{at}{R}\right)^2$$

Solving this equation we get, $$a = \frac{g\sin\theta}{1 + \frac{I}{mR^2}}$$ **Ans.**

Type 14. *Problems of accelerated pure rolling by finding angular acceleration 'α' about the bottommost axis.*

Concept

We have discussed in final touch points that :

$$\alpha = \frac{\tau_{ext}}{I}$$

can be applied from an inertial frame or about an axis passing through centre of mass (even if it is accelerated).

Same is the case about an axis passing through bottommost axis if accelerated pure rolling is taking place on a stationary ground.

Although the bottommost point is accelerated ($= R\omega^2$, towards centre), yet for symmetrical bodies net torque of pseudo forces on all particles of the rigid body about bottommost axis is zero. So,

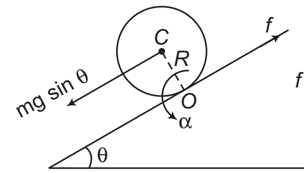
$$\alpha = \frac{\tau_{ext}}{I}$$

can be applied about this axis also.

Example 23 *In example 22, find linear acceleration 'a' of the body by calculating α about bottommost axis.*

Solution

$$\alpha = \frac{\tau_0}{I_0} = \frac{(mg\sin\theta)R}{(I_c + MR^2)}$$



But $I_c = I$

$\therefore$

$$\alpha = \frac{mg\,R\sin\theta}{I + mR^2}$$

Now, a_c or $a = R\alpha = \dfrac{mg\,R^2\sin\theta}{I + mR^2}$

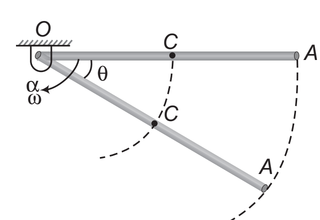
or

$$a = \frac{g\sin\theta}{1 + \dfrac{I}{mR^2}}$$ **Ans.**

Type 15. *Based on hinge force.*

Concept

A rod OA is hinged at O. It is released from the horizontal position as shown in figure. We have to find hinge force acting on the rod at a general angle θ.



We can see that motion of the rod is pure rotational about an axis passing through O. As the rod rotates downwards, its gravitational potential energy decreases and rotational kinetic energy increases. If the hinge is smooth, then we can apply the equation.

$$mgh = \frac{1}{2} I \omega^2 \qquad \text{...(i)}$$

Here, h = height fallen by centre of mass C of the rod.

and $$I = I_0$$

From here we can find ω^2.

At the same time, we can see that ω is increasing. So, there is an angular acceleration 'α' about O. Only two forces are acting on the rod, hinge force and weight.

Torque of hinge force about O is zero. Therefore, torque of 'mg' will only provide 'α'. Thus,

$$\alpha = \frac{\tau_{mg}}{I} \qquad \text{...(ii)}$$

Now, only two forces are acting on the rod, hinge force (say $\mathbf{F}$) and weight ($m\mathbf{g}$).

$\therefore$ $$\mathbf{F} + m\mathbf{g} = m\mathbf{a}_{\text{COM}} \text{ or } m\mathbf{a}_C$$

or $$\mathbf{F} = m\mathbf{a}_c - m\mathbf{g} \qquad \text{...(iii)}$$

By finding $\mathbf{a}_c$ we can find the hinge force $\mathbf{F}$ from Eq. (iii).

How to Solve a_c?

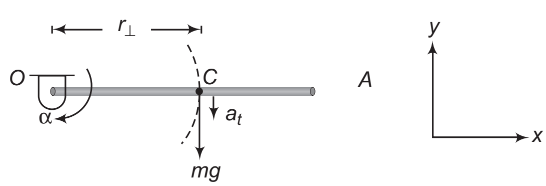
- In the figure, we can see that C is rotating in a circle with centre at O and radius $r = OC = \frac{l}{2}$, where l is the length of rod. In a circular motion, acceleration of a particle has two components.
- (i) radial $a_r = r\omega^2$
- (ii) tangential $a_t = r\alpha$
- ω^2 can be obtained from Eq. (i), α can be obtained from Eq. (ii). Writing all vector quantities in proper vector notations and then substituting in Eq.(iii) we can find the hinge force **F**.

Example 24 *In the figure given in the text if mass of the rod is 'm' then find hinge force.*

(a) Just after the rod is released from the horizontal position.

(b) When the rod becomes vertical.

Solution (a)



Just after the release, $\omega = 0$

$\therefore$ $$a_r = r\omega^2 = 0$$

$$\alpha = \frac{\tau_{mg}}{I} \qquad \text{(about } O\text{)}$$

$$= \frac{(mg)(r_\perp)}{I_0} = \frac{(mg)\left(\frac{l}{2}\right)}{(ml^2/3)} = \frac{3}{2}\frac{g}{l}$$

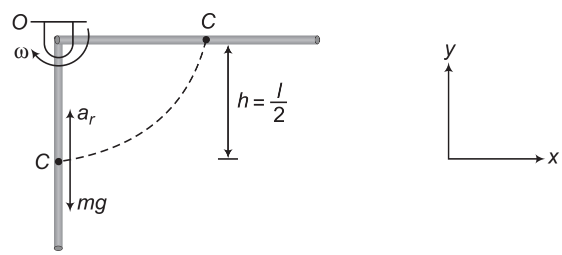
$\therefore$ $$a_t = r\alpha = \left(\frac{l}{2}\right)\left(\frac{3}{2}\frac{g}{l}\right) = \frac{3}{4}g$$

From Eq. (iii), the hinge force is

$$\mathbf{F} = m\,(\mathbf{a}_c - \mathbf{g})$$
$$= m\,[a_t\,(-\hat{\mathbf{j}}) - (-g\hat{\mathbf{j}})]$$
$$= m\left[-\frac{3}{4}\,g\hat{\mathbf{j}} + g\hat{\mathbf{j}}\right]$$
$$= \frac{mg}{4}\,\hat{\mathbf{j}}$$ **Ans.**

Therefore, hinge force is $\frac{mg}{4}$, in vertically upward direction.

(b)



When the rod becomes vertical height fallen by centre of mass is $h = \frac{l}{2}$

Therefore, from Eq. (i),

$$\omega^2 = \frac{2mgh}{I} \qquad \left(I = I_0 = \frac{ml^2}{3}\right)$$

$$= \frac{2mg\left(\frac{l}{2}\right)}{(ml^2/3)} = \left(\frac{3g}{l}\right)$$

$\therefore$ $$a_r = r\omega^2 = \left(\frac{l}{2}\right)\left(\frac{3g}{l}\right) = \frac{3}{2}\,g \qquad \text{(towards } O\text{)}$$

At this moment 'mg' also passes through O. Therefore, its torque about O is also zero. So, from Eq. (ii),

$$\alpha = 0 \quad \Rightarrow \quad a_t = r\alpha = 0$$

Now, substituting proper values in Eq. (iii), the hinge force is,

$$\mathbf{F} = m\,(\mathbf{a}_c - \mathbf{g})$$
$$= m\,[a_r\hat{\mathbf{j}} - (-\,g\hat{\mathbf{j}})]$$
$$= m\left[\left(\frac{3}{2}\,g\right)\hat{\mathbf{j}} + (g)\hat{\mathbf{j}}\right]$$
$$= \left(\frac{5}{2}\,mg\right)\hat{\mathbf{j}}$$ **Ans.**

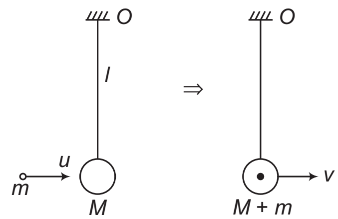
Therefore, hinge force is $\frac{5}{2}\,mg$ in vertically upward direction.

Type 16. *Collision Problems.*

Concept

There are mainly three types of collisions.

Type 1.



A ball of mass M is suspended by a light string of length 'l'. A bullet of mass 'm' strikes the ball with velocity u and sticks. This type of collision we have already discussed in the previous chapter.

The three important points in this collision are

(i) Velocity of combined mass after collision (say v) can be obtained by conservation of linear momentum or

$$p_f = p_i$$

$$\Rightarrow \qquad (M+m)v = mu$$

$$\text{or} \qquad v = \frac{mu}{M+m}$$

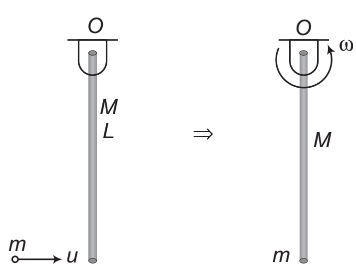
(ii) Mechanical energy is lost only during collision. This loss is given by

$$E_i - E_f \qquad (E = \text{mechanical energy})$$

$$= \frac{1}{2}mu^2 - \frac{1}{2}(M+m)v^2$$

(iii) After collision now the mechanical energy remains constant and the combined mass executes vertical circular motion. If $v \geq \sqrt{5gl}$ circle is completed. If $\sqrt{2gl} < v < \sqrt{5gl}$, string slacks in upper half of the circle and if $0 < v \leq \sqrt{2gl}$, combined mass oscillates in lower half of the circle.

Type 2.



A rod of mass M and length L is hinged at point O. A bullet of mass m moving with velocity u strikes the rod at its bottommost point and sticks. Just after collision the combined system (rod + bullet) starts rotating about the hinge point O with an angular speed ω.

So, the translational motion converts into rotational motion. The three important points in this collision are

(i) At the time of collision, a linear impulse acts on the system at point O from the hinge. Angular impulse of this linear impulse about O is zero (as $r_\perp = 0$). Therefore, angular momentum of the system about O remains constant or

$$L_i = L_f$$

$\therefore$ $$mur_\perp = I\omega$$

or $$mu\,L = (I_{\text{rod}} + I_{\text{Bullet}})\omega$$

$$= \left(\frac{ML^2}{3} + mL^2\right)\omega$$

From this equation, we can find 'ω'.

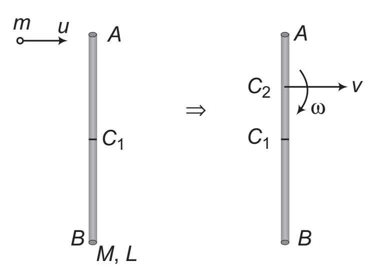
(ii) Mechanical energy is lost only during collision (not after that) and this loss is

$$E_i - E_f = \frac{1}{2}mu^2 - \frac{1}{2}I\omega^2$$

(iii) After the collision, as the system moves upwards its gravitational potential energy increases and rotational kinetic energy decreases. So, we can write :

decrease in rotational kinetic energy = increase in gravitational potential energy.

Type 3.



A rod of mass M and length L is lying on a smooth table. A bullet of mass 'm' and speed 'u' strikes the rod at A and sticks. C_1 is the centre of rod and C_2 is the centre of mass of system (rod + bullet).

Just after collision motion of the combined system is rotation (with angular velocity ω) and translation (with linear velocity v).

Thus, by the collision, translation motion converts into rotation plus translation motion. The three important points in this type of collision are:

(i) System is kept over a smooth horizontal table. So, net linear impulse on the system is zero. Therefore, v can be obtained by conservation of linear momentum or,

$$p_f = p_i$$

$\Rightarrow$ $$(M+m)v = mu$$

$\therefore$ $$v = \frac{mu}{M+m}$$

Since, net linear impulse on the system is zero. Therefore, angular impulse about any point is also zero. Hence, angular momentum of the system can be conserved about any point and ω can be obtained. But normally, we conserve it about point of impact (or A).

$\therefore \quad L_f = L_i$ (about A)

or $\quad (m+M)vr_\perp \pm I\omega = 0 \quad \Rightarrow \quad L_i = mur_\perp$

Here, $\quad = 0$ (as $r_\perp = 0$ from A)

$$I = I_{\text{rod}} + I_{\text{Bullet}}$$

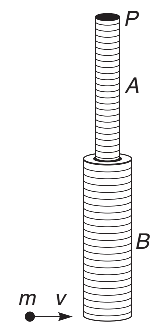
To find $r_\perp$ of v from A first we will have to find position of COM or C_2 of the combined system.

(ii) Mechanical energy is lost only during collision (not after that) and this loss is:

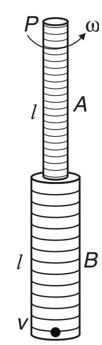
$$E_i - E_f = \left(\frac{1}{2}mu^2\right) - \left[\frac{1}{2}(M+m)v^2 + \frac{1}{2}I\omega^2\right]$$

(iii) Since, the system is kept over a smooth table, v and ω remain constant after the collision.

Example 25 *Two uniform rods A and B of length* 0.6 *m each and of masses* 0.01 *kg and* 0.02 *kg respectively are rigidly joined end to end. The combination is pivoted at the lighter end, P as shown in figure. Such that it can freely rotate about point P in a vertical plane. A small object of mass* 0.05 *kg, moving horizontally, hits the lower end of the combination and sticks to it. What should be the velocity of the object, so that the system could just be raised to the horizontal position?* **(JEE 1994)**



Solution System is free to rotate but not free to translate. During collision, net torque on the system (rod A + rod B + mass m) about point P is zero.



Therefore, angular momentum of system before collision

= angular momentum of system just after collision (about P).

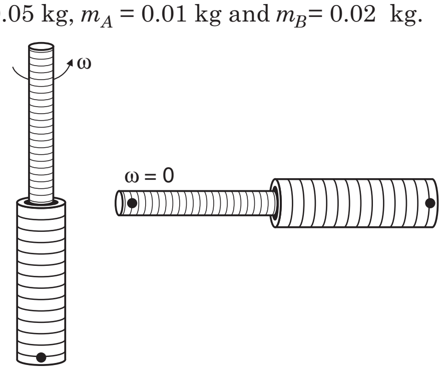
Let ω be the angular velocity of system just after collision, then

$$L_i = L_f \quad \Rightarrow \quad mv(2l) = I\omega \qquad \ldots\text{(i)}$$

Here, I = moment of inertia of system about P

$$= m(2l)^2 + m_A(l^2/3) + m_B\left[\frac{l^2}{12} + \left(\frac{l}{2} + l\right)^2\right]$$

Given, $l = 0.6$ m, $m = 0.05$ kg, $m_A = 0.01$ kg and $m_B = 0.02$ kg.



Substituting the values, we get

$$I = 0.09 \text{ kg- m}^2$$

Therefore, from Eq. (i)

$$\omega = \frac{2mvl}{I} = \frac{(2)\,(0.05)\,(v)(0.6)}{0.09}$$

$$\omega = 0.67v \qquad \text{...(ii)}$$

Now, after collision, mechanical energy will be conserved.

Therefore, decrease in rotational KE = increase in gravitational PE

or $$\frac{1}{2}\, I\omega^2 = mg\,(2l) + m_A\, g\left(\frac{l}{2}\right) + m_B g\,(l + \frac{l}{2})$$

or $$\omega^2 = \frac{gl\,(4m + m_A + 3\,m_B)}{I}$$

$$= \frac{(9.8)\,(0.6)\,(4 \times 0.05 + 0.01 + 3 \times 0.02)}{0.09}$$

$$= 17.64\ (\text{rad/s})^2$$

$\therefore$ $$\omega = 4.2 \text{ rad/s} \qquad \text{...(iii)}$$

Equating Eqs. (ii) and (iii), we get

$$v = \frac{4.2}{0.67} \text{ m/s} \quad \text{or} \quad v = 6.3 \text{ m/s}$$ **Ans.**

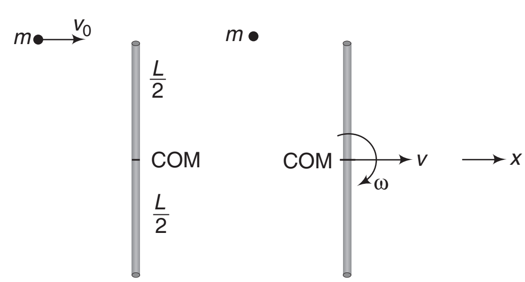
Example 26 *A rod AB of mass M and length L is lying on a horizontal frictionless surface. A particle of mass m travelling along the surface hits the end A of the rod with a velocity v_0 in a direction perpendicular to AB. The collision is elastic. After the collision the particle comes to rest.* **(JEE 2000)**

(a) Find the ratio m/M.

(b) A point P on the rod is at rest immediately after collision. Find the distance AP.

(c) Find the linear speed of the point P a time $\pi L/3v_0$ after the collision.

Solution (a) Suppose velocity of COM of the rod just after collision is v and angular velocity about COM is ω. Applying following three laws:



(1) External force on the system (rod + mass) in horizontal plane along x-axis is zero .

$\therefore$ Applying conservation of linear momentum in x-direction.

$$mv_0 = Mv \qquad \text{...(i)}$$

(2) Net torque on the system about COM of rod is zero.

$\therefore$ Applying conservation of angular momentum about COM of rod, we get $mv_0\left(\frac{L}{2}\right) = I\omega$

or $$mv_0 \frac{L}{2} = \frac{ML^2}{12}\omega$$

or $$mv_0 = \frac{ML\omega}{6} \qquad \text{...(ii)}$$

(3) Since, the collision is elastic, kinetic energy is also conserved.

$\therefore$ $$\frac{1}{2}mv_0^2 = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$

or $$mv_0^2 = Mv^2 + \frac{ML^2}{12}\omega^2 \qquad \text{...(iii)}$$

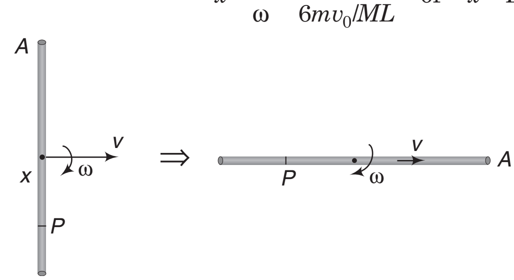
From Eqs. (i), (ii) and (iii), we get the following results

$$\frac{m}{M} = \frac{1}{4}$$ **Ans.**

$$v = \frac{mv_0}{M} \quad \text{and} \quad \omega = \frac{6\,mv_0}{ML}$$

(b) Point P will be at rest if $x\omega = v$

or $$x = \frac{v}{\omega} = \frac{mv_0/M}{6mv_0/ML} \quad \text{or} \quad x = L/6$$



$\therefore$ $$AP = \frac{L}{2} + \frac{L}{6} \quad \text{or} \quad AP = \frac{2}{3}L$$ **Ans.**

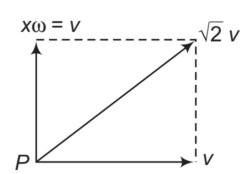
(c) After time $$t = \frac{\pi L}{3\,v_0}$$

angle rotated by rod, $\theta = \omega t = \frac{6mv_0}{ML} \cdot \frac{\pi L}{3v_0}$

$$= 2\pi\left(\frac{m}{M}\right) = 2\pi\left(\frac{1}{4}\right)$$

$\therefore$ $$\theta = \frac{\pi}{2}$$

Therefore, situation is as shown in figure.



$\therefore$ Resultant velocity of point P will be

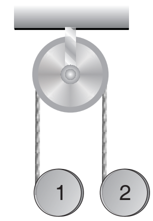
$$|\mathbf{v}_P| = \sqrt{2}v = \sqrt{2}\left(\frac{m}{M}\right)v_0$$

$$= \frac{\sqrt{2}}{4}v_0 = \frac{v_0}{2\sqrt{2}}$$

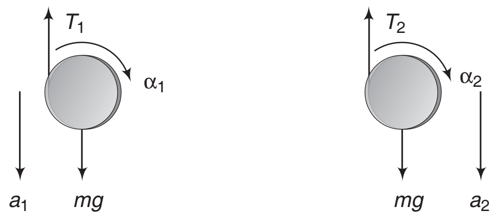
or $$|\mathbf{v}_P| = \frac{v_0}{2\sqrt{2}}$$ **Ans.**

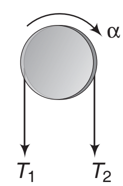
Miscellaneous Examples

Example 27 *A thread is wound around two discs on either sides. The pulley and the two discs have the same mass and radius. There is no slipping at the pulley and no friction at the hinge. Find out the accelerations of the two discs and the angular acceleration of the pulley.*



Solution Let R be the radius of the discs and T_1 and T_2 be the tensions in the left and right segments of the rope.





Acceleration of disc 1,

$$a_1 = \frac{mg - T_1}{m} \qquad \ldots\text{(i)}$$

Acceleration of disc 2,

$$a_2 = \frac{mg - T_2}{m} \qquad \ldots\text{(ii)}$$

Angular acceleration of disc 1, $$\alpha_1 = \frac{\tau}{I} = \frac{T_1 R}{\frac{1}{2} mR^2} = \frac{2T_1}{mR} \qquad \ldots\text{(iii)}$$

Similarly, angular acceleration of disc 2, $$\alpha_2 = \frac{2T_2}{mR} \qquad \ldots\text{(iv)}$$

Both α_1 and α_2 are clockwise.

Angular acceleration of pulley,

$$\alpha = \frac{(T_2 - T_1)R}{\frac{1}{2} mR^2} = \frac{2(T_2 - T_1)}{mR} \qquad \ldots\text{(v)}$$

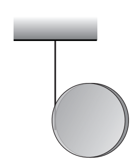
For no slipping, $$R\alpha_1 - a_1 = a_2 - R\alpha_2 = R\alpha \qquad \ldots\text{(vi)}$$

Solving these equations, we get

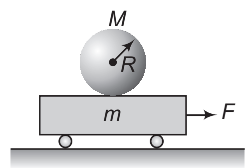
$$\alpha = 0 \qquad \text{and} \qquad a_1 = a_2 = \frac{2g}{3}$$ **Ans.**

Alternate Solution

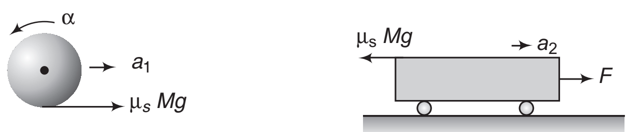
As both the discs are in identical situation, $T_1 = T_2$ and $\alpha = 0$. i.e. each of the discs falls independently and identically. Therefore, this is exactly similar to the problem shown in figure.



Example 28 *Determine the maximum horizontal force F that may be applied to the plank of mass m for which the solid sphere does not slip as it begins to roll on the plank. The sphere has a mass M and radius R. The coefficient of static and kinetic friction between the sphere and the plank are* μ_s *and* μ_k *respectively.*



Solution The free body diagrams of the sphere and the plank are as shown below:



Writing equations of motion

For sphere Linear acceleration

$$a_1 = \frac{\mu_s Mg}{M} = \mu_s g \quad \text{...(i)}$$

Angular acceleration

$$\alpha = \frac{(\mu_s Mg)R}{\frac{2}{5}MR^2}$$

$$= \frac{5}{2}\frac{\mu_s g}{R} \quad \text{...(ii)}$$

For plank Linear acceleration

$$a_2 = \frac{F - \mu_s Mg}{m} \quad \text{...(iii)}$$

For no slipping

$$a_2 = a_1 + R\alpha \quad \text{...(iv)}$$

Solving the above four equations, we get

$$F = \mu_s g\left(M + \frac{7}{2}m\right)$$

Thus, maximum value of F can be

$$\mu_s g\left(M + \frac{7}{2}m\right) \quad \textbf{Ans.}$$

Example 29 *A uniform disc of radius* r_0 *lies on a smooth horizontal plane. A similar disc spinning with the angular velocity* ω_0 *is carefully lowered onto the first disc. How soon do both discs spin with the same angular-velocity if the friction coefficient between them is equal to* μ?

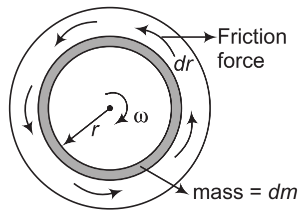
Solution From the law of conservation of angular momentum.

$$I\omega_0 = 2I\omega$$

Here, I = moment of inertia of each disc relative to common rotation axis

$$\therefore \qquad \omega = \frac{\omega_0}{2} = \text{steady state angular velocity}$$

The angular velocity of each disc varies due to the torque τ of the friction forces. To calculate τ, let us take an elementary ring with radii r and $r + dr$. The torque of the friction forces acting on the given ring is equal to



$$d\tau = (\text{friction force})\ (r_\perp) = [\mu(dm)g](r)$$

$$= \mu\left(\frac{mg}{\pi r_0^2}\right)(2\pi r dr)r$$

$$= \left(\frac{2\mu mg}{r_0^2}\right) r^2\, dr$$

where, m is the mass of each disc. Integrating this with respect to r between 0 and r_0, we get

$$\tau = \frac{2}{3}\mu mgr_0 = \text{constant}$$

$$\therefore \qquad \alpha = \frac{\tau}{I} = \frac{(2\mu mgr_0/3)}{(mr_0^2/2)}$$

$$= \frac{4\mu g}{3r_0} = \text{constant}$$

Now, angular speed of lower disc increases with this α from O to $\frac{\omega_0}{2}$ and α is constant.

$$\therefore \qquad \frac{\omega_0}{2} = \alpha t$$

or $$t = \frac{\omega_0}{2\alpha} = \frac{3r_0\omega_0}{8\mu g}$$ **Ans.**

Exercises

LEVEL 1

Assertion and Reason

Directions : *Choose the correct option.*

(*a*) *If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion.***

(*b*) *If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion.***

(*c*) *If **Assertion** is true, but the **Reason** is false.*

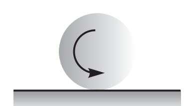
(*d*) *If **Assertion** is false but the **Reason** is true.*

1. Assertion : Moment of inertia of a rigid body about any axis passing through its centre of mass is minimum.

Reason : From theorem of parallel axis,

$$I = I_{cm} + Mr^2$$

2. Assertion : A ball is released on a rough ground in the condition shown in figure. It will start pure rolling after some time towards left side.

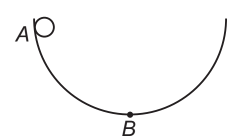


Reason : Friction will convert the pure rotational motion of the ball into pure rolling.

3. Assertion : A solid sphere and a hollow sphere are rolling on ground with same total kinetic energies. If translational kinetic energy of solid sphere is K, then translational kinetic energy of hollow sphere should be greater than K.

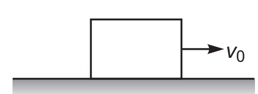
Reason : In case of hollow sphere rotational kinetic energy is less than its translational kinetic energy.

4. Assertion : A small ball is released from rest from point A as shown. If bowl is smooth, than ball will exert more pressure at point B, compared to the situation if bowl is rough.



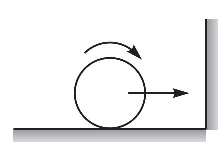
Reason : Linear velocity and hence, centripetal force in smooth situation is more.

5. Assertion : A cubical block is moving on a rough ground with velocity v_0. During motion net normal reaction on the block from ground will not pass through centre of cube. It will shift towards right.



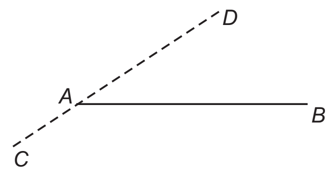
Reason : It is to keep the block in rotational equilibrium.

6. Assertion : A ring is rolling without slipping on a rough ground. It strikes elastically with a smooth wall as shown in figure. Ring will stop after some time while travelling in opposite direction.



Reason : After impact net angular momentum about an axis passing through bottommost point and perpendicular to plane of paper is zero.

7. Assertion : There is a thin rod AB and a dotted line CD. All the axes we are talking about are perpendicular to plane of paper. As we take different axes moving from A to D, moment of inertia of the rod may first decrease then increase.



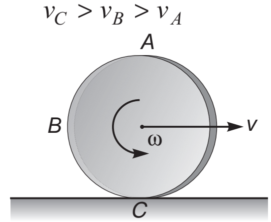
Reason : Theorem of perpendicular axis cannot be applied here.

8. Assertion : If linear momentum of a particle is constant, then its angular momentum about any point will also remain constant.

Reason : Linear momentum remains constant, if $\mathbf{F}_{\text{net}} = 0$ and angular momentum remains constant if $\tau_{\text{net}} = 0$.

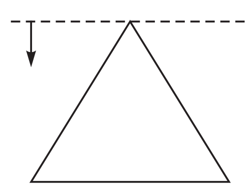
9. Assertion : In the figure shown, A, B and C are three points on the circumference of a disc. Let v_A, v_B and v_C are speeds of these three points, then

$$v_C > v_B > v_A$$



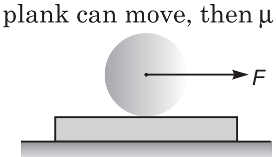
Reason : In case of rotational plus translational motion of a rigid body, net speed of any point (other than centre of mass) is greater than, less than or equal to the speed of centre of mass.

10. Assertion : There is a triangular plate as shown. A dotted axis is lying in the plane of slab. As the axis is moved downwards, moment of inertia of slab will first decrease then increase.



Reason : Axis is first moving towards its centre of mass and then it is receding from it.

11. Assertion : A horizontal force F is applied at the centre of solid sphere placed over a plank. The minimum coefficient of friction between plank and sphere required for pure rolling is μ_1 when plank is kept at rest and μ_2 when plank can move, then $\mu_2 < \mu_1$.



Reason : Work done by frictional force on the sphere in both cases is zero.

Objective Questions

Single Correct Option

1. The moment of inertia of a body does not depend on
(a) mass of the body
(b) the distribution of the mass in the body
(c) the axis of rotation of the body
(d) None of the above

2. The radius of gyration of a disc of radius 25 cm about a centroidal axis perpendicular to disc is
(a) 18 cm (b) 12.5 cm (c) 36 cm (d) 50 cm

3. A shaft initially rotating at 1725 rpm is brought to rest uniformly in 20s. The number of revolutions that the shaft will make during this time is
(a) 1680 (b) 575 (c) 287 (d) 627

4. A man standing on a platform holds weights in his outstretched arms. The system is rotated about a central vertical axis. If the man now pulls the weights inwards close to his body, then
(a) the angular velocity of the system will increase
(b) the angular momentum of the system will remain constant
(c) the kinetic energy of the system will increase
(d) All of the above

5. The moment of inertia of a uniform semicircular disc of mass M and radius r about a line perpendicular to the plane of the disc through the centre is
(a) Mr^2 (b) $\frac{1}{2}Mr^2$ (c) $\frac{1}{4}Mr^2$ (d) $\frac{2}{5}Mr^2$

6. Two bodies A and B made of same material have the moment of inertial in the ratio $I_A : I_B = 16 : 18$. The ratio of the masses $m_A : m_B$ is given by
(a) cannot be obtained (b) 2 : 3
(c) 1 : 1 (d) 4 : 9

7. When a sphere rolls down an inclined plane, then identity the correct statement related to the work done by friction force
(a) The friction force does positive translational work
(b) The friction force does negative rotational work
(c) The net work done by friction is zero
(d) All of the above

8. A circular table rotates about a vertical axis with a constant angular speed ω. A circular pan rests on the turn table (with the centre coinciding with centre of table) and rotates with the table. The bottom of the pan is covered with a uniform small thick layer of ice placed at centre of pan. The ice starts melting. The angular speed of the turn table
(a) remains the same
(b) decreases
(c) increases
(d) may increase or decrease depending on the thickness of ice layer

9. If R is the radius of gyration of a body of mass M and radius r, then the ratio of its rotational to translational kinetic energy in the rolling condition is
(a) $\frac{R^2}{R^2 + r^2}$ (b) $\frac{R^2}{r^2}$ (c) $\frac{r^2}{R^2}$ (d) 1

10. A solid sphere rolls down two different inclined planes of the same height but of different inclinations
(a) in both cases the speeds and time of descend will be same
(b) the speeds will be same but time of descend will be different
(c) the speeds will be different but time of descend will be same
(d) speeds and time of descend both will be different

11. For the same total mass, which of the following will have the largest moment of inertia about an axis passing through the centre of mass and perpendicular to the plane of the body
(a) a disc of radius R
(b) a ring of radius R
(c) a square lamina of side $2R$
(d) four rods forming a square of side $2R$

12. A disc and a solid sphere of same mass and radius roll down an inclined plane. The ratio of the friction force acting on the disc and sphere is
(a) $\frac{7}{6}$
(b) $\frac{5}{4}$
(c) $\frac{3}{2}$
(d) depends on angle of inclination

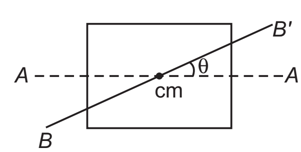
13. A horizontal disc rotates freely with angular velocity ω about a vertical axes through its centre. A ring, having the same mass and radius as the disc, is now gently placed coaxially on the disc. After some time, the two rotate with a common angular velocity. Then
(a) no friction exists between the disc and the ring
(b) the angular momentum of the system is conserved
(c) the final common angular velocity is $\frac{1}{2}\omega$
(d) All of the above

14. A solid homogeneous sphere is moving on a rough horizontal surface, partly rolling and partly sliding. During this kind of motion of the sphere
(a) total kinetic energy of the sphere is conserved
(b) angular momentum of the sphere about any point on the horizontal surface is conserved
(c) only the rotational kinetic energy about the centre of mass is conserved
(d) None of the above

15. A particle of mass $m = 3$ kg moves along a straight line $4y - 3x = 2$ where x and y are in metre, with constant velocity $v = 5\ \text{ms}^{-1}$. The magnitude of angular momentum about the origin is
(a) $12\ \text{kg m}^2\text{s}^{-1}$
(b) $6.0\ \text{kg m}^2\text{s}^{-1}$
(c) $4.5\ \text{kg m}^2\text{s}^{-1}$
(d) $8.0\ \text{kg m}^2\text{s}^{-1}$

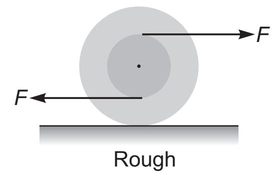
16. A solid sphere rolls without slipping on a rough horizontal floor, moving with a speed v. It makes an elastic collision with a smooth vertical wall. After impact,
(a) it will move with a speed v initially
(b) its motion will be rolling with slipping initially and its rotational motion will stop momentarily at some instant
(c) its motion will be rolling without slipping only after some time
(d) All of the above

17. The figure shows a square plate of uniform mass distribution. AA' and BB' are the two axes lying in the plane of the plate and passing through its centre of mass. If I_0 is the moment of inertia of the plate about AA' then its moment of inertia about the axis BB' is



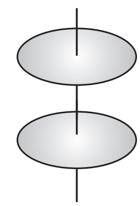
(a) I_0
(b) $I_0 \cos\theta$
(c) $I_0 \cos^2\theta$
(d) None of these

18. A spool is pulled horizontally on rough surface by two equal and opposite forces as shown in the figure. Which of the following statements are correct?



(a) The centre of mass moves towards left
(b) The centre of mass moves towards right
(c) The centre of mass remains stationary
(d) The net torque about the centre of mass of the spool is zero

19. Two identical discs are positioned on a vertical axis as shown in the figure. The bottom disc is rotating at angular velocity ω_0 and has rotational kinetic energy K_0. The top disc is initially at rest. It then falls and sticks to the bottom disc. The change in the rotational kinetic energy of the system is

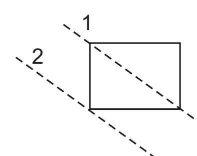


(a) $K_0/2$ (b) $-K_0/2$
(c) $-K_0/4$ (d) $K_0/4$

20. The moment of inertia of hollow sphere (mass M) of inner radius R and outer radius $2R$, having material of uniform density, about a diametric axis is
(a) $31\,MR^2/70$ (b) $43\,MR^2/90$
(c) $19\,MR^2/80$ (d) None of these

21. A rod of uniform cross-section of mass M and length L is hinged about an end to swing freely in a vertical plane. However, its density is non uniform and varies linearly from hinged end to the free end doubling its value. The moment of inertia of the rod, about the rotation axis passing through the hinge point is
(a) $\dfrac{2ML^2}{9}$ (b) $\dfrac{3ML^2}{16}$ (c) $\dfrac{7ML^2}{18}$ (d) None of these

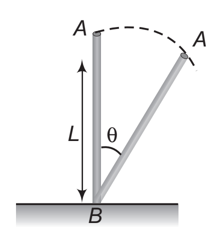
22. Let I_1 and I_2 be the moment of inertia of a uniform square plate about axes shown in the figure. Then, the ratio $I_1 : I_2$ is



(a) $1 : \dfrac{1}{7}$ (b) $1 : \dfrac{12}{7}$
(c) $1 : \dfrac{7}{12}$ (d) $1 : 7$

23. Moment of inertia of a uniform rod of length L and mass M, about an axis passing through $L/4$ from one end and perpendicular to its length is
(a) $\dfrac{7}{36}ML^2$ (b) $\dfrac{7}{48}ML^2$ (c) $\dfrac{11}{48}ML^2$ (d) $\dfrac{ML^2}{12}$

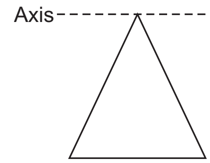
24. A uniform rod of length L is free to rotate in a vertical plane about a fixed horizontal axis through B. The rod begins rotating from rest. The angular velocity ω at angle θ is given as



(a) $\sqrt{\left(\dfrac{6g}{L}\right)}\sin\dfrac{\theta}{2}$ (b) $\sqrt{\left(\dfrac{6g}{L}\right)}\cos\dfrac{\theta}{2}$
(c) $\sqrt{\left(\dfrac{6g}{L}\right)}\sin\theta$ (d) $\sqrt{\left(\dfrac{6g}{L}\right)}\cos\theta$

25. Two particles of masses 1 kg and 2 kg are placed at a distance of 3m. Moment of inertia of the particles about an axis passing through their centre of mass and perpendicular to the line joining them is (in kg-m^2).
(a) 6 (b) 9
(c) 8 (d) 12

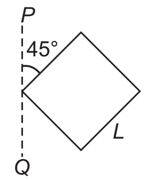
26. Find moment of inertia of a thin sheet of mass M in the shape of an equilateral triangle about an axis as shown in figure. The length of each side is L



(a) $ML^2/8$ (b) $3\sqrt{3}\, ML^2/8$

(c) $7ML^2/8$ (d) None of these

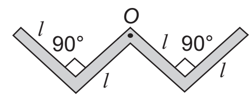
27. A square is made by joining four rods each of mass M and length L. Its moment of inertia about an axis PQ, in its plane and passing through one of its corner is



(a) $6\,ML^2$ (b) $\frac{4}{3}ML^2$

(c) $\frac{8}{3}ML^2$ (d) $\frac{10}{3}ML^2$

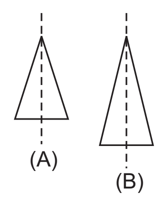
28. A thin rod of length $4l$, mass $4\,m$ is bent at the points as shown in the figure. What is the moment of inertia of the rod about the axis passing through O and perpendicular to the plane of the paper?



(a) $\frac{ml^2}{3}$ (b) $\frac{10\,ml^2}{3}$

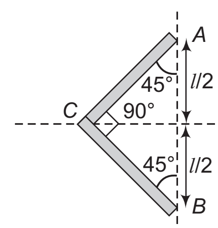
(c) $\frac{ml^2}{12}$ (d) $\frac{ml^2}{24}$

29. The figure shows two cones A and B with the conditions : $h_A < h_B$; $\rho_A > \rho_B$; $R_A = R_B$ $m_A = m_B$. Identify the correct statement about their axis of symmetry.



(a) Both have same moment of inertia
(b) A has greater moment of inertia
(c) B has greater moment of inertia
(d) Nothing can be said

30. Linear mass density of the two rods system, AC and CB is x. Moment of inertia of two rods about an axis passing through AB is



(a) $\frac{xl^3}{4\sqrt{3}}$ (b) $\frac{xl^3}{\sqrt{2}}$

(c) $\frac{xl^3}{4}$ (d) $\frac{xl^3}{6\sqrt{2}}$

Subjective Questions

1. If radius of the earth contracts to half of its present value without change in its mass, what will be the new duration of the day?

2. The radius of gyration of a uniform disc about a line perpendicular to the disc equals its radius R. Find the distance of the line from the centre.

3. Find the moment of inertia of a uniform square plate of mass M and edge a about one of its diagonals.

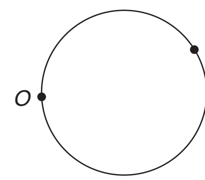
4. Moment of inertia of a uniform rod of mass m and length l is $\frac{7}{12}ml^2$ about a line perpendicular to the rod. Find the distance of this line from the middle point of the rod.

5. Two point masses m_1 and m_2 are joined by a weightless rod of length r. Calculate the moment of inertia of the system about an axis passing through its centre of mass and perpendicular to the rod.

6. Radius of gyration of a body about an axis at a distance 6 cm from its centre of mass is 10 cm. Find its radius of gyration about a parallel axis through its centre of mass.

7. A wheel rotates around a stationary axis so that the rotation angle θ varies with time as $\theta = at^2$, where $a = 0.2\ \text{rad/s}^2$. Find the magnitude of net acceleration of the point A at the rim at the moment $t = 2.5\ s$ if the linear velocity of the point A at this moment is $v = 0.65\ \text{m/s}$.

8. Particle P shown in figure is moving in a circle of radius $R = 10\,\text{cm}$ with linear speed $v = 2\,\text{m/s}$. Find the angular speed of particle about point O.



9. A particle of mass m is projected with velocity v at an angle θ with the horizontal. Find its angular momentum about the point of projection when it is at the highest point of its trajectory.

10. Linear mass density (mass/length) of a rod depends on the distance from one end (say A) as $\lambda_x = (\alpha x + \beta)$. Here, α and β are constants. Find the moment of inertia of this rod about an axis passing through A and perpendicular to the rod. Length of the rod is l.

11. When a body rolls, on a stationary ground, the acceleration of the point of contact is always zero. Is this statement true or false?

12. A solid sphere of mass m rolls down an inclined plane a height h. Find rotational kinetic energy of the sphere.

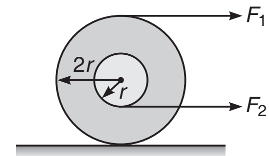
13. The topmost and bottommost velocities of a disc are v_1 and $v_2\ (< v_1)$ in the same direction. The radius is R. Find the value of angular velocity ω.

14. A circular lamina of radius a and centre O has a mass per unit area of kx^2, where x is the distance from O and k is a constant. If the mass of the lamina is M, find in terms of M and a, the moment of inertia of the lamina about an axis through O and perpendicular to the lamina.

15. A solid body starts rotating about a stationary axis with an angular acceleration $\alpha = (2.0 \times 10^{-2})\, t\ \text{rad/s}^2$, here, t is in seconds. How soon after the beginning of rotation will the total acceleration vector of an arbitrary point of the body form an angle $\theta = 60°$ with its velocity vector?

16. A ring of radius R rolls on a horizontal ground with linear speed v and angular speed ω. For what value of θ the velocity of point P is in vertical direction. $(v < R\omega)$

17. Two forces F_1 and F_2 are applied on a spool of mass M and moment of inertia I about an axis passing through its centre of mass. Find the ratio $\frac{F_1}{F_2}$, so that the force of friction is zero. Given that $I < 2Mr^2$.



18. A disc is placed on the ground. Friction coefficient is μ. What is the minimum force required to move the disc if it is applied at the topmost point?

19. A cube is resting on an inclined plane. If the angle of inclination is gradually increased, what must be the coefficient of friction between the cube and plane so that,

(a) cube slides before toppling? (b) cube topples before sliding?

20. A uniform disc of mass 20 kg and radius 0.5 m can turn about a smooth axis through its centre and perpendicular to the disc. A constant torque is applied to the disc for 3 s from rest and the angular velocity at the end of that time is $\frac{240}{\pi}$ rev/min. Find the magnitude of the torque. If the torque is then removed and the disc is brought to rest in t seconds by a constant force of 10 N applied tangentially at a point on the rim of the disc, find t.

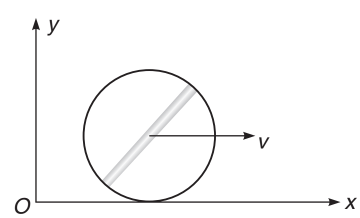
21. A uniform disc of mass m and radius R is rotated about an axis passing through its centre and perpendicular to its plane with an angular velocity ω_0. It is placed on a rough horizontal plane with the axis of the disc keeping vertical. Coefficient of friction between the disc and the surface is μ. Find

(a) the time when disc stops rotating,

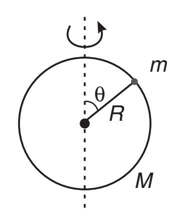
(b) the angle rotated by the disc before stopping.

22. A solid body rotates about a stationary axis according to the law $\theta = at - bt^3$, where $a = 6$ rad/s and $b = 2$ rad/s^3. Find the mean values of the angular velocity and acceleration over the time interval between $t = 0$ and the time, when the body comes to rest.

23. A rod of mass m and length $2R$ is fixed along the diameter of a ring of same mass m and radius R as shown in figure. The combined body is rolling without slipping along x-axis. Find the angular momentum about z-axis.



24. The figure shows a thin ring of mass $M = 1$ kg and radius $R = 0.4$ m spinning about a vertical diameter. (Take $I = \frac{1}{2}MR^2$). A small bead of mass $m = 0.2$ kg can slide without friction along the ring. When the bead is at the top of the ring, the angular velocity is 5 rad/s. What is the angular velocity when the bead slips halfway to $\theta = 45°$?



25. A horizontal disc rotating freely about a vertical axis makes 100 rpm. A small piece of wax of mass 10 g falls vertically on the disc and adheres to it at a distance of 9 cm from the axis. If the number of revolutions per minute is thereby reduced to 90. Calculate the moment of inertia of disc.

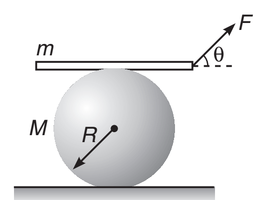
26. A man stands at the centre of a circular platform holding his arms extended horizontally with 4 kg block in each hand. He is set rotating about a vertical axis at 0.5 rev/s. The moment of inertia of the man plus platform is 1.6 kg-m^2, assumed constant. The blocks are 90 cm from the axis of rotation. He now pulls the blocks in toward his body until they are 15 cm from the axis of rotation. Find (a) his new angular velocity and (b) the initial and final kinetic energy of the man and platform. (c) how much work must the man do to pull in the blocks ?

27. A horizontally oriented uniform disc of mass M and radius R rotates freely about a stationary vertical axis passing through its centre. The disc has a radial guide along which can slide without friction a small body of mass m. A light thread running down through the hollow axle of the disc is tied to the body. Initially the body was located at the edge of the disc and the whole system rotated with an angular velocity ω_0. Then, by means of a force F applied to the lower end of the thread the body was slowly pulled to the rotation axis. Find :

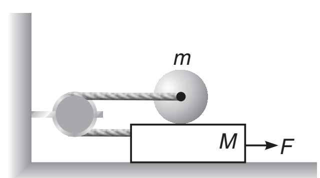
(a) the angular velocity of the system in its final state,

(b) the work performed by the force F.

28. Consider a cylinder of mass M and radius R lying on a rough horizontal plane. It has a plank lying on its top as shown in figure. A force F is applied on the plank such that the plank moves and causes the cylinder to roll. The plank always remains horizontal. There is no slipping at any point of contact. Calculate the acceleration of the cylinder and the frictional forces at the two contacts.



29. Find the acceleration of the cylinder of mass m and radius R and that of plank of mass M placed on smooth surface if pulled with a force F as shown in figure. Given that sufficient friction is present between cylinder and the plank surface to prevent sliding of cylinder.



30. A uniform rod AB of length $2l$ and mass m is rotating in a horizontal plane about a vertical axis through A, with angular velocity ω, when the mid-point of the rod strikes a fixed nail and is brought immediately to rest. Find the impulse exerted by the nail.

31. A uniform rod of length L rests on a frictionless horizontal surface. The rod is pivoted about a fixed frictionless axis at one end. The rod is initially at rest. A bullet travelling parallel to the horizontal surface and perpendicular to the rod with speed v strikes the rod at its centre and becomes embedded in it. The mass of the bullet is one-sixth the mass of the rod.

(a) What is the final angular velocity of the rod ?

(b) What is the ratio of the kinetic energy of the system after the collision to the kinetic energy of the bullet before the collision ?

32. A uniform rod AB of mass $3m$ and length $2l$ is lying at rest on a smooth horizontal table with a smooth vertical axis through the end A. A particle of mass $2m$ moves with speed $2u$ across the table and strikes the rod at its mid-point C. If the impact is perfectly elastic. Find the speed of the particle after impact if

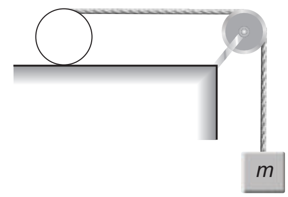
(a) it strikes the rod normally, (b) its path before impact was inclined at 60° to AC.

LEVEL 2

Objective Questions

Single Correct Option

1. In the given figure a ring of mass m is kept on a horizontal surface while a body of equal mass m is attached through a string, which is wounded on the ring. When the system is released, the ring rolls without slipping. Consider the following statement and choose the correct option.



(i) acceleration of the centre of mass of ring is $\frac{2g}{3}$

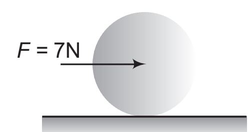
(ii) acceleration of hanging particle is $\frac{4g}{3}$

(iii) frictional force (on the ring) acts in forward direction

(iv) frictional force (on the ring) acts in backward direction

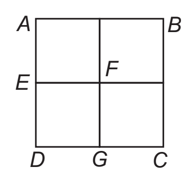
(a) only statements (i) and (ii) are correct
(b) only statements (ii) and (iii) are correct
(c) only statements (iii) and (iv) are correct
(d) None of these

2. A solid sphere of mass 10 kg is placed on a rough surface having coefficient of friction $\mu = 0.1$. A constant force $F = 7$ N is applied along a line passing through the centre of the sphere as shown in the figure. The value of frictional force on the sphere is



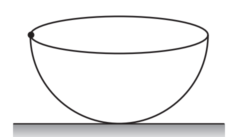
(a) 1 N
(b) 2 N
(c) 3 N
(d) 7 N

3. From a uniform square plate of side a and mass m, a square portion $DEFG$ of side $\frac{a}{2}$ is removed. Then, the moment of inertia of remaining portion about the axis AB is



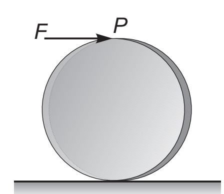
(a) $\frac{7ma^2}{16}$
(b) $\frac{3ma^2}{16}$
(c) $\frac{3ma^2}{4}$
(d) $\frac{9ma^2}{16}$

4. A small solid sphere of mass m and radius r starting from rest from the rim of a fixed hemispherical bowl of radius $R(>>r)$ rolls inside it without sliding. The normal reaction exerted by the sphere on the hemisphere when it reaches the bottom of hemisphere is



(a) $(3/7)\,mg$
(b) $(9/7)\,mg$
(c) $(13/7)\,mg$
(d) $(17/7)\,mg$

5. A uniform solid cylinder of mass m and radius R is placed on a rough horizontal surface. A horizontal constant force F is applied at the top point P of the cylinder so that it starts pure rolling. The acceleration of the cylinder is



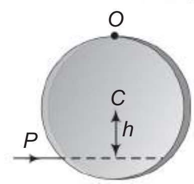
(a) $F/3m$
(b) $2F/3m$
(c) $4F/3m$
(d) $5F/3m$

6. In the above question, the frictional force on the cylinder is
(a) $F/3$ towards right
(b) $F/3$ towards left
(c) $2F/3$ towards right
(d) $2F/3$ towards left

7. A small pulley of radius 20 cm and moment of inertia $0.32\ \text{kg-m}^2$ is used to hang a 2 kg mass with the help of massless string. If the block is released, for no slipping condition acceleration of the block will be

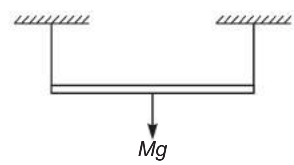
(a) $2\ \text{m/s}^2$
(b) $4\ \text{m/s}^2$
(c) $1\ \text{m/s}^2$
(d) $3\ \text{m/s}^2$

8. A uniform circular disc of radius R is placed on a smooth horizontal surface with its plane horizontal and hinged at circumference through point O as shown. An impulse P is applied at a perpendicular distance h from its centre C. The value of h so that the impulse due to hinge is zero, is



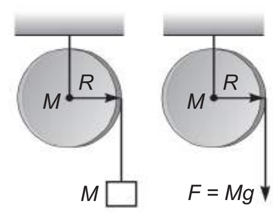
(a) R
(b) $R/2$
(c) $R/3$
(d) $R/4$

9. A rod is supported horizontally by means of two strings of equal length as shown in figure. If one of the string is cut. Then tension in other string at the same instant will



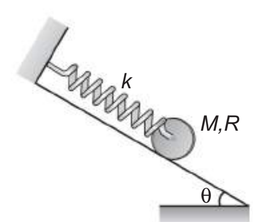
(a) remain unaffected
(b) increase
(c) decrease
(d) become equal to weight of the rod

10. The figure represents two cases. In first case a block of mass M is attached to a string which is tightly wound on a disc of mass M and radius R. In second case $F = Mg$. Initially, the disc is stationary in each case. If the same length of string is unwound from the disc, then



(a) same amount of work is done on both discs
(b) angular velocities of both the discs are equal
(c) both the discs have unequal angular accelerations
(d) All of the above

11. A uniform cylinder of mass M and radius R is released from rest on a rough inclined surface of inclination θ with the horizontal as shown in figure. As the cylinder rolls down the inclined surface, the maximum elongation in the spring of stiffness k is



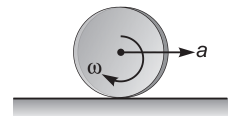
(a) $\dfrac{3}{4}\dfrac{Mg\sin\theta}{k}$
(b) $\dfrac{2\,Mg\sin\theta}{k}$
(c) $\dfrac{Mg\sin\theta}{k}$
(d) None of these

12. A uniform rod of mass m and length l rotates in a horizontal plane with an angular velocity ω about a vertical axis passing through one end. The tension in the rod at a distance x from the axis is
(a) $\dfrac{1}{2}m\omega^2 x$
(b) $\dfrac{1}{2}m\omega^2\left(1-\dfrac{x^2}{l}\right)$
(c) $\dfrac{1}{2}m\omega^2 l\left(1-\dfrac{x^2}{l^2}\right)$
(d) $\dfrac{1}{2}m\omega^2 l\left[1-\dfrac{x}{l}\right]$

13. A rod of length 1 m rotates in the xy plane about the fixed point O in the anticlockwise sense, as shown in figure with velocity $\omega = a + bt$ where $a = 10 \text{ rad s}^{-1}$ and $b = 5 \text{ rad s}^{-2}$. The velocity and acceleration of the point A at $t = 0$ is

(a) $+10\hat{\mathbf{i}}$ ms^{-1} and $+5\hat{\mathbf{i}}$ ms^{-2}
(b) $+10\hat{\mathbf{j}}$ ms^{-1} and $(-100\hat{\mathbf{i}} + 5\hat{\mathbf{j}})$ ms^{-2}
(c) $-10\hat{\mathbf{j}}$ ms^{-1} and $(100\hat{\mathbf{i}} + 5\hat{\mathbf{j}})$ ms^{-2}
(d) $-10\hat{\mathbf{j}}$ ms^{-1} and $-5\hat{\mathbf{j}}$ ms^{-2}

14. A ring of radius R rolls on a horizontal surface with constant acceleration a of the centre of mass as shown in figure. If ω is the instantaneous angular velocity of the ring, then the net acceleration of the point of contact of the ring with ground is

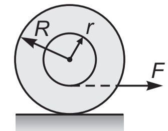


(a) zero
(b) $\omega^2 R$
(c) a
(d) $\sqrt{a^2 + (\omega^2 R)^2}$

15. The density of a rod AB increases linearly from A to B. Its midpoint is O and its centre of mass is at C. Four axes pass through A, B, O and C, all perpendicular to the length of the rod. The moments of inertia of the rod about these axes are I_A, I_B, I_O and I_C respectively. Then

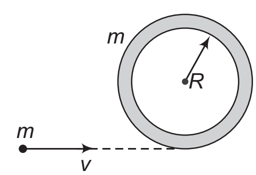
(a) $I_A > I_B$
(b) $I_C < I_B$
(c) $I_O > I_C$
(d) All of these

16. The figure shows a spool placed at rest on a horizontal rough surface. A tightly wound string on the inner cylinder is pulled horizontally with a force F. Identify the correct alternative related to the friction force f acting on the spool



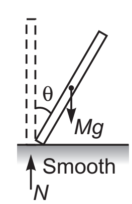
(a) f acts leftwards with $f < F$
(b) f acts leftwards but nothing can be said about its magnitude
(c) $f < F$ but nothing can be said about its magnitude
(d) None of the above

17. A circular ring of mass m and radius R rests flat on a horizontal smooth surface as shown in figure. A particle of mass m, and moving with a velocity v, collides inelastically ($e = 0$) with the ring. The angular velocity with which the system rotates after the particle strikes the ring is



(a) $\frac{v}{2R}$
(b) $\frac{v}{3R}$
(c) $\frac{2v}{3R}$
(d) $\frac{3v}{4R}$

18. A stationary uniform rod in the upright position is allowed to fall on a smooth horizontal surface. The figure shows the instantaneous position of the rod. Identify the correct statement.

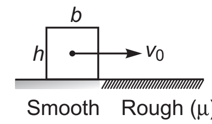


(a) normal reaction N is equal to Mg
(b) N does positive rotational work about the centre of mass
(c) a couple of equal and opposite forces acts on the rod
(d) All of the above

19. A thin uniform rod of mass m and length l is free to rotate about its upper end. When it is at rest. It receives an impulse J at its lowest point, normal to its length. Immediately after impact

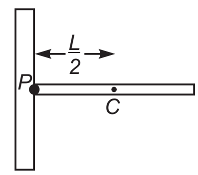
(a) the angular momentum of the rod is Jl
(b) the angular velocity of the rod is $3J/ml$
(c) the kinetic energy of the rod is $3J^2/2m$
(d) All of the above

20. A rectangular block of size $(b \times h)$ moving with velocity v_o enters on a rough surface where the coefficient of friction is μ as shown in figure. Identify the correct statement.



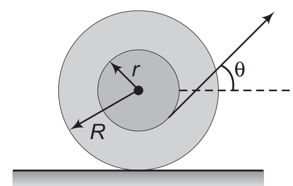
(a) The net torque acting on the block about its COM is $\mu\, mg \dfrac{h}{2}$ (clockwise)
(b) The net torque acting on the block about its COM is zero
(c) The net torque acting on the block about its COM is in the anticlockwise sense
(d) None of the above

21. A uniform rod of length L and mass m is free to rotate about a frictionless pivot at one end as shown in figure. The rod is held at rest in the horizontal position and a coin of mass m is placed at the free end. Now the rod is released. The reaction on the coin immediately after the rod starts falling is



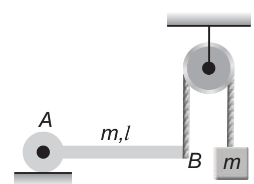
(a) $\dfrac{3\,mg}{2}$ (b) $2\,mg$
(c) zero (d) $\dfrac{mg}{2}$

22. A spool is pulled at an angle θ with the horizontal on a rough horizontal surface as shown in the figure. If the spool remains at rest, the angle θ is equal to



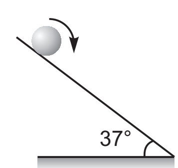
(a) $\cos^{-1}\left(\dfrac{R}{r}\right)$ (b) $\sin^{-1}\left(\sqrt{1-\dfrac{r^2}{R^2}}\right)$
(c) $\pi - \cos^{-1}\left(\dfrac{r}{R}\right)$ (d) $\sin^{-1}\left(\dfrac{r}{R}\right)$

23. Uniform rod AB is hinged at end A in horizontal position as shown in the figure. The other end is connected to a block through a massless string as shown. The pulley is smooth and massless. Mass of block and rod is same and is equal to m. Then acceleration of block just after release from this position is



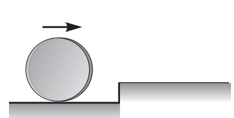
(a) $6\,g/13$ (b) $g/4$
(c) $3g/8$ (d) None of these

24. A cylinder having radius 0.4 m, initially rotating (at $t = 0$) with $\omega_0 = 54$ rad/s is placed on a rough inclined plane with $\theta = 37°$ having friction coefficient $\mu = 0.5$. The time taken by the cylinder to start pure rolling is ($g = 10$ m/s^2)



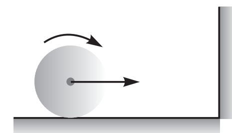
(a) 5.4 s (b) 2.4 s
(c) 1.4 s (d) None of these

25. A disc of mass M and radius R is rolling purely with center's velocity v_0 on a flat horizontal floor when it hits a step in the floor of height $R/4$. The corner of the step is sufficiently rough to prevent any slipping of the disc against itself. What is the velocity of the centre of the disc just after impact?



(a) $4v_0/5$ (b) $4v_0/7$
(c) $5v_0/6$ (d) None of these

26. A solid sphere is rolling purely on a rough horizontal surface (coefficient of kinetic friction = μ) with speed of centre = u. It collides inelastically with a smooth vertical wall at a certain moment, the coefficient of restitution being $\frac{1}{2}$. The sphere will begin pure rolling after a time



(a) $\frac{3u}{7\mu g}$ (b) $\frac{2u}{7\mu g}$ (c) $\frac{3u}{5\mu g}$ (d) $\frac{2u}{5\mu g}$

27. A thin hollow sphere of mass m is completely filled with non viscous liquid of mass m. When the sphere roll-on horizontal ground such that centre moves with velocity v, kinetic energy of the system is equal to

(a) mv^2 (b) $\frac{4}{3}mv^2$ (c) $\frac{4}{5}mv^2$ (d) None of these

28. A solid uniform disc of mass m rolls without slipping down a fixed inclined plank with an acceleration a. The frictional force on the disc due to surface of the plane is

(a) $\frac{1}{4}ma$ (b) $\frac{3}{2}ma$ (c) ma (d) $\frac{1}{2}ma$

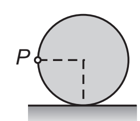
29. A uniform slender rod of mass m and length L is released from rest, with its lower end touching a frictionless horizontal floor. At the initial moment, the rod is inclined at an angle $\theta = 30°$ with the vertical. Then the value of normal reaction from the floor just after release will be

(a) $4mg/7$ (b) $5mg/9$
(c) $2mg/5$ (d) None of these

30. In the above problem, the initial acceleration of the lower end of the rod will be

(a) $g\sqrt{3}/4$ (b) $g\sqrt{3}/5$
(c) $3g\sqrt{3}/7$ (d) None of these

31. A disc of radius R is rolling purely on a flat horizontal surface, with a constant angular velocity. The angle between the velocity and acceleration vectors of point P is

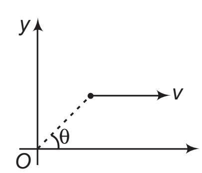


(a) zero (b) 45°
(c) $\tan^{-1}(2)$ (d) $\tan^{-1}(1/2)$

32. A straight rod AB of mass M and length L is placed on a frictionless horizontal surface. A force having constant magnitude F and a fixed direction starts acting at the end A. The rod is initially perpendicular to the force. The initial acceleration of end B is

(a) zero (b) $2F/M$
(c) $4F/M$ (d) None of these

33. A particle moves parallel to x-axis with constant velocity v as shown in the figure. The angular velocity of the particle about the origin O

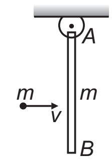


(a) remains constant
(b) continuously increases
(c) continuously decreases
(d) oscillates

34. A thin uniform rod of mass M and length L is hinged at its upper end, and released from rest from a horizontal position. The tension at a point located at a distance $L/3$ from the hinge point, when the rod becomes vertical, will be

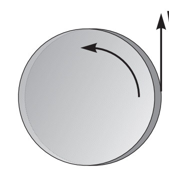
(a) $22\,Mg/27$ (b) $11\,Mg/13$
(c) $6\,Mg/11$ (d) $2\,Mg$

35. A uniform rod AB of length L and mass m is suspended freely at A and hangs vertically at rest when a particle of same mass m is fired horizontally with speed v to strike the rod at its mid point. If the particle is brought to rest after the impact. Then the impulsive reaction at A in horizontal direction is



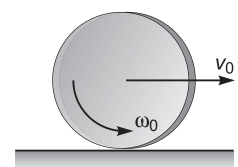
(a) $mv/4$ (b) $mv/2$
(c) mv (d) $2\,mv$

36. A child with mass m is standing at the edge of a merry go round having moment of inertia I, radius R and initial angular velocity ω as shown in the figure. The child jumps off the edge of the merry go round with tangential velocity v with respect to the ground. The new angular velocity of the merry go round is



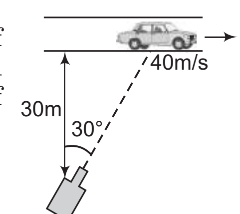
(a) $\sqrt{\dfrac{I\omega^2 - mv^2}{I}}$ (b) $\sqrt{\dfrac{(I + mR^2)\,\omega^2 - mv^2}{I}}$
(c) $\dfrac{I\omega - mvR}{I}$ (d) $\dfrac{(I + mR^2)\,\omega - mvR}{I}$

37. A disc of radius R is spun to an angular speed ω_0 about its axis and then imparted a horizontal velocity of magnitude $\dfrac{\omega_0 R}{4}$. The coefficient of friction is μ. The sense of rotation and direction of linear velocity are shown in the figure. The disc will return to its initial position



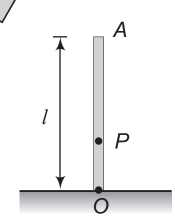
(a) if the value of $\mu < 0.5$
(b) irrespective of the value of μ
(c) if the value of $0.5 < \mu < 1$
(d) if $\mu > 1$

38. A racing car is travelling along a straight track at a constant velocity of 40 m/s. A fixed TV camera is recording the event as shown in figure. In order to keep the car in view, in the position shown, the angular velocity of camera should be



(a) 3 rad/s (b) 2 rad/s
(c) 4 rad/s (d) 1 rad/s

39. A uniform rod OA of length l, resting on smooth surface is slightly distributed from its vertical position. P is a point on the rod whose locus is a circle during the subsequent motion of the rod. Then the distance OP is equal to

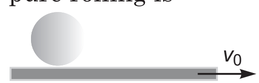


(a) $l/2$
(b) $l/3$
(c) $l/4$
(d) there is no such point

40. In the above question, the velocity of end O when end A hits the ground is
(a) zero
(b) along the horizontal
(c) along the vertical
(d) at some inclination to the ground ($\neq 90°$)

41. In the above question, the velocity of end A at the instant it hits the ground is
(a) $\sqrt{3gl}$ (b) $\sqrt{12gl}$
(c) $\sqrt{6gl}$ (d) None of these

42. A solid sphere of mass m and radius R is gently placed on a conveyer belt moving with constant velocity v_0. If coefficient of friction between belt and sphere is 2/7, the distance traveled by the centre of the sphere before it starts pure rolling is



(a) $\frac{v_0^2}{7g}$

(b) $\frac{2v_0^2}{49g}$

(c) $\frac{2v_0^2}{5g}$

(d) $\frac{2v_0^2}{7g}$

More than One Correct Options

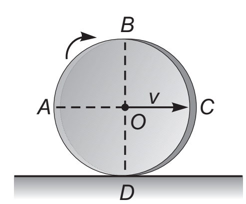
1. A mass m of radius r is rolling horizontally without any slip with a linear speed v. It then rolls up to a height given by $\frac{3}{4}\frac{v^2}{g}$

(a) the body is identified to be a disc or a solid cylinder
(b) the body is a solid sphere
(c) moment of inertia of the body about instantaneous axis of rotation is $\frac{3}{2}mr^2$
(d) moment of inertia of the body about instantaneous axis of rotation is $\frac{7}{5}mr^2$

2. Four identical rods each of mass m and length l are joined to form a rigid square frame. The frame lies in the xy plane, with its centre at the origin and the sides parallel to the x and y axes. Its moment of inertia about

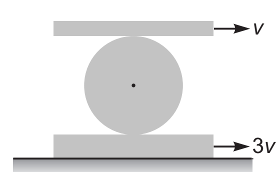
(a) the x-axis is $\frac{2}{3}ml^2$
(b) the z-axis is $\frac{4}{3}ml^2$
(c) an axis parallel to the z-axis and passing through a corner is $\frac{10}{3}ml^2$
(d) one side is $\frac{5}{3}ml^2$

3. A uniform circular ring rolls without slipping on a horizontal surface. At any instant, its position is as shown in the figure. Then



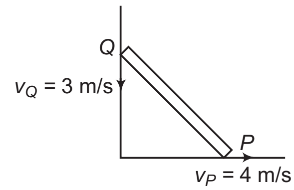
(a) section ABC has greater kinetic energy than section ADC
(b) section BC has greater kinetic energy than section CD
(c) section BC has the same kinetic energy as section DA
(d) the sections CD and DA have the same kinetic energy

4. A cylinder of radius R is to roll without slipping between two planks as shown in the figure. Then



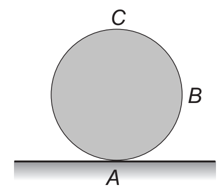
(a) angular velocity of the cylinder is $\frac{v}{R}$ counter clockwise
(b) angular velocity of the cylinder is $\frac{2v}{R}$ clockwise
(c) velocity of centre of mass of the cylinder is v towards left
(d) velocity of centre of mass of the cylinder is $2v$ towards right

5. A uniform rod of mass $m = 2$ kg and length $l = 0.5$ m is sliding along two mutually perpendicular smooth walls with the two ends P and Q having velocities $v_P = 4$ m/s and $v_Q = 3$ m/s as shown. Then



(a) The angular velocity of rod, $\omega = 10$ rad/s, counter clockwise
(b) The angular velocity of rod, $\omega = 5.0$ rad/s, counter clockwise
(c) The velocity of centre of mass of rod, $v_{cm} = 2.5$ m/s
(d) The total kinetic energy of rod, $K = \dfrac{25}{3}$ joule

6. A wheel is rolling without slipping on a horizontal plane with velocity v and acceleration a of centre of mass as shown in figure. Acceleration at



(a) A is vertically upwards
(b) B may be vertically downwards
(c) C cannot be horizontal
(d) a point on the rim may be horizontal leftwards

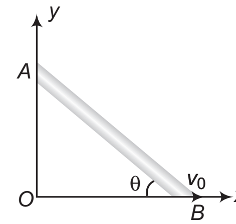
7. A uniform rod of length l and mass 2 m rests on a smooth horizontal table. A point mass m moving horizontally at right angles to the rod with velocity v collides with one end of the rod and sticks it. Then
(a) angular velocity of the system after collision is $\dfrac{2}{5}\dfrac{v}{l}$
(b) angular velocity of the system after collision is $\dfrac{v}{2l}$
(c) the loss in kinetic energy of the system as a whole as a result of the collision is $\dfrac{3}{10}mv^2$
(d) the loss in kinetic energy of the system as a whole as a result of the collision is $\dfrac{7mv^2}{24}$

8. A non-uniform ball of radius R and radius of gyration about geometric centre $= R/2$, is kept on a frictionless surface. The geometric centre coincides with the centre of mass. The ball is struck horizontally with a sharp impulse $= J$. The point of application of the impulse is at a height h above the surface. Then
(a) the ball with slip on surface for all cases
(b) the ball will roll purely if $h = 5R/4$
(c) the ball will roll purely if $h = 3R/2$
(d) there will be no rotation if $h = R$

9. A hollow spherical ball is given an initial push, up an incline of inclination angle α. The ball rolls purely. Coefficient of static friction between ball and incline $= \mu$. During its upwards journey
(a) friction acts up along the incline
(b) $\mu_{min} = (2 \tan \alpha)/5$
(c) friction acts down along the incline
(d) $\mu_{min} = (2 \tan \alpha)/7$

10. A uniform disc of mass m and radius R rotates about a fixed vertical axis passing through its centre with angular velocity ω. A particle of same mass m and having velocity of $2\omega R$ towards centre of the disc collides with the disc moving horizontally and sticks to its rim. Then,
(a) the angular velocity of the disc will become $\omega/3$
(b) the angular velocity of the disc will become $5\omega/3$
(c) the impulse on the particle due to disc is $\dfrac{\sqrt{37}}{3} m\omega R$
(d) the impulse on the particle due to disc is $2m\omega R$

11. The end B of the rod AB which makes angle θ with the floor is being pulled with a constant velocity v_0 as shown. The length of the rod is l.



(a) At $\theta = 37°$ velocity of end A is $\frac{4}{3} v_0$ downwards

(b) At $\theta = 37°$ angular velocity of rod is $\frac{5v_0}{3l}$

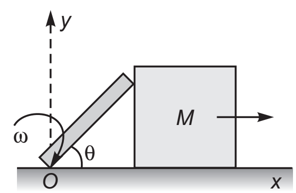
(c) Angular velocity of rod is constant

(d) Velocity of end A is constant

Comprehension Based Questions

Passage 1 (Q. Nos. 1 to 4)

A uniform rod of mass m and length l is pivoted at point O. The rod is initially in vertical position and touching a block of mass M which is at rest on a horizontal surface. The rod is given a slight jerk and it starts rotating about point O. This causes the block to move forward as shown. The rod loses contact with the block at $\theta = 30°$. All surfaces are smooth. Now answer the following questions.



1. The value of ratio M/m is

(a) 2 : 3 (b) 3 : 2 (c) 4 : 3 (d) 3 : 4

2. The velocity of block when the rod loses contact with the block is

(a) $\frac{\sqrt{3gl}}{4}$ (b) $\frac{\sqrt{5gl}}{4}$ (c) $\frac{\sqrt{6gl}}{4}$ (d) $\frac{\sqrt{7gl}}{4}$

3. The acceleration of centre of mass of rod, when it loses contact with the block is

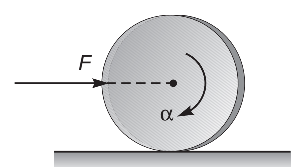
(a) $5g/4$ (b) $5g/2$ (c) $3g/2$ (d) $3g/4$

4. The hinge reaction at O on the rod when it loses contact with the block is

(a) $\frac{3\,mg}{4}(\hat{\mathbf{i}} + \hat{\mathbf{j}})$ (b) $\left(\frac{mg}{4}\right)\hat{\mathbf{j}}$ (c) $\left(\frac{mg}{4}\right)\hat{\mathbf{i}}$ (d) $\frac{mg}{4}(\hat{\mathbf{i}} + \hat{\mathbf{j}})$

Passage 2 (Q. Nos. 5 to 7)

Consider a uniform disc of mass m, radius r, rolling without slipping on a rough surface with linear acceleration 'a' and angular acceleration α due to an external force F as shown in the figure. Coefficient of friction is μ



5. The work done by the frictional force at the instant of pure rolling is

(a) $\frac{\mu mgat^2}{2}$ (b) $\mu mgat^2$ (c) $\mu mg \frac{at^2}{\alpha}$ (d) zero

6. The magnitude of frictional force acting on the disc is

(a) ma (b) μmg (c) $\frac{ma}{2}$ (d) zero

7. Angular momentum of the disc will be conserved about

(a) centre of mass

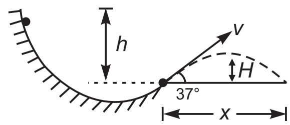
(b) point of contact

(c) a point at a distance $3R/2$ vertically above the point of contact

(d) a point at a distance $4R/3$ vertically above the point of contact

Passage 3 (Q. No. 8 to 10)

A tennis ball, starting from rest, rolls down the hill in the drawing. At the end of the hill the ball becomes airborne, leaving at an angle of 37° with respect to the ground. Treat the ball as a thin-walled spherical shell.



8. The velocity of projection v is

(a) $\sqrt{2gh}$ (b) $\sqrt{\frac{10}{7}gh}$ (c) $\sqrt{\frac{5}{7}gh}$ (d) $\sqrt{\frac{6}{5}gh}$

9. Maximum height reached by ball H above ground is

(a) $\frac{9h}{35}$ (b) $\frac{18h}{35}$ (c) $\frac{18h}{25}$ (d) $\frac{27h}{125}$

10. Range x of the ball is

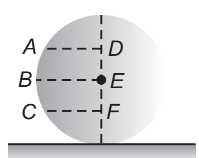
(a) $\frac{144}{125}h$ (b) $\frac{48}{25}h$ (c) $\frac{48}{35}h$ (d) $\frac{24}{7}h$

Match the Columns

1. A solid sphere, a hollow sphere and a disc of same mass and same radius are released from a rough inclined plane. All of them rolls down without slipping. On reaching the bottom of the plane, match the two columns.

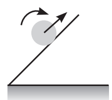
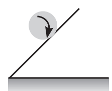
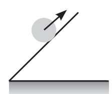
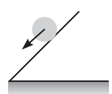
Column I	Column II
(a) time taken to reach the bottom	(p) maximum for solid sphere
(b) total kinetic energy	(q) maximum for hollow sphere
(c) rotational kinetic energy	(r) maximum for disc
(d) translational kinetic energy	(s) same for all

2. A solid sphere is placed on a rough ground as shown. E is the centre of sphere and $DE > EF$. We have to apply a linear impulse either at point A, B or C. Match the following two columns.

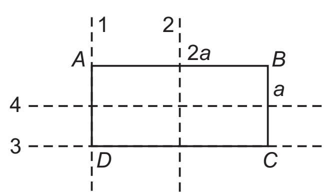


Column I	Column II
(a) Sphere will acquire maximum angular speed it impulse is applied at	(p) A
(b) Sphere will acquire maximum linear speed it impulse is applied at	(q) B
(c) Sphere can roll without slipping if impulse is applied at	(r) C
(d) Sphere can roll with forward slipping if impulse is applied at	(s) at any point A, B or C

3. The inclined surfaces shown in Column I are sufficiently rough. In Column II direction and magnitudes of frictional forces are mentioned. Match the two columns.

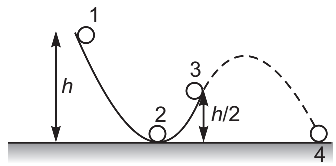
Column I	Column II
(a)  Rolling upwards	(p) upwards
(b)  Kept in rotating position	(q) downwards
(c)  Kept in translational position	(r) maximum friction will act
(d)  Kept in translational position	(s) required value of friction will act

4. A rectangular slab $ABCD$ have dimensions $a \times 2a$ as shown in figure. Match the following two columns.



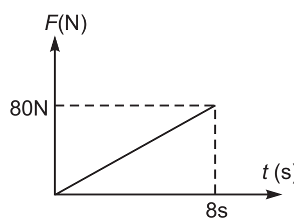
Column I	Column II
(a) Radius of gyration about axis-1	(p) $\dfrac{a}{\sqrt{12}}$
(b) Radius of gyration about axis-2	(q) $\dfrac{2a}{\sqrt{3}}$
(c) Radius of gyration about axis-3	(r) $\dfrac{a}{\sqrt{3}}$
(d) Radius of gyration about axis-4	(s) None

5. A small solid ball rolls down along sufficiently rough surface from 1 to 3 as shown in figure. From point-3 onwards it moves under gravity. Match the following two columns.



Column I	Column II
(a) Rotational kinetic energy of ball at point-2	(p) $\frac{1}{7}mgh$
(b) Translational kinetic energy of ball at point-3	(q) $\frac{2}{7}mgh$
(c) Rotational kinetic energy of ball at point-4	(r) $\frac{5}{7}mgh$
(d) Translational kinetic energy of ball at point-4	(s) None

6. A uniform disc of mass 10 kg, radius 1 m is placed on a rough horizontal surface. The coefficient of friction between the disc and the surface is 0.2. A horizontal time varying force is applied on the centre of the disc whose variation with time is shown in graph.



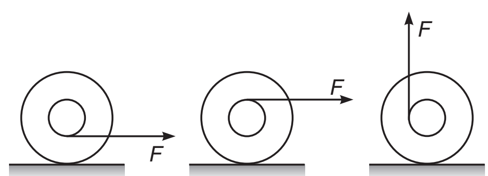
Column I	Column II
(a) Disc rolls without slipping	(p) at $t = 7$ s
(b) Disc rolls with slipping	(q) at $t = 3$ s
(c) Disc starts slipping at	(r) at $t = 4$ s
(d) Friction force is 10 N at	(s) None

7. Match the columns.

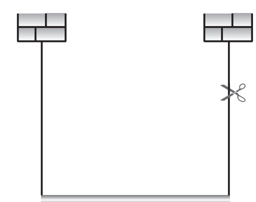
Column I	Column II
(a) Moment of inertia of a circular disc of mass M and radius R about a tangent parallel to plane of disc	(p) $\frac{MR^2}{2}$
(b) Moment of inertia of a solid sphere of mass M and radius R about a tangent	(q) $\frac{7}{5}MR^2$
(c) Moment of inertia of a circular disc of mass M and radius R about a tangent perpendicular to plane of disc	(r) $\frac{5}{4}MR^2$
(d) Moment of inertia of a cylinder of mass M and radius R about its axis	(s) $\frac{3}{2}MR^2$

Subjective Questions

1. Figure shows three identical yo-yos initially at rest on a horizontal surface. For each yo-yo the string is pulled in the direction shown. In each case there is sufficient friction for the yo-yo to roll without slipping. Draw the free-body diagram for each yo-yo. In what direction will each yo-yo rotate ?



2. A uniform rod of mass m and length l is held horizontally by two vertical strings of negligible mass, as shown in the figure.



(a) Immediately after the right string is cut, what is the linear acceleration of the free end of the rod ?

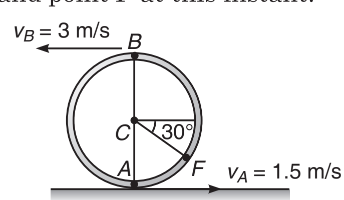
(b) Of the middle of the rod ?

(c) Determine the tension in the left string immediately after the right string is cut.

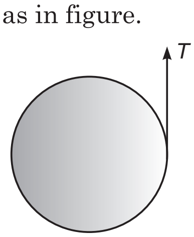
3. A solid disk is rolling without slipping on a level surface at a constant speed of 2.00 m/s. How far can it roll up a 30° ramp before it stops? (Take $g = 9.8\ \text{m/s}^2$)

4. A lawn roller in the form of a thin-walled hollow cylinder of mass M is pulled horizontally with a constant horizontal force F applied by a handle attached to the axle. If it rolls without slipping, find the acceleration and the friction force.

5. Due to slipping, points A and B on the rim of the disk have the velocities shown. Determine the velocities of the centre point C and point F at this instant.



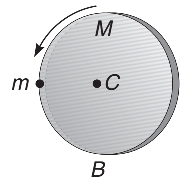
6. A uniform cylinder of mass M and radius R has a string wrapped around it. The string is held fixed and the cylinder falls vertically, as in figure.



(a) Show that the acceleration of the cylinder is downward with magnitude $a = \dfrac{2g}{3}$.

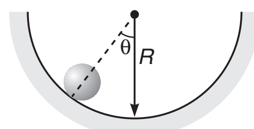
(b) Find the tension in the string.

7. A uniform disc of mass M and radius R is pivoted about the horizontal axis through its centre C. A point mass m is glued to the disc at its rim, as shown in figure. If the system is released from rest, find the angular velocity of the disc when m reaches the bottom point B.

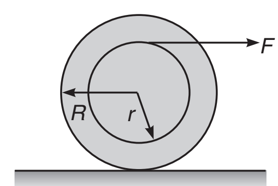


8. A disc of radius R and mass m is projected on to a horizontal floor with a backward spin such that its centre of mass speed is v_0 and angular velocity is ω_0. What must be the minimum value of ω_0 so that the disc eventually returns back ?

9. A ball of mass m and radius r rolls along a circular path of radius R. Its speed at the bottom $(\theta = 0°)$ of the path is v_0. Find the force of the path on the ball as a function of θ.



10. A heavy homogeneous cylinder has mass m and radius R. It is accelerated by a force F, which is applied through a rope wound around a light drum of radius r attached to the cylinder (figure). The coefficient of static friction is sufficient for the cylinder to roll without slipping.



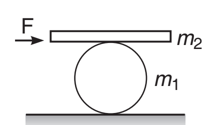
(a) Find the friction force.

(b) Find the acceleration a of the centre of the cylinder.

(c) Is it possible to choose r, so that a is greater than $\dfrac{F}{m}$? How ?

(d) What is the direction of the friction force in the circumstances of part (c) ?

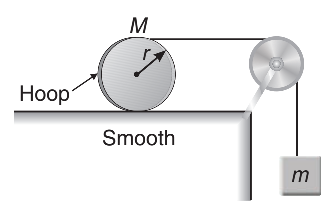
11. A man pushes a cylinder of mass m_1 with the help of a plank of mass m_2 as shown. There is no slipping at any contact. The horizontal component of the force applied by the man is F. Find :



(a) the acceleration of the plank and the centre of mass of the cylinder and

(b) the magnitudes and directions of frictional forces at contact points.

12. For the system shown in figure, $M = 1$ kg, $m = 0.2$ kg, $r = 0.2$ m. Calculate: $(g = 10 \text{ m/s}^2)$



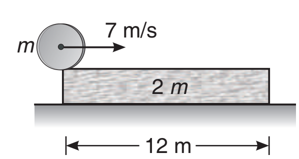
(a) the linear acceleration of hoop,

(b) the angular acceleration of the hoop of mass M and

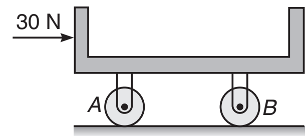
(c) the tension in the rope.

Note *Treat hoop as the ring. Assume no slipping between string and hoop.*

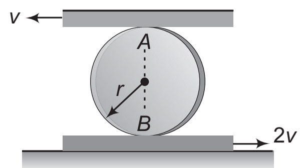
13. A cylinder of mass m is kept on the edge of a plank of mass $2m$ and length 12 m, which in turn is kept on smooth ground. Coefficient of friction between the plank and the cylinder is 0.1. The cylinder is given an impulse, which imparts it a velocity 7 m/s but no angular velocity. Find the time after which the cylinder falls off the plank. $(g = 10 \text{ m/s}^2)$



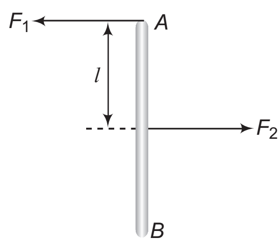
14. The 9 kg cradle is supported as shown by two uniform disks that roll without sliding at all surfaces of contact. The mass of each disk is $m = 6$ kg and the radius of each disk is $r = 80$ mm. Knowing that the system is initially at rest, determine the velocity of the cradle after it has moved 250 mm.



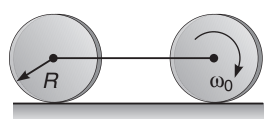
15. The disc of the radius r is confined to roll without slipping at A and B. If the plates have the velocities shown, determine the angular velocity of the disc.



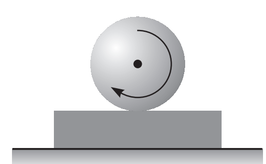
16. A thin uniform rod AB of mass $m = 1$ kg moves transnationally with acceleration $a = 2\ \text{m/s}^2$ due to two antiparallel forces F_1 and F_2. The distance between the points at which these forces are applied is equal to $l = 20$ cm. Besides, it is known that $F_2 = 5$ N. Find the length of the rod.



17. The assembly of two discs as shown in figure is placed on a rough horizontal surface and the front disc is given an initial angular velocity ω_0. Determine the final linear and angular velocity when both the discs start rolling. It is given that friction is sufficient to sustain rolling in the rear wheel from the starting of motion.



18. A horizontal plank having mass m lies on a smooth horizontal surface. A sphere of same mass and radius r is spined to an angular frequency ω_0 and gently placed on the plank as shown in the figure. If coefficient of friction between the plank and the sphere is μ. Find the distance moved by the plank till the sphere starts pure rolling on the plank. The plank is long enough.



19. A ball rolls without sliding over a rough horizontal floor with velocity $v_0 = 7$ m/s towards a smooth vertical wall. If coefficient of restitution between the wall and the ball is $e = 0.7$. Calculate velocity v of the ball long after the collision.

20. A uniform rod of mass m and length l rests on a smooth horizontal surface. One of the ends of the rod is struck in a horizontal direction at right angles to the rod. As a result the rod obtains velocity v_0. Find the force with which one-half of the rod will act on the other in the process of motion.

21. A sphere, a disk and a hoop made of homogeneous materials have the same radius (10 cm) and mass (3 kg). They are released from rest at the top of a 30° incline and roll down without slipping through a vertical distance of 2 m. ($g = 9.8 \text{ m/s}^2$)

(a) What are their speeds at the bottom ?
(b) Find the frictional force f in each case
(c) If they start together at $t = 0$, at what time does each reach the bottom ?

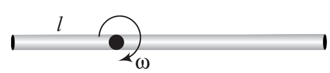
22. ABC is a triangular framework of three uniform rods each of mass m and length $2l$. It is free to rotate in its own plane about a smooth horizontal axis through A which is perpendicular to ABC. If it is released from rest when AB is horizontal and C is above AB. Find the maximum velocity of C in the subsequent motion.

23. A uniform stick of length L and mass M hinged at one end is released from rest at an angle θ_0 with the vertical. Show that when the angle with the vertical is θ, the hinge exerts a force F_r along the stick and F_t perpendicular to the stick given by $F_r = \frac{1}{2} Mg(5\cos\theta - 3\cos\theta_0)$ and $F_t = \frac{1}{4} Mg\sin\theta$

24. A uniform rod AB of mass $3m$ and length $4l$, which is free to turn in a vertical plane about a smooth horizontal axis through A, is released from rest when horizontal. When the rod first becomes vertical, a point C of the rod, where $AC = 3l$, strikes a fixed peg. Find the linear impulse exerted by the peg on the rod if

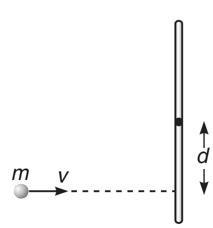
(a) the rod is brought to rest by the peg,
(b) the rod rebounds and next comes to instantaneous rest inclined to the downward vertical at an angle $\frac{\pi}{3}$ radian.

25. A uniform rod of length $4l$ and mass m is free to rotate about a horizontal axis passing through a point distant l from its one end. When the rod is horizontal, its angular velocity is ω as shown in figure. Calculate :

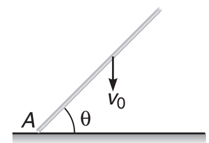


(a) reaction of axis at this instant,
(b) acceleration of centre of mass of the rod at this instant,
(c) reaction of axis and acceleration of centre mass of the rod when rod becomes vertical for the first time,
(d) minimum value of ω, so that centre of rod can complete circular motion.

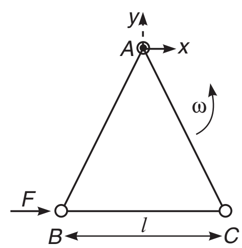
26. A stick of length l lies on horizontal table. It has a mass M and is free to move in any way on the table. A ball of mass m, moving perpendicularly to the stick at a distance d from its centre with speed v collides elastically with it as shown in figure. What quantities are conserved in the collision ? What must be the mass of the ball, so that it remains at rest immediately after collision?



27. A rod of length l forming an angle θ with the horizontal strikes a frictionless floor at A with its centre of mass velocity v_0 and no angular velocity. Assuming that the impact at A is perfectly elastic. Find the angular velocity of the rod immediately after the impact.

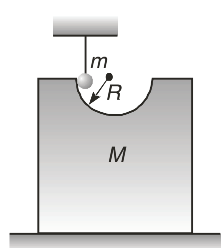


28. Three particles A, B and C, each of mass m, are connected to each other by three massless rigid rods to form a rigid, equilateral triangular body of side l. This body is placed on a horizontal frictionless table (x-y plane) and is hinged to it at the point A, so that it can move without friction about the vertical axis through A (see figure). The body is set into rotational motion on the table about A with a constant angular velocity ω.



(a) Find the magnitude of the horizontal force exerted by the hinge on the body.
(b) At time T, when the side BC is parallel to the x-axis, a force F is applied on B along BC (as shown). Obtain the x-component and the y-component of the force exerted by the hinge on the body, immediately after time T.

29. A semicircular track of radius $R = 62.5$ cm is cut in a block. Mass of block, having track, is $M = 1$ kg and rests over a smooth horizontal floor. A cylinder of radius $r = 10$ cm and mass $m = 0.5$ kg is hanging by thread such that axes of cylinder and track are in same level and surface of cylinder is in contact with the track as shown in figure. When the thread is burnt, cylinder starts to move down the track. Sufficient friction exists between surface of cylinder and track, so that cylinder does not slip.

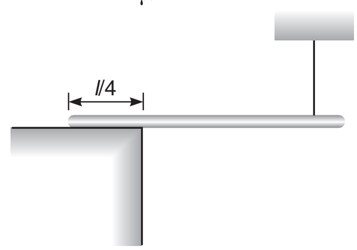


Calculate velocity of axis of cylinder and velocity of the block when it reaches bottom of the track. Also find force applied by block on the floor at that moment. ($g = 10$ m/s^2)

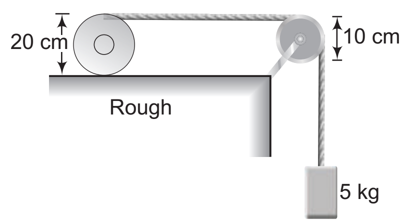
30. A uniform circular cylinder of mass m and radius r is given an initial angular velocity ω_0 and no initial translational velocity. It is placed in contact with a plane inclined at an angle α to the horizontal. If there is a coefficient of friction μ for sliding between the cylinder and plane. Find the distance the cylinder moves up before sliding stops. Also, calculate the maximum distance it travels up the plane. Assume $\mu > \tan\alpha$.

31. Show that if a rod held at angle θ to the horizontal and released, its lower end will not slip if the friction coefficient between rod and ground is greater than $\dfrac{3\sin\theta\cos\theta}{1+3\sin^2\theta}$.

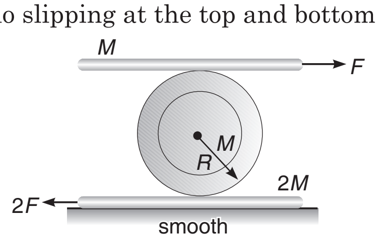
32. One-fourth length of a uniform rod of mass m and length l is placed on a rough horizontal surface and it is held stationary in horizontal position by means of a light thread as shown in the figure. The thread is then burnt and the rod starts rotating about the edge. Find the angle between the rod and the horizontal when it is about to slide on the edge. The coefficient of friction between the rod and the surface is μ.



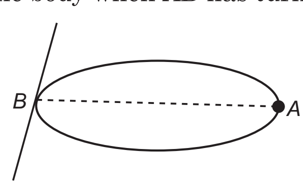
33. In figure the cylinder of mass 10 kg and radius 10 cm has a tape wrapped round it. The pulley weighs 100 N and has a radius 5 cm. When the system is released, the 5 kg mass comes down and the cylinder rolls without slipping. Calculate the acceleration and velocity of the mass as a function of time.



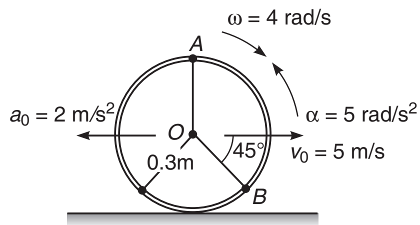
34. A cylinder is sandwiched between two planks. Two constant horizontal forces F and $2F$ are applied on the planks as shown. Determine the acceleration of the centre of mass of cylinder and the top plank, if there is no slipping at the top and bottom of cylinder.



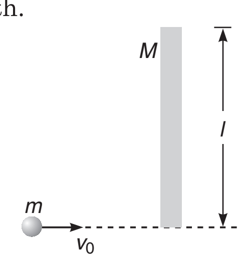
35. A ring of mass m and radius r has a particle of mass m attached to it at a point A. The ring can rotate about a smooth horizontal axis which is tangential to the ring at a point B diametrically opposite to A. The ring is released from rest when AB is horizontal. Find the angular velocity and the angular acceleration of the body when AB has turned through an angle $\dfrac{\pi}{3}$.



36. A hoop is placed on the rough surface such that it has an angular velocity $\omega = 4$ rad/s and an angular deceleration $\alpha = 5\,\text{rad/s}^2$. Also, its centre has a velocity of $v_0 = 5$ m/s and a deceleration $a_0 = 2\ \text{m/s}^2$. Determine the magnitude of acceleration of point B at this instant.



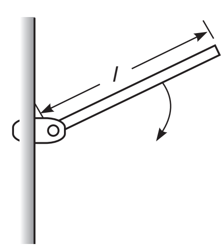
37. A boy of mass m runs on ice with velocity v_0 and steps on the end of a plank of length l and mass M which is perpendicular to his path.



(a) Describe quantitatively the motion of the system after the boy is on the plank. Neglect friction with the ice.

(b) One point on the plank is at rest immediately after the collision. Where is it?

38. A thin plank of mass M and length l is pivoted at one end. The plank is released at 60° from the vertical. What is the magnitude and direction of the force on the pivot when the plank is horizontal?



Answers

Introductory Exercise 12.1

1. $\dfrac{2}{\sqrt{3}}\,l$ **2.** 55 kg-m^2 **3.** $\dfrac{5}{3}ml^2$

4. About a diagonal, because the mass is more concentrated about a diagonal

5. (i) $\dfrac{8}{5}mr^2 + 2ma^2$ (ii) $\dfrac{8}{5}mr^2 + ma^2$ **6.** (a) $2Ml^2$ (b) $\dfrac{1}{3}Ml^2$

7. 0.5 kg-m^2 **8.** The one having the smaller density **9.** 0.43 kg-m^2 **10.** $\dfrac{\pi^2}{3}$

Introductory Exercise 12.2

1. $\dfrac{\pi}{30}$ rad s^{-1} **2.** $\omega = \dfrac{v}{2R}$ **3.** $(-\hat{\mathbf{k}})$ rad s^{-1} **4.** $\dfrac{v}{l}$, perpendicular to paper inwards

Introductory Exercise 12.3

1. $(-2\hat{\mathbf{i}} - 2\hat{\mathbf{k}})$ N-m **2.** 400 N-m (perpendicular to the plane of motion) **3.** 2.71 N-m **4.** $\dfrac{83}{20}$ N-m

Introductory Exercise 12.4

1. 8π rad-s^{-2}, (40π) rad-s^{-1} **2.** 100 rad **3.** 5 N-m **4.** 800 rad **5.** 0.87 N

6. (a) 4 rad-s^{-1}, -6 rad-s^{-2} (b) -12 rad-s^{-2} **7.** 20 s **8.** (a) 0.01 N-m (b) 10.13 N-m **9.** (a) 36 s (b) $12\sqrt{\dfrac{2}{k}}$

10. (a) 12.5 rad (b) 127.5 rad **11.** 9 rad, 1.43

Introductory Exercise 12.5

1. $ml^2\omega$ **2.** $4\sqrt{2}$ kg-m^{2}s^{-1} **3.** $\dfrac{mu^3\cos\alpha\sin^2\alpha}{2g}$ **4.** No **5.** $-\left(\dfrac{7}{5}mRv\right)\hat{\mathbf{k}}$ **6.** $\dfrac{1}{2}mRv$ (clockwise)

Introductory Exercise 12.6

1. $\dfrac{\omega_0 M}{M + 2m}$ **2.** Duration of day night will increase **3.** True **4.** Increase

Introductory Exercise 12.7

1. $\mathbf{v}_M = \dfrac{3}{2}v\hat{\mathbf{i}}$, $\mathbf{v}_N = v\hat{\mathbf{i}} - \dfrac{v}{2}\hat{\mathbf{j}}$, $\mathbf{v}_R = \dfrac{v}{2}\hat{\mathbf{i}}$, $\mathbf{v}_S = v\hat{\mathbf{i}} + \dfrac{v}{2}\hat{\mathbf{j}}$

2. $\mathbf{a}_M = (R\omega^2)\hat{\mathbf{j}}$, $\mathbf{a}_N = (R\omega^2)\hat{\mathbf{i}}$, $\mathbf{a}_R = -(R\omega^2)\hat{\mathbf{j}}$, $\mathbf{a}_S = -(R\omega^2)\hat{\mathbf{i}}$

Introductory Exercise 12.8

1. 25 J, 35 J **2.** True **3.** True

Introductory Exercise 12.9

1. $\mathbf{v}_A$ is zero. Rest three velocities are : $|\mathbf{v}_C| = 2v$, $|\mathbf{v}_B| = |\mathbf{v}_D| = \sqrt{2}v$

Introductory Exercise 12.10

1. False **2.** 72 N **3.** (a) $\dfrac{2}{7\sqrt{3}}$ (b) $\dfrac{25}{7}$ ms^{-2} (c) $\dfrac{30}{7}$ ms^{-2} **4.** (a) $g\sin\theta - \mu g\cos\theta$ (b) $\dfrac{5\mu g\cos\theta}{2R}$ **5.** Leftwards

Introductory Exercise 12.11

1. False **2.** (a) Same in both cases (b) Solid sphere (c) Solid sphere

Exercises

LEVEL 1

Assertion and Reason

1. (d) **2.** (b) **3.** (d) **4.** (a) **5.** (a) **6.** (a) **7.** (c) **8.** (b) **9.** (b) **10.** (a) **11.** (c)

Objective Questions

1. (d) **2.** (a) **3.** (c) **4.** (d) **5.** (b) **6.** (a) **7.** (c) **8.** (b) **9.** (b) **10.** (b)
11. (d) **12.** (a) **13.** (b) **14.** (b) **15.** (b) **16.** (d) **17.** (a) **18.** (b) **19.** (b) **20.** (d)
21. (c) **22.** (d) **23.** (b) **24.** (a) **25.** (a) **26.** (d) **27.** (c) **28.** (b) **29.** (a) **30.** (d)

Subjective Questions

1. $6h$ **2.** $\frac{R}{\sqrt{2}}$ **3.** $\frac{Ma^2}{12}$ **4.** $\frac{l}{\sqrt{2}}$ **5.** $I = \mu r^2$, where $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is called the reduced mass of two masses.

6. 8 cm **7.** $0.7\,\text{ms}^{-2}$ **8.** $10\,\text{rad-s}^{-1}$ **9.** $\frac{mv^3 \sin^2\theta \cos\theta}{2g}$ **10.** $I = \left(\frac{\alpha l^4}{4} + \frac{\beta l^3}{3}\right)$ **11.** False

12. $\frac{2}{7} mgh$ **13.** $\frac{v_1 - v_2}{2R}$ **14.** $\frac{2}{3} Ma^2$ **15.** 7 s **16.** $\pm \cos^{-1}\left(\frac{v}{R\omega}\right)$ **17.** $\frac{I + 2Mr^2}{4Mr^2 - I}$

18. $\lim_{F \to 0}$ can make the body move **19.** (a) $\mu < 1$ (b) $\mu > 1$

20. $\frac{20}{3}$ N-m, 4 s **21.** (a) $\frac{3\omega_0 R}{4\mu g}$ (b) $\frac{3\omega_0^2 R}{8\mu g}$

22. $\omega_{av} = 4$ rad/s, $\alpha_{av} = -6.0\,\text{rad-s}^{-2}$ **23.** $-\left(\frac{10}{3} mRv\right)\hat{\mathbf{k}}$ **24.** $\frac{25}{6}\,\text{rad-s}^{-1}$ **25.** $7.29 \times 10^{-4}\,\text{kg-m}^2$

26. (a) $14.3\,\text{rad-s}^{-1} = 2.27\,\text{rev-s}^{-1}$ (b) $E_i = 39.9$ J, $E_f = 181$ J (c) 141.1 J

27. (a) $\omega = \left(1 + \frac{2m}{M}\right)\omega_0$ (b) $\frac{1}{2} m\omega_0^2 R^2 \left(1 + \frac{2m}{M}\right)$ **28.** $\frac{4F\cos\theta}{3M + 8m}, \frac{3MF\cos\theta}{3M + 8m}, \frac{MF\cos\theta}{3M + 8m}$

29. $\frac{F}{M + 3m}$ each in opposite directions **30.** $\frac{4}{3} ml\omega$ **31.** (a) $\frac{2v}{9L}$ (b) $\frac{1}{9}$

32. (a) $\frac{2u}{3}$ (b) $\frac{2u}{\sqrt{3}}$

LEVEL 2

Objective Questions

1. (d) **2.** (b) **3.** (b) **4.** (d) **5.** (c) **6.** (a) **7.** (a) **8.** (b) **9.** (c) **10.** (c)
11. (b) **12.** (c) **13.** (b) **14.** (b) **15.** (d) **16.** (a) **17.** (b) **18.** (b) **19.** (d) **20.** (b)
21. (c) **22.** (b) **23.** (c) **24.** (d) **25.** (c) **26.** (a) **27.** (b) **28.** (d) **29.** (a) **30.** (c)
31. (b) **32.** (b) **33.** (c) **34.** (d) **35.** (a) **36.** (d) **37.** (b) **38.** (d) **39.** (c) **40.** (a)
41. (a) **42.** (a)

More than One Correct Options

1. (a,c) **2.** (all) **3.** (a,b,d) **4.** (a,d) **5.** (a,c,d) **6.** (all) **7.** (a,c)
8. (b,d) **9.** (a,b) **10.** (a,c) **11.** (a,b)

Comprehension Based Questions

1. (c) **2.** (a) **3.** (d) **4.** (b) **5.** (d) **6.** (c) **7.** (c) **8.** (d) **9.** (d) **10.** (a)

Match the Columns

1. (a) → q (b) → s (c) → q (d) → p
2. (a) → p (b) → s (c) → p (d) → r
3. (a) → p,s (b) → p,r (c) → q,r (d) → p,r
4. (a) → q (b) → r (c) → r (d) → p
5. (a) → q (b) → s (c) → p (d) → s
6. (a) → q,r (b) → p (c) → s (d) → q
7. (a) → r (b) → q (c) → s (d) → p

Subjective Questions

1. In each case in clockwise direction **2.** (a) $3g/2$, (b) $3g/4$ (c) $mg/4$ **3.** 0.612 m **4.** $\frac{F}{2M}$, $\frac{F}{2}$

5. 0.75 ms^{-1}, 1.98 ms^{-1} **6.** (b) $\frac{1}{3}Mg$ **7.** $\sqrt{\frac{4mg}{(2m+M)R}}$ **8.** $\frac{2v_0}{R}$

9. $f = \frac{2}{7}mg\sin\theta$, $N = \frac{mg}{7}(17\cos\theta - 10) + \frac{mv_0^2}{(R-r)}$

10. (a) $f = \frac{2}{3}\left(\frac{1}{2} - \frac{r}{R}\right)F$, assuming f opposite to F (b) $a = \left(\frac{2F}{3mR}\right)(R + r)$ (c) yes, if r is greater than $\frac{1}{2}R$.

(d) f in same direction as F. **11.** (a) $\frac{8F}{3m_1 + 8m_2}$, $\frac{4F}{3m_1 + 8m_2}$

(b) $\frac{3m_1F}{3m_1 + 8m_2}$ (between plank and cylinder) $\frac{m_1F}{3m_1 + 8m_2}$ (between cylinder and ground)

12. (a) 1.43 ms^{-2} (b) 7.15 rad-s^{-2} (c) 1.43 N **13.** 2.25 s **14.** 0.745 ms^{-1} (rightwards)

15. $\frac{3}{2}\frac{v}{r}$ (anticlockwise) **16.** 1 m **17.** $\frac{\omega_0 R}{6}$; $\frac{\omega_0}{6}$ **18.** $S = \frac{2\omega_0^2 r^2}{81\mu g}$ **19.** $v = 1.5\ ms^{-1}$ **20.** $\frac{9}{2}\frac{mv_0^2}{l}$

21. (a) Sphere, 5.29 ms^{-1}, disk 5.11 ms^{-1}, hoop 4.43 ms^{-1}
(b) Sphere 4.2 N, disk 4.9 N, hoop 7.36 N (c) Sphere, 1.51 s disk 1.56 s hoop 1.81 s

22. $2l\sqrt{\frac{g\sqrt{3}}{l}}$ **24.** (a) $\left(\frac{8}{3}m\right)\sqrt{3gl}$, (b) $\frac{4}{3}m\sqrt{6gl}(\sqrt{2} + 1)$

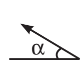
25. (a) $\frac{4}{7}mg\sqrt{1 + \left(\frac{7l\omega^2}{4g}\right)^2}$ (b) $\sqrt{\left(\frac{3g}{7}\right)^2 + (l\omega^2)^2}$ (c) $\left(\frac{13}{7}mg + ml\omega^2\right), \left(\frac{6g}{7} + l\omega^2\right)$ (d) $\sqrt{\frac{6g}{7l}}$

26. Linear momentum, angular momentum and kinetic energy, $\frac{Ml^2}{12d^2 + l^2}$ **27.** $\omega = \frac{6v_0}{l}\left(\frac{\cos\theta}{1 + 3\cos^2\theta}\right)$

28. (a) $\sqrt{3}\,ml\omega^2$ (b) $F_x = -\frac{F}{4}$, $F_y = \sqrt{3}\,ml\omega^2$ **29.** 2.0 ms^{-1}, 1.5 ms^{-1}, 16.67 N

30. $d_1 = \frac{r^2\omega_0^2(\mu\cos\alpha - \sin\alpha)}{2g(3\mu\cos\alpha - \sin\alpha)^2}$, $d_{max} = \frac{r^2\omega_0^2(\mu\cos\alpha - \sin\alpha)}{4g\sin\alpha(3\mu\cos\alpha - \sin\alpha)}$

32. $\theta = \tan^{-1}\left(\frac{4\mu}{13}\right)$ **33.** 3.6 ms^{-2}, $\frac{4gt}{11}$ **34.** $\frac{F}{26M}$, $\frac{21F}{26M}$ **35.** $\sqrt{\frac{6g\sqrt{3}}{11r}}$, $\frac{3g}{11r}$ **36.** 6.21 ms^{-2}

37. (b) $\frac{2l}{3}$ from the boy **38.** $\frac{\sqrt{10}}{4}Mg$, 

$\alpha = \tan^{-1}\left(\frac{1}{3}\right)$

13

Gravitation

Chapter Contents

13.1 Introduction

Why are planets, moon and the sun all nearly spherical? Why do some earth satellites circle the earth in 90 minutes, while the moon takes 27 days for the trip? And why don't satellites fall back to earth? The study of gravitation provides the answers for these and many related questions.

Gravitation is one of the four classes of interactions found in nature. These are :

(i) the gravitational force
(ii) the electromagnetic force
(iii) the strong nuclear force (also called the hadronic forces)
(iv) the weak nuclear forces.

Although, of negligible importance in the interactions of elementary particles, gravity is of primary importance in the interactions of large objects. It is gravity that holds the universe together.

In this chapter, we will learn the basic laws that govern gravitational interactions.

13.2 Newton's Law of Gravitation

Along with his three laws of motion, Newton published the law of gravitation in 1687. According to him; *"every particle of matter in the universe attracts every other particle with a force that is directly proportional to the product of the masses of the particles and inversely proportional to the square of the distance between them."*

Thus, the magnitude of the gravitational force F between two particles m_1 and m_2 placed at a distance r is,

$$F \propto \frac{m_1 m_2}{r^2} \quad \text{or} \quad \boxed{F = G\frac{m_1 m_2}{r^2}}$$

Here, G is a universal constant called gravitational constant whose magnitude is,

$$G = 6.67 \times 10^{-11} \text{ N-m}^2/\text{kg}^2$$

The direction of the force F is along the line joining the two particles.

Following three points are important regarding the gravitational force :

(i) Unlike the electrostatic force, it is independent of the medium between the particles.
(ii) It is conservative in nature.
(iii) It expresses the force between two point masses (of negligible volume). However, for external points of spherical bodies the whole mass can be assumed to be concentrated at its centre of mass.

Gravity

In Newton's law of gravitation, gravitation is the force of attraction between any two bodies. If one of the bodies is earth then the gravitation is called 'gravity'. Hence, gravity is the force by which earth attracts a body towards its centre. It is a special case of gravitation.

Extra Points to Remember

- Direct formula $F = \frac{Gm_1 m_2}{r^2}$ can be applied under following three conditions:

  (a) To find force between two point masses

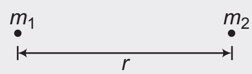


Fig. 13.1

  (b) To find force between two spherical bodies

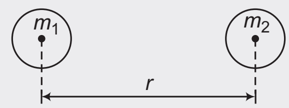


Fig. 13.2

  (c) To find force between a spherical body and a point mass.

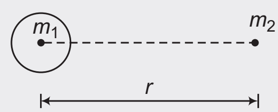


Fig. 13.3

- To find force between a point mass and a rod single integration is required. In this case, we cannot assume whole mass of the rod at its centre to find force between them. Thus,

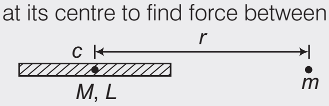


Fig. 13.4

$$F \neq \frac{GMm}{r^2}$$

- To find force between two rods double integration is required but normally, double integration is not asked in physics paper.

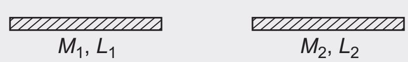


Fig. 13.5

- Two or more than two gravitational forces acting on a body can be added by vector addition method.

Example 13.1 *Three point masses 'm' each are placed at the three vertices of an equilateral triangle of side 'a'. Find net gravitational force on any one point mass.*

Solution We are finding net force on the point mass kept at O.

$$F = \frac{G(m)(m)}{a^2} = \frac{Gm^2}{a^2}$$

Since, the two forces are equal in magnitudes, therefore the resultant force will pass through the centre as shown in figure.

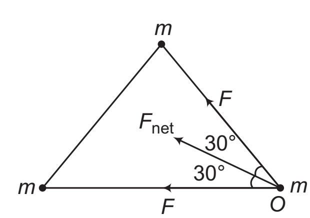


Fig. 13.6

$$F_{net} = \sqrt{F^2 + F^2 + 2(F)(F)\cos 60°}$$

$$= \sqrt{3}\,F$$

$$= \frac{\sqrt{3}\,Gm^2}{a^2}$$

Ans.

Example 13.2 *Four particles each of mass 'm' are placed at the four vertices of a square of side 'a'. Find net force on any one of the particle.*

Solution. We are finding net force on the particle at D.

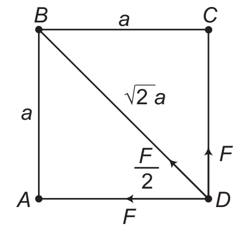


Fig. 13.7

$$F_{DC} = F_{DA} = \frac{G(m)(m)}{a^2} = \frac{Gm^2}{a^2} = F \text{ (say)}$$

$$F_{DB} = \frac{G(m)(m)}{(\sqrt{2}a)^2} = \frac{1}{2}\frac{Gm^2}{a^2} = \frac{F}{2}$$

Now, resultant of F_{DA} and F_{DC} is $\sqrt{2}F$ in the direction of DB.

$$\therefore \qquad F_{\text{net}} = \sqrt{2}F + \frac{F}{2} = \left(\sqrt{2} + \frac{1}{2}\right)F$$

$$= \left(\sqrt{2} + \frac{1}{2}\right)\frac{Gm^2}{a^2} \quad \text{(towards } DB\text{)}$$ **Ans.**

Example 13.3 *Six particles each of mass 'm' are placed at six vertices A, B, C, D, E and F of a regular hexagon of side 'a'. A seventh particle of mass 'M' is kept at centre 'O' of the hexagon.*

(a) Find net force on 'M'.

(b) Find net force on 'M' if particle at A is removed.

(c) Find net force on 'M' if particles at A and C are removed.

Solution (a) $F_A = F_B = F_C = F_D = F_E = F_F = \dfrac{GmM}{a^2} = F$ (say)

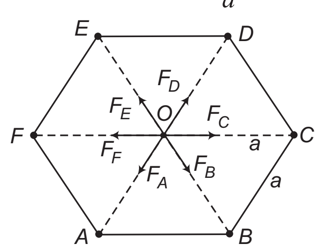


Fig. 13.8

So, net force will be zero, as three pairs of equal and opposite forces are acting on 'M' at O.

(b) When particle at A is removed then F_A is removed. So, there is no force to cancel F_D.

$$\therefore \qquad F_{\text{net}} = F_D = \frac{GMm}{a^2} \quad \text{(towards } D\text{)}$$ **Ans.**

(c) When particles at A and C are removed then, F_A and F_C are removed. F_B and F_E are still cancelled. So, net force is the resultant of two forces F_D and F_F of equal magnitudes acting at 120°. So, the resultant will pass through the centre or towards E. Magnitude of this resultant is

$$F_{\text{net}} = \sqrt{F^2 + F^2 + 2(F)(F)\cos 120^\circ}$$

$$= F = \frac{GMm}{a^2} \text{ (towards } E\text{)}$$ **Ans.**

Example 13.4 *Five particles each of mass 'm' are kept at five vertices of a regular pentagon. A sixth particle of mass 'M' is kept at centre of the pentagon 'O'. Distance between 'M' and 'm' is 'a'. Find*

(a) net force on 'M'

(b) magnitude of net force on 'M' if any one particle is removed from one of the vertices.

Solution (a) $F_A = F_B = F_C = F_D = F_E = \dfrac{GMm}{a^2} = F$ (say)

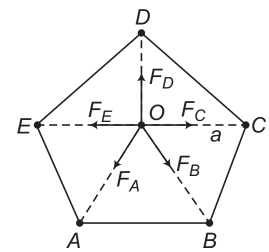


Fig. 13.9

Angle between two successive force vectors is $\theta = \dfrac{360°}{5} = 72°$.

When these five force vectors are added as per polygon law of vector addition we get another closed regular polygon as shown below.

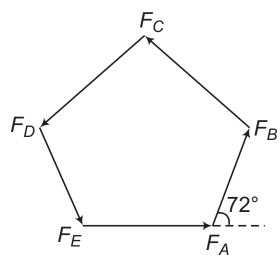


Fig. 13.10

Therefore, net resultant force on 'M' is zero.

Note *This result is true ($F_{net} = 0$) for any number of particles, provided masses at vertices are equal and polygon is regular.*

(b) When particle at A is removed, then F_A will be removed. So, magnitude of net force will be F as shown below:

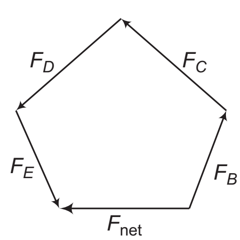


Fig. 13.11

$$\therefore \qquad F_{net} = F = \frac{GMm}{a^2} \qquad \textbf{Ans.}$$

Example 13.5 *A mass m is at a distance a from one end of a uniform rod of length l and mass M. Find the gravitational force on the mass due to the rod.*

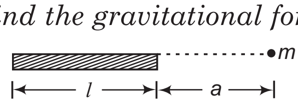


Fig. 13.12

Solution $dF = \dfrac{G\,(dM)\,m}{x^2} = \dfrac{G\left(\dfrac{M}{l}\cdot dx\right)m}{x^2}$

$\therefore \qquad F = \displaystyle\int_{x=a}^{x=(a+l)} dF = \dfrac{GMm}{a\,(l+a)}$ **Ans.**

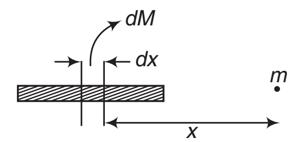


Fig. 13.13

Example 13.6 *A uniform ring of mass m is lying at a distance $\sqrt{3}\,a$ from the centre of a sphere of mass M just over the sphere (where a is the radius of the ring as well as that of the sphere). Find the magnitude of gravitational force between them.*

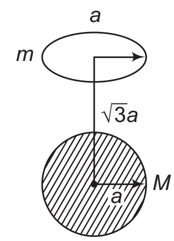


Fig. 13.14

Solution Net force on ring $= \displaystyle\int_{\text{whole ring}} dF \sin 60^\circ$

$$= \int_{\text{whole ring}} \frac{GM\,(dm)}{(2a)^2}\,\frac{\sqrt{3}}{2}$$

$$= \frac{\sqrt{3}\,GMm}{8a^2}$$

as $\displaystyle\int_{\text{whole ring}} dm = m$

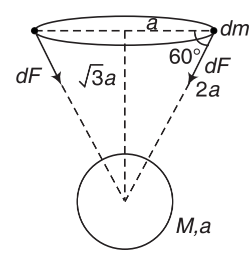


Fig. 13.15

INTRODUCTORY EXERCISE 13.1

1. Three uniform spheres each having a mass M and radius a are kept in such a way that each touches the other two. Find the magnitude of the gravitational force on any one of the spheres due to the other two.
2. Four particles having masses m, $2m$, $3m$ and $4m$ are placed at the four corners of a square of edge a. Find the gravitational force acting on a particle of mass m placed at the centre.
3. Two particles of masses 1.0 kg and 2.0 kg are placed at a separation of 50 cm. Assuming that the only forces acting on the particles are their mutual gravitation, find the initial accelerations of the two particles.
4. Three particles A, B and C, each of mass m, are placed in a line with $AB = BC = d$. Find the gravitational force on a fourth particle P of same mass, placed at a distance d from the particle B on the perpendicular bisector of the line AC.
5. Spheres of the same material and same radius r are touching each other. Show that gravitational force between them is directly proportional to r^4.

13.3 Acceleration due to Gravity

When a body is dropped from a certain height above the ground it begins to fall towards the earth under gravity. The acceleration produced in the body due to gravity is called the acceleration due to gravity. It is denoted by g . Its value close to the earth's surface is 9.8 m/s^2.

Suppose that the mass of the earth is M, its radius is R, then the force of attraction acting on a body of mass m close to the surface of earth is

$$F = \frac{GMm}{R^2}$$

According to Newton's second law, the acceleration due to gravity

$$\boxed{g = \frac{F}{m} = \frac{GM}{R^2}}$$

This expression is independent of m. If two bodies of different masses are allowed to fall freely, they will have the same acceleration, i.e. if they are allowed to fall from the same height, they will reach the earth simultaneously.

Variation in the Value of *g*

The value of g varies from place to place on the surface of earth. It also varies as we go above or below the surface of earth. Thus, value of g depends on following factors :

Shape of the Earth

The earth is not a perfect sphere. It is somewhat flat at the two poles. The equatorial radius is approximately 21 km more than the polar radius and since

$$g = \frac{GM}{R^2} \quad \text{or} \quad g \propto \frac{1}{R^2}$$

The value of g is minimum at the equator and maximum at the poles.

Height above the Surface of the Earth

The force of gravity on an object of mass m at a height h above the surface of earth is,

$$F = \frac{GMm}{(R+h)^2}$$

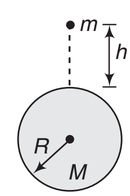


Fig. 13.16

$\therefore$ Acceleration due to gravity at this height will be,

$$g' = \frac{F}{m} = \frac{GM}{(R+h)^2}$$

This can also be written as,

$$g' = \frac{GM}{R^2\left(1+\frac{h}{R}\right)^2}$$

or

$$\boxed{g' = \frac{g}{\left(1+\frac{h}{R}\right)^2}} \quad \text{as} \quad \frac{GM}{R^2} = g$$

Thus, $$g' < g$$

i.e. the value of acceleration due to gravity g goes on decreasing as we go above the surface of earth. Further,

$$g' = g\left(1 + \frac{h}{R}\right)^{-2} \quad \text{or} \quad \boxed{g' \approx g\left(1 - \frac{2h}{R}\right) \quad \text{if} \quad h << R}$$

Depth below the Surface of the Earth

Let an object of mass m is situated at a depth d below the earth's surface. Its distance from the centre of earth is $(R - d)$. This mass is situated at the surface of the inner solid sphere and lies inside the outer spherical shell. According to Gauss theorem (you will study in class XII) the gravitational force of attraction on a mass inside a spherical shell is always zero. Therefore, the object experiences gravitational attraction only due to inner solid sphere.

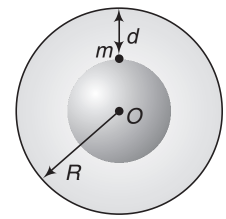


Fig. 13.17

The mass of this sphere is,

$$M' = \left(\frac{M}{4/3\,\pi R^3}\right)\frac{4}{3}\pi\,(R - d)^3$$

or $$M' = \frac{(R - d)^3}{R^3} \,.\, M$$

$$F = \frac{GM'm}{(R - d)^2} = \frac{GMm(R - d)}{R^3}$$

and $$g' = \frac{F}{m}$$

Substituting the values, we get $\boxed{g' = g\left(1 - \frac{d}{R}\right)}$ i.e. $g' < g$

Note *We can see from this equation that $g' = 0$ at $d = R$, i.e. acceleration due to gravity is zero at the centre of the earth.*

Thus, the variation in the value of g with r (the distance from the centre of earth) is as shown in Fig. 13.18.

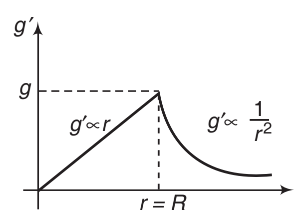


Fig. 13.18

For $r \le R$, $$g' = g\left(1 - \frac{d}{R}\right) = \frac{gr}{R}$$

as $$R - d = r \quad \text{or} \quad g' \propto r$$

For $r \ge R$, $$g' = \frac{g}{\left(1 + \frac{d}{R}\right)^2} = \frac{gR^2}{r^2}$$

as $$R + d = r \quad \text{or} \quad g' \propto \frac{1}{r^2}$$

Axial Rotation of the Earth

Let us consider a particle P at rest on the surface of the earth, at latitude ϕ. Then the pseudo force acting on the particle is $mr\omega^2$ in outward direction. The true acceleration g is acting towards the centre O of the earth. Thus, the effective acceleration g' is the resultant of g and $r\omega^2$ or

$$g' = \sqrt{g^2 + (r\omega^2)^2 + 2g(r\omega^2)\cos(180° - \phi)}$$

as angle between g and $r\omega^2$ is $(180° - \phi)$.

or $$g' = \sqrt{g^2 + r^2\omega^4 - 2gr\omega^2 \cos\phi} \qquad \text{...(i)}$$

Here, the term $r^2\omega^4$ comes out to be too small as $\omega = \frac{2\pi}{T} = \frac{2\pi}{24 \times 3600}$ rad/s is small. Hence, this term can be ignored. Also, $r = R\cos\phi$. Therefore, Eq. (i) can be written as

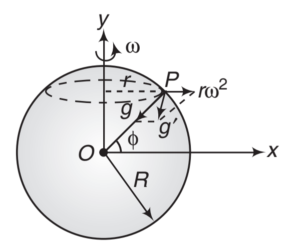


Fig. 13.19

$$g' = (g^2 - 2gR\omega^2 \cos^2\phi)^{1/2}$$

$$= g\left(1 - \frac{2R\omega^2\cos^2\phi}{g}\right)^{1/2}$$

$$\approx g\left(1 - \frac{R\omega^2\cos^2\phi}{g}\right)$$

Thus, $$\boxed{g' = g - R\omega^2\cos^2\phi}$$

Following conclusions can be drawn from the above discussion :

(i) The effective value of g is not truly vertical passing through the centre O.

(ii) The effect of centrifugal force due to rotation of earth is to reduce the effective value of g.

(iii) At equator $\phi = 0°$.

Therefore, $g' = g - R\omega^2$

and at poles $\phi = 90°$,

Therefore, $g' = g$

Thus, at equator g' is minimum while at poles g' is maximum.

Example 13.7 *Assuming earth to be a sphere of uniform mass density, how much would a body weigh half way down the centre of the earth, if it weighed 100 N on the surface?*

Solution Given, $mg = 100\text{ N} \Rightarrow g' = g\left(1 - \frac{d}{R}\right)$

$$d = \frac{R}{2} \Rightarrow \frac{d}{R} = \frac{1}{2}$$

$\therefore$ $$g' = g\left(1 - \frac{1}{2}\right) = \frac{g}{2}$$

$\therefore$ $$mg' = \frac{mg}{2} = \frac{100}{2} = 50\text{ N}$$ **Ans.**

Example 13.8 *Suppose the earth increases its speed of rotation. At what new time period will the weight of a body on the equator becomes zero? Take $g = 10\ m/s^2$ and radius of earth $R = 6400\ km$.*

Solution The weight will become zero, when

$$g' = 0 \quad \text{or} \quad g - R\omega^2 = 0 \qquad \text{(on the equator } g' = g - R\omega^2\text{)}$$

or
$$\omega = \sqrt{\frac{g}{R}}$$

$\therefore$
$$\frac{2\pi}{T} = \sqrt{\frac{g}{R}} \quad \text{or} \quad T = 2\pi\sqrt{\frac{R}{g}}$$

Substituting the values,
$$T = \frac{2\pi\sqrt{\dfrac{6400 \times 10^3}{10}}}{3600}\ \text{h}$$

or
$$T = 1.4\ \text{h}$$ **Ans.**

Thus, the new time period should be 1.4 h instead of 24 h for the weight of a body to be zero on the equator.

Example 13.9 *Draw g' versus d and g' versus h graph. Here, 'd' is depth below the surface of earth and h is the height from the surface of earth.*

Solution On the surface of earth,

$$g' = g$$

At depth 'd' below the surface of earth,

$$g' = g\left(1 - \frac{d}{R}\right)$$

i.e. g' decreases linearly with depth.

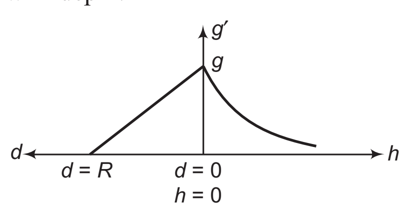


Fig. 13.20

At, $d = R$, $g' = 0$

At height h above the surface of earth.

$$g' = \frac{g}{\left(1 + \dfrac{h}{R}\right)^2}$$

With increase in the value of 'h' value of g' decreases but not linearly. At $h \to \infty$, $g' \to 0$.

The correct graph is as shown in Fig. 13.20.

Example 13.10 *At what depth below the surface of earth, value of acceleration due to gravity is same as the value at height $h = R$, where R is the radius of earth.*

Solution Given that, g' at depth $d = g'$ at height $h = R$

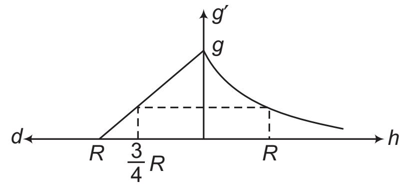


Fig. 13.21

$$\therefore \qquad g\left(1-\frac{d}{R}\right)=\frac{g}{\left(1+\frac{h}{R}\right)^2}$$

Substituting, $h = R$, we get

$$d = \frac{3}{4}R \qquad \textbf{Ans.}$$

From the graph we can see that value of g' at depth $d = \frac{3}{4}R$ and height $h = R$ are same.

INTRODUCTORY EXERCISE 13.2

1. Value of g on the surface of earth is 9.8 m/s^2. Find its value on the surface of a planet whose mass and radius both are two times that of earth.
2. Value of g on the surface of earth is 9.8 m/s^2. Find its value
   (a) at height $h = R$ from the surface,
   (b) at depth $d = \frac{R}{2}$ from the surface. (R = radius of earth)
3. Calculate the distance from the surface of the earth at which the acceleration due to gravity is the same below and above the surface of the earth.
4. Calculate the change in the value of g at altitude 45°. Take radius of earth $R = 6.37 \times 10^3$ km.
5. At what height from the surface of earth will the value of g be reduced by 36% from the value at the surface? Take radius of earth $R = 6400$ km.
6. Determine the speed with which the earth would have to rotate on its axis, so that a person on the equator would weigh $\frac{3}{5}$ th as much as at present. Take $R = 6400$ km.
7. A body is weighed by a spring balance to be 1000 N at the north pole. How much will it weight at the equator? Account for the earth's rotation only.
8. At what rate should the earth rotate so that the apparent g at the equator becomes zero ? What will be the length of the day in this situation?
9. Assuming earth to be spherical, at what height above the north pole, value of g is same as that on the earth's surface at equator?

13.4 Gravitational Field and Field Strength

The space around a mass or system of masses in which any other test mass experiences a gravitational force is called gravitational field. When this test mass is moved from one point to another point, some work is also done by this gravitational force.

Mathematically, gravitational field at any point can be defined by two physical quantities. One is vector quantity, called **gravitational field strength** or **intensity of gravitational field** and it is denoted by **E**. This is related to the gravitational force (a vector quantity) experienced by test mass in gravitational field. We also sometimes called it **gravitational field**.

The other physical quantity is **gravitational potential**. It is represented by V. This is related to the work done (a scalar quantity) by gravitational force in moving the test mass from one point to another point in the gravitational field.

Gravitational Field Strength (E)

The force experienced (both in magnitude and direction) by a unit test mass placed at a point in a gravitational field is called the **gravitational field strength** or **intensity of gravitational field** at that point. Thus,

$$\boxed{\mathbf{E} = \frac{\mathbf{F}}{m}}$$

SI unit of **E** is N/kg.

Note *The test mass to find field strength at some point should be infinitesimally small otherwise it will produce its own field and will disturb the original field.*

In Article 13.3, we have seen that acceleration due to gravity **g** is also $\frac{\mathbf{F}}{m}$. Hence, for the earth's gravitational field **g** and **E** are same. The E *versus* r (the distance from the centre of earth) graph are same as that of g' *versus* r graph.

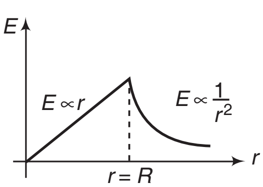


Fig. 13.22

Field due to a Point Mass

Suppose, a point mass M is placed at point O. We want to find the intensity of gravitational field **E** at a point P, a distance r from O. Magnitude of force **F** acting on a particle of mass m placed at P is,

$$F = \frac{GMm}{r^2}$$

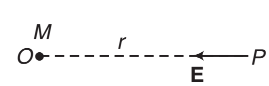


Fig. 13.23

$$\therefore \qquad E = \frac{F}{m} \quad \text{or} \quad \boxed{E = \frac{GM}{r^2}}$$

The direction of the force **F** and hence of **E** is from P to O as shown in Fig. 13.23.

Gravitational Field due to a Uniform Solid Sphere

Field at an External Point

A uniform sphere may be treated as a single particle of same mass placed at its centre for calculating the gravitational field at an external point. Thus,

$$\boxed{E(r) = \frac{GM}{r^2}} \quad \text{for} \quad r \geq R \quad \text{or} \quad E(r) \propto \frac{1}{r^2}$$

Here, r is the distance of the point from the centre of the sphere and R the radius of sphere.

Field at an Internal Point

The gravitational field due to a uniform sphere at an internal point is proportional to the distance of the point from the centre of the sphere. At the centre itself, it is zero and at surface it is $\frac{GM}{R^2}$, where R is the radius of the sphere. Thus,

$$\boxed{E(r) = \frac{GM}{R^3} r} \quad \text{for} \quad r \leq R \quad \text{or} \quad E(r) \propto r$$

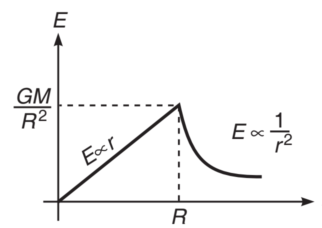


Fig. 13.24

Hence, E *versus* r graph is as shown in Fig. 13.24.

Field due to a Uniform Spherical Shell

At an External Point

For an external point the shell may be treated as a single particle of same mass placed at its centre. Thus, at an external point the gravitational field is given by,

$$\boxed{E(r) = \frac{GM}{r^2}} \quad \text{for} \quad r \geq R$$

at $\quad r = R \quad$ (the surface of shell)

$$E = \frac{GM}{R^2} \quad \text{and otherwise} \quad E \propto \frac{1}{r^2}$$

At an Internal Point

The field inside a uniform spherical shell is zero.

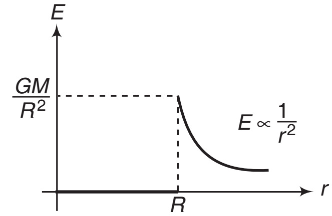


Fig. 13.25

Thus, E *versus* r graph is as shown in Fig. 13.25.

Field due to a Uniform Circular Ring at some Point on its Axis

Field strength at a point P on the axis of a circular ring of radius R and mass M is given by,

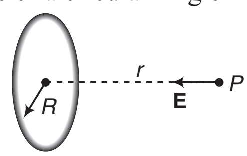


Fig. 13.26

$$\boxed{E(r)=\frac{GMr}{(R^2+r^2)^{3/2}}}$$

This is directed towards the centre of the ring. It is zero at the centre of the ring and maximum at $r=\frac{R}{\sqrt{2}}$ (can be obtained by putting $\frac{dE}{dr}=0$). Thus, E-r graph is as shown in Fig. 13.27.

The maximum value is $E_{\max}=\frac{2GM}{3\sqrt{3}R^2}$.

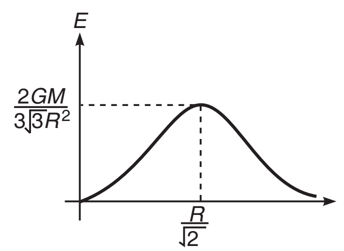


Fig. 13.27

13.5 Gravitational Potential

As we have discussed earlier also, this is a scalar quantity related to work done by gravitational force in moving a unit test mass in gravitational field from one point to another point.

Gravitational potential at any point is defined as the negative of work done by gravitational force in moving a unit test mass from infinity (where potential is assumed to be zero) to that point. Thus, potential at P is

$$\boxed{V_P=\frac{-W_{\infty\to P}}{m}} \qquad \text{(by gravitational force)}$$

It SI unit is J/kg.

Potential due to a Point Mass

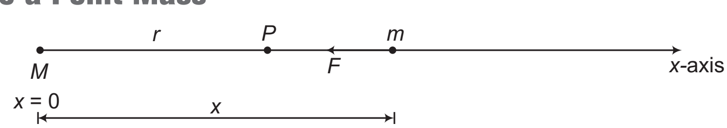


Fig. 13.28

Suppose a point mass 'M' is placed at origin ($x=0$). We wish to find gravitational potential at P, at a distance 'r' from M.

First of all we will calculate the work done by gravitational force in moving a test mass 'm' from infinity to P. Gravitational force on 'm' when it is at a distance 'x' from M is

$$F=-\frac{GMm}{x^2}$$

Here, negative sign implies that this force is towards 'M' or towards negative x-direction. This is a variable force (a function of x). Therefore, work done is

$$W = \int_{\infty}^{r} F dx = \int_{\infty}^{r} \left(-\frac{GMm}{x^2} \right) dx = \frac{GMm}{r}$$

Now, from the definition of potential,

$$V = -\frac{W}{m} = -\frac{GM}{r}$$

$$\therefore \qquad \boxed{V = -\frac{GM}{r}}$$

Potential due to a Uniform Solid Sphere

Potential at some External Point

The gravitational potential due to a uniform sphere at an external point is same as that due to a single particle of same mass placed at its centre. Thus,

$$\boxed{V(r) = -\frac{GM}{r}} \qquad r \geq R$$

At the surface, $\qquad r = R \quad \text{and} \quad V = -\frac{GM}{R}$

Potential at some Internal Point

At some internal point, potential at a distance r from the centre is given by,

$$\boxed{V(r) = -\frac{GM}{R^3}(1.5R^2 - 0.5r^2)} \qquad r \leq R$$

At $r = R$, $\quad V = -\frac{GM}{R}$

while at $r = 0$, $\quad V = -\frac{1.5\, GM}{R}$

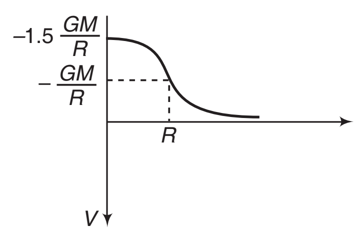


Fig. 13.29

i.e. at the centre of the sphere the potential is 1.5 times the potential at the surface. The variation of V *versus* r graph is as shown in Fig. 13.29.

Potential due to a Uniform Thin Spherical Shell

Potential at an External Point

To calculate the potential at an external point, a uniform spherical shell may be treated as a point mass of same magnitude at its centre. Thus, potential at a distance r is given by,

$$\boxed{V(r) = -\frac{GM}{r}} \qquad r \geq R$$

at $\qquad r = R, \quad V = -\frac{GM}{R}$

Potential at an Internal Point

The potential due to a uniform spherical shell is constant at any point inside the shell and this is equal to $-\frac{GM}{R}$. Thus, *V-r* graph for a spherical shell is as shown in Fig. 13.30.

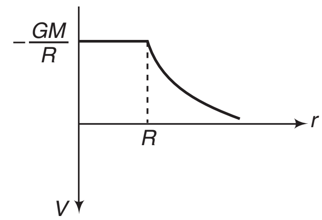


Fig. 13.30

Potential due to a Uniform Ring at some Point on its Axis

The gravitational potential at a distance r from the centre on the axis of a ring of mass M and radius R is given by,

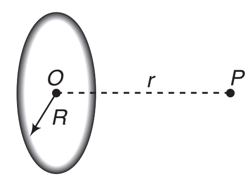


Fig. 13.31

$$\boxed{V(r) = -\frac{GM}{\sqrt{R^2 + r^2}}} \qquad 0 \le r \le \infty$$

At $r = 0$, $V = -\frac{GM}{R}$, i.e. at the centre of the ring gravitational potential is $-\frac{GM}{R}$.

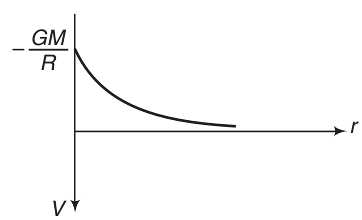


Fig. 13.32

The *V-r* graph is as shown in Fig. 13.32.

Example 13.11 *Three point masses 'm' each are kept at three vertices of a square of side 'a' as shown in figure. Find gravitational potential and field strength at point O.*

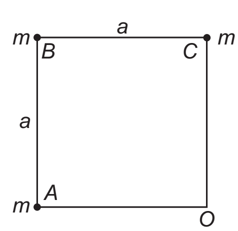


Fig. 13.33

Solution Gravitational potential is a scalar quantity.

$\therefore \quad V_O$ = scalar sum of potentials due to point masses at A, B and C.

$$= -\frac{Gm}{a} - \frac{Gm}{\sqrt{2}a} - \frac{Gm}{a}$$

$$= -\frac{Gm}{a}\left[2 + \frac{1}{\sqrt{2}}\right] \qquad \textbf{Ans.}$$

Gravitational field strength is a vector quantity. So, net field strength at O is the vector sum of three field strengths produced due to the three point masses at A, B and C.

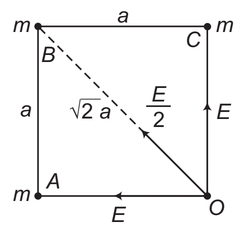


Fig. 13.34

Field strength due to A = field strength due to $C = \frac{Gm}{a^2}$ (towards the point masses)

$= E$ (say)

Field strength due to $B = \frac{Gm}{(\sqrt{2}a)^2} = \frac{Gm}{2a^2} = \frac{E}{2}$

Resultant of E and E is $\sqrt{2}E$ towards B.

$\frac{E}{2}$ is also towards B.

$$\therefore \qquad E_{\text{net}} = \sqrt{2}\,E + \frac{E}{2} = \left(\sqrt{2} + \frac{1}{2}\right)E$$

$$= \left(\sqrt{2} + \frac{1}{2}\right)\frac{Gm}{a^2} \qquad \text{(towards } B\text{)} \qquad \textbf{Ans.}$$

Example 13.12 *Four point masses each of mass 'm' are placed at four vertices A, B, C and D of a regular hexagon of side 'a' as shown in figure. Find gravitational potential and field strength at the centre O of the hexagon.*

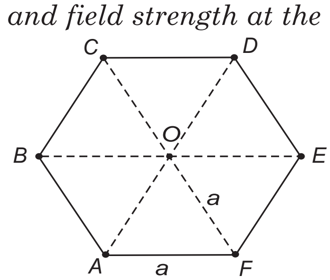


Fig. 13.35

Solution Gravitational potential is a scalar quantity. Therefore,

V_O = scalar sum of gravitational potentials produced by four point masses at A, B, C and D.

$$= -\frac{Gm}{a} - \frac{Gm}{a} - \frac{Gm}{a} - \frac{Gm}{a}$$

$$= -\frac{4\,Gm}{a} \qquad \textbf{Ans.}$$

Gravitational field strength is a vector quantity. So, it is a vector sum of four vectors of equal magnitudes.

$$E = \frac{Gm}{a^2} = E_A = E_B = E_C = E_D$$

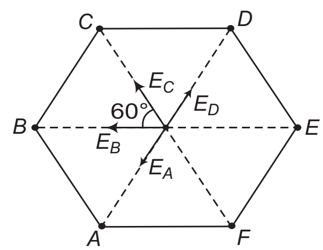


Fig. 13.36

E_A and E_D are cancelled. So, net field strength is a vector sum of E_B and E_C at angle 60°.

$$\therefore \qquad E_{\text{net}} = \sqrt{E^2 + E^2 + 2(E)(E)\cos 60°} = \sqrt{3}\,E$$

$$= \frac{\sqrt{3}\,Gm}{a^2} \qquad \textbf{Ans.}$$

Note *If* $E_1 = E_2 = E$ and θ *be the angle between them then,* $E_{\text{net}} = 2E\cos\left(\frac{\theta}{2}\right)$

This net field strength is along the bisector line of $\angle COB$, away from O, between E_C and E_B.

Example 13.13 *At what distance 'd' from the surface of a solid sphere of radius 'R',*

(a) potential is same as at a distance $\frac{R}{2}$ *from the centre ?*

(b) field strength is same as at a distance $\frac{R}{4}$ *from centre.*

Solution (a) Given, $V_{\text{outside}} = V_{\text{inside}}$

No such point will exist. Because potential at centre is $\frac{-1.5\,GM}{R}$. Potential at surface is $-\frac{GM}{R}$ and potential at infinity is zero. From centre to surface potential varies between $-\frac{1.5\,GM}{R}$ and $\frac{-GM}{R}$. From surface to infinity potential varies between $\frac{-GM}{R}$ and zero.

(b) Given, $$E_{\text{inside}} = E_{\text{outside}}$$

$$\therefore \qquad \frac{GM}{R^3}(r_1) = \frac{GM}{r_2^2}$$

Here, r_1 and r_2 are the distances from centre.

$$\therefore \qquad \frac{GM}{R^3}\left(\frac{R}{4}\right) = \frac{GM}{(R+d)^2}$$

Solving this equation, we get

$$d = R$$ **Ans.**

INTRODUCTORY EXERCISE 13.3

1. Two points masses '*m*' each are kept at the two vertices of an equilateral triangle of side '*a*' as shown in figure.

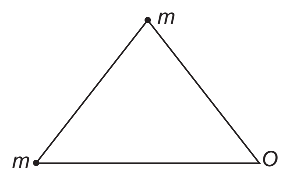


Fig. 13.37

Find gravitational potential and magnitude of field strength at *O*.

2. Five point masses '*m*' each are kept at five vertices of a regular pentagon. Distance of centre of pentagon from any one of the vertices is '*a*'. Find gravitational potential and field strength at centre.
3. In the above problem, if any one point mass is removed then what is gravitational potential and magnitude of field strength at centre?
4. A particle of mass *m* is placed at the centre of a uniform spherical shell of same mass and radius *R*. Find the gravitational potential at a distance $\frac{R}{2}$ from the centre.
5. A particle of mass 20 *g* experiences a gravitational force of 4.0 N along positive *x*-direction. Find the gravitational field at that point.

13.6 Relation between Gravitational Field and Potential

In gravitational field, field strength **E** and potential V are different at different points. So, they are functions of position. In cartesian co-ordinate system position of a point can be represented by three variable co-ordinates x, y and z. So, **E** and V are functions of three variables x, y and z. But in physics, we normally try to keep least number of variables. So, in one dimension (say along x-axis) position of a point can be represented by a single variable co-ordinates x.

So, E and V are functions of only one variable x. Similarly, in case of point mass or spherical bodies this single variable is r. Where r is the distance from point mass or distance from the centre of the spherical body.

Further, **E** and V are not two independent functions, but they are related to each other either by differentiation or integration.

Conversion of *V* Function into E Function

To convert V function into **E** function, differentiation is required. If there are more than one variables then partial differentiation is done and if there is only one variable then, direct differentiation is done.

More than one variables In this case,

$$\mathbf{E} = -\text{gradient } V = -\left[\frac{\partial V}{\partial x}\hat{\mathbf{i}} + \frac{\partial V}{\partial y}\hat{\mathbf{j}} + \frac{\partial V}{\partial z}\hat{\mathbf{k}}\right]$$

or
$$\boxed{\mathbf{E} = -\left[\frac{\partial V}{\partial x}\hat{\mathbf{i}} + \frac{\partial V}{\partial y}\hat{\mathbf{j}} + \frac{\partial V}{\partial z}\hat{\mathbf{k}}\right]}$$

Here, $\frac{\partial V}{\partial x}$ is called partial differentiation of V with respect to x. In this differentiation we differentiate V with respect to x, assuming other two variables y and z to be constant. Similarly $\frac{\partial V}{\partial y}$ and $\frac{\partial V}{\partial z}$.

Only one variable In this case,

$$E = -\frac{dV}{dx} = -\frac{dV}{dr} \quad \text{or} \quad E = (-\text{ slope of } V\text{-}x) \text{ or } (-\text{ Slope of } V\text{-}r \text{ graph})$$

Conversion of E Function into *V* Function

In this case, integration will be required. In integration, we will be required value of V at some given position (also called limit) to get complete V function. Otherwise an unknown constant of integration remains in the expression. One known limit is $V = 0$ at infinity. This limit may not be given in the question.

More than one variables In this case, $\boxed{dV = -\mathbf{E} \cdot d\mathbf{r}}$

or
$$\int_a^b dV = -\int_a^b \mathbf{E} \cdot d\mathbf{r}$$

$\Rightarrow$
$$V_b - V_a = -\int_a^b \mathbf{E} \cdot d\mathbf{r}$$

Here, **E** will be given in the question and $d\mathbf{r}$ is a standard vector given by

$$d\mathbf{r} = dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}} + dz\,\hat{\mathbf{k}}$$

Only one Variable

$$\boxed{\int dV = -\int E\,dr}$$

or
$$\int_a^b dV = -\int_a^b E\,dr$$

$\Rightarrow$
$$V_b - V_a = -\int_a^b E\,dr$$

Here, E as a function of r will be given in the question. We may also write the above equation as

$$\int dV = -\int E\,dx$$

where, E is a function of x.

Example 13.14 *Gravitational potential in x-y plane varies with x and y coordinates as*

$$V = x^2 y + 2xy$$

Find gravitational field strength **E**.

Solution Gravitational field strength is given by

$$\mathbf{E} = -\left[\frac{\partial V}{\partial x}\hat{\mathbf{i}} + \frac{\partial V}{\partial x}\hat{\mathbf{j}}\right]$$

For the given potential function

$$\frac{\partial V}{\partial x} = (2xy + 2y)$$

and
$$\frac{\partial V}{\partial y} = x^2 + 2x$$

$\therefore$
$$\mathbf{E} = -[(2xy + 2y)\hat{\mathbf{i}} + (x^2 + 2x)\hat{\mathbf{j}}]$$
Ans.

Example 13.15 *Gravitational potential at a distance 'r' from a point mass 'm' is*

$$V = -\frac{Gm}{r}$$

Find gravitational field strength at that point.

Solution $E = -\dfrac{dV}{dr}$

$$= -\frac{d}{dr}\left(-\frac{Gm}{r}\right) = -\frac{Gm}{r^2} \quad \text{or} \quad |E| = \frac{Gm}{r^2}$$

Negative sign implies that direction of **E** is towards the point mass.

Example 13.16 *Gravitational potential varies along x-axis as shown in figure.*

(a) Plot E versus x graph corresponding to given V-x graph.

(b) A mass of 2 kg is kept at x = 3 m. Find gravitational force on it.

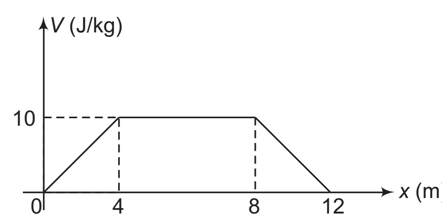


Fig. 13.38

Solution (a) $E = -\dfrac{dV}{dx} = -$ slope of V-x graph.

From $x = 0$ to $x = 4$ m

$$\text{Slope} = +\frac{10}{4} = +2.5 \text{ N/kg}$$

$\therefore$
$$E = -2.5 \text{ N/kg}$$

From $x = 4$ m to $x = 8$ m

$$\text{Slope} = 0$$

$\therefore$
$$E = 0$$

From $x = 8$ m to $x = 12$ m

$$\text{Slope} = -\frac{10}{4} = -2.5 \text{ N/kg}$$

$\therefore \quad E = +2.5$ N/kg

Therefore, E-x graph is as shown in figure.

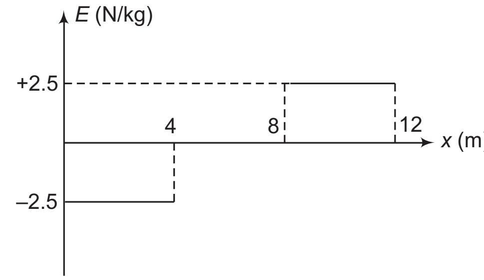


Fig. 13.39

(b) At $x = 3$ m, $E = -2.5$ N/kg

$\therefore$ Gravitational force

$$F = mE \qquad \left(\text{as } E = \frac{F}{m}\right)$$

$$= (2)(-2.5) = -5 \text{ N}$$ **Ans.**

Here, negative sign implies that this force is acting towards negative x-direction.

Example 13.17 *Gravitational field in x-y plane is given as*

$$\mathbf{E} = (2x\hat{\mathbf{i}} + 3y^2\hat{\mathbf{j}})\ N/kg$$

Find difference in gravitation potential between two points A and B, where co-ordinates of A and B are (2m, 4m) and (6m, 0).

Solution We know that, $\quad dV = -\mathbf{E} \cdot d\mathbf{r} \quad$ ...(i)

Here $\quad d\mathbf{r} = dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}}$

$\therefore \quad \mathbf{E} \cdot d\mathbf{r} = (2x\,dx + 3y^2\,dy)$

Now, Eq. (i) can be written as

$$\int_B^A dV = -\int_B^A (2x\,dx + 3y^2\,dy)$$

or

$$V_A - V_B = -[x^2 + y^3]_{(6\text{m},\ 0)}^{(2\text{m},\ 4\text{m})}$$

$$= -[\{(2)^2 + (4)^3\} - (6)^2 + (0)^3]$$

$$= -32 \text{ J/kg}$$ **Ans.**

INTRODUCTORY EXERCISE 13.4

1. The gravitational potential due to a mass distribution is $V = 3x^2y + y^3z$. Find the gravitational field.
2. Gravitational potential at $x = 2$ m is decreasing at a rate of 10 J/kg-m along the positive x-direction. It implies that the magnitude of gravitational field at $x = 2$ m is also 10 N/Kg. Is this statement true or false?
3. The gravitational potential in a region is given by, $V = 20(x + y)$ J/kg. Find the magnitude of the gravitational force on a particle of mass 0.5 kg placed at the origin.
4. The gravitational field in a region is given by $\mathbf{E} = (2\hat{\mathrm{i}} + 3\hat{\mathrm{j}})$ N/kg.

   Find the work done by the gravitational field when a particle of mass 1 kg is moved on the line $3y + 2x = 5$ from (1m, 1m) to (–2m, 3m).

13.7 Gravitational Potential Energy

The concept of potential energy has already been discussed in the chapter of work, energy and power. The word potential energy is defined only for a conservative force field. There we have discussed that the change in potential energy (dU) of a system corresponding to a conservative force is given by

$$dU = -\mathbf{F}\cdot d\mathbf{r} \quad \text{or} \quad \int_i^f dU = -\int_{\mathbf{r}_i}^{\mathbf{r}_f} \mathbf{F}\cdot d\mathbf{r} \quad \text{or} \quad U_f - U_i = -\int_{\mathbf{r}_i}^{\mathbf{r}_f} \mathbf{F}\cdot d\mathbf{r}$$

We generally choose the reference point at infinity and assume potential energy to be zero there, i.e. if we take $r_i = \infty$ (infinite) and $U_i = 0$, then we can write

$$\boxed{U = -\int_{\infty}^{\mathbf{r}} \mathbf{F}\cdot d\mathbf{r} = -W}$$

or potential energy of a body or system is negative of work done by the conservative forces in bringing it from infinity to the present position.

Gravitational Potential Energy of a two Particle System

The gravitational potential energy of two particles of masses m_1 and m_2 separated by a distance r is given by,

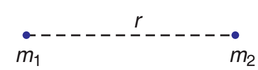


Fig. 13.40

$$\boxed{U = -\frac{Gm_1 m_2}{r}}$$

This is actually the negative of work done in bringing those masses from infinity to a distance r by the gravitational forces between them.

Gravitational Potential Energy for a System of Particles

The gravitational potential energy for a system of particles (say m_1, m_2, m_3 and m_4) is given by

$$\boxed{U = -G\left[\frac{m_4 m_3}{r_{43}} + \frac{m_4 m_2}{r_{42}} + \frac{m_4 m_1}{r_{41}} + \frac{m_3 m_2}{r_{32}} + \frac{m_3 m_1}{r_{31}} + \frac{m_2 m_1}{r_{21}}\right]}$$

Thus, for a n particle system there are $\dfrac{n(n-1)}{2}$ pairs and the potential energy is calculated for each pair and added to get the total potential energy of the system.

Gravitational Potential Energy of a Body on Earth's Surface

The gravitational potential energy of mass m in the gravitational field of mass M at a distance r from it is,

$$U = -\frac{GMm}{r}$$

The earth behaves for all external points as if its mass M were concentrated at its centre. Therefore, a mass m near earth's surface may be considered at a distance R (the radius of earth) from M. Thus, the potential energy of the system will be

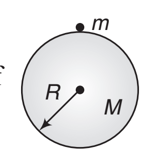


Fig. 13.41

$$U = -\frac{GMm}{R}$$

Difference in Potential Energy (ΔU)

Let us find the difference in potential energy in two positions shown in figure. The potential energy when the mass is on the surface of earth (at B) is,

$$U_B = -\frac{GMm}{R}$$

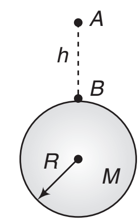


Fig. 13.42

and potential energy when the mass m is at height h above the surface of earth (at A) is,

$$U_A = -\frac{GMm}{R+h} \qquad (U_A > U_B)$$

$$\therefore \quad \Delta U = U_A - U_B$$

$$= -\frac{GMm}{R+h} - \left(-\frac{GMm}{R}\right)$$

$$= GMm\left(\frac{1}{R} - \frac{1}{R+h}\right)$$

$$= \frac{GMmh}{R(R+h)}$$

$$= \frac{GMmh}{R^2\left(1+\dfrac{h}{R}\right)} = \frac{mgh}{\left(1+\dfrac{h}{R}\right)} \qquad \left(\frac{GM}{R^2} = g\right)$$

$$\therefore \quad \boxed{\Delta U = \frac{mgh}{1+\dfrac{h}{R}}}$$

For $\quad h << R, \quad \Delta U \approx mgh$

Thus, mgh is the difference in potential energy (not the absolute potential energy), for $h << R$.

Extra Points to Remember

- **Maximum height attained by a particle**

  Suppose a particle of mass m is projected vertically upwards with a speed v and we want to find the maximum height h attained by the particle. Then we can use conservation of mechanical energy, i.e. Decrease in kinetic energy = increase in gravitational potential energy of particle.

  $$\therefore \quad \frac{1}{2}mv^2 = \Delta U$$

  or $$\frac{1}{2}mv^2 = \frac{mgh}{1+\dfrac{h}{R}}$$

  Solving this, we get $$\boxed{h = \frac{v^2}{2g - \dfrac{v^2}{R}}}$$

  From this, we can see that $h \approx \dfrac{v^2}{2g}$ if v is small.

Example 13.18 *Three masses of 1 kg, 2 kg and 3 kg are placed at the vertices of an equilateral triangle of side 1 m. Find the gravitational potential energy of this system.*
Take $G = 6.67 \times 10^{-11}$ $N\text{-}m^2/kg^2$

Solution $U = -G\left(\frac{m_3 m_2}{r_{32}} + \frac{m_3 m_1}{r_{31}} + \frac{m_2 m_1}{r_{21}}\right)$

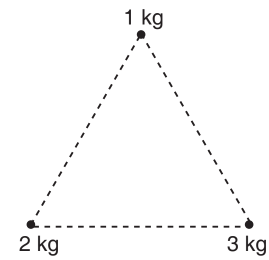


Fig. 13.43

Here, $r_{32} = r_{31} = r_{21} = 1.0$ m, $m_1 = 1\,\text{kg}$,
$m_2 = 2\,\text{kg}$
and $m_3 = 3\,\text{kg}$
Substituting the values, we get

$$U = -(6.67 \times 10^{-11})\left(\frac{3\times 2}{1} + \frac{3\times 1}{1} + \frac{2\times 1}{1}\right)$$

or $U = -7.337 \times 10^{-10}$ J **Ans.**

Example 13.19 *Eight particles of mass 'm' each are placed at the vertices of a cube of side 'a'. Find gravitational potential energy of this system.*

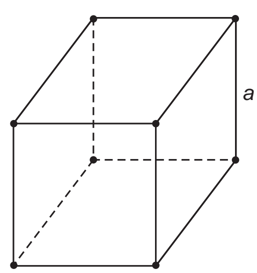


Fig. 13.44

Solution Total particles are $n = 8$

$\therefore$ Total number of pairs are $\frac{n(n-1)}{2}$ or 28.

In 12 pairs, distance between the particles is 'a'.
In 12 pairs, distance is $\sqrt{2}a$ and in remaining 4 pairs distance is $\sqrt{3}a$.

$$\therefore \quad U = 12\left(-\frac{Gmm}{a}\right) + 12\left(\frac{-Gmm}{\sqrt{2}a}\right) + 4\left(\frac{-Gmm}{\sqrt{3}a}\right)$$

$$= -\frac{Gm^2}{a}\left(12 + 6\sqrt{2} + \frac{4}{\sqrt{3}}\right)$$ **Ans.**

Example 13.20 *A particle of mass 'm' is raised from the surface of earth to a height $h = 2R$. Find work done by some external agent in this process. Here, R is the radius of earth and g the acceleration due to gravity on earth's surface.*

Solution No information is given about the change in kinetic energy of the particle. So, assuming change in kinetic energy to be zero.

$$\text{Work done by external agent} = \text{change in potential energy}$$

$$= \Delta U = \frac{mgh}{1+\dfrac{h}{R}}$$

Substituting $h = 2R$, we get, $\quad$ Work done $= \dfrac{2}{3} mgR$ **Ans.**

Example 13.21 *A particle is projected from the surface of the earth with an initial speed of 4.0 km/s. Find the maximum height attained by the particle. Radius of earth = 6400 km and $g = 9.8\ m/s^2$.*

Solution The maximum height attained by the particle is,

$$h = \frac{v^2}{2g - \dfrac{v^2}{R}}$$

Substituting the values, we have

$$h = \frac{(4.0\times 10^3)^2}{2\times 9.8 - \dfrac{(4.0\times 10^3)^2}{6.4\times 10^6}}$$

$$= 9.35\times 10^5 \text{ m} \quad \text{or} \quad h \approx 935 \text{ km}$$ **Ans.**

INTRODUCTORY EXERCISE 13.5

1. Two particles of masses 20 kg and 10 kg are initially at a distance of 1.0 m. Find the speeds of the particles when the separation between them decreases to 0.5 m, if only gravitational forces are acting.
2. Four particles each of mass m are kept at the four vertices of a square of side 'a'. Find gravitational potential energy of this system.
3. A particle of mass 'm' is raised to a height $h = R$ from the surface of earth. Find increase in potential energy. R = radius of earth. g = acceleration due to gravity on the surface of earth.
4. Show that if a body be projected vertically upward from the surface of the earth so as to reach a height nR above the surface
   (i) the increase in its potential energy is $\left(\dfrac{n}{n+1}\right) mgR$,
   (ii) the velocity with which it must be projected is $\sqrt{\dfrac{2ngR}{n+1}}$, where R is the radius of the earth and m the mass of body.
5. A projectile is fired vertically from the earth's surface with an initial speed of 10 km/s. Neglecting air drag, how high above the surface of earth will it go?
6. A particle is fired vertically upwards from earth's surface and it goes upto a maximum height of 6400 km. Find the initial speed of the particle.

13.8 Binding Energy

Total mechanical energy (potential + kinetic) of a closed system is negative. The modulus of this total mechanical energy is known as the binding energy of the system. This is the energy due to which system is closed or different parts of the system are bound to each other.

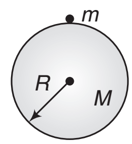


Fig. 13.45

Suppose the mass m is placed on the surface of earth. The radius of the earth is R and its mass is M. Then, the kinetic energy of the particle $K = 0$ and the potential energy is $U = -\frac{GMm}{R}$.

Therefore, the total mechanical energy is,

$$E = K + U = 0 - \frac{GMm}{R}$$

or $$E = -\frac{GMm}{R}$$

$\therefore$ $$\text{Binding energy} = |E| = \frac{GMm}{R}$$

It is due to this energy, the particle is attached with the earth. If minimum this much energy is given to the particle in any form (normally kinetic) the particle no longer remains attached to the earth. It goes out of the gravitational field of earth.

Escape Velocity

As we discussed above, the binding energy of a particle on the surface of earth kept at rest is $\frac{GMm}{R}$. If this much energy in the form of kinetic energy is supplied to the particle, it leaves the gravitational field of the earth. So, if v_e is the escape velocity of the particle, then

$$\frac{1}{2}mv_e^2 = \frac{GMm}{R} \quad \text{or} \quad v_e = \sqrt{\frac{2GM}{R}}$$

or $$\boxed{v_e = \sqrt{2gR}} \quad \text{as} \quad g = \frac{GM}{R^2}$$

Substituting the value of g (9.8 m/s^2) and R (6.4×10^6 m), we get

$$\boxed{v_e \approx 11.2 \text{ km/s}}$$

Thus, the minimum velocity needed to take a particle to infinity from the earth is called the escape velocity. On the surface of earth its value is 11.2 km/s.

Extra Points to Remember

- The value of escape velocity is 11.2 km/s from the surface of earth. From some height above the surface of earth this value will be less than 11.2 km/s.
- Escape velocity is independent of the direction in which it is projected. In the figure shown, body is given 11.2 km/s along three different paths. In each case, it will escape to infinity, but following different paths. For example, along path-1 it will follow a straight line.

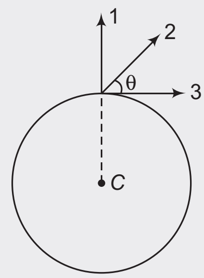


Fig. 13.46

- If velocity of a particle is v_e, then its total mechanical energy is zero. As the particle moves towards infinity its kinetic energy decreases and potential energy increases, but total mechanical energy remains constant. At any point

$$E = K + U = 0 \quad \Rightarrow \quad K = -U$$

For example, if $K = 100\,\text{J}$ on the surface of earth then $U = -100\,\text{J}$. At some height suppose K becomes 60 J, then U will become $-60\,\text{J}$. At infinity $K = 0$. So U is also zero. Hence, speed at infinity will be zero.

- If velocity of the particle is less than v_e then total mechanical energy is negative and it does not escape to infinity.
- If velocity of the particle is more than v_e then total mechanical energy is positive. Even at infinity some kinetic energy and speed are left in the particle. Although its potential energy becomes zero.

Example 13.22 *Calculate the escape velocity from the surface of moon. The mass of the moon is 7.4×10^{22} kg and radius $= 1.74 \times 10^6$ m.*

Solution Escape velocity from the surface of moon is $v_e = \sqrt{\dfrac{2GM_m}{R_m}}$

Substituting the values, we have

$$v_e = \sqrt{\frac{2 \times 6.67 \times 10^{-11} \times 7.4 \times 10^{22}}{1.74 \times 10^6}}$$

$$= 2.4 \times 10^3 \text{ m/s} = 2.4 \text{ km/s}$$ **Ans.**

Example 13.23 *Kinetic energy of a particle on the surface is E_0 and potential energy is $-\dfrac{E_0}{2}$.*

(a) Will the particle escape to infinity ?

(b) At some height its kinetic energy becomes $0.6E_0$. What is potential energy at this height ?

(c) If the particle escapes to infinity, what is kinetic energy of the particle at infinity?

Solution (a) Total mechanical energy

$$E = K + U = E_0 - \frac{E_0}{2} = 0.5E_0$$

Since, E is positive, particle will escape to infinity.

(b) Potential energy $U = E - K = 0.5E_0 - 0.6E_0 = -0.1E_0$ **Ans.**

(c) At infinity, $U = 0$

$\therefore$ $$K = E = 0.5E_0$$ **Ans.**

INTRODUCTORY EXERCISE 13.6

1. What is the kinetic energy needed to project a body of mass m from the surface of the earth to infinity? Radius of earth is R and acceleration due to gravity on earth's surface is g.
2. Mass and radius of a planet are two times the values of earth. What is the value of escape velocity from the surface of this planet ?
3. Kinetic energy of a particle on the surface of earth is E_0 and the potential energy is $-2E_0$.
   (a) Will the particle escape to infinity ?
   (b) What is the value of potential energy where speed of the particle becomes zero ?

13.9 Motion of Satellites

Just as the planets revolve around the sun, in the same way few celestial bodies revolve around these planets. These bodies are called 'Satellites'. For example moon is a satellite of earth. Artificial satellites are launched from the earth. Such satellites are used for telecommunication, weather forecast and other applications. The path of these satellites are elliptical with the centre of earth at a focus. However the difference in major and minor axes is so small that they can be treated as nearly circular for not too sophisticated calculations. Let us derive certain characteristics of the motion of satellites by assuming the orbit to be perfectly circular.

Orbital Speed

The necessary centripetal force to the satellite is being provided by the gravitational force exerted by the earth on the satellite. Thus,

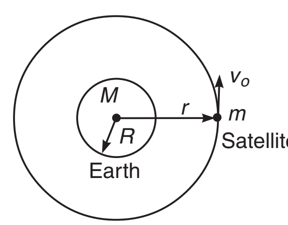


Fig. 13.47

$$\frac{mv_0^2}{r} = \frac{GMm}{r^2}$$

$$\therefore \qquad \boxed{v_0 = \sqrt{\frac{GM}{r}}} \quad \text{or} \quad v_0 \propto \frac{1}{\sqrt{r}}$$

Hence, the orbital speed (v_o) of the satellite decreases as the orbital radius (r) of the satellite increases. Further, the orbital speed of a satellite close to the earth's surface $(r \approx R)$ is,

$$\boxed{v_0 = \sqrt{\frac{GM}{R}} = \sqrt{gR} = \frac{v_e}{\sqrt{2}}}$$

Substituting $v_e = 11.2$ km/s $\Rightarrow$ $\boxed{v_0 = 7.9 \text{ km/s}}$

Period of Revolution

The period of revolution (T) is given by

$$T = \frac{2\pi r}{v_o} \quad \text{or} \quad T = \frac{2\pi r}{\sqrt{\frac{GM}{r}}}$$

or

$$T = 2\pi\sqrt{\frac{r^3}{GM}} \quad \text{or} \quad \boxed{T = 2\pi\sqrt{\frac{r^3}{gR^2}}} \qquad (\text{as } GM = gR^2)$$

Energy of Satellite

The potential energy of the system is

$$\boxed{U = -\frac{GMm}{r}}$$

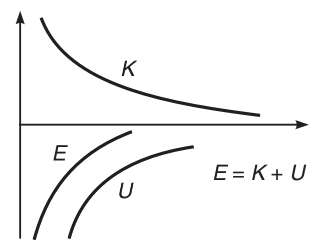


Fig. 13.48

The kinetic energy of the satellite is,

$$K = \frac{1}{2}mv_0^2 = \frac{1}{2}m\left(\frac{GM}{r}\right)$$

or

$$\boxed{K = \frac{1}{2}\frac{GMm}{r}}$$

The total energy is,

$$E = K + U = -\frac{GMm}{2r}$$

or

$$\boxed{E = -\frac{GMm}{2r}}$$

This energy is constant and negative, i.e. the system is closed. The farther the satellite from the earth the greater its total energy.

Extra Points to Remember

- $T = 2\pi\sqrt{\frac{r^3}{gR^2}} \Rightarrow T \propto r^{3/2}$ or $T^2 \propto r^3$ (which is also the Kepler's third law)
- Time period of a satellite very close to earth's surface $(r \approx R)$ is,

$$T = 2\pi\sqrt{\frac{R}{g}}$$

  Substituting the values, we get $T \approx 84.6$ min
- Suppose the height of a satellite is such that the time period of the satellite is 24 h and it moves in the same sense as the earth. The satellite will always be overhead a particular place on the equator. As seen from the earth, this satellite will appear to be stationary. Such a satellite is called a geostationary satellite. Putting $T = 24$ h in the expression of T, the radius of geostationary satellite comes out to be $r = 4.2 \times 10^4$ km. The height above the surface of earth is about 36000 km.
- The plane of orbit of a satellite always passes through the centre of the earth as force is always towards centre of earth. In case of geostationary satellite it is an equatorial plane (passing through equator)

Example 13.24 *As orbital radius r of a satellite is increased, state which of the following quantities will increase and which will decrease ?*

(i) Orbital speed *(ii) Time period*
(iii) Frequency *(iv) Angular speed*
(v) Kinetic energy *(vi) Potential energy*
(vii) Total mechanical energy

Solution (i) Orbital speed $v_o = \sqrt{\frac{GM}{r}}$ or $v_o \propto \frac{1}{\sqrt{r}}$.

Therefore, orbital speed will decrease.

(ii) $T = 2\pi\sqrt{\frac{r^3}{gR^2}}$ or $T \propto r^{3/2}$

Therefore, time period will increase.

(iii) Frequency, $f = \frac{1}{T}$

Time period is increasing. So, frequency will decrease.

(iv) Angular speed $\omega = \frac{2\pi}{T}$ or $\omega \propto \frac{1}{T}$

Time period is increasing. Hence, angular speed will decrease.

(v) Kinetic energy, $K = \frac{GMm}{2r}$ or $K \propto \frac{1}{r}$.

Therefore, kinetic energy will decrease.

(vi) Potential energy $U = -\frac{GMm}{r}$ or $U \propto -\frac{1}{r}$

Therefore, potential energy will increase.

(vii) Total mechanical energy, $E = -\frac{GMm}{2r}$ or $E \propto -\frac{1}{r}$

So, mechanical energy will also increase.

Example 13.25 *A geostationary satellite is orbiting the earth at a height of 6R above the surface of the earth where R is the radius of earth. The time period of another satellite at a distance of 3.5 R from the centre of the earth is .... hours.* **(JEE 1987)**

Solution $T \propto r^{\frac{3}{2}}$

$\therefore$ $$\frac{T_2}{T_1} = \left(\frac{r_2}{r_1}\right)^{\frac{3}{2}} \quad \text{or} \quad T_2 = \left(\frac{r_2}{r_1}\right)^{\frac{3}{2}} T_1$$

$$T_2 = \left(\frac{3.5R}{7R}\right)^{\frac{3}{2}}(24)\,\text{h} = 8.48\,\text{h} \quad (T_1 = 24\text{ h for geostationary satellite})$$

Example 13.26 *A spaceship is launched into a circular orbit close to the earth's surface. What additional velocity has now to be imparted to the spaceship in the orbit to overcome the gravitational pull. Radius of earth = 6400 km, g = 9.8 m/s².*

Solution The speed of the spaceship in a circular orbit close to the earth's surface is given by,

$$v_o = \sqrt{gR}$$

and escape velocity is given by, $v_e = \sqrt{2gR}$

∴ Additional velocity required to escape

$$v_e - v_o = \sqrt{2gR} - \sqrt{gR} = (\sqrt{2} - 1)\sqrt{gR}$$

Substituting the values of g and R, we get

$$v_e - v_o = 3.278 \times 10^3 \text{ m/s}$$ **Ans.**

Example 13.27 *What is the minimum energy required to launch a satellite of mass m from the surface of a planet of mass M and radius R in a circular orbit at an altitude of 2R?* **(JEE 2013, Main)**

(a) $\dfrac{5GmM}{6R}$ (b) $\dfrac{2GmM}{3R}$

(c) $\dfrac{GmM}{2R}$ (d) $\dfrac{GmM}{3R}$

Solution E = Energy of satellite – energy of mass on the surface of planet

$$= -\frac{GMm}{2r} - \left(-\frac{GMm}{R}\right)$$

Here, $r = R + 2R = 3R$

Substituting the values in above equation we get, $E = \dfrac{5GMm}{6R}$

The correct answer is (a)

.Example 13.28 *An artificial satellite is moving in a circular orbit around the earth with a speed equal to half the magnitude of escape velocity from the earth.* **(1990, 8M)**

(a) Determine the height of the satellite above the earth's surface.

(b) If the satellite is stopped suddenly in its orbit and allowed to fall freely onto the earth, find the speed with which it hits the surface of the earth.

Solution (a) Orbital speed of a satellite at distance r from centre of earth,

$$v_0 = \sqrt{\frac{GM}{r}} = \sqrt{\frac{GM}{R+h}} \quad \text{...(i)}$$

Given, $$v_0 = \frac{v_e}{2} = \frac{\sqrt{2GM/R}}{2} = \sqrt{\frac{GM}{2R}} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$h = R = 6400 \text{ km}$$

(b) Decrease in potential energy = increase in kinetic energy

or $$\frac{1}{2}mv^2 = \Delta U$$

$$\therefore \quad v = \sqrt{\frac{2(\Delta U)}{m}} = \sqrt{\frac{2\left(\frac{mgh}{1+h/R}\right)}{m}} = \sqrt{gR} \qquad (h = R)$$

$$= \sqrt{9.8 \times 6400 \times 10^3}$$

$$= 7919\,\text{m/s} = 7.9\,\text{km/s}$$ **Ans.**

INTRODUCTORY EXERCISE 13.7

1. Is it possible to have a geostationary satellite which always remains over New Delhi ?
2. Two satellites *A* and *B* revolve around a planet in two coplanar circular orbits in the same sense with radii 10^4 km and 2×10^4 km respectively. Time period of *A* is 28 hours. What is time period of another satellite?
3. Two satellites *A* and *B* of the same mass are orbiting the earth at altitudes *R* and 3*R* respectively, where *R* is the radius of the earth. Taking their orbits to be circular obtain the ratios of their kinetic and potential energies.
4. A satellite of mass 1000 kg is supposed to orbit the earth at a height of 2000 km above the earth's surface. Find (a) its speed in the orbit, (b) its kinetic energy, (c) the potential energy of the earth-satellite system and (d) its time period. Mass of the earth $= 6 \times 10^{24}$ kg.
5. A sky lab of mass 2×10^3 kg is first launched from the surface of earth in a circular orbit of radius 2*R* and then it is shifted from this circular orbit to another circular orbit of radius 3*R*. Calculate the energy required
   (a) to place the lab in the first orbit,
   (b) to shift the lab from first orbit to the second orbit. ($R = 6400$ km, $g = 10\,\text{m/s}^2$)

13.10 Kepler's Laws of Planetary Motion

Kepler discovered three empirical laws that accurately described the motion of the planets. The three laws may be stated as,

(i) Each planet moves in an elliptical orbit, with the sun at one focus of the ellipse. This law is also known as the **law of elliptical orbits** and obviously gives the shape of the orbits of the planets round the sun.

(ii) The radius vector, drawn from the sun to a planet, sweeps out equal areas in equal time, i.e. its areal velocity (or the area swept out by it per unit time) is constant. This is referred to as the **law of areas** and gives the relationship between the orbital speed of the planet and its distance from the sun.

(iii) The square of the planet's time period is proportional to the cube of the semi-major axis of its orbit. This is known as the **law of time period** and gives the relationship between the size of the orbit of a planet and its time of revolution.

Kepler did not know why the planets move in this way. Three generations later when Newton turned his attention to the motion of the planets, he discovered that each of Kepler's laws can be derived. They are consequences of Newton's law of motion and the law of gravitation.

Let us first consider the elliptical orbits described in Kepler's first law. Figure shows the geometry of the ellipse. The longest dimension is the major axis with half length *a*. This half length is called the semi-major axis.

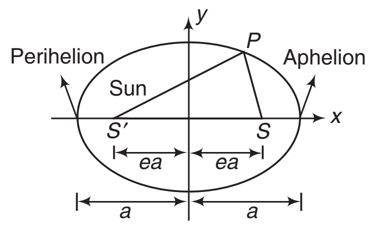


Fig. 13.49

$$SP + S'P = \text{constant}$$

Here, S and S' are the foci and P any point on the ellipse. The sun is at S and planet at P.

The distance of each focus from the centre of ellipse is ea, where e is the dimensionless number between 0 to 1 called the eccentricity. If $e = 0$, the ellipse is a circle. The actual orbits of the planets are nearly circular, their eccentricities range from 0.007 for Venus to 0.248 for Pluto. For earth $e = 0.017$. The point in the planet's orbit closest to the sun is the perihelion and the point most distant from the sun is aphelion.

Explanation of First Law

Newton was able to show that for a body acted on by an attractive force proportional to $\frac{1}{r^2}$, the only possible closed orbits are a circle or an ellipse. The open orbits must be parabolas or hyperbolas. He also showed that if total energy E is negative the orbit is an ellipse (or circle), if it is zero the orbit is a parabola and if E is positive the orbit is a hyperbola. Further, it was also shown that the orbits under the attractive force $F = \frac{K}{r^n}$, are stable for $n < 3$. Therefore, it follows that circular orbits will be stable for a force varying inversely as the distance or the square of the distance and will be unstable for the inverse cube (or a higher power) law.

Explanation of Second Law

$$PP' = v\,dt$$

$$P'M = (PP')\sin(180° - \theta) = PP'\sin\theta$$

$$= (v\sin\theta)\,dt$$

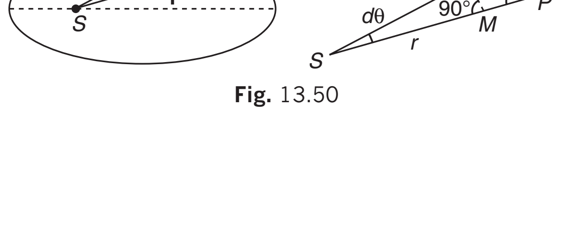


Fig. 13.50

Kepler's second law is shown in figure. In a small time interval dt, the line from the sun S to the planet P turns through an angle $d\theta$. The area swept out in this time interval is,

$$dA = \text{area of triangle shown in figure}$$
$$= \frac{1}{2}(\text{base})(\text{height})$$
$$= \frac{1}{2}(SP)(P'M)$$
$$= \frac{1}{2}(r)(v\sin\theta)\,dt$$

$\therefore$ Areal velocity $\dfrac{dA}{dt} = \dfrac{1}{2} rv\sin\theta$ ...(i)

Now, $rv\sin\theta$ is the magnitude of the vector product $\mathbf{r} \times \mathbf{v}$ which in turn is $\dfrac{1}{m}$ times the angular momentum $\mathbf{L} = \mathbf{r} \times m\mathbf{v}$ of the planet with respect to the sun. So we have,

$$\frac{dA}{dt} = \frac{1}{2m}|\mathbf{r} \times m\mathbf{v}| = \frac{L}{2m} \qquad \text{...(ii)}$$

or

$$\boxed{\frac{dA}{dt} = \frac{L}{2m} = \text{constant}}$$

Thus, Kepler's second law, that areal velocity is constant, means that angular momentum is constant. It is easy to see why the angular momentum of the planet must be constant. According to Newton's law the rate of change of $\mathbf{L}$ equals the torque of the gravitational force $\mathbf{F}$ acting on the planet,

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \mathbf{F} = \tau$$

Here, $\mathbf{r}$ is the radius vector of planet from the centre of the sun and the force $\mathbf{F}$ is directed from the planet towards the centre of the sun. So, these vectors always lie along the same line and their vector product $\mathbf{r} \times \mathbf{F}$ is zero. Hence, $\dfrac{d\mathbf{L}}{dt} = 0$ or $\mathbf{L} = \text{constant}$. Thus, from Eq. (ii) we can see that $\dfrac{dA}{dt} = \text{constant}$ if $\mathbf{L} = \text{constant}$. **Thus, second law is actually the law of conservation of angular momentum.**

Explanation of Third Law

In Article 13.9 we have already derived Kepler's third law for the particular case of circular orbits ($T^2 \propto r^3$). Newton was able to show that the same relationship holds for an elliptical orbit, with the orbit radius r replaced by semimajor axis a. Thus,

$$\boxed{T = \frac{2\pi a^{3/2}}{\sqrt{GM_s}}} \qquad \text{(elliptical orbit)}$$

Here, M_s is the mass of the sun.

Extra Points to Remember

- Most of the problems of planetary motion are solved by two conservation laws:
  (i) conservation of angular momentum about centre of the sun and
  (ii) conservation of mechanical (potential + kinetic) energy

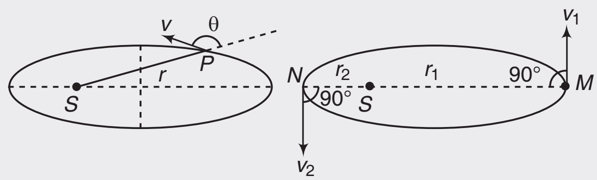


Fig. 13.51

Hence, the following two equations are used in most of the cases,

$$\boxed{\begin{aligned} mvr\sin\theta &= \text{constant} \\ \frac{1}{2}mv^2 - \frac{GMm}{r} &= \text{constant} \end{aligned}} \qquad \begin{aligned} &\ldots\text{(i)} \\ &\ldots\text{(ii)} \end{aligned}$$

At aphelion (or M) and perihelion (or N) positions $\theta = 90°$

Hence, Eq. (i) can be written as,

$$mvr \sin 90° = \text{constant}$$

or $$mvr = \text{constant} \qquad \ldots\text{(iii)}$$

Further, since mass of the planet (m) also remains constant, Eq. (i) can also be written as

$$vr \sin\theta = \text{constant} \qquad \ldots\text{(iv)}$$

or $$v_1 r_1 = v_2 r_2 \qquad (\theta = 90°)$$

$$r_1 > r_2$$

$\therefore$ $$v_1 < v_2$$

Example 13.29 *Name the physical quantities which remain constant in a planetary motion (in elliptical orbits).*

Solution Angular momentum about centre of sun and mechanical energy.

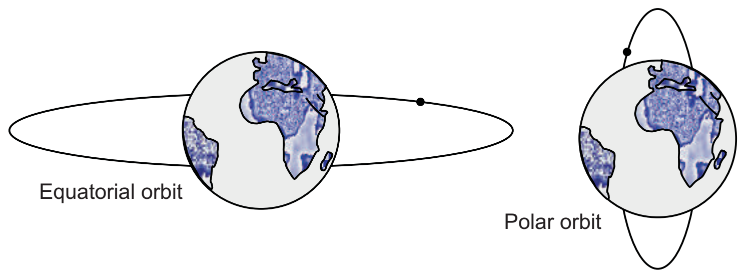
Example 13.30 *Consider a planet moving in an elliptical orbit round the sun. The work done on the planet by the gravitational force of the sun is zero in any small part of the orbit. Is this statement true or false ?*

Solution **False,** only at aphelion and perihelion positions **F** is perpendicular to d**S**. So, at these two positions work done by gravitational force is zero. At other points angle between **F** and d**S** is not 90°. So, work done is not zero. But, being a conservative force work done in a closed path (or in one full rotation) is zero.

Final Touch Points

Polar and Geostationary Satellites

1. **Satellites in low polar orbit** pass over the poles. They orbit between 100 km and 200 km above the Earth's surface, taking around 90 minutes to make each orbit. The earth spins beneath the satellites as it moves, so the satellite can scan the whole surface of the earth. Low orbit polar satellites have uses such as



- Monitoring the weather.
- Observing the earth's surface.
- Military uses including spying.

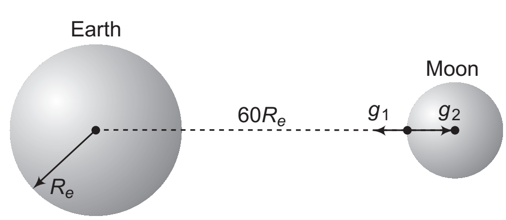
2. **Geostationary satellites** have a different trajectory to polar satellites. They are in orbit above the equator from west to east. The height of their orbit-36,000 km is just the right distance so that it takes them one day (24 hours) to make each orbit. This means that they stay in a fixed position over the earth's surface. A single geostationary satellite is on a line sight with about 40 percent of the earth's surface. Three such satellites, each separated by 120 degrees of longitude, can provide coverage of the entire planet. Geostationary satellites have uses such as:
   - communications- including satellites phones
   - global positioning or GPS.

   Geostationary satellites always appear in the same position when seen from the ground. This is why satellite television dishes can be bolted into one position and do not need to move.

3. Acceleration due to moon's gravity on moon's surface is $\frac{g_e}{6}$ because

$$\frac{M_m}{R_m^2} \approx \frac{1}{6}\frac{M_e}{R_e^2} \qquad \left(g = \frac{GM}{R^2}\right)$$

While acceleration due to earth's gravity on moon's surface is approximately $\frac{g_e}{(60)^2}$ or $\frac{g_e}{3600}$. This is because distance of moon from the earth's centre is approximately equal to 60 times the radius of earth and $g \propto \frac{1}{r^2}$. In the shown figure



.

$$g_1 = \frac{g_e}{(60)^2} \quad \text{while} \quad g_2 = \frac{g_e}{6}$$

4. Total energy of a closed system is always negative. For example, energy of planet-sun, satellite-earth or electron-nucleus system is always negative.

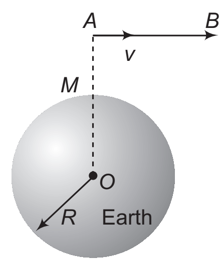
5. If the law of force obeys the inverse square law $\left(F \propto \frac{1}{r^2},\ F = \frac{-dU}{dr}\right)$

$$\boxed{K = \frac{|U|}{2} = |E|}$$

The same is true for electron-nucleus system because there also, the electrostatic force $F_e \propto \frac{1}{r^2}$.

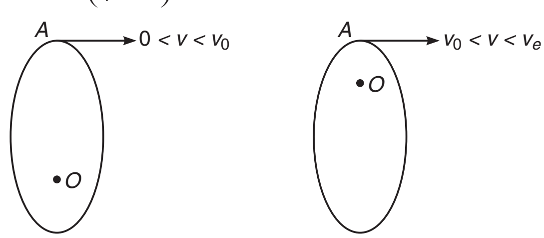
6. **Trajectory of a body projected from point *A* in the direction *AB* with different initial velocities :**

Let a body be projected from point A with velocity v in the direction AB. For different values of v the paths are different. Here, are the possible cases.

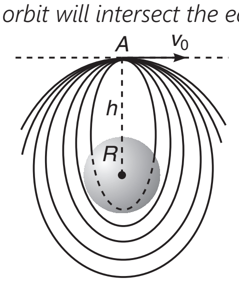


(i) If $v = 0$, path is a straight line from A to M.
(ii) If $0 < v < v_o$, path is an ellipse with centre O of the earth as a focus.
(iii) If $v = v_o$, path is a circle with O as the centre.
(iv) If $v_o < v < v_e$, path is again an ellipse with O as a focus.
(v) If $v = v_e$, body escapes from the gravitational pull of the earth and path is a parabola
(vi) If $v > v_e$, body again escapes but now the path is a hyperbola.

Here, v_o = orbital speed $\left(\sqrt{\frac{GM}{r}}\right)$ at A for a circular orbit and v_e = escape velocity from A.



Note 1. *From case (i) to (iv), total mechanical energy is negative. Hence, these are the closed orbits. For case (v), total energy is zero and for case (vi) total energy is positive. In these two cases orbits are open.*
2. *If v is not very large the elliptical orbit will intersect the earth and the body will fall back to earth.*



7. If $F \propto r^n$

then $T^2 \propto (r)^{1-n}$

and if $U \propto r^m$

then $T^2 \propto (r)^{2-m}$ (Applicable only for circular orbits)

8. $T = 2\pi\sqrt{\frac{R}{g}} \approx 84.6$ min comes in following four places in whole physics :

(i) If time period of rotation of earth becomes 84.6 min, effective value of g on equator becomes zero or we feel weightlessness on equator.

(ii) Time period of a satellite close to earth's surface is 84.6 min.

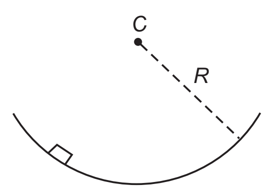
(iii) Time pendulum of a pendulum of infinite length is 84.6 min.

(iv) If a tunnel is dug along any chord of the earth and a particle is released from the surface of earth along this tunnel, then motion of this particle is simple harmonic and time period of this is also 84.6 min.

Note *(a) Points (iii) and (iv) come in the chapter of simple harmonic motion.*

(b) $T = 2\pi\sqrt{\frac{R}{g}}$ *is also the time period of small oscillations of a block inside a smooth spherical bowl of radius R.*

But this is not 84.6 min because here R is the radius of bowl not the radius of earth.



This expression can be compared with the time period of a pendulum $T = 2\pi\sqrt{\frac{l}{g}}$.

Solved Examples

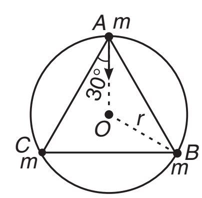
TYPED PROBLEMS

Type 1. *Two or more than two particles rotate in circular motion under their mutual gravitational attraction. We have to find speed of each particle.*

How to Solve?

Find net force on any one particle. It should come towards centre of the circle. This net force provides the necessary centripetal force.

Example 1 *Three particles each of mass m, are located at the vertices of an equilateral triangle of side a. At what speed must they move if they all revolve under the influence of their gravitational force of attraction in a circular orbit circumscribing the triangle while still preserving the equilateral triangle?*



Solution $\mathbf{F}_A = \mathbf{F}_{AB} + \mathbf{F}_{AC}$

$$= 2\left[\frac{Gm^2}{a^2}\right]\cos 30° = \sqrt{3}\left[\frac{Gm^2}{a^2}\right]$$

$$r = \frac{a}{\sqrt{3}},$$

Now $$\frac{mv^2}{r} = F \quad \text{or} \quad \frac{\sqrt{3}mv^2}{a} = \frac{\sqrt{3}Gm^2}{a^2}$$

$\therefore$ $$v = \sqrt{\frac{Gm}{a}}$$ **Ans.**

Example 2 *In the above problem, find total mechanical energy of the system.*

Solution Total mechanical energy,

$$E = U + K$$

where, U = potential energy of 3 identical pairs of masses 'm' each at a distance 'a'

$$= -3\left[\frac{Gmm}{a}\right] = -\frac{3Gm^2}{a}$$

K = kinetic energy of three particles

$$= 3\left[\frac{1}{2}mv^2\right] = \frac{3}{2}m\left(\sqrt{\frac{Gm}{a}}\right)^2 = \frac{3}{2}\frac{Gm^2}{a}$$

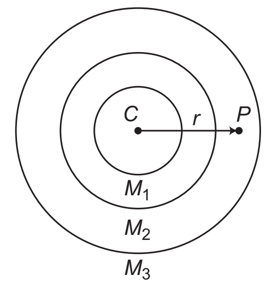
$\therefore$ $$E = U + K = -\frac{3Gm^2}{2a}$$ **Ans.**

Note *Total mechanical energy is negative, as the system is closed.*

Type 2. *Gravitational field and force on a mass due to spherical shells.*

Concept

According to Gauss theorem, net field strength at any point in the above situation is only due to the masses inside an imaginary spherical surface drawn at that point.



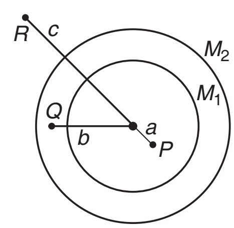
For example, field strength at P is

$$E_P = \frac{G(M_1 + M_2)}{r^2} \quad \text{(towards } C\text{)}$$

If a mass m is kept at P, then force on this mass is

$$F = mE_P = \frac{G(M_1 + M_2)m}{r^2} \quad \text{(towards } C\text{)}$$

Example 3 *Two concentric shells of masses M_1 and M_2 are concentric as shown. Calculate the gravitational force on m due to M_1 and M_2 at points P,Q and R.*



Solution At P, $F = 0$

At Q, $F = \dfrac{GM_1 m}{b^2}$

At R, $F = \dfrac{G(M_1 + M_2)m}{c^2}$

Type 3. *To find variation of time period of a satellite in circular orbit with its orbital radius 'r' if gravitational potential energy between two point masses varies as $U \propto r^n$ with negative sign.*

How to Solve?

From the relation between conservative force F and its potential energy U,

$$F = -\frac{dU}{dr}$$

First find the variation of F with r. Then, this force provides the necessary centripetal force. So, find variation of v with r by putting,

$$F = \frac{mv^2}{r}$$

Now, finally time period is given by $T = \dfrac{2\pi r}{v}$. So, find variation of T with r.

Example 4 *Gravitational potential energy between two points masses is*

$$U = -\frac{K\, m_1 m_2}{r^n}$$

where, K is a positive constant. With what power of 'r' time period of a satellite of mass 'm' varies in circular orbit if mass of planet is M ?

Solution $U = -\frac{KMm}{r^n}$

$$F = -\frac{dU}{dr} = \frac{KMmn}{r^{n+1}} \quad \text{or} \quad F \propto r^{-(n+1)}$$

Now, this force provides the necessary centripetal force.

$\therefore$ $$\frac{mv^2}{r} = \frac{KMm}{r^{n+1}} \quad \text{or} \quad v^2 \propto r^{-n}$$

$\therefore$ $$v \propto r^{\frac{-n}{2}}$$

Time period is given by

$$T = \frac{2\pi r}{v} \quad \text{or} \quad T \propto \frac{r}{v}$$

or $$T \propto \frac{r}{r^{\frac{-n}{2}}} \Rightarrow \therefore \quad T \propto r^{1+\frac{n}{2}}$$

Note *Under normal conditions*

$$U = -\frac{GMm}{r}$$

So, if we compare with the given equation then, $n = 1$.

Now, $$T \propto r^{1+\frac{n}{2}} \quad \text{or} \quad T \propto r^{\frac{3}{2}} \qquad \text{(for } n = 1\text{)}$$

Example 5 *Imagine a light planet revolving around a very massive star in a circular orbit of radius R with a period of revolution T. If the gravitational force of attraction between the planet and the star is proportional to $R^{-5/2}$, then*

(a) T^2 is proportional to R^2
(b) T^2 is proportional to $R^{7/2}$
(c) T^2 is proportional to $R^{3/2}$
(d) T^2 is proportional to $R^{3.75}$

Solution $\frac{mv^2}{R} \propto R^{-5/2}$

$\therefore$ $$v \propto R^{-3/4}$$

Now, $$T = \frac{2\pi R}{v} \quad \text{or} \quad T^2 \propto \left(\frac{R}{v}\right)^2$$

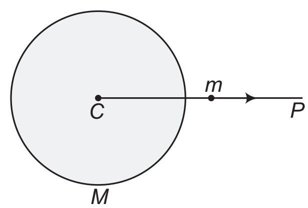
or $$T^2 \propto \left(\frac{R}{R^{-3/4}}\right)^2 \quad \text{or} \quad T^2 \propto R^{7/2}$$

$\therefore$ The correct answer is (b).

Type 4. *Based on conservation of mechanical energy.*

Concept

In the figure, $M >> m$, hence only the particle moves along the line CP. The spherical body remains at rest. The problem of any such type can be solved by energy conservation principle or



$$E_i = E_f$$

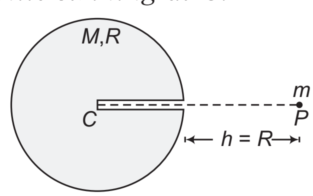
$$\Rightarrow \quad U_i + K_i = U_f + K_f$$

Here, $$U = -\frac{GMm}{r} \text{ for external points}$$

and $U = mV$ for internal points of sphere.

where, V is the gravitational potential due to M. In this type, we are using only one conservation law. Therefore, number of unknown should be only one.

Example 6 *There is a smooth tunnel upto centre C of a solid sphere of mass 'M' and radius R. A particle of mass $m\,(<< M)$ is released from point P along the line CP. Find velocity of 'm' while striking at C.*



Solution Using mechanical energy conservation equation.

$$E_C = E_P \quad \Rightarrow \quad K_C + U_C = K_P + U_P$$

$$\Rightarrow \quad \frac{1}{2} m v_C^2 + m V_C = 0 - \frac{GMm}{(R+R)} \qquad \ldots\text{(i)}$$

Here, V_C = potential at C due to mass $M = -\frac{3}{2}\frac{GM}{R}$

Substituting this value in Eq. (i) and then solving we get,

$$v_C = \sqrt{\frac{2GM}{R}} \qquad \textbf{Ans.}$$

Example 7 *A particle of mass 'm' is projected from the surface of earth with velocity $v = 2v_e$, where v_e is the value of escape velocity from the surface of earth. Find velocity of the particle on reaching to interstellar space (at infinity) in terms of v_e.*

Solution $v_e = \sqrt{\frac{2GM}{R}} \Rightarrow \frac{GM}{R} = \frac{v_e^2}{2}$ ...(i)

Using conservation of mechanical energy at the surface of earth and infinity.

We have, $K_f + U_f = K_i + U_i$

$\Rightarrow$ $\frac{1}{2}mv_\infty^2 + 0 = \frac{1}{2}m(2v_e)^2 - \frac{GMm}{R}$ ...(ii)

Substituting the value of $\frac{GM}{R} = \frac{v_e^2}{2}$

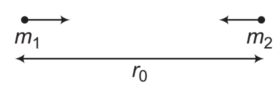
From Eq. (i) in Eq. (ii) we get, $v_\infty = \sqrt{3}\, v_e$

Alternate Method

On the surface of earth speed is two times the escape velocity. So, kinetic energy is four times. One kinetic energy is used in taking it to infinity. So, three kinetic energies are still left at infinity. In terms of speed it is $\sqrt{3}v_e$.

Type 5. *Based on conservation of mechanical energy and linear momentum.*

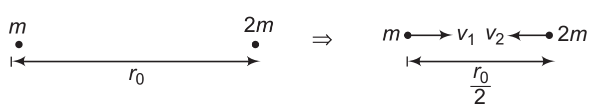
Concept



Two particles (or two spherical bodies) of masses m_1 and m_2 are free to move along the line joining them. This time their masses are comparable. Net force on the two particle system is zero.Therefore, linear momentum of the system may be conserved. Further, gravitational forces are conservative in nature.

Therefore, mechanical energy of the system is also conserved. Since, we are using two conservation laws, therefore number of unknowns are also two.

Example 8 *In the figure shown in the text, $m_1 = m$, $m_2 = 2m$ and initial distance between them is r_0. Find velocities of the masses when separation between them becomes $\frac{r_0}{2}$.*



Solution Let their velocities are v_1 and v_2. From conservation of linear momentum.

$$p_i = p_f$$

$\therefore$ $0 = mv_1 - 2mv_2$ ...(i)

From conservation of mechanical energy,

$$E_i = E_f$$

or $K_i + U_i = K_f + U_f$

or $0 - \frac{G(m)(2m)}{r_0} = \frac{1}{2}mv_1^2 + \frac{1}{2} \times 2m \times v_2^2 - \frac{G(m)(2m)}{(r_2/2)}$ ...(ii)

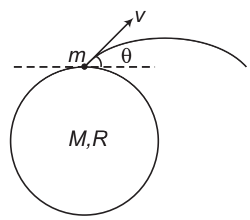
Solving Eqs. (i) and (ii), we get

$$v_1 = 2\sqrt{\frac{2Gm}{3r_0}},\ v_2 = \sqrt{\frac{2Gm}{3r_0}}$$ **Ans.**

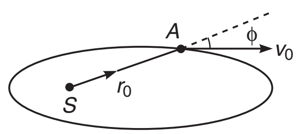
Type 6. *Based on conservation of angular momentum ($L = mvr \sin\theta$) and mechanical energy.*

Concept

In the figure shown, $M >> m$. Hence, the sphere of mass M is at rest and only the particle of mass 'm' will move. This particle is projected with velocity v at some angle θ from the surface as shown. Gravitational force on 'm' is always towards centre of 'M'. So, its torque about centre is always zero and angular momentum of particle about centre may be conserved. Further, the gravitational force, being a conservative force mechanical energy can also be conserved. We are applying two conservation laws, so number of unknown are also two.



Example 9 *Find the maximum and minimum distances of the planet A from the sun S, if at a certain moment of time it was at a distance r_0 and travelling with the velocity v_0, with the angle between the radius vector and velocity vector being equal to ϕ.*



Solution At minimum and maximum distances velocity vector ($\mathbf{v}$) makes an angle of 90° with radius vector. Hence, from conservation of angular momentum,

$$mv_0r_0 \sin\phi = mrv \quad \text{...(i)}$$

Here, m is the mass of the planet.

From energy conservation law, it follows that

$$\frac{mv_0^2}{2} - \frac{GMm}{r_0} = \frac{mv^2}{2} - \frac{GMm}{r} \quad \text{...(ii)}$$

Here, M is the mass of the sun.

Solving Eqs. (i) and (ii) for r, we get two values of r, one is r_{max} and another is r_{min}. So,

$$r_{max} = \frac{r_0}{2-K}\left(1 + \sqrt{1 - K(2-K)\sin^2\phi}\right)$$

and

$$r_{min} = \frac{r_0}{2-K}\left(1 - \sqrt{1 - K(2-K)\sin^2\phi}\right)$$

Here,

$$K = \frac{r_0^2 v_0^2}{GM}$$

Example 10 *A projectile of mass m is fired from the surface of the earth at an angle $\alpha = 60°$ from the vertical. The initial speed v_0 is equal to $\sqrt{\frac{GM_e}{R_e}}$. How high does the projectile rise ? Neglect air resistance and the earth's rotation.*

Solution Let v be the speed of the projectile at highest point and r_{max} its distance from the centre of the earth. Applying conservation of angular momentum and mechanical energy,

$$mv_0 \sin\alpha = mvr_{max} \sin 90°$$

or $$mv_0 \sin\alpha = mvr_{max} \quad \text{...(i)}$$

$$\frac{1}{2}mv_0^2 - \frac{GM_e m}{R_e} = \frac{1}{2}mv^2 - \frac{GM_e m}{r_{max}} \quad \text{...(ii)}$$

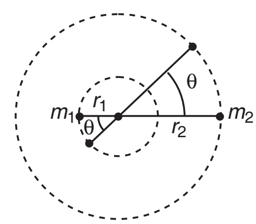
Solving these two equations with the given data we get,

$$r_{max} = \frac{3R_e}{2}$$

or the maximum height $$h_{max} = r_{max} - R_e = \frac{R_e}{2}$$ **Ans.**

Type 7. *Based on double star system.*

Concept



In motion of a planet round the sun we have assumed the mass of the sun to be too large in comparison to the mass of the planet. Under such situation the sun remains stationary and the planet revolves round the sun. If however masses of sun and planet are comparable and motion of sun is also to be considered, then both of them revolve around their centre of mass with same angular velocity but different linear speeds in the circles of different radii. The centre of mass remains stationary. This system of two stars is called a double star system.

We use following equations under this condition.

$$m_1 r_1 = m_2 r_2 \quad \text{...(i)}$$

$$m_1 r_1 \omega^2 = m_2 r_2 \omega^2 = \frac{Gm_1 m_2}{(r_1 + r_2)^2} \quad \text{...(ii)}$$

Solving these two equations, we can find that

$$\omega = \sqrt{\frac{GM}{r^3}} \quad \text{or} \quad T = \frac{2\pi r^{3/2}}{\sqrt{GM}}$$

Here, $M = m_1 + m_2$ and $r = r_1 + r_2$

Further, angular momentum of the system about COM

$$L = (I_1 + I_2)\,\omega = \left(\frac{m_1 m_2}{m_1 + m_2}\right) r^2\omega = \mu r^2 \omega$$

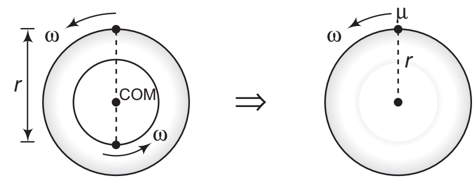
Kinetic energy of system, $K = \frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right) r^2\omega^2 = \frac{1}{2}\mu r^2\omega^2$

and moment of inertia of system,

$$I = \left(\frac{m_1 m_2}{m_1 + m_2}\right) r^2 = \mu r^2$$

Here, $$\mu = \frac{m_1 m_2}{m_1 + m_2} = \text{reduced mass}$$

Thus, the two bodies can be replaced by a single body whose mass is equal to reduced mass. This single body revolve in a circular orbit whose radius is equal to the distance between two bodies and centripetal force of circular motion is equal to force of interaction between two bodies for actual separation.



Example 11 *A planet of mass m_1 revolves round the sun of mass m_2. The distance between the sun and the planet is r. Considering the motion of the sun find the total energy of the system assuming the orbits to be circular.*

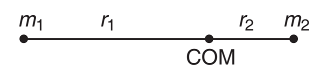
Solution Both the planet and the sun revolve around their centre of mass with same angular velocity (say ω)

$$r = r_1 + r_2 \quad \text{...(i)}$$

$$m_1 r_1 \omega^2 = m_2 r_2 \omega^2 = \frac{Gm_1 m_2}{r^2} \quad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we get

$$r_1 = r\left(\frac{m_2}{m_1 + m_2}\right)$$



$$r_2 = r\left(\frac{m_1}{m_1 + m_2}\right)$$

and $$\omega^2 = \frac{G(m_1 + m_2)}{r^3}$$

Now, total energy of the system is

$$E = \text{PE} + \text{KE} \quad \text{or} \quad E = -\frac{Gm_1 m_2}{r} + \frac{1}{2} m_1 r_1^2 \omega^2 + \frac{1}{2} m_2 r_2^2 \omega^2$$

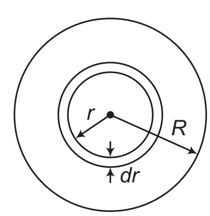
Substituting the values of r_1, r_2 and ω^2, we get

$$E = -\frac{Gm_1 m_2}{2r}$$ **Ans.**

Type 8. *To find gravitational field strength for spherical mass distribution when mass density is a function of r (not constant), where r is the distance from centre.*

Concept

Mass density of a solid sphere of radius R varies as $\rho = \rho_0 r^2$, where ρ_0 is a positive constant, we have to find variation of E with r.



According to Gauss theorem, the mass inside a spherical surface only contributes in the field strength on the surface of that sphere or at a distance 'r' from the centre

$$E = \frac{Gm_{\text{in}}}{r^2} \quad \text{...(i)}$$

where, m_{in} is the mass inside that sphere. Here, m_{in} will be obtained by integration as the mass density is not constant.

Let us take a small element of thickness dr at distance 'r'. Volume of this element.

$$dV = (4\pi r^2)dr$$

Mass density at distance r is

$$\rho = \rho_0 r^2$$

Therefore, small mass of this element

$$dm = \rho\, dV$$

or $$dm = (4\pi\rho_0 r^4)dr$$

∴ $$m_{in} = \int_0^r dm = \int_0^r (4\pi\rho_0 r^4)\, dr$$

or $$m_{in} = \frac{4\pi\rho_0}{5} r^5 \qquad \text{...(ii)}$$

Substituting this value of m_{in} in Eq. (i) we can find E.

Example 12 *In the problem discussed in the text, find E-r expressions for inside and outside points.*

Solution **Inside points ($r \le R$)**

Directly substituting value of m_{in} from Eq. (ii) in Eq. (i) we have,

$$E = \frac{G}{r^2}\left[\frac{4\pi\rho_0 r^5}{5}\right]$$

or $$E = \frac{4\pi G\rho_0 r^3}{5}$$

Outside points ($r \ge R$)

Mass is only upto $r = R$. So, substituting $r = R$ in Eq. (ii) we have,

$$m_{in} = \frac{4\pi\rho_0}{5} R^5$$

Now, substituting this value of m_{in} in Eq. (i), we have

$$E = \frac{G}{r^2}\left[\frac{4\pi\rho_0 R^5}{5}\right]$$

or $$E = \frac{4\pi\, G\rho_0 R^5}{5r^2}$$

Note *Direction of* **E** *is always towards the centre of the sphere.*

Type 9. *To draw E - r and V - r graphs due to two points masses, along the line joining two masses.*

Concept

(i) Expressions of E and V due to a point mass are

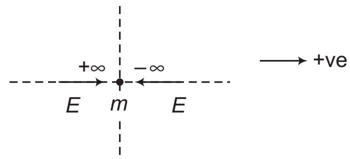
$$E = \frac{Gm}{r^2} \qquad \text{(towards the point mass)}$$

and $$V = -\frac{Gm}{r}$$

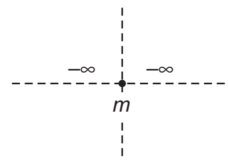
(ii) As $r \to \infty$, E and V both $\to 0$.

(iii) As $r \to 0$, $E \to \infty$ and $V \to -\infty$

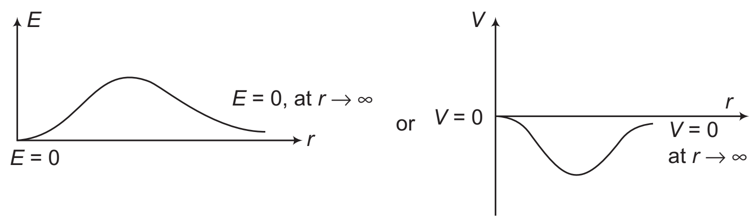
(iv) E is a vector quantity. On two sides of a point mass directions are different. So, on one side if value is $+\infty$ (just over the mass as $r \to 0$) and on its other side value will be $-\infty$.



(v) V is a scalar quantity. On both sides of the mass (as $r \to 0$) value will be $-\infty$.



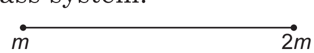
(vi) Between two zero values, we will get one maximum (or minimum value)



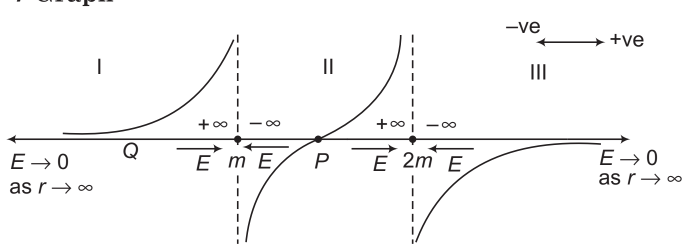
How to Solve?

Just write down the values at $r \to \infty$ and $r \to 0$, then draw the graph with the help of five points discussed above.

Example 13 Two points masses 'm' and $2m$ are kept at certain distance as shown in figure. Draw E - r and V - r graphs along the line joining them corresponding to given mass system.



Solution **E - r Graph**

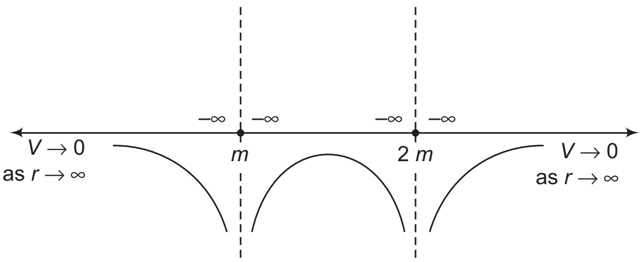


Region I Fields of m and $2m$ both are positive. Hence, net field at any point is also positive.

Region II Field at m is negative and field of $2m$ is positive. So, at some point P (nearer to m) two fields are equal and opposite and net field is zero.

Region III Fields of m and $2m$ both are negative. Hence, net field at any point is also negative.

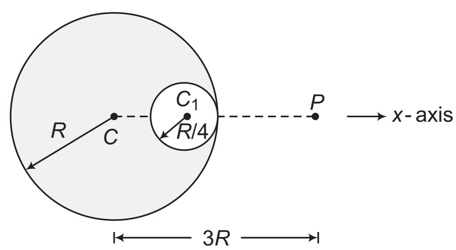
***V*-*r* Graph**



Gravitational potential is always negative on both sides. So, net potential at any point is always negative. Between two masses potential varies between $-\infty$ and $-\infty$. So, graph is as shown in figure.

Type 10. *To find value of E or V at some point due to a solid sphere having some cavity in it.*

How to Solve?



Suppose a solid sphere of mass M and radius R has a cavity of radius $\frac{R}{4}$ as shown in figure and we wish to find net gravitational field and potential at point P. Then,

$$\mathbf{E}_T = \mathbf{E}_R + \mathbf{E}_C$$

$\therefore$
$$\mathbf{E}_R = \mathbf{E}_T - \mathbf{E}_C \quad \ldots(\text{i})$$

Similarly,
$$V_R = V_T - V_C \quad \ldots(\text{ii})$$

Here, R stands for remaining mass, T for total mass and C for cavity.

Example 14 *In the problem discussed in the text, find the values of E and V at P due to the remaining mass.*

Solution Total mass is M of volume $\frac{4}{3}\pi R^3$. Therefore, mass of cavity of volume $\frac{4}{3}\pi\left(\frac{R}{4}\right)^3$ will be $\frac{M}{64}$.

Further,
$$C_1P = 2R + \frac{R}{4} = \frac{9R}{4}$$

Using Eq. (i),
$$\mathbf{E}_R = \frac{GM}{(3R)^2}(-\hat{\mathbf{i}}) - \frac{G\left(\frac{M}{64}\right)}{(9R/4)^2}(-\hat{\mathbf{i}})$$

Note *Field strength is always towards the centre.*

$$\therefore \qquad \mathbf{E}_R = \frac{35}{324}\frac{GM}{R^2}(-\hat{\mathbf{i}}) \qquad \textbf{Ans.}$$

$\therefore$ Field Strength is $\dfrac{35\,GM}{324\,R^2}$ towards C.

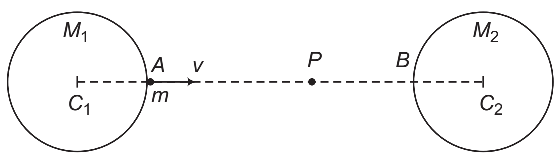
Using Eq. (ii), we have

$$V_R = -\frac{GM}{3R} - \left[\frac{-G\left(\frac{M}{64}\right)}{(9R/4)}\right]$$

$$= -\frac{47}{144}\frac{GM}{R} \qquad \textbf{Ans.}$$

Type 11. *To find minimum velocity required to project one particle from the surface of one planet to other planet.*

Concept



Suppose the particle of mass 'm' is projected from the surface of planet of mass M_1 and we wish to project it upto the surface of other planet. In between the two planets, there is a point P, where net field strength is zero. Or, net force on 'm' is zero.

From C_1 to P field strength of M_1 is stronger and net force on 'm' is towards C_1. At point P field strengths of both M_1 and M_2 are equal and opposite. Between C_2 and P field strengths of M_2 is stronger and net force on 'm' is towards C_2.

So, we have to project the particle only upto P. After P, it automatically moves towards the surface of B by the attraction of M_2.

How to Solve?

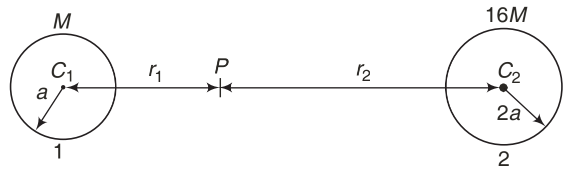
First find the point P, where net gravitational force on 'm' is zero. Then, apply energy conservation equation between A and P where,

$$v_A = \text{minimum velocity required and } v_P \text{ is tending to zero.}$$

Note *By applying energy conservation principle we can also find velocity at B, on reaching to the surface of B.*

Example 15 *Distance between the centres of two stars is $10a$. The masses of these stars are M and $16M$ and their radii a and $2a$ respectively. A body of mass m is fired straight from the surface of the larger star towards the surface of the smaller star. What should be its minimum initial speed to reach the surface of the smaller star? Obtain the expression in terms of G, M and a.* **(JEE 1996)**

Solution Let there are two stars 1 and 2 as shown below.

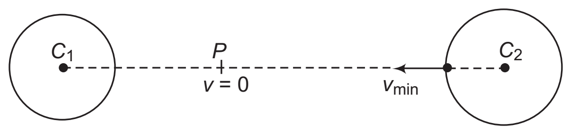


Let P is a point between C_1 and C_2, where gravitational field strength is zero or at P field strength due to star 1 is equal and opposite to the field strength due to star 2. Hence,

$$\frac{GM}{r_1^2} = \frac{G\,(16\,M)}{r_2^2} \quad \text{or} \quad \frac{r_2}{r_1} = 4 \quad \text{also} \quad r_1 + r_2 = 10\,a$$

$$\therefore \qquad r_2 = \left(\frac{4}{4+1}\right)(10\,a) = 8\,a \quad \text{and} \quad r_1 = 2a$$

Now, the body of mass m is projected from the surface of larger star towards the smaller one. Between C_2 and P it is attracted towards 2 and between C_1 and P it will be attracted towards 1. Therefore, the body should be projected to just cross point P because beyond that the particle is attracted towards the smaller star itself.



From conservation of mechanical energy $\frac{1}{2}mv^2_{\min}$

= Potential energy of the body at P – Potential energy at the surface of larger star.

$$\therefore \qquad \frac{1}{2}mv^2_{\min} = \left[-\frac{GMm}{r_1} - \frac{16\,GMm}{r_2}\right] - \left[-\frac{GMm}{10a - 2a} - \frac{16\,GMm}{2a}\right]$$

$$= \left[-\frac{GMm}{2a} - \frac{16\,GMm}{8a}\right] - \left[-\frac{GMm}{8a} - \frac{8\,GMm}{a}\right]$$

or $$\frac{1}{2}mv^2_{\min} = \left(\frac{45}{8}\right)\frac{GMm}{a} \Rightarrow \therefore\ v_{\min} = \frac{3\sqrt{5}}{2}\left(\sqrt{\frac{GM}{a}}\right)$$ **Ans.**

Type 12. *To find gravitational potential due to two or more than two spherical shells.*

Concept

At any point inside the shell (upto the surface).

$$V = -\frac{GM}{R} = \text{constant}$$

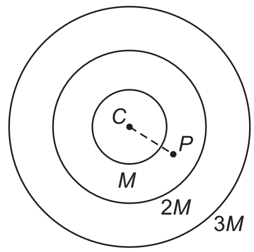
In the denominator, substitute R, radius of the shell.

At any point outside the shell,

$$V = -\frac{GM}{r} \quad \text{or} \quad V \propto -\frac{1}{r}$$

In the denominator, substitute r, actual distance of the point from the centre.

Example 16 *Three spherical shells of masses M, 2M and 3M have radii R, 3R and 4R as shown in figure. Find net potential at point P, where CP = 2R*



Solution Point P lies outside the innermost shell. So, in the denominator we will substitute actual distance of P from the centre or $r = CP = 2R$.

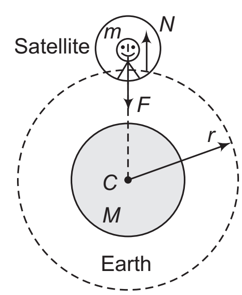
This point P lies inside the other two shells. So, we will substitute their radii, $3R$ and $4R$ in the denominator.

$$\therefore \qquad V_P = -\frac{GM}{2R} - \frac{G(2M)}{3R} - \frac{G(3M)}{4R} = -\frac{23\,GM}{12\,R} \qquad \textbf{Ans.}$$

Miscellaneous Examples

Example 17 *Explain the reason of weightlessness inside a satellite.*

Solution Feeling of weight is due to the normal reaction from the ground. Inside a satellite, this normal reaction becomes zero, which can be proved as given below.



Orbital speed is given by

$$v = \sqrt{\frac{GM}{r}} \qquad \ldots\text{(i)}$$

Two forces are acting on the person.

(i) F = gravitational force from earth

or
$$F = \frac{GMm}{r^2} \qquad \ldots\text{(ii)}$$

(ii) N = normal reaction

Person is also revolving in circular motion with same speed v. So, it needs a centripetal force.

$$\therefore \qquad F - N = \frac{mv^2}{r} \quad \text{or} \quad N = F - \frac{mv^2}{r}$$

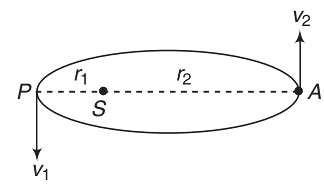
Substituting the values of F and v in Eqs. (i) and (ii) we have,

$$N = \frac{GMm}{r^2} - \frac{m}{r}\left(\sqrt{\frac{GM}{r}}\right)^2 = 0 \qquad \textbf{Hence Proved.}$$

Note *Moon is also an earth's satellite but there we don't feel weightlessness because mass of moon is large and we feel weight due to the gravitational force due to the moon (which is not insignificant like other satellite as its mass is less in comparison to the moon).*

Example 18 *Find the speeds of a planet of mass m in its perihelion and aphelion positions. The semimajor axis of its orbit is a, eccentricity is e and the mass of the sun is M. Also find the total energy of the planet in terms of the given parameters.*

Solution Let v_1 and v_2 be the speeds of the planet at perihelion and aphelion positions.



$$r_1 = a(1-e)$$

and $$r_2 = a(1+e) \quad \text{...(i)}$$

Applying conservation of angular momentum of the planet at P (perihelion) and A (aphelion)

$$mv_1 r_1 \sin 90° = mv_2 r_2 \sin 90°$$

or $$v_1 r_1 = v_2 r_2 \quad \text{...(ii)}$$

Applying conservation of mechanical energy in these two positions, we have

$$\frac{1}{2}mv_1^2 - \frac{GMm}{r_1} = \frac{1}{2}mv_2^2 - \frac{GMm}{r_2} \quad \text{...(iii)}$$

Solving Eqs. (i), (ii) and (iii), we get

$$v_1 = \sqrt{\frac{GM}{a}\left(\frac{1+e}{1-e}\right)} \quad \text{and} \quad v_2 = \sqrt{\frac{GM}{a}\left(\frac{1-e}{1+e}\right)}$$

Further, total energy of the planet

$$E = \frac{1}{2}mv_1^2 - \frac{GMm}{r_1} = \frac{1}{2}m\left[\frac{GM}{a}\left(\frac{1+e}{1-e}\right)\right] - \frac{GMm}{a(1-e)}$$

$$= \frac{GMm}{a(1-e)}\left[\left(\frac{1+e}{2}\right) - 1\right]$$

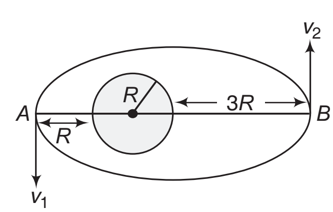
$$= \frac{GMm}{a(1-e)}\left(\frac{e-1}{2}\right) \quad \text{or} \quad E = -\frac{GMm}{2a}$$ **Ans.**

Example 19 *The minimum and maximum distances of a satellite from the centre of the earth are 2R and 4R respectively, where R is the radius of earth and M is the mass of the earth. Find*

(a) its minimum and maximum speeds,

(b) radius of curvature at the point of minimum distance.

Solution (a) Applying conservation of angular momentum



$$mv_1(2R) = mv_2(4R)$$

$$v_1 = 2v_2 \quad \text{... (i)}$$

From conservation of energy

$$\frac{1}{2}mv_1^2 - \frac{GMm}{2R} = \frac{1}{2}mv_2^2 - \frac{GMm}{4R} \quad \text{... (ii)}$$

Solving Eqs. (i) and (ii), we get

$$v_2 = \sqrt{\frac{GM}{6R}}, \quad v_1 = \sqrt{\frac{2GM}{3R}}$$

(b) If r is the radius of curvature at point A

$$\frac{mv_1^2}{r} = \frac{GMm}{(2R)^2}$$

$$r = \frac{4v_1^2R^2}{GM} = \frac{8R}{3} \qquad \text{(putting value of } v_1\text{)}$$

Example 20 *A planet of mass m revolves in elliptical orbit around the sun of mass M so that its maximum and minimum distances from the sun are equal to r_a and r_p respectively. Find the angular momentum of this planet relative to the sun.*

Solution Using conservation of angular momentum

$$mv_pr_p = mv_ar_a$$

As velocities are perpendicular to the radius vectors at apogee and perigee.

$$\Rightarrow \qquad v_pr_p = v_ar_a$$

Using conservation of energy,

$$-\frac{GMm}{r_p} + \frac{1}{2}mv_p^2 = \frac{-GMm}{r_a} + \frac{1}{2}mv_a^2$$

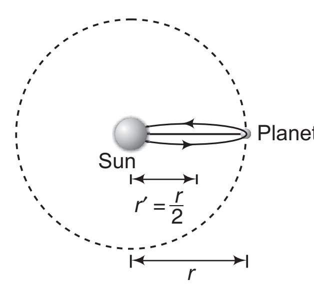
By solving, the above equations,

$$v_p = \sqrt{\frac{2GMr_a}{r_p(r_p + r_a)}}$$

$$L = mv_pr_p = m\sqrt{\frac{2GMr_pr_a}{(r_p + r_a)}}$$

Example 21 *If a planet was suddenly stopped in its orbit supposed to be circular, show that it would fall onto the sun in a time $\frac{\sqrt{2}}{8}$ times the period of the planet's revolution.*

Solution Consider an imaginary planet moving along a strongly extended flat ellipse, the extreme points of which are located on the planet's orbit and at the centre of the sun. The semi-major axis of the orbit of such a planet would apparently be half the semi-major axis of the planet's orbit. So, the time period of the imaginary planet T' according to Kepler's law will be given by



$$\left(\frac{T'}{T}\right) = \left(\frac{r'}{r}\right)^{\frac{3}{2}}$$

or $$T' = T\left(\frac{1}{2}\right)^{\frac{3}{2}} \qquad \left(\text{as} \quad r' = \frac{r}{2}\right)$$

$\therefore$ Time taken by the planet to fall onto the sun is

$$t = \frac{T'}{2} = \frac{T}{2}\left(\frac{1}{2}\right)^{\frac{3}{2}} \Rightarrow t = \frac{\sqrt{2}}{8}T$$

Example 22 *A satellite is revolving round the earth in a circular orbit of radius r and velocity v_0. A particle is projected from the satellite in forward direction with relative velocity $v = (\sqrt{5/4} - 1)\, v_0$. Calculate its minimum and maximum distances from earth's centre during subsequent motion of the particle.*

Solution $v_o = \sqrt{\dfrac{GM}{r}}$ = orbital speed of satellite ...(i)

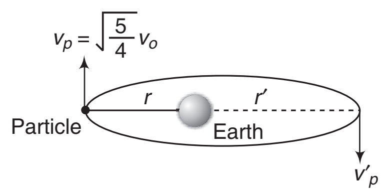
where, M = mass of earth.

Absolute velocity of particle would be

$$v_p = v + v_0 = \sqrt{\frac{5}{4}}\, v_0 = \sqrt{1.25}\, v_0 \qquad \text{...(ii)}$$

Since, v_p lies between orbital velocity and escape velocity, path of the particle would be an ellipse with r being the minimum distance.

Let r' be the maximum distance and v_p' its velocity at that moment.



Then, from conservation of angular momentum and conservation of mechanical energy, we get

$$m v_p r = m v_p' r' \qquad \text{...(iii)}$$

and
$$\frac{1}{2} m v_p^2 - \frac{GMm}{r} = \frac{1}{2} m v_p'^2 - \frac{GMm}{r'} \qquad \text{...(iv)}$$

Solving the above Eqs. (i), (ii), (iii) and (iv), we get

$$r' = \frac{5r}{3} \quad \text{and} \quad r$$

Hence, the maximum and minimum distances are $\dfrac{5r}{3}$ and r respectively.

Example 23 *An earth satellite is revolving in a circular orbit of radius a with velocity v_o. A gun is in the satellite and is aimed directly towards the earth. A bullet is fired from the gun with muzzle velocity $\dfrac{v_0}{2}$. Neglecting resistance offered by cosmic dust and recoil of gun, calculate maximum and minimum distance of bullet from the centre of earth during its subsequent motion.*

Solution Orbital speed of satellite is

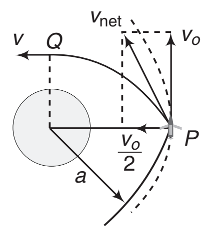
$$v_o = \sqrt{\frac{GM}{a}} \qquad \text{...(i)}$$

From conservation of angular momentum at P and Q, we have

$$m a v_o = m v r$$

or
$$v = \frac{a v_o}{r} \qquad \text{...(ii)}$$

From conservation of mechanical energy at P and Q, we have



$$\frac{1}{2}m\left(v_o^2+\frac{v_o^2}{4}\right)-\frac{GMm}{a}=\frac{1}{2}mv^2-\frac{GMm}{r}$$

or $$\frac{5}{8}v_o^2-\frac{GM}{a}=\frac{v^2}{2}-\frac{GM}{r}$$

Substituting values of v and v_o from Eqs. (i) and (ii), we get

$$\frac{5}{8}\frac{GM}{a}-\frac{GM}{a}=\frac{a^2}{r^2}\cdot\left(\frac{GM}{2a}\right)-\frac{GM}{r}$$

or $$-\frac{3}{8a}=\frac{a}{2r^2}-\frac{1}{r}$$

or $$-3r^2=4a^2-8ar$$

or $$3r^2-8ar+4a^2=0$$

or $$r=\frac{8a\pm\sqrt{64a^2-48a^2}}{6}$$

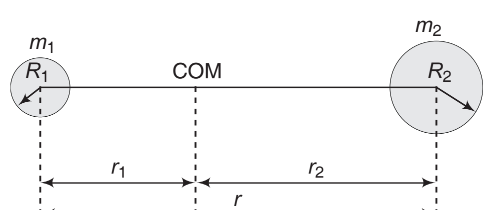
or $$r=\frac{8a\pm 4a}{6}$$

or $$r=2a \text{ and } \frac{2a}{3}$$

Hence, the maximum and minimum distances are $2a$ and $\frac{2a}{3}$ respectively.

Example 24 *Binary stars of comparable masses m_1 and m_2 rotate under the influence of each other's gravity with a time period T. If they are stopped suddenly in their motions, find their relative velocity when they collide with each other. The radii of the stars are R_1 and R_2 respectively. G is the universal constant of gravitation.*

Solution Both the stars rotate about their centre of mass (COM).



For the position of COM, $$\frac{r_1}{m_2}=\frac{r_2}{m_1}=\frac{r_1+r_2}{m_1+m_2}=\frac{r}{m_1+m_2} \qquad (r=r_1+r_2)$$

Also, $$m_1 r_1 \omega^2 = \frac{G m_1 m_2}{r^2}$$

or $$\omega^2 = \frac{G m_2}{r_1 r^2} \qquad \left(\omega = \frac{2\pi}{T}\right)$$

But, $$r_1 = \frac{m_2 r}{m_1 + m_2}$$

$\therefore$ $$\omega^2 = \frac{G(m_1 + m_2)}{r^3}$$

or $$r = \left\{\frac{G(m_1 + m_2)}{\omega^2}\right\}^{1/3} \qquad \text{...(i)}$$

Applying conservation of mechanical energy, we have

$$-\frac{G m_1 m_2}{r} = -\frac{G m_1 m_2}{(R_1 + R_2)} + \frac{1}{2}\mu v_r^2 \qquad \text{...(ii)}$$

Here, $$\mu = \text{reduced mass}$$
$$= \frac{m_1 m_2}{m_1 + m_2}$$

and v_r = relative velocity between the two stars.

From Eq. (ii), we find that

$$v_r^2 = \frac{2G m_1 m_2}{\mu}\left(\frac{1}{R_1 + R_2} - \frac{1}{r}\right)$$
$$= \frac{2G m_1 m_2}{\frac{m_1 m_2}{m_1 + m_2}}\left(\frac{1}{R_1 + R_2} - \frac{1}{r}\right)$$
$$= 2G(m_1 + m_2)\left(\frac{1}{R_1 + R_2} - \frac{1}{r}\right)$$

Substituting the value of r from Eq. (i), we get

$$v_r = \sqrt{2G(m_1 + m_2)\left[\frac{1}{R_1 + R_2} - \left\{\frac{4\pi^2}{G(m_1 + m_2)T^2}\right\}^{1/3}\right]}$$

Exercises

LEVEL 1

Assertion and Reason

Directions : *Choose the correct option.*

(*a*) *If both* ***Assertion*** *and* ***Reason*** *are true and the* ***Reason*** *is correct explanation of the* ***Assertion.***

(*b*) *If both* ***Assertion*** *and* ***Reason*** *are true but* ***Reason*** *is not the correct explanation of* ***Assertion.***

(*c*) *If* ***Assertion*** *is true, but the* ***Reason*** *is false.*

(*d*) *If* ***Assertion*** *is false but the* ***Reason*** *is true.*

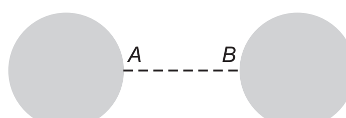
1. Assertion : When two masses come closer, their gravitational potential energy decreases.

Reason : Two masses attract each other.

2. Assertion : In moving from centre of a solid sphere to its surface, gravitational potential increases.

Reason : Gravitational field strength increases.

3. Assertion : There are two identical spherical bodies fixed in two positions as shown. While moving from A to B gravitational potential first increases then decreases.



Reason : At centre point of A and B field strength will be zero.

4. Assertion : If we plot potential *versus* x-coordinate graph along the x-axis, then field strength is zero where slope of V-x graph is zero.

Reason : If potential is function of x-only then

$$E = -\frac{dV}{dx}$$

5. Assertion : A particle is projected upwards with speed v and it goes to a height h. If we double the speed then it will move to height $4h$.

Reason : In case of earth, acceleration due to gravity g varies as

$$g \propto \frac{1}{r^2} \qquad \text{(for } r \geq R\text{)}$$

6. Assertion : In planetary motion angular momentum of planet about centre of sun remains constant. But linear momentum of system does not remain constant.

Reason : Net torque on planet about any point is zero.

7. Assertion : Plane of space satellite is always equatorial plane.

Reason : On the equator value of g is minimum.

8. **Assertion :** On satellites we feel weightlessness. Moon is also a satellite of earth. But we do not feel weightlessness on moon.

Reason : Mass of moon is considerable.

9. **Assertion :** Plane of geostationary satellites always passes through equator.

Reason : Geostationary satellites always lies above Moscow.

10. **Assertion :** It we double the circular radius of a satellite, then its potential energy, kinetic energy and total mechanical energy will become half.

Reason : Orbital speed of a satellite.

$$v \propto \frac{1}{\sqrt{r}}$$

where, r is its radius of orbit.

11. **Assertion :** If the radius of earth is decreased keeping its mass constant, effective value of g may increase or decrease at pole.

Reason : Value of g on the surface of earth is given by $g = \dfrac{GM}{R^2}$.

Objective Questions

Single Correct Option

1. A satellite orbiting close to the surface of earth does not fall down because the gravitational pull of earth

(a) is balanced by the gravitational pull of moon
(b) is balanced by the gravitational pull of sun
(c) provides the necessary acceleration for its motion along the circular path
(d) makes it weightless

2. For the planet-sun system identify the correct statement.

(a) the angular momentum of the planet is conserved about any point
(b) the total energy of the system is conserved
(c) the momentum of the planet is conserved
(d) All of the above

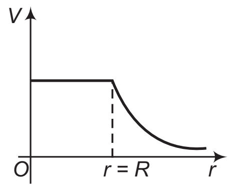
3. If the earth stops rotating about its axis, then the magnitude of gravity

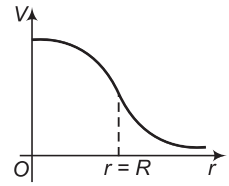
(a) increases everywhere on the surface of earth
(b) will increase only at the poles
(c) will not change at the poles
(d) All of the above

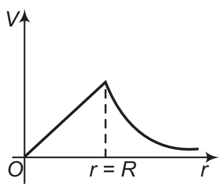
4. For a body to escape from earth, angle from horizontal at which it should be fired is

(a) 45°
(b) 0°
(c) 90°
(d) any angle

5. The correct variation of gravitational potential V with radius r measured from the centre of earth of radius R is given by

(a) 

(b) 

(c) 

(d) None of these

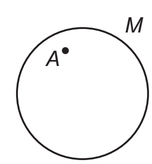
6. The Gauss' theorem for gravitational field may be written as

(a) $\oint \mathbf{g} \cdot d\mathbf{S} = \dfrac{m}{G}$ (b) $-\oint \mathbf{g} \cdot d\mathbf{S} = 4\pi mG$ (c) $\oint \mathbf{g} \cdot d\mathbf{S} = \dfrac{m}{4\pi G}$ (d) $-\oint \mathbf{g} \cdot d\mathbf{S} = \dfrac{m}{G}$

7. In the earth-moon system, if T_1 and T_2 are period of revolution of earth and moon respectively about the centre of mass of the system then

(a) $T_1 > T_2$ (b) $T_1 = T_2$ (c) $T_1 < T_2$ (d) Insufficient data

8. The figure shows a spherical shell of mass M. The point A is not at the centre but away from the centre of the shell. If a particle of mass m is placed at A, then

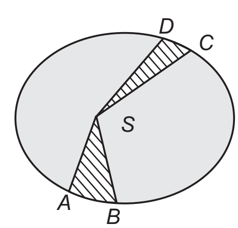


(a) it remains at rest
(b) it experiences a net force towards the centre
(c) it experiences a net force away from the centre
(d) None of the above

9. If the distance between the earth and the sun were reduced to half its present value, then the number of days in one year would have been

(a) 65 (b) 129
(c) 183 (d) 730

10. The figure represents an elliptical orbit of a planet around sun. The planet takes time T_1 to travel from A to B and it takes time T_2 to travel from C to D. If the area CSD is double that of area ASB, then



(a) $T_1 = T_2$
(b) $T_1 = 2T_2$
(c) $T_1 = 0.5T_2$
(d) Data insufficient

11. At what depth from the surface of earth the time period of a simple pendulum is 0.5% more than that on the surface of the Earth? (Radius of earth is 6400 km)

(a) 32 km (b) 64 km (c) 96 km (d) 128 km

12. If M is the mass of the earth and R its radius, the ratio of the gravitational acceleration and the gravitational constant is

(a) $\dfrac{R^2}{M}$ (b) $\dfrac{M}{R^2}$ (c) MR^2 (d) $\dfrac{M}{R}$

13. The height above the surface of earth at which the gravitational field intensity is reduced to 1% of its value on the surface of earth is

(a) $100R_e$ (b) $10R_e$ (c) $99R_e$ (d) $9R_e$

14. For a satellite orbiting close to the surface of earth the period of revolution is 84 min. The time period of another satellite orbiting at a height three times the radius of earth from its surface will be

(a) $(84)\,2\sqrt{2}$ min (b) 8 (84) min
(c) $(84)\,3\sqrt{3}$ min (d) 3 (84) min

15. The angular speed of rotation of earth about its axis at which the weight of man standing on the equator becomes half of his weight at the poles is given by

(a) 0.034 rad s^{-1} (b) 8.75×10^{-4} rad s^{-1}
(c) 1.23×10^{-2} rad s^{-1} (d) 7.65×10^{-7} rad s^{-1}

16. The height from the surface of earth at which the gravitational potential energy of a ball of mass m is half of that at the centre of earth is (where R is the radius of earth)

(a) $\frac{R}{4}$ (b) $\frac{R}{3}$ (c) $\frac{3R}{4}$ (d) $\frac{4R}{3}$

17. A body of mass m is lifted up from the surface of earth to a height three times the radius of the earth R. The change in potential energy of the body is

(a) $3mgR$ (b) $\frac{5}{4}mgR$
(c) $\frac{3}{4}mgR$ (d) $2mgR$

18. A satellite is revolving around earth in its equatorial plane with a period T. If the radius of earth suddenly shrinks to half without change in the mass. Then, the new period of revolution will be

(a) $8T$ (b) $2\sqrt{2}\,T$
(c) $2T$ (d) T

19. A planet has twice the density of earth but the acceleration due to gravity on its surface is exactly the same as that on the surface of earth. Its radius in terms of earth's radius R will be

(a) $R/4$ (b) $R/2$
(c) $R/3$ (d) $R/8$

20. The speed of earth's rotation about its axis is ω. Its speed is increased to x times to make the effective acceleration due to gravity equal to zero at the equator, then x is around ($g = 10$ ms^{-2}; $R = 6400$ km)

(a) 1 (b) 8.5
(c) 17 (d) 34

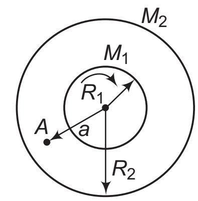
21. A satellite is seen every 6 h over the equator. It is known that it rotates opposite to that of earth's direction. Then, the angular velocity (in radian per hour) of satellite about the centre of earth will be

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{8}$

22. For a planet revolving around sun, if a and b are the respective semi-major and semi-minor axes, then the square of its time period is proportional to

(a) $\left(\frac{a+b}{2}\right)^3$ (b) $\left(\frac{a-b}{2}\right)^3$
(c) b^3 (d) a^3

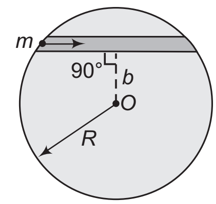
23. The figure represents two concentric shells of radii R_1 and R_2 and masses M_1 and M_2 respectively. The gravitational field intensity at the point A at distance a ($R_1 < a < R_2$) is



(a) $\frac{G(M_1 + M_2)}{a^2}$ (b) $\frac{GM_1}{a^2} + \frac{GM_2}{R_2^2}$

(c) $\frac{GM_1}{a^2}$ (d) zero

24. A straight tunnel is dug into the earth as shown in figure at a distance b from its centre. A ball of mass m is dropped from one of its ends. The time it takes to reach the other end is approximately



(a) 42 min (b) 84 min

(c) $84\left(\frac{b}{R}\right)$ min (d) $42\left(\frac{b}{R}\right)$ min

25. Three identical particles each of mass M are placed at the corners of an equilateral triangle of side l. The work done by external force to increase the side of triangle from l to $2l$ is

(a) $-\frac{3}{2}\frac{GM^2}{l}$ (b) $\frac{-3GM^2}{l}$

(c) $\frac{3}{2}\frac{GM^2}{l}$ (d) $\frac{3GM^2}{l}$

26. A particle is thrown vertically upwards from the surface of earth and it reaches to a maximum height equal to the radius of earth. The ratio of the velocity of projection to the escape velocity on the surface of earth is

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}$ (c) $\frac{1}{4}$ (d) $\frac{1}{2\sqrt{2}}$

27. The gravitational potential energy of a body at a distance r from the centre of earth is U. Its weight at a distance $2r$ from the centre of earth is

(a) $\frac{U}{r}$ (b) $\frac{U}{2r}$ (c) $\frac{U}{4r}$ (d) $\frac{U}{\sqrt{2}r}$

Subjective Questions

1. A particle of mass 1 kg is kept on the surface of a uniform sphere of mass 20 kg and radius 1.0 m. Find the work to be done against the gravitational force between them to take the particle away from the sphere.

2. What is the fractional decrease in the value of free-fall acceleration g for a particle when it is lifted from the surface to an elevation h? ($h << R$)

3. Two masses m_1 and m_2 at an infinite distance from each other are initially at rest, start interacting gravitationally. Find their velocity of approach when they are at a distance r apart.

4. If a satellite is revolving close to a planet of density ρ with period T, show that the quantity ρT^2 is a universal constant.

5. A satellite is revolving around a planet in a circular orbit. What will happen, if its speed is increased from v_0 to

(a) $\sqrt{1.5}\, v_0$ (b) $2v_0$

6. If the radius of the earth contracts to half of its present value without change in its mass, what will be the new duration of the day?

7. Two concentric spherical shells have masses m_1, m_2 and radii $R_1, R_2 (R_1 < R_2)$. Calculate the force exerted by this system on a particle of mass m, if it is placed at a distance $\frac{(R_1 + R_2)}{2}$ from the centre.

8. Find the force of attraction on a particle of mass m placed at the centre of a semicircular wire of length L and mass M.

9. A rocket is accelerated to speed $v = 2\sqrt{gR}$ near the earth's surface (R = radius of earth). Show that very far from earth its speed will be $v = \sqrt{2gR}$.

10. Two spheres one of mass M has radius R. Another sphere has mass $4M$ and radius $2R$. The centre to centre distance between them is $12R$. Find the distance from the centre of smaller sphere where

(a) net gravitational field is zero,

(b) net gravitational potential is half the potential on the surface of larger sphere.

11. A uniform solid sphere of mass M and radius a is surrounded symmetrically by a uniform thin spherical shell of equal mass and radius $2a$. Find the gravitational field at a distance (a) $\frac{3}{2}a$ from the centre, (b) $\frac{5}{2}a$ from the centre.

12. The density inside a solid sphere of radius a is given by $\rho = \rho_0 a/r$, where ρ_0 is the density at the surface and r denotes the distance from the centre. Find the gravitational field due to this sphere at a distance $2a$ from its centre.

13. Two neutron stars are separated by a distance of 10^{10} m. They each have a mass of 10^{30} kg and a radius of 10^5 m. They are initially at rest with respect to each other.

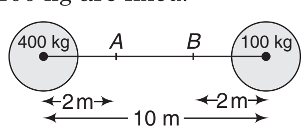
As measured from the rest frame, how fast are they moving when

(a) their separation has decreased to one-half its initial value,

(b) they are about to collide.

14. A mass m is taken to a height R from the surface of the earth and then is given a vertical velocity v. Find the minimum value of v, so that mass never returns to the surface of the earth. (Radius of earth is R and mass of the earth M).

15. In the figure masses 400 kg and 100 kg are fixed.



(a) How much work must be done to move a 1 kg mass from point A to point B?

(b) What is the minimum kinetic energy with which the 1 kg mass must be projected from A to the right to reach the point B?

16. Two identical stars of mass M orbit around their centre of mass. Each orbit is circular and has radius R, so that the two stars are always on opposite sides of the circle.

(a) Find the gravitational force of one star on the other.

(b) Find the orbital speed of each star and the period of the orbit.

(c) What minimum energy would be required to separate the two stars to infinity?

17. Consider two satellites A and B of equal mass, moving in the same circular orbit of radius r around the earth but in the opposite sense and therefore a collision occurs.

(a) Find the total mechanical energy $E_A + E_B$ of the two satellite-plus-earth system before collision.

(b) If the collision is completely inelastic, find the total mechanical energy immediately after collision. Describe the subsequent motion of the combined satellite.

18. In a certain binary star system, each star has the same mass as our sun. They revolve about their centre of mass. The distance between them is the same as the distance between earth and the sun. What is their period of revolution in years ?

19. (a) Does it take more energy to get a satellite upto 1500 km above earth than to put it in circular orbit once it is there.

(b) What about 3185 km?

(c) What about 4500 km? (Take $R_e = 6370$ km)

LEVEL 2

Objective Questions

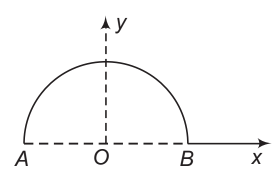
Single Correct Option

1. An artificial satellite of mass m is moving in a circular orbit at a height equal to the radius R of the earth. Suddenly due to internal explosion the satellite breaks into two parts of equal pieces. One part of the satellite stops just after the explosion. The increase in the mechanical energy of the system due to explosion will be
(Given, acceleration due to gravity on the surface of earth is g)

(a) mgR (b) $\frac{mgR}{2}$

(c) $\frac{mgR}{4}$ (d) $\frac{3mgR}{4}$

2. Gravitational field at the centre of a semicircle formed by a thin wire AB of mass M and length l is



(a) $\frac{GM}{l^2}$ along x-axis (b) $\frac{GM}{\pi l^2}$ along y-axis

(c) $\frac{2\pi GM}{l^2}$ along x-axis (d) $\frac{2\pi GM}{l^2}$ along y-axis

3. Four particles, each of mass M, move along a circle of radius R under the action of their mutual gravitational attraction. The speed of each particle is

(a) $\frac{GM}{R}$ (b) $\sqrt{2\sqrt{2}\frac{GM}{R}}$

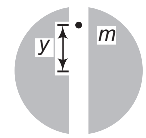
(c) $\sqrt{\frac{GM}{R}(2\sqrt{2}+1)}$ (d) $\sqrt{\frac{GM}{R}\frac{2\sqrt{2}+1}{4}}$

4. A projectile is fired from the surface of earth of radius R with a velocity kv_e (where v_e is the escape velocity from surface of earth and $k < 1$). Neglecting air resistance, the maximum height of rise from the centre of earth is

(a) $\dfrac{R}{k^2-1}$ (b) k^2R

(c) $\dfrac{R}{1-k^2}$ (d) kR

5. Suppose a vertical tunnel is dug along the diameter of earth, which is assumed to be a sphere of uniform mass density ρ. If a body of mass m is thrown in this tunnel, its acceleration at a distance y from the centre is given by



(a) $\dfrac{4\pi}{3} G\rho y m$ (b) $\dfrac{3}{4}\pi\rho y$

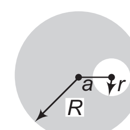
(c) $\dfrac{4}{3}\pi\rho y$ (d) $\dfrac{4}{3}\pi G\rho y$

6. A train of mass m moves with a velocity v on the equator from east to west. If ω is the angular speed of earth about its axis and R is the radius of the earth then the normal reaction acting on the train is

(a) $mg\left[1-\dfrac{(\omega R-2v)\,\omega}{g}-\dfrac{v^2}{Rg}\right]$ (b) $mg\left[1-2\dfrac{(\omega R-v)\,\omega}{g}-\dfrac{v^2}{Rg}\right]$

(c) $mg\left[1-\dfrac{(\omega R+2v)\,\omega}{g}-\dfrac{v^2}{Rg}\right]$ (d) $mg\left[1-2\dfrac{(\omega R-v)\,\omega}{g}-\dfrac{v^2}{Rg}\right]$

7. The figure represents a solid uniform sphere of mass M and radius R. A spherical cavity of radius r is at a distance a from the centre of the sphere. The gravitational field inside the cavity is



(a) non-uniform (b) towards the centre of the cavity

(c) directly proportional to a (d) All of these

8. If v_e is the escape velocity for earth when a projectile is fired from the surface of earth. Then, the escape velocity if the same projectile is fired from its centre is

(a) $\sqrt{\dfrac{3}{2}}\, v_e$ (b) $\dfrac{3}{2}\, v_e$

(c) $\sqrt{\dfrac{2}{3}}\, v_e$ (d) $\dfrac{2}{3}\, v_e$

9. If the gravitational field intensity at a point is given by $g = \dfrac{GM}{r^{2.5}}$. Then, the potential at a distance r is

(a) $\dfrac{-2GM}{3r^{1.5}}$ (b) $\dfrac{-GM}{r^{2.5}}$

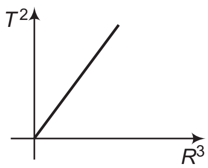
(c) $\dfrac{2GM}{3r^{1.5}}$ (d) $\dfrac{GM}{r^{3.5}}$

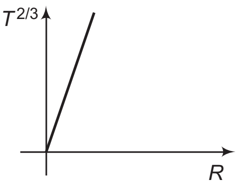
10. Three identical particles each of mass M move along a common circular path of radius R under the mutual interaction of each other. The velocity of each particle is

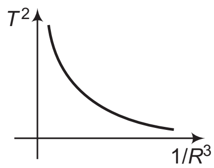
(a) $\sqrt{\dfrac{GM}{R}\sqrt{\dfrac{2}{3}}}$ (b) $\sqrt{\dfrac{GM}{\sqrt{3}R}}$

(c) $\sqrt{\dfrac{GM}{3R}}$ (d) $\sqrt{\dfrac{2}{3}\dfrac{GM}{R}}$

11. If T be the period of revolution of a planet revolving around sun in an orbit of mean radius R, then identify the **incorrect** graph.

(a) 

(b) 

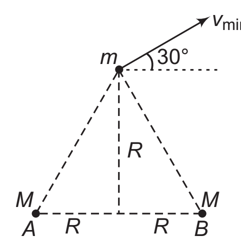
(c) 

(d) None of these

12. A person brings a mass of 1 kg from infinity to a point A. Initially, the mass was at rest but it moves at a speed of 3 m/s as it reaches A. The work done by the person on the mass is – 5.5 J. The gravitational potential at A is

(a) – 1 J/kg (b) – 4.5 J/kg (c) – 5.5 J/kg (d) – 10 J/kg

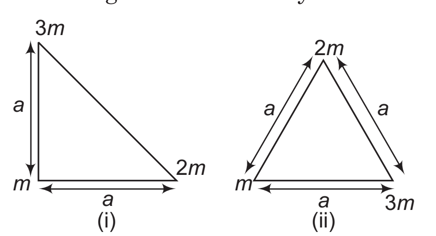
13. With what minimum speed should m be projected from point C in presence of two fixed masses M each at A and B as shown in the figure such that mass m should escape the gravitational attraction of A and B?



(a) $\sqrt{\frac{2GM}{R}}$ (b) $\sqrt{\frac{2\sqrt{2}GM}{R}}$

(c) $2\sqrt{\frac{GM}{R}}$ (d) $2\sqrt{2}\sqrt{\frac{GM}{R}}$

14. Consider two configurations of a system of three particles of masses m, $2m$ and $3m$. The work done by gravity in changing the configuration of the system from figure (i) to figure (ii) is



(a) zero (b) $\frac{6\,Gm^2}{a}\left\{1+\frac{1}{\sqrt{2}}\right\}$ (c) $\frac{6\,Gm^2}{a}\left\{1-\frac{1}{\sqrt{2}}\right\}$ (d) $\frac{6\,Gm^2}{a}\left\{2-\frac{1}{\sqrt{2}}\right\}$

15. A tunnel is dug along the diameter of the earth. There is a particle of mass m at the centre of the tunnel. Find the minimum velocity given to the particle so that is just reaches to the surface of the earth.(R = radius of earth)

(a) $\sqrt{\frac{GM}{R}}$

(b) $\sqrt{\frac{GM}{2R}}$

(c) $\sqrt{\frac{2GM}{R}}$

(d) it will reach with the help of negligible velocity

16. A body is projected horizontally from the surface of the Earth (radius = R) with a velocity equal to n times the escape velocity. Neglect rotational effects of the earth. The maximum height attained by the body from the earth's surface is $R/2$. Then, n must be

(a) $\sqrt{0.6}$ (b) $(\sqrt{3})/2$ (c) $\sqrt{0.4}$ (d) $1/2$

17. A tunnel is dug in the earth across one of its diameter. Two masses m and $2m$ are dropped from the two ends of the tunnel. The masses collide and stick each other. They perform SHM, the amplitude of which is (R = radius of earth)

(a) R (b) $R/2$ (c) $R/3$ (d) $2R/3$

18. There are two planets. The ratio of radius of the two planets is k but ratio of acceleration due to gravity of both planets is g. What will be the ratio of their escape velocity?

(a) $(kg)^{1/2}$ (b) $(kg)^{-1/2}$ (c) $(kg)^2$ (d) $(kg)^{-2}$

19. A body of mass 2 kg is moving under the influence of a central force whose potential energy is given by $U = 2r^3$ J. If the body is moving in a circular orbit of 5 m, its energy will be

(a) 625 J (b) 250 J (c) 500 J (d) 125 J

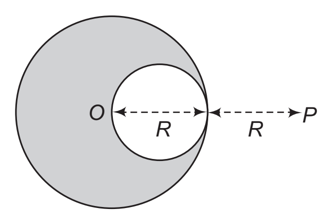
20. A research satellite of mass 200 kg circles the earth in an orbit of average radius $3R/2$, where R is the radius of the earth. Assuming the gravitational pull on the mass of 1 kg on the earth's surface to be 10 N, the pull on the satellite will be

(a) 1212 N (b) 889 N (c) 1280 N (d) 960 N

21. A satellite of mass m revolves around the earth of radius R at a height x from its surface. If g is the acceleration due to gravity on the surface of the earth, the orbital speed of the satellite is

(a) $\sqrt{gx}$ (b) $\sqrt{\dfrac{gR}{R-x}}$ (c) $\sqrt{\dfrac{gR^2}{R-x}}$ (d) $\sqrt{\dfrac{gR^2}{R+x}}$

22. A solid sphere of uniform density and radius R applies a gravitational force of attraction equal to F_1 on a particle placed at P, distance $2R$ from the centre O of the sphere. A spherical cavity of radius $R/2$ is now made in the sphere as shown in figure. The sphere with cavity now applies a gravitational force F_2 on same particle placed at P. The ratio F_2/F_1 will be



(a) 1/2 (b) 7/9
(c) 3 (d) 7

More than One Correct Options

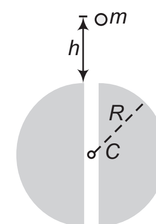
1. Three planets of same density have radii R_1, R_2 and R_3 such that $R_1 = 2R_2 = 3R_3$. The gravitational field at their respective surfaces are g_1, g_2 and g_3 and escape velocities from their surfaces are v_1, v_2 and v_3, then

(a) $g_1/g_2 = 2$ (b) $g_1/g_3 = 3$
(c) $v_1/v_2 = 1/4$ (d) $v_1/v_3 = 3$

2. For a geostationary satellite orbiting around the earth identify the necessary condition.

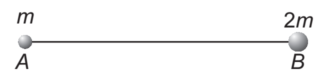
(a) it must lie in the equatorial plane of earth
(b) its height from the surface of earth must be 36000 km
(c) it period of revolution must be $2\pi\sqrt{\dfrac{R}{g}}$, where R is the radius of earth
(d) its period of revolution must be 24 hrs

3. A ball of mass m is dropped from a height h equal to the radius of the earth above the tunnel dug through the earth as shown in the figure. Choose the correct options. (Mass of earth = M)



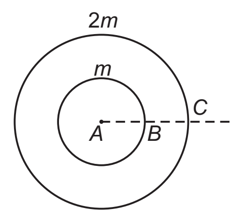
(a) Particle will oscillate through the earth to a height h on both sides
(b) Particle will execute simple harmonic motion
(c) Motion of the particle is periodic
(d) Particle passes the centre of earth with a speed $v = \sqrt{\dfrac{2GM}{R}}$

4. Two point masses m and $2m$ are kept at points A and B as shown. E represents magnitude of gravitational field strength and V the gravitational potential. As we move from A to B



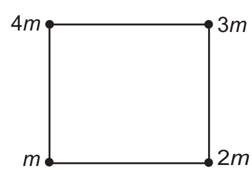
(a) E will first decrease then increases
(b) E will first increase then decrease
(c) V will first decrease then increase
(d) V will first increase then decrease

5. Two spherical shells have masses m and $2m$ as shown. Choose the correct options.



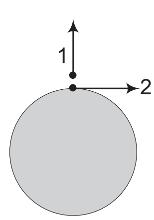
(a) Between A and B gravitational field strength is zero
(b) Between A and B gravitational potential is constant
(c) There will be two points one lying between B and C and other lying between C and infinity where gravitational field strength are same
(d) There will be a point between B and C where gravitational potential will be zero

6. Four point masses are placed at four corners of a square as shown. When positions of m and $2m$ are interchanged



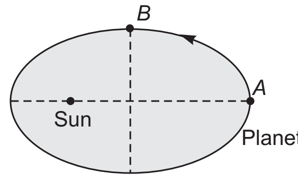
(a) gravitational field strength at centre will increase
(b) gravitational field strength at centre will decrease
(c) gravitational potential at centre will remain unchanged
(d) gravitational potential at centre will decrease

7. Two identical particles 1 and 2 are projected from surface of earth with same velocities in the directions shown in figure.



(a) Both the particles will stop momentarily (before striking with ground) at different times
(b) Particle-2 will rise upto lesser height compared to particle-2
(c) Minimum speed of particle-2 is more than that of particle-1
(d) Particle-1 will strike the ground earlier

8. A planet is moving round the sun in an elliptical orbit as shown. As the planet moves from A to B



(a) its kinetic energy will decrease
(b) its potential energy will remain unchanged
(c) its angular momentum about centre of sun will remain unchanged
(d) its speed is minimum at A

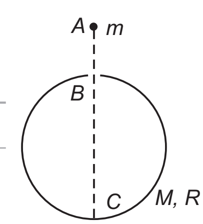
9. A satellite of mass m is just placed over the surface of earth. In this position mechanical energy of satellite is E_1. Now it starts orbiting round the earth in a circular path at height h = radius of earth. In this position, kinetic energy, potential energy and total mechanical energy of satellite are K_2, U_2 and E_2 respectively. Then

(a) $U_2 = \frac{E_1}{2}$ (b) $E_2 = \frac{E_1}{4}$ (c) $K_2 = -E_2$ (d) $K_2 = -\frac{U_2}{2}$

10. A satellite is revolving round the earth in circular orbit
(a) if mass of earth is made four times, keeping other factors constant, orbital speed of satellite will become two times
(b) corresponding to change in part (a), times period of satellite will remain half
(c) when value of G is made two times orbital speed increases and time period decreases
(d) G has no effect on orbital speed and time period

Match the Columns

1. There is a small hole in a spherical shell of mass M and radius R. A particle of mass m is dropped from point A as shown. Match the two columns for the situation shown in figure.



Column I	Column II
(a) Potential energy from A to B	(p) Continuously increases
(b) Potential energy from B to C	(q) Continuously decreases
(c) Speed of particle from B to C	(r) First increases then remains constant
(d) Acceleration of particle from A to C	(s) None of these

2. Five point masses m each are placed at five corners of a regular pentagon. Distance of any corner from centre is r. Match the following two columns.

Column I	Column II
(a) Gravitational field strength at centre	(p) Gm/r^2
(b) Gravitational potential at centre	(q) $4Gm/r$
(c) When one mass is removed gravitational field strength at centre	(r) zero
(d) When one mass is removed gravitational potential at centre	(s) None of these

3. | Potential | on the surface of a solid sphere is x and radius is y. Match the following two columns.

Column I	Column II
(a) Field strength at distance $2y$ from centre	(p) $\frac{x}{2y}$
(b) | Potential | at distance $\frac{y}{2}$ from centre	(q) $\frac{x}{2}$
(c) Field strength at distance $y/2$ from centre	(r) $\frac{x}{4y}$
(d) | Potential | at distance $2y$ from centre	(s) None

4. Match the following two columns.

Column I	Column II
(a) Work done is raising a mass m to a height $h = R$	(p) $\frac{1}{4}mgR$
(b) Kinetic energy of a satellite of mass m at height $h = R$	(q) mgR
(c) Difference in energies of two satellites each of mass m but one at height $h_1 = R$ and another of height $h_2 = 2R$	(r) $\frac{1}{2}mgR$
(d) Kinetic energy required to raise a particle of mass m to a height $h = R$ if projected vertically from surface of earth.	(s) None

5. Match the following two columns.

Column I	Column II
(a) Gravitational field strength is maximum at	(p) $r = 0$
(b) Gravitational field strength is zero at	(q) $r = R$
(c) Gravitational potential is minimum at	(r) $r = \frac{R}{\sqrt{2}}$
(d) Gravitational potential is zero at	(s) None of these

Here, r is distance from centre of a solid sphere or distance from centre of a ring along its axis.

Subjective Questions

1. Three particles of mass m each are placed at the three corners of an equilateral triangle of side a. Find the work which should be done on this system to increase the side of the triangle to $2a$.

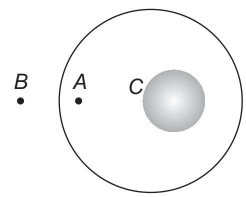
2. A man can jump vertically to a height of 1.5 m on the earth. Calculate the radius of a planet of the same mean density as that of the earth from whose gravitational field he could escape by jumping. Radius of earth is 6.41×10^6 m.

3. An artificial satellite is moving in a circular orbit around the earth with a speed equal to half the magnitude of escape velocity from the surface of earth. (Radius of earth = 6400 km)

(a) Determine the height of the satellite above the earth's surface.

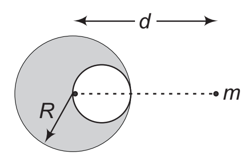
(b) If the satellite is stopped suddenly in its orbit and allowed to fall freely on the earth, find the speed with which it hits the surface of earth.

4. A uniform metal sphere of radius R and mass m is surrounded by a thin uniform spherical shell of same mass and radius $4R$. The centre of the shell C falls on the surface of the inner sphere. Find the gravitational fields at points A and B.

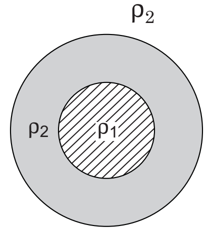


$AC = 3R, \quad BC = 5R$

5. Figure shows a spherical cavity inside a lead sphere. The surface of the cavity passes through the centre of the sphere and touches the right side of the sphere. The mass of the sphere before hollowing was M. With what gravitational force does the hollowed out lead sphere attract a particle of mass m that lies at a distance d from the centre of the lead sphere on the straight line connecting the centres of the spheres and of the cavity.



6. The density of the core of a planet is ρ_1 and that of the outer shell is ρ_2, the radii of the core and that of the planet are R and $2R$ respectively. The acceleration due to gravity at the surface of the planet is same as at a depth R. Find the ratio of $\frac{\rho_1}{\rho_2}$.



7. If a satellite is revolving around a planet of mass M in an elliptical orbit of semi-major axis a. Show that the orbital speed of the satellite when it is at a distance r from the focus will be given by

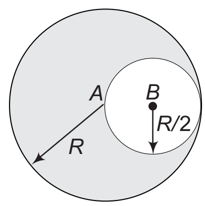
$$v^2 = GM\left[\frac{2}{r} - \frac{1}{a}\right]$$

8. A uniform ring of mass m and radius a is placed directly above a uniform sphere of mass M and of equal radius. The centre of the ring is at a distance $\sqrt{3}a$ from the centre of the sphere. Find the gravitational force exerted by the sphere on the ring.

9. Distance between the centres of two stars is $10a$. The masses of these stars are M and $16M$ and their radii a and $2a$ respectively. A body of mass m is fired straight from the surface of the larger star towards the smaller star. What should be its minimum initial speed to reach the surface of the smaller star? Obtain the expression in terms of G, M and a.

10. A smooth tunnel is dug along the radius of earth that ends at centre. A ball is released from the surface of earth along tunnel. Coefficient of restitution for collision between soil at centre and ball is 0.5. Calculate the distance travelled by ball just before second collision at centre. Given mass of the earth is M and radius of the earth is R.

11. Inside a fixed sphere of radius R and uniform density ρ, there is spherical cavity of radius $\frac{R}{2}$ such that surface of the cavity passes through the centre of the sphere as shown in figure. A particle of mass m_0 is released from rest at centre B of the cavity. Calculate velocity with which particle strikes the centre A of the sphere. Neglect earth's gravity. Initially sphere and particle are at rest.

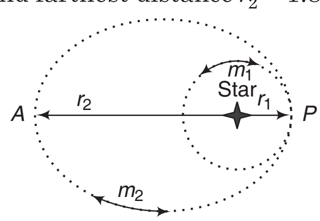


12. A ring of radius $R = 4$ m is made of a highly dense material. Mass of the ring is $m_1 = 5.4 \times 10^9$ kg distributed uniformly over its circumference. A highly dense particle of mass $m_2 = 6 \times 10^8$ kg is placed on the axis of the ring at a distance $x_0 = 3$ m from the centre. Neglecting all other forces, except mutual gravitational interaction of the two. Calculate

(i) displacement of the ring when particle is at the centre of ring, and

(ii) speed of the particle at that instant.

13. Two planets of equal mass orbit a much more massive star (figure). Planet m_1 moves in a circular orbit of radius 1×10^8 km with period 2 yr. Planet m_2 moves in an elliptical orbit with closest distance $r_1 = 1 \times 10^8$ km and farthest distance $r_2 = 1.8 \times 10^8$ km, as shown.



(a) Using the fact that the mean radius of an elliptical orbit is the length of the semi-major axis, find the period of m_2's orbit.

(b) Which planet has the greater speed at point P ? Which has the greater total energy ?

(c) Compare the speed of planet m_2 at P with that at A.

14. In a double star, two stars one of mass m_1 and another of mass m_2, with a separation d, rotate about their common centre of mass. Find

(a) an expression for their time period of revolution.

(b) the ratio of their kinetic energies.

(c) the ratio of their angular momenta about the centre of mass.

(d) the total angular momentum of the system.

(e) the kinetic energy of the system.

Answers

Introductory Exercise 13.1

1. $\dfrac{\sqrt{3}\,GM^2}{4a^2}$ **2.** $\dfrac{4\sqrt{2}\,Gm^2}{a^2}$ **3.** $a_1 = 5.3 \times 10^{-10}$ ms^{-2}, $a_2 = 2.65 \times 10^{-10}$ ms^{-2} **4.** $\left(\dfrac{\sqrt{2}+1}{\sqrt{2}}\right)\dfrac{Gm^2}{d^2}$ (along PB)

Introductory Exercise 13.2

1. 4.9 ms^{-2} **2.** (a) 2.45 ms^{-2} (b) 4.9 ms^{-2} **3.** $\dfrac{(\sqrt{5}-1)R}{2}$, where R is the radius of earth **4.** –0.0168 ms^{-2}

5. 1600 km **6.** 7.8×10^{-4} rad/s **7.** 997 N **8.** 1.237×10^{-3} rad/s, 84.6 min

9. Approximately 10 km

Introductory Exercise 13.3

1. $\dfrac{-2Gm}{a}, \dfrac{\sqrt{3}\,Gm}{a^2}$ **2.** $\dfrac{-5\,Gm}{a}$, zero **3.** $\dfrac{-4\,Gm}{a}, \dfrac{Gm}{a^2}$ **4.** $\dfrac{-3\,Gm}{R}$ **5.** 200 N/kg along $+x$ - direction

Introductory Exercise 13.4

1. $-[6xy\hat{\mathbf{i}} + (3x^2 + 3y^2z)\,\hat{\mathbf{j}} + y^3\,\hat{\mathbf{k}}]$ **2.** False **3.** $10\sqrt{2}$ N **4.** zero

Introductory Exercise 13.5

1. 2.1×10^{-5} ms^{-1} and 4.2×10^{-5} m/s **2.** $\dfrac{-Gm^2}{a}(4+\sqrt{2})$ **3.** $\dfrac{1}{2}\,mgR$ **5.** 2.51×10^4 km **6.** 7.9 km/s

Introductory Exercise 13.6

1. mgR **2.** 11.2 km/s **3.** (a) No (b) $-E_0$

Introductory Exercise 13.7

1. No **2.** $56\sqrt{2}$ h **3.** 2 : 1, 2 : 1

4. (a) 6.90 km/s (b) 2.38×10^{10} J (c) -4.76×10^{10} J with usual reference (d) $\left(\dfrac{21}{106}\right)^{3/2}(24)$ h

5. (a) 9.6×10^{10} J (b) 1.07×10^{10} J

Exercises

LEVEL 1

Assertion and Reason

1. (b) **2.** (b) **3.** (b) **4.** (d) **5.** (d) **6.** (d) **7.** (d) **8.** (a) **9.** (c) **10.** (b)
11. (d)

Single Correct Option

1. (c) **2.** (b) **3.** (c) **4.** (d) **5.** (d) **6.** (b) **7.** (b) **8.** (a) **9.** (b) **10.** (c)
11. (b) **12.** (b) **13.** (d) **14.** (b) **15.** (b) **16.** (b) **17.** (c) **18.** (d) **19.** (b) **20.** (c)
21. (c) **22.** (d) **23.** (c) **24.** (a) **25.** (c) **26.** (a) **27.** (c)

Subjective Questions

1. 1.334×10^{-9} J **2.** $\frac{dg}{g} = -2\left(\frac{h}{R}\right)$ **3.** $\sqrt{\frac{2G(m_1+m_2)}{r}}$

5. (a) Orbit will become elliptical (b) The satellite will escape

6. 6 h **7.** $F = \frac{4Gm_1m}{(R_1+R_2)^2}$ **8.** $\frac{2\pi GMm}{L^2}$ **10.** (a) $4R$ (b) $7.65\,R$ and $1.49\,R$

11. (a) $\frac{4GM}{9a^2}$ (towards the centre) (b) $\frac{8GM}{25a^2}$ (towards the centre) **12.** $\frac{\pi G\rho_0 a}{2}$

13. (a) 81.6 km/s (b) 1.8×10^4 kms^{-1} **14.** $v = \sqrt{\frac{GM}{R}}$

15. (a) 7.5×10^{-9} J (b) 8.17×10^{-9} J

16. (a) $F = \frac{GM^2}{4R^2}$ (b) $v = \sqrt{\frac{GM}{4R}}, T = \frac{4\pi R^{3/2}}{\sqrt{GM}}$ (c) $\frac{GM^2}{4R}$ **17.** (a) $\frac{-GMm}{r}$ (b) $\frac{-2GMm}{r}$

18. 0.71 yr **19.** (a) No (b) Same (c) Yes

LEVEL 2

Single Correct Option

1. (c)	**2.** (d)	**3.** (d)	**4.** (c)	**5.** (d)	**6.** (a)	**7.** (c)	**8.** (a)	**9.** (a)	**10.** (b)
11. (d)	**12.** (d)	**13.** (b)	**14.** (c)	**15.** (a)	**16.** (a)	**17.** (c)	**18.** (d)	**19.** (a)	**20.** (b)
21. (d)	**22.** (b)								

More than One Correct Options

1. (a,b,d)	**2.** (a,b,d)	**3.** (a,c,d)	**4.** (a,d)	**5.** (a,b,c)	**6.** (a,c)	**7.** (b,c,d)
8. (c,d)	**9.** (all)	**10.** (a,b,c)				

Match the Columns

1.	(a) → q	(b) → s	(c) → s	(d) → r
2.	(a) → r	(b) → s	(c) → p	(d) → s
3.	(a) → r	(b) → s	(c) → p	(d) → q
4.	(a) → r	(b) → p	(c) → s	(d) → r
5.	(a) → q,r	(b) → p	(c) → p	(d) → s

Subjective Questions

1. $\frac{3Gm^2}{2a}$ **2.** 3.1×10^3 m **3.** (a) 6400 km (b) 7.92 kms^{-1} **4.** $\frac{Gm}{16R^2}, \frac{61Gm}{900R^2}$

5. $\frac{GMm}{d^2}\left[1 - \frac{1}{8(1-R/2d)^2}\right]$ **6.** 7/3 **8.** $\frac{\sqrt{3}GMm}{8a^2}$ **9.** $\frac{3}{2}\sqrt{\frac{5GM}{a}}$ **10.** $d = 2R$

11. $\sqrt{\frac{2}{3}\pi G\rho R^2}$ **12.** (i) 0.3 m (ii) 18 cm/s

13. (a) 3.31 yr (b) m_2 has greater speed and greater total energy (c) $v_P = 1.8v_A$

14. (a) $2\pi\sqrt{\frac{d^3}{G(m_1+m_2)}}$ (b) $\frac{m_2}{m_1}$ (c) $\frac{m_2}{m_1}$ (d) $\mu\omega d^2$

(e) $\frac{1}{2}\mu\omega^2 d^2$, where μ is the reduced mass and ω the angular velocity.

14

Simple Harmonic Motion

Chapter Contents

14.1 Introduction

Following are given some general points regarding motion, periodic motion and simple harmonic motion (SHM):

1. In general, motion of a body (or its path) depends on two factors:
   (i) the nature of force (or acceleration) of the body and
   (ii) its velocity

   For example

   A constant force or constant acceleration always gives a straight line or parabolic path. If initial velocity is zero or parallel (or antiparallel) to constant acceleration then path is straight line. In all other cases, path is a parabola. For small height, acceleration due to gravity $(\mathbf{a} = \mathbf{g})$ is constant. So, path is either straight line or parabola.

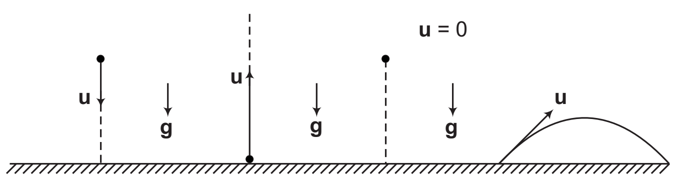


Fig. 14.1

   If force of constant magnitude is acting on a particle and its direction is always perpendicular to velocity, then path is circular motion in which speed is constant. This is also called uniform circular motion.

$$|\mathbf{F}_1| = |\mathbf{F}_2| = |\mathbf{F}_3| = |\mathbf{F}_4| = \frac{mv^2}{R} = \text{centripetal force}$$

$$|\mathbf{v}_1| = |\mathbf{v}_2| = |\mathbf{v}_3| = |\mathbf{v}_4| = v = \text{speed of the particle.}$$

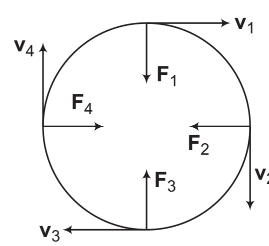


Fig. 14.2

   Now let us consider a particle free to move along x-axis, which is being acted upon by a force given by,

$$F = -kx^n$$

   Here, k is a positive constant.

   Now, following cases are possible depending on the value of n :

   (i) If n is an even integer (0, 2, 4, … etc), force is always along negative x-axis. If the particle is released from any position on the x-axis (except at $x = 0$) a force in negative direction of x-axis acts on it and it moves rectilinearly along negative x-axis.

   (ii) If n is an odd integer (1, 3, 5, … etc), force is along negative x-axis for $x > 0$, along positive x-axis for $x < 0$ and zero for $x = 0$. Thus, the particle will oscillate about stable equilibrium position (also called the mean position), $x = 0$. The force in this case is called the restoring force. Of these, if $n = 1$, i.e. $F = -kx$ the motion is said to be SHM.

2. In every oscillatory motion, there is one mean position (or stable equilibrium position) and two extreme positions.

3. Distance between mean position and the extreme position is called amplitude of oscillation A.

4. Oscillations does not start by itself. Normally the body has to be displaced from the mean position. In this displacement $F = -kx$ type force opposes the motion. So, work has to be done against this force which remains stored in the system in the form of mechanical energy. In the absence of any dissipative forces (like friction or viscous force) this mechanical energy remains constant. While moving from extreme positions to mean position potential energy decreases and kinetic energy increases but total mechanical energy remains constant. Similarly, in moving from mean position to extreme positions potential energy increases and kinetic energy decreases.
5. The more the initial displacement from the mean position, more is the amplitude, more is the initial work done and more is the mechanical energy given for oscillations.
6. Like any other motion, SHM is also a motion. So, basic characteristics of any motion (like $\mathbf{v} = \frac{d\mathbf{s}}{dt}$, $\mathbf{a} = \frac{d\mathbf{v}}{dt}$ etc.) can also be applied to SHM.
7. In most of the motions, displacement of the particle is measured from the starting point ($t = 0$). But in SHM displacement is normally measured from the mean position and it is denoted by x (or y).
8. **At mean position**, $x = 0$, $F = 0$, $a = 0$

   $v =$ maximum, kinetic energy = maximum and potential energy is minimum (not necessarily zero)

   At extreme positions, $x = \pm A$, $F =$ maximum, $a =$ maximum, kinetic energy $= 0$ and potential energy = maximum.
9. SHM is called simple harmonic motion because its mathematics is simple. Mathematics of periodic motion of higher powers ($n = 3, 5, 7$ etc in $F = -kx^n$) is slightly complex. This is the reason, we study only simple harmonic oscillatory motion at this stage.
10. In SHM, we shall discuss $x, A, v, a, F, K, U, E, T, f$ and ω. Here,

    $x =$ displacement from the mean position

    $A =$ amplitude of oscillations

    $v =$ velocity (or speed)

    $a =$ acceleration

    $F =$ force

    $K =$ kinetic energy

    $U =$ potential energy

    $E =$ total mechanical energy

    $T =$ time period of oscillations

    $f =$ frequency of oscillations and

    $\omega =$ angular frequency of oscillations.
11. SHM may be linear or angular. For example, motion of a pendulum for small amplitudes is angular SHM and motion of spring block system is linear SHM.
12. Like any other motion, equations of SHM can be written in terms of its displacement ($v^2 = u^2 + 2as$) or time ($v = u + at$). The only difference is, in most of the equations of SHM 'x' is measured from the mean position.

14.2 Displacement Equations of SHM

In most of the equations of SHM displacement will be represented by x, where $x =$ displacement from the mean position.

Force Equation As we have discussed earlier also,

$$\boxed{F = -Kx}$$

is the force required for a particle to execute SHM. This type of force is obtained when a block is attached with a spring. Therefore, motion of block with a spring is always SHM.

Following are given some important points in this equation :

(i) x varies between $+A$ and $-A$

(ii) F - x graph is a straight line passing through $x = 0$ or the mean position (not necessarily the origin)

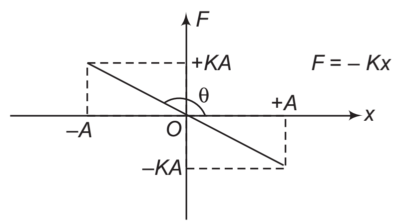


Fig. 14.3

At $x = \pm A$, $F = \mp KA$

(iii) $F = 0$ at $x = 0$ or at the mean position

(iv) $|F|_{max} = KA$ at $x = \pm A$ or at the extreme positions.

(v) $F \propto -x$

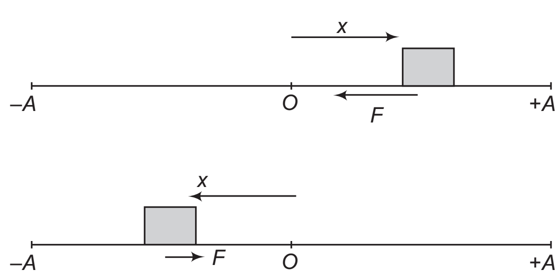


Fig. 14.4

Here, negative sign implies that direction of F and x are always opposite or direction of force is always towards the mean position. Hence, it is always restoring in nature.

Further, magnitude of force is proportional to magnitude of x. It means magnitude of force varies linearly with magnitude of x. If magnitude of x is doubled then magnitude of force will also become two times.

(vi) $\tan\theta =$ slope of F - x graph $= -K$

Acceleration Equation

$$\boxed{\begin{aligned} a &= \frac{F}{m} = -\frac{K}{m}x = -\omega^2 x \\ \omega^2 &= \frac{K}{m} \text{ or } \omega = \sqrt{\frac{K}{m}} \end{aligned}}$$

Here,

Following are given some important points in above two equations:

(i) $\omega = \sqrt{\frac{K}{m}}$ is called angular frequency of SHM. Later we will see that most of the equations of SHM can be derived from uniform circular motion. Angular frequency of SHM is similar to angular speed ω of uniform circular motion. At both places we can find time period T and frequency f from ω.

$$\omega = \sqrt{\frac{K}{m}} = \frac{2\pi}{T}$$

$\therefore$ $$\boxed{T = 2\pi\sqrt{\frac{m}{K}}}$$ (in SHM)

(ii) a - x graph is a straight line passing through $x = 0$ or the mean position.

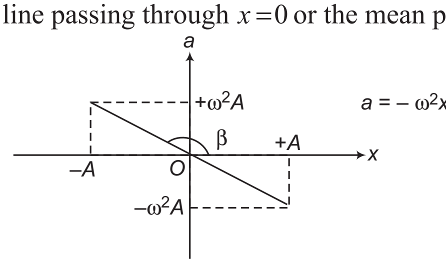


Fig. 14.5

$$\tan\beta = \text{Slope} = -\omega^2 \quad \Rightarrow \quad \text{At } x = \pm A,\ a = \mp\ \omega^2 A$$

(iii) $a = 0$ at $x = 0$, or at the mean position.

(iv) $|a|_{\max} = \omega^2 A$ at $x = \pm A$ or at extreme positions.

(v) $a \propto -x$

So, direction of 'a' is always opposite to the direction of x

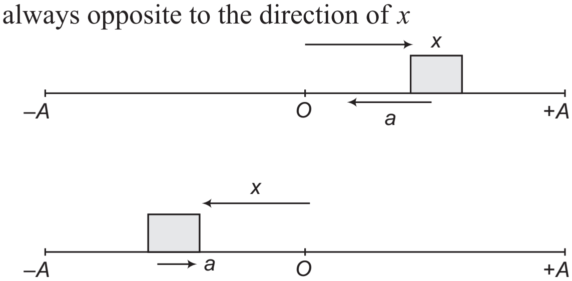


Fig. 14.6

or acceleration is always towards the mean position. Further, magnitude of 'a' is directly proportional to (linear variation) magnitude of 'x'.

Velocity Equation v - x graph in SHM is given by

$$\boxed{v = \pm\omega\sqrt{A^2 - x^2}}$$

Note *We will derive this v-x equation in example* 14.7.

Following are given some important points in this equation:

(i) The given equation can be written as

$$\frac{v^2}{\omega^2} + \frac{x^2}{(1)^2} = A^2$$

Therefore, v - x graph is an ellipse as shown below.

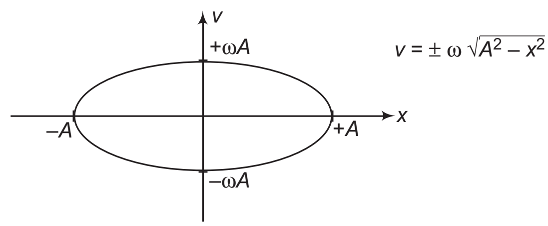


Fig. 14.7

(ii) $v = 0$ at $x = \pm A$ or at extreme positions.

(iii) $v = \pm\omega A$ or $|v|_{max} = \omega A$ at $x = 0$, or at the mean position.

(iv) For upper half of the ellipse velocity is positive (when the body is moving from $-A$ to $+A$). For lower half of the ellipse velocity is negative (when the body is moving from $+A$ to $-A$).

(v) At mean position (or $x = 0$), velocity is $+\omega A$ when the body is moving towards $+A$ and velocity is $-\omega A$ when the body is moving towards $-A$.

Energy Equations

Total Mechanical Energy

As we have discussed earlier also, at mean position potential energy is minimum (say U_0). This minimum potential energy U_0 may be zero, positive or negative. If not given in the question then we will take is zero. Now, suppose W is the work done in displacing the body from $x = 0$ to $x = A$. Then, total mechanical energy is now $E = W + U_0$. Let us start with an example of spring-block system kept at height h, so that the block has already a potential energy, $U_0 = mgh$ at mean position or natural length of spring where $F = 0$.

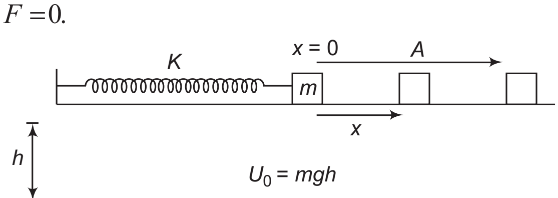


Fig. 14.8

Now, the block is displaced upto $x = A$. Work done (by external agent) in this displacement (against the spring force $-kx$) is $W = \frac{1}{2}KA^2$.

$\therefore$ Total mechanical energy $\qquad E = U_0 + W$

or $\qquad \boxed{E = U_0 + \frac{1}{2}KA^2 = U_0 + \frac{1}{2}m\omega^2 A^2} \qquad \left(\text{as } \omega^2 = \frac{K}{m} \text{ or } K = m\omega^2\right)$

Note *(i) In the above discussion, U_0 is positive as the spring block system is kept at some height. This value will be zero if the system is kept on ground. In some cases, it may be negative also.*

(ii) In the absence of any dissipative forces, this total mechanical energy remains constant at every point between $+A$ and $-A$.

At extreme positions, kinetic energy is zero. So, total mechanical energy is in the form of potential energy.

$$E = U = U_0 + \frac{1}{2}KA^2$$

or

$$U_0 + \frac{1}{2}m\omega^2 A^2$$

At mean position, potential energy is minimum or U_0 and the kinetic energy is $\frac{1}{2}KA^2$ or $\frac{1}{2}m\omega^2 A^2$.

Thus, U_0 is present at every point. This is initial work done $W\left(=\frac{1}{2}KA^2 \text{ or } \frac{1}{2}m\omega^2 A^2\right)$ which changes between potential and kinetic energies. This is also called energy of oscillation. At mean position, it completely converts into kinetic energy and at extreme positions it converts into potential energy.

Potential Energy From Fig. 14.8, we can see that, at a general displacement x (from the mean position) the potential energy is

$$U = U_0 + \text{Spring potential energy} \quad \text{or} \quad \boxed{U = U_0 + \frac{1}{2}Kx^2 = U_0 + \frac{1}{2}m\omega^2 x^2}$$

Kinetic Energy Here, let us call it T as we are using K for spring constant of spring.

At a general displacement x,

$$T = E - U$$

$$= \left(U_0 + \frac{1}{2}KA^2\right) - \left(U_0 + \frac{1}{2}Kx^2\right)$$

$$\therefore \qquad T = \frac{1}{2}K(A^2 - x^2) = \frac{1}{2}m\omega^2(A^2 - x^2)$$

Note *At some places kinetic energy may be represented by K and force constant by k.*

Alternate Method

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m[\pm\,\omega\sqrt{A^2 - x^2}\,]$$

$$= \frac{1}{2}m\omega^2(A^2 - x^2)$$

Graphs The three equations of total mechanical energy, potential energy and kinetic energy are

$$E = U_0 + \frac{1}{2} K A^2 = U_0 + \frac{1}{2} m\omega^2 A^2$$

$$U = U_0 + \frac{1}{2} K x^2 = U_0 + \frac{1}{2} m\omega^2 x^2$$

$$T = \frac{1}{2} K (A^2 - x^2) = \frac{1}{2} m\omega^2 (A^2 - x^2)$$

E is a constant function. U - x and T - x graphs are parabolas as shown in the Fig.14.9.

At mean position ($x = 0$)

$$E = U_0 + \frac{1}{2} KA^2$$

$$U = U_0 = \text{minimum value and}$$

$$T = \frac{1}{2} KA^2 = \text{maximum value.}$$

At extreme positions ($x = \pm A$)

$$E = U_0 + \frac{1}{2} KA^2$$

$$U = U_0 + \frac{1}{2} KA^2 = \text{maximum value}$$

and $T = 0$

The three graphs are as shown in figure.

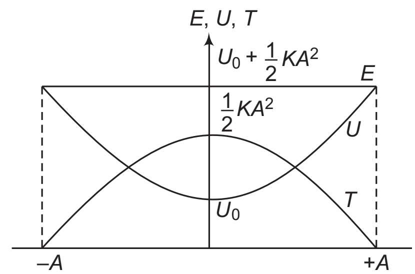


Fig. 14.9

Example 14.1 *Describe the motion of a particle acted upon by a force*

(*i*) $F = -2(x-2)^3$

(*ii*) $F = -2(x-2)^2$

(*iii*) $F = -2(x-2)$

Solution (i) $F = -2(x-2)^3$

$$F = 0 \quad \text{at} \quad x = 2$$

Force is along negative x-direction for $x > 2$ and it is along positive x-direction for $x < 2$. Thus, the motion of the particle is oscillatory (but not simple harmonic) about $x = 2$.

(ii) $F = 0$ for $x = 2$, but force is always along negative x-direction for any value of x except at $x = 2$. Thus, the motion of the particle is rectilinear along negative x-direction provided it is not kept at rest at $x = 2$.

(iii) Let, us take $x - 2 = X$, then the given force can be written as,

$$F = -2X$$

This is the equation of SHM. Hence, the particle oscillates simple harmonically about $X = 0$ or $x = 2$.

Example 14.2 *Maximum acceleration of a particle in SHM is 16 cm/s² and maximum velocity is 8 cm/s. Find time period and amplitude of oscillations.*

Solution
$$a_{max} = \omega^2 A = 16 \text{ cm/s} \quad \ldots\text{(i)}$$
$$u_{max} = \omega A = 8 \text{ cm/s} \quad \ldots\text{(ii)}$$

Solving Eqs. (i) and (ii), we get
$$A = 4 \text{ cm} \quad \textbf{Ans.}$$
and
$$\omega = 2 \text{ rad/s}$$
Now,
$$T = \frac{2\pi}{\omega} = \frac{2\pi}{2} = (\pi) \text{ s}$$
or
$$T \approx 3.14 \text{ s} \quad \textbf{Ans.}$$

Example 14.3 *F - x equation of a body in SHM is*
$$F + 4x = 0$$
Here, F is in newton and x in metre. Mass of the body is 1 kg. Find time period of oscillations.

Solution The given equation can be written as
$$F = -4x$$
Comparing this equation with the standard equation of SHM or
$$F = -Kx$$
we get
$$K = 4 \text{ N/m}$$
Now,
$$T = 2\pi\sqrt{\frac{m}{K}}$$
$$= 2\pi\sqrt{\frac{1}{4}} = (\pi)\text{s} \quad \textbf{Ans.}$$

Example 14.4 *A linear harmonic oscillator has a total mechanical energy of 200 J. Potential energy of it at mean position is 50 J. Find*
(i) the maximum kinetic energy,
(ii) the minimum potential energy,
(iii) the potential energy at extreme positions.

Solution At mean position, potential energy is minimum and kinetic energy is maximum. Hence,
$$U_{min} = 50 \text{ J} \quad \text{(at mean position)}$$
and
$$K_{max} = E - U_{min} = 200 - 50$$
$$= 150 \text{ J} \quad \text{(at mean position)}$$
At extreme positions kinetic energy is zero and potential energy is maximum
$$\therefore \quad U_{max} = E$$
$$= 200 \text{ J} \quad \text{(at extreme position)}$$

Example 14.5 *A particle executes SHM.*

(a) What fraction of total energy is kinetic and what fraction is potential when displacement is one half of the amplitude?

(b) At what value of displacement are the kinetic and potential energies equal?

Solution Here, let us take minimum potential energy at mean position

$$U_0 = 0$$

We know that $$E_{\text{total}} = \frac{1}{2} m\omega^2 A^2$$

$$\text{KE} = \frac{1}{2} m\omega^2 (A^2 - x^2) \quad \text{and} \quad U = \frac{1}{2} m\omega^2 x^2$$

(a) When $$x = \frac{A}{2}$$

$$\text{KE} = \frac{1}{2} m\omega^2 \left(\frac{3A^2}{4}\right)$$

$\Rightarrow$ $$\frac{\text{KE}}{E_{\text{total}}} = \frac{3}{4}$$ **Ans.**

At $x = \frac{A}{2}$, $$U = \frac{1}{2} m\omega^2 \left(\frac{A^2}{4}\right)$$

$\Rightarrow$ $$\frac{\text{PE}}{E_{\text{total}}} = \frac{1}{4}$$ **Ans.**

(b) Since, $$K = U$$

$$\frac{1}{2} m\omega^2 (A^2 - x^2) = \frac{1}{2} m\omega^2 x^2$$

or $$2x^2 = A^2 \quad \text{or} \quad x = \frac{A}{\sqrt{2}} = 0.707A$$ **Ans.**

Example 14.6 *The potential energy of a particle oscillating along x-axis is given as*

$$U = 20 + (x - 2)^2$$

Here, U is in joules and x in metres. Total mechanical energy of the particle is 36 J.

(a) State whether the motion of the particle is simple harmonic or not.

(b) Find the mean position.

(c) Find the maximum kinetic energy of the particle.

Solution (a) $F = -\dfrac{dU}{dx} = -2(x - 2)$

By assuming $x - 2 = X$, we have $$F = -2X$$

Since, $$F \propto -X$$

The motion of the particle is simple harmonic.

(b) The mean position of the particle is $X = 0$ or $x - 2 = 0$, which gives $x = 2$ m.

(c) Maximum kinetic energy of the particle is,

$$K_{max} = E - U_{min} = 36 - 20 = 16 \text{ J}$$

Note *U_{min} is 20 J at mean position or at $x = 2$ m.*

INTRODUCTORY EXERCISE 14.1

1. a - x equation of a particle in SHM is $a + 4x = 0$
   Here, a is in cm/s^2 and x in cm. Find time period in seconds.
2. At $x = \frac{A}{4}$, what fraction of the mechanical energy is potential? What fraction is kinetic? Assume potential energy to be zero at mean position.
3. A cart of mass 2.00 kg is attached to the end of a horizontal spring with force constant $k = 150$ N/m. The cart is displaced 15.0 cm from its equilibrium position and released. What are (a) the amplitude (b) the period (c) the frequency (d) the mechanical energy (e) the maximum velocity of the cart? Neglect friction.
4. A 0.5 kg body performs simple harmonic motion with a frequency of 2 Hz and an amplitude of 8 mm. Find the maximum velocity of the body, its maximum acceleration and the maximum restoring force to which the body is subjected.
5. Can we use the equation $v = u + at$ in SHM or not ?

14.3 Time Equations of SHM

In one dimensional motion, if acceleration as a function of time is given, then by integrating a - t equation we can make v - t equation. By further integrating v - t equation we can make s - t or x - t equation.

Meaning of General Solution

In one dimensional motion, if $a =$ constant then general solution of v - t equation is (which can be obtained by integrating a - t equation)

$$v = u + at$$

In this equation, u and a are constants but v and t are variables. In different problems, values of these constants u and a are different but basic nature of all v - t equations are same. Here, the basic nature is linear equation in v and t or straight line graph between v and t.

Standard Differential Equation of SHM

In article 14.2, we have seen that a - x equation of SHM is

$$a = -\omega^2 x$$

This equation can also be written as

$$\boxed{\frac{d^2x}{dt^2} = -\omega^2 x} \quad \text{...(i)}$$

This is standard second order differential equation of SHM. Solving this differential equation we can find x - t equation. But normally we do not solve a differential equation in physics at this stage. So, we can directly write general solution of Eq. (i). The general solution of Eq. (i) is

$$x = A\sin(\omega t \pm \phi) \quad \text{or} \quad x = A\cos(\omega t \pm \phi)$$

In these equations x and t are variables. But A, ω and ϕ are constants. In different problems, values of these constants are different but basic nature of all equations will be same. Here, the basic nature is sinusoidal (sine or cosine) x - t equation and x - t graph.

By differentiating x - t equation we can also make other equations. For example, if

$$x = A\sin(\omega t \pm \phi)$$

Then
$$v = \frac{dx}{dt} = \omega A\cos(\omega t \pm \phi)$$

and,
$$a = \frac{dv}{dt} = -\omega^2 A\sin(\omega t \pm \phi) = -\omega^2 x \qquad [\text{as } A\sin(\omega t \pm \phi) = x]$$

Time Period (T) and Frequency (f)

In the above equations, time period of oscillations is

$$T = \frac{2\pi}{\omega}$$

and frequency of oscillation is

$$f = \frac{1}{T} = \frac{\omega}{2\pi}$$

Phase Angle at Time t

In any trigonometric function its angle is very important. The value of function varies with this angle. For example, suppose we have a trigonometric function.

$$y = \sin\theta \text{ then, } y = 0 \text{ for } \theta = 0°, \; y = \frac{1}{2} \text{ for } \theta = 30°$$

So, value of y depends on the angle θ. This angle is called the phase angle. In x - t, v - t and a - t equations of SHM we have seen above that the phase angle is same, $(\omega t \pm \phi)$. This may be called the phase angle of x - t, v - t and a - t equations at time t. With the help of this angle values of x, v and a can be obtained at any time t. Here, x varies between $+A$ and $-A$, v varies between $+\omega A$ and $-\omega A$ and a varies between $+\omega^2 A$ and $-\omega^2 A$. If $(\omega t \pm \phi)$ is 30° at some instant, then according to above equations values of x, v and a are $+\frac{A}{2}$, $+\frac{\sqrt{3}\,\omega A}{2}$ and $-\frac{\omega^2 A}{2}$.

This condition is shown in following figure:

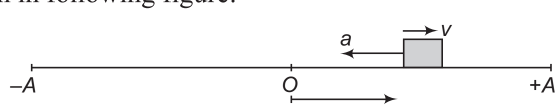


Fig. 14.10

$$x = +\frac{A}{2}, \; v = \frac{+\sqrt{3}\,\omega A}{2} \text{ and } a = \frac{-\omega^2 A}{2}$$

Initial Phase Angle

In the phase angle $\omega t \pm \phi$ if we substitute $t = 0$, then the angle is $\pm\phi$. This is called initial phase angle. With the help of initial phase angle we can determine initial values of x, v and a.

Dependence of Initial Phase Angle

Initial phase angle depends on following three factors :

(i) Whether we write sine equation or cosine equation.

(ii) Initial value of x (at $t = 0$)

(iii) Initial direction of velocity.

For example, in first figure the body starts

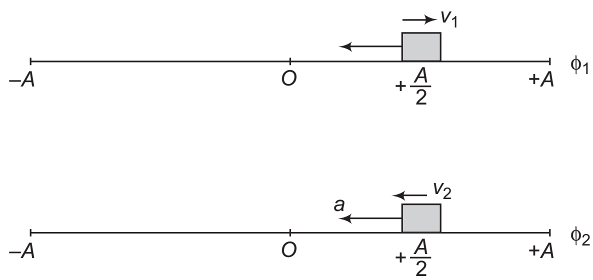


Fig. 14.11

($t = 0$, $u \neq 0$) from $x = +\frac{A}{2}$ with velocity $v_1 = \frac{\sqrt{3}\,\omega A}{2}$ and the initial phase angle is suppose ϕ_1 (corresponding to sine equation)

In second figure, body starts from the same point $x = +\frac{A}{2}$ but direction of velocity is opposite or $v_2 = -\frac{\sqrt{3}\,\omega A}{2}$ and the initial phase angle is suppose ϕ_2 (corresponding to sine equation). Then

$$\phi_1 \neq \phi_2$$

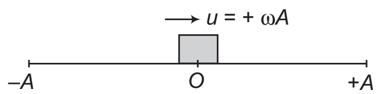
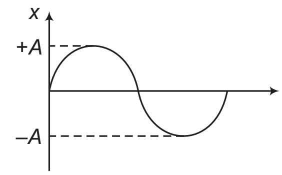
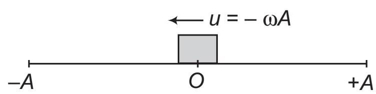
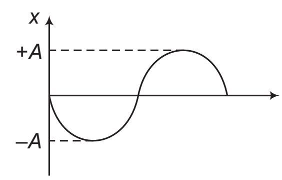
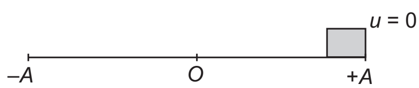
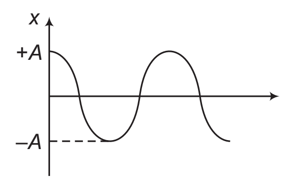
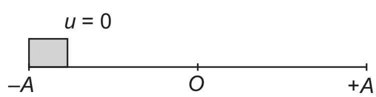
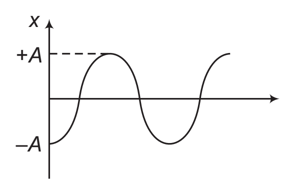
Similarly, if we wish to write cosine equations and their initial phase angles are suppose ϕ_1' and ϕ_2', then $\phi_1 \neq \phi_1'$ and $\phi_2 \neq \phi_2'$

Four Frequently used x - t Equations

Between $+A$ and $-A$ there may be infinite number of points from where the body may start its journey ($t = 0$ but u is zero only if, it starts from $+A$ or $-A$).

Further at every point (except at $+A$ and $-A$) two directions (positive x or negative x) of initial velocity can exist. So, infinite number of initial phase angles are possible or infinite number of x - t equations can be written. But following four initial conditions ($t = 0$) and corresponding four x - t equations are frequently used.

Table 14.1

Initial Conditions at $t=0$	x - t equation	x - t graph
	$x = A\sin\omega t$	
	$x = -A\sin\omega t$	
	$x = A\cos\omega t$	
	$x = -A\cos\omega t$	

Six Time Equations of SHM and their Graphs

Six time equations in SHM are x - t, v - t, a - t, E - t, U - t and T - t (T = kinetic energy). If one equation is given other five equations can be made. Let us start with x - t equation :

$$\boxed{x = A\sin\omega t} \qquad \text{...(i)}$$

According to this equation the body starts from mean positive, moving towards positive x-direction with initial velocity, $u = +\omega A$.

Now,
$$v = \frac{dx}{dt} = \frac{d}{dt}(A\sin\omega t) = \omega A\cos\omega t$$

or
$$\boxed{v = \omega A\cos\omega t} \qquad \text{...(ii)}$$

$$a = \frac{dv}{dt} = \frac{d}{dt}(\omega A\cos\omega t) = -\omega^2 A\sin\omega t$$

or
$$\boxed{a = -\omega^2 A\sin\omega t} \qquad \text{...(iii)}$$

$$\boxed{E = U_0 + \frac{1}{2}KA^2 = U_0 + \frac{1}{2}m\omega^2 A^2} \qquad \text{...(iv)}$$

$$U = U_0 + \frac{1}{2}Kx^2 = U_0 + \frac{1}{2}KA^2 \sin^2 \omega t$$

$$= U_0 + \frac{1}{2}m\omega^2 A^2 \sin^2 \omega t$$

or $$\boxed{U = U_0 + \frac{1}{2}KA^2 \sin^2 \omega t = U_0 + \frac{1}{2}m\omega^2 A^2 \sin^2 \omega t} \quad \text{...(v)}$$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\omega A \cos \omega t)^2$$

$$= \frac{1}{2}m\omega^2 A^2 \cos^2 \omega t = \frac{1}{2}KA^2 \cos^2 \omega t$$

or $$\boxed{T \text{ or KE} = \frac{1}{2}KA^2 \cos^2 \omega t = \frac{1}{2}m\omega^2 A^2 \cos^2 \omega t} \quad \text{...(vi)}$$

From the above six equations, we can draw following conclusions :

(i) $x-t$, $v-t$ and $a-t$ are sine or cosine functions of same ω. So, x, v and a oscillate sinusoidally with same time period $T = \frac{2\pi}{\omega}$.

(ii) U - t and T - t are $\sin^2$ and $\cos^2$ functions of time. But oscillation frequency of $\sin^2$ or $\cos^2$ functions is double of the oscillation frequency of sine and cosine functions. So, U and T oscillate with double the frequency of oscillations of x, v and a.

Minimum value of U is U_0 and maximum value of U is $U_0 + \frac{1}{2}KA^2$ or $U_0 + \frac{1}{2}m\omega^2 A^2$. Therefore, U oscillates between U_0 and $U_0 + \frac{1}{2}KA^2$. Similarly, minimum value of T is zero and the maximum value is $\frac{1}{2}KA^2$ or $\frac{1}{2}m\omega^2 A^2$.

Hence, T oscillates between 0 and $\frac{1}{2}KA^2$.

(iii) E does not oscillate because it is constant.

(iv) At $t = 0$

$x = 0$ (starts from mean position)

$v = \omega A$ (maximum velocity)

$a = 0$

$E = U_0 + \frac{1}{2}KA^2$

$U = U_0$ (minimum value)

and T or KE $= \frac{1}{2}KA^2$

or $\frac{1}{2}m\omega^2 A^2$ (maximum value)

(v) Average value of $\cos^2$ function in one time period is $\frac{1}{2}$. Therefore, from Eq. (vi) we can see that maximum value of kinetic energy is $\frac{1}{2}KA^2$ or $\frac{1}{2}m\omega^2 A^2$. But average value of kinetic energy in one time period is $\frac{1}{4}KA^2$ or $\frac{1}{4}m\omega^2 A^2$.

(vi) The six graphs are as shown below.

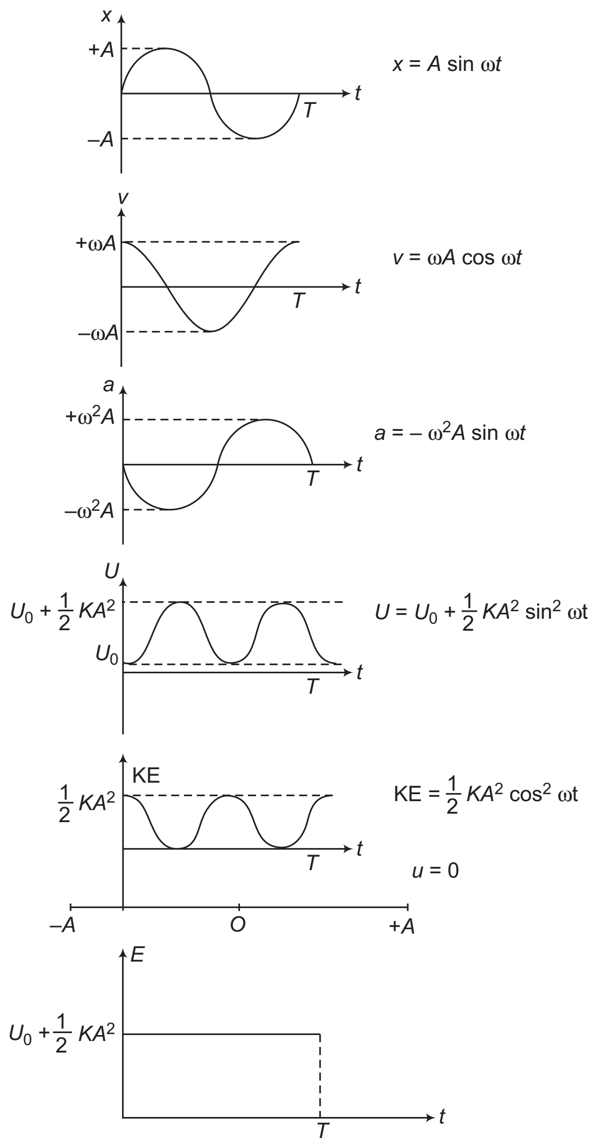


Fig. 14.12

In the first five graphs we can see that in time $T\,(=2\pi/\omega)$: x, v and a oscillate once but U and KE oscillate twice.

Note *(i) If the first equation, $x = A \sin \omega t$ is changed then other five equations (and their graphs also) will change. But the basic nature will remain same. Here the basic nature is that x, v and a oscillate sinusoidally with same time period and frequency. E does not oscillate. U and KE oscillate with double frequency.*

(ii) Phase difference between x - t and v - t equations or between v - t and a - t equations is $\frac{\pi}{2}$ or 90°. Phase difference between x - t and a - t equations is π or 180°.

List of Formulae of Articles 14.2 and 14.3 (with $U_0 = 0$)

Table 14.2

S.No.	Name of the equation	Expression of the equation	Remarks
1.	Displacement-time	$x = A\cos(\omega t + \phi)$	x varies between $+A$ and $-A$
2.	Velocity - time $\left(v = \frac{dx}{dt}\right)$	$v = -A\omega \sin(\omega t + \phi)$	v varies between $+A\omega$ and $-A\omega$
3.	Acceleration - time $\left(a = \frac{dv}{dt}\right)$	$a = -A\omega^2 \cos(\omega t + \phi)$	a varies between $+A\omega^2$ and $-A\omega^2$
4.	Kinetic energy - time $\left(\text{KE} = \frac{1}{2}mv^2\right)$	$\text{KE} = \frac{1}{2}mA^2\omega^2 \sin^2(\omega t + \phi)$	KE varies between 0 and $\frac{1}{2}mA^2\omega^2$
5.	Potential energy - time $\left(U = \frac{1}{2}m\omega^2 x^2\right)$	$\text{KE} = \frac{1}{2}m\omega^2 A^2 \cos^2(\omega t + \phi)$	U varies between $\frac{1}{2}mA^2\omega^2$ and 0
6.	Total energy - time ($E = \text{KE} + U$)	$E = \frac{1}{2}m\omega^2 A^2$	E is constant
7.	Velocity - displacement	$v = \pm\omega\sqrt{A^2 - x^2}$	$v = 0$ at $x = \pm A$ and at $x = 0$ $v = \pm A\omega$
8.	Acceleration - displacement	$a = -\omega^2 x$	$a = 0$ at $x = 0$ $a = \pm\omega^2 A$ at $x = \mp A$
9.	Kinetic energy - displacement	$\text{KE} = \frac{1}{2}m\omega^2(A^2 - x^2)$	$\text{KE} = 0$ at $x = \pm A$, $\text{KE} = \frac{1}{2}m\omega^2 A^2$ at $x = 0$
10.	Potential energy - displacement	$U = \frac{1}{2}m\omega^2 x^2$	$U = 0$ at $x = 0$, $U = \frac{1}{2}m\omega^2 A^2$ at $x = \pm A$
11.	Total energy - displacement	$E = \frac{1}{2}m\omega^2 A^2$	E is constant

Example 14.7 *From the time equations of SHM, prove the relation,*

$$v \pm \omega\sqrt{A^2 - x^2}$$

Solution Let, $x = A \sin \omega t$

Then,
$$\sin \omega t = \frac{x}{A}$$

Now,
$$v = \frac{dx}{dt} = \omega A \cos \omega t$$
$$= \omega A\sqrt{1 - \sin^2 \omega t}$$

$$= \omega A \sqrt{1 - \left(\frac{x}{A}\right)^2}$$

$\therefore$ $$v = \omega A \sqrt{\frac{A^2 - x^2}{A^2}}$$

or $$v = \pm \omega \sqrt{A^2 - x^2}$$ **Hence Proved.**

Example 14.8 *If a SHM is represented by the equation* $x = 10 \sin\left(\pi t + \frac{\pi}{6}\right)$ *in SI units, then determine its amplitude, time period and maximum velocity* v_{max} ?

Solution Comparing the above equation with

$$x = A \sin(\omega t + \phi), \text{ we get}$$

$$A = 10 \text{ m}$$ **Ans.**

$$\omega = (\pi)\frac{\text{rad}}{\text{s}} \quad \text{and} \quad \phi = \frac{\pi}{6}$$

$\because$ $$T = \frac{2\pi}{\omega} \quad \Rightarrow \quad T = 2\text{s}$$ **Ans.**

$$v_{max} = \omega A = (10\pi) \text{ m/s}$$ **Ans.**

Example 14.9 *A particle executes SHM with a time period of 4 s. Find the time taken by the particle to go directly from its mean position to half of its amplitude.*

Solution If the particle starts from the mean position in positive direction, then:

$$x = A \sin(\omega t)$$

or $$\frac{A}{2} = A \sin(\omega t)$$

or $$\frac{1}{2} = \sin(\omega t)$$

$$\omega t = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$t = \frac{\pi}{6\omega} = \frac{\pi T}{6(2\pi)}$$

as $$\omega = \frac{2\pi}{T}$$

$\Rightarrow$ $$t = \frac{T}{12} = \frac{4}{12} = \frac{1}{3} \text{s}$$ **Ans.**

Example 14.10 *A particle executes simple harmonic motion about the point $x = 0$. At time $t = 0$, it has displacement $x = 2$ cm and zero velocity. If the frequency of motion is $0.25\ s^{-1}$, find (a) the period, (b) angular frequency, (c) the amplitude, (d) maximum speed, (e) the displacement from the mean position at $t = 3$ s and (f) the velocity at $t = 3$ s.*

Solution (a) Period $T = \dfrac{1}{f}$

$$= \frac{1}{0.25\ s^{-1}} = 4\ s$$ **Ans.**

(b) Angular frequency $\omega = \dfrac{2\pi}{T}$

$$= \frac{2\pi}{4} = \frac{\pi}{2}\ \text{rad/s}$$

$$= 1.57\ \text{rad/s}$$ **Ans.**

(c) Amplitude is the maximum displacement from mean position. Hence, $A = 2 - 0 = 2$ cm.

(d) Maximum speed $v_{max} = A\,\omega$

$$= 2\cdot\frac{\pi}{2} = \pi\ \text{cm/s}$$

$$= 3.14\ \text{cm/s}$$ **Ans.**

(e) At $t = 0$, particle starts from extreme position.

$\therefore$ $$x = A\cos\omega t$$

$$= 2\cos\left(\frac{\pi}{2}t\right)$$

At $$t = 3\ \text{sec}$$

$$x = 0$$ **Ans.**

(f) Velocity at $x = 0$, is v_{max} i.e. 3.14 cm/s. **Ans.**

Example 14.11 *Find the period of the function,*

$$y = \sin\omega t + \sin 2\omega t + \sin 3\omega t$$

Solution The given function can be written as,

$$y = y_1 + y_2 + y_3$$

Here, $$y_1 = \sin\omega t,\ T_1 = \frac{2\pi}{\omega}$$

$$y_2 = \sin 2\omega t,\ T_2 = \frac{2\pi}{2\omega} = \frac{\pi}{\omega}$$

and $$y_3 = \sin 3\omega t,\ T_3 = \frac{2\pi}{3\omega}$$

$\therefore$ $$T_1 = 2T_2 \quad \text{and} \quad T_1 = 3T_3$$

So, the time period of the given function is T_1 or $\frac{2\pi}{\omega}$. **Ans.**

Because in time $T = \frac{2\pi}{\omega}$, first function completes one oscillation, the second function two oscillations and the third three.

INTRODUCTORY EXERCISE 14.2

1. A 2.0 kg particle undergoes SHM according to $x = 1.5 \sin\left(\frac{\pi t}{4} + \frac{\pi}{6}\right)$ (in SI units)
   (a) What is the total mechanical energy of the particle?
   (b) What is the shortest time required for the particle to move from $x = 0.5$ m to $x = -0.75$ m?
2. Given that the equation of motion of a mass is $x = 0.20 \sin(3.0t)$ m. Find the velocity and acceleration of the mass when the object is 5 cm from its equilibrium position. Repeat for $x = 0$.
3. A particle executes simple harmonic motion of amplitude A along the x-axis. At $t = 0$, the position of the particle is $x = \frac{A}{2}$ and it moves along the positive x-direction. Find the phase constant δ, if the equation is written as $x = A \sin(\omega t + \delta)$.
4. An object of mass 0.8 kg is attached to one end of a spring and the system is set into simple harmonic motion. The displacement x of the object as a function of time is shown in the figure. With the aid of the data, determine
   (a) the amplitude A of the motion,
   (b) the angular frequency ω,
   (c) the spring constant K,
   (d) the speed of the object at $t = 1.0$ s and
   (e) the magnitude of the object's acceleration at $t = 1.0$ s.

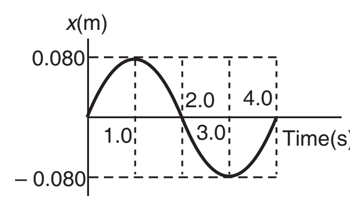


Fig. 14.13

5. The equation of motion of a particle started at $t = 0$ is given by $x = 5 \sin\left(20t + \frac{\pi}{3}\right)$, where x is in cm and t in sec. When does the particle (a) first come to rest, (b) first have zero acceleration, (c) first have maximum speed.
6. Describe the motion corresponding to x - t equation, $x = 10 - 4 \cos \omega t$.

14.4 Relation between Simple Harmonic Motion and Uniform Circular Motion

Consider a particle Q, moving on a circle of radius A with constant angular velocity ω. The projection of Q on a diameter BC is P. It is clear from the figure that as Q moves around the circle the projection P oscillates between B and C. The angle that the radius OQ makes with the x-axis is, $\theta = \omega t + \phi$. Here, ϕ is the angle made by the radius OQ with the x-axis at time $t = 0$. Further,

$$OP = OQ \cos \theta$$

or

$$x = A \cos(\omega t + \phi)$$

This is standard $x - t$ equation of SHM. Hence, P executes SHM. That is

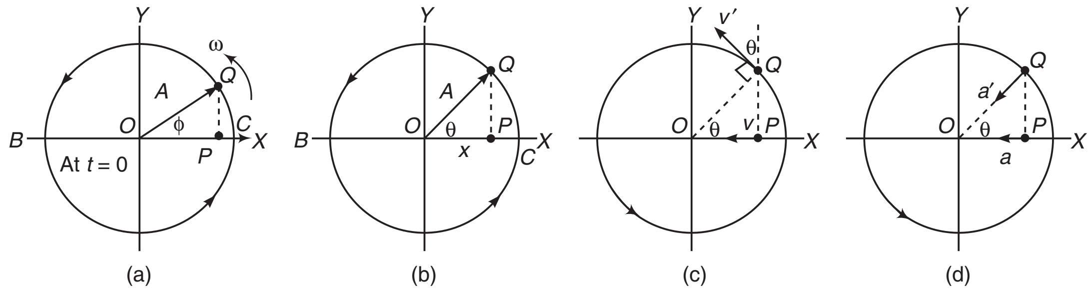


Fig. 14.14 Relation between SHM and uniform circular motion. (b) Position, (c) velocity and (d) acceleration

When a particle moves with uniform circular motion, its projection on a diameter moves with SHM.

The velocity of Q is perpendicular to OQ and has a magnitude of velocity $v' = \omega A$. The component of v' along the x-axis is,

$$v = -v' \sin\theta \quad \text{or} \quad v = -\omega A \sin(\omega t + \phi)$$

which is also the velocity of P. The acceleration of Q is centripetal and has a magnitude, $a' = \omega^2 A$.

The component of a' along the x-axis is

$$a = -a' \cos\theta \quad \text{or} \quad a = -\omega^2 A \cos(\omega t + \phi)$$

Which again coincides with the acceleration of P.

Note *v = velocity of SHM of particle P* $= \dfrac{dx}{dt} = \dfrac{d}{dt}[A\cos(\omega t + \phi)] = -\omega A \sin(\omega t + \phi)$

a = acceleration of SHM of particle $= \dfrac{dv}{dt} = -\omega^2 A\cos(\omega t + \phi)$

14.5 Methods of Finding Time Period of a Simple Harmonic Motion

If F - x or a - x equations are known then we can easily find the time period of SHM. For example, F - x equation is of type $F = -Kx$

and time period in this case is $T = 2\pi\sqrt{\dfrac{m}{K}}$

Similarly, a - x equation is of type

$$a = -\omega^2 x$$

$$\therefore \qquad \omega = \sqrt{\left|\frac{a}{x}\right|} = \frac{2\pi}{T}$$

or

$$T = 2\pi\sqrt{\left|\frac{x}{a}\right|} = 2\pi\sqrt{\left|\frac{\text{displacement}}{\text{acceleration}}\right|}$$

In angular SHM, we will be required $\tau - \theta$ or $\alpha - \theta$ equations.

Now, there are following two methods of finding F - x and a - x equations (or $\tau - \theta$ and $\alpha - \theta$ equations).

Restoring Force or Torque Method

The following steps are usually followed in this method:

Step 1 Find the stable equilibrium position which usually is also known as the mean position. Net force or torque on the particle in this position is zero. Potential energy is minimum.

Step 2 Displace the body from its mean position by a small displacement x (in case of a linear SHM) or θ (in case of an angular SHM).

Step 3 Find net force or torque in this displaced position.

Step 4 Show that this force or torque has a tendency to bring the body back to its mean position and magnitude of force or torque is proportional to displacement, i.e.

$$F \propto -x \quad \text{or} \quad F = -Kx \qquad \ldots\text{(i)}$$

or

$$\tau \propto -\theta \quad \text{or} \quad \tau = -K\theta \qquad \ldots\text{(ii)}$$

This force or torque is also known as restoring force or restoring torque.

Step 5 Find linear acceleration by dividing Eq. (i) by mass m or angular acceleration by dividing Eq. (ii) by moment of inertia I. Hence,

$$a = -\frac{K}{m}.x = -\omega^2 x \quad \text{or} \quad \alpha = -\frac{K}{I}\theta = -\omega^2\theta$$

Step 6 Finally,

$$\omega = \sqrt{\left|\frac{a}{x}\right|} \quad \text{or} \quad \sqrt{\left|\frac{\alpha}{\theta}\right|}$$

or

$$\frac{2\pi}{T} = \sqrt{\left|\frac{a}{x}\right|} \quad \text{or} \quad \sqrt{\left|\frac{\alpha}{\theta}\right|}$$

$\therefore$

$$T = 2\pi\sqrt{\left|\frac{x}{a}\right|} \quad \text{or} \quad 2\pi\sqrt{\left|\frac{\theta}{\alpha}\right|}$$

Energy Method

Repeat steps-1 and step-2 as in method-1. Find the total mechanical energy (E) in the displaced position. Since, mechanical energy in SHM remains constant.

$$\frac{dE}{dt} = 0$$

By differentiating the energy equation with respect to time and substituting $\frac{dx}{dt} = v$, $\frac{d\theta}{dt} = \omega$, $\frac{dv}{dt} = a$, and $\frac{d\omega}{dt} = \alpha$ we come to step-5 or we directly get $F - x$ and $a - x$ equations (or $\tau - \theta$ or $x - \theta$ equation).

The remaining procedure is same.

Note *(i) E usually consists of following terms :*

(a) Gravitational PE (b) Elastic PE (c) Electrostatic PE

(d) Rotational KE and (e) Translational KE

(ii) For gravitational PE, choose the reference point $(h = 0)$ at mean position.

Now, let us take few examples of finding time period (T) of certain simple harmonic motions.

The Simple Pendulum

An example of SHM is the motion of a pendulum. A simple pendulum is defined as a particle of mass m suspended from a point O by a string of length l and of negligible mass.

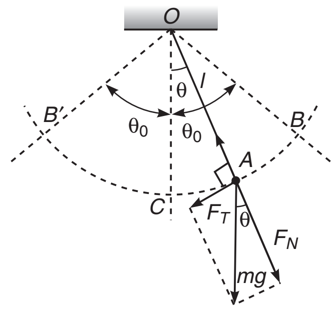


Fig. 14.15

When the particle is pulled aside to position B, so that the string makes an angle θ_0 with the vertical OC and then released, the pendulum will oscillate between B and the symmetric position B'. The oscillatory motion is due to the tangential component F_T of the weight mg of the particle. This force F_T is maximum at B and B' and zero at C. Thus, we can write,

$$F_T = -mg\sin\theta$$

Here, minus sign appears because it is opposite to the displacement.

$$x = CA$$

∴ $$ma_T = -mg\sin\theta \qquad \ldots\text{(i)}$$

Here, $$a_T = l\alpha \qquad \left(\text{where, } \alpha = \frac{d^2\theta}{dt^2}\right)$$

and $$\sin\theta \approx \theta \quad \text{for small oscillations}$$

∴ $$ml\alpha = -mg\theta$$

Since, α is proportional to $-\theta$. Hence, motion is simple harmonic.

or $$\alpha = -\left(\frac{g}{l}\right)\theta \quad \text{or} \quad \left|\frac{\theta}{\alpha}\right| = \frac{l}{g}$$

∴ $$T = 2\pi\sqrt{\left|\frac{\theta}{\alpha}\right|} \quad \text{or} \quad \boxed{T = 2\pi\sqrt{\frac{l}{g}}}$$

Note *that the period is independent of the mass of the pendulum.*

Energy Method

Let us derive the same expression by energy method. Suppose ω be the angular velocity of particle at angular displacement θ about point O. Then, total mechanical energy of particle in position A is,

$$E = \frac{1}{2}I\omega^2 + mg(h_A - h_C)$$

or $$E = \frac{1}{2}(ml^2)\omega^2 + mgl(1 - \cos\theta)$$

E is constant, therefore, $\frac{dE}{dt} = 0$ or $0 = ml^2\omega\left(\frac{d\omega}{dt}\right) + mgl\sin\theta\left(\frac{d\theta}{dt}\right)$

Putting $\frac{d\theta}{dt} = \omega$, $\frac{d\omega}{dt} = \alpha$ and $\sin\theta \approx \theta$, we get the same expression

$$\alpha = -\left(\frac{g}{l}\right)\theta$$

$$\therefore \qquad T = 2\pi\sqrt{\left|\frac{\theta}{\alpha}\right|} \quad \text{or} \quad \boxed{T = 2\pi\sqrt{\frac{l}{g}}}$$

Spring-block System

Suppose a mass m is attached to the free end of a massless spring of spring constant k, with its other end fixed to a rigid support.

If the mass be displaced through a distance x, as shown, a linear restoring force,

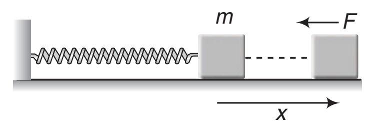


Fig. 14.16

$$F = -kx \qquad \ldots\text{(i)}$$

starts acting on the mass, tending to bring it back into its original position. The negative sign simply indicates that it is directed oppositely to the displacement of the mass.

Eq. (i) can be written as,

$$ma = -kx \qquad \ldots\text{(ii)}$$

or

$$\left|\frac{x}{a}\right| = \frac{m}{k}$$

$$\therefore \qquad T = 2\pi\sqrt{\left|\frac{x}{a}\right|} \quad \text{or} \quad T = 2\pi\sqrt{\frac{m}{k}}$$

Energy Method

The time period of the spring-block system can also be obtained by the energy method. Let v be the speed of the mass in displaced position. Then total mechanical energy of the spring-block system is.

$$E = \text{kinetic energy of the block} + \text{elastic potential energy}$$

or

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

Since, E is constant

$$\frac{dE}{dt} = 0 \quad \text{or} \quad 0 = mv\left(\frac{dv}{dt}\right) + kx\left(\frac{dx}{dt}\right)$$

Substituting,

$$\frac{dx}{dt} = v \quad \text{and} \quad \frac{dv}{dt} = a$$

We have,

$$ma = -kx$$

$$\therefore \qquad T = 2\pi\sqrt{\left|\frac{x}{a}\right|} = 2\pi\sqrt{\frac{m}{k}}$$

Example 14.12 *F-x equation of a body of mass 2 kg in SHM is*

$$F + 8x = 0$$

Here, F is in newton and x in metre. Find time period of oscillations.

Solution The given equation can be written as, $F = -8x$

Comparing with the standard equation of SHM, $F = -Kx$ we have,

$$K = 8\,\text{N/m}$$

$$\therefore \qquad T = 2\pi\sqrt{\frac{m}{K}}$$

$$= 2\pi\sqrt{\frac{2}{8}}$$

$$= (\pi)\,\text{s} = 3.14\,\text{s}$$ **Ans.**

Example 14.13 *Acceleration of a particle in SHM at displacement $x = 10$ cm (from the mean position is $a = -2.5$ cm/s^2). Find time period of oscillations.*

Solution Time period is given by

$$T = 2\pi\sqrt{\left|\frac{x}{a}\right|}$$

Substituting the values we have,

$$T = 2\pi\sqrt{\frac{10}{2.5}} = (4\pi)\,\text{s}$$ **Ans.**

Example 14.14 *Find time period of a vertical spring-block system by both methods.*

Solution **Force Method**

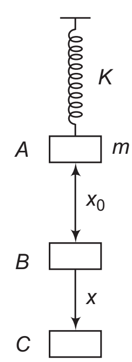


Fig. 14.17

Position *A* In this position, spring is in its natural length. But net force on block is not zero. Its *mg* is downwards.

Position *B* This is equilibrium position. Net force on block is zero. Spring force kx_0 in upward direction is equal to mg force in downward direction.

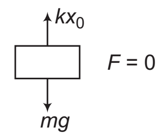


Fig. 14.18

Position *C* This is displaced position.

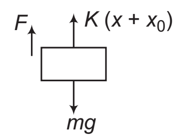


Fig. 14.19

Net restoring force is upward direction or towards the mean position is

$$F = -[k(x + x_0) - mg] = -kx \quad (\text{as } kx_0 = mg)$$

Since, $F = -kx$

$$T = 2\pi\sqrt{\frac{m}{k}}$$

Energy Method Taking $h = 0$, at the mean position B, total mechanical energy in displaced position C is

$$E = \frac{1}{2}mv^2 + \frac{1}{2}k(x + x_0)^2 - mg\,x$$

As $E =$ constant $\frac{dE}{dt} = 0$

or $$\frac{1}{2}m(2v)\frac{dv}{dt} + \frac{1}{2}k[2(x + x_0)]\frac{dx}{dt} - mg\frac{dx}{dt} = 0$$

Substituting, $\frac{dx}{dt} = v$, $\frac{dv}{dt} = a$, $kx_0 = mg$ and $ma = F$, the above equation also converts into,

$$F = -kx$$

$\therefore$ $$T = 2\pi\sqrt{\frac{m}{k}}$$

Example 14.15 *A block with a mass of 3.00 kg is suspended from an ideal spring having negligible mass and stretches the spring by 0.2 m.*

(a) What is the force constant of the spring?

(b) What is the period of oscillation of the block if it is pulled down and released?

Solution (a) In equilibrium,

$$kl = mg \qquad (l = \text{extension in spring})$$

$\therefore$ $$k = \frac{mg}{l} \qquad \dots\text{(i)}$$

Substituting the proper values, we have

$$k = \frac{(3.00)(9.8)}{0.2}$$

$$= 147 \text{ N/m}$$ **Ans.**

(b) From Eq. (i)

$$\frac{m}{k} = \frac{l}{g}$$

$\therefore$

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{l}{g}}$$

$$= 2\pi\sqrt{\frac{0.2}{9.8}} = 0.897 \text{ s}$$ **Ans.**

INTRODUCTORY EXERCISE 14.3

1. a - x equation of a body in SHM is $a + 16x = 0$. Here, x is in cm and a in cm/s^2. Find time period of oscillations.
2. A mass M, attached to a spring, oscillates with a period of 2 s. If the mass is increased by 4 kg, the time period increases by one second. Assuming that Hooke's law is obeyed, find the initial mass M.
3. Three masses of 500 g, 300 g and 100 g are suspended at the end of a spring as shown and are in equilibrium. When the 500 g mass is removed suddenly, the system oscillates with a period of 2 s. When the 300 g mass is also removed, it will oscillate with period T. Find the value of T.

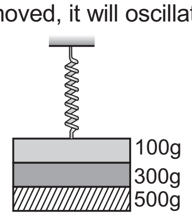


Fig. 14.20

4. A particle executes simple harmonic motion. Its instantaneous acceleration is given by $a = -px$, where p is a positive constant and x is the displacement from the mean position. Find angular frequency of oscillations.
5. A mass M is suspended from a spring of negligible mass. The spring is pulled a little then released, so that the mass executes simple harmonic motion of time period T. If the mass is increased by m, the time period becomes $\frac{5T}{3}$. Find the ratio of m/M.
6. The length of a simple pendulum is decreased by 21%. Find the percentage change in its time period.
7. A particle executes SHM on a straight line path. The amplitude of oscillation is 2 cm. When the displacement of the particle from the mean position is 1 cm, the numerical value of magnitude of acceleration is equal to the numerical value of magnitude of velocity. Find the frequency of SHM (in Hz).

14.6 Vector Method of Combining Two or More Simple Harmonic Motions

A simple harmonic motion is produced when a force (called restoring force) proportional to the displacement acts on a particle. If a particle is acted upon by two such forces the resultant motion of the particle is a combination of two simple harmonic motions. Suppose the two individual motions are represented by,

$$x_1 = A_1 \sin \omega t$$

and
$$x_2 = A_2 \sin (\omega t + \phi)$$

Both the simple harmonic motions have same angular frequency ω.

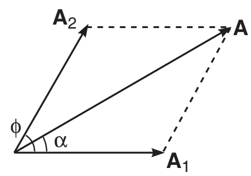


Fig. 14.21

The resultant displacement of the particle is given by,

$$\begin{aligned} x &= x_1 + x_2 \\ &= A_1 \sin \omega t + A_2 \sin (\omega t + \phi) \\ &= A \sin (\omega t + \alpha) \end{aligned}$$

Here,
$$A = \sqrt{A_1^2 + A_2^2 + 2A_1 A_2 \cos \phi}$$

and
$$\tan \alpha = \frac{A_2 \sin \phi}{A_1 + A_2 \cos \phi}$$

Thus, we can see that this is similar to the vector addition. The same method of vector addition can be applied to the combination of more than two simple harmonic motions.

Example 14.16 *Find the displacement equation of the simple harmonic motion obtained by combining the motions.*

$$x_1 = 2 \sin \omega t, \quad x_2 = 4 \sin \left(\omega t + \frac{\pi}{6}\right) \quad \textit{and} \quad x_3 = 6 \sin \left(\omega t + \frac{\pi}{3}\right)$$

Solution The resultant equation is,

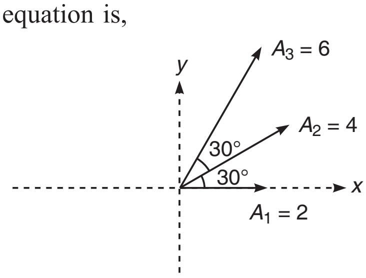


Fig. 11.22

$$x = A \sin(\omega t + \phi)$$

$$\Sigma A_x = 2 + 4\cos 30^\circ + 6\cos 60^\circ = 8.46$$

and $$\Sigma A_y = 4\sin 30^\circ + 6\cos 30^\circ = 7.2$$

$\therefore$ $$A = \sqrt{(\Sigma A_x)^2 + (\Sigma A_y)^2}$$

$$= \sqrt{(8.46)^2 + (7.2)^2}$$

$$= 11.25$$

and $$\tan\phi = \frac{\Sigma A_y}{\Sigma A_x}$$

$$= \frac{7.2}{8.46} = 0.85$$

or $$\phi = \tan^{-1}(0.85)$$

$$= 40.4^\circ$$

Thus, the displacement equation of the combined motion is,

$$x = 11.25 \sin(\omega t + \phi)$$

where, $$\phi = 40.4^\circ$$ **Ans.**

INTRODUCTORY EXERCISE 14.4

1. A particle is subjected to two simple harmonic motions of the same frequency and direction. The amplitude of the first motion is 4.0 cm and that of the second is 3.0 cm. Find the resultant amplitude if the phase difference between the two motions is

   (a) 0° (b) 60° (c) 90° (d) 180°

2. A particle is subjected to two simple harmonic motions.

   $$x_1 = 4.0 \sin(100\pi t) \quad \text{and} \quad x_2 = 3.0 \sin\left(100\pi t + \frac{\pi}{3}\right)$$

   Find
   (a) the displacement at $t = 0$
   (b) the maximum speed of the particle and
   (c) the maximum acceleration of the particle.

3. Three simple harmonic motions of equal amplitudes A and equal time periods in the same direction combine. The phase of the second motion is 60° ahead of the first and the phase of the third motion is 60° ahead of the second. Find the amplitude of the resultant motion.

4. A particle is subjected to two simple harmonic motions in the same direction having equal amplitudes and equal frequency. If the resultant amplitude is equal to the amplitude of the individual motions. Find the phase difference between the individual motions.

Final Touch Points

1. Simple Pendulum

(i) Time period $T = 2\pi\sqrt{\frac{l}{g}}$ is applicable only for small oscillations.

(ii) If the time period of a simple pendulum is 2 s, it is called **seconds pendulum.** Length of this pendulum is 1 m.

(iii) If length of the pendulum is large, g no longer remains vertical but will be directed towards the centre of the earth and expression for time period is given by,

$$T = 2\pi\sqrt{\frac{1}{g\left(\frac{1}{l}+\frac{1}{R}\right)}}$$

Here, R is the radius of earth. From this expression we can see that,

(a) if $l \ll R$, $\frac{1}{l} >> \frac{1}{R}$ and $T = 2\pi\sqrt{\frac{l}{g}}$

(b) as $l \to \infty, \frac{1}{l} \to 0$ and $T = 2\pi\sqrt{\frac{R}{g}}$ and substituting the value of R and g, we get $T = 84.6$ min.

(iv) Time period of a simple pendulum depends on acceleration due to gravity 'g' $\left(\text{as } T \propto \frac{1}{\sqrt{g}}\right)$ so take $|\mathbf{g}_{\text{eff}}|$ in $T = 2\pi\sqrt{\frac{l}{g}}$. Following two cases are possible :

(a) If a simple pendulum is in a carriage which is accelerating with acceleration $\mathbf{a}$, then

$$\mathbf{g}_{\text{eff}} = \mathbf{g} - \mathbf{a}$$

e.g. if the acceleration $\mathbf{a}$ is upwards, then

$$|\mathbf{g}_{\text{eff}}| = g + a \quad \text{and} \quad T = 2\pi\sqrt{\frac{l}{g+a}}$$

If the acceleration $\mathbf{a}$ is downwards, then (with $g > a$)

$$|\mathbf{g}_{\text{eff}}| = g - a \quad \text{and} \quad T = 2\pi\sqrt{\frac{l}{g-a}}$$

If the acceleration $\mathbf{a}$ is in horizontal direction, then

$$|\mathbf{g}_{\text{eff}}| = \sqrt{a^2 + g^2}$$

In a freely falling lift $g_{\text{eff}} = 0$ and $T = \infty$ i.e. the pendulum will not oscillate.

(b) If in addition to gravity one additional force $\mathbf{F}$, (e.g. electrostatic force $\mathbf{F}_e$) is also acting on the bob, then in that case,

$$\mathbf{g}_{\text{eff}} = \mathbf{g} + \frac{\mathbf{F}}{m}$$

Here, m is the mass of the bob.

(v) Due to change in temperature, length of pendulum and so the time period will change. If $\Delta\theta$ is the increase in temperature then,

$$l' = l(1 + \alpha\Delta\theta) \quad \text{or} \quad \frac{l'}{l} = 1 + \alpha\Delta\theta$$

$$\therefore \quad \frac{T'}{T} = \sqrt{\frac{l'}{l}} = (1 + \alpha\Delta\theta)^{1/2} \approx \left(1 + \frac{1}{2}\alpha\Delta\theta\right)$$

$\therefore$ $$\frac{T'}{T} - 1 \approx \frac{1}{2}\alpha\Delta\theta$$

or $$\frac{T' - T}{T} = \frac{1}{2}\alpha\Delta\theta$$

or $$\Delta T = \frac{1}{2}T\alpha\Delta\theta$$

Note *In case of a pendulum clock, time is lost if T increases and gained if T decreases. Time lost or gained in time t is given by,*

$$\Delta t = \frac{\Delta T}{T'} \cdot t$$

e.g. if $T = 2$ s, $T' = 3$ s, then $\Delta T = 1$ s

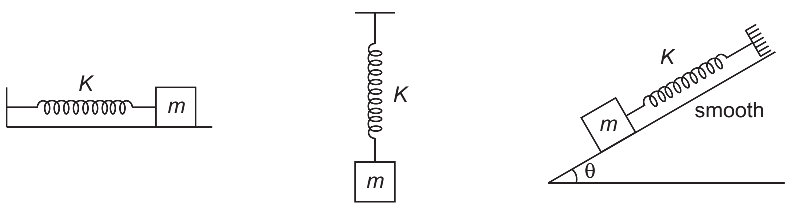
$\therefore$ *Time lost by the clock in 1 h.*

$$\Delta t = \frac{1}{3} \times 3600 = 1200 \text{ s}$$

(vi) Above the surface of earth, blow the surface of earth, in moving from pole to equator and on moon value of g decreases, therefore time period of pendulum will increase.

2. Spring-block System

(i)



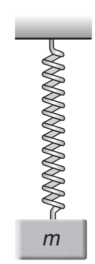
(a) (b) (c)

In all three figures shown above, restoring force in displaced position is $F = -kx$. Therefore, time period is,

$$T = 2\pi\sqrt{\frac{m}{K}}$$

The only difference is, their mean positions are different. In the first figure, mean position is at the natural length of spring. In the second figure, mean position is obtained after an extension of $x_0 = \frac{mg}{K}$ (as $kx_0 = mg$) and in the third figure after an extension of $x_0 = \frac{mg\sin\theta}{K}$.

(ii) In case of a vertical spring-block system, time period can also be written as,



$$\boxed{T = 2\pi\sqrt{\frac{l}{g}}}$$

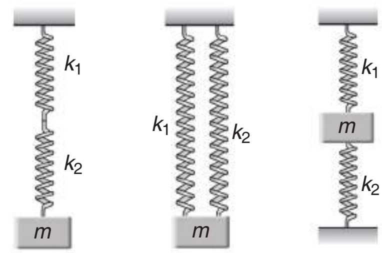
Here, l = extension in the spring when the mass m is suspended from the spring. This can be seen as under :

$$kl = mg \quad \text{(in equilibrium position)}$$

$\therefore$ $$\frac{m}{k} = \frac{l}{g}$$

or $$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{l}{g}}$$

(iii) **Equivalent force constant (k)** If a spring block system is constructed by using two springs and a mass, the following three situations are possible:



(a) (b) (c)

Refer Fig. (a)

In this case,

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$$

or
$$k = \frac{k_1 k_2}{k_1 + k_2}$$

$\therefore$
$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m(k_1 + k_2)}{k_1 k_2}}$$

Fig. (b) and (c)

In both the cases,

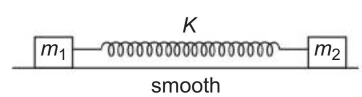
$$k = k_1 + k_2$$

or
$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{k_1 + k_2}}$$

(iv) If spring has a mass m_s and a mass m is suspended from it, then time period is given by,

$$T = 2\pi\sqrt{\frac{m + \frac{m_s}{3}}{k}}$$

(v) If the spring connected with two masses m_1 and m_2 is compressed or elongated by x_0 and then left for oscillations then both blocks execute SHM with same time period (and therefore same frequency) but different amplitudes.



This time period is

$$T = 2\pi\sqrt{\frac{\mu}{K}}$$

Here, μ is called their reduced mass given by

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \text{ or } \mu = \frac{m_1 m_2}{m_1 + m_2}$$

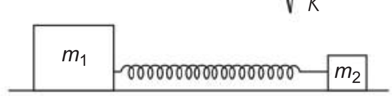
Their amplitudes of oscillations are in inverse ratio of their masses or

$$A \propto \frac{1}{m} \quad \Rightarrow \quad \frac{A_1}{A_2} = \frac{m_2}{m_1}$$

$\Rightarrow$
$$A_1 = \left(\frac{m_2}{m_1 + m_2}\right)x_0 \quad \text{and} \quad A_2 = \left(\frac{m_1}{m_1 + m_2}\right)x_0$$

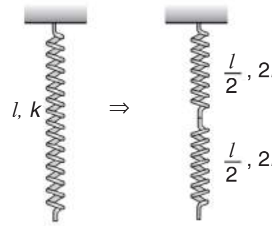
Note *If $m_1 >> m_2$, then $\frac{1}{m_1} << \frac{1}{m_2}$*

$\therefore$
$$\frac{1}{\mu} \approx \frac{1}{m_2} \quad \textit{or} \quad \mu \approx m_2 \quad \textit{and} \quad T = 2\pi\sqrt{\frac{m_2}{K}}$$



So, in this case we can understand that m_1 is almost stationary. Only m_2 will oscillate.

(vi) The force constant (k) of a spring is inversely proportional to the length of the spring. i.e.



$$k \propto \frac{1}{\text{length of spring}}$$

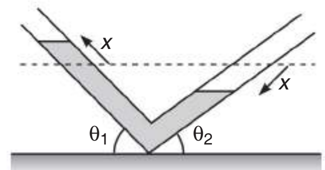
This can be visualized as:

A spring of length l and spring constant k can be supposed to be made up by two springs in series, of length $\frac{l}{2}$ and force constant $2k$. In series,

$$k_{\text{eff}} = \frac{k_1 k_2}{k_1 + k_2} = \frac{(2k)(2k)}{2k + 2k} = k$$

3. Oscillations of a fluid column

Initially the level of liquid in both the columns is same. The area of cross-section of the tube is uniform. If the liquid is depressed by x in one limb, it will rise by x along the length of the tube is the other limb. Here, the restoring force is provided by the hydrostatic pressure difference.



$\therefore$
$$F = -(\Delta P)A = -(h_1 + h_2)\rho g A$$
$$= -\rho g A(\sin\theta_1 + \sin\theta_2)\, x$$

Let, m be the mass of the liquid in the tube. Then,

$$ma = -\rho g A(\sin\theta_1 + \sin\theta_2)\, x$$

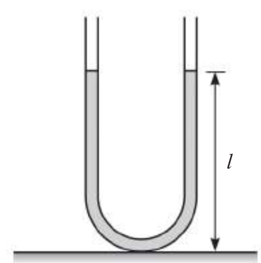
Since, F or a is proportional to $-x$, the motion of the liquid column is simple harmonic in nature, time period of which is given by,

$$T = 2\pi\sqrt{\left|\frac{x}{a}\right|}$$

or
$$\boxed{T = 2\pi\sqrt{\frac{m}{\rho g A(\sin\theta_1 + \sin\theta_2)}}}$$

Note *For a U-tube, if the liquid is filled to a height l, $\theta_1 = 90° = \theta_2$ and $m = 2(lA\rho)$*

So,
$$T = 2\pi\sqrt{\frac{l}{g}}$$



Thus, we see that the expression $T = 2\pi\sqrt{\frac{l}{g}}$ comes in picture at three places.

(i) Time period of a simple pendulum for small oscillations.
(ii) Time period of a spring-block system in vertical position.
(iii) Time period of a liquid column in a U-tube filled to a height l.
But l has different meanings at different places.

4. Lissajous figures Suppose two forces act on a particle, the first alone would produce a simple harmonic motion in x-direction given by,

$$x = a\sin\omega t$$

and the second would produce a simple harmonic motion in y-direction given by,

$$y = b\sin(\omega t + \phi)$$

The amplitudes a and b may be different and their phases differ by ϕ. The frequencies of the two simple harmonic motions are assumed to be equal. The resultant motion of the particle is a combination of the two simple harmonic motions.

Depending on the value of ϕ and relation between a and b, the particle follows different paths. These different paths are called Lissajous figures. Given below are few special cases :

***Case* 1 (When $\phi = 0°$)** When the phase difference between two simple harmonic motions is 0°, i.e.

$$x = a \sin \omega t \quad \Rightarrow \quad \sin \omega t = \frac{x}{a} \qquad \ldots(i)$$

$$y = b \sin \omega t \quad \Rightarrow \quad \sin \omega t = \frac{y}{b} \qquad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$\frac{x}{a} = \frac{y}{b} \quad \text{or} \quad y = \left(\frac{b}{a}\right) x$$

which is equation of a straight line with slope $\frac{b}{a}$. Thus, the path of the particle is a straight line. As a special case $y = x$ if $a = b$ or slope is 1.

***Case* 2 $\left(\text{when } \phi = \frac{\pi}{2}\right)$** When the phase difference is $\frac{\pi}{2}$ i.e.

$$x = a \sin \omega t \quad \Rightarrow \quad \sin \omega t = \frac{x}{a} \qquad \ldots(iii)$$

$$y = b \sin\left(\omega t + \frac{\pi}{2}\right) = b \cos \omega t$$

$$\Rightarrow \quad \cos \omega t = \frac{y}{b} \qquad \ldots(iv)$$

Squaring and adding, Eqs. (iii) and (iv), we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

which is an **ellipse.** Again as a special case, the above equation reduces to,

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$$

or

$$x^2 + y^2 = a^2 \qquad (\text{for } a = b)$$

This is an equation of a **circle.**

5. Free and damped oscillations

We know that in reality, a spring won't oscillate forever with constant amplitude. These constant amplitude oscillations (which will really occur in vacuum) are called **"free oscillations"**. Frictional forces will diminish the amplitude of oscillation until eventually the system comes to rest.

We will now add frictional forces to the mass and spring. Imagine that the mass was put in a viscous liquid. To incorporate friction, we can just say that there is a frictional force that's proportional to the velocity of the mass. This is a pretty good approximation for a body moving at a low velocity in air, or in a liquid. So we say the frictional force $f_r = -bv$. The constant b depends on the kind of liquid and the shape of the mass. The negative sign, just says that the force is in the opposite direction to the body's motion. Let's add this frictional force in to the equation $F_{\text{net}} = ma$

$$-kx - bv = ma \qquad \ldots(i)$$

In terms of derivatives,

$$m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = 0 \qquad \ldots(ii)$$

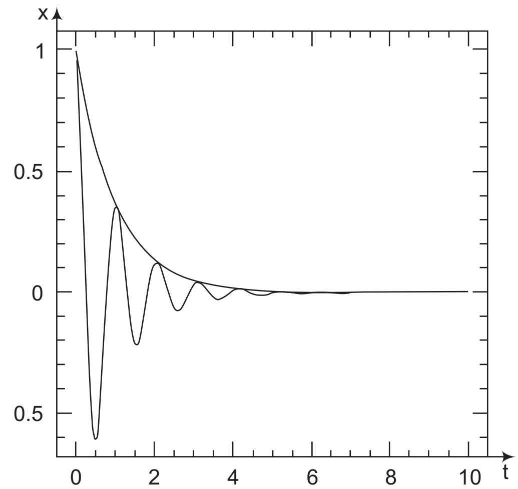
This is a differential equation. In the solution of this type of differential equation instead of the amplitude being constant, it is decaying with time.

$$\boxed{A(t) = A_0 e^{-\alpha t}}$$

Here, α is a positive constant.

So, $\boxed{x(t) = A(t)\cos(\omega t + \delta) = A_0 e^{-\alpha t}\cos(\omega t + \delta)}$

Here's a plot of an example of such a function.



These types of oscillations which eventually come to end are called **"damped oscillations".** There are further three types of damped oscillations, namely

(i) Under damped
(ii) Critically damped and
(iii) Over damped oscillations

But their detailed discussion is out of our syllabus at this stage.

Natural Frequency (or Characteristic Frequency)

That is the frequency at which a system would oscillate by itself if displaced. The natural frequency of a spring-mass system is $f_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$, where k is the spring constant and m is the mass of the object attached to the spring.

Forced Oscillations

A periodic force at a given frequency (called driving frequency f_d) is applied to an oscillating system of natural frequency f_0. At the beginning (transient stage), there is a mixture of two kinds of oscillations, one has the frequency f_0 and the other has f_d. The former will gradually die out because of the damping effect. Eventually (at the steady state) the system settles down with oscillation at the frequency of the driving force (f_d).

Resonance

When the driving frequency is at the same frequency as the natural frequency of the oscillator, the amplitude of oscillation is at its greatest. When this happens the energy of the oscillator becomes a maximum. This is called a condition of resonance.

Resonance and its Consequences

The fact that there is an amplitude peak at driving frequencies close to the natural frequency of the system is called **resonance.** Physics is full of examples of resonance ; building up the oscillations of a child on a swing by pushing with a frequency equal to the swing's natural frequency is one. Inexpensive loudspeakers often have an unwanted boom or buzz when a musical note happens to coincide with the resonant frequency of the speaker cone. Resonance also occurs in electric circuits ; a tuned circuit in a radio or television receiver responds strongly to waves having frequencies near its resonant frequency, and this fact is used to select a particular station and reject the others.

Resonance in mechanical systems can be destructive. A company of soldiers once destroyed a bridge by marching across it in step ; the frequency of their steps was close to the natural vibration frequency of the bridge and the resulting oscillation had large enough amplitude to tear the bridge apart. Ever since, marching soldiers have been ordered to break step before crossing a bridge.

Nearly everyone has seen the film of the collapse of the Tacoma Narrows suspension bridge in 1940.

Solved Examples

TYPED PROBLEMS

Type 1. *According to given x - t equation we have to find initial conditions (at $t = 0$).*

Concept

The method discussed here is a general method which can be applied for any type of motion.

How to Solve?

Differentiate the given x-t equation and find v-t equation. Now, put $t = 0$ in both x-t and v-t equations.

Example 1 *x - t equation of a particle executing SHM is*

$$x = A\cos(\omega t - 45°)$$

Find the point from where particle starts its journey and the direction of its initial velocity.

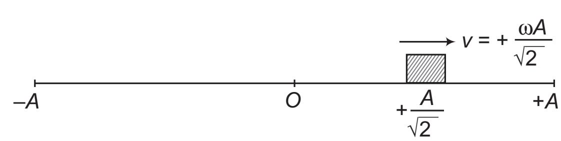
Solution Given,

$$x = A\cos(\omega t - 45°) \quad \ldots\text{(i)}$$

$$v = \frac{dx}{dt} = -\omega A \sin(\omega t - 45°) \quad \ldots\text{(ii)}$$

Putting $t = 0$ in both equations we have,

$$x = +\frac{A}{\sqrt{2}} \quad \text{and} \quad v = +\frac{\omega A}{\sqrt{2}}$$



At $t = 0$

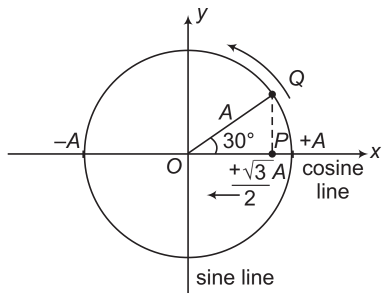
So, the particle starts from $x = +\frac{A}{\sqrt{2}}$ with velocity in positive direction.

Note *By further differentiating v-t equation, we can make a-t equation and by putting $t = 0$, we can also find initial value of acceleration.*

Type 2. *First Use of reference circle (to make x- t equation corresponding to given initial conditions).*

How to Solve?

For the given SHM particle P on the diameter we have to find its circular motion particle Q on the circle according to given initial conditions. The following points will help you in finding the particle Q.



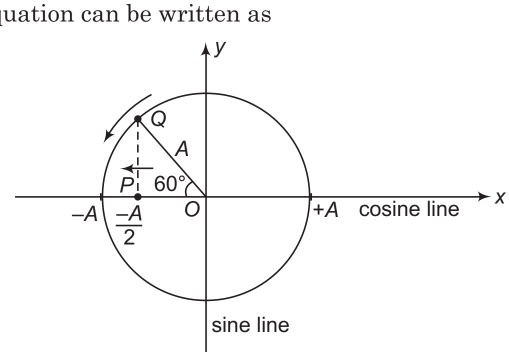
- Particle Q can rotate, either clockwise or anticlockwise. But in all problems, we will take it anticlockwise.
- We can draw the perpendicular on any diameter but in all problems we will draw the perpendicular lying along x-axis.
- If initial velocity of SHM particle P is negative then Q lies in upper half of the reference circle. Because, when Q moves in upper half of the circle (anticlockwise) P moves from $+A$ to $-A$ along negative x-direction. Similarly, if P has an initial velocity in positive direction then Q is in lower half of the circle.
- If initial value of x is positive then Q lies to the right of O (between 0 and $+A$) and if x is negative, then Q lies to the left of O (between 0 and $-A$).
- If initial value of x is suppose $+\frac{\sqrt{3}}{2}A$ and velocity is negative then circular particle Q is in the position as shown in figure.

  This particle is 120° ahead (in anticlockwise direction) or 240° behind (in clockwise direction) from sine line or it is 30° ahead or 330° behind the cosine line. Hence, the x-t equation corresponding to given initial conditions may be

$$
\begin{aligned}
x &= A\sin(\omega t + 120°) \\
&= A\sin(\omega t - 240°) \\
&= A\cos(\omega t + 30°) \\
&= A\cos(\omega t - 330°)
\end{aligned}
$$

Example 2 *A particle in SHM starts its journey (at $t=0$) from $x=-\frac{A}{2}$ in negative direction. Write x-t equation corresponding to given conditions. Angular frequency of oscillations is ω.*

Solution For the given initial conditions, particle Q on reference circle is as shown in figure. This is 210° ahead or 150° behind the sine line or 120° ahead or 240° behind the cosine line. Therefore, the x-t equation can be written as



$$x = A\sin(\omega t + 210°) = A\sin(\omega t - 150°) = A\cos(\omega t + 120°) = A\cos(\omega t - 240°)$$

Type 3. *Second use of reference circle (to find the shortest time in moving the particle from one point to another point between $+A$ and $-A$).*

Concept

Linear speed of particle in SHM is not constant (more near mean positive and less near mean extreme positions)

So, we cannot apply

$$\text{time} = \frac{\text{distance}}{\text{speed}}$$

but the angular speed of its corresponding particle on reference circle is constant.

Therefore, $$t = \frac{\theta}{\omega} \quad (\theta \text{ in radian})$$

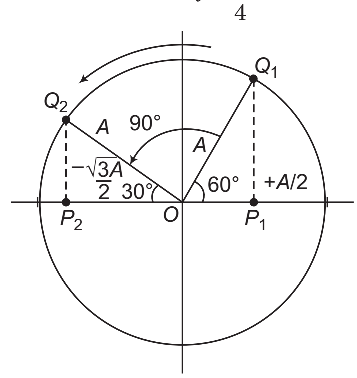
can be applied for that particle. The following table will help you in solving such problems.

Angle rotated in radian	Angle rotated in degrees	Time taken
2π	360°	T
π	180°	$T/2$
$\pi/3$	120°	$T/3$
$\pi/2$	90°	$T/4$
$\pi/4$	45°	$T/8$
$\pi/6$	30°	$T/12$

Example 3 *In terms of time period of oscillations T, find the shortest time in moving a particle from* $+\frac{A}{2}$ *to* $-\frac{\sqrt{3}}{2}$.

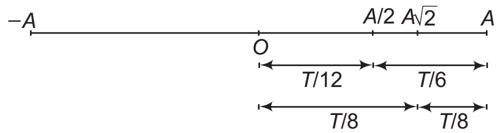
Solution As shown in figure, as the SHM particle P moves from P_1 to P_2, its corresponding particle Q on reference circle rotates from Q_1 to Q_2 an angle of 90°. From the above table, we can see that, in rotating an angle of 90° the time taken is $\frac{T}{4}$.

$$t = \frac{T}{4}$$ **Ans.**



Note *Some, standard results which are frequently asked are given in following table:*

From	To	Time taken
O	$A/2$	$T/12$
$A/2$	A	$T/6$
O	$A/\sqrt{2}$	$T/8$
$A/\sqrt{2}$	A	$T/8$

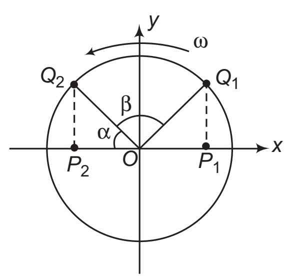


Near the mean position, speed of particle is more. Therefore, it covers first half of the distance (from 0 to $A/2$) in a shorter time compared to the second half (from $A/2$ to A).

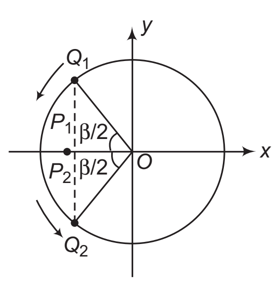
Type 4. *Third Use of reference circle (finding time of collision of two particles executing SHM along same straight line with same ω and A but different initial conditions).*

How to Solve?

Corresponding to initial conditions of SHM particles P_1 and P_2, first find their particles Q_1 and Q_2 on the reference circle.



Now, we can see that the particles Q_1 and Q_2 will never collide as they are rotating in the same direction (anticlockwise) with same ω and T. But the perpendiculars from Q_1 and Q_2 on our diameter along x-axis will collide when this diameter becomes the bisector line of the constant angle β (between OQ_1 and OQ_2) as shown below.



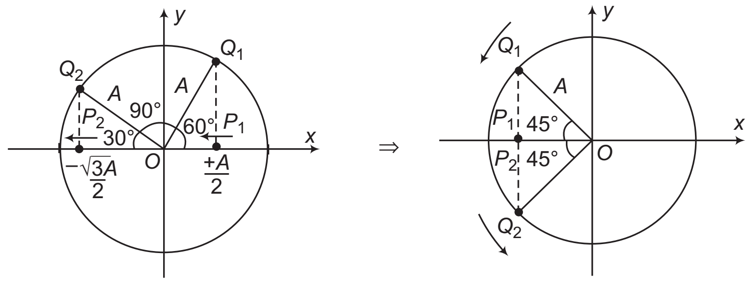
Now, either of the two particles has rotated an angle.

$$\theta = \alpha + (\beta/2)$$

∴ Time taken is
$$t = \frac{\theta}{\omega}$$

Example 4 *Two SHM particles P_1 and P_2 start from $+\frac{A}{2}$ and $-\frac{\sqrt{3}A}{2}$, both in negative directions. Find the time (in terms of T) when they collide. Both particles have same ω, A and T and they execute SHM along the same line.*

Solution



In the first figure, we have shown two particles Q_1 and Q_2 corresponding to given initial conditions of P_1 and P_2. In the second figure, we have shown the moment when the SHM particles are colliding at $x = -A\cos 45° = -\frac{A}{\sqrt{2}}$.

We can see that either of the particles Q_1 and Q_2 has rotated an angle $\theta = 75°$ or $\frac{5\pi}{12}$. So, the time taken is

$$t = \frac{\theta}{\omega} = \frac{(5\pi/12)}{(2\pi/T)} = \left(\frac{5}{24}\right)T$$

Note *In the above problem, the constant angle between OQ_1 and OQ_2 is 90°. So, the constant phase difference between two SHM particles corresponding to given initial conditions is also 90°.*

Type 5. *Spring Block Systems.*

Concept

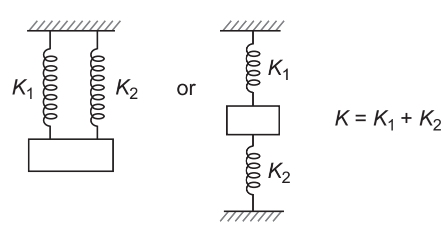
There are a lot of problems based on spring-block system. The main concepts are:

(i) $T = 2\pi\sqrt{\frac{m}{K}}$ and $\omega = \frac{2\pi}{T} = \sqrt{\frac{K}{m}}$

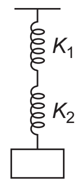
(ii) Force constant of spring,

$$K \propto \frac{1}{\text{Length of spring}}$$

(iii)



(iv)



$$\frac{1}{K}=\frac{1}{K_1}+\frac{1}{K_2}$$

$$\text{or } K=\frac{K_1K_2}{K_1+K_2}$$

Let us take some examples based on the above concepts.

Example 5 *A mass is suspended separately by two springs and the time periods in the two cases are T_1 and T_2. Now the same mass is connected in parallel ($K=K_1+K_2$) with the springs and the time is suppose T_P. Similarly time period in series is T_S, then find the relation between T_1, T_2 and T_P in the first case and T_1, T_2 and T_S in the second case.*

Solution $T=2\pi\sqrt{\frac{m}{K}}$

$\therefore$ $$T=\frac{\alpha}{\sqrt{K}}$$ (where, $2\pi\sqrt{m}=\alpha=$ constant)

$\Rightarrow$ $$K=\frac{\alpha^2}{T^2} \quad \text{...(i)}$$

or $$\frac{1}{K}=\frac{T^2}{\alpha^2} \quad \text{...(ii)}$$

First case In parallel,

$$K=K_1+K_2$$

Using Eq. (i), $$\frac{\alpha^2}{T_P^2}=\frac{\alpha^2}{T_1^2}+\frac{\alpha^2}{T^2}$$

$\therefore$ $$T_P^{-2}=T_1^{-2}+T_2^{-2}$$ **Ans.**

This is the desired relation.

Second case In series, $$\frac{1}{K}=\frac{1}{K_1}+\frac{1}{K_2}$$

Now, using Eq. (ii) we have, $$\frac{T_s^2}{\alpha^2}=\frac{T_1^2}{\alpha^2}+\frac{T_2^2}{\alpha^2}$$

$\therefore$ $$T_s^2=T_1^2+T_2^2$$ **Ans.**

This is the desired relation in this case.

Example 6 *Time period of a block with a spring is T_0. Now, the spring is cut in two parts in the ratio 2 : 3. Now find the time period of same block with the smaller part of the spring.*

Solution $T=2\pi\sqrt{\frac{m}{K}}$ $\Rightarrow$ $$T\propto\frac{1}{\sqrt{K}}$$

But $$K\propto\frac{1}{l}$$ ($l=$ length of spring)

$\therefore$ $$T\propto\sqrt{l} \quad \text{...(i)}$$

Now, suppose total length of the spring is $5l$ and it is cut in two parts of lengths $2l$ and $3l$. Eq. (i) can be written as

$$\frac{T_{2l}}{T_{5l}} = \sqrt{\frac{2l}{5l}} = \sqrt{\frac{2}{5}}$$

$\therefore$
$$T_{2l} = \left(\sqrt{\frac{2}{5}}\right) T_{5l} = \left(\sqrt{\frac{2}{5}}\right) T_0$$ **Ans.**

Example 7 *With the assumption of no slipping, determine the mass m of the block which must be placed on the top of a 6 kg cart in order that the system period is 0.75 s. What is the minimum coefficient of static friction μ_s for which the block will not slip relative to the cart if the cart is displaced 50 mm from the equilibrium position and released? Take $(g = 9.8\ m/s^2)$.*

Solution (a)
$$T = 2\pi\sqrt{\frac{m+6}{600}} \qquad \left(T = 2\pi\sqrt{\frac{m}{K}}\right)$$

or
$$0.75 = 2\pi\sqrt{\frac{m+6}{600}}$$

$\therefore$
$$m = \frac{(0.75)^2 \times 600}{(2\pi)^2} - 6$$
$$= 2.55 \text{ kg}$$ **Ans.**

(b) Maximum acceleration of SHM is,

$$a_{\max} = \omega^2 A \qquad (A = \text{amplitude})$$

i.e. maximum force on mass 'm' is $m\omega^2 A$ which is being provided by the force of friction between the mass and the cart. Therefore,

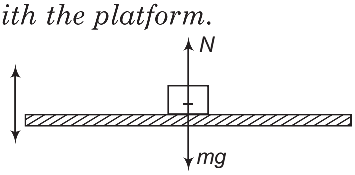
$$\mu_s mg \geq m\omega^2 A \quad \text{or} \quad \mu_s \geq \frac{\omega^2 A}{g}$$

or
$$\mu_s \geq \left(\frac{2\pi}{T}\right)^2 \cdot \frac{A}{g} \quad \text{or} \quad \mu_s \geq \left(\frac{2\pi}{0.75}\right)^2 \left(\frac{0.05}{9.8}\right) \qquad (A = 50 \text{ mm})$$

or
$$\mu_s \geq 0.358$$

Thus, the minimum value of μ_s should be 0.358. **Ans.**

Example 8 *A block is kept over a horizontal platform, executing vertical SHM of angular frequency ω. Find maximum amplitude of oscillations, so that the block does not leave contact with the platform.*



Solution At mean position, acceleration, $a = 0$

$\Rightarrow$ $$N = mg \qquad (N = \text{normal reaction})$$

Below the mean position, acceleration is upwards (towards the mean position)

$\therefore$ $$N - mg = ma \quad \text{or} \quad N = m(g + a)$$

Above the mean position, acceleration is downwards (towards the mean position)

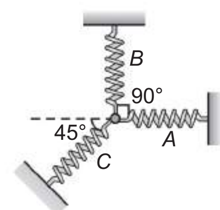
$\therefore$ $$mg - N = ma \quad \text{or} \quad N = m(g - a)$$

In this case, N decreases as 'a' increases. At the extreme position acceleration is maximum $(= \omega^2 A)$, so N is minimum. When the block leaves contact with the plank N becomes zero.

$\therefore$ $$0 = m(g - \omega^2 A) \quad \text{or} \quad A = \frac{g}{\omega^2}$$

Therefore, the maximum value of amplitude A for not leaving the contact is $\frac{g}{\omega^2}$.

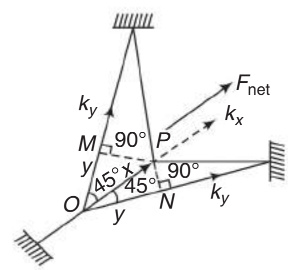
Example 9 *A particle of mass m is attached with three springs A, B and C of equal force constants k as shown in figure. The particle is pushed slightly against the spring C and released. Find the time period of oscillation.*



Solution $OP = x$

$$\angle POM = \angle PON \approx 45°$$

$$y = x \cos 45° = \frac{x}{\sqrt{2}}$$



Net restoring force,

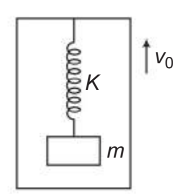
$$F_{net} = -[kx + 2ky \cos 45°]$$

$$= -(2k)x \qquad \left(\text{as } y = \frac{x}{\sqrt{2}}\right)$$

$\therefore$ $$k_e = 2k$$

Now, $$T = 2\pi\sqrt{\frac{m}{k_e}} = 2\pi\sqrt{\frac{m}{2k}}$$

Example 10 *A spring block system is kept inside a lift moving with a constant velocity v_0 as shown in figure. Block is in equilibrium and at rest with respect to lift. The lift is suddenly stopped at time $t = 0$. Taking upward direction as the positive direction, write x-t equation of the block.*



Solution Lift is suddenly stopped but block will not stop instantly. It will have same velocity v_0 at $t = 0$. But this is the velocity at its mean position (as it was at rest w.r.t lift). So, this is also its maximum velocity of SHM.

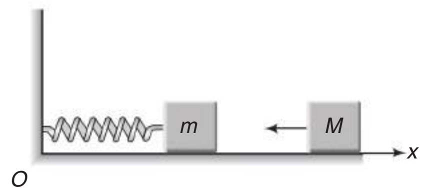
$\therefore$ $$v_0 = v_{max} = \omega A \quad \Rightarrow \quad A = \frac{v_0}{\omega} \qquad \left(\text{where, } \omega = \sqrt{\frac{K}{m}}\right)$$

Now, at $t = 0$, block is at mean position and has a velocity v_0 in upward or positive direction. Therefore, the x-t equation is

$$x = A \sin \omega t$$ **Ans.**

where, $$A = \frac{v_0}{\omega} \text{ and } \omega = \sqrt{\frac{K}{m}}$$

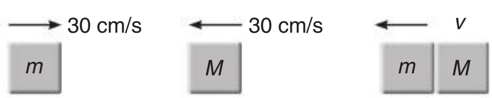
Example 11 *One end of an ideal spring is fixed to a wall at origin O and axis of spring is parallel to x-axis. A block of mass $m = 1$ kg is attached to free end of the spring and it is performing SHM. Equation of position of the block in co-ordinate system shown in figure is $x = 10 + 3\sin(10t)$. Here, t is in second and x in cm. Another block of mass $M = 3$ kg, moving towards the origin with velocity 30 cm/s collides with the block performing SHM at $t = 0$ and gets stuck to it. Calculate*



(a) new amplitude of oscillations,
(b) new equation for position of the combined body,
(c) loss of energy during collision. Neglect friction.

Solution (a) Initially, $\omega^2 = \dfrac{k}{m}$

$$\therefore \qquad k = m\omega^2 = (1)\,(10)^2 = 100 \text{ N/m}$$



At $t = 0$, block of mass m is at mean position ($x = 10$ cm) and moving towards positive x-direction with velocity $A\omega$ or 30 cm/s.

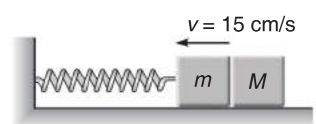
From conservation of linear momentum,

$$(M + m)v = M(30) - m(30)$$

Substituting the values, we have

$v =$ velocity of combined mass just after collision = 15 cm/s or 0.15 m/s

From conservation of mechanical energy,



$$\frac{1}{2}(M + m)v^2 = \frac{1}{2}k(A')^2$$

Here, A' = new amplitude of oscillation of combined mass

$$\therefore \qquad A' = \left(\sqrt{\frac{M+m}{k}}\right) v = \left(\frac{4}{100}\right)^{1/2} (0.15) = 0.03 \text{ m}$$

or $\qquad A' = 3$ cm **Ans.**

(b) New angular frequency $\omega' = \sqrt{\dfrac{k}{M+m}} = \sqrt{\dfrac{100}{4}} = 5$ rad/s

$$\therefore \qquad x' = 10 - 3\sin 5t$$ **Ans.**

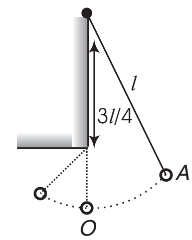
(c) Loss of mechanical energy $E_f - E_i$

$$= \frac{1}{2}(1)(0.3)^2 + \frac{1}{2}(3)(0.3)^2 - \frac{1}{2}(4)(0.15)^2$$

$$= 0.135 \text{ J}$$ **Ans.**

Type 6. *Time period of a periodic motion which is not completely simple harmonic.*

Example 12 *A pendulum has a period T for small oscillations. An obstacle is placed directly beneath the pivot, so that only the lowest one-quarter of the string can follow the pendulum bob when it swings to the left of its resting position. The pendulum is released from rest at a certain point. How long will it take to return to that point again ? In answering this question, you may assume that the angle between the moving string and the vertical stays small throughout the motion.*

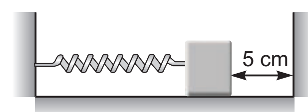


Solution Half the oscillation is completed with length l and rest half with length $\frac{l}{4}$.

$$T = \frac{T_1}{2} + \frac{T_2}{2} = \frac{1}{2}\left[2\pi\sqrt{\frac{l}{g}} + 2\pi\sqrt{\frac{\frac{l}{4}}{g}}\right]$$

$$= \frac{3}{4}\left[2\pi\sqrt{\frac{l}{g}}\right] = \frac{3}{4}T, \text{ where } T = 2\pi\sqrt{\frac{l}{g}}$$ **Ans.**

Example 13 *A block of mass 100 g attached to a spring of stiffness 100 N/m is lying on a frictionless floor as shown. The block is moved to compress the spring by 10 cm and released. If the collision with the wall is elastic then find the time period of oscillations.*



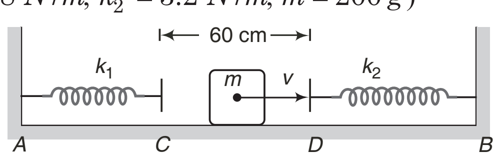
Solution The given distance on RHS 5 cm is $\frac{A}{2}$ and from 0 to $\frac{A}{2}$ time taken is $\frac{T}{12}$.

$$T = t_{\text{LHS}} + t_{\text{RHS}} = \frac{T}{2} + 2\left(\frac{T}{12}\right)$$

$$= \frac{2}{3}T = \frac{2}{3}(2\pi)\sqrt{\frac{m}{k}}$$

$$= \frac{4\pi}{3}\sqrt{\frac{0.1}{100}} = 0.133 \text{ s}$$ **Ans.**

Example 14 *Two light springs of force constants k_1 and k_2 and a block of mass m are in one line AB on a smooth horizontal table such that one end of each spring is fixed on rigid supports and the other end is free as shown in the figure. The distance CD between the free ends of the spring is 60 cm. If the block moves along AB with a velocity 120 cm/s in between the springs, calculate the period of oscillation of the block. (Take $k_1 = 1.8$ N/m, $k_2 = 3.2$ N/m, $m = 200$ g)* **(JEE 1985)**



Solution Between C and D block will move with constant speed of 120 cm/s. Therefore, period of oscillation will be (starting from C).

$$T = t_{CD} + \frac{T_2}{2} + t_{DC} + \frac{T_1}{2}$$

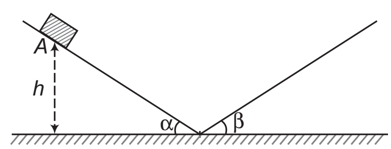
Here, $$T_1 = 2\pi\sqrt{\frac{m}{k_1}} \text{ and } T_2 = 2\pi\sqrt{\frac{m}{k_2}}$$

and $$t_{CD} = t_{DC} = \frac{60}{120} = 0.5 \text{ s}$$

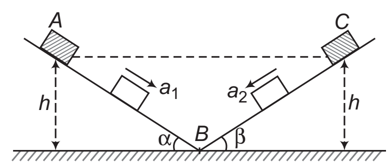
$\therefore$ $$T = 0.5 + \frac{2\pi}{2}\sqrt{\frac{0.2}{3.2}} + 0.5 + \frac{2\pi}{2}\sqrt{\frac{0.2}{1.8}}$$

$$T = 2.82 \text{ s}$$ **Ans.**

Example 15 *A block is released from point A as shown in figure. All surfaces are smooth and there is no loss of mechanical energy anywhere. Find the time period of oscillations of block.*



Solution As there is no loss of mechanical energy, the block rises upto the same height 'h' on other side also.



$$AB = \frac{h}{\sin\alpha},\ BC = \frac{h}{\sin\beta}$$

$$a_1 = g\sin\alpha,\ a_2 = g\sin\beta$$

$$u_A = u_C = 0$$

Using $$t = \sqrt{\frac{2s}{a}}$$ (with $u = 0$)

$\therefore$ Time period of oscillations,

$$T = 2t_{AB} + 2t_{CB} = 2(t_{AB} + t_{CB})$$

$$= 2\left[\sqrt{\frac{2 \times \frac{h}{\sin\alpha}}{g\sin\alpha}} + \sqrt{\frac{2 \times \frac{h}{\sin\beta}}{g\sin\beta}}\right]$$

$$= 2\sqrt{\frac{2h}{g}}\,[\text{cosec}\,\alpha + \text{cosec}\,\beta]$$ **Ans.**

Type 7. *When point of suspension of the pendulum has an acceleration* $\mathbf{a}$.

Concept

As we have discussed in the theory, time period in this case is

$$T = 2\pi\sqrt{\frac{l}{g_e}}$$

where, $$\mathbf{g}_e = \mathbf{g} - \mathbf{a} \text{ or } \mathbf{g} + (-\mathbf{a}) \text{ and } g_e = |\mathbf{g}_e|$$

Example 16 *A simple pendulum of length l is suspended from the ceiling of a cart which is sliding without friction on an inclined plane of inclination θ. What will be the time period of the pendulum?*

Solution Here, point of suspension has an acceleration. $a = g\sin\theta$ (down the plane). Further, $\mathbf{g}$ can be resolved into two components $g\sin\theta$ (along the plane) and $g\cos\theta$ (perpendicular to plane).

$$\therefore \qquad \mathbf{g}_{\text{eff}} = \mathbf{g} - \mathbf{a} = \mathbf{g} + (-\mathbf{a})$$

Resultant of $\mathbf{g}$ and $-\mathbf{a}$ will be $g\cos\theta$.

$$\therefore \qquad g_{\text{eff}} = g\cos\theta \qquad \text{(perpendicular to plane)}$$

$$\therefore \qquad T = 2\pi\sqrt{\frac{l}{|\mathbf{g}_{\text{eff}}|}} = 2\pi\sqrt{\frac{l}{g\cos\theta}} \qquad \textbf{Ans.}$$

Note *If $\theta = 0°$, $T = 2\pi\sqrt{\frac{l}{g}}$ which is quiet obvious.*

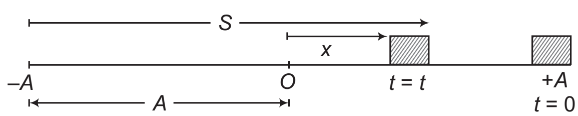
Type 8. *When displacement is not measured from the mean position.*

Concept

In most of the x-t equations of SHM, x is measured from the mean position. In this type of problems, we have to write S-t equation. But S here is not measured from the mean position. The method can be best explained by the following example:

Example 17 *A block in SHM starts from $+A$ position. Write S-t equation of the block, if S is measured from $-A$.*

Solution



When the block starts from $+A$ and x at any time 't' is measured from the mean position, then

$$x = A\cos\omega t$$

In the figure, we can see that

$$S = A + x \quad \text{or} \quad S = A + A\cos\omega t \quad \text{or} \quad S = A(1+\cos\omega t) \qquad \textbf{Ans.}$$

Check From the equation we can see that,

$$S = 2A \text{ at } t = 0$$

$$S = A \text{ at } t = \frac{T}{4} \text{ and } \omega = \frac{2\pi}{T} \Rightarrow \omega t = \frac{\pi}{2}$$

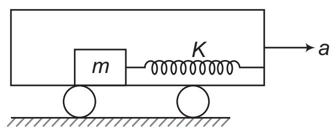
$$S = 0 \text{ at } T = \frac{T}{2} \text{ and } \omega = \frac{2\pi}{T} \Rightarrow \omega t = \pi$$

Type 9. *Based on the concept of pseudo force.*

Concept

If a block is observed from an accelerating or non-inertial frame of reference, a pseudo force $(= -m\mathbf{a})$ has to be applied on the block. Here, 'm' is the mass of the block and $\mathbf{a}$ the acceleration of frame of reference.

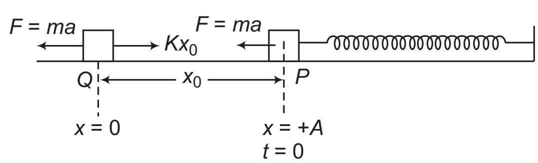
Example 18 *A spring block system is kept inside the smooth surface of a trolley as shown in figure.*



At $t = 0$, trolley is given an acceleration 'a' in the direction shown in figure.
Write S - t equation of the block
(a) with respect to trolley.
(b) with respect to ground.

Note *S has to be measured from the starting point (at $t = 0$) not from the mean position of SHM with respect to trolley.*

Solution (a) Trolley is an accelerating or non-inertial frame. So, a pseudo force $F = ma$ has to be applied on the block towards left.



Here, P or natural length of the spring is not the mean position (of SHM of the block with respect to trolley).

Let mean position (where $\mathbf{F}_{\text{net}} = 0$) is obtained after an extension of x_0. Then

$$ma = kx_0 \;\Rightarrow\; x_0 = \frac{ma}{k}$$

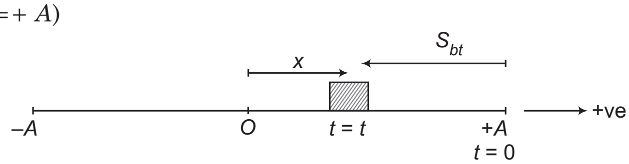
This x_0 is also the amplitude of SHM of block with respect to trolley.

$\therefore$ $$A = x_0 = \frac{ma}{k}$$

Further, at $t = 0$, block is at $x = +A$.

Therefore, x - t equation of block is, $\quad x = A\cos\omega t \quad$ (where, $\omega = \sqrt{\frac{k}{m}}$)

But in this equation x is measured from the mean position and we have to measure S from the starting point $(= +A)$



From the figure we can see that,

S_{bt} = displacement of block with respect to trolley.

$$= -(A - x)$$
$$= -(A - A\cos\omega t)$$
$$S_{bt} = (A\cos\omega t - A)$$ **Ans.**

(b) Displacement of trolley at time 't' is

$$S_t = \frac{1}{2}at^2$$

Now $$S_{bt} = S_b - S_t$$

$\therefore$ $$S_b = S_{bt} + S_t$$

or S_b = displacement of block with respect to ground.

or $$S_b = (A\cos\omega t - A) + \frac{1}{2}at^2$$ **Ans.**

Type 10. *Distance travelled in a given time interval.*

Concept

Linear speed of particle in SHM is not constant. Therefore, we cannot apply,

distance = speed × time

How to Solve?

From the given x-t equation, find the starting point, amplitude and time period of oscillations.

Example 19 *x-t equation of a particle in SHM is*

$$x = (4\,cm)\cos\left(\frac{\pi}{2}t\right)$$

Here, t is in seconds. Find the distance travelled by the particle in first three seconds.

Solution $A = 4$ cm

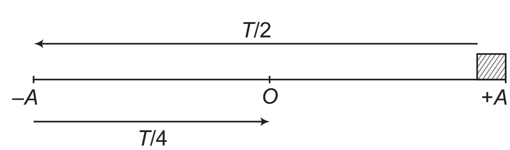
$$\omega = \left(\frac{\pi}{2}\right)\text{rad/s}$$

$\therefore$ $$T = \frac{2\pi}{T} = 4\,\text{sec}$$

The given equation is $x = A\cos\omega t$.

Therefore, the particle starts from $x = +A$.

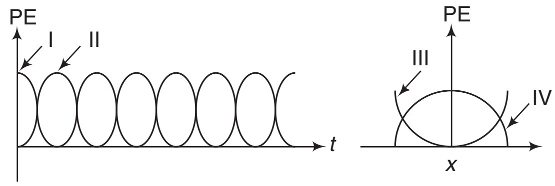
The given time $t = 3$ sec is $\frac{3T}{4}$. So, the particle moves from $+A$ to $-A$ and then from $-A$ to O.



So, distance travelled in the given time interval is $3A$ or 12 cm. **Ans.**

Type 11. *Graph Based problems.*

Example 20 *For a particle executing SHM, the displacement x is given by $x = A \cos \omega t$. Identify the graph which represents the variation of potential energy (PE) as a function of time t and displacement x.* **(JEE 2003)**

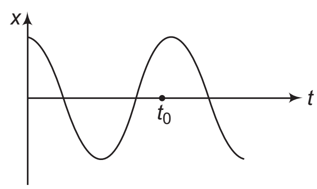


(a) I, III (b) II, IV (c) II, III (d) I, IV

Solution Potential energy is minimum (in this case zero) at mean position ($x = 0$) and maximum at extreme positions ($x = + A$).

At time $t = 0$, $x = A$. Hence, PE should be maximum. Therefore, graph I is correct. Further in graph III, PE is minimum at $x = 0$. Hence, this is also correct.

Example 21 *x - t graph of a particle in SHM is*



At time $t = t_0$, what are the signs of v and a of the particle?

Solution $v = \dfrac{dx}{dt}$ = slope of x - t graph.

At time $t = t_0$, slope is positive. So, velocity is positive.

$$a \propto -x$$

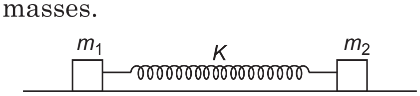
At time $t = t_0$, x is positive. Hence, acceleration is negative because 'a' and 'x' are always of opposite signs.

Note *At time t_0, velocity is positive and acceleration is negative. So, speed of the particle at this instant is decreasing or the particle is moving towards the extreme position.*

Type 12. *Based on two body oscillator system.*

Concept

As discussed in the theory, if the spring is compressed or elongated by x_0 and released then both blocks execute SHM with same time period T (and ω also) but different amplitudes in the inverse ratio of their masses.



$$T = 2\pi\sqrt{\frac{\mu}{k}}$$

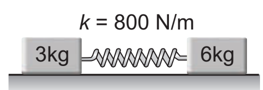
Here, $$\mu = \text{reduced mass} = \frac{m_1 m_2}{m_1 + m_2}$$

$$A \propto \frac{1}{m} \quad \text{or} \quad \frac{A_1}{A_2} = \frac{m_2}{m_1}$$

$\therefore$ $$A_1 = \left(\frac{m_2}{m_1 + m_2}\right) x_0$$

and $$A_2 = \left(\frac{m_1}{m_1 + m_2}\right) x_0$$

Example 22 *The system shown in the figure can move on a smooth surface. They are initially compressed by 6 cm and then released, then choose the correct options.*



(a) *The system performs, SHM with time period* $\frac{\pi}{10}$ *s*

(b) *The block of mass 3 kg perform SHM with amplitude 4 cm*

(c) *The block of mass 6 kg will have maximum momentum of 2.40 kg-m/s*

(d) *The time periods of two blocks are in the ratio of* $1 : \sqrt{2}$

Solution μ = Reduced mass

$$= \frac{m_1 m_2}{m_1 + m_2} = 2 \text{ kg}$$

$$T = 2\pi\sqrt{\frac{\mu}{k}}$$

$$= 2\pi\sqrt{\frac{2}{800}} = \frac{\pi}{10} \text{ s} = T_3 = T_6$$

$$A \propto \frac{1}{m}$$

$\therefore$ $$\frac{A_3}{A_6} = \frac{6}{3} = \frac{2}{1}$$

$\therefore$ $$A_3 = 4 \text{ cm} \quad \text{and} \quad A_6 = 2 \text{ cm}$$

$$(P_6)_{\max} = m_6 \, (v_6)_{\max}$$

$$= m_6 \, (\omega A_6)$$

$$= (6)\left(\frac{2\pi}{\pi/10}\right)(2 \times 10^{-2})$$

$$= 2.4 \text{ kg-m/s}$$

$\therefore$ The correct options are (a, b, c)

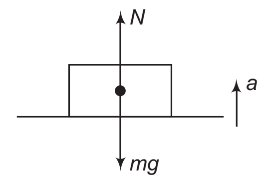
Type 13. *Change in variable force method for finding time period of SHM.*

Concept

(i) Two types of forces may act on a body: constant and variable.

Weight of a body (for small heights) and the upthrust force on a solid when it is completely immersed in a liquid are constant forces.

Spring force, upthrust force when the body is partially immersed in a liquid, tension and normal reaction are variable forces. Tension and normal reaction seem to be constant forces but actually they are variable forces.



In the figure shown,

$N = mg$ if $a = 0$

$N = m(g + a)$ if a is upwards and

$N = m(g - a)$ if a is downwards.

So, normal reaction N is not a constant force.

(ii) In equilibrium, summation of all constant and variable forces is zero.

(iii) When displaced from the mean position, constant forces do not contribute in the net restoring force (therefore in time period). They can only change the mean position.

So, just forget the constant forces. Only change in variable force becomes the net restoring force.

$\therefore$ $$F_{net} = -|\Delta \mathbf{F}| = \text{net restoring force.}$$

where, $\mathbf{F}$ is the variable force.

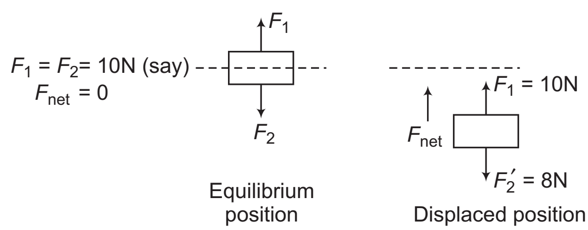
(iv) If two or more than two variable forces are acting on a body and both either increase or decrease (when displaced from the mean position), then

$$\mathbf{F}_{net} = -[|\Delta \mathbf{F}_1| + |\Delta \mathbf{F}_2|]$$

If one variable force increases and the other decreases then,

$$\mathbf{F}_{net} = -[|\Delta \mathbf{F}_1| \sim |\Delta \mathbf{F}_2|]$$

Let us understand this concept with an example.



In the above figures, F_1 is a constant force and F_2 is a variable force. In equilibrium,

$$F_1 = F_2 = 10\,\text{N (say)}$$

or $$F_{net} = 0$$

When displaced from the mean position, F_1 being a constant force remains 10 N.

But, suppose F_2 becomes $F_2' = 8\,\text{N}$.

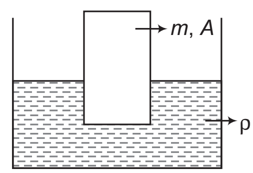
Then, net force is $2\,\text{N}(= F_1 - F_2')$ in upward direction or towards the mean position. We can see that this net force is also equal to change (here decrease) in variable force F_2.

How to Solve?

- Consider a mean or equilibrium position of the body.
- Displace the body from the mean position and find the change(s) in variable force(s) and calculate the net restoring force by the method discussed above. This net restoring force should be of type

$$F = -kx \Rightarrow \text{Now, } T = 2\pi\sqrt{\frac{m}{k}}$$

Example 23 *A plank of mass 'm' and area of cross-section A is floating in a non-viscous liquid of density ρ. When displaced slightly from the mean position, it starts oscillating. Prove that oscillations are simple harmonic and find its time period.*



Solution Two forces are acting on the plank: weight and upthrust. Weight is a constant force. So, forget it. Upthrust (if partially immersed) is a variable force. When displaced downwards by a distance x, then

Net restoring force $F_{\text{net}} = -$ [change or increase in variable force upthrust]

$= -$ [(extra immersed volume)(density of liquid) (g)]

$= -[(Ax)(\rho)(g)]$

This force is of type $F = -kx$.

So, motion is simple harmonic, where $k = \rho A g$

$$\therefore \quad T = 2\pi\sqrt{\frac{m}{k}} \quad \text{or} \quad T = 2\pi\sqrt{\frac{m}{\rho A g}} \qquad \textbf{Ans.}$$

Note *(i) Upthrust force is given by*

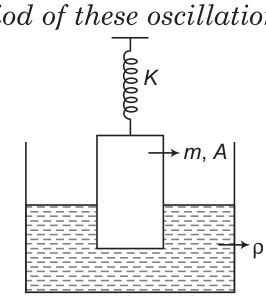
$$F = (\text{immersed volume of solid})(\text{density of liquid})(g)$$

$$= V_i \rho g \Rightarrow \therefore \quad \Delta F = (\Delta V_i)\rho g \qquad [\text{as } \rho \text{ and } g \text{ are constant}]$$

(ii) In the above problem, upthrust is just like a spring force of force constant:

$$k = \rho A g$$

Example 24 *A plank of mass 'm' and area of cross-section A is floating as shown in figure. When slightly displaced from mean position, plank starts oscillations. Find time period of these oscillations.*



Solution Three forces are acting on the plank :

(i) weight (ii) spring force

(iii) upthrust

Weight is a constant force. So, it will not contribute in the time period. Spring force and upthrust are variable forces. When displaced downwards from the mean position, both forces increase.

So, their force constants are additive

$$\therefore \qquad k_e = k + \rho A g$$

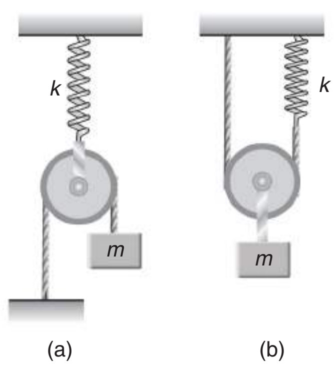
$$\therefore \qquad T = 2\pi\sqrt{\frac{m}{k_e}}$$

$$= 2\pi\sqrt{\frac{m}{k + \rho A g}} \qquad \textbf{Ans.}$$

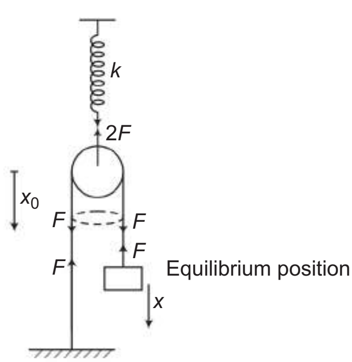
Note *If the plank oscillates with fully immersed position, then upthrust also becomes a constant force. So, the only varibale force remains kx. Therefore,*

$$T = 2\pi\sqrt{\frac{m}{k}}$$

Example 25 *Figure shows a system consisting of a massless pulley, a spring of force constant k and a block of mass m. If the block is slightly displaced vertically down from its equilibrium position and released, find the period of its vertical oscillation in cases (a) and (b).*



Solution (a)



Two forces are acting on the block :

(i) weight

(ii) tension

Weight is a constant force which does not contribute in time period. Tension is a variable force. Let tension increases by F when displaced from the equilibrium position by x. This increase in tension is the change in variable force. So, this is also the net restoring force. The distribution of this increase in tension F is as shown in figure.

Spring gets an extra stretching force $2F$. Let x_0 is the extra extension in the spring by this extra force $2F$. Then,

$$2F = kx_0$$

or $$x_0 = \frac{2F}{k} \quad \text{...(i)}$$

Now, we can see that by this extra extension of spring, pulley will also come down by the same distance x_0 and

$$x = 2x_0 = 2\left(\frac{2F}{k}\right)$$

$\Rightarrow$ $$F = \frac{k}{4}x$$

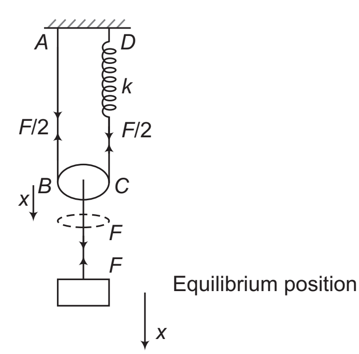
But F is also the net restoring force on block. So, we can write

$$F = -\left(\frac{k}{4}\right)x$$

$\therefore$ $$T = 2\pi\sqrt{\frac{m}{k/4}} \quad (\text{as } k_e = k/4)$$

$$= 2\pi\sqrt{\frac{4m}{k}}$$ **Ans.**

(b) Using the same concepts as discussed above,



$$\frac{F}{2} = kx_0$$

or $$x_0 = \frac{F}{2k}$$

But this time,

x_0 = extra extension on spring.

= extra increase in length $ABCD$

= $2x$

or $$\frac{F}{2k} = 2x$$

or $$F = 4kx$$

Due to the restoring nature of F we can write,

$$F = -(4k)x$$

$\therefore$ $$T = 2\pi\sqrt{\frac{m}{4k}} \quad (\text{as } k_e = 4k)$$

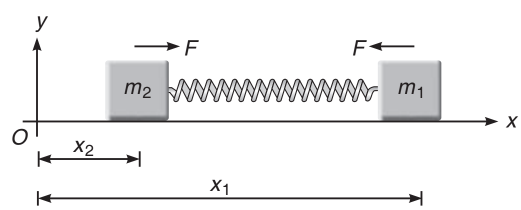
Miscellaneous Examples

Example 29 *For a two body oscillator system, prove the relation,*

$$T = 2\pi\sqrt{\frac{\mu}{k}}$$

where, $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$ = *reduced mass*

Solution A system of two bodies connected by a spring so that both are free to oscillate simple harmonically along the length of the spring constitutes a two body harmonic oscillator.



Suppose, two masses m_1 and m_2 are connected by a horizontal massless spring of force constant k, so as to be free to oscillate along the length of the spring on a frictionless horizontal surface.

Let l_0 be the natural length of the spring and let x_1 and x_2 be the coordinates of the two masses at any instant of time. Then,

Extension of the spring $\quad x = (x_1 - x_2) - l_0 \quad$ ...(i)

For $x > 0$, the spring force $F = kx$ acts on the two masses in the directions shown in above figure.

Thus, we can write

$$m_1 \frac{d^2x_1}{dt^2} = -kx \quad \text{...(ii)}$$

$$m_2 \frac{d^2x_2}{dt^2} = kx \quad \text{...(iii)}$$

Multiplying Eq. (ii) by m_2 and Eq. (iii) by m_1 and subtracting the latter from the former, we have

$$m_1 m_2 \frac{d^2x_1}{dt^2} - m_1 m_2 \frac{d^2x_2}{dt^2} = -(m_2 + m_1)\,kx$$

or

$$m_1 m_2 \frac{d^2(x_1 - x_2)}{dt^2} = -kx\,(m_1 + m_2)$$

or

$$\left(\frac{m_1 m_2}{m_1 + m_2}\right) \frac{d^2}{dt^2}(x_1 - x_2) = -kx \quad \text{...(iv)}$$

Differentiating Eq. (i), twice with respect to time, we have

$$\frac{d^2x}{dt^2} = \frac{d^2}{dt^2}(x_1 - x_2)$$

Also, $$\frac{m_1 m_2}{m_1 + m_2} = \mu = \text{ reduced mass of the two blocks}$$

Substituting these values in Eq. (iv), we have

$$\mu \frac{d^2x}{dt^2} = -kx$$

or $$\mu a_r = -kx \qquad \left(\text{Here, } a_r = \frac{d^2x}{dt^2} = \frac{d^2x_1}{dt^2} - \frac{d^2x_2}{dt^2} = \text{Relative acceleration}\right)$$

This, is the standard differential equation of SHM. Time period of which is

$$T = 2\pi \sqrt{\left|\frac{x}{a_r}\right|} \quad \text{or} \quad T = 2\pi \sqrt{\frac{\mu}{k}}$$

Example 30 *Two particles move parallel to x-axis about the origin with the same amplitude and frequency. At a certain instant they are found at distance $\frac{A}{3}$ from the origin on opposite sides but their velocities are found to be in the same direction. What is the phase difference between the two ?*

Solution Let equations of two SHM be

$$x_1 = A \sin \omega t \qquad \ldots \text{(i)}$$

$$x_2 = A \sin (\omega t + \phi) \qquad \ldots \text{(ii)}$$

Give that $$\frac{A}{3} = A \sin \omega t$$

and $$-\frac{A}{3} = A \sin (\omega t + \phi)$$

Which gives $$\sin \omega t = \frac{1}{3} \qquad \ldots \text{(iii)}$$

$$\sin (\omega t + \phi) = -\frac{1}{3} \qquad \ldots \text{(iv)}$$

From Eq. (iv),

$$\sin \omega t \cos \phi + \cos \omega t \sin \phi = -\frac{1}{3}$$

$\Rightarrow$ $$\frac{1}{3} \cos \phi + \sqrt{1 - \frac{1}{9}} \sin \phi = -\frac{1}{3}$$

Solving this equation, we get

or $$\cos \phi = -1, \frac{7}{9}$$

$\Rightarrow$ $$\phi = \pi \quad \text{or} \quad \cos^{-1}\left(\frac{7}{9}\right)$$

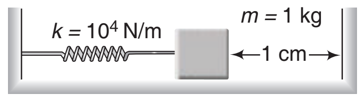
Differentiating Eqs. (i) and (ii), we obtain

$$v_1 = A\omega \cos \omega t \quad \text{and} \quad v_2 = A\omega \cos (\omega t + \phi)$$

If we put $\phi = \pi$, we find v_1 and v_2 are of opposite signs. Hence, $\phi = \pi$ is not acceptable.

$\therefore$ $$\phi = \cos^{-1}\left(\frac{7}{9}\right)$$ **Ans.**

Example 31 *For the arrangement shown in figure, the spring is initially compressed by 3 cm. When the spring is released the block collides with the wall and rebounds to compress the spring again.*



(a) *If the coefficient of restitution is $\frac{1}{\sqrt{2}}$, find the maximum compression in the spring after collision.*

(b) *If the time starts at the instant when spring is released, find the minimum time after which the block becomes stationary.*

Solution (a) Velocity of the block just before collision,

$$\frac{1}{2} m v_0^2 + \frac{1}{2} k x^2 = \frac{1}{2} k x_0^2$$

or
$$v_0 = \sqrt{\frac{k}{m}(x_0^2 - x^2)}$$

Here, $x_0 = 0.03$ m, $x = 0.01$ m, $k = 10^4$ N/m, $m = 1$ kg

$\therefore$
$$v_0 = 2\sqrt{2} \text{ m/s}$$

After collision,
$$v = ev_0 = \frac{1}{\sqrt{2}} 2\sqrt{2} = 2 \text{ m/s}$$

Maximum compression in the spring is

$$\frac{1}{2} k x_m^2 = \frac{1}{2} k x^2 + \frac{1}{2} m v^2$$

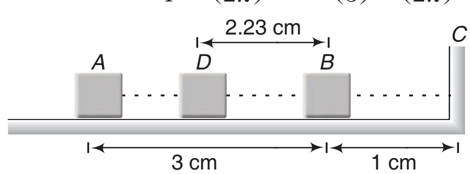
or
$$x_m = \sqrt{x^2 + \frac{m}{k} v^2}$$

$$= \sqrt{(0.01)^2 + \frac{(1)(2)^2}{10^4}} \text{ m} = 2.23 \text{ cm}$$ **Ans.**

(b) In the case of spring-mass system, since the time period is independent of the amplitude of oscillation.

$$\text{Time} = t_{AB} + t_{BC} + t_{CB} + t_{BD}$$

$$= \frac{T_0}{4} + \left(\frac{T_0}{2\pi}\right) \sin^{-1}\left(\frac{1}{3}\right) + \left(\frac{T_0}{2\pi}\right) \sin^{-1}\left(\frac{1}{2.23}\right) + \frac{T_0}{4}$$

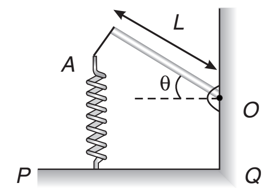


Here,
$$T_0 = 2\pi\sqrt{\frac{m}{k}}$$

Substituting the values, we get

$$\text{Total time} = \sqrt{\frac{m}{k}}\left[\pi + \sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{1}{2.23}\right)\right]$$ **Ans.**

Example 32 *A long uniform rod of length L and mass M is free to rotate in a horizontal plane about a vertical axis through its one end 'O'. A spring of force constant k is connected between one end of the rod and PQ. When the rod is in equilibrium it is parallel to PQ.*



(a) *What is the period of small oscillation that result when the rod is rotated slightly and released?*

(b) *What will be the maximum speed of the displaced end of the rod, if the amplitude of motion is θ_0?*

Solution (a) Restoring torque about 'O' due to elastic force of the spring

$$\tau = -FL = -kyL \qquad (F = ky)$$

$$\tau = -kL^2\theta \qquad (\text{as } y = L\theta)$$

$$\tau = I\alpha = \frac{1}{3}ML^2\frac{d^2\theta}{dt^2}$$

$$\frac{1}{3}ML^2\frac{d^2\theta}{dt^2} = -kL^2\theta$$

$$\frac{d^2\theta}{dt^2} = -\frac{3k}{M}\theta$$

$$\omega = \sqrt{\frac{3k}{M}}$$

$$\Rightarrow \qquad T = 2\pi\sqrt{\frac{M}{3k}} \qquad \textbf{Ans.}$$

(b) In angular SHM, maximum angular velocity

$$\left(\frac{d\theta}{dt}\right)_{\max} = \theta_0\,\omega = \theta_0\sqrt{\frac{3k}{M}}$$

$$v = r\left(\frac{d\theta}{dt}\right)$$

So ,

$$v_{\max} = L\left(\frac{d\theta}{dt}\right)_{\max} = L\theta_0\sqrt{\frac{3k}{M}} \qquad \textbf{Ans.}$$

Example 33 *A block with a mass of 2 kg hangs without vibrating at the end of a spring of spring constant 500 N/m, which is attached to the ceiling of an elevator. The elevator is moving upwards with an acceleration $\frac{g}{3}$. At time $t = 0$, the acceleration suddenly ceases.*

(a) *What is the angular frequency of oscillation of the block after the acceleration ceases?*

(b) *By what amount is the spring stretched during the time when the elevator is accelerating?*

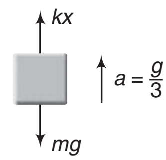
(c) *What is the amplitude of oscillation and initial phase angle observed by a rider in the elevator in the equation, $x = A\sin(\omega t + \phi)$? Take the upward direction to be positive. Take $g = 10.0\ m/s^2$.*

Solution (a) Angular frequency

$$\omega = \sqrt{\frac{k}{m}} \quad \text{or} \quad \omega = \sqrt{\frac{500}{2}}$$

or $$\omega = 15.81 \text{ rad/s}$$ **Ans.**

(b) Equation of motion of the block (while elevator is accelerating) is,



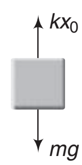
$$kx - mg = ma = m\frac{g}{3}$$

$\therefore$ $$x = \frac{4mg}{3k}$$

$$= \frac{(4)(2)(10)}{(3)(500)} = 0.053 \text{ m}$$

or $$x = 5.3 \text{ cm}$$ **Ans.**

(c) (i) In equilibrium when the elevator has zero acceleration, the equation of motion is,



$$kx_0 = mg$$

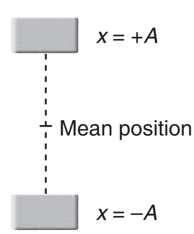
or $$x_0 = \frac{mg}{k} = \frac{(2)(10)}{500}$$

$$= 0.04 \text{ m} = 4 \text{ cm}$$

$\therefore$ Amplitude $$A = x - x_0 = 5.3 - 4.0$$

$$= 1.3 \text{ cm}$$ **Ans.**

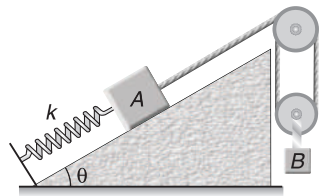
(ii) At time $t = 0$, block is at $x = -A$. Therefore, substituting $x = -A$ and $t = 0$ in equation,



$$x = A \sin(\omega t + \phi)$$

We get initial phase $$\phi = \frac{3\pi}{2}$$ **Ans.**

Example 34 *Calculate the angular frequency of the system shown in figure. Friction is absent everywhere and the threads, spring and pulleys are massless. Given that $m_A = m_B = m$.*



Solution Let x_0 be the extension in the spring in equilibrium. Then, equilibrium of A and B give,

$$T = kx_0 + mg \sin\theta \quad \text{...(i)}$$

and $$2T = mg \quad \text{...(ii)}$$

Here, T is the tension in the string. Now, suppose A is further displaced by a distance x from its mean position and v be its speed at this moment. Then, B lowers by $\frac{x}{2}$ and speed of B at this instant will be $\frac{v}{2}$. Total energy of the system in this position will be,

$$E = \frac{1}{2} k (x + x_0)^2 + \frac{1}{2} m_A v^2 + \frac{1}{2} m_B \left(\frac{v}{2}\right)^2 + m_A g h_A - m_B g\, h_B$$

or $$E = \frac{1}{2} k (x + x_0)^2 + \frac{1}{2} mv^2 + \frac{1}{8} mv^2 + mgx \sin\theta - mg \frac{x}{2}$$

or $$E = \frac{1}{2} k(x + x_0)^2 + \frac{5}{8} mv^2 + mgx \sin\theta - mg \frac{x}{2}$$

Since, E is constant, $$\frac{dE}{dt} = 0$$

or $$0 = k(x + x_0) \frac{dx}{dt} + \frac{5}{4} mv \left(\frac{dv}{dt}\right) + mg\,(\sin\theta)\left(\frac{dx}{dt}\right) - \frac{mg}{2}\left(\frac{dx}{dt}\right)$$

Substituting, $$\frac{dx}{dt} = v \quad \Rightarrow \quad \frac{dv}{dt} = a$$

and $$kx_0 + mg \sin\theta = \frac{mg}{2} \quad \text{[From Eqs. (i) and (ii)]}$$

We get, $$\frac{5}{4} m\, a = -\, kx$$

Since, $$a \propto -\, x$$

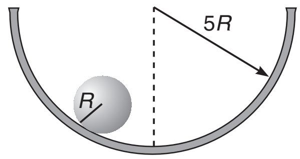
Motion is simple harmonic, time period of which is,

$$T = 2\pi \sqrt{\left|\frac{x}{a}\right|}$$

$$= 2\pi \sqrt{\frac{5m}{4k}}$$

$$\therefore \quad \omega = \frac{2\pi}{T} = \sqrt{\frac{4k}{5m}} \quad \textbf{Ans.}$$

Example 35 *A solid sphere (radius = R) rolls without slipping in a cylindrical through (radius = 5R). Find the time period of small oscillations.*



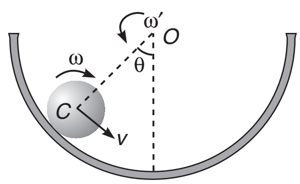
Solution For pure rolling to take place,

$$v = R\omega$$

ω' = angular velocity of COM of sphere C about O

$$= \frac{v}{4R} = \frac{R\omega}{4R} = \frac{\omega}{4}$$

$$\therefore \qquad \frac{d\omega'}{dt} = \frac{1}{4}\frac{d\omega}{dt}$$



or
$$\alpha' = \frac{\alpha}{4}$$

$$\alpha = \frac{a}{R} \text{ for pure rolling}$$

where,
$$a = \frac{g\sin\theta}{1 + \frac{I}{mR^2}}$$

$$= \frac{5g\sin\theta}{7}$$

as,
$$I = \frac{2}{5}mR^2$$

$$\therefore \qquad \alpha' = \frac{5g\sin\theta}{28R}$$

For small $\theta, \sin\theta \approx \theta$, being restoring in nature,

$$\alpha' = -\frac{5g}{28R}\theta$$

$$\therefore \qquad T = 2\pi\sqrt{\left|\frac{\theta}{\alpha'}\right|}$$

$$= 2\pi\sqrt{\frac{28R}{5g}}$$ **Ans.**

Example 36 *Consider the earth as a uniform sphere of mass M and radius R. Imagine a straight smooth tunnel made through the earth which connects any two points on its surface. Show that the motion of a particle of mass m along this tunnel under the action of gravitation would be simple harmonic. Hence, determine the time that a particle would take to go from one end to the other through the tunnel.*

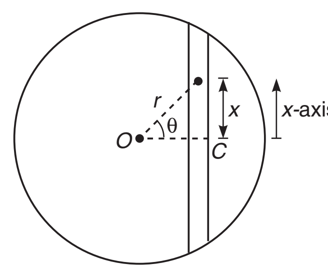
Solution Suppose at some instant, the particle is at radial distance r from centre of earth O. Since, the particle is constrained to move along the tunnel, we define its position as distance x from C. Hence, equation of motion of the particle is,

$$ma_x = F_x$$

The gravitational force on mass m at distance r is,

$$F = \frac{GMmr}{R^3} \qquad \text{(towards } O\text{)}$$

Therefore,



$$F_x = -F \sin\theta$$

$$= -\frac{GMmr}{R^3}\left(\frac{x}{r}\right)$$

$$= -\frac{GMm}{R^3} \cdot x$$

Since, $F_x \propto -x$, motion is simple harmonic in nature. Further,

$$ma_x = -\frac{GMm}{R^3} \cdot x$$

or

$$a_x = -\frac{GM}{R^3} \cdot x$$

$\therefore$ Time period of oscillation is,

$$T = 2\pi \sqrt{\left|\frac{x}{a_x}\right|}$$

$$= 2\pi \sqrt{\frac{R^3}{GM}}$$

The time taken by particle to go from one end to the other is $\frac{T}{2}$.

$\therefore$

$$t = \frac{T}{2}$$

$$= \pi \sqrt{\frac{R^3}{GM}}$$ **Ans.**

Exercises

LEVEL 1

Assertion and Reason

Directions : *Choose the correct option.*

(*a*) *If both* ***Assertion*** *and* ***Reason*** *are true and the* ***Reason*** *is correct explanation of the* ***Assertion.***

(*b*) *If both* ***Assertion*** *and* ***Reason*** *are true but* ***Reason*** *is not the correct explanation of* ***Assertion.***

(*c*) *If* ***Assertion*** *is true, but the* ***Reason*** *is false.*

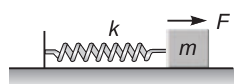
(*d*) *If* ***Assertion*** *is false but the* ***Reason*** *is true.*

1. Assertion : In $x = A\cos\omega t$, x is the displacement measured from extreme position.
Reason : In the above equation $x = A$ at time $t = 0$.

2. Assertion : A particle is under SHM along the x-axis. Its mean position is $x = 2$, amplitude is $A = 2$ and angular frequency ω. At $t = 0$, particle is at origin, then x-co-ordinate *versus* time equation of the particle will be $x = -2\cos\omega t + 2$.

Reason : At $t = 0$, particle is at rest.

3. Assertion : A spring block system is kept over a smooth surface as shown in figure. If a constant horizontal force F is applied on the block it will start oscillating simple harmonically.



Reason : Time period of oscillation is less than $2\pi\sqrt{\dfrac{m}{k}}$.

4. Assertion : Time taken by a particle in SHM to move from $x = A$ to $x = \dfrac{\sqrt{3}A}{2}$ is same as the time taken by the particle to move from $x = \dfrac{\sqrt{3}A}{2}$ to $x = \dfrac{A}{2}$.

Reason : Corresponding angles rotated in the reference circle are same in the given time intervals.

5. Assertion : Path of a particle in SHM is always a straight line.

Reason : All straight line motions are not simple harmonic.

6. Assertion : In spring block system if length of spring and mass of block both are halved, then angular frequency of oscillations will remain unchanged.

Reason : Angular frequency is given by $\omega = \sqrt{\dfrac{k}{m}}$

7. Assertion : All small oscillations are simple harmonic in nature.

Reason : Oscillations of spring block system are always simple harmonic whether amplitude is small or large.

8. **Assertion :** In $x = A\cos\omega t$, the dot product of acceleration and velocity is positive for time interval $0 < t < \dfrac{\pi}{2\omega}$.

Reason : Angle between them is 0°.

9. **Assertion :** For a given simple harmonic motion displacement (from the mean position) and acceleration have a constant ratio.

Reason : $T = 2\pi\sqrt{\left|\dfrac{\text{displacement}}{\text{acceleration}}\right|}$

10. **Assertion :** We can call circular motion also as simple harmonic motion.

Reason : Angular velocity in uniform circular motion and angular frequency in simple harmonic motion have the same meanings.

Objective Questions

Single Correct Option

1. A simple harmonic oscillation has an amplitude A and time period T. The time required to travel from $x = A$ to $x = \dfrac{A}{2}$ is

(a) $\dfrac{T}{6}$ (b) $\dfrac{T}{4}$ (c) $\dfrac{T}{3}$ (d) $\dfrac{T}{12}$

2. The potential energy of a particle executing SHM varies sinusoidally with frequency f. The frequency of oscillation of the particle will be

(a) $\dfrac{f}{2}$ (b) $\dfrac{f}{\sqrt{2}}$ (c) f (d) $2f$

3. For a particle undergoing simple harmonic motion, the velocity is plotted against displacement. The curve will be

(a) a straight line (b) a parabola
(c) a circle (d) an ellipse

4. A simple pendulum is made of bob which is a hollow sphere full of sand suspended by means of a wire. If all the sand is drained out, the period of the pendulum will

(a) increase (b) decrease
(c) remain same (d) become erratic

5. Two simple harmonic motions are given by $y_1 = a\sin\left[\left(\dfrac{\pi}{2}\right)t + \phi\right]$ and $y_2 = b\sin\left[\left(\dfrac{2\pi}{3}\right)t + \phi\right]$. The phase difference between these after 1 s is

(a) zero (b) $\pi/2$
(c) $\pi/4$ (d) $\pi/6$

6. A particle starts performing simple harmonic motion. Its amplitude is A. At one time its speed is half that of the maximum speed. At this moment the displacement is

(a) $\dfrac{\sqrt{2}\,A}{3}$ (b) $\dfrac{\sqrt{3}\,A}{2}$
(c) $\dfrac{2\,A}{\sqrt{3}}$ (d) $\dfrac{3\,A}{\sqrt{2}}$

7. Which of the following is not simple harmonic function?

(a) $y = a \sin 2\omega t + b \cos^2 \omega t$
(b) $y = a \sin \omega t + b \cos 2\omega t$
(c) $y = 1 - 2 \sin^2 \omega t$
(d) $y = (\sqrt{a^2 + b^2}) \sin \omega t \cos \omega t$

8. The displacement of a particle varies according to the relation $y = 4(\cos \pi t + \sin \pi t)$. The amplitude of the particle is

(a) 8 units
(b) 2 units
(c) 4 units
(d) $4\sqrt{2}$ units

9. Two pendulums X and Y of time periods 4 s and 4.2 s are made to vibrate simultaneously. They are initially in same phase. After how many vibrations of X, they will be in the same phase again?

(a) 30
(b) 25
(c) 21
(d) 26

10. A mass M is suspended from a massless spring. An additional mass m stretches the spring further by a distance x. The combined mass will oscillate with a period

(a) $2\pi \sqrt{\left\{\dfrac{(M+m)\,x}{mg}\right\}}$
(b) $2\pi \sqrt{\left\{\dfrac{mg}{(M+m)\,x}\right\}}$
(c) $2\pi \sqrt{\left\{\dfrac{(M+m)}{mgx}\right\}}$
(d) $\dfrac{\pi}{2} \sqrt{\left\{\dfrac{mg}{(M+m)\,x}\right\}}$

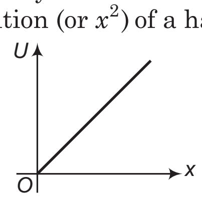
11. Two bodies P and Q of equal masses are suspended from two separate massless springs of force constants k_1 and k_2 respectively. If the two bodies oscillate vertically such that their maximum velocities are equal. The ratio of the amplitude of P to that of Q is

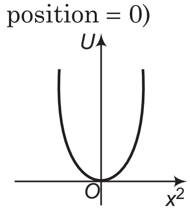
(a) $\sqrt{\dfrac{k_1}{k_2}}$
(b) $\dfrac{k_1}{k_2}$
(c) $\sqrt{\dfrac{k_2}{k_1}}$
(d) $\dfrac{k_2}{k_1}$

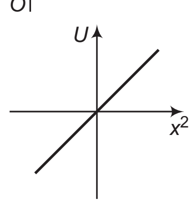
12. A disc of radius R is pivoted at its rim. The period for small oscillations about an axis perpendicular to the plane of disc is

(a) $2\pi\sqrt{\dfrac{R}{g}}$
(b) $2\pi\sqrt{\dfrac{2R}{g}}$
(c) $2\pi\sqrt{\dfrac{2R}{3g}}$
(d) $2\pi\sqrt{\dfrac{3R}{2g}}$

13. Identify the correct variation of potential energy U as a function of displacement x from mean position (or x^2) of a harmonic oscillator (U at mean position = 0)

(a) 

(b) 

(c) 

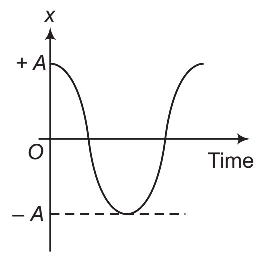
(d) None of these

14. If the length of a simple pendulum is equal to the radius of the earth, its time period will be

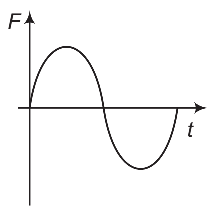
(a) $2\pi\sqrt{R/g}$ (b) $2\pi\sqrt{R/2g}$

(c) $2\pi\sqrt{2R/g}$ (d) infinite

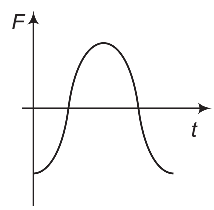
15. The displacement-time (x-t) graph of a particle executing simple harmonic motion is shown in figure. The correct variation of net force F acting on the particle as a function of time is



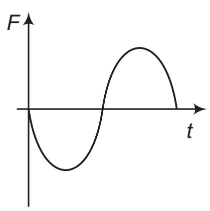
(a)



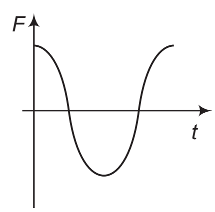
(b)



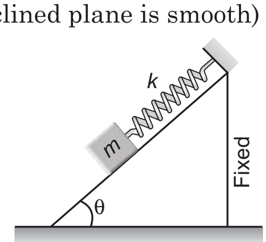
(c)



(d)



16. In the figure shown the time period and the amplitude respectively, when m is left from rest when spring is relaxed are (the inclined plane is smooth)



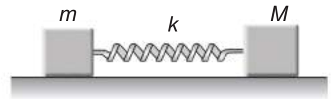
(a) $2\pi\sqrt{\dfrac{m}{k}}, \dfrac{mg\sin\theta}{k}$ (b) $2\pi\sqrt{\dfrac{m\sin\theta}{k}}, \dfrac{2\,mg\sin\theta}{k}$

(c) $2\pi\sqrt{\dfrac{m}{k}}, \dfrac{mg\cos\theta}{k}$ (d) None of these

17. The equation of motion of a particle of mass 1 g is $\dfrac{d^2x}{dt^2} + \pi^2 x = 0$, where x is displacement (in m) from mean position. The frequency of oscillation is (in Hz)

(a) 1/2 (b) 2 (c) $5\sqrt{10}$ (d) $1/5\sqrt{10}$

18. The spring as shown in figure is kept in a stretched position with extension x when the system is released. Assuming the horizontal surface to be frictionless, the frequency of oscillation is



(a) $\frac{1}{2\pi}\sqrt{\left[\frac{k(M+m)}{Mm}\right]}$
(b) $\frac{1}{2\pi}\sqrt{\left[\frac{mM}{k(M+m)}\right]}$
(c) $\frac{1}{2\pi}\sqrt{\left[\frac{kM}{m+M}\right]}$
(d) $\frac{1}{2\pi}\sqrt{\left[\frac{km}{M+m}\right]}$

19. The mass and diameter of a planet are twice those of earth. What will be the period of oscillation of a pendulum on this planet. It is a second's pendulum on earth?

(a) $\sqrt{2}$ s
(b) $2\sqrt{2}$ s
(c) $\frac{1}{\sqrt{2}}$ s
(d) $\frac{1}{2\sqrt{2}}$ s

20. The resultant amplitude due to superposition of three simple harmonic motions $x_1 = 3\sin\omega t$, $x_2 = 5\sin(\omega t + 37°)$ and $x_3 = -15\cos\omega t$ is

(a) 18
(b) 10
(c) 12
(d) None of these

21. Two SHMs $s_1 = a\sin\omega t$ and $s_2 = b\sin\omega t$ are superimposed on a particle. The s_1 and s_2 are along the directions which makes 37° to each other

(a) the particle will perform SHM
(b) the path of particle is straight line
(c) Both (a) and (b) are correct
(d) Both (a) and (b) are wrong

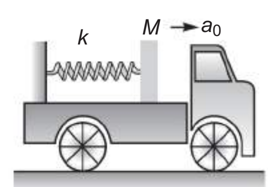
22. The amplitude of a particle executing SHM about O is 10 cm. Then

(a) when the KE is 0.64 times of its maximum KE, its displacement is 6 cm from O
(b) its speed is half the maximum speed when its displacement is half the maximum displacement
(c) Both (a) and (b) are correct
(d) Both (a) and (b) are wrong

23. A particle is attached to a vertical spring and is pulled down a distance 4 cm below its equilibrium and is released from rest. The initial upward acceleration is 0.5 ms^{-2}. The angular frequency of oscillation is

(a) 3.53 rad/s
(b) 0.28 rad/s
(c) 1.25 rad/s
(d) 0.08 rad/s

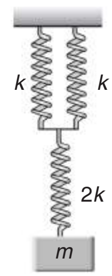
24. A block of mass 1 kg is kept on smooth floor of a truck. One end of a spring of force constant 100 N/m is attached to the block and other end is attached to the body of truck as shown in the figure. At $t = 0$, truck begins to move with constant acceleration 2 m/s^2. The amplitude of oscillation of block relative to the floor of truck is



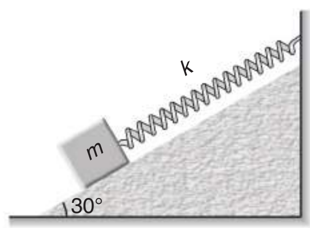
(a) 0.06 m
(b) 0.02 m
(c) 0.04 m
(d) 0.03 m

Subjective Questions

1. Find the period of oscillation of the system shown in figure.



2. A block of mass 0.2 kg is attached to a massless spring of force constant 80 N/m as shown in figure. Find the period of oscillation. Take $g = 10$ m/s^2. Neglect friction.



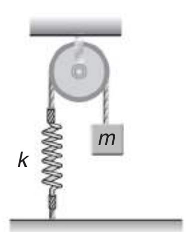
3. A body of weight 27 N hangs on a long spring of such stiffness that an extra force of 9 N stretches the spring by 0.05 m. If the body is pulled downward and released, what is the period ?

4. A clock with an iron pendulum keeps correct time at 20°C. How much time will it lose or gain in a day if the temperature changes to 40°C. Thermal coefficient of linear expansion $\alpha = 0.000012$ per°C.

5. A 50 g mass hangs at the end of a massless spring. When 20 g more are added to the end of the spring, it stretches 7.0 cm more. (a) Find the spring constant. (b) If the 20 g are now removed, what will be the period of the motion ?

6. An object suspended from a spring exhibits oscillations of period T. Now, the spring is cut in half and the two halves are used to support the same object, as shown in figure. Show that the new period of oscillation is $T/2$.

7. The string, the spring and the pulley shown in figure are light. Find the time period of the mass m.



8. A simple pendulum with a solid metal bob has a period T. What will be the period of the same pendulum if it is made to oscillate in a non-viscous liquid of density one-tenth of the metal of the bob?

9. A particle moves under the force $F(x) = (x^2 - 6x)$ N, where x is in metres. For small displacements from the origin what is the force constant in the simple harmonic motion approximation?

10. The initial position and velocity of a body moving in SHM with period $T = 0.25$ s are $x = 5.0$ cm and $v = 218$ cm/s. What are the amplitude and phase constant of the motion?

11. A point particle of mass 0.1 kg is executing SHM of amplitude 0.1 m. When the particle passes through the mean position, its kinetic energy is 8×10^{-3} J. Write down the equation of motion of this particle when the initial phase of oscillation is 45°.

12. Potential energy of a particle in SHM along x-axis is given by

$$U = 10 + (x - 2)^2$$

Here, U is in joule and x in metre. Total mechanical energy of the particle is 26 J. Mass of the particle is 2 kg. Find

(a) angular frequency of SHM,

(b) potential energy and kinetic energy at mean position and extreme position,

(c) amplitude of oscillation,

(d) x-coordinates between which particle oscillates.

13. A simple pendulum is taken at a place where its separation from the earth's surface is equal to the radius of the earth. Calculate the time period of small oscillations if the length of the string is 1.0 m. Take $g = \pi^2$ m/s^2 at the surface of the earth.

14. A solid cylinder of mass $M = 10$ kg and cross-sectional area $A = 20$ cm^2 is suspended by a spring of force constant $k = 100$ N/m and hangs partially immersed in water. Calculate the period of small oscillations of the cylinder.

15. A simple pendulum of length l and mass m is suspended in a car that is moving with constant speed v around a circle of radius r. Find the period of oscillation and equilibrium position of the pendulum.

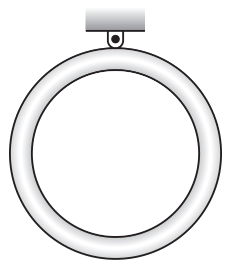
16. A body of mass 0.10 kg is attached to a vertical massless spring with force constant 4.0×10^3 N/m. The body is displaced 10.0 cm from its equilibrium position and released. How much time elapses as the body moves from a point 8.0 cm on one side of the equilibrium position to a point 6.0 cm on the same side of the equilibrium position?

17. A body of mass 200 g is in equilibrium at $x = 0$ under the influence of a force $F(x) = (-100x + 10x^2)$ N.

(a) If the body is displaced a small distance from equilibrium, what is the period of its oscillations?

(b) If the amplitude is 4.0 cm, by how much do we error in assuming that $F(x) = -kx$ at the end points of the motion.

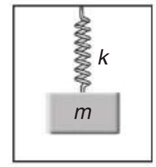
18. A ring of radius r is suspended from a point on its circumference. Determine its angular frequency of small oscillations.



19. A spring mass system is hanging from the ceiling of an elevator in equilibrium. The elevator suddenly starts accelerating upwards with acceleration a. Find

(a) the frequency and

(b) the amplitude of the resulting SHM.



20. A body makes angular simple harmonic motion of amplitude $\pi / 10$ rad and time period 0.05 s. If the body is at a displacement $\theta = \pi / 10$ rad at $t = 0$, write the equation giving angular displacement as a function of time.

21. A particle executes simple harmonic motion of period 16 s. Two seconds later after it passes through the centre of oscillation its velocity is found to be 2 m/s. Find the amplitude.

22. A simple pendulum consists of a small sphere of mass m suspended by a thread of length l. The sphere carries a positive charge q. The pendulum is placed in a uniform electric field of strength E directed vertically upwards. With what period will pendulum oscillate if the electrostatic force acting on the sphere is less than the gravitational force?

Hint: Electrostatic force is given by $\mathbf{F} = q\mathbf{E}$

23. Find the period of oscillation of a pendulum of length l if its point of suspension is

(a) moving vertically up with acceleration a.

(b) moving vertically down with acceleration a ($< g$).

(c) falling freely under gravity

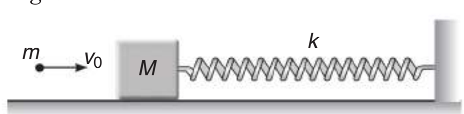
(d) moving horizontally with acceleration a.

24. A block with mass M attached to a horizontal spring with force constant k is moving with simple harmonic motion having amplitude A_1. At the instant when the block passes through its equilibrium position a lump of putty with mass m is dropped vertically on the block from a very small height and sticks to it.

(a) Find the new amplitude and period.

(b) Repeat part (a) for the case in which the putty is dropped on the block when it is at one end of its path.

25. A bullet of mass m strikes a block of mass M. The bullet remains embedded in the block. Find the amplitude of the resulting SHM.



26. An annular ring of internal and outer radii r and R respectively oscillates in a vertical plane about a horizontal axis perpendicular to its plane and passing through a point on its outer edge. Calculate its time period and show that the length of an equivalent simple pendulum is $\frac{3R}{2}$ as $r \to 0$ and $2R$ as $r \to R$.

27. A body of mass 200 g oscillates about a horizontal axis at a distance of 20 cm from its centre of gravity. If the length of the equivalent simple pendulum is 35 cm, find its moment of inertia about the point of suspension.

28. Show that the period of oscillation of simple pendulum at depth h below earth's surface is inversely proportional to $\sqrt{R - h}$, where R is the radius of earth. Find out the time period of a second pendulum at a depth $R/2$ from the earth's surface ?

29. The period of a particle in SHM is 8 s. At $t = 0$ it is in its equilibrium position.

(a) Compare the distance travelled in the first 4 s and the second 4 s.

(b) Compare the distance travelled in the first 2 s and the second 2 s.

30. (a) The motion of the particle in simple harmonic motion is given by $x = a \sin \omega t$. If its speed is u, when the displacement is x_1 and speed is v, when the displacement is x_2, show that the amplitude of the motion is

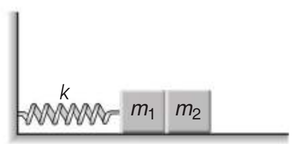
$$A = \left[\frac{v^2 x_1^2 - u^2 x_2^2}{v^2 - u^2}\right]^{1/2}$$

(b) A particle is moving with simple harmonic motion in a straight line. When the distance of the particle from the equilibrium position has the values x_1 and x_2, the corresponding values of velocity are u_1 and u_2, show that the period is

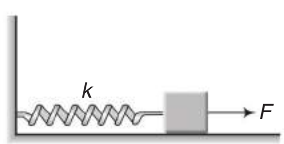
$$T = 2\pi \left[\frac{x_2^2 - x_1^2}{u_1^2 - u_2^2}\right]^{1/2}$$

31. Show that the combined spring energy and gravitational energy for a mass m hanging from a light spring of force constant k can be expressed as $U_0 + \frac{1}{2} ky^2$, where y is the distance above or below the equilibrium position and U_0 is constant.

32. The masses in figure slide on a frictionless table. m_1 but not m_2, is fastened to the spring. If now m_1 and m_2 are pushed to the left, so that the spring is compressed a distance d, what will be the amplitude of the oscillation of m_1 after the spring system is released ?



33. The spring shown in figure is unstretched when a man starts pulling on the cord. The mass of the block is M. If the man exerts a constant force F, find

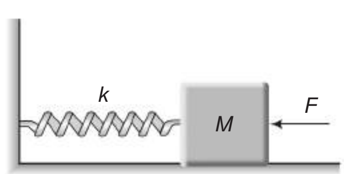


(a) the amplitude and the time period of the motion of the block,

(b) the energy stored in the spring when the block passes through the equilibrium position and

(c) the kinetic energy of the block at this position.

34. In figure, $k = 100$ N/m, $M = 1$ kg and $F = 10$ N.



(a) Find the compression of the spring in the equilibrium position.

(b) A sharp blow by some external agent imparts a speed of 2 m/s to the block towards left. Find the sum of the potential energy of the spring and the kinetic energy of the block at this instant.

(c) Find the time period of the resulting simple harmonic motion.

(d) Find the amplitude.

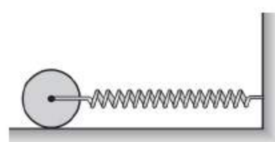
(e) Write the potential energy of the spring when the block is at the left extreme.

(f) Write the potential energy of the spring when the block is at the right extreme.

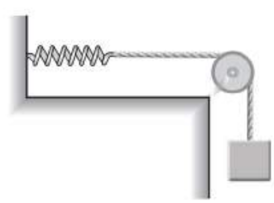
The answers of (b), (e) and (f) are different. Explain why this does not violate the principle of conservation of energy?

35. Pendulum A is a physical pendulum made from a thin, rigid and uniform rod whose length is d. One end of this rod is attached to the ceiling by a frictionless hinge, so that the rod is free to swing back and forth. Pendulum B is a simple pendulum whose length is also d. Obtain the ratio $\frac{T_A}{T_B}$ of their periods for small angle oscillations.

36. A solid cylinder of mass m is attached to a horizontal spring with force constant k. The cylinder can roll without slipping along the horizontal plane. (See the accompanying figure.) Show that the centre of mass of the cylinder executes simple harmonic motion with a period $T = 2\pi\sqrt{\frac{3m}{2k}}$, if displaced from mean position.



37. A cord is attached between a 0.50 kg block and a spring with force constant $k = 20$ N/m. The other end of the spring is attached to the wall and the cord is placed over a pulley ($I = 0.60\ MR^2$) of mass 5.0 kg and radius 0.50 m. (See the accompanying figure). Assuming no slipping occurs, what is the frequency of the oscillations when the body is set into motion?



38. Two linear SHM of equal amplitudes A and frequencies ω and 2ω are impressed on a particle along x and y-axes respectively. If the initial phase difference between them is $\pi/2$. Find the resultant path followed by the particle.

39. A particle is subjected to two simple harmonic motions given by

$$x_1 = 2.0\sin(100\,\pi t) \quad \text{and} \quad x_2 = 2.0\sin(120\,\pi\,t + \pi/3)$$

where, x is in cm and t in second. Find the displacement of the particle at

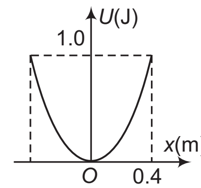
(a) $t = 0.0125$, (b) $t = 0.025$.

LEVEL 2

Objective Questions

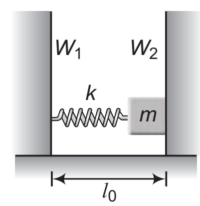
Single Correct Option

1. A particle of mass 2 kg moves in simple harmonic motion and its potential energy U varies with position x as shown. The period of oscillation of the particle is



(a) $\frac{2\pi}{5}$ s
(b) $\frac{2\sqrt{2}\pi}{5}$ s
(c) $\frac{\sqrt{2}\pi}{5}$ s
(d) $\frac{4\pi}{5}$ s

2. In the figure shown, a spring mass system is placed on a horizontal smooth surface in between two vertical rigid walls W_1 and W_2. One end of spring is fixed with wall W_1 and other end is attached with mass m which is free to move. Initially, spring is tension free and having natural length l_0. Mass m is compressed through a distance a and released. Taking the collision between wall W_2 and mass m as elastic and K as spring constant, the average force exerted by mass m on wall W_2 in one oscillation of block is



(a) $\frac{2aK}{\pi}$
(b) $\frac{2ma}{\pi}$
(c) $\frac{aK}{\pi}$
(d) $\frac{2aK}{m}$

3. Two simple harmonic motions are represented by the following equations $y_1 = 40 \sin \omega t$ and $y_2 = 10(\sin \omega t + c \cos \omega t)$. If their displacement amplitudes are equal, then the value of c (in appropriate units) is

(a) $\sqrt{13}$
(b) $\sqrt{15}$
(c) $\sqrt{17}$
(d) 4

4. A particle executes simple harmonic motion with frequency 2.5 Hz and amplitude 2 m. The speed of the particle 0.3 s after crossing, the equilibrium position is

(a) zero
(b) 2π m/s
(c) 4π m/s
(d) π m/s

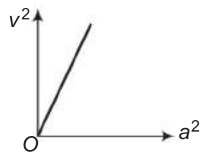
5. A particle oscillates simple harmonically with a period of 16 s. Two second after crossing the equilibrium position its velocity becomes 1 m/s. The amplitude is

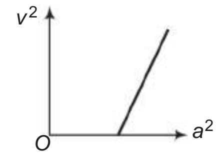
(a) $\frac{\pi}{4}$ m
(b) $\frac{8\sqrt{2}}{\pi}$ m
(c) $\frac{8}{\pi}$ m
(d) $\frac{4\sqrt{2}}{\pi}$ m

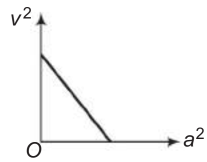
6. A seconds pendulum is suspended from the ceiling of a trolley moving horizontally with an acceleration of 4 m/s^2. Its period of oscillation is

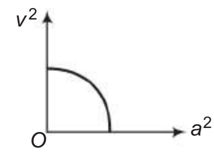
(a) 1.90 s
(b) 1.70 s
(c) 2.30 s
(d) 1.40 s

7. A particle is performing a linear simple harmonic motion. If the instantaneous acceleration and velocity of the particle are a and v respectively, identify the graph which correctly represents the relation between a and v.

(a) 

(b) 

(c) 

(d) 

8. In a vertical U-tube a column of mercury oscillates simple harmonically. If the tube contains 1 kg of mercury and 1 cm of mercury column weighs 20 g, then the period of oscillation is

(a) 1 s (b) 2 s
(c) $\sqrt{2}$ s (d) Insufficient data

9. A solid cube of side a and density ρ_0 floats on the surface of a liquid of density ρ. If the cube is slightly pushed downward, then it oscillates simple harmonically with a period of

(a) $2\pi\sqrt{\frac{\rho_0}{\rho}\frac{a}{g}}$ (b) $2\pi\sqrt{\frac{\rho}{\rho_0}\frac{a}{g}}$

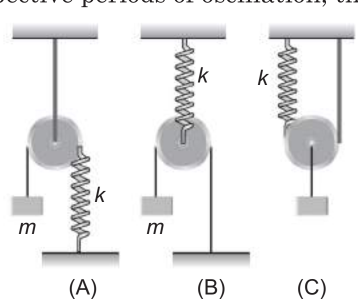
(c) $2\pi\sqrt{\frac{a}{\left(1-\frac{\rho}{\rho_0}\right)g}}$ (d) $2\pi\sqrt{\frac{a}{\left(1+\frac{\rho}{\rho_0}\right)g}}$

10. A uniform stick of length l is mounted so as to rotate about a horizontal axis perpendicular to the stick and at a distance d from the centre of mass. The time period of small oscillations has a minimum value when d/l is

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{\sqrt{12}}$

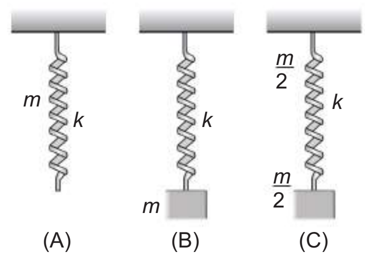
(c) $\frac{1}{\sqrt{3}}$ (d) $\frac{1}{\sqrt{6}}$

11. Three arrangements of spring-mass system are shown in figures (A), (B) and (C). If T_1, T_2 and T_3 represent the respective periods of oscillation, then correct relation is



(a) $T_1 > T_2 > T_3$ (b) $T_3 > T_2 > T_1$ (c) $T_2 > T_1 > T_3$ (d) $T_2 > T_3 > T_1$

12. Three arrangements are shown in figure.



(a) A spring of mass m and stiffness k

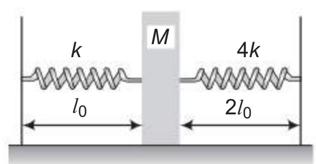
(b) A block of mass m attached to massless spring of stiffness k

(c) A block of mass $\frac{m}{2}$ attached to a spring of mass $\frac{m}{2}$ and stiffness k

If T_1, T_2 and T_3 represent the period of oscillation in the three cases respectively, then identify the correct relation.

(a) $T_1 < T_2 < T_3$
(b) $T_1 < T_3 < T_2$
(c) $T_1 > T_3 > T_2$
(d) $T_3 < T_1 < T_2$

13. A block of mass M is kept on a smooth surface and touches the two springs as shown in the figure but not attached to the springs. Initially springs are in their natural length. Now, the block is shifted $(l_0/2)$ from the given position in such a way that it compresses a spring and released. The time-period of oscillation of mass will be



(a) $\frac{\pi}{2}\sqrt{\frac{M}{k}}$
(b) $2\pi\sqrt{\frac{M}{5k}}$
(c) $\frac{3\pi}{2}\sqrt{\frac{M}{k}}$
(d) $\pi\sqrt{\frac{M}{2k}}$

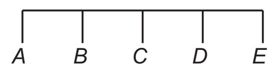
14. A particle moving on x-axis has potential energy $U = 2 - 20x + 5x^2$ joule along x-axis. The particle is released at $x = -3$. The maximum value of x will be (x is in metre)

(a) 5 m
(b) 3 m
(c) 7 m
(d) 8 m

15. A block of mass m, when attached to a uniform ideal spring with force constant k and free length L executes SHM. The spring is then cut in two pieces, one with free length $n\,L$ and other with free length $(1-n)\,L$. The block is also divided in the same fraction. The smaller part of the block attached to longer part of the spring executes SHM with frequency f_1. The bigger part of the block attached to smaller part of the spring executes SHM with frequency f_2. The ratio f_1/f_2 is

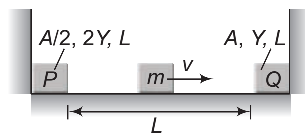
(a) 1
(b) $\frac{n}{1-n}$
(c) $\frac{1+n}{n}$
(d) $\frac{n}{1+n}$

16. A body performs simple harmonic oscillations along the straight line $ABCDE$ with C as the midpoint of AE. Its kinetic energies at B and D are each one fourth of its maximum value. If $AE = 2R$, the distance between B and D is



(a) $\frac{\sqrt{3}}{2}R$ (b) $\frac{R}{\sqrt{2}}$ (c) $\sqrt{3}\,R$ (d) $\sqrt{2}R$

17. In the given figure, two elastic rods P and Q are rigidly joined to end supports. A small mass m is moving with velocity v between the rods. All collisions are assumed to be elastic and the surface is given to be smooth. The time period of small mass m will be (A = area of cross section, Y = Young's modulus, L = length of each rod)



(a) $\frac{2L}{v} + 2\pi\sqrt{\frac{mL}{AY}}$ (b) $\frac{2L}{v} + 2\pi\sqrt{\frac{2mL}{AY}}$

(c) $\frac{2L}{v} + \pi\sqrt{\frac{mL}{AY}}$ (d) $\frac{2L}{v}$

18. A particle executes SHM of period 1.2 s and amplitude 8 cm. Find the time it takes to travel 3 cm from the positive extremity of its oscillation. [$\cos^{-1}(5/8) = 0.9$ rad]

(a) 0.28 s (b) 0.32 s (c) 0.17 s (d) 0.42 s

19. A wire frame in the shape of an equilateral triangle is hinged at one vertex so that it can swing freely in a vertical plane, with the plane of the triangle always remaining vertical. The side of the frame is $1/\sqrt{3}$ m. The time period in seconds of small oscillations of the frame will be ($g = 10$ m/s^2)

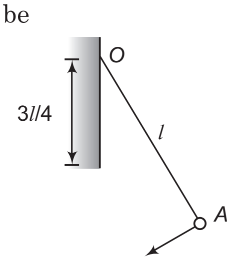
(a) $\pi/\sqrt{2}$ (b) $\pi/\sqrt{3}$

(c) $\pi/\sqrt{6}$ (d) $\pi/\sqrt{5}$

20. A particle moves along the x-axis according to $x = A\,[1 + \sin\omega t]$. What distance does is travel in time interval from $t = 0$ to $t = 2.5\pi/\omega$?

(a) $4A$ (b) $6A$ (c) $5A$ (d) $3A$

21. A small bob attached to a light inextensible thread of length l has a periodic time T when allowed to vibrate as a simple pendulum. The thread is now suspended from a fixed end O of a vertical rigid rod of length $3l/4$. If now the pendulum performs periodic oscillations in this arrangement, the periodic time will be

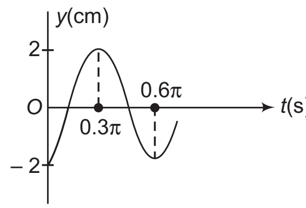


(a) $3T/4$ (b) $4T/5$ (c) $2T/3$ (d) $5T/6$

22. A stone is swinging in a horizontal circle of diameter 0.8 m at 30 rev/min. A distant light causes a shadow of the stone on a nearly wall. The amplitude and period of the SHM for the shadow of the stone are

(a) 0.4 m, 4s
(b) 0.2 m, 2s
(c) 0.4 m, 2s
(d) 0.8 m, 2s

23. Part of SHM is graphed in the figure. Here, y is displacement from mean position. The correct equation describing the SHM is



(a) $y = 4\cos(0.6\,t)$
(b) $y = 2\sin\left(\frac{10}{3}t - \frac{\pi}{2}\right)$
(c) $y = 2\sin\left(\frac{\pi}{2} - \frac{10}{3}t\right)$
(d) $y = 2\cos\left(0.6\,t + \frac{\pi}{2}\right)$

24. A particle performs SHM with a period T and amplitude a. The mean velocity of particle over the time interval during which it travels a distance $a/2$ from the extreme position is

(a) $6\,a/T$
(b) $2\,a/T$
(c) $3\,a/T$
(d) $a/2T$

25. A man of mass 60 kg is standing on a platform executing SHM in the vertical plane. The displacement from the mean position varies as $y = 0.5\sin(2\pi ft)$. The value of f, for which the man will feel weightlessness at the highest point, is (y in metre)

(a) $g/4\pi$
(b) $4\pi g$
(c) $\frac{\sqrt{2g}}{2\pi}$
(d) $2\pi\sqrt{2g}$

26. A particle performs SHM on a straight line with time period T and amplitude A. The average speed of the particle between two successive instants, when potential energy and kinetic energy become same is

(a) $\frac{A}{T}$
(b) $\frac{4\sqrt{2}A}{T}$
(c) $\frac{2A}{T}$
(d) $\frac{2\sqrt{2}A}{T}$

27. The time taken by a particle performing SHM to pass from point A to B where its velocities are same is 2 s. After another 2 s it returns to B. The ratio of distance OB to its amplitude (where O is the mean position) is

(a) $1:\sqrt{2}$
(b) $(\sqrt{2}-1):1$
(c) $1:2$
(d) $1:2\sqrt{2}$

28. A particle is executing SHM according to the equation $x = A\cos\omega t$. Average speed of the particle during the interval $0 \le t \le \frac{\pi}{6\omega}$ is

(a) $\frac{\sqrt{3}A\omega}{2}$
(b) $\frac{\sqrt{3}A\omega}{4}$
(c) $\frac{3\,A\omega}{\pi}$
(d) $\frac{3\,A\omega}{\pi}(2-\sqrt{3})$

More than One Correct Options

1. A simple pendulum with a bob of mass m is suspended from the roof of a car moving with horizontal acceleration a

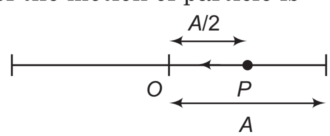
(a) The string makes an angle of $\tan^{-1}(a/g)$ with the vertical

(b) The string makes an angle of $\sin^{-1}\left(\dfrac{a}{g}\right)$ with the vertical

(c) The tension in the string is $m\sqrt{a^2+g^2}$

(d) The tension in the string is $m\sqrt{g^2-a^2}$

2. A particle starts from a point P at a distance of $A/2$ from the mean position O and travels towards left as shown in the figure. If the time period of SHM, executed about O is T and amplitude A then the equation of the motion of particle is



(a) $x = A\sin\left(\dfrac{2\pi}{T}t+\dfrac{\pi}{6}\right)$
(b) $x = A\sin\left(\dfrac{2\pi}{T}t+\dfrac{5\pi}{6}\right)$

(c) $x = A\cos\left(\dfrac{2\pi}{T}t+\dfrac{\pi}{6}\right)$
(d) $x = A\cos\left(\dfrac{2\pi}{T}t+\dfrac{\pi}{3}\right)$

3. A spring has natural length 40 cm and spring constant 500 N/m. A block of mass 1 kg is attached at one end of the spring and other end of the spring is attached to a ceiling. The block is released from the position, where the spring has length 45 cm

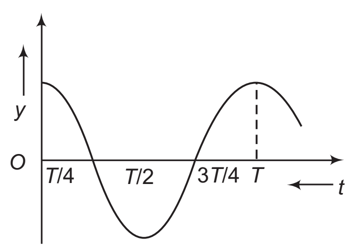
(a) the block will perform SHM of amplitude 5 cm

(b) the block will have maximum velocity $30\sqrt{5}$ cm/s

(c) the block will have maximum acceleration 15 m/s^2

(d) the minimum elastic potential energy of the spring will be zero

4. The displacement-time graph of a particle executing SHM is shown in figure. Which of the following statements is/are true?



(a) The velocity is maximum at $t = T/2$

(b) The acceleration is maximum at $t = T$

(c) The force is zero at $t = 3T/4$

(d) The kinetic energy equals the total oscillation energy at $t = T/2$

5. For a particle executing SHM, $x =$ displacement from mean position, $v =$ velocity and $a =$ acceleration at any instant, then

(a) v-x graph is a circle
(b) v-x graph is an ellipse

(c) a-x graph is a straight line
(d) a-x graph is a circle

6. The acceleration of a particle is $a = -100x + 50$. It is released from $x = 2$. Here, a and x are in SI units
   (a) the particle will perform SHM of amplitude 2 m
   (b) the particle will perform SHM of amplitude 1.5 m
   (c) the particle will perform SHM of time period 0.63 s
   (d) the particle will have a maximum velocity of 15 m/s

7. Two particles are performing SHM in same phase. It means that
   (a) the two particles must have same distance from the mean position simultaneously
   (b) two particles may have same distance from the mean position simultaneously
   (c) the two particles must have maximum speed simultaneously
   (d) the two particles may have maximum speed simultaneously

8. A particle moves along y-axis according to the equation
   y (in cm) $= 3\sin 100\pi t + 8\sin^2 50\pi t - 6$
   (a) the particle performs SHM
   (b) the amplitude of the particle's oscillation is 5 cm
   (c) the mean position of the particle is at $y = -2$ cm
   (d) the particle does not perform SHM

Comprehension Based Questions

Passage (Q Nos. 1 to 2)

A 2 kg block hangs without vibrating at the bottom end of a spring with a force constant of 400 N/m. The top end of the spring is attached to the ceiling of an elevator car. The car is rising with an upward acceleration of 5 m/s^2 when the acceleration suddenly ceases at time $t = 0$ and the car moves upward with constant speed ($g = 10\ m/s^2$)

1. What is the angular frequency of oscillation of the block after the acceleration ceases?
   (a) $10\sqrt{2}$ rad/s
   (b) 20 rad/s
   (c) $20\sqrt{2}$ rad/s
   (d) 32 rad/s

2. The amplitude of the oscillation is
   (a) 7.5 cm
   (b) 5 cm
   (c) 2.5 cm
   (d) 1 cm

Match the Columns

1. For the x-t equation of a particle in SHM along x-axis, match the following two columns.

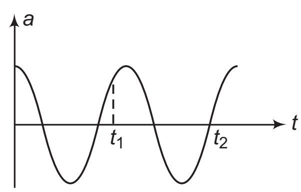
$$x = 2 + 2\cos\omega t$$

Column I	Column II
(a) Mean position	(p) $x = 0$
(b) Extreme position	(q) $x = 2$
(c) Maximum potential energy at	(r) $x = 4$
(d) Zero potential energy at	(s) Can't tell

2. Potential energy of a particle at mean position is 4 J and at extreme position is 20 J. Given that amplitude of oscillation is A. Match the following two columns.

Column I	Column II
(a) Potential energy at $x = \frac{A}{2}$	(p) 18 J
(b) Kinetic energy at $x = \frac{A}{4}$	(q) 16 J
(c) Kinetic energy at $x = 0$	(r) 8 J
(d) Kinetic energy at $x = \frac{A}{2}$	(s) None

3. Acceleration-time graph of a particle in SHM is as shown in figure. Match the following two columns.



Column I	Column II
(a) Displacement of particle at t_1	(p) zero
(b) Displacement of particle at t_2	(q) positive
(c) Velocity of particle at t_1	(r) negative
(d) Velocity of particle at t_2	(s) maximum

4. Mass of a particle is 2 kg. Its displacement-time equation in SHM is

$$x = 2\sin(4\pi t) \qquad \text{(SI Units)}$$

Match the following two columns for 1 second time interval.

Column I	Column II
(a) Speed becomes 30 m/s	(p) two times
(b) Velocity becomes + 10 m/s	(q) four times
(c) Kinetic energy becomes 400 J	(r) one time
(d) Acceleration becomes $-100\ \text{m/s}^2$	(s) None

5. x-t equation of a particle in SHM is, $x = 4 + 6\sin \pi t$. Match the following tables corresponding to time taken in moving from

Column I	Column II
(a) $x = 10$ m to $x = 4$ m	(p) $\frac{1}{3}$ second
(b) $x = 10$ m to $x = 7$ m	(q) $\frac{1}{2}$ second
(c) $x = 7$ m to $x = 1$ m	(r) 1 second
(d) $x = 10$ m to $x = -2$ m	(s) None

Subjective Questions

1. A 1 kg block is executing simple harmonic motion of amplitude 0.1 m on a smooth horizontal surface under the restoring force of a spring of spring constant 100 N/m. A block of mass 3 kg is gently placed on it at the instant it passes through the mean position. Assuming that the two blocks move together. Find the frequency and the amplitude of the motion.

2. Two particles are in SHM along same line. Time period of each is T and amplitude is A. After how much time will they collide if at time $t = 0$. (a) first particle is at $x_1 = +\frac{A}{2}$ and moving towards positive x-axis and second particle is at $x_2 = -\frac{A}{\sqrt{2}}$ and moving towards negative x-axis, (b) rest information are same as mentioned in part (a) except that particle first is also moving towards negative x-axis.

3. A particle that hangs from a spring oscillates with an angular frequency of 2 rad/s. The spring particle system is suspended from the ceiling of an elevator car and hangs motionless (relative to the elevator car) as the car descends at a constant speed of 1.5 m/s. The car then stops suddenly. (a) With what amplitude does the particle oscillate ? (b) What is the equation of motion for the particle ? (Choose upward as the positive direction)

4. A 2 kg mass is attached to a spring of force constant 600 N/m and rests on a smooth horizontal surface. A second mass of 1 kg slides along the surface toward the first at 6 m/s.

(a) Find the amplitude of oscillation if the masses make a perfectly inelastic collision and remain together on the spring. What is the period of oscillation ?

(b) Find the amplitude and period of oscillation if the collision is perfectly elastic.

(c) For each case, write down the position x as a function of time t for the mass attached to the spring, assuming that the collision occurs at time $t = 0$. What is the impulse given to the 2 kg mass in each case?

5. A block of mass 4 kg hangs from a spring of force constant $k = 400$ N/m. The block is pulled down 15 cm below equilibrium and released. How long does it take the block to go from 12 cm below equilibrium (on the way up) to 9 cm above equilibrium?

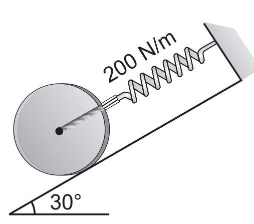
6. A plank with a body of mass m placed on it starts moving straight up according to the law $y = a(1 - \cos \omega t)$, where y is the displacement from the initial position, $\omega = 11$ rad/s. Find

(a) The time dependence of the force that the body exerts on the plank.

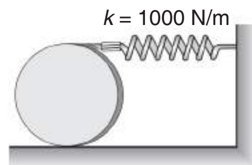
(b) The minimum amplitude of oscillation of the plank at which the body starts falling behind the plank.

7. A particle of mass m free to move in the x-y plane is subjected to a force whose components are $F_x = -kx$ and $F_y = -ky$, where k is a constant. The particle is released when $t = 0$ at the point (2, 3). Prove that the subsequent motion is simple harmonic along the straight line $2y - 3x = 0$.

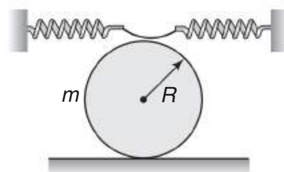
8. Determine the natural frequency of vibration of the 100 N disk. Assume the disk does not slip on the inclined surface.



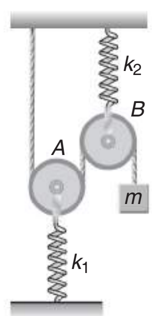
9. The disk has a weight of 100 N and rolls without slipping on the horizontal surface as it oscillates about its equilibrium position. If the disk is displaced, by rolling it counterclockwise 0.4 rad, determine the equation which describes its oscillatory motion when it is released.



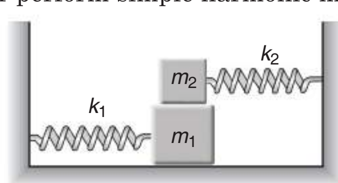
10. A solid uniform cylinder of mass m performs small oscillations due to the action of two springs of stiffness k each (figure). Find the period of these oscillations in the absence of sliding.



11. A block of mass m is attached to one end of a light inextensible string passing over a smooth light pulley B and under another smooth light pulley A as shown in the figure. The other end of the string is fixed to a ceiling. A and B are held by springs of spring constants k_1 and k_2. Find angular frequency of small oscillations of the system.



12. In the shown arrangement, both the springs are in their natural lengths. The coefficient of friction between m_2 and m_1 is μ. There is no friction between m_1 and the surface. If the blocks are displaced slightly, they together perform simple harmonic motion. Obtain

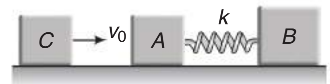


(a) Frequency of such oscillations.

(b) The condition if the frictional force on block m_2 is to act in the direction of its displacement from mean position.

(c) If the condition obtained in (b) is met, what can be maximum amplitude of their oscillations?

13. Two blocks A and B of masses $m_1 = 3$ kg and $m_2 = 6$ kg respectively are connected with each other by a spring of force constant $k = 200$ N/m as shown in figure. Blocks are pulled away from each other by $x_0 = 3$ cm and then released. When spring is in its natural length and blocks are moving towards each other, another block C of mass $m = 3$ kg moving with velocity $v_0 = 0.4$ m/s (towards right) collides with A and gets stuck to it. Neglecting friction, calculate



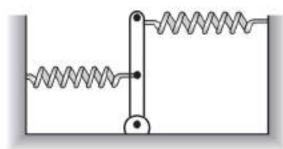
(a) velocities v_1 and v_2 of the blocks A and B respectively just before collision and their angular frequency.

(b) velocity of centre of mass of the system, after collision,

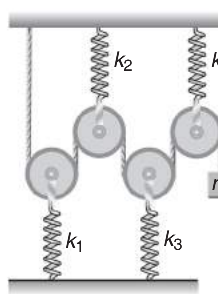
(c) amplitude of oscillations of combined body,

(d) loss of energy during collision.

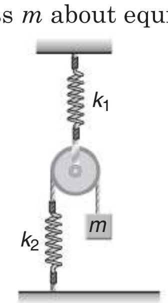
14. A rod of length l and mass m, pivoted at one end, is held by a spring at its mid-point and a spring at far end. The springs have spring constant k. Find the frequency of small oscillations about the equilibrium position.



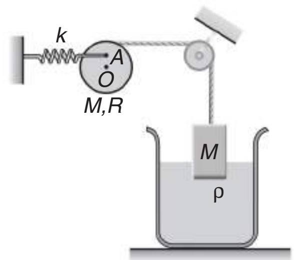
15. In the arrangement shown in figure, pulleys are light and springs are ideal. k_1, k_2, k_3 and k_4 are force constants of the springs. Calculate period of small vertical oscillations of block of mass m.



16. A light pulley is suspended at the lower end of a spring of constant k_1, as shown in figure. An inextensible string passes over the pulley. At one end of string a mass m is suspended, the other end of the string is attached to another spring of constant k_2. The other ends of both the springs are attached to rigid supports, as shown. Neglecting masses of springs and any friction, find the time period of small oscillations of mass m about equilibrium position.

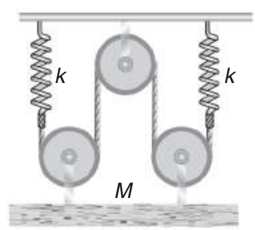


17. Figure shows a solid uniform cylinder of radius R and mass M, which is free to rotate about a fixed horizontal axis O and passes through centre of the cylinder. One end of an ideal spring of force constant k is fixed and the other end is hinged to the cylinder at A. Distance OA is equal to $\frac{R}{2}$. An inextensible thread is wrapped round the cylinder and passes over a smooth, small pulley. A block of equal mass M and having cross sectional area A is suspended from free end of the thread. The block is partially immersed in a non-viscous liquid of density ρ.



If in equilibrium, spring is horizontal and line OA is vertical, calculate frequency of small oscillations of the system.

18. Find the natural frequency of the system shown in figure. The pulleys are smooth and massless.



Answers

Introductory Exercise 14.1

1. π **2.** $\frac{1}{16}, \frac{15}{16}$ **3.** (a) 15.0 cm (b) 0.726 s (c) 1.38 Hz (d) 1.69 J (e) 1.30 m/s

4. 0.101 m/s, 1.264 m/s^2, 0.632 N **5.** No

Introductory Exercise 14.2

1. (a) 1.39 J (b) 1.1 s **2.** $\pm$ 0.58 m/s, – 0.45 ms^2, $\pm$ 0.60 m/s, zero **3.** $\frac{\pi}{6}$

4. (a) 0.08 m (b) 1.57 rad/s (c) 1.97 N/m (d) Zero (e) 0.197 m/s^2

5. (a) $\frac{\pi}{120}$ sec (b) $\frac{\pi}{30}$ sec (c) $\frac{\pi}{30}$ sec **6.** See the hints.

Introductory Exercise 14.3

1. $\left(\frac{\pi}{2}\right)$ sec **2.** 3.2 kg **3.** 1 sec **4.** $\sqrt{p}$ **5.** $\frac{16}{9}$ **6.** 11 % **7.** $\frac{\sqrt{3}}{2\pi}$

Introductory Exercise 14.4

1. (a) 7.0 cm (b) 6.1 cm (c) 5.0 cm (d) 1.0 cm

2. (a) 2.6 unit (b) 1917unit (c) 6.0×10^5unit **3.** $2A$ **4.** $\frac{2\pi}{3}$

Exercises

LEVEL 1

Assertion and Reason

1. (d) **2.** (b) **3.** (c) **4.** (a) **5.** (d) **6.** (d) **7.** (d) **8.** (a) **9.** (a) **10.** (d)

Objective Questions

1. (a) **2.** (a) **3.** (d) **4.** (c) **5.** (d) **6.** (b) **7.** (b) **8.** (d) **9.** (c) **10.** (a)
11. (c) **12.** (d) **13** (c) **14.** (b) **15.** (b) **16.** (a) **17.** (a) **18.** (a) **19.** (b) **20.** (d)
21. (c) **22.** (a) **23.** (a) **24.** (b)

Subjective Questions

1. $T = 2\pi\sqrt{\frac{m}{k}}$ **2.** 0.314 s **3.** 0.78 s **4.** The clock will lose 10.37s **5.** (a) 2.8 N/m (b) 0.84 s

7. $T = 2\pi\sqrt{\frac{m}{k}}$ **8.** $\left(\sqrt{\frac{10}{9}}\right)T$ **9.** 6.0 N/m **10.** 10.0 cm, $\frac{\pi}{6}$ rad **11.** $y = (0.1 \text{ m}) \sin\left[(4\text{s}^{-1})t + \frac{\pi}{4}\right]$

12. (a) 1 rad/s (b) $U_{\text{mean}} = 10$ J, $K_{\text{mean}} = 16$ J, $U_{\text{extreme}} = 26$ J, $K_{\text{extreme}} = 0$ (c) 4 m (d) $x = 6$ m and $x = -2$ m

13. 4 s **14.** 1.8 s

15. $2\pi\sqrt{\frac{l}{\left\{g^2 + \left(\frac{v^2}{r}\right)^2\right\}^{1/2}}}$ and inclined to the vertical at an angle $\theta = \tan^{-1}\left(\frac{v^2}{rg}\right)$ away from the centre

16. 1.4×10^{-3} s **17.** (a) 0.28 s (b) 0.4% **18.** $\sqrt{\frac{g}{2r}}$ **19.**(a) $\frac{1}{2\pi}\sqrt{\frac{k}{m}}$ (b) $\frac{ma}{k}$ **20.** $\theta = \left(\frac{\pi}{10} \text{ rad}\right)\cos[(40\pi\text{s}^{-1})t]$

21. 7.2 m **22.** $2\pi\sqrt{\dfrac{l}{g-\dfrac{qE}{m}}}$ **23.** (a) $2\pi\sqrt{\dfrac{l}{(g+a)}}$ (b) $2\pi\sqrt{\dfrac{l}{(g-a)}}$ (c) Infinite (d) $2\pi\sqrt{\dfrac{l}{(g^2+a^2)^{1/2}}}$

24. (a) $A_2 = A_1\sqrt{\dfrac{M}{M+m}}$, $T = 2\pi\sqrt{\dfrac{M+m}{k}}$ (b) $A_2 = A_1$, $T = 2\pi\sqrt{\dfrac{M+m}{k}}$ **25.** $\dfrac{mv_0}{\sqrt{k(M+m)}}$

26. $T = 2\pi\sqrt{\dfrac{\dfrac{3R}{2}+\dfrac{r^2}{2R}}{g}}$ **27.** 1.4×10^5 g-cm^2 **28.** $2\sqrt{2}$ s **29.** (a) equal (b) equal

32. $A = d\sqrt{\dfrac{m_1}{m_1+m_2}}$ **33.** (a) $\dfrac{F}{k}$, $2\pi\sqrt{\dfrac{M}{k}}$ (b) $\dfrac{F^2}{2k}$ (c) $\dfrac{F^2}{2k}$

34. (a) 10 cm (b) 2.5 J (c) $\dfrac{\pi}{5}$ sec (d) 20 cm (e) 4.5 J (f) 0.5 J **35.** 0.816 **37.** 0.38 Hz

38. Parabola, $y = A\left(1-\dfrac{2x^2}{A^2}\right)$ **39.** (a) –2.41 cm (b) 0.27 cm

LEVEL 2

Single Correct Option

1. (d) **2.** (a) **3.** (b) **4.** (a) **5.** (b) **6.** (a) **7.** (c) **8.** (a) **9.** (a) **10.** (b)
11. (c) **12.** (b) **13.** (c) **14.** (c) **15.** (a) **16.** (c) **17.** (a) **18.** (c) **19.** (d) **20.** (c)
21. (a) **22.** (c) **23.** (b) **24.** (c) **25.** (c) **26.** (b) **27.** (a) **28.** (d)

More than One Correct Options

1. (a,c) **2.** (b,d) **3.** (b,c,d) **4.** (b, c) **5.** (b,c) **6.** (b,c, d) **7.** (b,c) **8.** (a, b, c)

Comprehension Based Questions

1. (a) **2.** (c)

Match the Columns

1. (a) → q (b) → p,r (c) → p,r (d) → s
2. (a) → r (b) → s (c) → q (d) → s
3. (a) → r (b) → p (c) → r (d) → r,s
4. (a) → s (b) → q (c) → s (d) → q
5. (a) → q (b) → p (c) → p (d) → r

Subjective Questions

1. 0.8 Hz, 0.05 m **2.** (a) $\dfrac{19}{48}T$ (b) $\dfrac{11}{48}T$ **3.** (a) $A = 0.75$ m (b) $x = -0.75\sin 2t$

4. (a) 14.1 cm, 0.44 s (b) 23 cm, 0.36 s (c) $x = \pm$ (14.1cm) $\sin(10\sqrt{2}\,t)$, $x = \pm$ (23 cm) $\sin(10\sqrt{3}\,t)$, 4 N-s, 8 N-s

5. $\dfrac{\pi}{20}$ s = 0.157 s **6.** (a) $N = m(g + a\omega^2\cos\omega t)$ (b) 8.1 cm **8.** 0.56 Hz **9.** $\theta = 0.4\cos(16.16\,t)$

10. $T = \dfrac{\pi}{2}\sqrt{\dfrac{3m}{k}}$ **11.** $\sqrt{\dfrac{k_1k_2}{4m(k_1+k_2)}}$ **12.** (a) $\dfrac{1}{2\pi}\sqrt{\dfrac{k_1+k_2}{m_1+m_2}}$ (b) $\dfrac{k_1}{k_2} < \dfrac{m_1}{m_2}$ (c) $\dfrac{\mu(m_1+m_2)m_2g}{m_1k_2-m_2k_1}$

13. (a) 0.2 m/s, 0.1 m/s, 10 rad/s (b) 0.1 m/s (towards right) (c) 4.8 cm (d) 0.03 J

14. $\dfrac{1}{2\pi}\sqrt{\dfrac{15k}{4m}}$ **15.** $T = 4\pi\sqrt{m\left(\dfrac{1}{k_1}+\dfrac{1}{k_2}+\dfrac{1}{k_3}+\dfrac{1}{k_4}\right)}$ **16.** $T = 2\pi\sqrt{\dfrac{m(4k_2+k_1)}{k_1k_2}}$

17. $f = \dfrac{1}{2\pi}\sqrt{\dfrac{k+4A\rho g}{6M}}$ **18.** $\dfrac{1}{\pi}\sqrt{\dfrac{2k}{M}}$

15

Elasticity

Chapter Contents

15.1 Introduction

The properties of material under the action of external deforming forces are very essential, for an engineer, to enable him, in designing him all types of structures and machines.

Whenever a load is attached to a thin hanging wire it elongates and the load moves downwards (sometimes through a negligible distance). The amount by which the wire elongates depends upon the amount of load and the nature of wire material. Cohesive force, between the molecules of the hanging wire offer resistance against the deformation, and the force of resistance increases with the deformation. The process of deformation stops when the force of resistance is equal to the external force (i.e. the load attached). Sometimes the force of resistance offered by the molecules is less than the external force. In such a case, the deformation continues until the wire breaks.

Thus, we may conclude that if some external deforming force is applied to a body it has two effects on it, namely :

(i) deformation of the body,

(ii) internal resistance (restoring) forces are developed.

15.2 Elasticity

As we have already discussed that whenever a single force (or a system of forces) acts on a body it undergoes some deformation and the molecules offer some resistance to the deformation. When the external force is removed, the force of resistance also vanishes and the body returns back to its original shape. But it is only possible if the deformation is within a certain limit. Such a limit is called elastic limit. This property of materials of returning back to their original position is called the elasticity.

A body is said to be perfectly elastic if it returns back completely to its original shape and size after removing the external force. If a body remains in the deformed state and does not even partially regain its original shape after the removal of the deforming forces, it is called a perfectly inelastic or plastic body. Quite often, when the external forces are removed, the body partially regains the original shape. Such bodies are partially elastic. If the force acting on the body is increased and the deformation exceeds the elastic limit, the body loses to some extent, its property of elasticity. In this case, the body will not return to its original shape and size even after removal of the external force. Some deformation is left permanently.

15.3 Stress and Strain

Stress

When an external force is applied to a body then at each cross-section of the body an internal restoring force is developed which tends to restore the body to its original state. The internal restoring force per unit area of cross-section of the deformed body is called stress. It is usually denoted by σ (sigma).

Thus,
$$\boxed{\text{Stress } (\sigma) = \frac{\text{Restoring force}}{\text{Area}}}$$

Note *In equilibrium, when further deformation stops, the restoring force is equal to the external force (or the suspended load from a hanging wire). Therefore, the stress is also sometime called the external deforming force per unit area.*

Strain

When the size or shape of a body is changed under an external force, the body is said to be strained. The change occurred in the unit size of the body is called strain. Usually, it is denoted by ε. Thus,

$$\boxed{\varepsilon = \frac{\Delta x}{x}}$$

Here, Δx is the change (may be in length, volume etc.) and x the original value of the quantity in which change has occurred. For example, when the length of a suspended wire increases under an applied load, the value of strain is,

$$\boxed{\varepsilon = \frac{\Delta l}{l}}$$

15.4 Hooke's Law and the Modulus of Elasticity

According to Hooke's law,

"For small deformation, the stress in a body is proportional to the corresponding strain." i.e.

$$\text{stress} \propto \text{strain} \quad \text{or} \quad \boxed{\text{stress} = (E)\,(\text{strain})}$$

Here, $E = \dfrac{\text{stress}}{\text{strain}}$ is a constant called the modulus of elasticity. Now, depending upon the nature of deforming force applied on the body, stress, strain and hence modulus of elasticity are classified in following three types:

Young's Modulus of Elasticity (Y)

When a wire is acted upon by two equal and opposite forces in the direction of its length, the length of the body is changed. The change in length per unit length $\left(\dfrac{\Delta l}{l}\right)$ is called the **longitudinal strain** and the restoring force (which is equal to the applied force in equilibrium) per unit area of cross section of the wire is called the **longitudinal stress**.

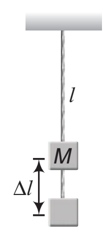


Fig. 15.1

For small change in the length of the wire, the ratio of the longitudinal stress to the corresponding strain is called the Young's modulus of elasticity (Y) of the wire. Thus,

$$Y = \frac{F/A}{\Delta l/l} \quad \text{or} \quad \boxed{Y = \frac{Fl}{A\Delta l}}$$

Let there be a wire of length l and radius r. Its one end is clamped to a rigid support and a mass M is attached at the other end. Then,

$$F = Mg \quad \text{and} \quad A = \pi r^2$$

Substituting in above equation, we have

$$\boxed{Y = \frac{Mgl}{(\pi r^2)\Delta l}}$$

Bulk Modulus of Elasticity (*B*)

When a uniform pressure (normal force) is applied all over the surface of a body, the volume of the body changes. The change in volume per unit volume of the body is called the '**volume strain**' and the normal force acting per unit area of the surface (pressure) is called the normal stress or **volume stress**. For small strains, the ratio of the volume stress to the volume strain is called the 'bulk modulus' of the material of the body. It is denoted by *B*. Then,

$$\boxed{B = \frac{-p}{\Delta V/V} \quad \text{or} \quad \frac{-\Delta p}{(\Delta V/V)}}$$

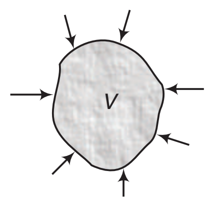


Fig. 15.2

Here, negative sign implies that, when the pressure increases volume decreases and *vice-versa*.

Compressibility

The reciprocal of the bulk modulus of the material of a body is called the 'compressibility' of that material. Thus,

$$\boxed{\text{Compressibility} = \frac{1}{B}}$$

Modulus of Rigidity (η)

When a body is acted upon by an external force tangential to a surface of the body, the opposite surface being kept fixed, it suffers a change in shape, its volume remaining unchanged. Then, the body is said to be sheared.

The ratio of the displacement of a layer in the direction of the tangential force and the distance of the layer from the fixed surface is called the **shearing strain** and the tangential force acting per unit area of the surface is called the "**shearing stress**".

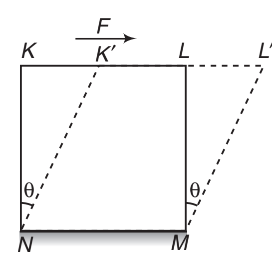


Fig. 15.3

For small strain the ratio of the shearing stress to the shearing strain is called the "modulus of rigidity" of the material of the body. It is denoted by η.

Thus, $$\eta = \frac{F/A}{KK'/KN}$$

Here, $$\frac{KK'}{KN} = \tan\theta \approx \theta$$

∴ $$\eta = \frac{F/A}{\theta} \quad \text{or} \quad \boxed{\eta = \frac{F}{A\theta}}$$

Extra Points to Remember

- If a spring is stretched or compressed by an amount Δl, the restoring force produced in it is,

$$F_s = k\,\Delta l \qquad \ldots(\text{i})$$

Here, k = force constant of spring

Similarly, if a wire is stretched by an amount Δl, the restoring force produced in it is,

$$\boxed{F = \left(\frac{YA}{l}\right)\Delta l} \qquad \ldots(\text{ii})$$

as,
$$Y = \frac{F/A}{\Delta l/l}$$

Comparing Eqs. (i) and (ii), we can see that force constant of a wire is,

$$k = \frac{YA}{l} \qquad \ldots(\text{iii})$$

i.e. a wire is just like a spring of force constant $\frac{YA}{l}$. So, all formulae which we use in case of a spring can be applied to a wire also.

From Eq. (iii), we may also conclude that force constant of a spring is inversely proportional to the length of the spring l or

$$\boxed{k \propto \frac{1}{l}}$$

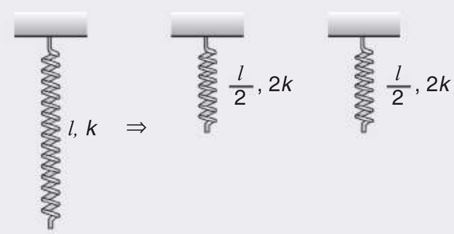


Fig. 15.4

i.e. if a spring is cut into two equal pieces its force constant is doubled.

Example 15.1 *Determine the elongation of the steel bar 1 m long and 1.5 cm² cross-sectional area when subjected to a pull of* 1.5×10^4 *N.*
(Take $Y = 2.0 \times 10^{11}\ N/m^2$*)*

Solution $Y = \dfrac{F/A}{\Delta l/l}$

$\therefore$
$$\Delta l = \frac{Fl}{AY}$$

Substituting the values,

$$\Delta l = \frac{(1.5 \times 10^4)(1.0)}{(1.5 \times 10^{-4})(2.0 \times 10^{11})}$$
$$= 0.5 \times 10^{-3}\ \text{m}$$

or
$$\Delta l = 0.5\ \text{mm}$$
Ans.

Example 15.2 *The bulk modulus of water is* $2.3 \times 10^9\ N/m^2$.

(a) Find its compressibility in the units atm^{-1}.

(b) How much pressure in atmospheres is needed to compress a sample of water by 0.1%?

Solution Here, $B = 2.3 \times 10^9\ \text{N/m}^2$

$$= \frac{2.3 \times 10^9}{1.01 \times 10^5}$$

$$= 2.27 \times 10^4 \text{ atm}$$

(a) Compressibility $= \frac{1}{B} = \frac{1}{2.27 \times 10^4} = 4.4 \times 10^{-5}\ \text{atm}^{-1}$ **Ans.**

(b) Here, $\frac{\Delta V}{V} = -\,0.1\% = -\,0.001$

Required increase in pressure,

$$\Delta p = B \times \left(-\frac{\Delta V}{V}\right)$$

$$= 2.27 \times 10^4 \times 0.001$$

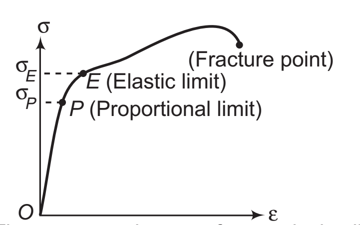
$$= 22.7 \text{ atm}$$ **Ans.**

INTRODUCTORY EXERCISE 15.1

1. Two wires A and B of same dimensions are stretched by same amount of force. Young's modulus of A is twice that of B. Which wire will get more elongation?
2. A rod 100 cm long and of 2 cm $\times$ 2 cm cross-section is subjected to a pull of 1000 kg force. If the modulus of elasticity of the material is 2.0×10^6 kg/cm^2, determine the elongation of the rod.
3. A cast iron column has internal diameter of 200 mm. What should be the minimum external diameter so that it may carry a load of 1.6 MN without the stress exceeding 90 N/mm^2?
4. Find the dimensions of stress, strain and modulus of elasticity.

15.5 The Stress-Strain Curve

A plot of longitudinal stress (either tensile or compressive) *versus* longitudingal strain for a typical solid is shown in figure. The strain is directly proportional to the applied stress for values of stress upto σ_P. In this linear region, the material returns to its original size when the stress is removed. Point P is known as the proportional limit of the solid. For stresses between σ_P and σ_E, where point E is called the elastic limit, the material also returns to its original size.



The stress-strain curve for a typical solid

Fig. 15.5

However, notice that stress and strain are not proportional in this region. For deformations beyond the elastic limit, the material does not return to its original size when the stress is removed, it is permanently distorted. Finally, further stretching beyond the elastic limit leads to the eventual fracture of the solid. The proportionality constant for linear region or the slope of stress-strain curve in this curve is called the Young's modulus of elasticity Y.

15.6 Potential Energy Stored in a Stretched Wire

When a wire is stretched, work is done against the inter atomic forces. This work is stored in the wire in the form of elastic potential energy. Suppose on applying a force F on a wire of length l, the increase in length is Δl. The area of cross-section of the wire is A. The potential energy stored in the wire should be,

$$\boxed{U = \frac{1}{2} k (\Delta l)^2}$$

Here, $$k = \frac{YA}{l}$$

$\therefore$ $$U = \frac{1}{2} \frac{YA}{l} (\Delta l)^2$$

Elastic potential energy per unit volume of the wire (also called energy density) is,

$$u = \frac{U}{\text{volume}} \quad \text{or} \quad u = \frac{\frac{1}{2}\frac{YA}{l}(\Delta l)^2}{Al}$$

or $$u = \frac{1}{2}\left(\frac{\Delta l}{l}\right)\left(Y \cdot \frac{\Delta l}{l}\right) \quad \text{or} \quad u = \frac{1}{2}(\text{strain})\,(Y \times \text{strain})$$

or $$\boxed{u = \frac{1}{2}(\text{strain}) \times (\text{stress})}$$

15.7 Thermal Stresses or Strains

Whenever there is some increase or decrease in the temperature of the body, it causes the body to expand or contract. If the body is allowed to expand or contract freely, with the rise or fall of the temperature, no stresses are induced in the body. But if the deformation of the body is prevented, some stresses are induced in the body. Such stresses are called thermal stresses or temperature stresses. The corresponding strains are called thermal strains or temperature strains.

Consider a rod AB fixed at two supports as shown in figure.

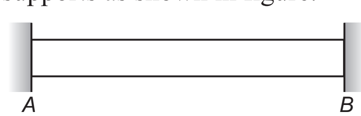


Fig. 15.6

Let l = length of rod

A = area of cross-section of the rod

Y = Young's modulus of elasticity of the rod

and α = thermal coefficient of linear expansion of the rod

Let the temperature of the rod is increased by an amount t. The length of the rod would had increased by an amount Δl, if it were not fixed at two supports. Here

$$\Delta l = l\alpha t$$

But the rod is fixed at the supports. Hence a compressive strain will be produced in the rod. Because at the increased temperature, the natural length of the rod is $l+\Delta l$, while being fixed at two supports its actual length is l. Hence, thermal strain

$$\varepsilon = \frac{\Delta l}{l+\Delta l} \approx \frac{\Delta l}{l} = \frac{l\alpha t}{l} = \alpha t$$

or $$\boxed{\varepsilon = \alpha t}$$

Therefore, thermal stress $$\sigma = Y\varepsilon \qquad \text{(stress} = Y \times \text{strain)}$$

or $$\boxed{\sigma = Y\alpha t}$$

or force on the supports,

$$\boxed{F = \sigma A = YA\alpha t}$$

This force F is in the direction shown below.

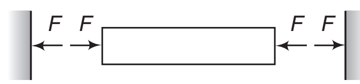


Fig. 15.7

Example 15.3 *A steel wire 4.0 m in length is stretched through 2.0 mm. The cross-sectional area of the wire is 2.0 mm^2. If Young's modulus of steel is 2.0×10^{11} N/m^2. Find*

(a) the energy density of wire,

(b) the elastic potential energy stored in the wire.

Solution Here, $l = 4.0$ m, $\Delta l = 2\times 10^{-3}$ m, $A = 2.0\times 10^{-6}$ m^2, $Y = 2.0\times 10^{11}$ N/m^2

(a) The energy density of stretched wire

$$U = \frac{1}{2} \times \text{stress} \times \text{strain}$$

$$= \frac{1}{2} \times Y \times (\text{strain})^2$$

$$= \frac{1}{2} \times 2.0\times 10^{11} \times \left(\frac{(2\times 10^{-3})}{4}\right)^2$$

$$= 0.25\times 10^5 = 2.5\times 10^4 \text{ J/m}^3 \qquad \textbf{Ans.}$$

(b) Elastic potential energy = energy density × volume

$$= 2.5\times 10^4 \times (2.0\times 10^{-6}) \times 4.0 \text{ J}$$

$$= 20\times 10^{-2} = 0.20 \text{ J} \qquad \textbf{Ans.}$$

Example 15.4 *A rubber cord has a cross-sectional area 1 mm^2 and total unstretched length 10.0 cm. It is stretched to 12.0 cm and then released to project a missile of mass 5.0 g. Taking Young's modulus Y for rubber as 5.0×10^8 N/m^2. Calculate the velocity of projection.*

Solution Equivalent force constant of rubber cord

$$k = \frac{YA}{l} = \frac{(5.0\times 10^8)(1.0\times 10^{-6})}{(0.1)} = 5.0\times 10^3 \text{ N/m}$$

Now, from conservation of mechanical energy,

elastic potential energy of cord = kinetic energy of missile

$$\therefore \qquad \frac{1}{2}k(\Delta l)^2 = \frac{1}{2}mv^2 \quad \Rightarrow \quad \therefore \quad v = \left(\sqrt{\frac{k}{m}}\right)\Delta l$$

$$= \left(\sqrt{\frac{5.0\times 10^3}{5.0\times 10^{-3}}}\right)(12.0-10.0)\times 10^{-2} = 20 \text{ m/s}$$ **Ans.**

Note *Following assumptions have been made in this problem :*

(*i*) *k has been assumed constant, even though it depends on the length* (l).

(*ii*) *The whole of the elastic potential energy is converting into kinetic energy of missile.*

INTRODUCTORY EXERCISE 15.2

1. Find the dimensions of energy density.

2. (a) A wire 4 m long and 0.3 mm in diameter is stretched by a force of 100 N. If extension in the wire is 0.3 mm, calculate the potential energy stored in the wire.

(b) Find the work done in stretching a wire of cross-section 1 mm^2 and length 2 m through 0.1 mm. Young's modulus for the material of wire is 2.0×10^{11} N/m^2.

Extra Points to Remember

- Modulus of elasticity E (whether, it is Y, B or η) is given by

$$E = \frac{\text{stress}}{\text{strain}}$$

Following conclusions can be made from the above expression :

(i) $E \propto$ stress (for same strain), i.e. if we want the equal amount of strain in two different materials, the one which needs more stress is having more E.

(ii) $E \propto \frac{1}{\text{strain}}$ (for same stress), i.e. if the same amount of stress is applied on two different materials, the one having the less strain is having more E. Rather we can say that the material which offers more resistance to the external forces is having greater value of E. So, we can see that modulus of elasticity of steel is more than that of rubber or

$$\boxed{E_{\text{steel}} > E_{\text{rubber}}}$$

(iii) E = stress for unit strain $\left(\frac{\Delta x}{x} = 1 \text{ or } \Delta x = x\right)$, i.e. suppose the length of a wire is 2 m, then the Young's modulus of elasticity (Y) is the stress applied on the wire to stretch the wire by the same length of 2 m.

- The material which has smaller value of Y is more ductile, i.e. it offers less resistance in framing it into a wire. Similarly, the material having the smaller value of B is more malleable. Thus, for making wire we will more concentrate on Y.

- A solid will have all the three modulii of elasticity Y, B and η. But in case of a liquid or a gas only B can be defined, as a liquid or a gas cannot be framed into a wire or no shear force can be applied on them.
- For a liquid or a gas,

$$\boxed{B = \left(\frac{-dp}{dV/V}\right) \quad \text{or} \quad \left(-\frac{\Delta p}{\Delta V/V}\right)}$$

So, instead of p we are more interested in change in pressure dp or Δp.

- In case of a gas, bulk modulus is process dependent and is given by,

$$\boxed{B = xp}$$

in the process pV^x = constant

For example, for $x = 1$, or pV = constant (isothermal process), $B = p$.

i.e., isothermal bulk modulus of a gas (denoted by B_T) is equal to the pressure of the gas at that instant of time or

$$\boxed{B_T = p}$$

Similarly, for $x = \gamma = \dfrac{C_p}{C_V}$ or pV^γ = constant (adiabatic process), $B = \gamma p$.

i.e., adiabatic bulk modulus of a gas (denoted by B_s) is equal to γ times the pressure of the gas at that instant of time or $B_s = \gamma p$

In general for a gas, $B \propto p$ in all processes.

Physically this can be explained as under:

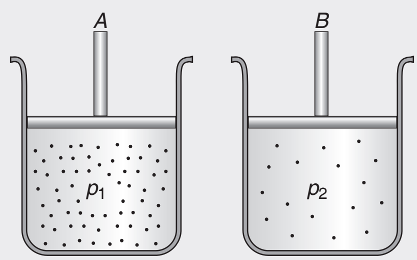


Fig. 15.8

Suppose we have two containers A and B. Some gas is filled in both the containers. But the pressure in A is more than the pressure in B, i.e. $p_1 > p_2$

So, bulk modulus of A should be more than the bulk modulus of B, or $B_1 > B_2$

and this is quiet obvious, because it is more difficult to compress the gas in chamber A, i.e. it provides more resistance to the external forces. And as we have discussed earlier also, the modulus of elasticity is more for a material which offers more resistance to external forces.

- When a pressure is applied on a substance, its density is changed. The change in density can be calculated as under :

$$\rho = \frac{\text{mass}}{\text{volume}} \qquad (\rho = \text{density})$$

or

$$\rho \propto \frac{1}{V} \qquad (\text{mass = constant})$$

$$\frac{\rho'}{\rho} = \frac{V}{V'} = \frac{V}{V + dV}$$

or

$$\rho' = \rho\left(\frac{V}{V + dV}\right)$$

$$= \rho\left(\frac{V}{V - (dp/B)V}\right) \qquad \left(\text{as } B = -\frac{dp}{dV/V}\right)$$

$$\rho' = \frac{\rho}{1 - \frac{dp}{B}}$$

From this expression, we can see that ρ' increases as pressure is increased (dp is positive) and *vice-versa*.

We can also write the above equation as

$$\rho' = \rho\left(1 - \frac{dp}{B}\right)^{-1}$$

or $$\rho' \approx \rho\left(1 + \frac{dp}{B}\right) \qquad (\text{if } dp << B)$$

or $$\rho' - \rho = \Delta\rho = \rho\frac{(dp)}{B}$$

$\therefore$ $$\Delta\rho = \rho\left(\frac{\Delta p}{B}\right)$$

- In the figure shown

Work done by gravity is

$$W = (Mg)\Delta l \qquad \dots\text{(i)}$$

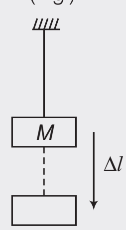


Fig. 15.9

But potential energy stored in the stretched wire is

$$U = \left(\frac{\text{energy}}{\text{volume}} \text{ or energy density}\right)(\text{volume})$$

$$= \left[\frac{1}{2}(\text{stress}) \times (\text{strain})\right][\text{volume}]$$

$$= \frac{1}{2}\left(\frac{Mg}{A}\right)\left(\frac{\Delta l}{l}\right)(Al)$$

or $$U = \frac{1}{2}(Mg)(\Delta l) \qquad \dots\text{(ii)}$$

From Eqs. (i) and (ii), we can see that half of the work done by gravity is stored as potential energy in stretched wire and rest half (or 50%) is dissipated in the form of heat, sound etc. during stretching.

Solved Examples

TYPED PROBLEMS

Type 1. *Based on change in length of a wire.*

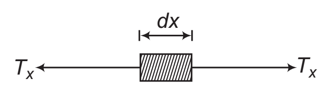
Concept

We have discussed in article 15.4 that change in length is given by $\Delta l = \dfrac{Fl}{AY}$

Here, F is the internal restoring force in the wire. So, this is also the tension in the wire. Therefore, we can also write the above equation as

$$\Delta l = \frac{Tl}{AY}$$

Now, the concept is, if tension is uniform then this equation can be applied directly to find the change in length. If tension is non-uniform then we will find change in length by integration.



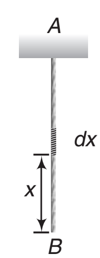
In that case, $$\Delta l = \int dl = \int \frac{T_x \cdot dx}{AY}$$

Here, T_x is the tension at some intermediate point x and

$dl = \dfrac{T_x\,dx}{AY}$ is the change in length in a small element dx, due to the tension T_x at this point.

Note *To find T_x at some intermediate point (if it is non-uniform) students can refer the chapter 'Laws of motion'.*

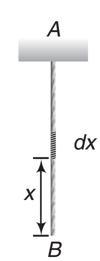
Example 1 *A bar of mass m and length l is hanging from point A as shown in figure. Find the increase in its length due to its own weight. The Young's modulus of elasticity of the wire is Y and area of cross-section of the wire is A.*



Solution Consider a small section dx of the bar at a distance x from B.

Tension in the bar at this point is $\quad T_x = W_x = \left(\dfrac{mg}{l}\right)x$

Elongation in section dx will be $\quad dl = \dfrac{T_x dx}{AY} = \left(\dfrac{mg}{lAY}\right)x\,dx$

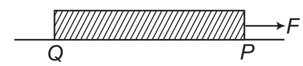


Total elongation in the bar can be obtained by integrating this expression from $x = 0$ to $x = l$.

$\therefore$ $$\Delta l = \int_{x=0}^{x=l} dl = \left(\frac{mg}{lAY}\right)\int_0^l x\,dx$$

or $$\Delta l = \frac{mgl}{2AY}$$ **Ans.**

Example 2 *A rod PQ of mass m, area of cross section A, length l and Young's modulus of elasticity Y is lying on a smooth table as shown in figure. A force F is applied at P. Find*



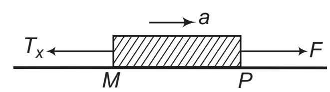
(a) tension at a distance x from end P,

(b) longitudinal stress at this point,

(c) total change in length and

(d) total strain in the rod.

Solution (a) Acceleration of the rod, $a = \frac{F}{m}$



$$F - T_x = (m_{PM})\, a = \left(\frac{m}{l}x\right)\left(\frac{F}{m}\right)$$

$\therefore$ $$T_x = F\left(1 - \frac{x}{l}\right)$$ **Ans.**

(b) Stress $$\sigma = \frac{F}{A} = \frac{T_x}{A} = \frac{F}{A}\left(1 - \frac{x}{l}\right)$$ **Ans.**

(c) Change in length $$\Delta l = \int_0^l \frac{T_x dx}{AY}$$

$$= \int_0^l \frac{F\left(1 - \frac{x}{l}\right) dx}{AY}$$

$$\Delta l = \frac{Fl}{2AY}$$ **Ans.**

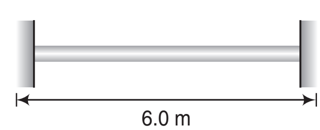
(d) Strain $$= \frac{\Delta l}{l} = \frac{F}{2AY}$$ **Ans.**

Type 2. *Based on thermal stresses.*

Concept

As discussed in article 15.7, if temperature of a rod is increased or decreased, it has a tendency to expand or contract. If rod is not fixed from its two ends and it is free to expand or contract then no stresses will be developed in it. If rod is fixed from its two ends then it means rod is not allowed to expand or contract. So stresses are developed. If temperature is decreased, then its natural length is less but it is fixed from two ends. So, we can assume that it has been stretched by Δl from its natural length $l - \Delta l (\approx l)$. So, tensile stresses will be developed. Similarly, compressive stresses are developed if temperature is increased.

Example 3 *A steel rod of length* 6.0 *m and diameter* 20 *mm is fixed between two rigid supports. Determine the stress in the rod, when the temperature increases by* $80°C$ *if*



(a) the ends do not yield *(b) the ends yield by* 1 *mm.*

Take $Y = 2.0 \times 10^6\ kg/cm^2$ *and* $\alpha = 12 \times 10^{-6}$ *per°C.*

Solution Given, length of the rod $l = 6 \text{ m} = 600 \text{ cm}$

Diameter of the rod $d = 20 \text{ mm} = 2 \text{ cm}$

Increase in temperature $t = 80°\text{C}$

Young's modulus

$$Y = 2.0 \times 10^6 \text{ kg/cm}^2$$

and thermal coefficient of linear expansion $\alpha = 12 \times 10^{-6}$ per°C

(a) When the ends do not yield

Let, $\sigma_1 =$ stress in the rod

Using the relation $\sigma = \alpha t Y$

$$\therefore \quad \sigma_1 = (12 \times 10^{-6})(80)(2 \times 10^6)$$
$$= 1920 \text{ kg/cm}^2$$ **Ans.**

(b) When the ends yield by 1 mm.

Increase in length due to increase in temperature $\Delta l = l\alpha t$

of this 1 mm or 0.1 cm is allowed to expand. Therefore, net compression in the rod

$$\Delta l_{\text{net}} = (l\alpha t - 0.1)$$

or compressive strain in the rod,

$$\varepsilon = \frac{\Delta l_{\text{net}}}{l} = \left(\alpha t - \frac{0.1}{l}\right)$$

$$\therefore \quad \text{stress } \sigma_2 = Y\varepsilon = Y\left(\alpha t - \frac{0.1}{l}\right)$$

Substituting the values,

$$\sigma_2 = 2 \times 10^6 \left(12 \times 10^{-6} \times 80 - \frac{0.1}{600}\right)$$
$$= 1587 \text{ kg/cm}^2$$ **Ans.**

Note *For more examples of thermal stresses, students can refer the topic thermal expansion.*

Type 3. *Based on breaking stress.*

Concept

Every material has a limit of maximum stress which can be applied to this. This is called breaking stress. Material breaks beyond this.

Now, the maximum force which can be applied across it depends on its area of cross-section.

$$\sigma_m = \frac{F_m}{A} \qquad (\sigma_m = \text{breaking stress})$$

$$\therefore \quad \text{Maximum force } F_m = (\sigma_m)A$$

Example 4 *If the elastic limit of copper is* $1.5 \times 10^8 \, N/m^2$, *determine the minimum diameter a copper wire can have under a load of* 10.0 *kg, if its elastic limit is not to be exceeded.*

Solution $\dfrac{F}{A_{\min}} = \sigma_m$

$\therefore$ $$\frac{F}{\left(\dfrac{\pi d_{\min}^2}{4}\right)} = \sigma_m \quad \text{or} \quad d_{\min} = \sqrt{\frac{4F}{\pi\,\sigma_m}}$$

$$= \sqrt{\frac{4 \times 10 \times 9.8}{3.14 \times 1.5 \times 10^8}} = 9.1 \times 10^{-4}\ \text{m} = 0.91\ \text{mm}$$ **Ans.**

Example 5 *Find the greatest length of steel wire that can hang vertically without breaking. Breaking stress of steel* = $8.0 \times 10^8\ N/m^2$. *Density of steel* = $8.0 \times 10^3\ kg/m^3$. *Take* $g = 10\ m/s^2$.

Solution Let l be the length of the wire that can hang vertically without breaking. Then, the stretching force on it is equal to its own weight. If therefore, A is the area of cross-section and ρ the density, then

$$\text{Maximum stress } (\sigma_m) = \frac{\text{weight}}{A} \qquad \left(\text{Stress} = \frac{\text{force}}{\text{area}}\right)$$

or $$\sigma_m = \frac{(Al\rho)g}{A}$$

$\therefore$ $$l = \frac{\sigma_m}{\rho g}$$

Substituting the values $$l = \frac{8.0 \times 10^8}{(8.0 \times 10^3)(10)} = 10^4\ \text{m}$$ **Ans.**

Type 4. *Based on change in density due to change in pressure.*

Concept

As discussed in extra touch points, if extra pressure Δp is applied on a body its volume decreases without change in its mass. So, its density increases. If $\Delta p << B$, where B is the bulk modulus, then increase in density is given by $\Delta\rho = \dfrac{\rho\Delta p}{B}$

Example 6 *Bulk modulus of water is* $2.3 \times 10^9\ N/m^2$. *Taking average density of water* $\rho = 10^3\ kg/m^3$, *find increase in density at a depth of* 1*km*. *Take* $g = 10\ m/s^2$

Solution Pressure increases with depth of a liquid. At a depth 'h' below the water surface increase in pressure is given by $\Delta p = \rho g h$

Using the equation, $$\Delta\rho = \rho\,\frac{\Delta p}{B}$$

we get $$\Delta\rho = \frac{\rho\,(\rho g h)}{B} = \frac{\rho^2 g h}{B}$$

Substituting the values we have,

$$\Delta\rho = \frac{(10^3)^2 (10)\,(10^3)}{2.3 \times 10^9} = 4.33\ \text{kg/m}^3$$ **Ans.**

Miscellaneous Examples

Example 7 *The pressure of a medium is changed from* 1.01×10^5 *Pa to* 1.165×10^5 *Pa and change in volume is 10% keeping temperature constant. The bulk modulus of the medium is*

(a) 204.8×10^5 *Pa* (b) 102.4×10^5 *Pa* (c) 51.2×10^5 *Pa* (d) 1.55×10^5 *Pa*

Solution From the definition of bulk modulus, $B = \dfrac{-\Delta p}{(\Delta V / V)}$

Substituting the values, we have

$$B = \frac{(1.165 - 1.01) \times 10^5}{(10/100)} = 1.55 \times 10^5 \text{ Pa}$$

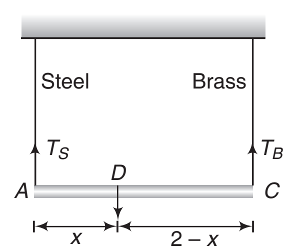
Therefore, the correct option is (d).

Example 8 *A light rod of length* 2.00 *m is suspended from the ceiling horizontally by means of two vertical wires of equal length tied to its ends. One of the wires is made of steel and is of cross-section* 10^{-3} m^2 *and the other is of brass of cross-section* 2×10^{-3} m^2*. Find out the position along the rod at which a weight may be hung to produce,*

(a) equal stresses in both wires *(b) equal strains on both wires.*

Young's modulus for steel is 2×10^{11} N/m^2 *and for brass is* 10^{11} N/m^2.

Solution (a) Given, stress in steel = stress in brass



$$\therefore \quad \frac{T_S}{A_S} = \frac{T_B}{A_B}$$

$$\therefore \quad \frac{T_S}{T_B} = \frac{A_S}{A_B}$$

$$= \frac{10^{-3}}{2 \times 10^{-3}} = \frac{1}{2} \qquad \text{...(i)}$$

As the system is in equilibrium, taking moments about D, we have

$$T_S \cdot x = T_B (2 - x)$$

$$\therefore \quad \frac{T_S}{T_B} = \frac{2 - x}{x} \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we get $x = 1.33$ m **Ans.**

(b) Strain $= \dfrac{\text{Stress}}{Y}$

Given, strain in steel = strain in brass

$$\therefore \quad \frac{T_S / A_S}{Y_S} = \frac{T_B / A_B}{Y_B}$$

$$\therefore \quad \frac{T_S}{T_B} = \frac{A_S Y_S}{A_B Y_B} = \frac{(1 \times 10^{-3})(2 \times 10^{11})}{(2 \times 10^{-3})(10^{11})} = 1 \qquad \text{...(iii)}$$

From Eqs. (ii) and (iii), we have $x = 1.0$ m **Ans.**

Example 9 *A sphere of radius* $0.1\,m$ *and mass* 8π *kg is attached to the lower end of a steel wire of length* 5.0 *m and diameter* 10^{-3} *m. The wire is suspended from* 5.22 *m high ceiling of a room. When the sphere is made to swing as a simple pendulum, it just grazes the floor at its lowest point. Calculate the velocity of the sphere at the lowest position. Young's modulus of steel is* $1.994 \times 10^{11}\,N/m^2$.

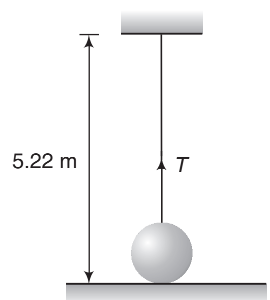
Solution Let Δl be the extension of wire when the sphere is at mean position. Then, we have

$$l + \Delta l + 2r = 5.22$$

or

$$\begin{aligned}\Delta l &= 5.22 - l - 2r \\ &= 5.22 - 5 - 2 \times 0.1 \\ &= 0.02 \text{ m}\end{aligned}$$

Let T be the tension in the wire at mean position during oscillations, then



$$Y = \frac{T/A}{\Delta l/l}$$

$\therefore$

$$T = \frac{YA\Delta l}{l} = \frac{Y\pi r^2 \Delta l}{l}$$

Substituting the values, we have

$$T = \frac{(1.994 \times 10^{11}) \times \pi \times (0.5 \times 10^{-3})^2 \times 0.02}{5}$$

$$= 626.43 \text{ N}$$

The equation of motion at mean position is,

$$T - mg = \frac{mv^2}{R} \qquad \ldots\text{(i)}$$

Here, $R = 5.22 - r = 5.22 - 0.1 = 5.12$ m

and $m = 8\pi$ kg $= 25.13$ kg

Substituting the proper values in Eq. (i), we have

$$(626.43) - (25.13 \times 9.8) = \frac{(25.13)\,v^2}{5.12}$$

Solving this equation, we get

$$v = 8.8 \text{ m/s}$$ **Ans.**

Example 10 *A thin ring of radius R is made of a material of density* ρ *and Young's modulus Y. If the ring is rotated about its centre in its own plane with angular velocity* ω, *find the small increase in its radius.*

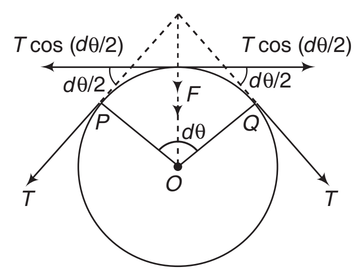
Solution Consider an element PQ of length dl. Let T be the tension and A the area of cross-section of the wire.

$$\text{Mass of element } dm = \text{volume} \times \text{density}$$
$$= A(dl)\rho$$

The component of T, towards the centre provides the necessary centripetal force to portion PQ.

$$\therefore \qquad F = 2T \sin\left(\frac{d\theta}{2}\right) = (dm)R\omega^2 \qquad \text{...(i)}$$

For small angles
$$\sin\frac{d\theta}{2} \approx \frac{d\theta}{2} = \frac{(dl/R)}{2}$$



or
$$d\theta = \frac{dl}{R}$$

Substituting in Eq. (i), we have

$$T \cdot \frac{dl}{R} = A(dl)\rho R\omega^2$$

or
$$T = A\rho\omega^2 R^2$$

Let ΔR be the increase in radius.

$$\text{Longitudinal strain} = \frac{\Delta l}{l} = \frac{\Delta(2\pi R)}{2\pi R} = \frac{\Delta R}{R}$$

Now,
$$Y = \frac{T/A}{\Delta R/R}$$

$$\therefore \qquad \Delta R = \frac{TR}{AY}$$
$$= \frac{(A\rho\omega^2 R^2)R}{AY}$$

or
$$\Delta R = \frac{\rho\omega^2 R^3}{Y} \qquad \textbf{Ans.}$$

Exercises

LEVEL 1

Assertion and Reason

Directions : *Choose the correct option.*

(*a*) *If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion.***

(*b*) *If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion.***

(*c*) *If **Assertion** is true, but the **Reason** is false.*

(*d*) *If **Assertion** is false but the **Reason** is true.*

1. Assertion : Steel is more elastic than rubber.

Reason : For same strain, steel requires more stress to be produced in it.

2. Assertion : If pressure is increased, bulk modulus of gases will increase.

Reason : With increase in pressure, temperature of gas also increases.

3. Assertion : From the relation $Y = \dfrac{Fl}{A\Delta l}$, we can say that, if length of a wire is doubled, its Young's modulus of elasticity will also becomes two times.

Reason : Modulus of elasticity is a material property.

4. Assertion : Bulk modulus of elasticity can be defined for all three states of matter, solid liquid and gas.

Reason : Young's modulus is not defined for liquids and gases.

5. Assertion : Every wire is like a spring, whose spring constant, $K \propto \dfrac{1}{l}$ where l is length of wire.

Reason : It follows from the relation

$$K = \frac{YA}{l}$$

6. Assertion : Ratio of stress and strain is always constant for a substance.

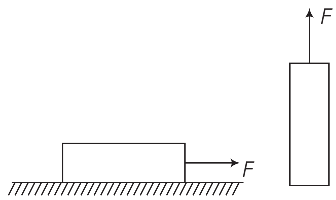
Reason : This ratio is called modulus of elasticity.

7. Assertion : Ratio of isothermal bulk modulus and adiabatic bulk modulus for a monoatomic gas at a given pressure is $\dfrac{3}{5}$.

Reason : This ratio is equal to

$$\gamma = \frac{C_p}{C_V}.$$

8. Assertion: A uniform elastic rod lying on smooth horizontal surface is pulled by a constant horizontal force of magnitude F as shown in the figure. Another identical elastic rod is pulled vertically upwards by a constant vertical force of magnitude F as shown in the figure. The extensions in both the rods will be same.



Reason: In a uniform elastic rod, the extension depends only on forces acting at the ends of rods.

9. Assertion: Identical springs of steel and copper are equally stretched. More work will be done on the steel spring.

Reason: Steel is more elastic than copper.

Objective Questions

Single Correct Option

1. The bulk modulus for an incompressible liquid is

(a) zero (b) unity
(c) infinity (d) between 0 and 1

2. The Young's modulus of a wire of length (L) and radius (r) is Y. If the length is reduced to $\frac{L}{2}$ and radius $\frac{r}{2}$, then its Young's modulus will be

(a) $\frac{Y}{2}$ (b) Y
(c) $2Y$ (d) $4Y$

3. The maximum load that a wire can sustain is W. If the wire is cut to half its value, the maximum load it can sustain is

(a) W (b) $\frac{W}{2}$
(c) $\frac{W}{4}$ (d) $2W$

4. Identify the case when an elastic metal rod does not undergo elongation

(a) it is pulled with a constant acceleration on a smooth horizontal surface
(b) it is pulled with constant velocity on a rough horizontal surface
(c) it is allowed to fall freely
(d) All of the above

5. Vessel of $1 \times 10^{-3}\ m^3$ volume contains an oil. If a pressure of $1.2 \times 10^5\ N/m^2$ is applied on it, then volume decreases by $0.3 \times 10^{-3}\ m^3$. The bulk modulus of oil is

(a) $6 \times 10^{10}\ N/m^2$ (b) $4 \times 10^5\ N/m^2$
(c) $2 \times 10^7\ N/m^2$ (d) $1 \times 10^6\ N/m^2$

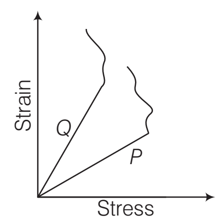
6. A load of 4 kg is suspended from a ceiling through a steel wire of length 20 m and radius 2 mm. It is found that the length of the wire increases by 0.031 mm, as equilibrium is achieved. If $g = 3.1 \times \pi$ ms^{-2}, the value of Young's modulus of the material of the wire (in Nm^{-2}) is

(a) 2×10^{12}
(b) 4×10^{11}
(c) 2×10^{11}
(d) 0.02×10^{9}

7. A wire of length 1m and radius 1mm is subjected to a load. The extension is x. The wire is melted and then drawn into a wire of square cross-section of side 2 mm. Its extension under the same load will be

(a) $\frac{\pi^2 x}{8}$
(b) $\frac{\pi^2 x}{16}$
(c) $\frac{\pi^2 x}{2}$
(d) $\frac{x}{2\pi}$

8. Figure shows the stress-strain curve of two metals P and Q. From the graph, it can be concluded that



(a) P has greater Young's modulus and lesser ductility
(b) Q has greater Young's modulus and lesser ductility
(c) P has greater Young's modulus and greater ductility
(d) Q has greater Young's modulus and greater ductility

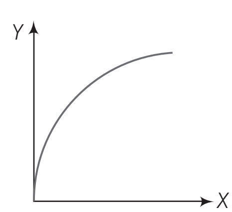
9. The bulk modulus of water is 2.0×10^9 N/m^2. The pressure required to increase the density of water by 0.1% is

(a) 2.0×10^3 N/m^2
(b) 2.0×10^6 N/m^2
(c) 2.0×10^5 N/m^2
(d) 2.0×10^7 N/m^2

10. If the work done in stretching a wire by 1 mm is 2 J, then work necessary for stretching another wire of same material but with double radius of cross-section and half of the length by 1 mm is

(a) 8 J
(b) 16 J
(c) 4 J
(d) 32 J

11. The graph shows the behaviour of a steel wire in the region for which the wire obeys Hooke's law. The graph is a parabola. The variables X and Y-axes, respectively can be [stress (σ), strain (ε) and elastic potential energy (U)]



(a) U, σ
(b) U, ε
(c) σ, ε
(d) None of these

12. Depth of sea is maximum at Mariana Trench in West Pacific Ocean. Trench has a maximum depth of about 11 km. At bottom of trench water column above it exerts 1000 atm pressure. Percentage change in density of sea water at such depth will be around

(Given, $B = 2 \times 10^9 \text{ Nm}^{-2}$ and $p_{\text{atm}} = 1 \times 10^5 \text{ Nm}^{-2}$)

(a) about 5% (b) about 10%
(c) about 3% (d) about 7%

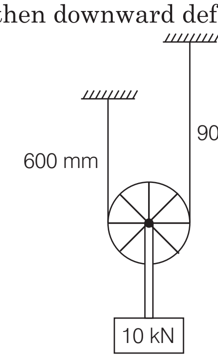
13. One end of a horizontal thick copper wire of length $2L$ and radius $2R$ is welded to an end of another horizontal thin copper wire of length L and radius R. When the arrangement is stretched by applying forces at two ends, the ratio of the elongation in the thin wire to that in the thick wire is

(a) 0.25 (b) 0.50
(c) 2.00 (d) 4.00

14. An air filled balloon is at a depth of 1km below the water level in an ocean. The normal stress of the balloon (in Pa) is
(Given, $\rho_{\text{water}} = 10^3 \text{ kgm}^{-3}$, $g = 9.8 \text{ ms}^{-2}$ and $p_{\text{atm}} = 10^5 \text{ Pa}$)

(a) 10^6 (b) 9.9×10^5
(c) 9.9×10^7 (d) 9.9×10^6

15. A load of 10 kN is supported from a pulley, which in turn is supported by a rope of cross-sectional area 10^3 mm^2 and modulus of elasticity 10^3 Nmm^{-2} as shown in the figure. Neglecting friction at the pulley, then downward deflection of the load (in mm) is



(a) 3.75 (b) 4.25 (c) 2.75 (d) 4.00

16. A body of mass 3.14 kg is suspended from one end of a wire of length 10 m. The radius of cross-section of the wire is changing uniformly from 5×10^{-4} m at the top (i.e. point of suspension) to 9.8×10^{-4} m at the bottom. Young's modulus of elasticity is $2 \times 10^{11} \text{ N/m}^2$. The change in length of the wire is

(a) 4×10^{-3} m (b) 3×10^{-3} m
(c) 10^{-3} m (d) 2×10^{-3} m

Subjective Questions

1. What is the density of lead under a pressure of $2.0 \times 10^8 \text{ N/m}^2$, if the bulk modulus of lead is $8.0 \times 10^9 \text{ N/m}^2$ and initially the density of lead is 11.4 g/cm^3?

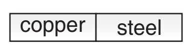
2. A cylindrical steel wire of 3 m length is to stretch no more than 0.2 cm when a tensile force of 400 N is applied to each end of the wire. What minimum diameter is required for the wire ?
$Y_{\text{steel}} = 2.1 \times 10^{11} \text{ N/m}^2$

3. The elastic limit of a steel cable is $3.0 \times 10^8\ \text{N/m}^2$ and the cross-section area is 4 cm^2. Find the maximum upward acceleration that can be given to a 900 kg elevator supported by the cable if the stress is not to exceed one-third of the elastic limit.

4. Find the increment in the length of a steel wire of length 5 m and radius 6 mm under its own weight. Density of steel = 8000 kg/m^3 and Young's modulus of steel = $2 \times 10^{11}\ \text{N/m}^2$. What is the energy stored in the wire ? (Take $g = 9.8\ \text{m/s}^2$)

5. Two wires shown in figure are made of the same material which has a breaking stress of $8 \times 10^8\ \text{N/m}^2$. The area of cross-section of the upper wire is 0.006 cm^2 and that of the lower wire is 0.003 cm^2. The mass $m_1 = 10\ \text{kg}$, $m_2 = 20$ kg and the hanger is light. Find the maximum load that can be put on the hanger without breaking a wire. Which wire will break first if the load is increased ? (Take $g = 10\ \text{m/s}^2$)

6. A steel wire and a copper wire of equal length and equal cross-sectional area are joined end to end and the combination is subjected to a tension. Find the ratio of



(a) the stresses developed in the two wires,
(b) the strains developed. (Y of steel $= 2 \times 10^{11}\ \text{N/m}^2$ and Y of copper $= 1.3 \times 10^{11}\ \text{N/m}^2$)

7. Calculate the approximate change in density of water in a lake at a depth of 400 m below the surface. The density of water at the surface is 1030 kg/m^3 and bulk modulus of water is $2 \times 10^9\ \text{N/m}^2$.

8. In taking a solid ball of rubber from the surface to the bottom of a lake of 180 m depth, reduction in the volume of the ball is 0.1%. The density of water of the lake is $1 \times 10^3\ \text{kg/m}^3$. Determine the value of the bulk modulus of elasticity of rubber. ($g = 9.8\ \text{m/s}^2$)

LEVEL 2

Single Correct Option

1. A wire elongates by l units, when a load w is suspended from it. If the wire gets over a pulley (equally on both the sides) and two weights w each are hung at the two ends, the elongation of wire (in units) will be

(a) zero
(b) $\dfrac{l}{2}$
(c) l
(d) $2l$

2. Maximum stress that can be applied to a wire which supports on elevator is σ. Mass of elevator is m and it is moved upwards with an acceleration of $g/2$. Minimum diameter of wire (Neglecting weight of wire) must be

(a) $\sqrt{\dfrac{2mg}{\pi\sigma}}$ (b) $\sqrt{\dfrac{3mg}{2\pi\sigma}}$

(c) $\sqrt{\dfrac{5mg}{2\pi\sigma}}$ (d) $\sqrt{\dfrac{6mg}{\pi\sigma}}$

3. A bob of mass 10 kg is attached to a wire 0.3 m long. Its breaking stress is 4.8×10^7 N/m^2. The area of cross-section of the wire is 10^{-6} m^2. What is the maximum angular velocity with which it can be rotated in a horizontal circle?

(a) 8 rad/s (b) 4 rad/s

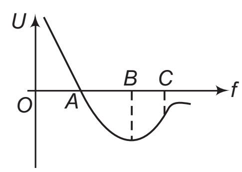
(c) 2 rad/s (d) 1 rad/s

4. A uniform steel rod of cross-sectional area A and length L is suspended so that it hangs vertically. The stress at the middle point of the rod is

(a) $\dfrac{1}{2}\rho gL$ (b) $\dfrac{1}{4}\rho gL$

(c) ρgL (d) None of these

5. The potential energy U of diatomic molecules as a function of separation r is shown in figure. Identify the correct statement.



(a) The atoms are in equilibrium if $r = OA$

(b) The force is repulsive only if r lies between A and B

(c) The force is attractive if r lies between A and B

(d) The atoms are in equilibrium if $r = OB$

6. The length of a steel wire is l_1 when the stretching force is T_1 and l_2 when the stretching force is T_2. The natural length of the wire is

(a) $\dfrac{l_1T_1 + l_2T_2}{T_1 + T_2}$ (b) $\dfrac{l_2T_1 + l_1T_2}{T_1 + T_2}$

(c) $\dfrac{l_2T_1 - l_1T_2}{T_1 - T_2}$ (d) $\dfrac{l_1T_1 - l_2T_2}{T_1 - T_2}$

7. A mass m is suspended from a wire. Change in length of the wire is Δl. Now the same wire is stretched to double its length and the same mass is suspended from the wire. The change in length in this case will become (it is assumed that elongation in the wire is within the proportional limit)

(a) Δl (b) $2\Delta l$ (c) $4\Delta l$ (d) $8\Delta l$

8. A uniform metal rod fixed at its ends of 2 mm^2 cross-section is cooled from 40°C to 20°C. The coefficient of the linear expansion of the rod is 12×10^{-6} per degree celsius and its Young's modulus of elasticity is 10^{11} N/m^2. The energy stored per unit volume of the rod is

(a) 2880 J/m^3 (b) 1500 J/m^3

(c) 5760 J/m^3 (d) 1440 J/m^3

9. A rod of length 1000 mm and coefficient of linear expansion $\alpha = 10^{-4}$ per degree celsius is placed in horizontal smooth surface symmetrically between fixed walls separated by 1001 mm. The Young's modulus of rod is 10^{11} N/m^2. If the temperature is increased by 20°C, then the stress developed in the rod is (in N/m^2)

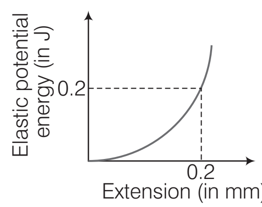
(a) 10^5 (b) 10^8
(c) 10^7 (d) 10^6

10. A uniform elastic plank moves due to a constant force F_0 applied at one end whose area is S. The Young's modulus of the plank is Y. The strain produced in the direction of force is

(a) $\dfrac{F_0}{2SY}$ (b) $\dfrac{F_0}{SY}$ (c) $\dfrac{2F_0}{SY}$ (d) $\dfrac{\sqrt{2}F_0}{SY}$

More than One Correct Options

1. Figure shows the graph of elastic potential energy (U) stored *versus* extension, for a steel wire ($Y = 2 \times 10^{11}$ Pa) of volume 200 cc. If area of cross-section A and original length L, then



(a) $A = 10^{-4}$ m^2 (b) $A = 10^{-3}$ m^2 (c) $L = 1.5$ m (d) $L = 2$ m

2. A metal wire of length L, area of cross-section A and Young's modulus Y is stretched by a variable force F such that F is always slightly greater than the elastic forces of resistance in the wire. When the elongation of the wire is l

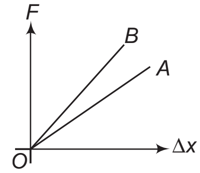
(a) the work done by F is $\dfrac{YAl^2}{2L}$

(b) the work done by F is $\dfrac{YAl^2}{L}$

(c) the elastic potential energy stored in the wire is $\dfrac{YAl^2}{2L}$

(d) the elastic potential energy stored in the wire is $\dfrac{YAl^2}{4L}$

3. Two wires A and B of same length are made of same material. The figure represents the load F *versus* extension Δx graph for the two wires. Then

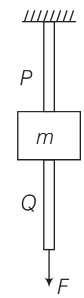


(a) The cross sectional area of A is greater than that of B
(b) The elasticity of B is greater than that of A
(c) The cross-sectional area of B is greater than that of A
(d) The elasticity of A is greater than that of B

4. A body of mass M is attached to the lower end of a metal wire, whose upper end is fixed. The elongation of the wire is l.

(a) Loss in gravitational potential energy of M is Mgl
(b) The elastic potential energy stored in the wire is Mgl
(c) The elastic potential energy stored in the wire is $\frac{1}{2}Mgl$
(d) Heat produced is $\frac{1}{2}Mgl$

5. Two light wires P and Q shown in the figure are made of same material and have radii r_P and r_Q, respectively. The block between them has a mass m. When the force $F = \frac{mg}{3}$, then one of the wires breaks. Choose the correct option(s).



(a) P breaks, if $r_P = r_Q$
(b) P breaks, if $r_P < 2r_Q$
(c) Either P or Q may break, if $r_P = 2r_Q$
(d) To predict, which wire will break, the lengths of P and Q must be known

6. Two wires A and B have equal lengths and are made of the same material, but diameter of wire A is twice that of wire B. Then, for a given load,

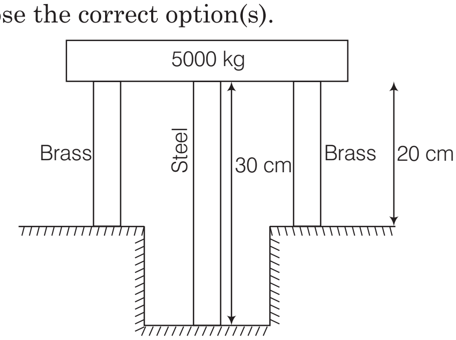
(a) the extension of B will be four times that of A
(b) the extensions of A and B will be equal
(c) the strain in B is four times that in A
(d) the strains in A and B will be equal

7. A light rod of length 2 m is suspended from a ceiling horizontally by means of two vertical wires of equal length tied to its ends. One of the wires is made of steel and is of cross-section 10^{-3} m^2 and the other is of brass of cross-section 2×10^{-3} m^2. x is the distance from steel wire end, at which a weight may be hung. $Y_{\text{steel}} = 2 \times 10^{11}$ Pa and $Y_{\text{brass}} = 10^{11}$ Pa

Which of the following statement(s) is/are correct?

(a) $x = 1.2$ m, if the strains of both the wires are to be equal
(b) $x = 1.42$ m, if the stresses of both the wires are to be equal
(c) $x = 1$ m, if the strains of both the wires are to be equal
(d) $x = 1.33$ m, if the stresses of both the wires are to be equal

8. A steel rod of cross-sectional area 16 cm^2 and two brass rods each of cross-sectional area 10 cm^2 together support a load of 5000 kg as shown in the figure. (Given, $Y_{\text{steel}} = 2 \times 10^6$ kg cm^{-2} and $Y_{\text{brass}} = 10^6$ kg cm^{-2}). Choose the correct option(s).

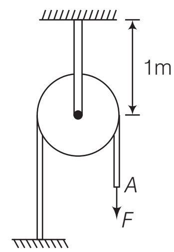


(a) Stress in brass rod = 121 kg cm^{-2}
(b) Stress in steel rod = 161 kg cm^{-2}
(c) Stress in brass rod = 141 kg cm^{-2}
(d) Stress in steel rod = 141 kg cm^{-2}

Comprehension Based Questions

Passage 1 (Q. Nos. 1 to 3)

The axle of a pulley of mass 1 kg is attached to the end of an elastic string of length 1 m, cross-sectional area 10^{-3} m^2 and Young's modulus 2×10^5 Nm^{-2}, whose other end is fixed to the ceiling. A rope of negligible mass is placed on the pulley such that its left end is fixed to the ground and its right end is hanging freely, from the pulley which is at rest in equilibrium. Now, the free end A of the rope is subjected to pulling with constant force $F = 10\,N$. Friction between the rope and the pulley can be neglected. (Given, $g = 10\,ms^{-2}$)



1. The elongation of the string before applying force is
(a) 50 cm (b) 5 cm (c) 0.5 cm (d) 0.05 cm

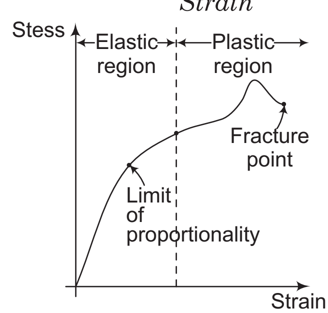
2. The greatest elongation of the string is
(a) 35 cm (b) 30 cm (c) 25 cm (d) 20 cm

3. The maximum displacement of point A after applying F is
(a) 70 cm (b) 60 cm (c) 40 cm (d) 50 cm

Passage 2 (Q. Nos. 4 to 6)

On gradual loading, stress-strain relationship for a metal wire is as follows.

Within proportionality limit, stress ∝ strain or, $\frac{Stress}{Strain}$ *= a constant for the material of wire.*



4. Two wires of same material have length and radius (L, r) and $\left(2L, \frac{r}{2}\right)$. The ratio of their Young's modulii is
(a) 1 : 2 (b) 2 : 3
(c) 2 : 1 (d) 1: 1

5. Just on crossing the yield region, the material will have
(a) increased and breaking stress (b) reduced and breaking stress
(c) constant stress (d) None of these

6. If $\frac{\text{Stress}}{\text{Strain}}$ is x in elastic region and y in the region of yield, then

(a) $x > y$ (b) $x = y$ (c) $x < y$ (d) $x = 2y$

Match the Columns

1. Match the following two columns. (dimension wise)

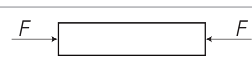
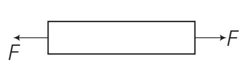
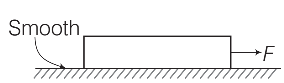
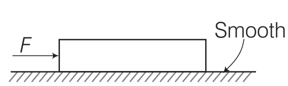
Column I	Column II
(a) Stress	(p) coefficient of friction
(b) Strain	(q) relative density
(c) Modulus of elasticity	(r) energy density
(d) Force constant of a wire	(s) None

2. A wire of length l, area of cross section A and Young's modulus of elasticity Y is stretched by a longitudinal force F. The change in length is Δl. Match the following two columns.

Note *In column I, corresponding to every option, other factors remain constant.*

Column I	Column II
(a) F is increased	(p) Δl will increase
(b) l is increased	(q) stress will increase
(c) A is increased	(r) Δl will decrease
(d) Y is increased	(s) stress will decrease

3. In Column I, a uniform bar of uniform cross-sectional area under the application of forces is shown in the figure and in Column II, some effects/phenomena are given. Match the two columns.

Column I	Column II
(a) 	(p) Uniform stresses are developed in the rod.
(b) 	(q) Non-uniform stresses are developed in the rod.
(c) 	(r) Compressive stresses are developed in the rod.
(d) 	(s) Tensile stresses are developed in the rod.

Subjective Questions

1. A solid sphere of radius R made of a material of bulk modulus B is surrounded by a liquid in a cylindrical container. A massless piston of area A (the area of container is also A) floats on the surface of the liquid. When a mass M is placed on the piston to compress the liquid, fractional change in radius of the sphere is $\frac{Mg}{\alpha AB}$. Find the value of α.

2. A 0.1 kg mass is suspended from a wire of negligible mass. The length of the wire is 1 m and its cross-sectional area is 4.9×10^{-7} m^2. If the mass is pulled a little in the vertically downward direction and released, it performs SHM with angular frequency 140 rad s^{-1}. If the Young's modulus of the material of the wire is $p \times 10^9$ Nm^{-2}, find the value of p.

3. A wire having a length L and cross-sectional area A is suspended at one of its ends from a ceiling. Density and Young's modulus of material of the wire are ρ and Y, respectively. Its strain energy due to its own weight is $\dfrac{\rho^2 g^2 A L^3}{\alpha Y}$. Find the value of α.

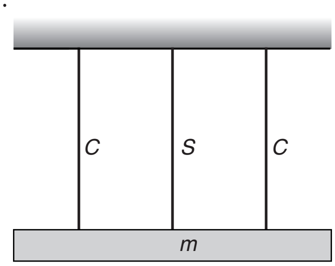
4. A wire of length 3 m, diameter 0.4 mm and Young's modulus 8×10^{10} N/m^2 is suspended from a point and supports a heavy cylinder of volume 10^{-3} m^3 at its lower end. Find the decrease in length when the metal cylinder is immersed in a liquid of density 800 kg/m^3.

5. A sphere of radius 10 cm and mass 25 kg is attached to the lower end of a steel wire of length 5 m and diameter 4 mm which is suspended from the ceiling of a room. The point of support is 521 cm above the floor. When the sphere is set swinging as a simple pendulum, its lowest point just grazes the floor. Calculate the velocity of the ball at its lowest position ($Y_{\text{steel}} = 2 \times 10^{11}$ N/m^2).

6. A uniform ring of radius R and made up of a wire of cross-sectional radius r is rotated about its axis with a frequency f. If density of the wire is ρ and Young's modulus is Y. Find the fractional change in radius of the ring.

7. A 6 kg weight is fastened to the end of a steel wire of unstretched length 60 cm. It is whirled in a vertical circle and has an angular velocity of 2 rev/s at the bottom of the circle. The area of cross-section of the wire is 0.05 cm^2. Calculate the elongation of the wire when the weight is at the lowest point of the path. Young's modulus of steel = 2×10^{11} N/m^2.

8. A homogeneous block with a mass m hangs on three vertical wires of equal length arranged symmetrically. Find the tension of the wires if the middle wire is of steel and the other two are of copper. All the wires have the same cross-section. Consider the modulus of elasticity of steel to be double than that of copper.



9. A uniform copper bar of density ρ, length L, cross-sectional area S and Young's modulus Y is moving horizontally on a frictionless surface with constant acceleration a_0. Find
(a) the stress at the centre of the wire,
(b) total elongation of the wire.

10. A 5 m long cylindrical steel wire with radius 2×10^{-3} m is suspended vertically from a rigid support and carries a bob of mass 100 kg at the other end. If the bob gets snapped, calculate the change in temperature of the wire ignoring radiation losses. (Take $g = 10$ m/s^2)
(For the steel wire, Young's modulus = 2.1×10^{11} N/m^2; Density = 7860 kg/m^3;
Specific heat = 420 J/kg-°C).

Answers

Introductory Exercise 15.1

1. Wire B **2.** 0.0125 cm **3.** 250.2 mm **4.** $[ML^{-1}T^{-2}]$, $[M^0L^0T^0]$, $[ML^{-1}T^{-2}]$

Introductory Exercise 15.2

1. $[ML^{-1}T^{-2}]$ **2.** (a) 0.015 J (b) 5.0×10^{-4} J

Exercises

LEVEL 1

Assertion and Reason

1. (a) **2.** (c) **3.** (d) **4.** (b) **5.** (a) **6.** (d) **7.** (c) **8.** (c) **9.** (a)

Objective Questions

1. (c) **2.** (b) **3.** (a) **4.** (c) **5.** (b) **6.** (a) **7.** (b) **8.** (a) **9.** (b) **10.** (b)
11. (b) **12.** (a) **13** (c) **14.** (d) **15.** (a) **16.** (c)

Subjective Questions

1. 11.69 g/cm^3 **2.** 1.91mm **3.** 34.64 m/s^2 **4.** $\Delta l = 4.9 \times 10^{-6}$m, $U = 5.43 \times 10^{-5}$J
5. Load = 14 kg, lower string **6.** (a) 1 (b) Ratio = $\frac{13}{20}$ **7.** 2.0 kg/m^3 **8.** $B = 1.76 \times 10^9$ N/m^2

LEVEL 2

Single Correct Option

1. (c) **2.** (d) **3.** (b) **4.** (a) **5.** (d) **6.** (c) **7.** (c) **8.** (a) **9.** (b) **10.** (a)

More than One Correct Options

1. (a, d) **2.** (a, c) **3.** (c) **4.** (a, c, d) **5.** (a, b,c) **6.** (a, c) **7.** (c, d) **8.** (a, b)

Comprehension Based Questions

1. (b) **2.** (c) **3.** (d) **4.** (d) **5.** (b) **6.** (a)

Match the Columns

1.(a) → r (b) → p, q (c) → r (d) → s
2.(a) → p, q (b) → p (c) → r, s (d) → r
3.(a) → p, r (b) → p, s (c) → q, s (d) → q, r

Subjective Questions

1. 3 **2.** 4 **3.** 6 **4.** $\Delta l = 2.34 \times 10^{-3}$ m **5.** $v = 31.23$ m/s **6.** $\frac{4\pi^2 f^2 \rho R^2}{Y}$ **7.** 3.8×10^{-4} m
8. $T_c = \frac{mg}{4}$, $T_s = 2T_c$ **9.** (a) $\frac{1}{2}L\rho a_0$ (b) $\frac{1}{2}\frac{\rho a_0 L^2}{Y}$ **10.** 4.568×10^{-3}°C

16

Fluid Mechanics

Chapter Contents

16.1 Definition of a Fluid

Fluid mechanics deals with the behaviour of fluids at rest and in motion. A fluid is a substance that deforms continuously under the application of a shear (tangential) stress no matter how small the shear stress may be.

Thus, fluids comprise the liquid and gas (or vapour) phases of the physical forms in which matter exists. The distinction between a fluid and the solid state of matter is clear if you compare fluid and solid behaviour. A solid deforms when a shear stress is applied but it does not continue to increase with time. However, if a shear stress is applied to a fluid, the deformation continues to increase as long as the stress is applied. We may alternatively define a fluid as a substance that **cannot sustain a shear stress when at rest.**

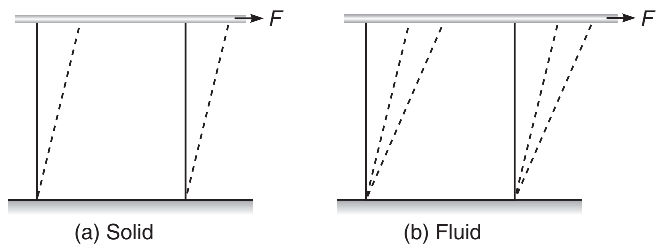


Fig. 16.1 Behaviour of a solid and a fluid, under the action of a constant shear force

Ideal Fluid

An ideal liquid is incompressible and non-viscous in nature. An incompressible liquid means the density of the liquid is constant, it is independent of the variations in pressure. A non-viscous liquid means that, parts of the liquid in contact do not exert any tangential force on each other. Thus, there is no friction between the adjacent layers of a liquid. The force by one part of the liquid on the other part is **perpendicular to the surface of contact.**

16.2 Density of a Liquid

Density (ρ) of any substance is defined as the mass per unit volume or

$$\rho = \frac{\text{mass}}{\text{volume}} \quad \text{or} \quad \boxed{\rho = \frac{m}{V}}$$

Relative Density (RD)

In case of a liquid, sometimes an another term **Relative Density** (RD) is defined. It is the ratio of density of the substance to the density of water at 4°C. Hence,

$$\boxed{\text{RD} = \frac{\text{Density of substance}}{\text{Density of water at } 4^\circ\text{C}}}$$

RD is a pure ratio. So, it has no units. It is also sometimes referred as specific gravity. Density of water at 4°C in CGS is 1 g/cm^3. Therefore, numerically the RD and density of substance (in CGS) are equal. In SI units, the density of water at 4°C is 1000 kg/m^3.

Density of a Mixture of two or more Liquids

Here, we have two cases:

Case 1 Suppose two liquids of densities ρ_1 and ρ_2 having masses m_1 and m_2 are mixed together. Then, the density of the mixture will be

$$\rho = \frac{\text{Total mass}}{\text{Total volume}} = \frac{(m_1 + m_2)}{(V_1 + V_2)}$$

$$= \frac{(m_1 + m_2)}{\left(\dfrac{m_1}{\rho_1} + \dfrac{m_2}{\rho_2}\right)}$$

If $m_1 = m_2$, then $$\rho = \frac{2\rho_1\rho_2}{\rho_1 + \rho_2}$$

Case 2 If two liquids of densities ρ_1 and ρ_2 having volumes V_1 and V_2 are mixed, then the density of the mixture is,

$$\rho = \frac{\text{Total mass}}{\text{Total volume}} = \frac{m_1 + m_2}{V_1 + V_2}$$

$$= \frac{\rho_1 V_1 + \rho_2 V_2}{V_1 + V_2}$$

If $V_1 = V_2$, then $$\rho = \frac{\rho_1 + \rho_2}{2}$$

Effect of Temperature on Density

As the temperature of a liquid is increased, the mass remains the same while the volume is increased and hence, the density of the liquid decreases $\left(\text{as } \rho \propto \dfrac{1}{V}\right)$. Thus,

$$\frac{\rho'}{\rho} = \frac{V}{V'} = \frac{V}{V + dV} = \frac{V}{V + V\gamma\Delta\theta}$$

or $$\frac{\rho'}{\rho} = \frac{1}{1 + \gamma\Delta\theta}$$

Here, $\gamma =$ thermal coefficient of volume expansion

and $\Delta\theta =$ rise in temperature

$\therefore$ $$\boxed{\rho' = \frac{\rho}{1 + \gamma\Delta\theta}}$$

We can also write $$\rho' = \rho(1 + \gamma\Delta\theta)^{-1}$$

or $$\boxed{\rho' \approx \rho(1 - \gamma\Delta\theta)} \qquad (\text{if } \gamma\Delta\theta << 1)$$

Effect of Pressure on Density

As pressure is increased, volume decreases and hence density will increase. Thus,

$$\rho \propto \frac{1}{V}$$

$$\therefore \qquad \frac{\rho'}{\rho} = \frac{V}{V'} = \frac{V}{V + dV} = \frac{V}{V - \left(\frac{dp}{B}\right)V}$$

or
$$\frac{\rho'}{\rho} = \frac{1}{1 - \frac{dp}{B}}$$

Here, $dp =$ change in pressure

and $B =$ bulk modulus of elasticity of the liquid

Therefore,
$$\boxed{\rho' = \frac{\rho}{1 - \frac{dp}{B}}}$$

$$\Rightarrow \qquad \rho' = \rho\left(1 - \frac{dp}{B}\right)^{-1}$$

or
$$\rho' \approx \rho\left(1 + \frac{dp}{B}\right) \qquad \text{(if } dp << B\text{)}$$

Further,
$$\boxed{\rho' - \rho = \Delta\rho = \frac{\rho(dp)}{B} \text{ or } \frac{\rho(\Delta p}{B}}$$

Example 16.1 *Relative density of an oil is* 0.8. *Find the absolute density of oil in CGS and SI units.*

Solution Density of oil (in CGS) = (RD) g/cm^3

$= 0.8 \text{ g/cm}^3$

$= 800 \text{ kg/m}^3$ **Ans.**

16.3 Pressure in a Fluid

When a fluid (either liquid or gas) is at rest, it exerts a force perpendicular to any surface in contact with it, such as a container wall or a body immersed in the fluid. While the fluid as a whole is at rest, the molecules that makes up the fluid are in motion, the force exerted by the fluid is due to molecules colliding with their surroundings.

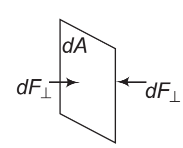


Fig. 16.2

If we think of an imaginary surface within the fluid, the fluid on the two sides of the surface exerts equal and opposite forces on the surface, otherwise the surface would accelerate and the fluid would not remain at rest.

Consider a small surface of area dA centered on a point on the fluid, the normal force exerted by the fluid on each side is $dF_\perp$. The pressure p is defined at that point as the normal force per unit area, i.e.

$$\boxed{p = \frac{dF_\perp}{dA}}$$

If the pressure is the same at all points of a finite plane surface with area A, then

$$p = \frac{F_\perp}{A}$$

where, $F_\perp$ is the normal force on one side of the surface. The SI unit of pressure is pascal, where

$$1 \text{ pascal} = 1 \text{ Pa} = 1.0 \text{ N/m}^2$$

One unit used principally in meterology is the Bar which is equal to 10^5 Pa.

$$1 \text{ Bar} = 10^5 \text{ Pa}$$

Atmospheric Pressure (p_0)

It is pressure of the earth's atmosphere. This changes with weather and elevation. Normal atmospheric pressure at sea level (an average value) is 1.013×10^5 Pa. Thus,

$$1 \text{ atm} = 1.013 \times 10^5 \text{ Pa}$$

Note *Fluid pressure acts perpendicular to any surface in the fluid no matter how that surface is oriented. Hence, pressure has no intrinsic direction of its own, it is a scalar. By contrast, force is a vector with a definite direction.*

Absolute Pressure and Gauge Pressure

The excess pressure above atmospheric pressure is usually called gauge pressure and the total pressure is called absolute pressure. Thus,

$$\boxed{\text{Gauge pressure} = \text{absolute pressure} - \text{atmospheric pressure}}$$

Absolute pressure is always greater than or equal to zero. While gauge pressure can be negative also.

Variation in Pressure with Depth

If the weight of the fluid can be neglected, the pressure in a fluid is the same throughout its volume. But often the fluid's weight is not negligible and under such condition pressure increases with increasing depth below the surface.

Let us now derive a general relation between the pressure p at any point in a fluid at rest and the elevation y of that point. We will assume that the density ρ and the acceleration due to gravity g are the same throughout the fluid. If the fluid is in equilibrium, every volume element is in equilibrium.

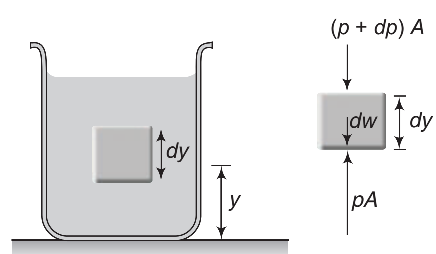


Fig. 16.3

Consider a thin element of fluid with height dy. The bottom and top surfaces each have area A, and they are at elevations y and $y + dy$ above some reference level where $y = 0$. The weight of the fluid element is

$$dw = (\text{volume})\,(\text{density})\,(g) = (A\,dy)(\rho)(g) \quad \text{or} \quad dw = \rho g A\,dy$$

What are the other forces in y-direction on this fluid element? Call the pressure at the bottom surface p, the total y-component of upward force is pA. The pressure at the top surface is $p + dp$ and the total y-component of downward force on the top surface is $(p + dp)A$. The fluid element is in equilibrium, so the total y-component of force including the weight and the forces at the bottom and top surfaces must be zero.

$$\Sigma F_y = 0$$

$\therefore$ $$pA - (p + dp)A - \rho g A\, dy = 0$$

or $$\boxed{\frac{dp}{dy} = -\rho g} \qquad \ldots\text{(i)}$$

This equation shows that when y increases, p decreases, i.e. as we move upward in the fluid, pressure decreases.

If p_1 and p_2 be the pressures at elevations y_1 and y_2 and if ρ and g are constant, then integrating Eq. (i), we get

$$\int_{p_1}^{p_2} dp = -\rho g \int_{y_1}^{y_2} dy$$

or $$\boxed{p_2 - p_1 = -\rho g (y_2 - y_1)} \qquad \ldots\text{(ii)}$$

It is often convenient to express Eq. (ii) in terms of the depth below the surface of a fluid. Take point 1 at depth h below the surface of fluid and let p represents pressure at this point. Take point 2 at the surface of the fluid, where the pressure is p_0 (subscript zero for zero depth). The depth of point 1 below the surface is,

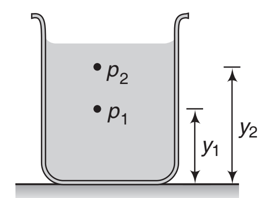


Fig. 16.4

$$h = y_2 - y_1$$

and Eq. (ii) becomes $$p_0 - p = -\rho g (y_2 - y_1) = -\rho g h$$

$\therefore$ $$\boxed{p = p_0 + \rho g h} \qquad \ldots\text{(iii)}$$

Thus, pressure increases linearly with depth, if ρ and g are uniform. A graph between p and h is shown below.

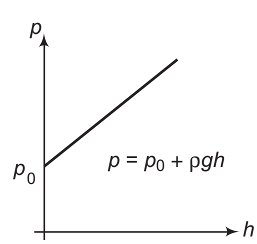


Fig. 16.5

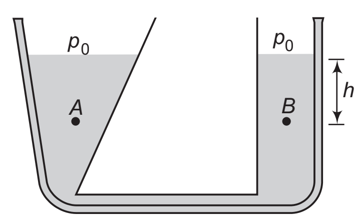


Fig. 16.6

Further, the pressure is the same at any two points at the same level in the fluid. The shape of the container does not matter.

Pascal's Law

Pascal's law or the **principle of transmission of fluid-pressure** is a principle in fluid mechanics that states that pressure exerted anywhere in a confined incompressible fluid is transmitted equally in all directions throughout the fluid.

The simplest instance of this is stepping on a balloon; the balloon bulges out on all sides under the foot and not just on one side. This is precisely what Pascal's law is all about – the air which is the fluid in this case, was confined by the balloon and you applied pressure with your foot causing it to get displaced uniformly.

A well known application of Pascal's law is the hydraulic lift used to support or lift heavy objects. It is schematically illustrated in figure.

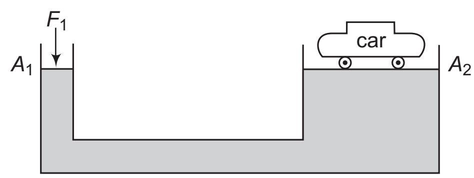


Fig. 16.7

A piston with small cross section area A_1 exerts a force F_1 on the surface of a liquid such as oil. The applied pressure $p = \frac{F_1}{A_1}$ is transmitted through the connecting pipe to a larger piston of area A_2. The applied pressure is the same in both cylinders, so

$$p = \frac{F_1}{A_1} = \frac{F_2}{A_2}$$

or

$$F_2 = \frac{A_2}{A_1} . F_1$$

Now, since $A_2 > A_1$, therefore, $F_2 > F_1$. Thus, hydraulic lift is a force multiplying device with a multiplication factor equal to the ratio of the areas of the two pistons. Dentist's chairs, car lifts and jacks, many elevators and hydraulic brakes all use this principle.

Extra Points to Remember

- At same point on a fluid pressure is same in all directions. In the figure,

$$p_1 = p_2 = p_3 = p_4$$

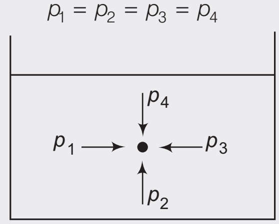


Fig. 16.8

- Forces acting on a fluid in equilibrium have to be perpendicular to its surface. Because it cannot sustain the shear stress.

- In the **same liquid** pressure will be same at all points at the same level (provided their speeds are same).

  For example, in the figure $p_1 \neq p_2$, $p_3 = p_4$ and $p_5 = p_6$

  Further, $p_3 = p_4$

  $\therefore$ $P_0 + \rho_1 g h_1 = P_0 + \rho_2 g h_2$

  or $\rho_1 h_1 = \rho_2 h_2$ or $\boxed{h \propto \frac{1}{\rho}}$

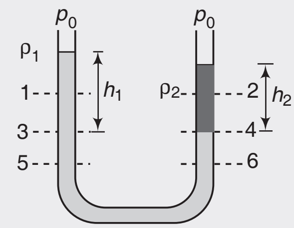


Fig. 16.9

- **Barometer** It is a device used to measure atmospheric pressure.

  In principle, any liquid can be used to fill the barometer, but mercury is the substance of choice because its great density makes possible an instrument of reasonable size.

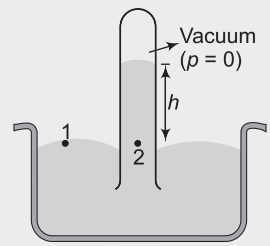


Fig. 16.10

$$p_1 = p_2$$

Here, p_1 = atmospheric pressure (p_0)

and $p_2 = 0 + \rho g h = \rho g h$

Here, ρ = density of mercury

$\therefore$ $p_0 = \rho g h$

Thus, the mercury barometer reads the atmospheric pressure (p_0) directly from the height of the mercury column. For example, if the height of mercury in a barometer is 760 mm, then atmospheric pressure will be

$$p_0 = \rho g h = (13.6 \times 10^3)(9.8)(0.760) = 1.01 \times 10^5 \text{ N/m}^2$$

- **Manometer** It is a device used to measure the pressure of a gas inside a container.

  The U-shaped tube often contains mercury.

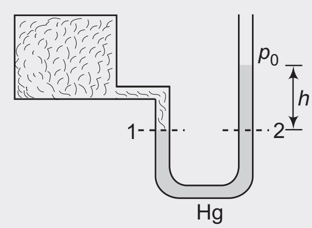


Fig. 16.11

$$\boxed{p_1 = p_2}$$

Here, p_1 = pressure of the gas in the container (p)

and p_2 = atmospheric pressure (p_0) + $\rho g h$

$\therefore$ $\boxed{p = p_0 + h\rho g}$

This can also be written as $\boxed{p - p_0 = \text{gauge pressure} = h\rho g}$

Here, ρ is the density of the liquid used in U-tube.

Thus, by measuring h we can find absolute (or gauge) pressure in the vessel.

- **Free body diagram of a liquid** The free body diagram of the liquid (showing the vertical forces only) is shown in Fig. 16.12 (b). For the equilibrium of liquid,

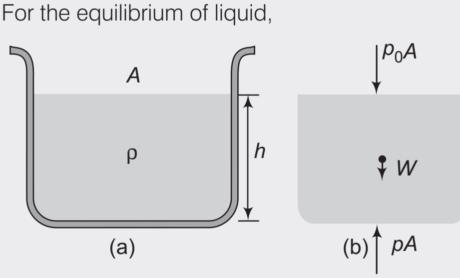


Fig. 16.12

net downward force = net upward force

$\therefore$ $p_0 A + W = pA$ here, $W = \rho g h A$ $\therefore$ $p = p_0 + \rho g h$

Example 16.2 *For the arrangement shown in the figure, what is the density of oil ?*

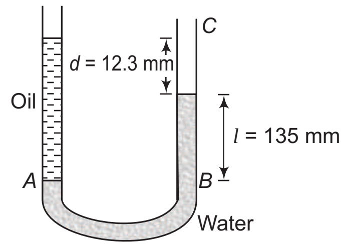


Fig. 16.13

Solution $p_B = p_A$

$\therefore$ $$p_0 + \rho_w gl = p_0 + \rho_{oil}\,(l + d)\,g$$

$\Rightarrow$ $$\rho_{oil} = \frac{\rho_w l}{l + d} = \frac{(1000)\,(135)}{(135 + 12.3)} = 916 \text{ kg/m}^3$$ **Ans.**

Example 16.3 *A U-tube of uniform cross-sectional area and open to the atmosphere is partially filled with mercury. Water is then poured into both arms. If the equilibrium configuration of the tube is as shown in figure with $h_2 = 1.0$ cm, determine the value of h_1.*

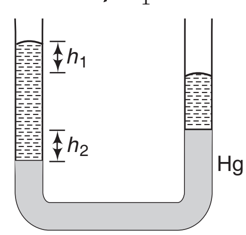


Fig. 16.14

Solution $p_1 = p_2$

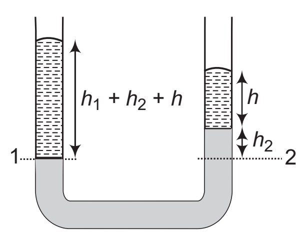


Fig. 16.15

$\therefore$ $$p_0 + \rho_w g\,(h_1 + h_2 + h) = p_0 + \rho_w gh + \rho_{Hg}\,gh_2$$

$\therefore$ $$h_1 = \frac{(\rho_{Hg} - \rho_w)\,h_2}{\rho_w}$$

$$= \frac{(13.6 - 1)\,(1.0)}{1}$$

$$= 12.6\,\text{cm}$$ **Ans.**

Example 16.4 *A U-shaped tube open to the air at both ends contains some mercury. A quantity of water is carefully poured into the left arm of the U-shaped tube until the vertical height of the water column is 15.0 cm.*

(a) What is the gauge pressure at the water mercury interface ?

(b) Calculate the vertical distance h from the top of the mercury in the right hand arm of the tube to the top of the water in the left-hand arm.

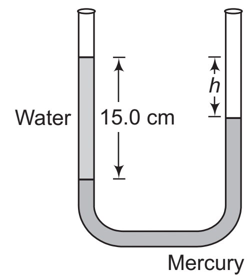


Fig. 16.16

Solution (a) Gauge pressure $= \rho_w g h_w$

$$= (10^3)(9.8)(0.15)$$

$$= 1470\ \text{N/m}^2$$ **Ans.**

(b) Let us calculate pressure or two sides at the level of water and mercury interface.

$$p_0 + \rho_w g\, h_w = p_0 + \rho_{Hg}\, g h_{Hg}$$

$$\rho_w h_w = \rho_{Hg}\, h_{Hg}$$

$\therefore$ $$(1)(15) = (13.6)(15 - h)$$

$$h = 13.9\ \text{cm}$$ **Ans.**

Example 16.5 *For the system shown in figure, the cylinder on the left, at L, has a mass of 600 kg and a cross-sectional area of 800 cm^2. The piston on the right, at S, has cross-sectional area 25 cm^2 and negligible weight. If the apparatus is filled with oil $(\rho = 0.78\ g/cm^3)$, what is the force F required to hold the system in equilibrium?*

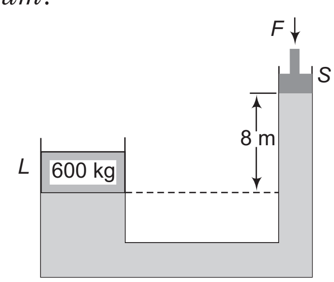


Fig. 16.17

Solution Let A_1 = area of cross-section on LHS and

A_2 = area of cross-section on RHS

Equating the pressure on two sides of the dotted line.

Then, $$\frac{F}{A_2} + \rho g h = \frac{Mg}{A_1} \qquad (M = 600\ \text{kg})$$

$\therefore$ $$F = A_2\left[\frac{Mg}{A_1} - \rho g h\right]$$

$$= (25 \times 10^{-4})\left[\frac{600 \times 9.8}{800 \times 10^{-4}} - 780 \times 9.8 \times 8\right]$$

$$= 31\ \text{N}$$ **Ans.**

INTRODUCTORY EXERCISE 16.1

1. Water and oil are poured into the two limbs of a U-tube containing mercury. The interfaces of the mercury and the liquids are at the same height in both limbs.

Determine the height of the water column h_1 if that of the oil $h_2 = 20$ cm. The density of the oil is 0.9.

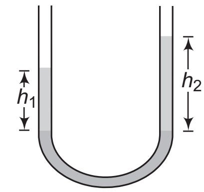


Fig. 16.18

2. The liquids shown in figure in the two arms are mercury (specific gravity $= 13.6$) and water. If the difference of heights of the mercury columns is 2 cm, find the height h of the water column.

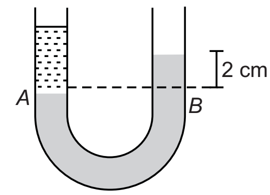


Fig. 16.19

3. The heights of mercury surfaces in the two arms of the manometer shown in Fig. 16.20 are 2 cm and 8 cm. Atmospheric pressure is $1.01 \times 10^5 \text{Nm}^{-2}$.

Find (a) the pressure of the gas in the cylinder and (b) the pressure of mercury at the bottom of the U tube.

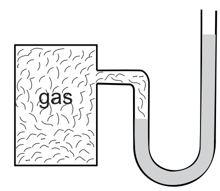


Fig. 16.20

4. A cylinderical vessel containing a liquid is closed by a smooth piston of mass m as shown in the Fig. 16.21. The area of cross section of the piston is A. If the atmospheric pressure is p_0, find the pressure of the liquid just below the piston.

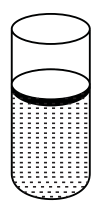


Fig. 16.21

5. The area of cross section of the two arms of a hydraulic press are 1 cm^2 and 10 cm^2 respectively (Fig. 16.22). A force of 5 N is applied on the water in the thinner arm. What force should be applied on the water in the thicker arm so that the water may remain in equilibrium?

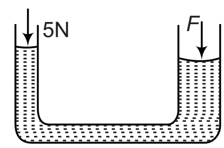


Fig. 16.22

6. The area of cross section of the wider tube shown in Fig. (16.23) is 900 $\textbf{cm}^2$. If the body standing on the position weighs 45 kg, find the difference in the levels of water in the two tubes.

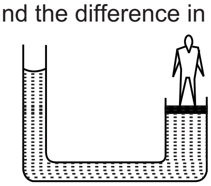


Fig. 16.23

16.4 Pressure Difference in Accelerating Fluids

If fluids are at rest then pressure does not change in horizontal direction.

It changes only in vertical direction.

At a height difference 'h', difference in pressure (or change in pressure) is

Pressure increases with depth. So, $\Delta p = +\rho gh$ in moving downwards and $\Delta p = -\rho gh$ in moving upwards.

If fluids are accelerated then pressure changes in both horizontal and vertical directions.

In Horizontal Direction

(i) Pressure decreases in the direction of acceleration.

(ii) At a horizontal distance x change in pressure is

(a = horizontal acceleration)

Take $\Delta p = +\rho ax$, while moving in opposite direction of acceleration. Because pressure increases in the opposite direction of acceleration and take $\Delta P = -\rho ax$ while moving in the direction of acceleration.

In Vertical Direction

Instead of 'g' in the equation, $\Delta p = \pm \rho gh$ take effective value of g or g_e. Thus,

Here, $g_e = g$ if vertical acceleration of fluid is zero.

$g_e = g + a$ if vertical acceleration of fluid 'a' is upwards and

$g_e = g - a$ if vertical acceleration 'a' is downwards.

In equation form, all above statements can be explained as below.

Consider a liquid kept at rest in a beaker as shown in Fig. 16.24 (a). In this case, we know that pressure do not change in horizontal direction (x-direction), it decreases upwards along y-direction. So, we can write the equations,

$$\boxed{\frac{dp}{dx} = 0 \qquad \text{and} \qquad \frac{dp}{dy} = -\rho g} \qquad \ldots\text{(i)}$$

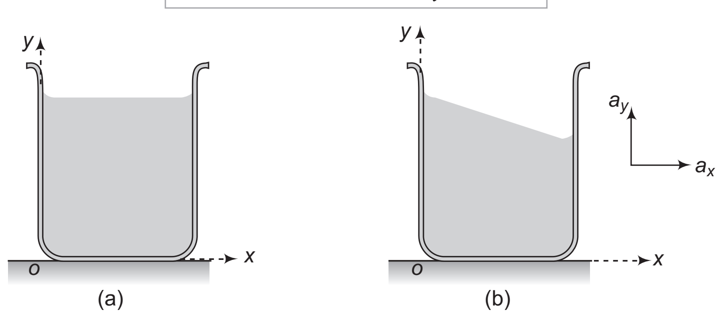


Fig. 16.24

But, suppose the beaker is accelerated and it has components of acceleration a_x and a_y in x and y directions respectively, then the pressure decreases along both x and y-directions. The above equations in that case become,

$$\boxed{\begin{aligned} \frac{dp}{dx} &= -\rho a_x \\ \text{and} \quad \frac{dp}{dy} &= -\rho(g + a_y) \end{aligned}} \quad \ldots\text{(ii)}$$

These equations can be derived as under :

Consider a beaker filled with some liquid of density ρ accelerating upwards with an acceleration a_y along positive y-direction. Let us draw the free body diagram of a small element of fluid of area A and length dy as shown in figure.

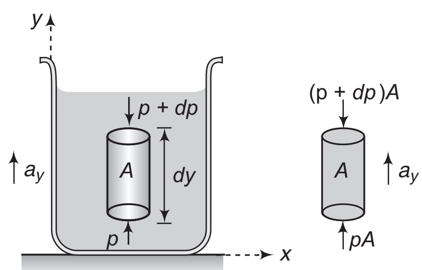


Fig. 16.25

Equation of motion for this fluid element is,

$$pA - W - (p + dp)A = (\text{mass})(a_y) \quad \text{or} \quad -W - (dp)A = (A\rho\, dy)(a_y)$$

$$\text{or} \quad -(A\rho g\, dy) - (dp)A = (A\rho\, dy)(a_y) \qquad \text{or} \quad \frac{dp}{dy} = -\rho(g + a_y)$$

Similarly, if the beaker moves along positive x-direction with an acceleration a_x, the equation of motion for the fluid element shown in figure is,

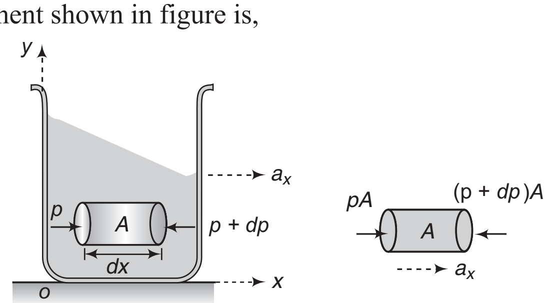


Fig. 16.26

$$pA - (p + dp)A = (\text{mass})(a_x)$$

$$\text{or} \quad -(dp)A = (A\rho\, dx)a_x$$

$$\text{or} \quad \frac{dp}{dx} = -\rho a_x$$

Note (i) $\frac{dp}{dx} = -\rho\, a_x$, $\Delta p = -\rho\, a_x x$ = *pressure difference in a horizontal distance x.*

Here, negative sign implies that pressure decreases with x or in the direction of acceleration.

(ii) *In Fig. 16.26, pressure on left hand side of the fluid element should be more than the pressure on right hand side of this element. This is because this element is accelerated only due to this pressure difference.*

(iii) $\frac{dp}{dy} = -\rho\,(g + a_y)$

$\Rightarrow \qquad \Delta p = -\rho\,(g + a_y)\, y$

or $\qquad \Delta p = -\rho g_e h$

Here, negative sign implies that pressure decreases with 'y' or in moving upwards.

Free Surface of a Liquid Accelerated in Horizontal Direction

Consider a liquid placed in a beaker which is accelerating horizontally with an acceleration 'a'. Let A and B be two points in the liquid at a separation x in the same horizontal line. As we have seen in this case

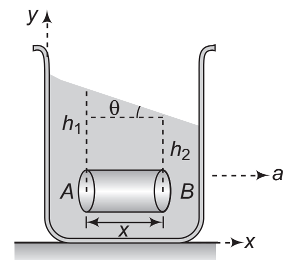


Fig. 16.27

$$\frac{dp}{dx} = -\rho a$$

or $\qquad dp = -\rho a\, dx$

Integrating this with proper limits, we get

$$p_A - p_B = \rho a x \qquad \text{...(iii)}$$

Further, $\qquad p_A = p_0 + \rho g h_1$

and $\qquad p_B = p_0 + \rho g h_2$

Substituting in Eq. (iii), we get

$$\rho g (h_1 - h_2) = \rho a x$$

$$\therefore \qquad \frac{h_1 - h_2}{x} = \frac{a}{g} = \tan\theta$$

$$\therefore \qquad \boxed{\tan\theta = \frac{a}{g}}$$

Pressure Difference in Rotating Fluids

In a rotating fluid (also accelerating) pressure increases in moving away from the rotational axis. At a distance 'x' from the rotational axis, pressure difference is

$$\boxed{\Delta p = \pm \frac{\rho\,\omega^2 x^2}{2}}$$

Take $\Delta p = + \frac{\rho\,\omega^2 x^2}{2}$ in moving away from the rotational axis, as pressure increases in this direction and take $\Delta p = - \frac{\rho\,\omega^2 x^2}{2}$ in moving towards the rotational axis.

Proof

Suppose that liquid of density ρ kept inside a tube of area of cross-section A is rotating with an angular velocity 'ω' as shown.

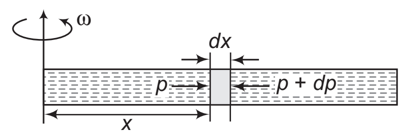


Fig. 16.28

Consider a small element of length 'dx' at a distance x from the axis of rotation. Mass of this element is,

$$dm = (\text{density})\,(\text{volume})$$

or
$$dm = (\rho\, A dx)$$

This element is rotating in a circle of radius 'x'. So, this is accelerated towards centre with a centripetal acceleration

$$a = x\omega^2 \qquad (\text{as } a = R\,\omega^2)$$

To provide this acceleration, pressure on right hand side of the element should be more.

∴
$$(p + dp)A - (p)A = (dm)\,a = (\rho\, A dx)\,(x\omega^2)$$

or
$$dp = (\rho\, x\,\omega^2)\,dx$$

∴
$$\Delta p = \int_0^x (\rho\omega^2)\, x dx$$

or
$$\Delta p = \frac{\rho\,\omega^2 x^2}{2}$$

Hence proved.

Example 16.6 *A closed container shown in figure is filled with water* $(\rho = 10^3\, kg/m^3)$

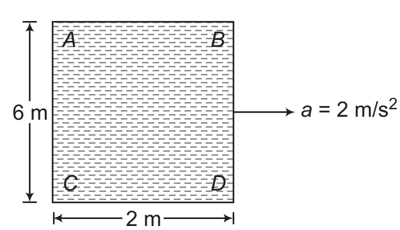


Fig. 16.29

This is accelerated in horizontal direction with an acceleration, $a = 2\, m/s^2$. *Find (a)* $p_C - p_D$ *and (b)* $p_A - p_D$

Solution (a) In horizontal direction, pressure decreases in the direction of acceleration.

Thus,
$$p_C > p_D$$

or
$$p_C - p_D = +\rho\, ax$$

Substituting the values, we have

$$p_C - p_D = (10^3)(2)(2)$$

or $$p_C - p_D = 4.0 \times 10^3 \text{ N/m}^2$$ **Ans.**

(b) In vertical direction, pressure increases with depth.

$\therefore$ $$p_C > p_A$$

or $$p_A - p_C = -\rho g h \qquad (g_e = g)$$
$$= -(10^3)(10)(6)$$
$$= -60 \times 10^3 \text{ N/m}^2$$

Now, $$p_A - p_D = (p_A - p_C) + (p_C - p_D)$$
$$= (-60 \times 10^3) + (4.0 \times 10^3)$$
$$= -56 \times 10^3 \text{ N/m}^2$$
$$= -5.6 \times 10^4 \text{ N/m}^2$$ **Ans.**

Note *Container is closed. So, nowhere inside the container pressure is atmospheric pressure p_0.*

Example 16.7 *A closed tube is filled with*

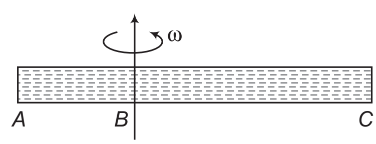


Fig. 16.30

$$AB = 2 \text{ m}$$
$$BC = 4 \text{ cm}$$

water $(\rho = 10^3 \ kg/m^3)$. *It is rotating about an axis shown in figure with an angular velocity* $\omega = 2 \ rad/s$. *Find,* $p_A - p_C$.

Solution Pressure decreases in moving towards the axis of rotation and increases in moving away from the axis $\left(\Delta p = \pm \dfrac{\rho \omega^2 x^2}{2}\right)$

$\therefore$ $$p_A > p_B \text{ and } p_B < p_C$$
$$p_A - p_C = (p_A - p_B) + (p_B - p_C)$$
$$= \left(+\frac{\rho\omega^2 x_1^2}{2}\right) + \left(\frac{-\rho\omega^2 x_2^2}{2}\right)$$

Here, $$x_1 = AB = 2 \text{ m}$$

and $$x_2 = BC = 4 \text{ m}$$

Substituting the values we have,

$$p_A - p_C = \frac{(10^3)(2)^2(2)^2}{2} - \frac{(10^3)(2)^2(4)^2}{2}$$

$$= -2.4 \times 10^3 \text{ N/m}^2$$ **Ans.**

Example 16.8 *A liquid of density* ρ *is in a bucket that spins with angular velocity 'ω' as shown in figure. Prove that the free surface of the liquid has a parabolic shape. Find equation of this.*

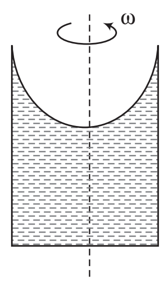


Fig. 16.31

Solution In the figure shown, suppose the coordinates of point M are (x, y) with respect to the coordinate axes.

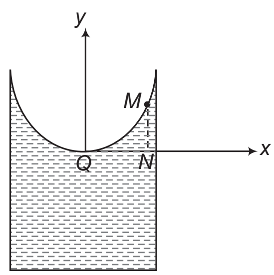


Fig. 16.32

Then, $MN = y, NQ = x$

Points M and Q are open to atmosphere.

∴ $p_M = p_Q =$ atmospheric pressure p_0

Now, $p_M - p_N = -\rho g y$

⇒ $p_0 - p_N = -\rho g y$ ...(i)

$$p_N - p_Q = \frac{\rho \omega^2 x^2}{2}$$

∴ $p_N - p_0 = \dfrac{\rho \omega^2 x^2}{2}$ ...(ii)

Adding Eqs. (i) and (ii), we have

$$0 = -\rho g y + \frac{\rho \omega^2 x^2}{2}$$

$$y = \frac{\omega^2 x^2}{2g}$$

This is the required equation of free surface of the liquid and we can see that this is an equation of a parabola.

INTRODUCTORY EXERCISE 16.2

1. In example 16.6, which point has the maximum pressure and which has the minimum pressure.
2. A cubical closed vessel of side 5 m filled with a liquid is accelerated with an acceleration a. Find the value of a so that pressure at mid point M of AC is equal to pressure at N.

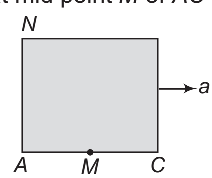


Fig. 16.33

3. Water ($\rho = 10^3$ kg/m^3) is filled in tube AB as shown in figure. $\omega = 10$ rad/s. Tube is open at end A. Atmospheric pressure is $p_0 = 10^5$ N/m^2. Find absolute pressure at end B.

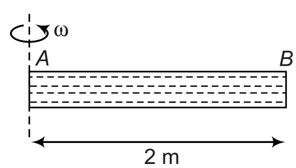


Fig. 16.34

16.5 Archimedes' Principle

If a heavy object is immersed in water, it seems to weightless than when it is in air. This is because the water exerts an upward force called **buoyant force.** It is equal to the weight of the fluid displaced by the body.

A body wholly or partially submerged in a fluid is buoyed up by a force equal to the weight of the displaced fluid.

This result is known as **Archimedes' principle.**

Thus, the magnitude of buoyant force (F) is given by,

$$\boxed{F = V_i \rho_L g}$$

Here, V_i = immersed volume of solid, ρ_L = density of liquid

and g = acceleration due to gravity

Proof

Consider an arbitrarily shaped body of volume V placed in a container filled with a fluid of density ρ_L. The body is shown completely immersed, but complete immersion is not essential to the proof. To begin with, imagine the situation before the body was immersed. The region now occupied by the body was filled with fluid, whose weight was $V\rho_L g$. Because the fluid as a whole was in hydrostatic equilibrium, the net upwards force (due to difference in pressure at different depths) on the fluid in that region was equal to the weight of the fluid occupying that region.

Now, consider what happens when the body has displaced the fluid. The pressure at every point on the surface of the body is unchanged from the value at the same location when the body was not present. This is because the pressure at any point depends only on the depth of that point below the fluid surface. Hence, the net force exerted by the surrounding fluid on the body is exactly the same as that exerted on the region before the body was present. But we know the latter to be $V\rho_L g$, the weight of the displaced fluid. Hence, this must also be the buoyant force exerted on the body. Archimedes' principle is thus proved.

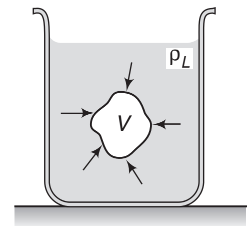


Fig. 16.35

Law of Floatation

Consider an object of volume V and density ρ_S floating in a liquid of density ρ_L. Let V_i be the volume of object immersed in the liquid.

For equilibrium of object,

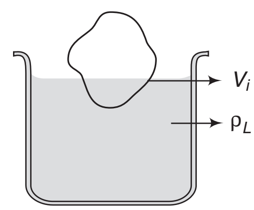


Fig. 16.36

$$\text{Weight} = \text{Upthrust}$$

$$\therefore \quad V\rho_S g = V_i \rho_L g$$

$$\therefore \quad \frac{V_i}{V} = \frac{\rho_S}{\rho_L} \quad \ldots\text{(i)}$$

This is the fraction of volume immersed in liquid.

Percentage of volume immersed in liquid $= \frac{V_i}{V} \times 100 = \frac{\rho_S}{\rho_L} \times 100$

Three possibilities may now arise:

(i) If $\rho_S < \rho_L$, only fraction of body will be immersed in the liquid. This fraction will be given by the above equation.

(ii) If $\rho_S = \rho_L$, the whole of the rigid body will be immersed in the liquid. Hence, the body remains floating in the liquid wherever it is left.

(iii) If $\rho_S > \rho_L$, the body will sink.

Buoyant Force in Accelerating Fluids

Suppose a body is dipped inside a liquid of density ρ_L placed in an elevator moving with an acceleration **a**. The buoyant force F in this case becomes,

$$\boxed{F = V\rho_L g_{\text{eff}}} \qquad (V = \text{immersed volume of solid or } V_i)$$

Here, $$g_{\text{eff}} = |\mathbf{g} - \mathbf{a}|$$

For example, if the lift has an upward acceleration a, the value of g_{eff} is $g + a$ and if it has a downward acceleration a, the g_{eff} is $g - a$. In a freely falling lift g_{eff} is zero (as $a = g$) and hence, net buoyant force is zero. This is why, in a freely falling vessel filled with some liquid, the air bubbles do not rise up (which otherwise move up due to buoyant force). The above result can be derived as follows.

Suppose a body is dipped inside a liquid of density ρ_L in an elevator moving up with an acceleration a. As was done earlier also, replace the body into the liquid by the same liquid of equal volume. The replaced liquid is at rest with respect to the elevator. Thus, this replaced liquid is also moving up with an acceleration a together with the rest of the liquid.

The forces acting on the replaced liquid are,

(i) the buoyant force F and

(ii) the weight mg of the substituted liquid.

From Newton's second law,

$$F - mg = ma \quad \text{or} \quad F = m(g + a)$$

Here, $m = V\rho_L$

$$\therefore \qquad F = V\rho_L (g + a) = V\rho_L g_{\text{eff}}$$

where $g_{\text{eff}} = g + a$

Extra Points to Remember

- In weight of a solid, take total volume of solid, density of solid and g. In upthrust (or buoyant force) take immersed volume of solid, density of liquid and g_{eff}. Thus,

$$\boxed{\begin{aligned} W &= V\rho_s g \\ F &= V_i \rho_L g_{\text{eff}} \end{aligned}}$$

- Upthrust force also makes a pair of equal and opposite forces. On solid it is upwards and on liquid, it is downwards.

Example 16.9 *Density of ice is 900 kg/m^3. A piece of ice is floating in water of density 1000 kg/m^3. Find the fraction of volume of the piece of ice outside the water.*

Solution Let V be the total volume and V_i the volume of ice piece immersed in water. For equilibrium of ice piece,

$$\text{weight} = \text{upthrust}$$

$$\therefore \qquad V\rho_i g = V_i \rho_w g$$

Here, $\rho_i = \text{density of ice} = 900 \text{ kg/m}^3$

and $\rho_w = \text{density of water} = 1000 \text{ kg/m}^3$

Substituting in above equation, we get

$$\frac{V_i}{V} = \frac{900}{1000} = 0.9$$

i.e. the fraction of volume outside the water,

$$f = 1 - 0.9 = 0.1$$ **Ans.**

Example 16.10 *A metallic sphere floats in an immiscible mixture of water $(\rho_w = 10^3\ kg/m^3)$ and a liquid $(\rho_L = 13.5 \times 10^3\ kg/m^3)$ such that its $\frac{4}{5}$ th volume is in water and $\frac{1}{5}$ th volume in the liquid. Find the density of metal.*

Solution Total upthrust = weight of metal sphere

$$\therefore \quad \left(\frac{4}{5}V\right)(10^3)g + \left(\frac{1}{5}V\right)(13.5 \times 10^5)g = (V)(\rho_{\text{metal}})(g)$$

$$\therefore \quad \rho_{\text{metal}} = 3.5 \times 10^3\ \text{kg/m}^3$$ **Ans.**

Example 16.11 *A block of mass 1 kg and density $0.8\ g/cm^3$ is held stationary with the help of a string as shown in figure. The tank is accelerating vertically upwards with an acceleration $a = 1.0\ m/s^2$. Find*

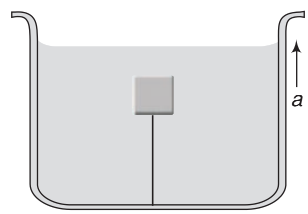


Fig. 16.37

(a) the tension in the string,
(b) if the string is now cut find the acceleration of block.
(Take $g = 10\ m/s^2$ and density of water $= 10^3\ kg/m^3$).

Solution (a) Free body diagram of the block is shown in Fig. 16.38.

In the figure,

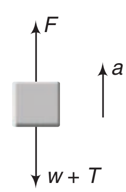


Fig. 16.38

$$F = \text{upthrust force}$$
$$= V\rho_\omega (g + a)$$
$$= \left(\frac{\text{mass of block}}{\text{density of block}}\right)\rho_\omega\ (g + a)$$
$$= \left(\frac{1}{800}\right)(1000)(10 + 1) = 13.75\ \text{N}$$
$$w = mg = 10\ \text{N}$$

Equation of motion of the block is,

$$F - T - w = ma$$

$$\therefore \quad 13.75 - T - 10 = 1 \times 1$$

$$\therefore \quad T = 2.75 \text{ N}$$ **Ans.**

(b) When the string is cut, $T = 0$

$$\therefore \quad a = \frac{F - w}{m} = \frac{13.75 - 10}{1} = 3.75 \text{ m/s}^2$$ **Ans.**

Example 16.12 *The tension in a string holding a solid block below the surface of a liquid (of density greater than that of solid) as shown in figure is T_0 when the system is at rest. What will be the tension in the string if the system has an upward acceleration a ?*

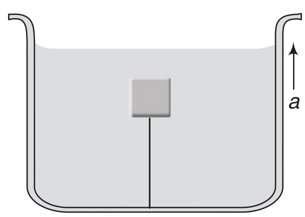


Fig. 16.39

Solution Let m be the mass of block.

Initially for the equilibrium of block,

$$F = T_0 + mg \quad \text{...(i)}$$

Here, F is the upthrust on the block.

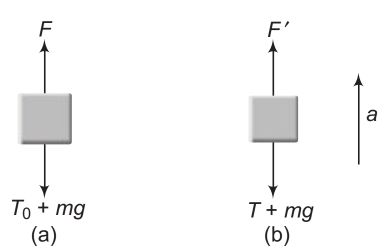


Fig. 16.40

When the lift is accelerated upwards, g_{eff} becomes $g + a$ instead of g. Hence,

$$F' = F\left(\frac{g + a}{g}\right) \quad \text{...(ii)}$$

From Newton's second law,

$$F' - T - mg = ma \qquad \ldots\text{(iii)}$$

Solving Eqs. (i), (ii) and (iii), we get

$$T = T_0\left(1 + \frac{a}{g}\right) \qquad \textbf{Ans.}$$

INTRODUCTORY EXERCISE 16.3

1. A block of material has a density ρ_1 and floats three-fourth submerged in a liquid of unknown density. Show that the density ρ_2 of the unknown liquid is given by $\rho_2 = \frac{4}{3}\rho_1$.
2. A block of wood weighing 71.2 N and of specific gravity 0.75 is tied by a string to the bottom of a tank of water in order to have the block totally immersed. What is the tension in the string?
3. A beaker when partly filled with water has total mass 20.00 g. If a piece of metal with density 3.00 g/cm^3 and volume 1.00 cm^3 is suspended by a thin string, so that it is submerged in the water but does not rest on the bottom of the beaker, how much does the beaker then appear to weigh if it is resting on a scale?
4. A small block of wood of density 0.4×10^3 kg/m^3 is submerged in water at a depth of 2.9 m. Find
   (a) the acceleration of the block towards the surface when the block is released and
   (b) the time for the block to reach the surface. Ignore viscosity.

16.6 Flow of Fluids

Steady Flow

If the velocity of fluid particles at any point does not vary with time, the flow is said to be steady. Steady flow is also called streamlined or laminar flow. The velocity at different points may be different. Hence, in the figure,

$$\mathbf{v}_1 = \text{constant},\ \mathbf{v}_2 = \text{constant},\ \mathbf{v}_3 = \text{constant but } \mathbf{v}_1 \neq \mathbf{v}_2 \neq \mathbf{v}_3$$

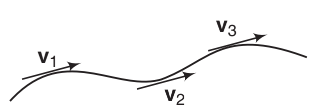


Fig 16.41

Principle of Continuity

It states that, when an incompressible and non-viscous liquid flows in a stream lined motion through a tube of non-uniform cross-section, then the product of the area of cross section and the velocity of flow is same at every point in the tube.

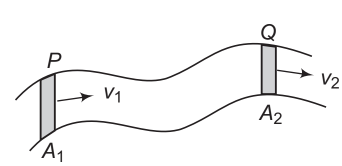


Fig. 16.42

Thus, $\boxed{A_1 v_1 = A_2 v_2}$ or $Av = \text{constant}$ or $v \propto \frac{1}{A}$

This is basically the law of conservation of mass in fluid dynamics.

Proof

Let us consider two cross sections P and Q of area A_1 and A_2 of a tube through which a fluid is flowing. Let v_1 and v_2 be the speeds at these two cross sections. Then, being an incompressible fluid, mass of fluid going through P in a time interval Δt = mass of fluid passing through Q in the same interval of time Δt.

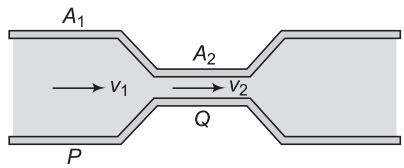


Fig. 16.43

$\therefore$
$$A_1 v_1 \rho \Delta t = A_2 v_2 \rho\, \Delta t$$
or
$$A_1 v_1 = A_2 v_2 \qquad \textbf{Proved.}$$

Therefore, the velocity of the liquid is smaller in the wider parts of the tube and larger in the narrower parts.

or
$$v_2 > v_1 \quad \text{as} \quad A_2 < A_1$$

Note *The product Av is the volume flow rate* $\frac{dV}{dt}$*, the rate at which volume crosses a section of the tube.*

Hence,

$$\boxed{\frac{dV}{dt} = \text{volume flow rate} = Av}$$

The mass flow rate is the mass flow per unit time through a cross-section. This is equal to density (ρ) times the volume flow rate $\frac{dV}{dt}$.

We can generalize the continuity equation for the case in which the fluid is not incompressible. If ρ_1 and ρ_2 are the densities at sections 1 and 2 then,

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2$$

So, this is the continuity equation for a compressible fluid.

Bernoulli's Equation

Bernoulli's equation relates the pressure, flow speed and height for flow of an ideal (incompressible and non-viscous) fluid. The pressure of a fluid depends on height as in the static situation and it also depends on the speed of flow.

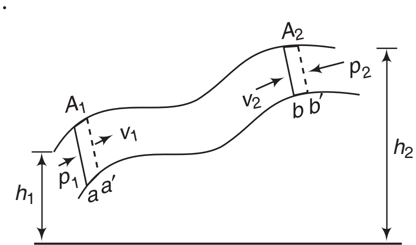


Fig. 16.44

The dependence of pressure on speed can be understood from the continuity equation. When an incompressible fluid flows along a tube with varying cross section, its speed must change, and so, an element of fluid must have an acceleration. If the tube is horizontal, the force that causes this acceleration has to be applied by the surrounding fluid. This means that the pressure must be different in regions of different cross section.

When a horizontal flow tube narrows and a fluid element speeds up, it must be moving towards a region of lower pressure in order to have a net forward force to accelerate it. If the elevation also changes, this causes additional pressure difference.

To derive Bernoulli's equation, we apply the work-energy theorem to the fluid in a section of the fluid element. Consider the element of fluid that at some initial time lies between two cross sections *a* and *b*. The speeds at the lower and upper ends are v_1 and v_2. In a small time interval, the fluid that is initially at *a* moves to a' a distance $aa' = ds_1 = v_1\ dt$ and the fluid that is initially at *b* moves to b' a distance $bb' = ds_2 = v_2\ dt$. The cross-section areas at the two ends are A_1 and A_2 as shown. The fluid is incompressible, hence, by the continuity equation, the volume of fluid dV passing through any cross-section during time dt is the same.

That is, $$dV = A_1 ds_1 = A_2 ds_2$$

Work Done on the Fluid Element

Let us calculate the work done on this fluid element during time interval *dt*. The pressure at the two ends are p_1 and p_2, the force on the cross section at *a* is $p_1 A_1$ and the force at *b* is $p_2 A_2$. The net work done dW on the element by the surrounding fluid during this displacement is,

$$dW = p_1 A_1 ds_1 - p_2 A_2 ds_2 = (p_1 - p_2)dV \qquad \text{...(i)}$$

The second term is negative, because the force at *b* opposes the displacement of the fluid.

This work dW is due to forces other than the conservative force of gravity, so it equals the change in total mechanical energy (kinetic plus potential). The mechanical energy for the fluid between sections *a* and *b* does not change.

Change in Potential Energy

At the beginning of *dt* the potential energy for the mass between *a* and a' is $dmgh_1 = \rho dVgh_1$. At the end of *dt* the potential energy for the mass between *b* and b' is $dmgh_2 = \rho dVgh_2$. The net change in potential energy dU during *dt* is,

$$dU = \rho(dV)g(h_2 - h_1) \qquad \text{...(ii)}$$

Change in Kinetic Energy

At the beginning of *dt* the fluid between *a* and a' has volume $A_1 ds_1$, mass $\rho A_1 ds_1$ and kinetic energy $\frac{1}{2}\rho(A_1 ds_1)v_1^2$. At the end of *dt* the fluid between *b* and b' has kinetic energy $\frac{1}{2}\rho(A_2 ds_2)v_2^2$. The net change in kinetic energy dK during time *dt* is,

$$dK = \frac{1}{2}\rho(dV)(v_2^2 - v_1^2) \qquad \text{...(iii)}$$

Combining Eqs. (i), (ii) and (iii) in the energy equation,

$$dW = dK + dU$$

We obtain,

$$(p_1 - p_2)dV = \frac{1}{2}\rho dV(v_2^2 - v_1^2) + \rho(dV)g(h_2 - h_1)$$

or $$p_1 - p_2 = \frac{1}{2}\rho(v_2^2 - v_1^2) + \rho g(h_2 - h_1) \quad \text{...(iv)}$$

This is Bernoulli's equation. It states that the work done on a unit volume of fluid by the surrounding fluid is equal to the sum of the changes in kinetic and potential energies per unit volume that occur during the flow. We can also express Eq. (iv) in a more convenient form as,

$$\boxed{p_1 + \rho g h_1 + \frac{1}{2}\rho v_1^2 = p_2 + \rho g h_2 + \frac{1}{2}\rho v_2^2} \quad \text{Bernoulli's equation}$$

The subscripts 1 and 2 refer to any two points along the flow tube, so we can also write

$$\boxed{p + \rho g h + \frac{1}{2}\rho v^2 = \text{constant}}$$

Extra Points to Remember

- In Bernoulli's equation, there are three terms; p, $\frac{1}{2}\rho v^2$ and $\rho g h$. Under following three cases, this equation reduces to a two term Bernoulli.

Case 1 If all points are open to atmosphere then pressure at every point may be assumed to be constant $(= p_0)$ and the Bernoulli equation can be written as,

$$\frac{1}{2}\rho v^2 + \rho g h = \text{constant}$$

At greater heights 'h', speed 'v' will be less as ρ and g are constants.

Case 2 If the liquid is passing through a pipe of uniform cross-section, then from continuity equation (Av = constant), speed v is same at all points. Therefore, the Bernoulli equation becomes,

$$p + \rho g h = \text{constant}$$

or $$p_1 + \rho g h_1 = p_2 + \rho g h_2$$

or $$p_1 - p_2 = \rho g (h_2 - h_1)$$

This is the pressure relation we have already derived for a fluid at rest or pressure decreases with height of liquid and increases with depth of liquid.

Case 3 If a liquid is flowing in a horizontal pipe, then height 'h' of the liquid at every point may be assumed to be constant. So, the two term Bernoulli becomes,

$$p + \frac{1}{2}\rho v^2 = \text{constant}$$

From this equation, we may conclude that pressure decreases at a point where speed increases.

Example 16.13 *Water is flowing through a horizontal tube of non-uniform cross section. At a place, the radius of the tube is* 1.0 cm *and the velocity of water is* 2 m/s. *What will be the velocity of water where the radius of the pipe is* 2.0 cm?

Solution Using equation of continuity, $A_1 v_1 = A_2 v_2$

$$v_2 = \left(\frac{A_1}{A_2}\right) v_1$$

or
$$v_2 = \left(\frac{\pi r_1^2}{\pi r_2^2}\right) v_1 = \left(\frac{r_1}{r_2}\right)^2 v_1$$

Substituting the values, we get

$$v_2 = \left(\frac{1.0 \times 10^{-2}}{2.0 \times 10^{-2}}\right)^2 (2)$$

or
$$v_2 = 0.5\,\text{m/s}$$ **Ans.**

Example 16.14 *Calculate the rate of flow of glycerine of density 1.25×10^3 kg/m^3 through the conical section of a pipe, if the radii of its ends are 0.1 m and 0.04 m and the pressure drop across its length is 10 N/m^2.*

Solution From continuity equation,

$$A_1 v_1 = A_2 v_2$$

or
$$\frac{v_1}{v_2} = \frac{A_2}{A_1} = \frac{\pi r_2^2}{\pi r_1^2} = \left(\frac{r_2}{r_1}\right)^2$$

$$= \left(\frac{0.04}{0.1}\right)^2 = \frac{4}{25} \qquad \ldots\text{(i)}$$

From Bernoulli's equation,

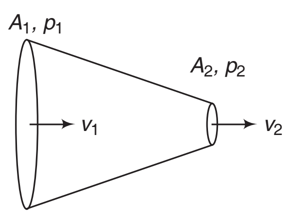


Fig. 16.45

$$p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2 \text{ or } v_2^2 - v_1^2 = \frac{2(p_1 - p_2)}{\rho}$$

or
$$v_2^2 - v_1^2 = \frac{2 \times 10}{1.25 \times 10^3} = 1.6 \times 10^{-2}\ \text{m}^2/\text{s}^2 \qquad \ldots\text{(ii)}$$

Solving Eqs. (i) and (ii), we get $v_2 \approx 0.128\,\text{m/s}$

∴ Rate of volume flow through the tube

$$Q = A_2 v_2 = (\pi r_2^2) v_2$$
$$= \pi (0.04)^2 (0.128)$$
$$= 6.43 \times 10^{-4}\ \text{m}^3/\text{s}$$ **Ans.**

Example 16.15 *Water is flowing smoothly through a closed-pipe system. At one point the speed of the water is 3.0 m/s, while at another point 1.0 m higher the speed is 4.0 m/s. If the pressure is 20 kPa at the lower point, what is the pressure at the upper point ? What would the pressure at the upper point be if the water were to stop flowing and the pressure at the lower point were 18 kPa ?*

Solution (i)
$$p_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = p_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2$$
$$(20 \times 10^3) + \frac{1}{2} \times 10^3 \times (3)^2 + 0 = p_2 + \frac{1}{2} \times 10^3 \times (4)^2 + 10^3 \times 10 \times 1$$
$$\therefore \quad p_2 = 6.5 \times 10^3 \text{ N/m}^2$$
$$= 6.5 \text{ kPa}$$
Ans.

(ii) Again applying the same equation, we have
$$(18 \times 10^3) + 0 + 0 = p_2 + 0 + (10^3)(10)(1)$$
$$\Rightarrow \quad p_2 = 8 \times 10^3 \text{ N/m}^2$$
$$= 8 \text{ kPa}$$
Ans.

INTRODUCTORY EXERCISE 16.4

1. Water flows through a tube shown in figure. The areas of cross section at A and B are 1 cm^2 and 0.5 cm^2 respectively. The height difference between A and B is 5 cm. If the speed of water at A is 10 cms^{-1}, find (a) the speed at B and (b) the difference in pressures at A and B.

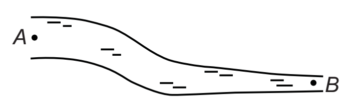


Fig. 16.46

2. Water flows through a horizontal tube of variable cross section as shown in figure. The area of cross section at A and B are 4 mm^2 and 2 mm^2 respectively. If 1 cc of water enters per second through A, find (a) the speed of water at A, (b) the speed of water at B and (c) the pressure difference $p_A - p_B$.

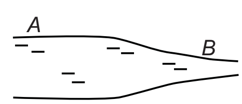


Fig. 16.47

3. Water from a tap emerges vertically downwards with an initial speed of 1.0 m/s. The cross-sectional area of tap is 10^{-4} m^2. Assume that the pressure is constant throughout the stream of water and that the flow is steady, the cross-sectional area of stream 0.15 m below the tap is **(JEE 1998)**

   (a) 5.0×10^{-4} m^2
   (b) 1.0×10^{-4} m^2
   (c) 5.0×10^{-5} m^2
   (d) 2.0×10^{-5} m^2

4. A horizontal pipeline carries water in a streamline flow. At a point along the pipe, where the cross-sectional area is 10 cm^2, the water velocity is 1 ms^{-1} and the pressure is 2000 Pa. The pressure of water at another point where the cross-sectional area is 5 cm^2, is ......Pa. (Density of water = 10^3 kg-m^3) **(JEE 1994)**

16.7 Application Based on Bernaulli's Equation

Atomizer or Spray Gun

Fig. 16.48 shows the essential parts of spray gun. When the piston is pressed, the air rushes out of the horizontal tube B decreasing the pressure to p_2 which is less than the pressure p_1 in the container. As a result, the liquid rises up in the vertical tube A. When it collides with the high speed air in tube B, it breaks up into a fine spray. Filter pumps, bunsen burner and sprayers used for perfumes or to spray insecticides work on the same principle.

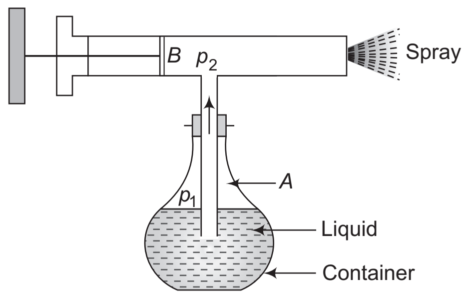


Fig. 16.48

Principle of Lifting of an Aircraft

We all know that airplanes fly. This is a result of a lift force acting on the wings of the aircraft. Lift can be generated when an asymmetric object moves through a fluid. The asymmetry of the object requires the fluid particles to travel different distances along different paths as they pass around the object. The molecules will either speed up or slowdown in order to remain even with the fluid molecules on the other side. For example, in Fig. 16.49, the air particles passing along the top of the body must travel a greater distance than those travelling below.

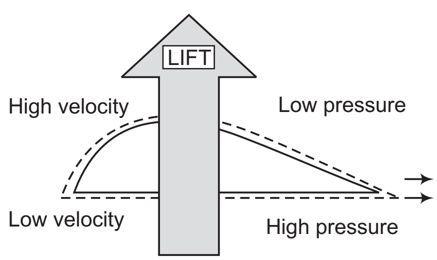


Fig. 16.49. The principle of lift of an aircraft

In order to arrive at the other side of the foil at the same time, these particles must travel more quickly than those travelling below the foil. The difference in the velocities of the fluid molecules results in a pressure difference according to the Bernoulli principle: high velocities are associated with relatively low pressure regions, while low velocities are associated with relatively high pressure regions. This pressure difference produces a force in upward direction.

Magnus Effect and Spinning of a Ball

Spinning objects can also generate force. When an object spins in a fluid medium, the fluid boundary layer spins along with the object. This results in a difference in velocity on two sides. A pressure difference is created from one side of the object to the other and the object curves towards the area of low pressure. This influence of rotation of an object on its flight path is termed the magnus effect. This magnus force changes the path of a rotating cricket ball. This is called spinning of ball.

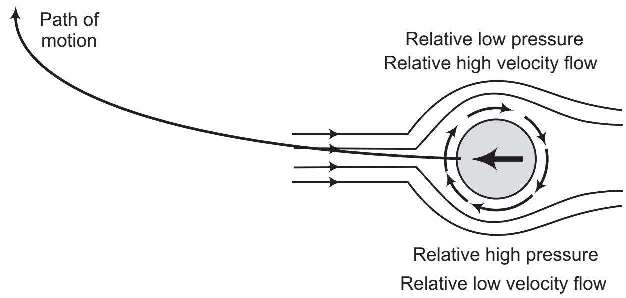


Fig. 16.50. The magnus effect

Blowing Off the Roof during Wind Storm

During certain wind storm or cyclone, the roofs of some houses are blown off without damaging the other parts of the house. The high wind blowing over the roof creates a low pressure p_2 in accordance with Bernoulli's principle. The pressure p_1 below the roof is equal to the atmospheric pressure which is larger than p_2. The difference of pressure $(p_1 - p_2)$ causes an upward thrust and the roof is lifted up. Once the roof is lifted up, it is blown off with the wind.

Venturimeter

Figure shows a venturimeter used to measure flow speed in a pipe of non-uniform cross-section. We apply Bernoulli's equation to the wide and narrow parts of the pipe, with $h_1 = h_2$

$$p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2$$

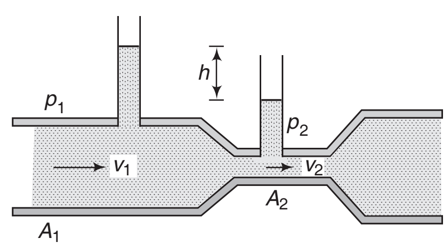


Fig. 16.51

From the continuity equation $v_2 = \dfrac{A_1 v_1}{A_2}$

Substituting and rearranging, we get

$$p_1 - p_2 = \frac{1}{2}\rho v_1^2 \left(\frac{A_1^2}{A_2^2} - 1\right) \qquad \ldots\text{(i)}$$

Because A_1 is greater than A_2, v_2 is greater than v_1 and hence the pressure p_2 is less than p_1. A net force to the right accelerates the fluid as it enters the narrow part of the tube (called throat) and a net force to the left slows as it leaves.

The pressure difference is also equal to ρgh, where h is the difference in liquid level in the two tubes. Substituting in Eq. (i), we get

$$\boxed{v_1 = \sqrt{\frac{2gh}{\left(\frac{A_1}{A_2}\right)^2 - 1}}}$$

Note *The discharge or volume flow rate can be obtained as,*

$$\frac{dV}{dt} = A_1 v_1 = A_1 \sqrt{\frac{2gh}{\left(\frac{A_1}{A_2}\right)^2 - 1}}$$

Speed of Efflux

Suppose, the surface of a liquid in a tank is at a height h from the orifice O on its sides, through which the liquid issues out with velocity v. The speed of the liquid coming out is called the speed of efflux. If the dimensions of the tank be sufficiently large, the velocity of the liquid at its surface may be taken to be zero. Applying Bernoulli's equation at the surface and just outside the orifice.

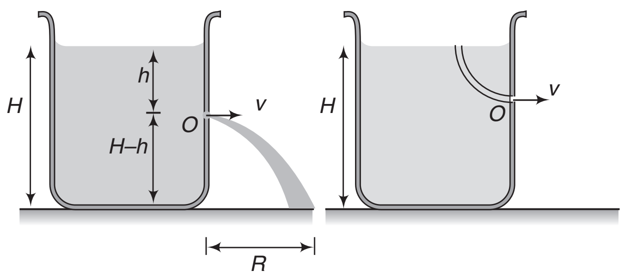


Fig. 16.52

$$\frac{1}{2}\rho v^2 + \rho gh + p = \text{constant}$$

with $h = 0$ at the orifice, we have

$$\rho gh + p_0 = \frac{1}{2}\rho v^2 + p_0 \quad \text{or} \quad \boxed{v = \sqrt{2gh}}$$

Evangelista Torricelli show that this velocity is the same as the liquid will attain in falling freely through the vertical height (h) from the surface to the orifice. This is known as **Torricelli's theorem** and may be stated as, "*The velocity of efflux of a liquid issuing out of an orifice is the same as it would attain if allowed to fall freely through the vertical height between the liquid surface and orifice.*"

Range (R)

Let us find the range R on the ground.

Considering the vertical motion of the liquid,

$$(H - h) = \frac{1}{2}gt^2 \quad \text{or} \quad \boxed{t = \sqrt{\frac{2(H-h)}{g}}}$$

Now, considering the horizontal motion,

$$R = vt$$

or $$R = (\sqrt{2gh})\left(\sqrt{\frac{2(H-h)}{g}}\right)$$

or $$\boxed{R = 2\sqrt{h(H-h)}}$$

From the expression of R, following conclusions can be drawn,

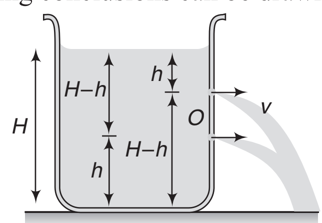


Fig. 16.53

(i) $R_h = R_{H-h}$ as $R_h = 2\sqrt{h(H-h)}$ and $R_{H-h} = 2\sqrt{(H-h)h}$

This is shown in Fig. 16.53.

(ii) R is maximum at $h = \frac{H}{2}$ and $R_{\max} = H$.

Proof $$R^2 = 4(Hh - h^2)$$

For R to be maximum, $\frac{dR^2}{dh} = 0$ or $H - 2h = 0$ or $h = \frac{H}{2}$

That is, R is maximum at $h = \frac{H}{2}$

and $$R_{\max} = 2\sqrt{\frac{H}{2}\left(H - \frac{H}{2}\right)} = H$$ **Hence Proved.**

Time Taken to Empty a Tank

We are here interested in finding the time required to empty a tank if a hole is made at the bottom of the tank.

Consider a tank filled with a liquid of density ρ upto a height H. A small hole of area of cross section a is made at the bottom of the tank. The area of cross-section of the tank is A.

Let at some instant of time the level of liquid in the tank is y. Velocity of efflux at this instant of time would be,

$$v = \sqrt{2gy}$$

Now, at this instant volume of liquid coming out of the hole per second is $\left(\frac{dV_1}{dt}\right)$.

Volume of liquid coming down in the tank per second is $\left(\frac{dV_2}{dt}\right)$.

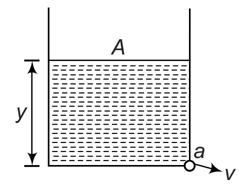


Fig. 16.54

$$\frac{dV_1}{dt} = \frac{dV_2}{dt}$$

$\therefore$ $$av = A\left(-\frac{dy}{dt}\right)$$

$\therefore$ $$a\sqrt{2gy} = A\left(-\frac{dy}{dt}\right)$$

or $$\int_0^t dt = -\frac{A}{a\sqrt{2g}} \int_H^0 y^{-1/2}\, dy \qquad \text{...(i)}$$

$\therefore$ $$t = \frac{2A}{a\sqrt{2g}} [\sqrt{y}]_0^H$$

$\therefore$ $$\boxed{t = \frac{A}{a}\sqrt{\frac{2H}{g}}}$$

Example 16.16 *Water flows through a horizontal tube as shown in figure . If the difference of heights of water column in the vertical tubes is 2 cm and the areas of cross-section at A and B are* 4cm^2 *and* $2\,\text{cm}^2$ *respectively. Find the rate of flow of water across any section.*

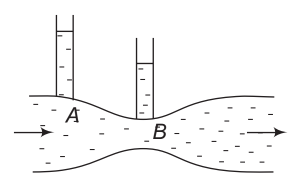


Fig. 16.55

Solution Applying Bernoulli's equation at A and B

$$p_A + \frac{1}{2}\rho v_A^2 + \rho g h_A = p_B + \frac{1}{2}\rho v_B^2 + \rho g h_B \qquad (h_A = h_B)$$

or $$\frac{1}{2}\rho v_B^2 - \frac{1}{2}\rho v_A^2 = \rho g \ (h_A - h_B)$$

or $$v_B^2 - v_A^2 = 2g\,(h_A - h_B)$$

$$= 2 \times 10 \times 0.02$$

or $$v_B^2 - v_A^2 = 0.4 \text{ m}^2/\text{s}^2 \qquad \text{...(i)}$$

$$v_A A_A = v_B A_B$$

or $$4v_A = 2v_B$$

$\therefore$ $$v_B = 2v_A \qquad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we get

$$v_A = 0.363 \text{ m/s}$$

Volume flow rate $= v_A A_A$

$$= (0.365)\,(4 \times 10^{-4})$$

$$= 1.46 \times 10^{-4} \text{ m}^3/\text{s} = 146\,\text{cm}^3/\text{s}$$ **Ans.**

Example 16.17 *Water flows through the tube as shown in figure. The areas of cross-section of the wide and the narrow portions of the tube are 5 cm^2 and 2 cm^2 respectively. The rate of flow of water through the tube is 500 cm^3/s. Find the difference of mercury levels in the U-tube.*

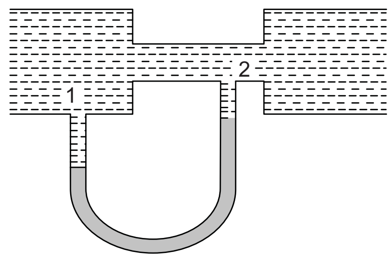


Fig. 16.56

Solution Applying continuity equation

$$A_1 v_1 = A_2 v_2 = \text{rate of flow of water}$$

$$5v_1 = 2v_2 = 500\,\text{cm}^3/\text{s}$$

$$\therefore \quad v_1 = 100\text{ cm/s} = 1.0\text{ m/s}$$

$$v_2 = 250\,\text{cm/s} = 2.5\text{ m/s}$$

Now, applying Bernoulli's equation at 1 and 2

$$p_1 + \frac{1}{2}\rho_w v_1^2 + \rho_w g h_1 = p_2 + \frac{1}{2}\rho_w v_2^2 + \rho_w g h_2 \qquad (h_1 = h_2)$$

$$\Rightarrow \quad \frac{1}{2}\rho_w (v_2^2 - v_1^2) = p_1 - p_2 = \rho_{\text{Hg}} g h_{\text{Hg}}$$

or

$$\frac{1}{2}\rho_w (v_2^2 - v_1^2) = \rho_{\text{Hg}}\, g\, h_{\text{Hg}}$$

$$\therefore \quad h_{\text{Hg}} = \frac{\rho_w (v_2^2 - v_1^2)}{2\rho_{\text{Hg}}\, g}$$

$$= \frac{10^3\,(6.25 - 1)}{2 \times 13.6 \times 10^3 \times 9.8}$$

$$= 0.0196\,\text{m}$$

$$= 1.96\,\text{cm}$$ **Ans**

Example 16.18 *A tank is filled with a liquid upto a height H. A small hole is made at the bottom of this tank. Let t_1 be the time taken to empty first half of the tank and t_2 the time taken to empty rest half of the tank. Then find $\frac{t_1}{t_2}$.*

Solution Substituting the proper limits in Eq. (i), derived in the theory, we have

$$\int_0^{t_1} dt = -\frac{A}{a\sqrt{2g}} \int_H^{H/2} y^{-1/2}\, dy$$

or $$t_1 = \frac{2A}{a\sqrt{2g}} [\sqrt{y}]_{H/2}^{H}$$

or $$t_1 = \frac{2A}{a\sqrt{2g}} \left[\sqrt{H} - \sqrt{\frac{H}{2}}\right]$$

or $$t_1 = \frac{A}{a}\sqrt{\frac{H}{g}}\,(\sqrt{2}-1) \qquad \text{...(ii)}$$

Similarly, $$\int_0^{t_2} dt = -\frac{A}{a\sqrt{2g}} \int_{H/2}^{0} y^{-1/2}\, dy$$

or $$t_2 = \frac{A}{a}\sqrt{\frac{H}{g}} \qquad \text{...(iii)}$$

From Eqs. (ii) and (iii), we get

$$\frac{t_1}{t_2} = \sqrt{2} - 1$$

or $$\frac{t_1}{t_2} = 0.414$$ **Ans.**

Note *From the above example we can see that $t_1 < t_2$. This is because initially the pressure is high and the liquid comes out with greater speed.*

INTRODUCTORY EXERCISE 16.5

1. There is a small hole at the bottom of tank filled with water. If total pressure at the bottom is 3 atm (1 atm = 10^5 Nm^{-2}), then find the velocity of water flowing from hole.

2. Liquid is filled in a container upto a height of H. A small hole is made at the bottom of the tank. Time taken to empty from H to $\frac{H}{3}$ is t_0. Find the time taken to empty the tank from $\frac{H}{3}$ to zero.

3. Water is filled in a cylindrical container to a height of 3 m. The ratio of the cross-sectional area of the orifice and the beaker is 0.1. The square of the speed of the liquid coming out from the orifice is ($g = 10\,\text{m/s}^2$) **(JEE 2005)**

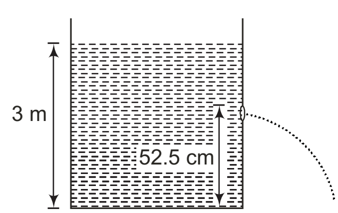


Fig. 16.57

(a) $50\ \text{m}^2/\text{s}^2$
(b) $50.5\ \text{m}^2/\text{s}^2$
(c) $51\text{m}^2/\text{s}^2$
(d) $52\ \text{m}^2/\text{s}^2$

16.8 Viscosity

Viscosity is internal friction in a fluid. Viscous forces oppose the motion of one portion of a fluid relative to the other.

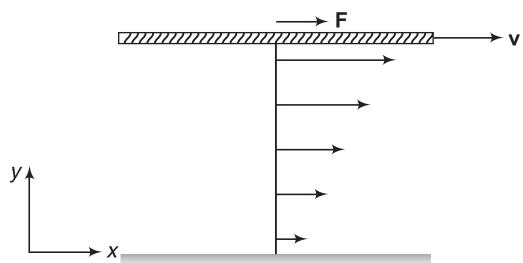


Fig. 16.58

The simplest example of viscous flow is motion of a fluid between two parallel plates.

The bottom plate is stationary and the top plate moves with constant velocity **v**. The fluid in contact with each surface has same velocity at that surface. The flow speeds of intermediate layers of fluid increase uniformly from bottom to top, as shown by arrows. So, the fluid layers slide smoothly over one another.

According to Newton, the frictional force F (or viscous force) between two layers depends upon the following factors,

(i) Force F is directly proportional to the area (A) of the layers in contact, i.e.

$$F \propto A$$

(ii) Force F is directly proportional to the velocity gradient $\left(\frac{dv}{dy}\right)$ between the layers. Combining these two, we have

$$F \propto A\frac{dv}{dy} \quad \text{or} \quad \boxed{F = -\eta A\frac{dv}{dy}}$$

Here, η is constant of proportionality and is called **coefficient of viscosity.** Its value depends on the nature of the fluid. The negative sign in the above equation shows that the direction of viscous force F is opposite to the direction of relative velocity of the layer.

The SI unit of η is N-s/m^2. It is also called decapoise or pascal second. Thus,

$$1 \text{ decapoise} = 1 \text{ N-s/m}^2 = 1 \text{ Pa-s} = 10 \text{ poise}$$

The SI unit of viscosity is sometimes referred to as the poiseuille (symbol Pl).

Thus, $$1 \text{ Pl} = 1 \text{ N-s/m}^2$$

Dimensions of η are $[ML^{-1}T^{-1}]$.

Coefficient of viscosity of water at 10°C is $\eta = 1.3 \times 10^{-3}$ N-s/m^2. Experiments show that coefficient of viscosity of a liquid decreases as its temperature rises.

Stoke's Law and Terminal Velocity

When an object moves through a fluid, it experiences a viscous force which acts in opposite direction of its velocity. The mathematics of the viscous force for an irregular object is difficult, we will consider here only the case of a small sphere moving through a fluid.

The formula for the viscous force on a sphere was first derived by the English physicist G. Stokes in 1843. According to him, a spherical object of radius r moving at velocity v experiences a viscous force given by

$$\boxed{F = 6\pi\eta r v} \qquad (\eta = \text{coefficient of viscosity})$$

This law is called **Stoke's law.**

Terminal Velocity (v_T)

Consider a small sphere falling from rest through a large column of viscous fluid. The forces acting on the sphere are,

(i) Weight w of the sphere acting vertically downwards

(ii) Upthrust F_t acting vertically upwards

(iii) Viscous force F_v acting vertically upwards, i.e. in a direction opposite to velocity of the sphere.

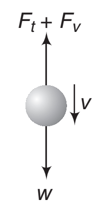


Fig. 16.59

Initially, $\quad F_v = 0 \quad$ (as $v = 0$)

and $\quad w > F_t$

and the sphere accelerates downwards. As the velocity of the sphere increases, F_v increases. Eventually a stage in reached when

$$w = F_t + F_v \qquad \ldots(i)$$

After this net force on the sphere is zero and it moves downwards with a constant velocity called **terminal velocity** (v_T).

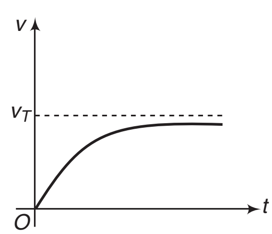


Fig. 16.60

Substituting proper values in Eq. (i) we have,

$$\frac{4}{3}\pi r^3 \rho g = \frac{4}{3}\pi r^3 \sigma g + 6\pi\eta r v_T \qquad \ldots(ii)$$

Here, $\quad \rho$ = density of sphere, $\quad \sigma$ = density of fluid

and $\quad \eta$ = coefficient of viscosity of fluid

From Eq. (ii), we get $\quad \boxed{v_T = \frac{2}{9}\frac{r^2(\rho - \sigma)g}{\eta}}$

Figure shows the variation of the velocity v of the sphere with time.

Extra Points to Remember

- Terminal velocity $v_T \propto r^2$
- If the fluid is air, then its density σ is negligible compared to density of sphere. So, in that case upthrust will be zero and terminal velocity will be

$$v_T = \frac{2}{9}\frac{r^2 \rho g}{\eta}$$

In this case, weight is equal to the viscous force when terminal velocity is attained.

- If density of fluid is greater than density of sphere ($\sigma > \rho$) then terminal velocity comes out to be negative.

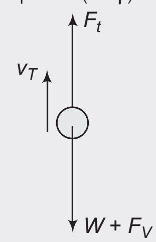


Fig. 16.61

So, in this case terminal velocity is upwards. In the beginning upthrust is greater than the weight. Viscous force in this case will be downwards.

$$v_T = \frac{2}{9}\frac{r^2(\sigma - \rho)g}{\eta}$$

When terminal velocity is attained,

$$F_t = W + F_v$$

This is the reason why, air bubbles rise up in water.

Example 16.19 *A plate of area 2 m^2 is made to move horizontally with a speed of 2 m/s by applying a horizontal tangential force over the free surface of a liquid. If the depth of the liquid is 1 m and the liquid in contact with the bed is stationary. Coefficient of viscosity of liquid is 0.01 poise. Find the tangential force needed to move the plate.*

Solution Velocity gradient $= \frac{\Delta v}{\Delta y} = \frac{2-0}{1-0} = 2\,\frac{\text{m/s}}{\text{m}}$

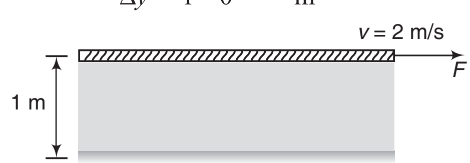


Fig. 16.62

From, Newton's law of viscous force,

$$|F| = \eta A \frac{\Delta v}{\Delta y}$$

$$= (0.01 \times 10^{-1})(2)(2)$$

$$= 4 \times 10^{-3}\ \text{N}.$$

So, to keep the plate moving, a force of 4×10^{-3} N must be applied. **Ans.**

Example 16.20 *Two spherical raindrops of equal size are falling vertically through air with a terminal velocity of 1 m/s. What would be the terminal speed if these two drops were to coalesce to form a large spherical drop?*

Solution $$v_T \propto r^2 \qquad \text{...(i)}$$

Let r be the radius of small rain drops and R the radius of large drop.

Equating the volumes, we have

$$\frac{4}{3}\pi R^3 = 2\left(\frac{4}{3}\pi r^3\right)$$

$$\therefore \quad R = (2)^{1/3}.r \quad \text{or} \quad \frac{R}{r} = (2)^{1/3}$$

$$\therefore \quad \frac{v_T'}{v_T} = \left(\frac{R}{r}\right)^2 = (2)^{2/3}$$

$$\therefore \quad v_T' = (2)^{2/3} v_T = (2)^{2/3}(1.0) \text{ m/s}$$

$$= 1.587 \text{ m/s}$$ **Ans.**

Example 16.21 *With what terminal velocity will an air bubble 0.8 mm in diameter rise in a liquid of viscosity 0.15 N-s/m² and specific gravity 0.9 ? Density of air is 1.293 kg/m³.*

Solution The terminal velocity of the bubble is given by,

$$v_T = \frac{2}{9}\frac{r^2(\rho-\sigma)g}{\eta}$$

Here, $r = 0.4 \times 10^{-3}$ m, $\sigma = 0.9 \times 10^3$ kg/m³, $\rho = 1.293$ kg/m³, $\eta = 0.15$ N-s/m²

and $g = 9.8$ m/s²

Substituting the values, we have

$$v_T = \frac{2}{9} \times \frac{(0.4 \times 10^{-3})^2 (1.293 - 0.9 \times 10^3) \times 9.8}{0.15}$$

$$= -0.0021 \text{ m/s}$$

or $$v_T = -0.21 \text{cm}$$ **Ans.**

Note *Here, negative sign implies that the bubble will rise up.*

Example 16.22 *A spherical ball of radius 3.0×10^{-4} m and density 10^4 kg/m³ falls freely under gravity through a distance h before entering a tank of water. If after entering the water the velocity of the ball does not change, find h. Viscosity of water is 9.8×10^{-6} N-s/m².*

Solution Before entering the water the velocity of ball is $\sqrt{2gh}$. If after entering the water this velocity does not change then this value should be equal to the terminal velocity. Therefore,

$$\sqrt{2gh} = \frac{2}{9}\frac{r^2(\rho-\sigma)g}{\eta}$$

$$\therefore \qquad h = \frac{\left\{\frac{2}{9}\frac{r^2(\rho-\sigma)g}{\eta}\right\}^2}{2g}$$

$$= \frac{2}{81} \times \frac{r^4(\rho-\sigma)^2 g}{\eta^2}$$

$$= \frac{2}{81} \times \frac{(3\times10^{-4})^4(10^4-10^3)^2\times 9.8}{(9.8\times10^{-6})^2}$$

$$= 1.65\times10^3 \text{ m} \qquad \textbf{Ans.}$$

Example 16.23 *A small sphere falls from rest in a viscous liquid. Due to friction, heat is produced. Find the relation between the rate of production of heat and the radius of the sphere at terminal velocity.* **(JEE 2004)**

Solution Terminal velocity $v_T = \frac{2r^2g}{9\eta}(\rho_S - \rho_L)$

and viscous force $F = 6\pi\eta r v_T$

Rate of production of heat (power) : as viscous force is the only dissipative force. Hence,

$$\frac{dQ}{dt} = Fv_T = (6\pi\eta r v_T)(v_T)$$

$$= 6\pi\eta r v_T^2$$

$$= 6\pi\eta r\left\{\frac{2}{9}\frac{r^2 g}{\eta}(\rho_S-\rho_L)\right\}^2$$

$$= \frac{8\pi g^2}{27\eta}(\rho_S-\rho_L)^2 r^5 \quad \text{or} \quad \frac{dQ}{dt} \propto r^5 \qquad \textbf{Ans.}$$

INTRODUCTORY EXERCISE 16.6

1. A typical riverborne silt particle has a radius of 20 μm and a density of 2×10^3 kg/m^3. The viscosity of water is 1.0 mPl. Find the terminal speed with which such a particle will settle to the bottom of a motionless volume of water.
2. Two equal drops of water are falling through air with a steady velocity v. If the drops coalesced, what will be the new velocity ?
3. A large wooden plate of area 10 m^2 floating on the surface of a river is made to move horizontally with a speed of 2 m/s by applying a tangential force. If the river is 1 m deep and the water in contact with the bed is stationary, find the tangential force needed to keep the plate moving. Coefficient of viscosity of water at the temperature of the river $= 10^{-2}$ poise.
4. The velocity of water in a river is 18 km/h near the surface. If the river is 5 m deep, find the shearing stress between the horizontal layers of water. The coefficient of viscosity of water $= 10^{-2}$ poise.

16.9 Surface Tension

A needle can made to float on a water surface if it is placed there carefully. The forces that support the needle are not buoyant forces but are due to surface tension. The surface of a liquid behaves like a membrane under tension. The molecules of the liquid exert attractive forces on each other. There is zero net force on a molecule inside the volume of the liquid.But a surface molecule has a net force towards inside of the liquid. Thus, the liquid tends to minimize its surface area, just as a stretched membrane does.

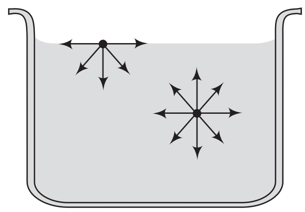


Fig. 16.63

Freely falling raindrops are spherical because a sphere has a smaller surface area for a given volume than any other shape. Hence, the surface tension can be defined as the *property of a liquid at rest by virtue of which its free surface behaves like a stretched membrane under tension and tries to occupy as small area as possible.*

Let an imaginary line *AB* be drawn in any direction in a liquid surface. The surface on either side of this line exerts a pulling force on the surface on the other side. This force is at right angles to the line *AB*. The magnitude of this force per unit length of *AB* is taken as a measure of the surface tension of the liquid. Thus, if *F* be the total force acting on either side of the line *AB* of length *L,* then the surface tension is given by,

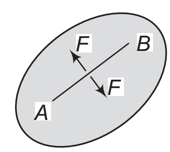


Fig. 16.64

$$T = \frac{F}{L}$$

Hence, the surface tension of a liquid is defined as the force per unit length in the plane of the liquid surface, acting at right angles on either side of an imaginary line drawn on that surface.

Examples of Surface Tension

Example 1 *Take a ring of wire and dip it in a soap solution. When the ring is taken out, a soap film is formed. Place a loop of thread gently on the soap film. Now, prick a hole inside the loop. The thread is radially pulled by the film surface outside and it takes a circular shape.*

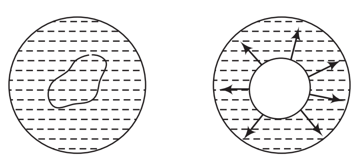


Fig. 16.65

Reason Before the pricking, there were surfaces both inside and outside the thread loop. Surfaces on both sides pull it equally and the net force is zero. Once the surface inside was punctured, the outside surface pulled the thread to take the circular shape so that area outside the loop becomes minimum (because for given perimeter area of circle is maximum).

Example 2 *A piece of wire is bent into a U-shape and a second piece of wire slides on the arms of the U. When the apparatus is dipped into a soap solution and removed, a liquid film is formed. The film exerts a surface tension force on the slider and if the frame is kept in a horizontal position, the slider quickly slides towards the closing arm of the frame. If the frame is kept vertical, one can have some weight to keep it in equilibrium. This shows that the soap surface in contact with the slider pulls it parallel to the surface.*

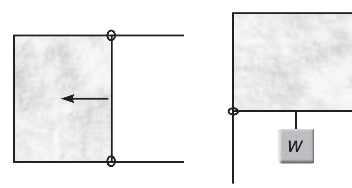


Fig. 16.66

Example 3 ***Needle supported on water surface*** *Take a greased needle of steel on a piece of blotting paper and place it gently over the water surface. Blotting paper soaks water and soon sinks down but the needle keeps floating. The floating needle causes a little depression. The forces F and F due to surface tension of the curved surface are inclined as shown in Fig.*16.67. *The vertical components of these two forces support the weight of the needle.*

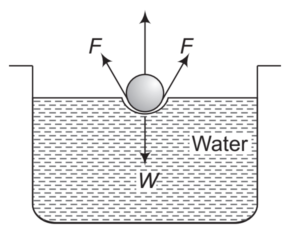


Fig. 16.67

Example 4 *Small mercury droplets are spherical and larger ones tend to flatten.*

Reason Small mercury droplets are spherical because the forces of surface tension tend to reduce their area to a minimum value and a sphere has minimum surface area for a given volume.

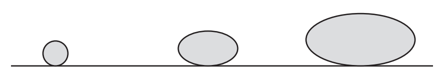


Fig. 16.68

Larger drops of mercury are flattened due to the large gravitational force acting on them. Here the shape is such that the sum of the gravitational potential energy and the surface potential energy must be minimum. Hence the centre of gravity moves down as low as possible. This explains flattening of the larger drops.

Example 5 *The hair of a painting brush cling together when taken out of water.*

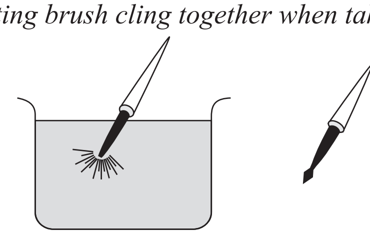


Fig. 16.69

Reason This is because the water films formed on them tend to contract to minimum area.

Note *The surface tension of a particular liquid usually decreases as temperature increases. To wash clothing thoroughly, water must be forced through the tiny spaces between the fibers. This requires increasing the surface area of the water ,which is difficult to do because of surface tension. Hence, hot water and soapy water is better for washing.*

Surface Energy

When the surface area of a liquid is increased, the molecules from the interior rise to the surface. This requires work against force of attraction of the molecules just below the surface. This work is stored in the form of potential energy. Thus, the molecules in the surface have some additional energy due to their position. This additional energy per unit area of the surface is called 'surface energy'. The surface energy is related to the surface tension as discussed below.

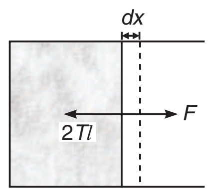


Fig. 16.70

Let a liquid film be formed on a wire frame and a straight wire of length l can slide on this wire frame as shown in figure. The film has two surfaces and both the surfaces are in contact with the sliding wire and hence, exert forces of surface tension on it. If T be the surface tension of the solution, each surface will pull the wire parallel to itself with a force Tl. Thus, net force on the wire due to both the surfaces is $2Tl$. One has to apply an external force F equal and opposite to it to keep the wire in equilibrium.

Thus,

$$F = 2Tl$$

Now, suppose the wire is moved through a small distance dx, the work done by the force is,

$$dW = F\ dx = (2Tl)\ dx$$

But $(2l)\ (dx)$ is the total increase in area of both the surfaces of the film. Let it be dA. Then,

$$dW = T\ dA$$

or $$\boxed{T = \frac{dW}{dA} \quad \text{or} \quad \frac{\Delta W}{\Delta A}}$$

Thus, the surface tension T can also be defined as the work done in increasing the surface area by unity.

Further, since there is no change in kinetic energy, the work done by the external force is stored as the potential energy of the new surface.

$\therefore$ $$\boxed{T = \frac{dU}{dA} \quad \text{or} \quad \frac{\Delta U}{\Delta A}} \qquad \text{(as } dW = dU\text{)}$$

Thus, the surface tension of a liquid is equal to the surface energy per unit surface area.

Excess Pressure inside a Bubble or Liquid Drop

Surface tension causes a pressure difference between the inside and outside of a soap bubble or a liquid drop.

Excess Pressure Inside Soap Bubble

A soap bubble consists of two spherical surface films with a thin layer of liquid between them. Because of surface tension, the film tend to contract in an attempt to minimize their surface area. But as the bubble contracts, it compresses the inside air, eventually increasing the interior pressure to a level that prevents further contraction.

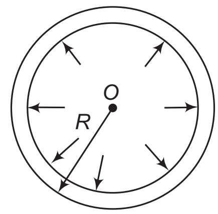


Fig. 16.71

We can derive an expression for the excess pressure inside a bubble in terms of its radius R and the surface tension T of the liquid.

Each half of the soap bubble is in equilibrium. The lower half is shown in figure. The forces at the flat circular surface where this half joins the upper half are

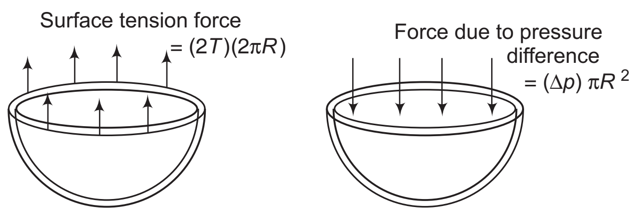


Fig. 16.72

(a) The upward force of surface tension. The total surface tension force for each surface (inner and outer) is $T(2\pi R)$, for a total of $(2T)(2\pi R)$.

(b) downward force due to pressure difference.

The magnitude of this force is $(\Delta p)(\pi R^2)$. In equilibrium these two forces have equal magnitude.

$\therefore$ $$(2T)(2\pi R) = (\Delta P)(\pi R^2)$$

or $$\boxed{\Delta p = \frac{4T}{R}}$$

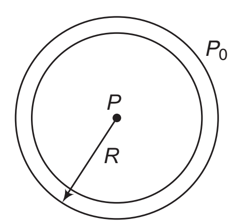


Fig. 16.73

Note *Suppose, the pressure inside the air bubble is P, then*

$$p - p_0 = \frac{4T}{R}$$

Excess Pressure Inside a Liquid Drop

A liquid drop has only one surface film. Hence, the surface tension force is $T(2\pi R)$, half that for a soap bubble. Thus, in equilibrium,

$$T(2\pi R) = \Delta p(\pi R^2)$$

or $$\boxed{\Delta p = \frac{2T}{R}}$$

Note (i) *If we have an air bubble inside a liquid, a single surface is formed. There is air on the concave side and liquid on the convex side. The pressure in the concave side (that is in the air) is greater than the pressure in the convex side (that is in the liquid) by an amount* $\frac{2T}{R}$.

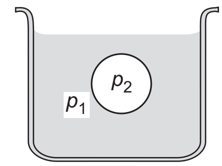


Fig. 16.74

$$\therefore \qquad \boxed{p_2 - p_1 = \frac{2T}{R}}$$

The above expression has been written by assuming P_1 *to be constant from all sides of the bubble. For small size bubbles this can be assumed.*

(ii) *From the above discussion, we can make a general statement. The pressure on the concave side of a spherical liquid surface is greater than the convex side by* $\frac{2T}{R}$.

Extra Points to Remember

- **Radius of curvature of a curve :**

  To describe the shape of a curved surface or interface, it is necessary to know the radii of curvature to a curve at some point. Consider the curve *AB* as shown in figure:

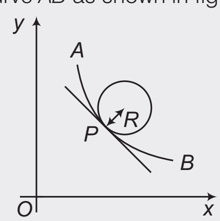


Fig. 16.75

  Let *P* be a point on this curve. The radius of curvature *R* of *AB* at *P* is defined as the radius of the circle which is tangent to the curve at point *P*.

- **Principal radii of curvature of a surface :**

  The spherical and cylindrical surfaces are rather simple cases for mathematical treatment. In many other cases however, the shapes are more complicated.

  Let us now consider a curved surface. At each point on a given surface, two radii of curvature (which are denoted by r_1 and r_2) are required to describe the shape.

  If we want to determine these radii at any point (say *P*), the normal to the surface at this point is drawn and a plane is constructed through the surface containing the normal. This will intersect the surface in a plane curve. The radius of curvature of the curve at point *P* is denoted by r_1. An infinite number of such planes can be constructed each of which intersects the surface at *P*. For each of these planes, a radius of curvature can be obtained.

  If we construct a second plane through the surface, containing the normal and perpendicular to the first plane, the second line of intersection and hence the second radius of curvature at point *P* (i.e., r_2) is obtained. These two radii define the curvature at *P* completely. It can be shown that $(1/r_1 + 1/r_2)$ called mean radius of curvature of the surface is constant, which is independent of the choice of the planes.

  An infinite set of such pairs of radii is possible. For standardization, the first plane is rotated around the normal until the radius of curvature in that plane reaches minimum. The other radius of curvature is therefore maximum. These are the principal radii of curvature (denoted by R_1 and R_2).

- **Young-Laplace Equation :**

  There exists a difference in pressure across a curved surface which is a consequence of surface tension. The pressure is greater on the concave side. The Laplace equation relates the pressure difference to the shape of the Young surface.

This difference in pressure is given by

$$\boxed{\Delta p = T\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}$$

The simplified forms of spherical, cylindrical and planar surfaces are given below

For a spherical surface:

$$R_1 = R_2 = R \text{ (say), therefore } \Delta p = \frac{2T}{R}$$

For a cylindrical surface:

$$R_1 = R \text{ and } R_2 = \infty, \text{ therefore } \Delta p = \frac{T}{R}$$

For a planer surface:

$$R_1 = R_2 = \infty, \text{ therefore } \Delta p = 0$$

- As we know that molecules of a liquid reach to its surface after struggling with the net inward force acting on them from other liquid molecules. So, we can say that the surface molecules have some extra energy compared to inner molecules and every system has a tendency to keep its energy minimum. So, a liquid also has a tendency to keep its energy minimum by putting least number of molecules on the surface or by making its surface area minimum. This property of a liquid is called surface tension.
- Surface tension is a property of a liquid. It does not depend on the surface area.

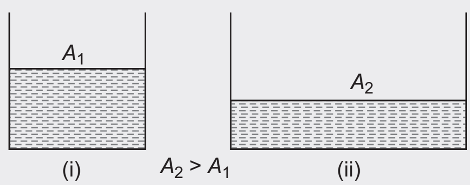


Fig. 16.76

For example, both containers have the same liquid under same conditions. $A_2 > A_1$ but surface tension will be same in both cases.

Example 16.24 *How much work will be done in increasing the diameter of a soap bubble from 2 cm to 5 cm ? Surface tension of soap solution is* 3.0×10^{-2} *N/m*.

Solution Soap bubble has two surfaces. Hence,

$$W = T\,\Delta A \qquad \left(T = \frac{\Delta W}{\Delta A}\right)$$

Here, $\Delta A = 2[4\pi\{(2.5 \times 10^{-2})^2 - (1.0 \times 10^{-2})^2\}]$

$$= 1.32 \times 10^{-2} \text{ m}^2$$

$\therefore$ $W = (3.0 \times 10^{-2})(1.32 \times 10^{-2})$ J

$$= 3.96 \times 10^{-4} \text{ J}$$ **Ans.**

Example 16.25 *Calculate the energy released when* 1000 *small water drops each of same radius* 10^{-7} *m coalesce to form one large drop. The surface tension of water is* 7.0×10^{-2} *N/m.*

Solution Let r be the radius of smaller drops and R of bigger one. Equating the initial and final volumes, we have

$$\frac{4}{3}\pi R^3 = (1000)\left(\frac{4}{3}\pi r^3\right)$$

or $$R = 10r = (10)(10^{-7})\text{ m}$$

or $$R = 10^{-6}\text{ m}$$

Further, the water drops have only one free surface. Therefore,

$$\Delta A = 4\pi R^2 - (1000)(4\pi r^2)$$
$$= 4\pi\ [(10^{-6})^2 - (10^3)(10^{-7})^2]$$
$$= -36\pi(10^{-12})\text{ m}^2$$

Here, negative sign implies that surface area is decreasing. Hence, energy released in the process.

$$U = T|\Delta A| = (7\times 10^{-2})(36\pi\times 10^{-12})\text{ J}$$
$$= 7.9\times 10^{-12}\text{ J}$$ **Ans.**

Example 16.26 *What should be the pressure inside a small air bubble of 0.1 mm radius situated just below the water surface. Surface tension of water $= 7.2\times 10^{-2}$ N/m and atmospheric pressure $= 1.013\times 10^5$ N/m^2.*

Solution Surface tension of water $T = 7.2\times 10^{-2}$ N/m

Radius of air bubble $R = 0.1\text{ mm} = 10^{-4}$ m

The excess pressure inside the air bubble is given by,

$$p_2 - p_1 = \frac{2T}{R}$$

∴ Pressure inside the air bubble,

$$p_2 = p_1 + \frac{2T}{R}$$

Substituting the values, we have

$$p_2 = (1.013\times 10^5) + \frac{(2\times 7.2\times 10^{-2})}{10^{-4}}$$
$$= 1.027\times 10^5\text{ N/m}^2$$ **Ans.**

Example 16.27 *Two separate air bubbles (radii 0.004 m and 0.002 m) formed of the same liquid (surface tension 0.07 N/m) come together to form a double bubble. Find the radius and the sense of curvature of the internal film surface common to both the bubbles.*

Solution $p_1 = p_0 + \dfrac{4T}{r_1} \Rightarrow p_2 = p_0 + \dfrac{4T}{r_2}$

$$r_2 < r_1$$

∴ $$p_2 > p_1$$

i.e. pressure inside the smaller bubble will be more. The excess pressure

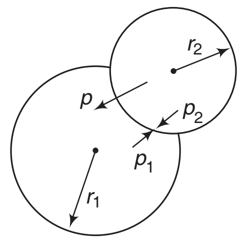


Fig. 16.77

$$p = p_2 - p_1 = 4T\left(\frac{r_1 - r_2}{r_1 r_2}\right) \quad \text{...(i)}$$

This excess pressure acts from concave to convex side, the interface will be concave towards smaller bubble and convex towards larger bubble. Let R be the radius of interface then,

$$p = \frac{4T}{R} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$R = \frac{r_1 r_2}{r_1 - r_2} = \frac{(0.004)(0.002)}{(0.004 - 0.002)}$$

$$= 0.004 \text{ m}$$ **Ans**

Example 16.28 *Under isothermal condition two soap bubbles of radii r_1 and r_2 coalesce to form a single bubble of radius r. The external pressure is p_0. Find the surface tension of the soap in terms of the given parameters.*

Solution As mass of the air is conserved,

$\therefore$ $$n_1 + n_2 = n \quad \text{(as } pV = nRT\text{)}$$

$\therefore$ $$\frac{p_1V_1}{RT_1} + \frac{p_2V_2}{RT_2} = \frac{pV}{RT}$$

As temperature is constant,

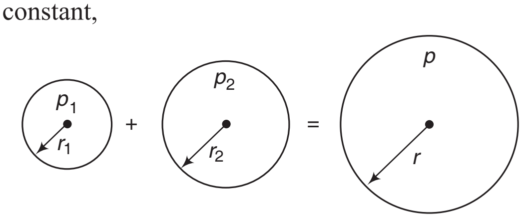


Fig. 16.78

$$T_1 = T_2 = T$$

$\therefore$ $$p_1V_1 + p_2V_2 = pV$$

$\therefore$ $$\left(p_0 + \frac{4S}{r_1}\right)\left(\frac{4}{3}\pi r_1^3\right) + \left(p_0 + \frac{4S}{r_2}\right)\left(\frac{4}{3}\pi r_2^3\right) = \left(p_0 + \frac{4S}{r}\right)\left(\frac{4}{3}\pi r^3\right)$$

Solving, this we get

$$S = \frac{p_0(r^3 - r_1^3 - r_2^3)}{4(r_1^2 + r_2^2 - r^2)}$$ **Ans.**

Note *To avoid confusion with the temperature, surface tension here is represented by S.*

Example 16.29 *Assume that a drop of liquid evaporates by decrease in its surface energy, so that its temperature remains unchanged. What should be the minimum radius of the drop for this to be possible? The surface tension is T, density of liquid is ρ and L is its latent heat of vaporization* **(2013 Main)**

(a) $\frac{\rho L}{T}$ (b) $\sqrt{\frac{T}{\rho L}}$ (c) $\frac{T}{\rho L}$ (d) $\frac{2T}{\rho L}$

Solution Decrease in surface energy = heat required in vaporization.

$\therefore$ $$T(dS) = L(dm)$$

$\therefore$ $$T(2)(4\pi r)\,dr = L(4\pi r^2\,dr)\rho$$

$\therefore$ $$r = \frac{2T}{\rho L}$$

The correct options is (d).

INTRODUCTORY EXERCISE 16.7

1. A mercury drop of radius 1 cm is sprayed into 10^6 droplets of equal size. Calculate the increase in surface energy if surface tension of mercury is 35×10^{-3} N/m.
2. A water film is made between two straight parallel wires of length 10 cm each and at a distance of 0.5 cm from each other. If the distance between the wires is increased by 1 mm, how much work will be done? Surface tension of water $= 7.2 \times 10^{-2}$ N/m
3. A soap bubble of radius R has been formed at normal temperature and pressure under isothermal conditions. Compute the work done. The surface tension of soap solution is T.
4. A small air bubble of radius 'r' is at a depth 'h' below the water surface (density of water $= \rho$). Surface tension of water is T, atmospheric pressure is p_0. Find pressure inside the air bubble for the condition $r << h$.

16.10 Capillary Rise or Fall

Cohesive and Adhesive Forces

The force of attraction between the molecules of the same substance is called cohesion.

In case of solids, the force of cohesion is very large and due to this solids have definite shape and size. On the other hand, the force of cohesion in case of liquids is weaker than that of solids. Hence, liquids do not have definite shape but have definite volume. The force of cohesion is negligible in case of gases. Because of this fact, gases have neither fixed shape nor volume.

Examples

(i) Two drops of liquid coalesce into one when brought in mutual contact because of the cohesive force.

(ii) It is difficult to separate two sticky plates of glass wetted with water because a large force has to be applied against the cohesive force between the molecules of water.

(iii) It is very difficult to break a drop of mercury into small droplets because of large cohesive force between mercury molecules.

The force of attraction between molecules of different substance is called adhesion.

Examples

(i) Adhesive force enables us to write on the black board with a chalk.

(ii) Due to force of adhesion, water wets the glass plate.

(iii) Fevicol and gum are used in gluing two surfaces together because of adhesive force.

Contact Angle

When a liquid surface touches a solid surface, the shape of the liquid surface near the contact is generally curved. When a glass plate is immersed in water, the surface near the plate becomes concave. On the other hand, if glass plate is immersed in mercury, the surface is depressed near the plate.

The angle between the tangent planes at the solid surface and the liquid at the contact is called the contact angle. In this case the tangent plane to the solid surface is to be drawn towards the liquid and the tangent plane to the liquid is to be drawn away from the solid.

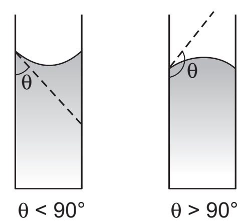


Fig. 16.79

Those liquids which wet the walls of the container (say in case of water and glass) have meniscus concave upwards and their values of angle of contact is less than 90° (also called acute angle). However, those liquids which don't wet the walls of the container (say in case of mercury and glass) have meniscus convex upwards and their value of angle of contact is greater than 90° (also called obtuse angle). The angle of contact of mercury with glass is about 140°, whereas the angle of contact of water with glass is about 8°. But, for pure water, the angle of contact θ with glass is taken as 0°.

Shape of Liquid Meniscus

When the adhesive force (P) between solid and liquid molecules is more than the cohesive force (Q) between liquid-liquid molecules (as with water and glass), shape of the meniscus is concave and the angle of contact θ is less than 90°. In this case, the liquid wets or adheres to the solid surface. The resultant (R) of P and Q passes through the solid.

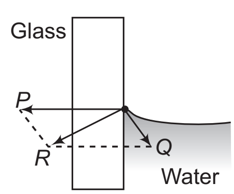


Fig. 16.80

On the other hand when $P < Q$ (as with glass and mercury), shape of the meniscus is convex and the angle of contact $\theta > 90°$. The resultant (R) of P and Q in this case passes through the liquid.

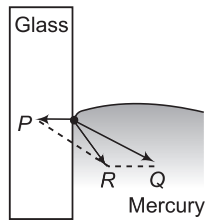


Fig. 16.81

Let us now see why the liquid surface bends near the contact with a solid. A liquid in equilibrium can not sustain tangential stress. The resultant force on any small part of the surface layer must be perpendicular to the surface at that point. Basically three forces are acting on a small part of the liquid surface near its contact with solid. These forces are,

(i) P, attraction due to the molecule of the solid surface near it

(ii) Q, attraction due to liquid molecules near this part and

(iii) The weight w of the part considered.

We have considered very small part, so weight of that part can be ignored for better understanding. As we have seen in the last figures, to make the resultant (R) of P and Q perpendicular to the liquid surface the surface becomes curved (convex or concave).

Capillarity

Surface tension causes elevation or depression of the liquid in a narrow tube. This effect is called capillarity. When a glass capillary tube (A tube of very small diameter is called a capillary tube) open at both ends is dipped vertically in water the water in the tube will rise above the level of water in the vessel as shown in figure (a). In case of mercury, the liquid is depressed in the tube below the level of mercury in the vessel as shown in figure (b).

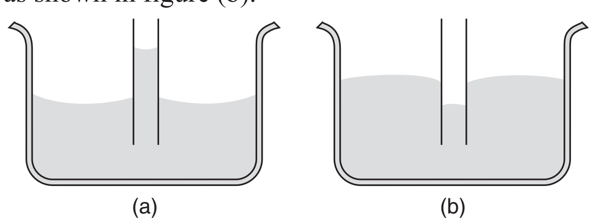


Fig. 16.82

When the contact angle is less than 90° the liquid rises in the tube. For a non-wetting liquid angle of contact is greater than 90° and the surface is depressed, pulled down by the surface tension forces.

Explanation

When a capillary tube is dipped in water, the water meniscus inside the tube is concave. The pressure just below the meniscus is less than the pressure just above it by $\dfrac{2T}{R}$, where T is the surface tension of water and R is the radius of curvature of the meniscus. The pressure on the surface of water is p_0, the atmospheric pressure. The pressure just below the plane surface of water outside the tube is also

p_0, but that just below the meniscus inside the tube is $p_0 - \frac{2T}{R}$. We know that pressure at all points in the same level of water must be the same. Therefore, to make up the deficiency of pressure $\frac{2T}{R}$ below the meniscus water begins to flow from outside into the tube. The rising of water in the capillary stops at a certain height h. In this position, the pressure of water column of height h becomes equal to $\frac{2T}{R}$, i.e.

$$h\rho g = \frac{2T}{R}$$

or

$$h = \frac{2T}{R\rho g}$$

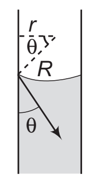


Fig. 16.83

If r is the radius of the capillary tube and θ the angle of contact, then

$$R = \frac{r}{\cos\theta}$$

$$\therefore \qquad \boxed{h = \frac{2T\cos\theta}{r\rho g}}$$

Alternative Proof for the Formula of Capillary Rise

As we have already seen, when the contact angle is less than 90°, the total surface tension force just balances the extra weight of the liquid in the tube.

The water meniscus in the tube is along a circle of circumference $2\pi r$ which is in contact with the glass. Due to the surface tension of water, a force equal to T per unit length acts at all points of the circle. If the angle of contact is θ, then this force is directed inward at an angle θ from the wall of the tube.

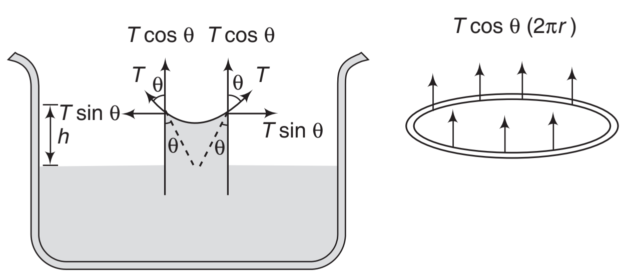


Fig. 16.84

In accordance with Newton's third law, the tube exerts an equal and opposite force T per unit length on the circumference of the water meniscus. This force which is directed outward, can be resolved into two components $T\cos\theta$ per unit length acting vertically upward and $T\sin\theta$ per unit length acting horizontally outward. Considering the entire circumference $2\pi r$, for each horizontal component $T\sin\theta$ there is an equal and opposite component and the two neutralise each other. The vertical components being in the same direction are added up to give a total upward force $(2\pi r)(T\cos\theta)$. It is this force which supports the weight of the water column so raised.

Thus,

$$(T\cos\theta)(2\pi r) = \text{Weight of the liquid column.}$$
$$= (\pi r^2 \rho g h) \qquad \ldots\text{(i)}$$
$$\therefore \qquad h = \frac{2T\cos\theta}{r\rho g}$$

The result has following notable features :

(i) If the contact angle θ is greater than 90°, the term $\cos\theta$ is negative and hence, h is negative. The expression then gives the depression of the liquid in the tube.

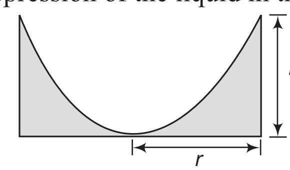


Fig. 16.85

(ii) The **correction** due to weight of the liquid contained in the meniscus can be made for contact angle $\theta = 0°$. The meniscus is then hemispherical. The volume of the shaded part is

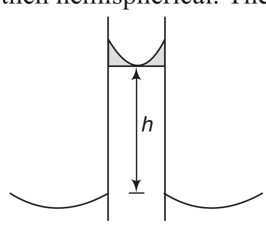


Fig. 16.86

$$V = (\pi r^2)(r) - \frac{1}{2}\left(\frac{4}{3}\pi r^3\right) = \frac{1}{3}\pi r^3$$

The weight of this shaded part is $\frac{1}{3}\pi r^3 \rho g$.

Therefore, we can write Eq. (i) as,

$$(T\cos 0°)(2\pi r) = \pi r^2 \rho g h + \frac{1}{3}\pi r^3 \rho g$$

or
$$\boxed{h = \frac{2T}{r\rho g} - \frac{r}{3}}$$

(iii) Suppose a capillary tube is held vertically in a liquid which has a concave meniscus, then capillary rise is given by,

$$h = \frac{2T\cos\theta}{r\rho g} = \frac{2T}{R\rho g} \qquad \left(\text{as } R = \frac{r}{\cos\theta}\right)$$

or
$$hR = \frac{2T}{\rho g} \qquad \ldots\text{(ii)}$$

When the length of the tube is greater than h, the liquid rises in the tube, so as to satisfy the above relation. But if the length of the tube is insufficient (i.e. less than h) say h', the liquid does not emerge in the form of a fountain from the upper end (because it will violate the law of conservation of energy) but the angle made by the liquid surface and hence, the R changes in such a way that the force $2\pi rT\cos\theta$ equals the weight of the liquid raised. Thus,

$$2\pi rT\cos\theta' = \pi r^2\rho gh'$$

$$h' = \frac{2T\cos\theta'}{r\rho g}$$

or $$h' = \frac{2T}{R'\rho g}$$

or $$h'R' = \frac{2T}{\rho g} \qquad \text{...(iii)}$$

From Eqs. (ii) and (iii), we have $$hR = h'R' = \frac{2T}{\rho g}$$

Practical Applications of Capillarity

1. The oil in a lamp rises in the wick by capillary action.
2. The tip of nib of a pen is split up, to make a narrow capillary so that the ink rises upto the tin or nib continuously.
3. If one end of the towel dips into a bucket of water and the other end hangs over the bucket, the towel soon becomes wet throughout, due to capillary action.
4. Ink is absorbed by the blotter due to capillary action.

Example 16.30 *A capillary tube whose inside radius is 0.5 mm is dipped in water having surface tension 7.0×10^{-2} N/m. To what height is the water raised above the normal water level? Angle of contact of water with glass is 0°. Density of water is 10^3 kg/m³ and $g = 9.8$ m/s².*

Solution $$h = \frac{2T\cos\theta}{r\rho g}$$

Substituting the proper values, we have

$$h = \frac{(2)(7.0\times10^{-2})\cos 0^\circ}{(0.5\times10^{-3})(10^3)(9.8)}$$

$$= 2.86\times10^{-2}\ \text{m} = 2.86\ \text{cm}$$ **Ans.**

Example 16.31 *A glass tube of radius 0.4 mm is dipped vertically in water. Find upto what height the water will rise in the capillary ? If the tube in inclined at an angle of 60° with the vertical, how much length of the capillary is occupied by water ? Surface tension of water = 7.0×10^{-2} N/m, density of water = 10^3 kg/m³.*

Solution For glass-water, angle of contact $\theta = 0°$.

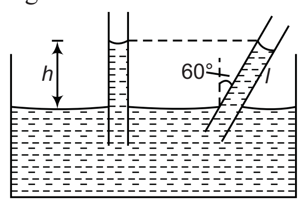


Fig. 16.87

Now, $$h = \frac{2T\cos\theta}{r\rho g}$$

$$= \frac{(2)(7.0\times 10^{-2})\cos 0°}{(0.4\times 10^{-3})(10^3)(9.8)}$$

$$= 3.57\times 10^{-2}\ \text{m} = 3.57\,\text{cm}$$ **Ans.**

$$l = \frac{h}{\cos 60°} = \frac{3.57}{\frac{1}{2}} = 7.14\,\text{cm}$$ **Ans.**

Example 16.32 *Mercury has an angle of contact of 120° with glass. A narrow tube of radius 1.0 mm made of this glass is dipped in a trough containing mercury. By what amount does the mercury dip down in the tube relative to the liquid surface outside. Surface tension of mercury at the temperature of the experiment is 0.5 N/m and density of mercury is 13.6×10^3 kg/m^3. (Take $g = 9.8\ m/s^2$).*

Solution $$h = \frac{2T\cos\theta}{r\rho g}$$

Substituting the values, we get $$h = \frac{2\times 0.5\times \cos 120°}{10^{-3}\times 13.6\times 10^3\times 9.8}$$

$$= -\,3.75\times 10^{-3}\ \text{m}$$

or $$h = -\,3.75\ \text{mm}$$ **Ans.**

Note *Here, negative sign implies that mercury suffers capillary depression.*

Example 16.33 *If a 5 cm long capillary tube with 0.1 mm internal diameter open at both ends is slightly dipped in water having surface tension 75 dyne cm^{-1}, state whether (a) water will rise half way in the capillary, (b) Water will rise up to the upper end of capillary and (c) water will overflow out of the upper end of capillary? Explain your answer.*

Solution Given that surface tension of water, $T = 75$ dyne/cm

Radius $r = \frac{0.1}{2}\ \text{mm} = 0.05\ \text{mm} = 0.005\ \text{cm}$

Density $\rho = 1\ \text{gm/cm}^3$, angle of contact, $\theta = 0°$

Let h be the height to which water rises in the capillary tube. Then

$$h = \frac{2T\cos\theta}{r\rho g} = \frac{2 \times 75 \times \cos 0°}{0.005 \times 1 \times 981}\ \text{cm}$$

$$= 30.58\ \text{cm}$$

But length of capillary tube is $h' = 5\,\text{cm}$

(a) Because $h > \frac{h'}{2}$ therefore the first possibility does not exist.

(b) Because the tube is of insufficient length therefore the water will rise upto the upper end of the tube.

(c) The water will not overflow of the upper end of the capillary. It will rise only up to the upper end of the capillary.

The liquid meniscus will adjust its radius of curvature R' in such a way that

$$R'h' = Rh \qquad \left[\because hR = \frac{2T}{\rho g} = \text{constant}\right]$$

where, R is the radius of curvature that the liquid meniscus would possess if the capillary tube were of sufficient length.

$$\therefore \qquad R' = \frac{Rh}{h'} = \frac{rh}{h'} = \frac{0.005 \times 30.58}{5} = 0.0306\,\text{cm} \qquad \left[\because R = \frac{r}{\cos\theta} = \frac{r}{\cos 0°} = r\right]$$

INTRODUCTORY EXERCISE 16.8

1. Water rises in a capillary tube to a height of 2.0 cm. In another capillary tube whose radius is one third of it, how much the water will rise?
2. Water rises up in a glass capillary upto a height of 9.0 cm, while mercury falls down by 3.4 cm in the same capillary. Assume angles of contact for water glass and mercury glass 0° and 135° respectively. Determine the ratio of surface tension of mercury and water ($\cos 135° = -0.71$).
3. A tube of insufficient length is immersed in water (surface tension $= 0.7$ N/m) with 1 cm of it projecting vertically upwards outside the water. What is the radius of meniscus ?
   Given, radius of tube $= 1\,\text{mm}$.
4. A capillary tube is dipped in a liquid. Let pressures at points A, B and C be p_A, p_B and p_C respectively, then

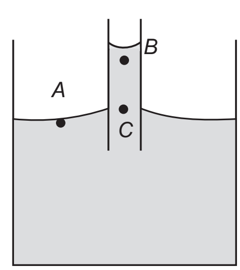


Fig. 16.88

(a) $p_A = p_B = p_C$ (b) $p_A = p_B < p_C$ (c) $p_A = p_C < p_B$ (d) $p_A = p_C > p_B$

Final Touch Points

1. Application on Surface Tension

(i) When the detergent materials are added to liquids, the angle of contact decreases and hence the wettability increases. On the other hand, when water proofing material is added to a fabric it increases the angle of contact, making the fabric water-repellant.

(ii) Surface tension of all lubricating oils and paints is kept low so that they spread over a large area.

(iii) Oil spreads over the surface of water because the surface tension of oil is less than the surface tension of cold water.

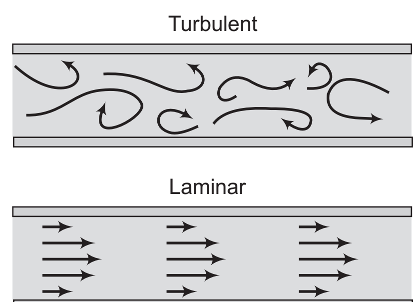
2. Effect of Temperature and Impurities on Surface Tension

The surface tension of a liquid decreases with the rise in temperature and *vice-versa*. Surface tension becomes zero at a critical temperature. It is for this reason that hot soup tastes better. Surface tension of a liquid changes appreciably with addition of impurities. For example, surface tension of water increases with addition of highly soluble substances like Nacl, $ZnSO_4$ etc. On the other hand surface tension of water gets reduced with addition of sparingly soluble substances like phenol, soap etc.

3. Laminar and Turbulent Flow, Reynolds Number

When a liquid flowing in a pipe is observed carefully, it will be seen that the pattern of flow becomes more disturbed as the velocity of flow increases. Perhaps this phenomenon is more commonly seen in a river or stream. When the flow is slow the pattern is smooth, but when the flow is fast, eddies develop and swirl in all directions.

At the low velocities, flow is calm. This is called "laminar flow".



In a series of experiments, **Reynolds** showed this by injecting a thin stream of dye into the fluid and finding that it ran in a smooth stream in the direction of the flow at low speeds. As the velocity of flow increased, he found that the smooth line of dye was broken up, at high velocities, the dye was rapidly mixed into the disturbed flow of the surrounding fluid. This is called **"turbulent flow"**.

After many experiments **Reynolds** saw that the expression

$$\frac{\rho u d}{\eta}$$

where, ρ = density, u = mean velocity, d = diameter and η = viscosity would help in predicting the change in flow type. If the value is less than about 2000 then flow is laminar, if greater than 4000 then turbulent and in between these two in the transition zone.

This value is known as the Reynolds number, R_e.

$$R_e = \frac{\rho u d}{\eta}$$

Laminar flow: $R_e < 2000$, Transitional flow: $2000 < R_e < 4000$, Turbulent flow: $R_e > 4000$

SI Units of Reynolds Number

$$\rho = \text{kg/m}^3,\ u = \text{m/s},\ d = m,\ \eta = \text{Ns/m}^2 = \text{kg/ms}$$

$$R_e = \frac{\rho u d}{\eta} = \frac{(\text{kg/m}^3)(\text{m/s})(\text{m})}{(\text{kg/ms})} = 1$$

i.e. it has no units. A quantity that has no units is known as a non-dimensional (or dimensionless) quantity. Thus, the Reynolds number, R_e is a non-dimensional number.

Solved Examples

TYPED PROBLEMS

Type 1. *Based on Law of Floatation.*

Concept

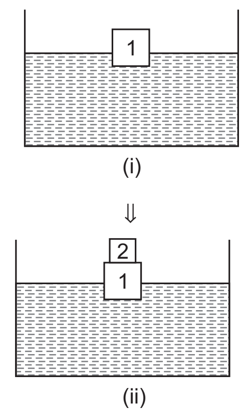
Whenever a block floats in a liquid (density of block should be less than density of liquid), there is one single equation.

Weight of solid = Upthrust on solid

$\therefore \qquad V\rho_s g = V_i \rho_l g$

Here, V is total volume of solid and V_i is immersed volume of solid.

$\therefore \qquad \boxed{\dfrac{V_i}{V} = f_i = \dfrac{\rho_s}{\rho_l}}$



Here, f_1 is immersed fraction of volume of solid.

Further, in figure (i) immersed volume is less and in figure (ii) immersed volume is more. Upthrust on immersed volume in figure (i) is equal to weight of block-1 and upthrust on immersed volume in figure (ii) is equal to weight of both the blocks 1 and 2. In other words, we can also say that upthrust on extra immersed volume in figure (ii) is equal to extra weight. In equation form, we can write as

$$m_1 g = (V_i)_1 \rho_l g \quad \text{and} \quad (m_1 + m_2) g = (V_i)_2 \rho_l g$$

or $$m_2 g = [(V_i)_2 - (V_i)_1] \rho_l g \quad \text{or} \quad (\Delta m) g \quad \text{or} \quad \Delta w = (\Delta V_i) \rho_l g$$

Note *If fluid is accelerated, then in the expression of upthrust g is replaced be g_{eff}.*

Example 1 *A block of wood floats in a bucket of water placed in a lift. Will the block sink more or less if the lift starts accelerating up?*

Solution Under nromal conditions, fraction of volume immersed under floating condition is

$$f_1 = \frac{\rho_s}{\rho_l} \qquad \ldots\text{(i)}$$

If the lift starts accelerating up, then

$$\text{upthrust} - \text{weight} = ma$$

$\therefore \qquad V_i \rho_l (g + a) - V\rho_s g = V\rho_s a$

Solving this equation, we get

$$\frac{V_i}{V} = f_2 = \frac{\rho_s}{\rho_l} \qquad \ldots\text{(ii)}$$

$$f_1 = f_2$$

So, the block neither sinks less nor more.

Note *From this example, we can make a general concept that by the acceleration of container, fraction (or percentage of volume immersed) does not change.*

Example 2 *A raft of wood (density $=600 kg/m^3$) of mass 120 kg floats in water. How much weight can be put on the raft to make it just sink?*

Solution Weight of raft + external weight = upthrust on 100% volume of raft

$$\therefore \qquad (120 + m)\, g = \left(\frac{120}{600}\right) \times 10^3 \times g$$

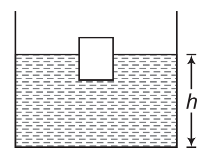
$$\therefore \qquad m = 80 \text{ kg}$$

Ans.

Note *Here immersed fraction used in upthrust is the total volume of solid* $\left(= \frac{m}{\rho} = \frac{120}{600}\, m^3\right)$.

Type 2. *Level Problems.*

Concept



Let us first note down the following four results:

(i) If ice is floating in water, then after melting of ice the level 'h' remains unchanged.

(ii) In floating to floating condition level 'h' remains unchanged.

For example Suppose some wooden blocks are floating in water kept inside a boat. If these wooden blocks are thrown into the water, then level will remain unchanged. This is because under both the conditions wooden blocks are floating.

(iii) In the floating to sink condition, level 'h' decreases.

For example In the above case if wooden blocks are replaced by stone pieces then level will fall. Because, initially stone pieces were floating (with the help of boat) but eventually those pieces will sink.

(iv) A solid is floating in a liquid of density ρ_1. After sometime, the solid melts and density (of solid) after melting (in liquid state) is suppose ρ_2, then there are following three cases :

Case 1 If $\rho_2 = \rho_1$, then level remains unchanged. Note that this is also result (i).

Case 2 If $\rho_2 < \rho_1$, then level will increase.

Case 3 If $\rho_2 > \rho_1$, then level will decrease.

Example 3 *A piece of ice is floating in a glass vessel filled with water. Then prove that level of water in the vessel remains unchanged after melting of ice .*

Note *This is result (i).*

Solution Let m be the mass of ice piece floating in water.

In equilibrium, weight of ice piece = upthrust

$$\text{or} \qquad mg = V_i \rho_w g$$

$$\text{or} \qquad V_i = \frac{m}{\rho_w} \qquad \text{...(i)}$$

Here, V_i is the volume of ice piece immersed in water.

When the ice melts, let V be the volume of water formed by m mass of ice. Then,

$$V = \frac{m}{\rho_w} \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we see that

$$V_i = V$$

Hence, the level will not change. **Ans.**

Example 4 *A piece of ice having a stone frozen in it floats in a glass vessel filled with water. How will the level of water in the vessel change when the ice melts?*

Note *This is result (iii).*

Solution Let, m_1 = mass of ice, m_2 = mass of stone

ρ_S = density of stone and ρ_w = density of water

In equilibrium, when the piece of ice floats in water,

weight of (ice + stone) = upthrust

or $$(m_1 + m_2)g = V_i \rho_w g$$

$\therefore$ $$V_i = \frac{m_1}{\rho_w} + \frac{m_2}{\rho_w} \qquad \text{...(i)}$$

Here, V_i = Volume of ice immersed

When the ice melts, m_1 mass of ice converts into water and stone of mass m_2 is completely submerged.

Volume of water formed by m_1 mass of ice,

$$V_1 = \frac{m_1}{\rho_w}$$

Volume of stone (which is also equal to the volume of water displaced)

$$V_2 = \frac{m_2}{\rho_S}$$

Since, $\rho_S > \rho_w$

Therefore, $V_1 + V_2 < V_i$

or the level of water will decrease. **Ans.**

Example 5 *A solid floats in a liquid of different material. Carry out an analysis to see whether the level of liquid in the container will rise or fall when the solid melts.*

Note *This is result (iv).*

Solution Let M = Mass of the floating solid

ρ_2 = density of liquid formed by the melting of the solid

ρ_1 = density of the liquid in which the solid is floating

The mass of liquid displaced by the solid is M. Hence, the volume of liquid displaced is $\frac{M}{\rho_1}$.

When the solid melts, the volume occupied by it is $\frac{M}{\rho_2}$. Hence, the level of liquid in container will rise or fall according as

$$\frac{M}{\rho_2} > \text{ or } < \frac{M}{\rho_1} \quad \text{i.e.} \quad \rho_2 < \text{ or } > \rho_1$$

There will be no change in the level if $\rho_1 = \rho_2$. In case of ice, floating in water $\rho_1 = \rho_2$ and hence, the level of water remains unchanged when ice melts.

Type 3. *Based on apparent weight or change in weight of a solid inside a liquid.*

Concept

If density of a solid is greater than density of liquid then it cannot float, it will sink. In this case, we hold the solid inside the liquid. Upthrust acts on its 100% volume and the solid is felt lighter. The equations are as under.

$w_{\text{app}} = w_{\text{actual}}$ – Upthrust on 100% volume or change in weight $\Delta w =$ Upthrust $= V\rho_l g$ ...(i)

From Eq. (i) we come across following two results:

(i) If the liquid in which solid is immersed, is water, then

$$\frac{\text{Weight in air}}{\text{Decrease in weight}} = \text{Relative density of body (RD)}$$

This can be shown as under

$$\frac{\text{Weight in air}}{\text{Decrease in weight}} = \frac{\text{Weight in air}}{\text{Upthrust in water}} = \frac{V\rho_S g}{V\rho_w g}$$

$$= \frac{\rho_S}{\rho_w} = \text{RD}$$

Hence proved.

(ii) Change in weight Δw is directly proportional to density of liquid or relative density of liquid. Thus,

$$\Delta w \propto \rho_l \text{ or } (\text{RD})_l \quad \Rightarrow \quad \frac{\Delta w_1}{\Delta w_2} = \frac{(\rho_l)_1}{(\rho_l)_2} = \frac{(\text{RD})_{l_1}}{(\text{RD})_{l_2}}$$

Note *In this case, 100% volume of solid remains immersed in the liquid, So V and g are same in two liquids in Eq. (i).*

Example 6 *A metallic sphere weighs 210 g in air, 180 g in water and 120 g in an unknown liquid. Find the density of metal and of liquid.*

Solution Relative density of metal

$$= \frac{\text{weight in air}}{\text{change in weight of water}} = \frac{210}{210 - 180} = 7$$

$\therefore$ Density of metal = 7 g/cm^2

Change in weight in a liquid = upthrust in liquid

$$= (V_{\text{solid}})(\rho_{\text{liquid}})\,g$$

or $$\Delta w \propto \rho_{\text{liquid}}$$

$\therefore$ $$\frac{\Delta w_l}{\Delta w_w} = \frac{\rho_l}{\rho_w}$$

or $$\rho_l = \frac{\Delta w_l}{\Delta w_w}\rho_w$$

$$= \left(\frac{210-120}{210-180}\right)(1)\ \text{gm/cm}^3$$

$$= 3\ \text{g/cm}^3 \qquad \textbf{Ans.}$$

Example 7 *An ornament weighing 50 g in air weighs only 46 g is water. Assuming that some copper is mixed with gold to prepare the ornament. Find the amount of copper in it. Specific gravity of gold is 20 and that of copper is 10.*

Solution Let m be the mass of the copper in ornament. Then, mass of gold in it is $(50 - m)$.

Volume of copper $$V_1 = \frac{m}{10} \qquad \left(\text{volume} = \frac{\text{mass}}{\text{density}}\right)$$

and volume of gold $$V_2 = \frac{50-m}{20}$$

When immersed in water ($\rho_w = 1\ \text{g/cm}^3$)

Decrease in weight = upthrust

$\therefore$ $$(50-46)g = (V_1 + V_2)\rho_w g$$

or $$4 = \frac{m}{10} + \frac{50-m}{20} \quad \text{or} \quad 80 = 2m + 50 - m$$

$$m = 30\ \text{g} \qquad \textbf{Ans.}$$

Example 8 *An iron casting containing a number of cavities weighs 6000 N in air and 4000 N in water. What is the volume of the cavities in the casting? Density of iron is $7.87\ g/cm^3$.*
Take $g = 9.8\ m/s^2$ and density of water $= 10^3\ kg/m^3$.

Solution Let v be the volume of cavities and V the volume of solid iron. Then,

$$V = \frac{\text{mass}}{\text{density}} = \left(\frac{6000/9.8}{7.87\times10^3}\right) = 0.078\ \text{m}^3$$

Further, decrease in weight = upthrust

$\therefore$ $$(6000-4000) = (V+v)\rho_w g$$

or $$2000 = (0.078 + v)\times10^3\times9.8$$

or $$0.078 + v \approx 0.2$$

$\therefore$ $$v = 0.12\ \text{m}^3 \qquad \textbf{Ans.}$$

Type 4. *Based on pressure equation.*

Concept

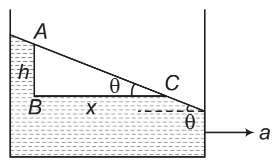
(i) If we wish to find pressure difference between two points A and B, then start from A, write pressure p_A and reach upto point B. Keep on writing increase or decrease in pressure and finally write pressure equal to p_B.

(ii) If fluids are at rest then there is no change in pressure in horizontal direction. In vertical direction write $(+\rho gh)$ while moving in downward direction and $(-\rho gh)$ while moving in upward direction (ρ = density of liquid).

(iii) If the fluid is accelerated, write $(+\rho ax)$ in horizontal direction opposite to horizontal acceleration 'a' and $(-\rho ax)$ in the same direction of acceleration. Here, x is the horizontal distance. In vertical direction, write $(\pm \rho g_e h)$. Here, g_e is the effective value of acceleration due to gravity.

(iv) If the fluid is rotating, write $\left(+\rho \frac{\omega^2 x^2}{2}\right)$ while moving away from the axis and $\left(-\frac{\rho\omega^2 x^2}{2}\right)$ in moving towards the axis.

Example 9 *A liquid kept in a container has a horizontal acceleration 'a' as shown in figure.*



Using the pressure equation along the path ABC find the angle θ.

Solution Writing pressure equation along the path ABC we have

$$p_A + \rho gh - \rho ax = p_C \qquad (\rho = \text{density of liquid})$$

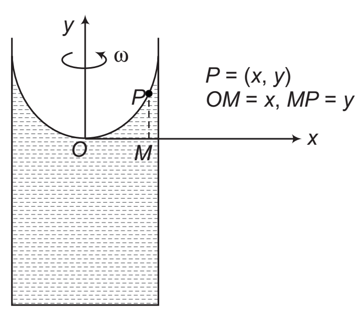
Points A and C are open to atmosphere.

$\therefore$ $$p_A = p_C = p_0 = \text{atmosphere pressure}$$

$\Rightarrow$ $$\rho gh - \rho ax = 0 \quad \text{or} \quad \frac{h}{x} = \frac{a}{g}$$

$\Rightarrow$ $$\tan\theta = \frac{a}{g} \quad \text{or} \quad \theta = \tan^{-1}\left(\frac{a}{g}\right)$$ **Ans.**

Example 10 *A liquid of density 'ρ' is rotated with an angular speed 'ω' as shown in figure. Using the pressure equation concept find the equation of free surface of the liquid.*



Solution Writing pressure equation along the path PMO.

$$p_P + \rho gy - \frac{\rho\omega^2 x^2}{2} = p_O$$

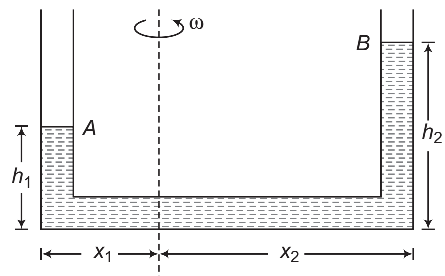
Points P and O are open to atmosphere.

So, pressures at these points are same (= atmospheric pressure). Therefore, the above equation becomes,

$$\rho gy - \frac{\rho\omega^2 x^2}{2} = 0 \quad \text{or} \quad y = \left(\frac{\omega^2}{2g}\right)x^2$$

This is the required parabolic equation of free surface of liquid.

Example 11



A liquid of density ρ is rotated by an angular speed ω as shown in figure, Using the concept of pressure equation, find a relation between h_1, h_2, x_1 and x_2.

Solution Writing pressure equation between points A and B, we have

$$p_A + \rho g h_1 - \frac{\rho\omega^2 x_1^2}{2} + \frac{\rho\omega^2 x_2^2}{2} - \rho g h_2 = p_B$$

Points A and B are open to atmosphere.

Therefore,

$$p_A = p_B = p_0 = \text{atmospheric pressure.}$$

Substituting the values in above equation we have,

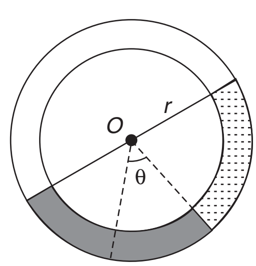
$$\rho g h_1 - \frac{\rho\omega^2 x_1^2}{2} + \frac{\rho\omega^2 x_2^2}{2} - \rho g h_2 = 0$$

On simplifying we get,

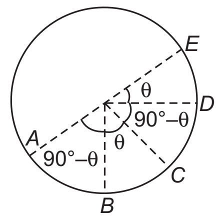
$$(h_2 - h_1) = \frac{\omega^2}{2g}\left(x_2^2 - x_1^2\right)$$

This is the desired relation between h_1, h_2, x_1 and x_2.

Example 12 *A small uniform tube is bent into a circle of radius r whose plane is vertical. Equal volumes of two fluids whose densities are ρ and $\sigma (\rho > \sigma)$ fill half the circle. Find the angle that the radius passing through the interface makes with the vertical.*



Solution $h_{AB} = r - r\cos(90° - \theta) = r - r\sin\theta$



$$h_{BC} = r - r\cos\theta$$
$$h_{CD} = r\sin(90° - \theta) = r\cos\theta$$
$$h_{DE} = r\sin\theta$$

Writing pressure equation between points A and E we have

$$p_A + (r - r\sin\theta)\rho g - (r - r\cos\theta)\rho g - (r\cos\theta)(\sigma)\, g - (r\sin\theta)\,\sigma\, g = p_E$$

But $\qquad p_A = p_E = p_{\text{gas}}$

Solving this equation, we get

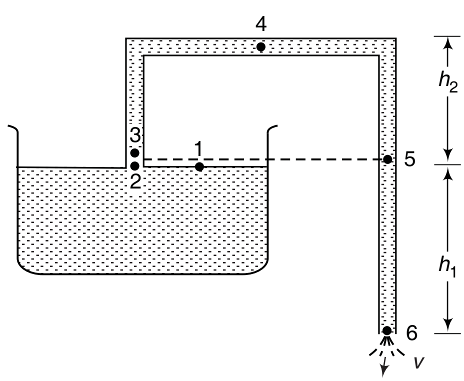
$$\tan\theta = \left(\frac{\rho - \sigma}{\rho - \sigma}\right)$$ **Ans.**

Type 5. *Based on concept of syphon.*

Concept

In the figure shown, $\qquad v_1 \approx v_2 = 0$

$$p_1 = p_2 = p_6 = p_0$$



Note *Point 3 is just above point 2. $v_2 = 0$ but $v_3 = v$*

If area of cross-section of pipe is uniform, then from continuity equation,

$$v_3 = v_4 = v_5 = v_6 = v \text{ (say)}$$

Applying Bernoulli's equation at 1 (or 2), 3, 4, 5 and 6, we have

$$p_0 + 0 + 0 = p_3 + \frac{1}{2}\rho v^2 + 0 = p_4 + \frac{1}{2}\rho v^2 + \rho g h_2$$
$$= p_5 + \frac{1}{2}\rho v^2 + 0 = p_0 + \frac{1}{2}\rho v^2 - \rho g h_1$$

From this equation, following conclusions can be made:

(i) $p_1 = p_2 = p_6 = p_0$

(ii) $p_3 = p_5 < p_0$

(iii) $v_1 = v_2 = 0$

(vi) $v_3 = v_4 = v_5 = v_6 = v$

(v) $v = \sqrt{2gh_1}$ so, h_1 should be greater than zero

(vi) $p_4 = p_0 - \rho g (h_1 + h_2)$

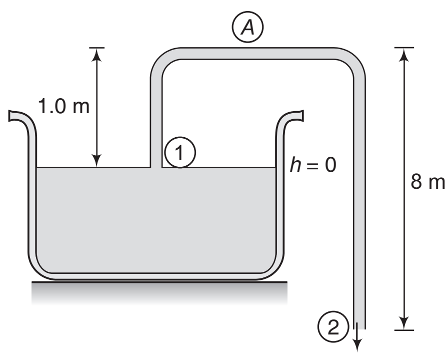
From the last equation we can see that p_4 decreases as $(h_1 + h_2)$ increases. Minimum value of pressure is at 4 or p_4 and this minimum value is zero and this will occur at,

$$0 = p_0 - \rho g (h_1 + h_2)_{\max}$$

Thus, $(h_1 + h_2)_{\max} = \dfrac{p_0}{\rho g}$ and simultaneously $h_1 > 0$

Thus, syphon will work when $h_1 > 0$ and $(h_1 + h_2) < \dfrac{p_0}{\rho g}$.

Example 13 *The U-tube acts as a water siphon. The bend in the tube is 1 m above the water surface. The tube outlet is 7 m below the water surface. The water issues from the bottom of the siphon as a free jet at atmospheric pressure. Determine the speed of the free jet and the minimum absolute pressure of the water in the bend. Given atmospheric pressure* $= 1.01 \times 10^5 \, N/m^2$, $g = 9.8 \, m/s^2$ *and density of water* $= 10^3 \, kg/m^3$.



Solution (a) Applying Bernoulli's equation between points (1) and (2)

$$p_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = p_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2$$

Since, area of reservoir >> area of pipe

$$v_1 \approx 0,$$

also $p_1 = p_2 =$ atmospheric pressure

So,
$$v_2 = \sqrt{2g(h_1 - h_2)}$$
$$= \sqrt{2 \times 9.8 \times 7}$$
$$= 11.7 \text{ m/s}$$

Ans.

(b) The minimum pressure in the bend will be at A. Therefore, applying Bernoulli's equation between (1) and (A)

$$p_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = p_A + \frac{1}{2}\rho v_A^2 + \rho g h_A$$

Again, $v_1 \approx 0$ and from continuity equation $v_A = v_2$

or $$p_A = p_1 + \rho g(h_1 - h_A) - \frac{1}{2}\rho v_2^2$$

Therefore, substituting the values, we have

$$p_A = (1.01 \times 10^5) + (1000)(9.8)(-1) - \frac{1}{2} \times (1000)(11.7)^2$$

$$= 2.27 \times 10^4 \text{ N/m}^2$$ **Ans.**

Type 6. *Based on pressure force and its torque.*

Concept

$$p = \frac{F}{A} \Rightarrow F = pA$$

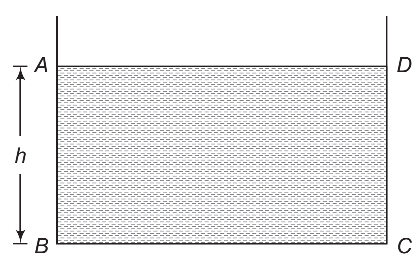
Let us call this force as the pressure force. Now, this force is calculated on a surface. If surface is horizontal then pressure is uniform at all points.

So, $F = pA$ can be applied directly. For calculation of torque, point of application of force is required. In the above case, point of application of force may be assumed at geometrical centre of the surface.

If the surface is vertical or inclined, pressure is non-uniform (it increases with depth) so pressure force and its torque can be obtained by integration. After finding force and torque by integration, we can also find point of application of this force by the relation.

$$r_\perp = \frac{\tau}{F} \qquad (\text{as } \tau = F \times r_1)$$

Example 14 *A liquid of density ρ is filled upto a height of 'h' in a container as shown in figure. Base of the container is a square of side L. Ignoring the atmospheric pressure find*



(a) pressure force F_1 on its base.

(b) torque of force F_1 about an axis passing through C and perpendicular to plane of paper.

(c) pressure force F_2 on the vertical side wall DC.

(d) torque of force F_2 about the same axis mentioned in part (b).

(e) point of application of force F_2.

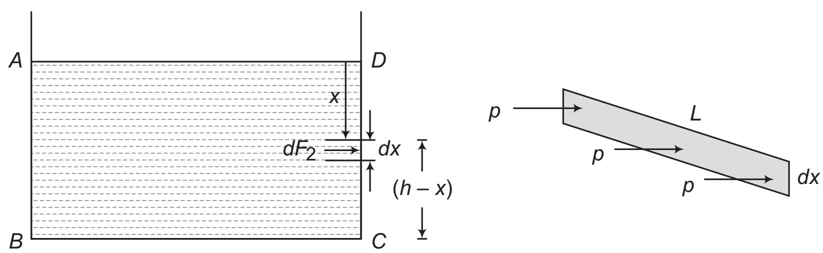
Solution (a) Base is horizontal. So, we can directly apply

$$F_1 = pA = (\rho\, gh)\,(L)\,(L)$$
$$= \rho\, ghL^2 \qquad \textbf{Ans.}$$

(b) Point of application of F_1 is at geometrical centre of base or at a perpendicular distance $\frac{L}{2}$ from the axis mentioned in the question.

$\therefore$
$$\tau_{F_1} = F_1 \times r_\perp = (\rho\, ghL^2)\left(\frac{L}{2}\right)$$
$$= \frac{1}{2}\rho\, gh\, L^3 \qquad \textbf{Ans.}$$

(c) and (d) : Side wall is vertical. Pressure is non-uniform. So, force and torque both will be obtained by integration.



Pressure at depth x,

$$p = \rho\, gx$$

Area of small element shown in figure is

$$dA = L\,(dx)$$

$\therefore$
$$dF_2 = (p)(dA) = (\rho\, gx)(L\, dx)$$

Perpendicular distance of this small force dF_2 from the axis mentioned in the question is $(h - x)$. Therefore, small torque of force dF_2 is

$$d\,\tau = (dF_2)\,(h - x)$$
$$= (\rho\, gx)(h - x)\,(L\, dx)$$

Now,
$$F_2 = \int_{x=0}^{x=h} dF_2$$

and
$$\tau_{F_2} = \int_{x=0}^{x=h} d\tau$$

Substituting the values and then integrating, we get

$$F_2 = \frac{\rho\, gLh^2}{2}$$

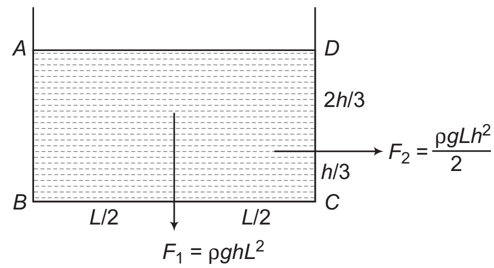
and
$$\tau_{F_2} = \frac{\rho\, gLh^3}{6}$$

(e) For point of application of F_2, we can apply

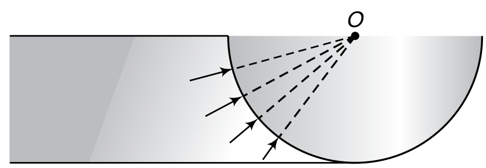
$$r_\perp = \frac{\tau_{F_2}}{F_2} \qquad \text{(from } C\text{)}$$
$$= \frac{(\rho\, gLh^3/6)}{(\rho\, gh^2L/2)} = \frac{h}{3}$$

Note that point of application of F_2 is below the centre, as pressure is not uniform. It is increasing with depth.

F_1 and F_2 and their points of application are shown in figure below.



Note



In the figure shown, torque of hydrostatic force about point O, the centre of a semicylindrical (or hemispherical) gate is zero as the hydrostatic force at all points passes through point O.

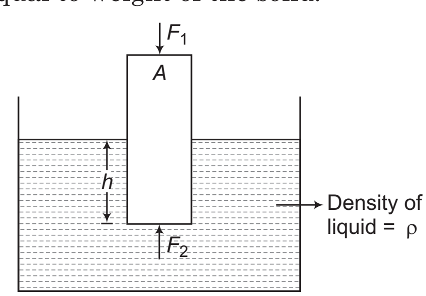
Type 7. *Based on the concept of pressure force and upthrust.*

Concept

(i) Pressure in a fluid (gas or a liquid) increases with depth. At a height (or depth) difference of h change in pressure is $\Delta p = \pm \rho g h$.

Here, ρ is the density of fluid. Now, density of air is almost negligible, so for small height differences it is almost constant $(= p_0)$. Therefore, value of atmospheric pressure is almost same everywhere. But this is not the case with a liquid, whose density is not negligible.

(ii) If a solid is floating in a liquid then net pressure force (including $p_0 A$) on this solid is upwards. This force is called upthrust and this is numerically equal to $V_i \rho_l g$ and in equilibrium this is equal to weight of the solid.

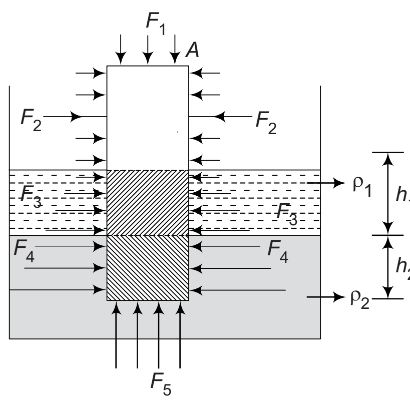


A plank of area of cross-section A is floating in a liquid of density ρ as shown in figure.

Net upward pressure force $= F_2 - F_1 = p_2 A - p_1 A$

$$= (p_0 + \rho gh) - p_0 A$$
$$= (hA)\rho g$$
$$= V_i \rho g \qquad (hA = \text{immersed volume})$$
$$= \text{upthrust}$$
$$= \text{weight of plank in equilibrium}$$

(iii) In some cases when a plank floats in two or more than two liquids then net force by a liquid on a plank is zero but immersed volume of plank on this liquid contributes in the upthrust.



A plank of area of cross-section A is floating in two liquids. h_1 height of the plank is immersed in first liquid and h_2 height in second liquid. Atmospheric pressures is constant at small height differences. So F_1 and F_2 forces shown in the figure can be obtained directly (= pressure × area). Inside the liquids pressure increases with depth. So, F_3 and F_4 will be obtained by integration as they are acting on vertical surfaces.

F_5 is acting on a horizontal surface so this force can also be obtained directly.

In the above figure

Net force on the plank by atmosphere $= F_1 = p_0 A$

Net force on the plank by liquid -1 = 0

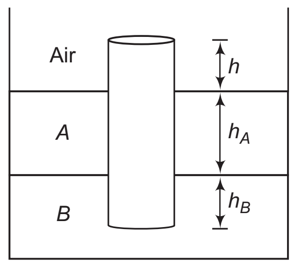
Net force on the plank by liquid - 2 $= F_5 = pA = (p_0 + \rho_1 gh_1 + \rho_2 gh_2) A$

Net upthrust pressure force on the plank $= F_5 - F_1$

$$= (h_1 A)\rho_1 g + (h_2 A)\rho_2 g$$
$$= (V_i)_1 \rho_1 g + (V_i)_2 \rho_2 g \qquad (V_i = \text{immersed volume})$$
$$= U_1 + U_2 \qquad (U = \text{upthrust})$$
$$= U_{\text{total}}$$
$$= \text{Weight of plank in equilibrium.}$$

In this case, we can see that net force on plank by liquid-1 is zero, but immersed volume of plank in liquid-1 contributes in the upthrust $(= U_1)$.

Example 15 *A uniform solid cylinder of density* $0.8\ g/cm^3$ *floats in equilibrium in a combination of two non-mixing liquids A and B with its axis vertical. The densities of the liquids A and B are* $0.7\ g/cm^3$ *and* $1.2\ g/cm^3$*, respectively. The height of liquid A is* $h_A = 1.2$ *cm. The length of the part of the cylinder immersed in liquid B is* $h_B = 0.8$ *cm.* **(JEE 2002)**

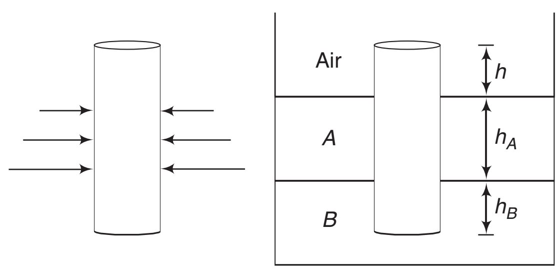


(a) *Find the total force exerted by liquid A on the cylinder.*

(b) *Find h, the length of the part of the cylinder in air.*

(c) *The cylinder is depressed in such a way that its top surface is just below the upper surface of liquid A and is then released. Find the acceleration of the cylinder immediately after it is released.*

Solution (a) Liquid A is applying the hydrostatic force on cylinder uniformly from all the sides. So, net force is zero.



(b) In equilibrium

Weight of cylinder = Net upthrust on the cylinder

Let s be the area of cross-section of the cylinder, then

$$\text{weight} = (s)\,(h + h_A + h_B)\,\rho_{\text{cylinder}}\, g$$

and upthrust on the cylinder

$$= \text{upthrust due to liquid } A + \text{ upthrust due to liquid } B$$

$$= sh_A\rho_A g + sh_B\rho_B g$$

Equating these two,

$$s\,(h + h_A + h_B)\,\rho_{\text{cylinder}}\, g = sg(h_A\rho_A + h_B\rho_B)$$

or

$$(h + h_A + h_B)\,\rho_{\text{cylinder}} = h_A\rho_A + h_B\rho_B$$

Substituting,

$$h_A = 1.2 \text{ cm}, h_B = 0.8 \text{ cm and } \rho_A = 0.7 \text{ g/cm}^3$$

$$\rho_B = 1.2 \text{ g/cm}^3 \text{ and } \rho_{\text{cylinder}} = 0.8 \text{ g/cm}^3$$

In the above equation, we get $h = 0.25$ cm **Ans.**

(c) Net upward force = extra upthrust = $sh\rho_B g$

$\therefore$ Net acceleration $a = \dfrac{\text{Force}}{\text{mass of cylinder}}$

or $$a = \frac{sh\rho_B g}{s(h + h_A + h_B)\rho_{\text{cylinder}}}$$

or $$a = \frac{h\rho_B g}{(h + h_A + h_B)\rho_{\text{cylinder}}}$$

Substituting the values of h, h_A h_B, ρ_B and ρ_{cylinder},

we get, $a = \dfrac{g}{6}$ (upwards)

Type 8. *Based on surface tension force.*

Concept

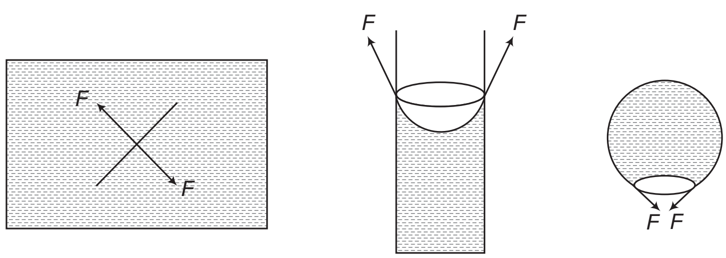
Surface tension is given by

$$T = \frac{F}{l}$$

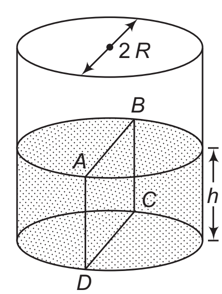
$\therefore$ Surface tension force, $F = Tl$

This force acts only on a line on free surface of the liquid. This force has a tendency to decrease the free surface of the liquid.

Following are given some figures for the direction of this force.



Example 16 Water is filled up to a height h in a beaker of radius R as shown in the figure. The density of water is ρ, the surface tension of water is T and the atmospheric pressure is p_0. Consider a vertical section $ABCD$ of the water column through a diameter of the beaker. The force on water on one side of this section by water on the other side of this section has magnitude **(JEE 2007)**



(a) $|2p_0Rh + \pi R^2\rho gh - 2RT|$

(b) $|2p_0Rh + R\rho gh^2 - 2RT|$

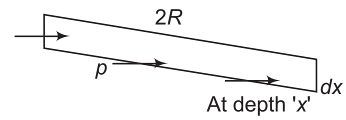
(c) $|p_0\pi R^2 + R\rho gh^2 - 2RT|$

(d) $|p_0\pi R^2 + R\rho gh^2 + 2RT|$

Solution Force from right hand side liquid on left hand side liquid.

(i) Due to surface tension force $= 2RT$ (towards right)

(ii) Due to liquid pressure force



$$= \int_{x=0}^{x=h} (p_0 + \rho g x)(2R \cdot dx)$$

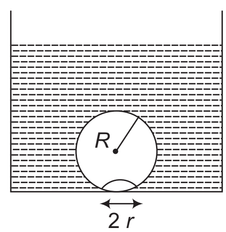
$$= (2p_0 Rh + R\rho g h^2) \quad \text{(towards left)}$$

$\therefore$ Net force is $|2p_0 Rh + R\rho g h^2 - 2RT|$

$\therefore$ The correct option is (b).

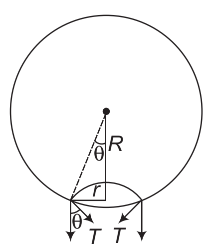
Example 17 *On heating water, bubbles beings formed at the bottom of the vessel detach and rise. Take the bubbles to be spheres of radius R and making a circular contact of radius r with the bottom of the vessel. If r <<R and the surface tension of water is T, value of r just before bubbles detach is (density of water is* ρ_w*)*

(JEE 2012 Main)



(a) $R^2\sqrt{\dfrac{\rho_w g}{3T}}$ (b) $R^2\sqrt{\dfrac{\rho_w g}{6T}}$ (c) $R^2\sqrt{\dfrac{\rho_w g}{T}}$ (d) $R^2\sqrt{\dfrac{3\rho_w g}{T}}$

Solution The bubble will detach if,



Buoyant force $\geq$ Surface tension force

$$\frac{4}{3}\pi R^3 \rho_w g \geq \int T \times dl \sin\theta$$

$$(\rho_w)\left(\frac{4}{3}\pi R^3\right) g \geq (T)(2\pi r)\sin\theta$$

Here, $\sin\theta = \dfrac{r}{R}$

Solving, $r = \sqrt{\dfrac{2\rho_w R^4 g}{3T}} = R^2\sqrt{\dfrac{2\rho_w g}{3T}}$

No option matches with the correct answer.

Passage (Ex. 18 to 20)

When liquid medicine of density ρ is to be put in the eye, it is done with the help of a dropper. As the bulb on the top of the dropper is pressed, a drop forms at the opening of the dropper. We wish to estimate the size of the drop.

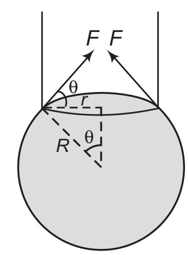
We first assume that the drop formed at the opening is spherical because that requires a minimum increase in its surface energy. To determine the size, we calculate the net vertical force due to the surface tension T when the radius of the drop is R. When this force becomes smaller than the weight of the drop, the drop gets detached from the dropper.

Example 18 *If the radius of the opening of the dropper is r, the vertical force due to the surface tension on the drop of radius R (assuming r << R) is*

(a) $2\pi rT$ (b) $2\pi RT$

(c) $\dfrac{2\pi r^2 T}{R}$ (d) $\dfrac{2\pi R^2 T}{r}$

Solution Vertical force due to surface tension,



$$F_v = F\sin\theta$$
$$= (T)(2\pi r)\,(r/R)$$
$$= \frac{2\pi r^2 T}{R}$$

∴ Correct option is (c).

Example 19 *If* $r = 5\times 10^{-4}\ m$, $\rho = 10^3\ kg\ m^{-3}$, $g = 10\ ms^{-2}$, $T = 0.11\ Nm^{-1}$, *the radius of the drop when it detaches from the dropper is approximately*

(a) $1.4\times 10^{-3}\ m$ (b) $3.3\times 10^{-3}\ m$

(c) $2.0\times 10^{-3}\ m$ (d) $4.1\times 10^{-3}\ m$

Solution $\dfrac{2\pi r^2 T}{R} = mg = \dfrac{4}{3}\pi R^3 \cdot \rho \cdot g$

∴ $$R^4 = \frac{3r^2 T}{2\rho g} = \frac{3\times(5\times 10^{-4})^2\,(0.11)}{2\times 10^3\times 10}$$
$$= 4.125\times 10^{-12}\ \text{m}^4$$

∴ $$R = 1.425\times 10^{-3}\ \text{m}$$
$$\approx 1.4\times 10^{-3}\ \text{m}$$

∴ Correct option is (a).

Example 20 *After the drop detaches, its surface energy is*

(a) $1.4\times 10^{-6}\ J$ (b) $2.7\times 10^{-6}\ J$ (c) $5.4\times 10^{-6}\ J$ (d) $8.1\times 10^{-9}\ J$

Solution Surface energy,

$$E = (4\pi R^2)\,T$$
$$= (4\pi)\,(1.4\times 10^{-3})^2\,(0.11)$$
$$= 2.7\times 10^{-6}\ \text{J}$$

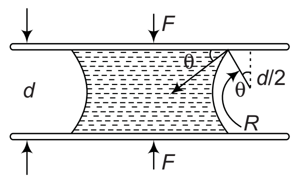
∴ Correct option is (b).

Type 9. *Based on force between two glass plates.*

Concept

When a small drop of water is placed between two glass plates put face to face, it forms a thin cylindrical film which is concave outward along its boundary. In article 16.9, we have seen that pressure difference on two sides of a cylindrical surface is

$$\Delta p = \frac{T}{R} \qquad \text{...(i)}$$



If d be the distance between the two plates and θ the angle of contact for water and glass, then, from the figure, $\cos\theta = \frac{\frac{1}{2}d}{R}$ or $\frac{1}{R} = \frac{2\cos\theta}{d}$.

Substituting for $\frac{1}{R}$ in Eq. (i), we get $\Delta p = \frac{2T}{d}\cos\theta$.

Angle θ can be taken zero for water and glass, i.e. $\cos\theta = 1$. Thus, the upper plate is pressed downward by the atmospheric pressure minus $\frac{2T}{d}$. Hence, the resultant downward pressure acting on the upper plate is $\frac{2T}{d}$. If A be the area of plate wetted by the film, the resultant force F pressing the upper plate downward is given by

$$F = \text{resultant pressure } \times \text{area } = \frac{2TA}{d}.$$

Example 21 *A drop of water of volume* $0.05\ cm^3$ *is pressed between two glass plates, as a consequence of which, it spreads and occupies an area of* $40\ cm^2$. *If the surface tension of water is 70 dyne/cm, find the normal force required to separate out the two glass plates in newton.*

Solution We have discussed above,

$$F = \frac{2AT}{d} = \frac{2A^2T}{Ad}$$

But, $Ad = \text{volume}$

$\therefore$
$$F = \frac{2A^2T}{V}$$

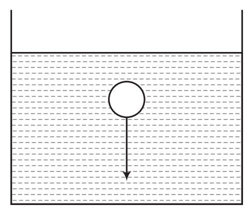
Substituting the values we get,

$$F = \frac{2 \times (40 \times 10^{-4})^2 \times (70 \times 10^{-3})}{0.05 \times 10^{-6}}$$

$$= 45\ \text{N}$$ **Ans.**

Miscellaneous Examples

Example 22 *A ball of volume V and density ρ_1 is moved downwards by a distance 'd' in liquid of density ρ_2. Find total change in potential energy of the system.*



Solution Decrease in potential energy of the ball.

$$= m_1 gh \qquad (m_1 = \text{mass of ball})$$

$$= (V\rho_1) gd$$

or $$\Delta U_1 = -V\rho_1 gd$$

When V volume of solid comes down, then it is replaced by V volume of liquid.

$\therefore$ Increase in potential energy of liquid :

$$= m_2 gh \qquad (m_2 = \text{mass of liquid of volume } V)$$

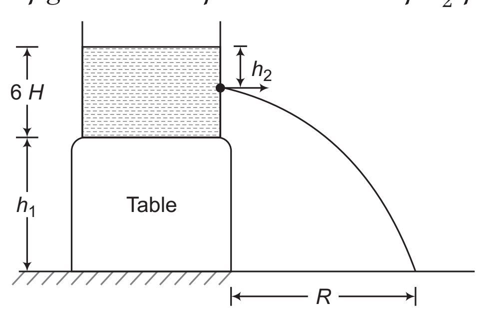
$$= (V\rho_2) gd$$

$\therefore$ $$\Delta U_2 = +V\rho_2 gd$$

Total change in potential energy,

$$\Delta U = \Delta U_1 + \Delta U_2 = V(\rho_2 - \rho_1) gd$$ **Ans.**

Example 23 *In the figure shown find the value of h_2 for maximum range R if*



(a) $h_1 = 4H$ *(b) $h_1 = 8H$*

Also find the value of this maximum range in both cases.

Solution (a) $h_1 + 6H = 4H + 6H = 10H = h$ (say)

Maximum range will be obtained from

$$h_2 = \frac{h}{2} = 5H$$ **Ans.**

and this maximum range will be,

$$R_{max} = h = 10\,H \qquad \textbf{Ans.}$$

(b) In this case,

$$h = h_1 + 6H = 8H + 6H = 14H$$

$\frac{h}{2}$ or $7H$ point lies on the table. So, maximum range will be obtained from the bottommost point of the liquid container or $h_2 = 6H$ **Ans.**

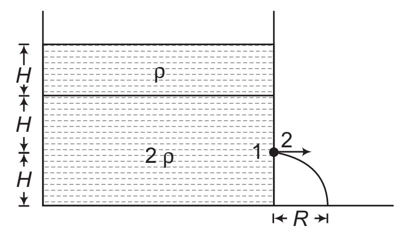
and this maximum range will be

$$R = 2\sqrt{h_2(h - h_2)}$$

$$= 2\sqrt{6H \times 8H} = 8\sqrt{3}H \qquad \textbf{Ans.}$$

Note *From top to centre range first increases and then from centre to bottom range then decreases. Therefore, in the second case range will be maximum from bottommost point of the liquid container.*

Example 24 *In the figure shown,*



find the range R.

Solution Applying Bernoulli's equation at points 1 and 2.

$$p_1 + \frac{1}{2}(2\rho)\,v_1^2 + (2\rho)\,gh_1 = p_2 + \frac{1}{2}(2\rho)\,v_2^2 + (2\rho)\,gh_2$$

Here, $\quad v_1 \approx 0,\ v_2 = v,\ h_1 = h_2,\ p_2 = p_0$

and $\quad p_1 = p_0 + \rho\,gH + 2\rho gH$

$$= p_0 + 3\rho gH$$

Substituting in the above equation we have,

$$p_0 + 3\rho gH = p_0 + \rho\,v^2$$

$\therefore \quad v = \sqrt{3\,gH}$

Now, this velocity is horizontal. So, time taken by the liquid to fall to the ground is free fall time or

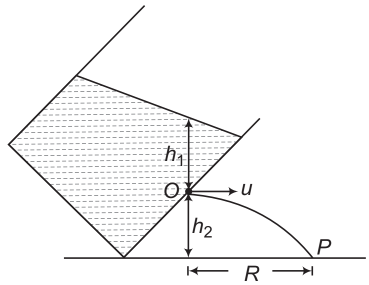
$$t = \sqrt{\frac{2h}{g}} \qquad \text{(where, } h = H\text{)}$$

or $\quad t = \sqrt{\frac{2H}{g}}$

$\therefore \quad R = vt = (\sqrt{3gH})\left(\sqrt{\frac{2H}{g}}\right)$

$$= \sqrt{6}H \qquad \textbf{Ans.}$$

Example 25 *In the figure shown*



find v, t_{op} and R.

Solution $v = \sqrt{2 \times g \times \text{distance of orifice from top surface of liquid}}$ **Ans.**

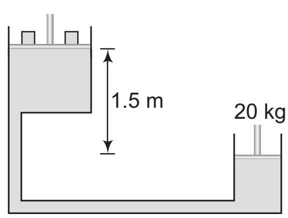
$$= \sqrt{2gh_1}$$

$$t_{op} = \text{free fall time} \qquad \text{(as } v \text{ is horizontal)}$$

$$= \sqrt{\frac{2h_2}{g}} \qquad \text{(where, } h_2 = \text{free fall height)}$$

$$R = v\,t_{op} = (\sqrt{2gh_1})\left(\sqrt{\frac{2h_2}{g}}\right) = 2\sqrt{h_1 h_2}$$ **Ans.**

Example 26 *Figure shows a hydraulic press with the larger piston of diameter 35 cm at a height of 1.5 m relative to the smaller piston of diameter 10 cm. The mass on the smaller piston is 20 kg. What is the force exerted on the load by the larger piston? The density of oil in the press is 750 kg/m^3. (Take $g = 9.8$ m/s^2)*



Solution Pressure on the smaller piston $= \dfrac{20 \times 9.8}{\pi \times (5 \times 10^{-2})^2}$ N/m^2

Pressure on the larger piston $= \dfrac{F}{\pi \times (17.5 \times 10^{-2})^2}$ N/m^2

The difference between the two pressures $= h\rho g$

where $h = 1.5$ m

and $\rho = 750$ kg/m^3

Thus, $$\frac{20 \times 9.8}{\pi \times (5 \times 10^{-2})^2} - \frac{F}{\pi \times (17.5 \times 10^{-2})^2} = 1.5 \times 750 \times 9.8$$

which gives, $F = 1.3 \times 10^3$ N **Ans.**

Note *Atmospheric pressure is common to both pistons and has been ignored.*

Example 27 *A glass full of water upto a height of* 10 *cm has a bottom of area* 10 cm^2*, top of area* 30 cm^2 *and volume* 1 *litre.*

(a) *Find the force exerted by the water on the bottom.*

(b) *Find the resultant force exerted by the sides of the glass on the water.*

(c) *If the glass is covered by a jar and the air inside the jar is completely pumped out, what will be the answers to parts (a) and (b).*

(d) *If a glass of different shape is used, provided the height, the bottom area, the top area and the volume are unchanged, will the answers to parts (a) and (b) change.*

Take $g = 10\ m/s^2$*, density of water* $= 10^3\ kg/m^3$ *and atmospheric pressure* $= 1.01 \times 10^5\ N/m^2$

Solution (a) Force exerted by the water on the bottom

$$F_1 = (p_0 + \rho g h) A_1 \qquad \text{...(i)}$$

Here, p_0 = atmospheric pressure $= 1.01 \times 10^5$ N/m^2

ρ = density of water $= 10^3$ k/gm^3

$g = 10$ m/s^2, $h = 10$ cm $= 0.1$ m

and A_1 = area of base $= 10$ cm^2 $= 10^{-3}$ m^2

Substituting in Eq. (i), we get

$$F_1 = (1.01 \times 10^5 + 10^3 \times 10 \times 0.1) \times 10^{-3}$$

or $F_1 = 102$ N (downwards) **Ans.**

(b) Force exerted by atmosphere on water

$$F_2 = (p_0) A_2$$

Here, A_2 = area of top $= 30$ cm^2 $= 3 \times 10^{-3}$ m^2

$\therefore$ $F_2 = (1.01 \times 10^5)(3 \times 10^{-3})$

$= 303$ N (downwards)

Force exerted by bottom on the water

$$F_3 = -F_1 \quad \text{or} \quad F_3 = 102 \text{ N (upwards)}$$

weight of water W = (volume)(density)(g) $= (10^{-3})(10^3)(10)$

$= 10$ N (downwards)

Let F be the force exerted by side walls on the water (upwards). Then, for equilibrium of water

Net upward force = net downward force

or $F + F_3 = F_2 + W$

$\therefore$ $F = F_2 + W - F_3 = 303 + 10 - 102$

or $F = 211$ N (upwards) **Ans.**

(c) If the air inside the jar is completely pumped out,

$F_1 = (\rho g h) A_1$ (as $p_0 = 0$)

$= (10^3)(10)(0.1)(10^{-3}) = 1$ N (downwards) **Ans.**

In this case, $F_2 = 0$

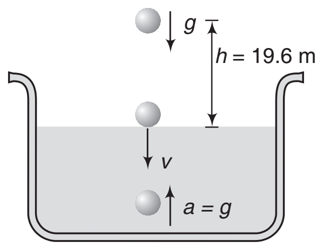
and $F_3 = 1$ N (upwards)

$\therefore$ $F = F_2 + W - F_3 = 0 + 10 - 1 = 9$ N (upwards) **Ans.**

(d) No, the answer will remain the same. Because the answers depend upon p_0, ρ, g, h, A_1 and A_2

Example 28 *A solid ball of density half that of water falls freely under gravity from a height of 19.6 m and then enters water. Upto what depth will the ball go. How much time will it take to come again to the water surface ? Neglect air resistance and viscosity effects in water. (Take $g = 9.8\ m/s^2$)*

Solution $v = \sqrt{2gh} = \sqrt{2 \times 9.8 \times 19.6} = 19.6$ m/s



Let ρ be the density of ball and 2ρ the density of water. Net retardation inside the water,

$$a = \frac{\text{upthrust} - \text{weight}}{\text{mass}}$$

$$= \frac{V(2\rho)g - V(\rho)(g)}{V(\rho)}$$

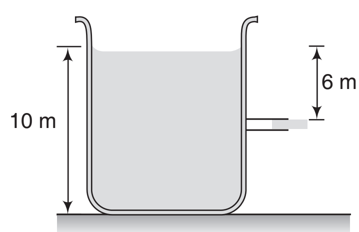
$$= g = 9.8 \text{ m/s}^2$$

Hence, the ball will go upto the same depth 19.6 m below the water surface. **Ans.**

Further, time taken by the ball to come back to water surface is,

$$t = 2\left(\frac{v}{a}\right) = 2\left(\frac{19.6}{9.8}\right) = 4 \text{ s}$$ **Ans.**

Example 29 *A fresh water on a reservoir is 10 m deep. A horizontal pipe 4.0 cm in diameter passes through the reservoir 6.0 m below the water surface as shown in figure. A plug secures the pipe opening.*



(a) Find the friction force between the plug and pipe wall.

(b) The plug is removed. What volume of water flows out of the pipe in 1 h? Assume area of reservoir to be too large.

Solution (a) Force of friction

= pressure difference on the sides of the plug × area of cross section of the plug

$= (\rho g h) A$

$= (10^3)(9.8)(6.0)(\pi)(2 \times 10^{-2})^2$

$= 73.9$ N **Ans.**

(b) Assuming the area of the reservoir to be too large,

Velocity of efflux

$$v = \sqrt{2gh} = \text{constant}$$

$$\therefore \qquad v = \sqrt{2 \times 9.8 \times 6} = 10.84 \text{ m/s}$$

Volume of water coming out per sec,

$$\frac{dV}{dt} = Av$$

$$= \pi(2 \times 10^{-2})^2 (10.84)$$

$$= 1.36 \times 10^{-2} \text{ m}^3/\text{s}$$

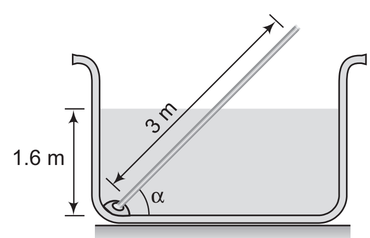
$\therefore$ The volume of water flowing through the pipe in 1 h

$$V = \left(\frac{dV}{dt}\right) t$$

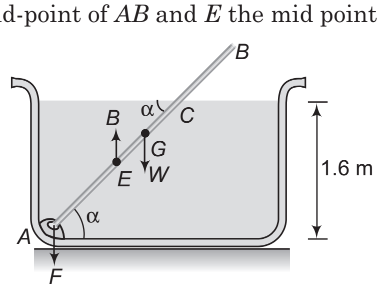
$$= (1.36 \times 10^{-2})(3600)$$

$$= 48.96 \text{ m}^3$$ **Ans.**

Example 30 *A wooden rod weighing 25 N is mounted on a hinge below the free surface of water as shown. The rod is 3 m long and uniform in cross section and the support is 1.6 m below the free surface. At what angle* α *rod is in equilibrium? The cross-section of the rod is* 9.5×10^{-4} m^2 *in area. Density of water is* $1000 \; kg/m^3$. *Assume buoyancy to act at centre of immersion.* $g = 9.8 \; m/s^2$. *Also find the reaction on the hinge in this position.*



Solution Let G be the mid-point of AB and E the mid point of AC (i.e. the centre of buoyancy)



$$AC = 1.6 \text{ cosec } \alpha$$

$$\text{Volume of } AC = (1.6 \times 9.5 \times 10^{-4}) \text{ cosec } \alpha$$

Weight of water displaced by AC

$$= (1.6 \times 9.5 \times 10^{-4} \times 10^3 \times 9.8) \text{ cosec } \alpha$$

$$= 14.896 \text{ cosec } \alpha$$

Hence, the buoyant force is 14.896 cosec α acting vertically upwards at E. While the weight of the rod is 25 N acting vertically downwards at G. Taking moments about A,

$$(14.896 \operatorname{cosec} \alpha)(AE \cos \alpha) = (25)(AG \cos \alpha)$$

or $$(14.896 \operatorname{cosec} \alpha)\left(\frac{1.6 \operatorname{cosec} \alpha}{2}\right) = 25 \times \frac{3}{2}$$

or $$\sin^2 \alpha = 0.32$$

∴ $$\sin \alpha = 0.56$$

or $$\alpha = 34.3^\circ$$ **Ans.**

Further, let F be the reaction at hinge in vertically downward direction. Then, considering the translatory equilibrium of rod in vertical direction we have,

$$F + \text{weight of the rod} = \text{upthrust}$$

∴ $$F = \text{upthrust} - \text{weight of the rod}$$
$$= 14.896 \operatorname{cosec}(34.3^\circ) - 25$$
$$= 26.6 - 25$$

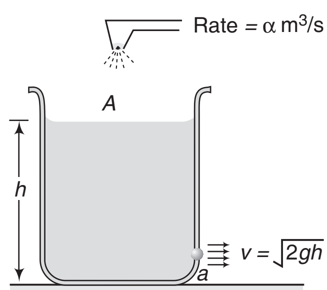
∴ $$F = 1.6 \text{ N} \quad \text{(downwards)}$$ **Ans.**

Example 31 *A cylindrical tank of base area A has a small hole of area a at the bottom. At time $t = 0$, a tap starts to supply water into the tank at a constant rate α m^3/s.*

(a) what is the maximum level of water h_{max} in the tank?

(b) find the time when level of water becomes h ($< h_{max}$).

Solution (a) Level will be maximum when



Rate of inflow of water = rate of outflow of water

i.e. $$\alpha = av$$

or $$\alpha = a\sqrt{2gh_{max}}$$

⇒ $$h_{max} = \frac{\alpha^2}{2ga^2}$$ **Ans.**

(b) Let at time t, the level of water be h. Then,

$$A\left(\frac{dh}{dt}\right) = \alpha - a\sqrt{2gh} \quad \text{or} \quad \int_0^h \frac{dh}{\alpha - a\sqrt{2gh}} = \int_0^t \frac{dt}{A}$$

Solving this, we get

$$t = \frac{A}{ag}\left[\frac{\alpha}{a} \ln\left\{\frac{\alpha - a\sqrt{2gh}}{\alpha}\right\} - \sqrt{2gh}\right]$$ **Ans.**

Exercises

LEVEL 1

Assertion and Reason

Directions : *Choose the correct option.*

(*a*) *If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion.***

(*b*) *If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion.***

(*c*) *If **Assertion** is true, but the **Reason** is false.*

(*d*) *If **Assertion** is false but the **Reason** is true.*

1. Assertion : Pressure is a vector quantity.

Reason : Pressure $P = \dfrac{F}{A}$. Here F, the force is a vector quantity.

2. Assertion : Surface tension $\left(T = \dfrac{F}{l}\right)$ is not a vector quantity.

Reason : Direction of force is specified.

3. Assertion : At depth h below the water surface pressure is p. Then at depth $2h$ pressure will be $2p$. (Ignore density variation).

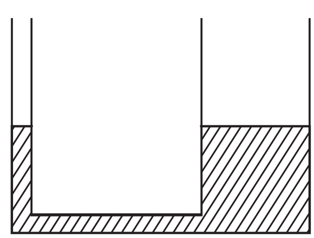
Reason : With depth pressure increases linearly.

4. Assertion : Weight of solid in air is w and in water is $\dfrac{2w}{3}$. Then relative density of solid is 3.0.

Reason : Relative density of any solid is given by

$$\text{RD} = \frac{\text{Weight in air}}{\text{Change in weight in water}}$$

5. Assertion : Water is filled in a U-tube of different cross-sectional area on two sides as shown in figure. Now equal amount of oil ($RD = 0.5$) is poured on two sides. Level of water on both sides will remain unchanged.

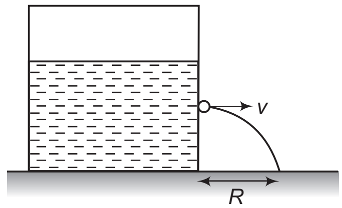


Reason : Same weight of oil poured on two sides will produce different pressures.

6. Assertion : An ideal fluid is flowing through a pipe. Speed of fluid particles is more at places where pressure is low.

Reason : Bernoulli's theorem can be derived from work-energy theorem.

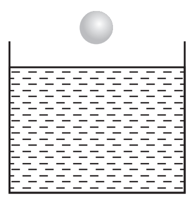
7. Assertion : In the figure shown v and R will increase if pressure above the liquid surface inside the chamber is increased.



Reason : Value of v or R is independent of density of liquid.

8. Assertion : A ball is dropped from a certain height above the free surface of an ideal fluid. When the ball enters the liquid it may accelerate or retard.

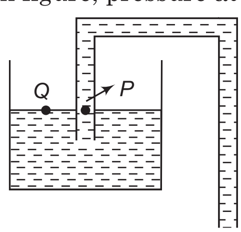
Reason : Ball accelerates or retards it all depends on the density of ball and the density of liquid.



9. Assertion : On moon, barometer height will be six times compared to the height on earth.

Reason : Value of g on moon's surface is $\frac{1}{6}$ the value of g on earth's surface.

10. Assertion : In the siphon shown in figure, pressure at P is equal to atmospheric pressure.



Reason : Pressure at Q is atmospheric pressure and points P and Q are at same levels.

11. Assertion : Force of buoyancy due to atmosphere on a small body is almost zero (or negligible).

Reason : If a body is completely submerged in a fluid, then buoyant force is zero.

Objective Questions

Single Correct Option

1. When a sphere falling in a viscous fluid attains a terminal velocity, then

(a) the net force acting on the sphere is zero
(b) the drag force balances the buoyant force
(c) the drag force balances the weight of the sphere
(d) the buoyant force balances the weight and drag force

2. Which one of the following represents the correct dimensions of the quantity : $x = \frac{\eta}{\rho}$, where $\eta =$ coefficient of viscosity and $\rho =$ the density of a liquid?

(a) $[ML^{-2}T^{-1}]$ (b) $[ML^{-4}T^{-2}]$
(c) $[ML^{-5}T^{-2}]$ (d) $[M^0L^2T^{-1}]$

3. Viscosity of liquids

(a) increases with increase in temperature
(b) is independent of temperature
(c) decreases with decrease in temperature
(d) decreases with increase in temperature

4. At critical temperature, the surface tension of a liquid

(a) is zero
(b) is infinity
(c) is same as that at any other temperature
(d) cannot be determined

5. A liquid will not wet the surface of a solid if the angle of contact is

(a) 0° (b) 45°
(c) 60° (d) >90°

6. The lower end of a capillary tube touches a liquid whose angle of contact is 110°,the liquid

(a) rises into the tube
(b) falls in the tube
(c) may rise or fall inside
(d) neither rises nor falls inside the tube

7. Two water droplets combine to form a large drop. In this process energy is

(a) liberated
(b) absorbed
(c) neither liberated nor absorbed
(d) sometimes liberated and sometimes absorbed

8. A number of small drops of mercury adiabatically coalesce to form a single drop. The temperature of the drop will

(a) increase (b) remain same
(c) decrease (d) depend on size

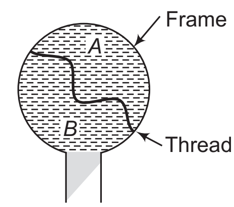
9. Two soap bubbles in vacuum of radius 3 cm and 4 cm coalesce to form a single bubble under isothermal conditions. Then the radius of bigger bubble is

(a) 7 cm (b) $\frac{12}{7}$ cm
(c) 12 cm (d) 5 cm

10. A small ball (mass m) falling under gravity in a viscous medium experiences a drag force proportional to the instantaneous speed u such that $F_{\text{drag}} = ku$. Then the terminal speed of ball within viscous medium is

(a) $\frac{k}{mg}$ (b) $\frac{mg}{k}$
(c) $\sqrt{\frac{mg}{k}}$ (d) None of these

11. A thread is tied slightly loose to a wire frame as in figure and the frame is dipped into a soap solution and taken out. The frame is completely covered with the film. When the portion A is punctured with a pin, the thread



(a) becomes concave towards A
(b) becomes convex towards A
(c) either (a) or (b) depending on the size of A with respect to B
(d) remains in the initial position

12. A steel ball of mass m falls in a viscous liquid with terminal velocity v, then the steel ball of mass 8 m will fall in the same liquid with terminal velocity

(a) v (b) $4v$ (c) $8v$ (d) $16\sqrt{2}v$

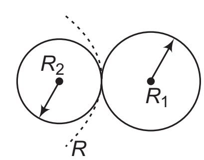
13. A liquid flows between two parallel plates along the x-axis. The difference between the velocity of two layers separated by the distance dy is dv. If A is the area of each plate, then Newton's law of viscosity may be written as

(a) $F = -\eta A \dfrac{dv}{dx}$ (b) $F = +\eta A \dfrac{dv}{dx}$
(c) $F = -\eta A \dfrac{dv}{dy}$ (d) $F = +\eta A \dfrac{dv}{dy}$

14. The work done to split a liquid drop of radius R into N identical drops is (take σ as the surface tension of the liquid)

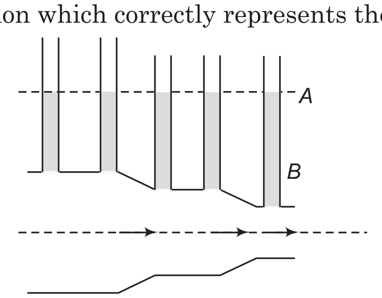
(a) $4\pi R^2(N^{1/3} - 1)\,\sigma$ (b) $4\pi R^2 N\sigma$
(c) $4\pi R^2(N^{1/2} - 1)$ (d) None of these

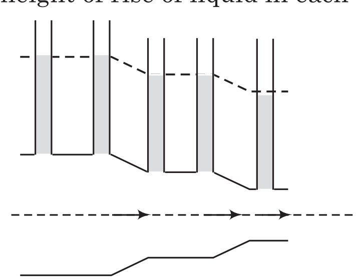
15. Two soap bubbles of different radii R_1 and $R_2 (< R_1)$ coalesce to form an interface of radius R as shown in figure. The correct value of R is

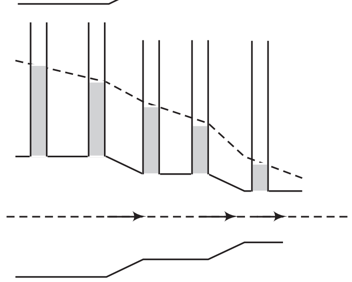


(a) $R = R_1 - R_2$ (b) $R = \dfrac{R_1 + R_2}{2}$
(c) $\dfrac{1}{R} = \dfrac{1}{R_2} - \dfrac{1}{R_1}$ (d) $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$

16. A viscous liquid flows through a horizontal pipe of varying cross-sectional area. Identify the option which correctly represents the variation of height of rise of liquid in each vertical tube

(a)


(b)


(c)


(d) None of these

17. The terminal velocity of a rain drop is 30 cm/s. If the viscosity of air is 1.8×10^{-5} Nsm^{-2}. The radius of rain drop is

(a) 1 μm (b) 0.5 mm
(c) 0.05 mm (d) 1 mm

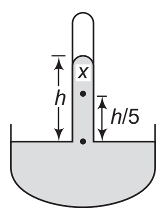
18. If a capillary tube is dipped and the liquid levels inside and outside the tube are same, then the angle of contact is

(a) zero (b) 90°
(c) 45° (d) Cannot be obtained

19. Uniform speed of 2 cm diameter ball is 20 cm/s in a viscous liquid. Then, the speed of 1 cm diameter ball in the same liquid is

(a) 5 cms^{-1} (b) 10 cms^{-1}
(c) 40 cms^{-1} (d) 80 cms^{-1}

20. The height of mercury barometer is h when the atmospheric pressure is 10^5 Pa.
The pressure at x in the shown diagram is



(a) 10^5 Pa (b) 0.8×10^5 Pa
(c) 0.2×10^5 Pa (d) 120×10^5 Pa

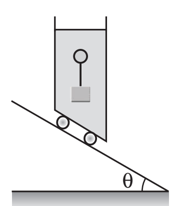
21. A body floats in water with its one-third volume above the surface. The same body floats in a liquid with one-third volume immersed. The density of the liquid is

(a) 9 times more than that of water
(b) 2 times more than that of water
(c) 3 times more than that of water
(d) 1.5 times more than that of water

22. A piece of ice is floating in a beaker containing thick sugar solution of water. As the ice melts, the total level of the liquid

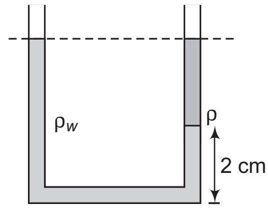
(a) increases (b) decreases
(c) remains unchanged (d) insufficient data

23. A body floats in completely immersed condition in water as shown in figure. As the whole system is allowed to slide down freely along the inclined surface, the magnitude of buoyant force



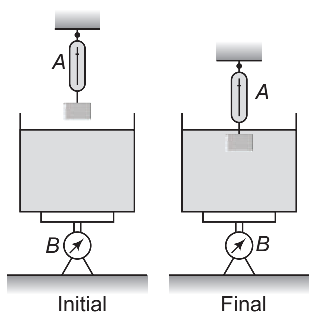
(a) remains unchanged (b) increases
(c) decreases (d) becomes zero

24. The figure represents a U-tube of uniform cross-section filled with two immiscible liquids. One is water with density ρ_w and the other liquid is of density ρ. The liquid interface lies 2 cm above the base. The relation between ρ and ρ_w is



(a) $\rho = \rho_w$ (b) $\rho = 1.02\,\rho_w$ (c) $\rho = 1.2\,\rho_w$ (d) None of these

25. For the arrangement shown in figure, initially the balance A and B reads F_1 and F_2 respectively and $F_1 > F_2$. Finally when the block is immersed in the liquid then the readings of balance A and B are f_1 and f_2 respectively. Identify the statement which is **not always** (where, F is some force) **correct** statement.



(a) $f_1 > f_2$ (b) $F_1 + F > F_2 + F$
(c) $f_1 + f_2 = F_1 + F_2$ (d) None of these

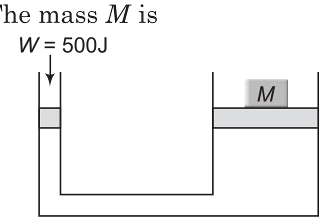
26. When a tap is closed, the manometer attached to the pipe reads 3.5×10^5 Nm^{-2}. When the tap is opened, the reading of manometer falls to 3.0×10^5 Nm^{-2}. The velocity of water in the pipe is

(a) 0.1 ms^{-1} (b) 1 ms^{-1}
(c) 5 ms^{-1} (d) 10 ms^{-1}

27. A balloon of mass M descends with an acceleration a_0. The mass that must be thrown out in order to give the balloon an equal upward acceleration will be

(a) $\dfrac{Ma_0}{g}$ (b) $\dfrac{2Ma_0}{g}$ (c) $\dfrac{2Ma_0}{g + a_0}$ (d) $\dfrac{M\,(g + a_0)}{a_0}$

28. The hydraulic press shown in the figure is used to raise the mass M through a height of 0.5 cm by performing 500 J of work at the small piston. The diameter of the large piston is 10 cm, while that of the smaller one is 2 cm. The mass M is



(a) 100 kg (b) 10^6 kg (c) 10^3 kg (d) None of these

29. When equal volumes of two substances are mixed, the specific gravity of the mixture is 4. When equal weights of the same substances are mixed, the specific gravity of the mixture is 3. The specific gravities of the two substances could be

(a) 6 and 2 (b) 3 and 4
(c) 2.5 and 3.5 (d) 5 and 3

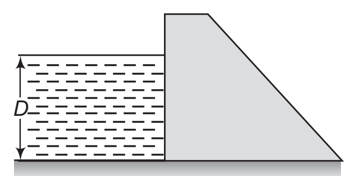
30. A block of ice of total area A and thickness 0.5 m is floating in water. In order to just support a man of mass 100 kg, the area A should be (the specific gravity of ice is 0.9)

(a) 2.2 m^2 (b) 1.0 m^2
(c) 0.5 m^2 (d) None of these

31. A piece of gold ($\rho = 19.3\ g/cm^3$) has a cavity in it. It weights 38.2 g in air and 36.2 g in water. The volume of the cavity in gold is

(a) 0.2 cm^3 (b) 0.04 cm^3
(c) 0.02 cm^3 (d) 0.01 cm^3

32. Water stands at a depth D behind the vertical upstream face of a dam as shown in the figure. The force exerted on the dam by water per unit width is

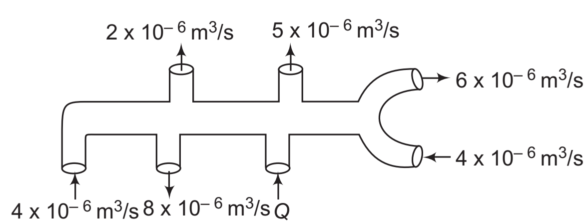


(a) $\frac{1}{3}\rho g D^2$ (b) $\frac{1}{2}\rho g D^2$
(c) $\frac{1}{3}\rho g D$ (d) $\frac{1}{2}(\rho g D)^2$

33. The volume of a liquid flowing per second out of an orifice at the bottom of a tank does not depend upon

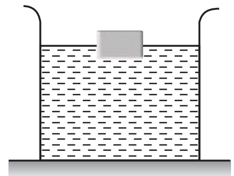
(a) the height of the liquid above the orifice
(b) the acceleration due to gravity
(c) the density of the liquid
(d) the area of the orifice

34. The pipe shows the volume flow rate of an ideal liquid at certain time and its direction. What is the value of Q in m^3/s? (Assume steady state and equal area of cross section at each opening)



(a) 10×10^{-6} (b) 11×10^{-6}
(c) 13×10^{-6} (d) 18×10^{-6}

35. A uniform cube of mass M is floating on the surface of a liquid with three fourth of its volume immersed in the liquid (density = ρ). The length of the side of the cube is equal to



(a) $(4M/3\rho)^{2/3}$ (b) $(M/3\rho)^{2/3}$ (c) $(M/4\rho)^{2/3}$ (d) None of these

36. Water rises to a height of 10 cm in a certain capillary tube. An another identical tube when dipped in mercury the level of mercury is depressed by 3.42 cm. Density of mercury is 13.6 g/cc. The angle of contact for water in contact with glass is 0° and mercury in contact with glass is 135°. The ratio of surface tension of water to that of Hg is

(a) 1 : 3 (b) 1 : 4 (c) 1 : 5.5 (d) 1 : 6.5

37. A capillary glass tube records a rise of 20 cm when dipped in water. When the area of cross-section of the tube is reduced to half of the former value, water will rise to a height of

(a) $10\sqrt{2}$ cm (b) 10 cm (c) 20 cm (d) $20\sqrt{2}$ cm

38. A cylindrical vessel open at the top is 20 cm high and 10 cm in diameter. A circular hole of cross sectional area 1 cm^2 is cut at the centre of the bottom of the vessel. Water flows from a tube above it into the vessel at the rate of 10^2 cm^3/s. The height of water in the vessel under steady state is (Take $g = 10$ m/s^2)

(a) 20 cm (b) 15 cm (c) 10 cm (d) 5 cm

39. A horizontal pipeline carries water in a streamline flow. At a point along the tube where the cross sectional area is 10^{-2} m^2, the water velocity is 2 m/s and the pressure is 8000 Pa. The pressure of water at another point where cross sectional area is 0.5×10^{-2} m^2 is

(a) 4000 Pa (b) 1000 Pa (c) 2000 Pa (d) 3000 Pa

40. Eight spherical rain drops of the same mass and radius are falling down with a terminal speed of 6 cms^{-1}. If they coalesce to form one big drop, what will be its terminal speed? Neglect the buoyancy due to air

(a) 1.5 cms^{-1} (b) 6 cms^{-1} (c) 24 cms^{-1} (d) 32 cms^{-1}

41. In a surface tension experiment with a capillary tube water rises upto 0.1 m. If the same experiment is repeated in an artificial satellite, which is revolving around the earth; water will rise in the capillary tube upto a height of

(a) 0.1 m (b) 0.2 m (c) 0.98 m (d) full length of tube

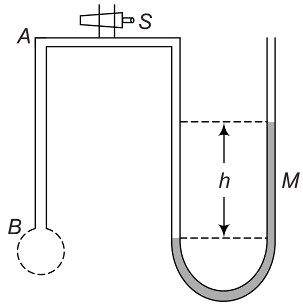
42. Two unequal soap bubbles are formed one on each side of a tube closed in the middle by a tap. What happens when the tap is opened to put the two bubbles in communication ?

(a) No air passes in any direction as the pressures are the same on two sides of the tap
(b) Larger bubble shrinks and smaller bubble increases in size till they become equal in size
(c) Smaller bubble gradually collapses and the bigger one increases in size
(d) None of the above

43. A capillary tube of radius R is immersed in water and water rises in it to a height h. Mass of water in capillary tube is M. If the radius of the tube is doubled, mass of water that will rise in the capillary tube will be

(a) $2M$ (b) M (c) $\frac{M}{2}$ (d) $4M$

44. A tube of fine bore AB is connected to a manometer M as shown. The stop cock S controls the flow of air. AB is dipped into a liquid whose surface tension is T. On opening the stop cock for a while, a bubble is formed at B and the manometer level is recorded, showing a difference h in the levels in the two arms, if p be the density of manometer liquid and r the radius of curvature of the bubble, then the surface tension T of the liquid is given by



(a) $\rho\, hrg$ (b) $2\rho hgr$ (c) $4\rho hrg$ (d) $\frac{rh\rho g}{4}$

45. A vessel whose bottom has round holes with a diameter of $d = 0.1$ mm is filled with water. The maximum height of the water level h at which the water does not flow out, will be (The water does not wet the bottom of the vessel). [ST of water = 70 dyne/cm]

(a) $h = 24.0$ cm (b) $h = 25.0$ cm (c) $h = 26.0$ cm (d) $h = 28.0$ cm

46. A large number of liquid drops each of radius 'a' coalesce to form a single spherical drop of radius b. The energy released in the process is converted into kinetic energy of the big drop formed. The speed of big drop will be

(a) $\sqrt{\frac{6T}{\rho}\left[\frac{1}{a}-\frac{1}{b}\right]}$ (b) $\sqrt{\frac{4T}{\rho}\left[\frac{1}{a}-\frac{1}{b}\right]}$

(c) $\sqrt{\frac{8T}{\rho}\left[\frac{1}{a}-\frac{1}{b}\right]}$ (d) $\sqrt{\frac{5T}{\rho}\left[\frac{1}{a}-\frac{1}{b}\right]}$

47. A glass capillary tube (closed from top) of inner diameter 0.28 mm is lowered vertically into water in a vessel. The pressure in the capillary tube so that water level in the tube is same as that in the vessel in N/m^2 is (surface tension of water $= 0.7 N/m$ and atmospheric pressure $= 10^5 N/m^2$)

(a) 10^3 (b) 99×10^3

(c) 100×10^3 (d) 101×10^3

48. A thin wire is bent in the form of a ring of diameter 3.0 cm. The ring is placed horizontally on the surface of soap solution and then raised up slowly. Upward force necessary to break the vertical film formed between the ring and the solution is

(a) $6\pi T$ dyne (b) $2\pi T$ dyne

(c) $4\pi T$ dyne (d) $3\pi T$ dyne

49. One end of a glass capillary tube with a radius $r = 0.05$ cm is immersed into water to a depth of $h = 2$ cm. Excess pressure required to blow an air bubble out of the lower end of the tube will be (S.T of water = 70 dyne/cm). Take $g = 980\ cm/s^2$

(a) 2840 dyne/cm^2 (b) 5840 dyne/cm^2

(c) 7840 dyne/cm^2 (d) 4760 dyne/cm^2

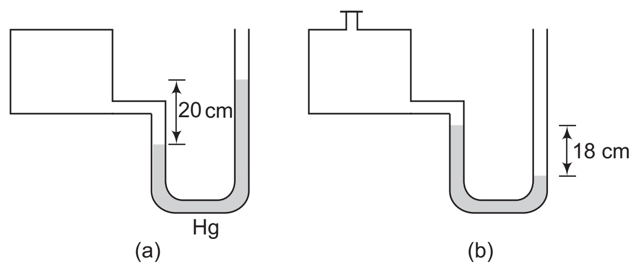
Subjective Questions

Note *Question numbers 16, 17 and 29 are based on flow of viscous liquids. Engineering aspirants can skip those problems.*

1. A body of weight w_1 when floats in water displaces an amount of water w_2. Then $w_1 < w_2$. Is this statement true or false?
2. A weightless balloon is filled with water. What will be its apparent weight when weighed in water?
3. Two vessels A and B have same base area. Equal volumes of a liquid are poured in the two vessels to different heights h_A and $h_B (> h_A)$. In which vessel, the force on the base of vessel will be more?
4. The work done in blowing a bubble of volume V is W, then what is the work done in blowing a soap bubble of volume $2V$?
5. A man is sitting in a boat which is floating in a pond. If the man drinks some water from the pond, how will the level of water in the pond change?
6. A metal ball weighs 0.096 N . When suspended in water it has an apparent weight of 0.071 N. Find the density of the metal.
7. A block of wood has a mass of 25 g. When a 5 g metal piece with a volume of 2 cm^3 is attached to the bottom of the block, the wood barely floats in water. What is the volume V of the wood?
8. What is the minimum volume of a block of wood (density = 850 kg/m^3) if it is to hold a 50 kg woman entirely above the water when she stands on it ?
9. A cubical block of ice floating in water has to support a metal piece weighing 0.5 kg. What can be the minimum edge of the block so that it does not sink in water? Specific gravity of ice = 0.9.
10. When a cube of wood floats in water, 60% of its volume is submerged. When the same cube floats in an unknown fluid 85% of its volume is submerged. Find the densities of wood and the unknown fluid.
11. A glass tube of radius 0.8 cm floats vertical in water, as shown in figure. What mass of lead pellets would cause the tube to sink a further 3 cm?

12. A hydraulic automobile lift is designed to lift cars with a maximum mass of 3000 kg. The area of cross section of the piston carrying the load is 425 cm^2. What maximum pressure would the smaller piston have to bear?
13. A U-tube contains water and methylated spirit separated by mercury. The mercury columns in the two arms are in level with 10.0 cm of water in one arm and 12.5 cm of spirit in the other. What is the relative density of sprit?
14. In the above question, if 15.0 cm of water and spirit each are further, poured into the respective arms of the tube, what is the difference in the levels of mercury in the two arms? (Relative density of mercury = 13.6)

15. A manometer reads the pressure of a gas in an enclosure as shown in figure (a) When some of the gas is removed by a pump, the manometer reads as in (b). The liquid used in the manometers is mercury and the atmospheric pressure is 76 cm of mercury.



(i) Give the absolute and gauge pressure of the gas in the enclosure for cases (a) and (b) in units of cm of mercury.

(ii) How would the levels change in case (b) if 13.6 cm of water are poured into the right limb of the manometer?

16. Water at 20°C is flowing in a pipe of radius 20.0 cm. The viscosity of water at 20°C is 1.005 centipoise. If the water's speed in the centre of the pipe is 3.00 m/s, what is water's speed:

(a) 10.0 cm from the centre of the pipe (half way between the centre and the walls)

(b) at the walls of the pipe?

17. Water at 20°C is flowing in a horizontal pipe that is 20.0 m long. The flow is laminar and the water completely fills the pipe. A pump maintains a gauge pressure of 1400 Pa, at a large tank at one end of the pipe. The other end of the pipe is open to the air, The viscosity of water at 20°C is 1.005 poise.

(a) If the pipe has diameter 8.0 cm, what is the volume flow rate?

(b) What gauge pressure must the pump provide to achieve the same volume flow rate for a pipe with a diameter of 4.0 cm?

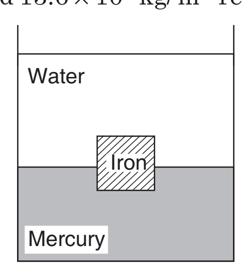
(c) For pipe in part (a) and the same gauge pressure maintained by the pump, what does the volume flow rate become if the water is at a temperature of 60°C (the viscosity of water at 60°C is 0.469 poise)?

18. An irregular piece of metal weighs 10.00 g in air and 8.00 g when submerged in water.

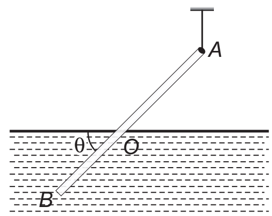
(a) Find the volume of the metal and its density.

(b) If the same piece of metal weighs 8.50 g when immersed in a particular oil, what is the density of the oil ?

19. A tank contains water on top of mercury. A cube of iron, 60 mm along each edge, is sitting upright in equilibrium in the liquids. Find how much of it is in each liquid. The densities of iron and mercury are 7.7×10^3 kg/ m^3 and 13.6×10^3 kg/ m^3 respectively.



20. A uniform rod AB, 4 m long and weighing 12 kg, is supported at end A, with a 6 kg lead weight at B. The rod floats as shown in figure with one-half of its length submerged. The buoyant force on the lead mass is negligible as it is of negligible volume. Find the tension in the cord and the total volume of the rod.

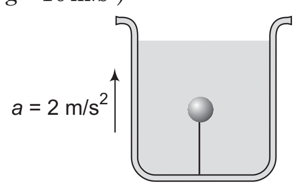


21. A solid sphere of mass $m = 2$ kg and density $\rho = 500\,\text{kg/m}^3$ is held stationary relative to a tank filled with water. The tank is accelerating upward with acceleration 2 m/s^2. Calculate

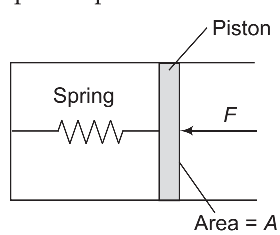
(a) Tension in the thread connected between the sphere and the bottom of the tank.

(b) If the thread snaps, calculate the acceleration of sphere with respect to the tank.

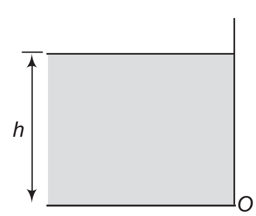
(Density of water $= 1000\ \text{kg/m}^3$, $g = 10\,\text{m/s}^2$)



22. The pressure gauge shown in figure has a spring for which $k = 60$ N/m and the area of the piston is $0.50\,\text{cm}^2$. Its right end is connected to a closed container of gas at a gauge pressure of 30 kPa. How far will the spring be compressed if the region containing the spring is (a) in vacuum and (b) open to the atmosphere ? Atmospheric pressure is 101 kPa.



23. Water stands at a depth h behind the vertical face of a dam. It exerts a resultant horizontal force on the dam tending to slide it along its foundation and a torque tending to overturn the dam about the point O. Find



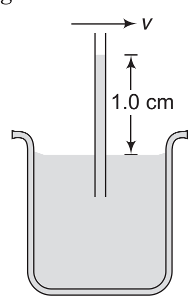
(a) horizontal force, (b) torque about O, (c) the height at which the resultant force would have to act to produce the same torque, l = cross-sectional length and ρ = density of water.

24. Mercury is poured into a U-tube in which the cross-sectional area of the left-hand limb is three times smaller than that of the right one. The level of the mercury in the narrow limb is a distance $l = 30\,\text{cm}$ from the upper end of the tube. How much will the mercury level rise in the right-hand limb if the left one is filled to the top with water ?

25. A water barrel stands on a table of height h. If a small hole is punched in the side of the barrel at its base, it is found that the resultant stream of water strikes the ground at a horizontal distance R from the barrel. What is the depth of water in the barrel ?

26. A pump is designed as a horizontal cylinder with a piston of area A and an outlet orifice of area a arranged near the cylinder axis. Find the velocity of out flow of the liquid from the pump if the piston moves with a constant velocity under the action of a constant force F. The density of the liquid is ρ.

27. When air of density $1.3\ \text{kg/m}^3$ flows across the top of the tube shown in the accompanying figure, water rises in the tube to a height of 1.0 cm. What is the speed of the air ?



28. The area of cross-section of a large tank is $0.5\ \text{m}^2$. It has an opening near the bottom having area of cross-section $1\ \text{cm}^2$. A load of 20 kg is applied on the water at the top. Find the velocity of the water coming out of the opening at the time when the height of water level is 50 cm above the bottom. (Take $g = 10\ \text{m/s}^2$)

29. What is the pressure drop (in mm Hg) in the blood as it passes through a capillary 1 mm long and $2\,\mu\text{m}$ in radius if the speed of the blood through the centre of the capillary is 0.66 mm/s ? (The viscosity of whole blood is 4×10^{-3} Pl).

30. A glass capillary sealed at the upper end is of length 0.11 m and internal diameter 2×10^{-5} m. The tube is immersed vertically into a liquid of surface tension 5.06×10^{-2} N/m. To what length has the capillary to be immersed so that the liquid levels inside and outside the capillary become the same ? What will happen to the water levels inside the capillary if the seal is now broken ?

31. A film of water is formed between two straight parallel wires each 10 cm long and at a separation 0.5 cm. Calculate the work required to increase 1 mm distance between them. Surface tension of water = 72×10^{-3} N/m.

32. A barometer contains two uniform capillaries of radii 1.44×10^{-3} m and 7.2×10^{-4} m. If the height of the liquid in the narrow tube is 0.2 m more than that in the wide tube, calculate the true pressure difference. Density of liquid $= 10^3\ \text{kg/m}^3$, surface tension $= 7.2 \times 10^{-3}$ N/m and $g = 9.8\ \text{m/s}^2$.

33. A liquid of specific gravity 1.5 is observed to rise 3.0 cm in a capillary tube of diameter 0.50 mm and the liquid wets the surface of the tube. Calculate the excess pressure inside a spherical bubble of 1.0 cm diameter blown from the same liquid. Angle of contact = 0°.

34. A glass U-tube is such that the diameter of one limb is 3.0 mm and that of the other is 6.0 mm. The tube is inverted vertically with the open ends below the surface of water in a beaker. What is the difference between the heights to which water rises in the two limbs? Surface tension of water is $0.07\,\text{Nm}^{-1}$. Assume that the angle of contact between water and glass is 0°.

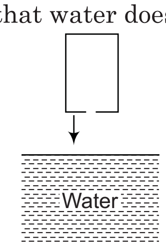
35. A minute spherical air bubble is rising slowly through a column of mercury contained in a deep jar. If the radius of the bubble at a depth of 100 cm is 0.1 mm, calculate its depth where its radius is 0.126 mm, given that the surface tension of mercury is 567 dyne/cm. Assume that the atmospheric pressure is 76 cm of mercury.

36. If a number of little droplets of water, each of radius r, coalesce to form a single drop of radius R, show that the rise in temperature will be given be

$$\frac{3T}{J}\left(\frac{1}{r}-\frac{1}{R}\right)$$

where, T is the surface tension of water and J is the mechanical equivalent of heat.

37. An empty container has a circular hole of radius r at its bottom. The container is pushed into water very slowly as shown. To what depth the lower surface of container (from surface of water) can be pushed into water such that water does not flow into the container ?

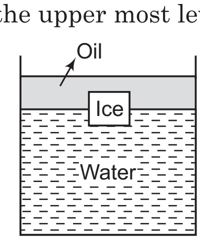


LEVEL 2

Objective Questions

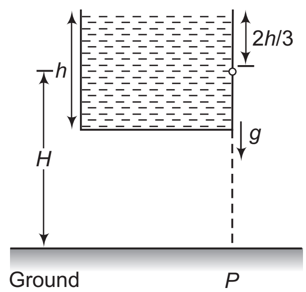
Single Correct Option

1. An ice cube is floating in water above which a layer of lighter oil is poured. As the ice melts completely, the level of interface and the upper most level of oil will respectively



(a) rise and fall (b) fall and rise
(c) not change and no change (d) not change and fall

2. An open vessel full of water is falling freely under gravity. There is a small hole in one face of the vessel as shown in the figure. The water which comes out from the hole at the instant when hole is at height H above the ground, strikes the ground at a distance of x from P.



Which of the following is correct for the situation described?

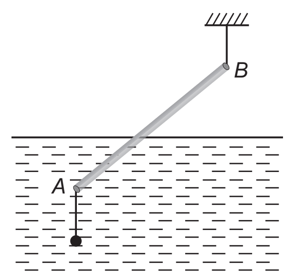
(a) The value of x is $2\sqrt{\dfrac{2hH}{3}}$

(b) The value of x is $\sqrt{\dfrac{4hH}{3}}$

(c) The value of x can't be computed from information provided

(d) The question is irrevalent as no water comes out from the hole

3. A uniform rod AB, 12 m long weighing 24 kg, is supported at end B by a flexible light string and a lead weight (of very small size) of 12 kg attached at end A.



The rod floats in water with one-half of its length submerged. For this situation, mark out the correct statement.
[Take $g = 10 \text{ m/s}^2$, density of water = 1000 kg/m^3]

(a) The tension in the string is 36 g

(b) The tension in the string is 12 g

(c) The volume of the rod is $6.4 \times 10^{-2} \text{ m}^3$

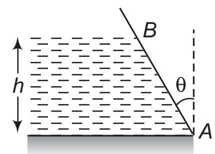
(d) The point of application of the buoyancy force is passing through C (centre of mass of rod)

4. A water hose pipe of cross-sectional area 5 cm^2 is used to fill a tank of 120 L. It has been observed that it takes 2 min to fill the tank. Now, a nozzle with an opening of cross-sectional area 1 cm^2 is attached to the hose. The nozzle is held so that water is projected horizontally from a point 1 m above the ground. The horizontal distance over which the water can be projected is (Take $g = 10 \text{ m/s}^2$)

(a) 3 m (b) 8 m

(c) 4.47 m (d) 8.64 m

5. The height of water in a vessel is h. The vessel wall of width b is at an angle θ to the vertical. The net force exerted by the water on the wall is

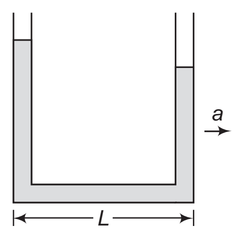


(a) $\frac{1}{3}\rho b h^2 g \cos\theta$ (b) $\frac{1}{2} b h^2 \rho g$ (c) $\frac{1}{2}\rho b h^2 g \sec\theta$ (d) zero

6. A body of density ρ is dropped from rest from a height h into a lake of density $\sigma\,(\sigma > \rho)$. The maximum depth the body sinks inside the liquid is (neglect viscous effect of liquid)

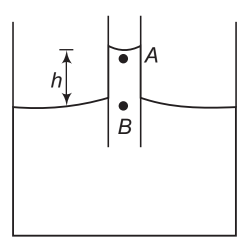
(a) $\frac{h\rho}{\sigma - \rho}$ (b) $\frac{h\sigma}{\sigma - \rho}$ (c) $\frac{h\rho}{\sigma}$ (d) $\frac{h\sigma}{\rho}$

7. A liquid stands at the plane level in the U-tube when at rest. If areas of cross-section of both the limbs are equal, what will be the difference in heights h of the liquid in the two limbs of U-tube, when the system is given an acceleration a in horizontal direction towards right as shown?



(a) $\frac{Lg}{a}$ (b) $\frac{La}{g}$ (c) $\frac{Lg^2}{a^2}$ (d) zero

8. A liquid of density ρ and surface tension σ rises in a capillary tube of inner radius R. The angle of contact between the liquid and the glass is θ. The point A lies just below the meniscus in the tube and the point B lies at the outside level of liquid in the beaker as shown in figure. The pressure at A is



(a) $p_B - \rho g h$ (b) $p_B - \frac{2\sigma\cos\theta}{R}$ (c) $p_{\text{atm}} - \frac{2\sigma\cos\theta}{R}$ (d) All of these

9. A large open tank has two holes in the wall. One is a square hole of side L at a depth h from the top and the other is a circular hole of radius R at a depth $4h$ from the top. When the tank is completely filled with water, quantities of water flowing out per second from both holes are the same. Then R is equal to

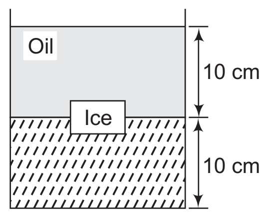
(a) $\frac{L}{\sqrt{2\pi}}$ (b) $2\pi L$ (c) L (d) $\frac{L}{2\pi}$

10. Two identical cylindrical vessels with their bases at the same level each, contain a liquid of density ρ. The area of either base is A but in one vessel the liquid height is h_1 and in the other liquid height is $h_2 (h_2 < h_1)$. If the two vessels are connected, the work done by gravity in equalizing the levels is

(a) $\frac{1}{2}(h_1 - h_2)^2 A\rho g$ (b) $\frac{1}{2}(h_1 + h_2) A\rho g$

(c) $\frac{1}{2}(h_1^2 - h_2^2) A\rho g$ (d) $\frac{1}{4}(h_1 - h_2)^2 A\rho g$

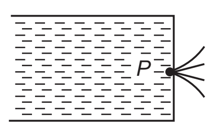
11. A cubical block of side 10 cm floats at the interface of an oil and water as shown in the figure. The density of oil is 0.6 g cm^{-3} and the lower face of ice cube is 2 cm below the interface. The pressure above that of the atmosphere at the lower face of the block is



(a) 200 Pa (b) 620 Pa

(c) 900 Pa (d) 800 Pa

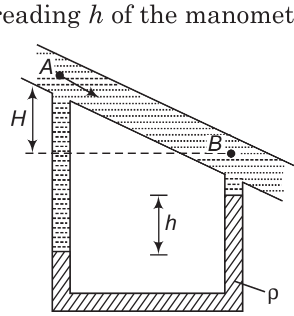
12. A leakage begins in water tank at position P as shown in the figure. The initial gauge pressure (pressure above that of the atmosphere) at P was 5×10^5 N/m^2. If the density of water is 1000 kg/m^3 the initial velocity with which water gushes out is approximately



(a) 3.2 ms^{-1} (b) 32 ms^{-1}

(c) 28 ms^{-1} (d) 2.8 ms^{-1}

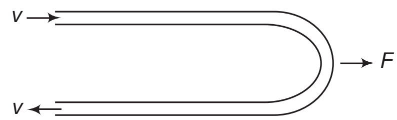
13. The figure shows a pipe of uniform cross-section inclined in a vertical plane. A U-tube manometer is connected between the points A and B. If the liquid of density ρ_0 flows with velocity v_0 in the pipe. Then the reading h of the manometer is



(a) $h = 0$ (b) $h = \frac{v_0^2}{2g}$

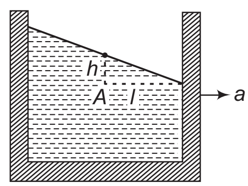
(c) $h = \frac{\rho_0}{\rho}\left(\frac{v_0^2}{2g}\right)$ (d) $h = \frac{\rho_0 H}{\rho - \rho_0}$

14. A horizontal tube of uniform cross-sectional area A is bent in the form of U as shown in figure. If the liquid of density ρ enters and leaves the tube with velocity v , then the external force F required to hold the bend stationary is



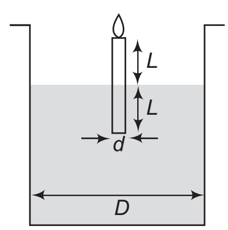
(a) $F = 0$
(b) $\rho A v^2$
(c) $2\rho A v^2$
(d) $\frac{1}{2}\rho A v^2$

15. A rectangular container moves with an acceleration a along the positive direction as shown in figure. The pressure at the point A in excess of the atmospheric pressure p_0 is (take ρ as the density of liquid)



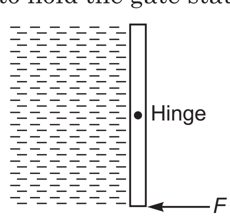
(a) ρgh
(b) ρal
(c) $\rho\ (gh + al)$
(d) Both (a) and (b)

16. A candle of diameter d is floating on a liquid in a cylindrical container of diameter D $(D >> d)$ as shown in figure. It it is burning at the rate of 2 cm/h. Then, the top of the candle will



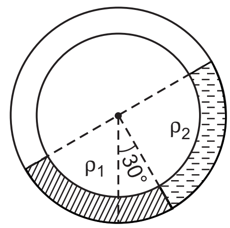
(a) remain at the same height
(b) fall at the rate of 1 cm/h
(c) fall at the rate of 2 cm/h
(d) go up at the rate of 1 cm/h

17. A square gate of size 1 m × 1 m is hinged at its mid point. A fluid of density ρ fills the space to the left of the gate. The force F required to hold the gate stationary is



(a) $\frac{\rho g}{3}$
(b) $\frac{1}{2}\rho g$
(c) $\frac{\rho g}{6}$
(d) None of these

18. A thin uniform circular tube is kept in a vertical plane. Equal volumes of two immiscible liquids whose densities are ρ_1 and ρ_2 fill half of the tube as shown. In equilibrium the radius passing through the interface makes an angle of 30° with vertical. The ratio of densities (ρ_1/ρ_2) is equal to

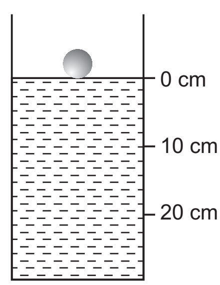


(a) $\dfrac{\sqrt{3}-1}{2-\sqrt{3}}$ (b) $\dfrac{\sqrt{3}+1}{2+\sqrt{3}}$ (c) $\dfrac{\sqrt{3}-1}{\sqrt{3}+1}$ (d) $\dfrac{\sqrt{3}+1}{\sqrt{3}-1}$

19. A plate moves normally with the speed v_1 towards a horizontal jet of water of uniform area of cross-section. The jet discharges water at the rate of volume V per second at a speed of v_2. The density of water is ρ. Assume that water splashes along the surface of the plate ar right angles to the original motion. The magnitude of the force acting on the plate due to the jet of water is

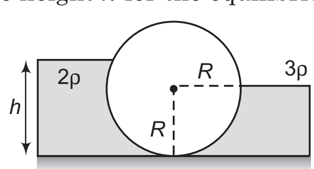
(a) $\rho V v_1$ (b) $\rho\left(\dfrac{V}{v_2}\right)(v_1+v_2)^2$ (c) $\dfrac{\rho V}{v_1+v_2}(v_1)^2$ (d) $\rho V(v_1+v_2)$

20. A spherical ball of density ρ and radius 0.003 m is dropped into a tube containing a viscous fluid up to the 0 cm mark as shown in the figure. Viscosity of the fluid = 1.26 N-s/m^2 and its density $\rho_L = \dfrac{\rho}{2} = 1260\ \text{kg/m}^3$. Assume that the ball reaches a terminal speed at 10 cm mark. The time taken by the ball to travel the distance between the 10 cm and 20 cm mark is ($g = 10\ \text{m/s}^2$)



(a) 2 s (b) 1 s
(c) 0.5 s (d) 5 s

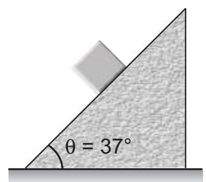
21. In the figure shown, the heavy cylinder (radius R) resting on a smooth surface separates two liquids of densities 2ρ and 3ρ. The height h for the equilibrium of cylinder must be



(a) $3R/2$ (b) $R\sqrt{\dfrac{3}{2}}$
(c) $R\sqrt{2}$ (d) None of these

22. A U-tube having horizontal arm of length 20 cm, has uniform cross-sectional area = 1 cm^2. It is filled with water of volume 60 cc. What volume of a liquid of density 4 g/cc should be poured from one side into the U-tube so that no water is left in the horizontal arm of the tube?
(a) 60 cc (b) 45 cc (c) 50 cc (d) 35 cc

23. A cubical block of side a and density ρ slides over a fixed inclined plane with constant velocity v. There is a thin film of viscous fluid of thickness t between the plane and the block. Then the coefficient of viscosity of the film will be



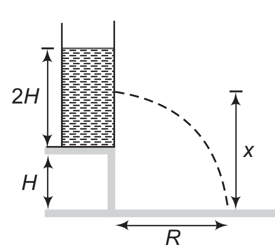
(a) $\frac{3\rho agt}{5v}$ (b) $\frac{4\rho agt}{5v}$ (c) $\frac{\rho agt}{v}$ (d) None of these

24. A spring balance reads 10 kg when a bucket of water is suspended from it. What will be the reading of the balance when an iron piece of mass 7.2 kg suspended by a string is immersed with half its volume inside the water in the bucket? Relative density of iron is 7.2.
(a) 10 kg (b) 10.5 kg (c) 13.6 kg (d) 17.2 kg

25. Three points A, B and C on a steady flow of a non-viscous and incompressible fluid are observed. The pressure, velocity and height of the points A, B and C are (2, 3, 1), (1, 2, 2) and (4, 1, 2) respectively. Density of the fluid is 1 kgm^{-3} and all other parameters are given in SI units. Then which of the following is correct? ($g = 10\ ms^{-2}$).
(a) Points A and B lie on the same stream line
(b) Points B and C lie on the same stream line
(c) Points C and A lie on the same stream line
(d) None of the above

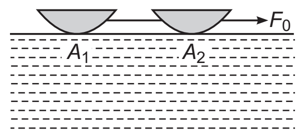
26. A body of density ρ is dropped from rest from height h (from the surface of water) into a lake of density of water σ ($\sigma > \rho$). Neglecting all dissipative effects, the acceleration of body while it is in the lake is
(a) $g\left(\frac{\sigma}{\rho}-1\right)$ upwards (b) $g\left(\frac{\sigma}{\rho}-1\right)$ downwards
(c) $g\left(\frac{\sigma}{\rho}\right)$ upwards (d) $g\left(\frac{\sigma}{\rho}\right)$ downwards

27. A tank is filled up to a height $2H$ with a liquid and is placed on a platform of height H from the ground. The distance x from the ground where a small hole is punched to get the maximum range R is



(a) H (b) 1.25 H (c) 1.5 H (d) 2 H

28. Two boats of base areas A_1 and A_2, connected by a string are being pulled by an external force F_0. The viscosity of water is η and depth of the water body is H. When the system attains a constant speed, the tension in the thread will be



(a) $F_0\left(\frac{A_1}{A_2}\right)$ (b) $F_0\frac{A_2}{(A_1+A_2)}$ (c) $F_0\frac{A_1}{(A_1+A_2)}$ (d) $F_0\left(\frac{A_2}{A_1}\right)$

29. A U-tube is partially filled with water. Oil which does not mix with water is next poured into one side, until water rises by 25 cm on the other side. If the density of oil is 0.8 g/cm^3, the oil level will stand higher than the water level by

(a) 6.25 cm (b) 12.50 cm (c) 31.75 cm (d) 25 cm

30. There is a horizontal film of soap solution. On it a thread is placed in the form of a loop. The film is punctured inside the loop and the thread becomes a circular loop of radius R. If the surface tension of the soap solution be T, then the tension in the thread will be

(a) $\pi R^2/T$ (b) π^2R^2T
(c) $2\pi RT$ (d) $2RT$

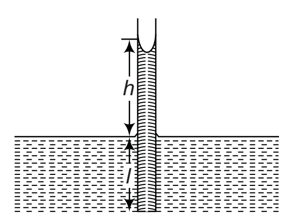
31. A thin metal disc of radius r floats on a liquid surface and bends the surface downwards along the perimeter making an angle θ with vertical edge of the disc. If the disc displaces a weight of liquid w and surface tension of liquid is T, then the weight of metal disc is

(a) $2\pi rT + w$ (b) $2\pi rT\cos\theta - w$
(c) $2\pi rT\cos\theta + w$ (d) $w - 2\pi rT\cos\theta$

32. The radii of the two columns is U-tube are r_1, and $r_2 (> r_1)$. When a liquid of density ρ (angle of contact is 0°) is filled in it, the level difference of liquid in two arms is h. The surface tension of liquid is
(g = acceleration due to gravity)

(a) $\frac{\rho g h r_1 r_2}{2(r_2-r_1)}$ (b) $\frac{\rho gh(r_2-r_1)}{2r_1r_2}$
(c) $\frac{2(r_2-r_1)}{\rho gh r_1 r_2}$ (d) $\frac{\rho gh}{2(r_2-r_1)}$

33. Water rises to a height h in a capillary tube lowered vertically into water to a depth l as shown in the figure. The lower end of the tube is now closed, the tube is then taken out of the water and opened again. The length of the water column remaining in the tube will be



(a) $2h$ if $l \geq h$ and $l + h$ if $l \leq h$
(b) h if $l \geq h$ and $l + h$ if $l \leq h$
(c) $4h$ if $l \geq h$ and $l - h$ if $l \leq h$
(d) $\frac{h}{2}$ if $l \geq h$ and $l + h$ if $l \leq h$

34. Two parallel glass plates are dipped partly in the liquid of density 'd' keeping them vertical. If the distance between the plates is 'x', Surface tension for liquid is T and angle of contact is θ then rise of liquid between the plates due to capillary will be

(a) $\frac{T\cos\theta}{xd}$ (b) $\frac{2T\cos\theta}{xdg}$ (c) $\frac{2T}{xdg\cos\theta}$ (d) $\frac{T\cos\theta}{xdg}$

35. Two identical spherical soap bubbles collapses. If V is the consequent change in volume of the contained air, S is the change in the total surface area and T is the surface tension of the soap solution, then (if p_0 is atmospheric pressure and assume temperature to remain same in all the bubbles).

(a) $3p_0V + 4ST = 0$ (b) $4p_0V + 3ST = 0$
(c) $p_0V + 4TS = 0$ (d) $4p_0V + ST = 0$

36. A cylinder with a movable piston contains air under a pressure p_1 and a soap bubble of radius 'r'. The pressure p_2 to which the air should be compressed by slowly pushing the piston into the cylinder for the soap bubble to reduce its size by half will be (The surface tension is σ, and the temperature T is maintained constant).

(a) $\left[8p_1 + \frac{24\sigma}{r}\right]$ (b) $\left[4p_1 + \frac{24\sigma}{r}\right]$
(c) $\left[2p_1 + \frac{24\sigma}{r}\right]$ (d) $\left[2p_1 + \frac{24\sigma}{r}\right]$

37. A thin metal ring of internal radius 8 cm and external radius 9 cm is supported horizontally from the pan of a balance so that it comes in contact with water in a glass vessel, If is found that an extra weight of 7.48 g is required to pull the ring out of water. The surface tension of water is ($g = 10\ \text{m/s}^2$)

(a) 80×10^{-3} N/m (b) 25×10^{-3} N/m
(c) 45×10^{-3} N/m (d) 70×10^{-3} N/m

More than One Correct Options

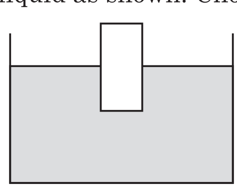
1. A large wooden plate of area 10 m^2 floating on the surface of a river is made to move horizontally with a speed of 2 m/s by applying a tangential force. River is 1 m deep and the water in contact with the bed is stationary. Then choose correct statement(s).
(coefficient of viscosity of water = 10^{-3} N-s/m^2)

(a) velocity gradient is $2\ \text{s}^{-1}$
(b) velocity gradient is $1\ \text{s}^{-1}$
(c) force required to keep the plate moving with constant speed is 0.02 N
(d) force required to keep the plate moving with constant speed is 0.01 N

2. Choose the correct options.

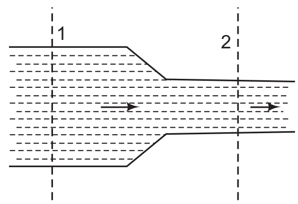
(a) Viscosity of liquids increases with temperature
(b) Viscosity of gases increases with temperature
(c) Surface tension of liquids decreases with temperature
(d) For angle of contact $\theta = 0°$, liquid neither rises nor falls on capillary

3. A plank is floating in a non-viscous liquid as shown. Choose the correct options.



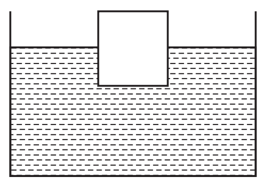
(a) Equilibrium of plank is stable in vertical direction
(b) For small oscillations of plank in vertical direction motion is simple harmonic
(c) Even if oscillations are large, motion is simple harmonic till it is not fully immersed
(d) On vertical displacement motion is periodic but not simple harmonic

4. A non-viscous incompressible liquid is flowing from a horizontal pipe of non-uniform cross section as shown. Choose the correct options.



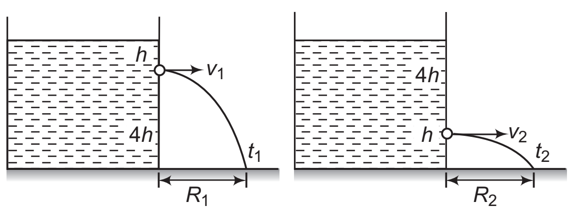
(a) speed of liquid at section-2 is more
(b) volume of liquid flowing per second from section-2 is more
(c) mass of liquid flowing per second at both the sections is same
(d) pressure at section-2 is less

5. A plank is floating in a liquid as shown. Fraction f of its volume is immersed. Choose the correct options.



(a) If the system is taken to a place where atmospheric pressure is more, f will increase
(b) In above condition f will remain unchanged
(c) If temperature is increased and expansion of only liquid is considered f will increase
(d) If temperature is increased and expansion of only plank is considered f will decrease

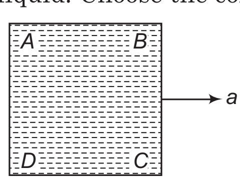
6. In two figures,



(a) $v_1/v_2 = \frac{1}{2}$ (b) $t_1/t_2 = 2$

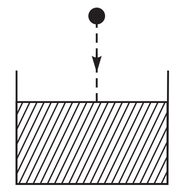
(c) $R_1/R_2 = 1$ (d) $v_1/v_2 = \frac{1}{4}$

7. A liquid is filled in a container as shown in figure. Container is accelerated towards right. There are four points A, B, C and D in the liquid. Choose the correct options.



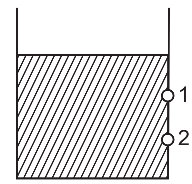
(a) $p_A > p_B$ (b) $p_C > p_A$
(c) $p_D > p_B$ (d) $p_A > p_C$

8. A ball of density ρ is dropped from a height on the surface of a non-viscous liquid of density 2ρ. Choose the correct options.



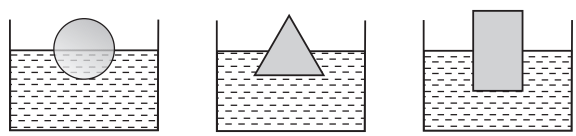
(a) Motion of ball is periodic but not simple harmonic
(b) Acceleration of ball in air and in liquid are equal
(c) Magnitude of upthrust in the liquid is two times the weight of ball
(d) Net force on ball in air and in liquid are equal and opposite

9. Two holes 1 and 2 are made at depths h and $16h$ respectively. Both the holes are circular but radius of hole-1 is two times.



(a) Initially equal volumes of liquid will flow from both the holes in unit time
(b) Initially more volume of liquid will flow from hole-2 per unit time
(c) After some time more volume of liquid will flow from hole-1 per unit time
(d) After some time more volume of liquid will flow from hole-2 per unit time

10. A solid sphere, a cone and a cylinder are floating in water. All have same mass, density and radius. Let f_1, f_2 and f_3 are the fraction of their volumes inside the water and h_1, h_2 and h_3 are the depths inside water. Then

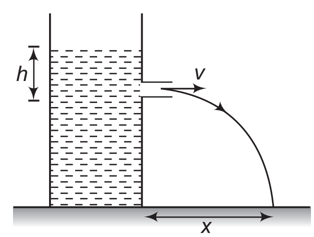


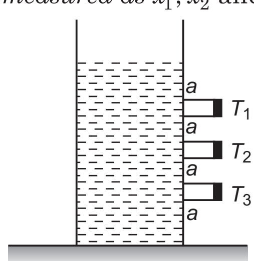
(a) $f_1 = f_2 = f_3$
(b) $f_3 > f_2 > f_1$
(c) $h_3 < h_1$
(d) $h_3 < h_2$

Comprehension Based Questions

Passage 1 (Q. Nos. 1 to 3)

The spouting can is something used to demonstrate the variation of pressure with depth. When the corks are removed from the tubes in the side of the can, water flows out with a speed that depends on the depth. In a certain can, three tubes T_1, T_2 and T_3 are set at equal distances 'a' above the base of the can. When water contained in this can is allowed to come out of the tubes, the distances on the horizontal surface are measured as x_1, x_2 and x_3.





1. Speed of efflux is

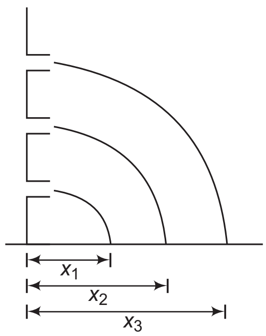
(a) $\sqrt{3gh}$ (b) $\sqrt{2gh}$ (c) $\sqrt{gh}$ (d) $\frac{1}{2}\sqrt{2gh}$

2. Distance x_3 is given by

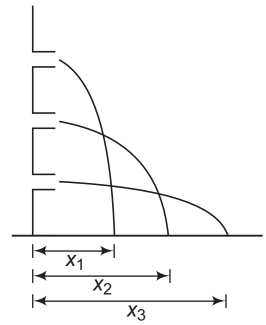
(a) $\sqrt{3}\ a$ (b) $\sqrt{2}\ a$ (c) $\frac{1}{2}\sqrt{3}\ a$ (d) $2\sqrt{3}\ a$

3. The correct sketch is

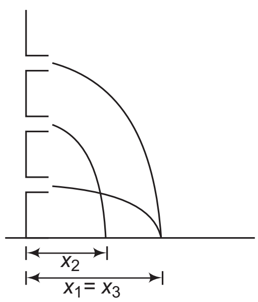
(a)



(b)



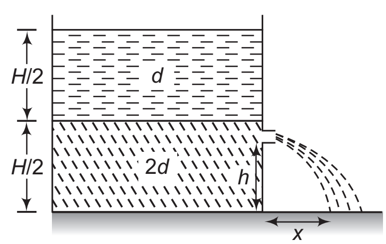
(c)



(d) None of these

Passage 2 (Q. Nos. 4 to 7)

A container of large uniform cross-sectional area A resting on a horizontal surface, holds two immiscible, non-viscous and incompressible liquids of densities d and $2d$ *each of height* $H/2$ *as shown in the figure. The lower density liquid is open to the atmosphere having pressure* P_0. *A homogeneous solid cylinder of length* $L(L < H/2)$ *and cross-sectional area* $A/5$ *is immersed such that it floats with its axis vertical at the liquid-liquid interface with length* $L/4$ *in the denser liquid.*



The cylinder is then removed and the original arrangement is restored. A tiny hole of area $s(s << A)$ *is punched on the vertical side of the container at a height* $h(h < H/2)$. *As a result of this, liquid starts flowing out of the hole with a range* x *on the horizontal surface.*

4. The density D of the material of the floating cylinder is

(a) $5d/4$ (b) $3d/4$ (c) $4d/5$ (d) $4d/3$

5. The total pressure with cylinder, at the bottom of the container is

(a) $p_0 + \dfrac{(6L + H)}{4} dg$ (b) $p_0 + \dfrac{(L + 6H)}{4} dg$

(c) $p_0 + \dfrac{(L + 3H)}{4} dg$ (d) $p_0 + \dfrac{(L + 2H)}{4} dg$

6. The initial speed of efflux without cylinder is

(a) $v=\sqrt{\frac{g}{3}[3H+4h]}$ (b) $v=\sqrt{\frac{g}{2}[4H-3h]}$

(c) $v=\sqrt{\frac{g}{2}[3H-4h]}$ (d) None of these

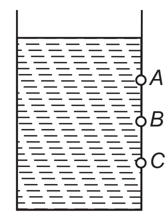
7. The initial value of x is

(a) $\sqrt{(3H+4h)\,h}$ (b) $\sqrt{(3h+4H)h}$

(c) $\sqrt{(3H-4h)\,h}$ (d) $\sqrt{(3H-3h)\,h}$

Match the Columns

1. Three holes A, B and C are made at depths 1m, 2 m and 5 m as shown. Total height of liquid in the container is 8 m. Let v is the speed with which liquid comes out of the hole and R the range on ground. Match the following two columns.

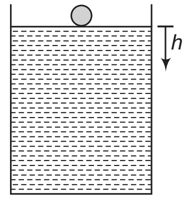


Column I	Column II
(a) v is maximum for	(p) hole A
(b) v is minimum for	(q) hole B
(c) R is maximum for	(r) hole C
(d) R is minimum for	(s) will depend on density of liquid

2. Match the following two columns.

Column I	Column II
(a) When temperature is increased	(p) Upthrust on a floating solid of constant volume will increase
(b) When density of liquid is increased	(q) Upthrust on a floating solid of constant volume will decrease
(c) When density of solid is increased	(r) Viscosity of gas will decrease
(d) When atmospheric pressure is increased	(s) None

3. A ball of density ρ is released from the surface of a liquid whose density varies with depth h as, $\rho_l=\alpha\,h$. Here α is a positive constant. Match the following two columns. (liquid is ideal)

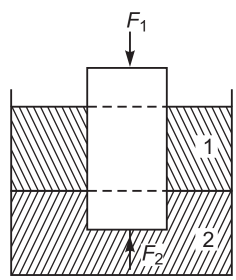


Column I	Column II
(a) Upthrust on ball	(p) will continuously decrease
(b) Speed of ball	(q) will continuously increase
(c) Net force on ball	(r) first increase then decrease
(d) Gravitational potential energy of ball	(s) first decrease then increase

4. Match the following two columns.

Column I	Column II
(a) Surface tension	(p) $[ML^{-1}T^{-2}]$
(b) Coefficient of viscosity	(q) $[L^3T^{-1}]$
(c) Energy density	(r) $[MT^{-2}]$
(d) Volume flow rate	(s) $[ML^{-1}T^{-1}]$

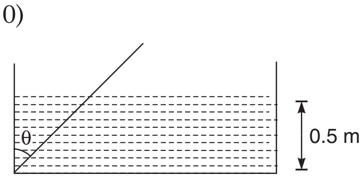
5. A cylinder of weight W is floating in two liquids as shown in figure. Net force on cylinder from top is F_1 and force on cylinder from the bottom is F_2. Match the following two columns.



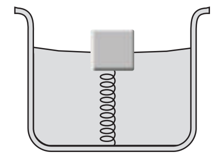
Column I	Column II
(a) Net force on cylinder from liquid-1	(p) Zero
(b) $F_2 - F_1$	(q) W
(c) Net force on cylinder from liquid-2	(r) Net upthrust
(d) Net force on cylinder from atmosphere	(s) None of these

Subjective Questions

1. A wooden plank of length 1 m and uniform cross-section is hinged at one end to the bottom of a tank as shown in figure. The tank is filled with water upto a height of 0.5 m. The specific gravity of the plank is 0.5. Find the angle θ that the plank makes with the vertical in the equilibrium position. (Exclude the case θ = 0)

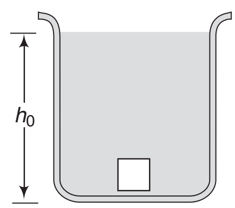


2. A cubical block of wood of edge 3 cm floats in water. The lower surface of the cube just touches the free end of a vertical spring fixed at the bottom of the pot. Find the maximum weight that can be put on the block without wetting the new weight. Density of wood = 800 kg/m^3 and spring constant of the spring = 50 N/m. (Take $g = 10$ m/s^2)

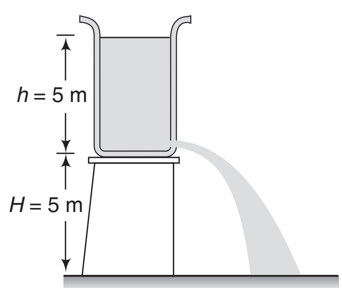


3. Figure shows a container having liquid of variable density. The density of liquid varies as $\rho = \rho_0\left(4 - \frac{3h}{h_0}\right)$. Here, h_0 and ρ_0 are constants and h is measured from bottom of the container.

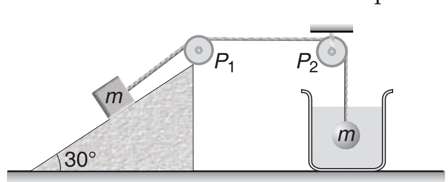
A solid block of small dimensions whose density is $\frac{5}{2}\rho_0$ and mass m is released from bottom of the tank. Prove that the block will execute simple harmonic motion. Find the frequency of oscillation.



4. A cylindrical tank 1 m in radius rests on a platform 5 m high. Initially the tank is filled with water to a height of 5 m. A plug whose area is 10^{-4}m^2 is removed from an orifice on the side of the tank at the bottom. Calculate (a) initial speed with which the water flows from the orifice, (b) initial speed with which water strikes the ground, (c) time taken to empty the tank to half its original value. ($g = 10$ m/ s^2)

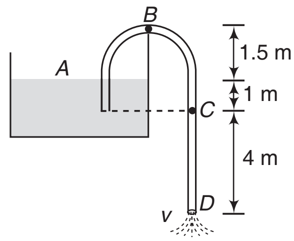


5. A block of mass m is kept over a fixed smooth wedge. Block is attached to a sphere of same mass through fixed massless pullies P_1 and P_2. Sphere is dipped inside the water as shown. If specific gravity of material of sphere is 2. Find the acceleration of sphere.



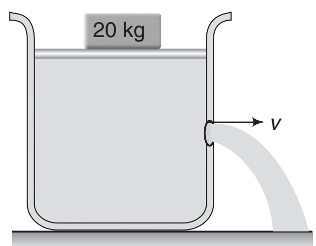
6. A cubic body floats on mercury with 0.25 fraction of its volume below the surface. What fraction of the volume of the body will be immersed in the mercury if a layer of water poured on top of the mercury covers the body completely ?

7. A siphon tube is discharging a liquid of specific gravity 0.9 from a reservoir as shown in the figure.

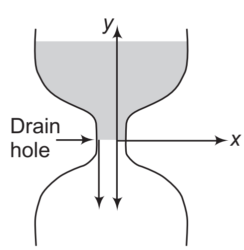


(a) Find the velocity of the liquid through the siphon.
(b) Find the pressure at the highest point B.
(c) Find the pressure at point C.

8. A long cylindrical tank of cross-sectional area 0.5 m^2 is filled with water. It has a small hole at a height 50 cm from the bottom. A movable piston of cross-sectional area almost equal to 0.5 m^2 is fitted on the top of the tank such that it can slide in the tank freely. A load of 20 kg is applied on the top of the water by piston, as shown in the figure. Calculate the speed of the water jet with which it hits the surface when piston is 1 m above the bottom. (Ignore the mass of the piston).



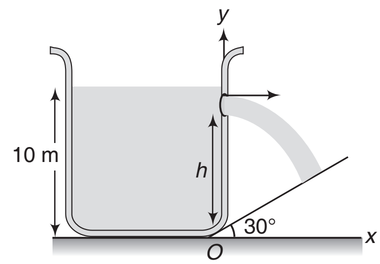
9. The shape of an ancient water clock jug is such that water level descends at a constant rate at all times. If the water level falls by 4 cm every hour, determine the shape of the jar, i.e. specify x as a function of y. The radius of drain hole is 2 mm and can be assumed to be very small compared to x.



10. A spring is attached to the bottom of an empty swimming pool, with the axis of the spring oriented vertically. An 8.00 kg block of wood ($\rho = 840\ kg/m^3$) is fixed to the top of the spring and compresses it. Then the pool is filled with water, completely covering the block. The spring is now observed to be stretched twice as much as it had been compressed. Determine the percentage of the block's total volume that is hollow. Ignore any air in the hollow space.

11. A rectangular tank of height 10 m filled with water, is placed near the bottom of a plane inclined at an angle 30° with horizontal. At height h from bottom a small hole is made (as shown in figure) such that the stream coming out from hole, strikes the inclined plane normally.

Calculate h.



12. A ball of density d is dropped onto a horizontal solid surface. It bounces elastically from the surface and returns to its original position in a time t_1. Next, the ball is released and it falls through the same height before striking the surface of a liquid of density d_L.

(a) If $d < d_L$, obtain an expression (in terms of d, t_1 and d_L) for the time t_2 the ball takes to come back to the position from which it was released.

(b) Is the motion of the ball simple harmonic ?

(c) If $d = d_L$, how does the speed of the ball depend on its depth inside the liquid ? Neglect all frictional and other dissipative forces. Assume the depth of the liquid to be large.

13. There is an air bubble of radius 1.0 mm in a liquid of surface tension 0.075 N/m and density 1000 kg/m^3. The bubble is at a depth of 10 cm below the free surface. By what amount is the pressure inside the bubble greater than the atmospheric pressure ? (Take g = 9.8 m/s^2)

14. A metal sphere of radius 1 mm and mass 50 mg falls vertically in glycerine. Find

(a) the viscous force exerted by the glycerine on the sphere when the speed of the sphere is 1 cm/s,

(b) the hydrostatic force exerted by the glycerine on the sphere and (c) the terminal velocity with which the sphere will move down without acceleration. Density of glycerine = 1260 kg/m^3 and its coefficient of viscosity at room temperature = 8.0 poise.

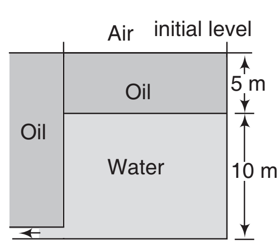
15. A wire forming a loop is dipped into soap solution and taken out, so that a film of soap solution is formed. A loop of 6.28 cm long thread is gently put on the film and the film is pricked with a needle inside the loop. The thread loop takes the shape of a circle. Find the tension in the thread. Surface tension of soap solution = 0.030 N/m.

16. A cylindrical vessel is filled with water upto a height of 1 m. The cross-sectional area of the orifice at the bottom is (1/400) that of the vessel.

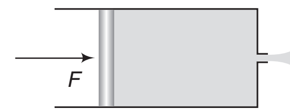
(a) What is the time required to empty the tank through the orifice at the bottom?

(b) What is the time required for the same amount of water to flow out if the water level in tank is maintained always at a height of 1 m from orifice?

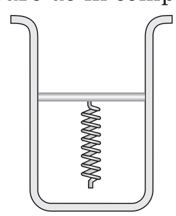
17. A tank having a small circular hole contains oil on top of water. It is immersed in a large tank of the same oil. Water flows through the hole. What is the velocity of this flow initially? When the flow stops, what would be the position of the oil-water interface in the tank from the bottom. The specific gravity of oil is 0.5.



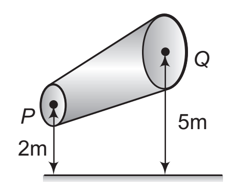
18. What work should be done in order to squeeze all water from a horizontally located cylinder (figure) during the time t by means of a constant force acting on the piston? The volume of water in the cylinder is equal to V, the cross-sectional area of the orifice is s, with s being considerably less than the piston area. The friction and viscosity are negligibly small. Density of water is ρ.



19. A cylinder is fitted with a piston, beneath which is a spring, as in the figure. The cylinder is open at the top. Friction is absent. The spring constant of the spring is 3600 N/m. The piston has a negligible mass and a radius of 0.025 m. (a) When air beneath the piston is completely pumped out, how much does the atmospheric pressure cause the spring to compress? (b) How much work does the atmospheric pressure do in compressing the spring?



20. A non-viscous liquid of constant density 1000 kg/m^3 flows in a streamline motion along a tube of variable cross-section. The tube is kept inclined in the vertical plane as shown in the figure. The area of cross-section of the tube at two points P and Q at heights of 2 m and 5 m are respectively 4×10^{-3} m^2 and 8×10^{-3} m^2. The velocity of the liquid at point P is 1 m/s. Find the work done per unit volume by the pressure and the gravity forces as the fluid flows from point P to Q. Take $g = 9.8 \text{ m/s}^2$.



21. A glass plate of length 10 cm, breadth 1.54 cm and thickness 0.20 cm weighs 8.2 gm in air. It is held vertically with the long side horizontal and the lower half under water. Find the apparent weight of the plate. Surface tension of water = 73 dyne per cm, $g = 980 \text{ cm/sec}^2$.

22. Two narrow bores of diameters 3.0 mm and 6.0 mm are joined together to form a U-shaped tube open at both ends. If the U-tube contains water, what is the difference in its levels in the two limbs of the tube? Surface tension of water at the temperature of the experiment is 7.3×10^{-2} Nm^{-1}. Take the angle of contact to be zero, and density of water to be $1.0 \times 10^3 \text{ kg/m}^3$ ($g = 9.8 \text{ ms}^{-2}$).

23. Two identical soap bubbles each of radius r and of the same surface tension T combine to form a new soap bubble of radius R. The two bubbles contain air at the same temperature. If the atmospheric pressure is p_0 then find the surface tension T of the soap solution in terms of p_0, r and R. Assume process is isothermal.

Note *Students are advised to attempt question numbers 24 and 25 after studying the chapter of electrostatics in class XII.*

24. A soap bubble of radius r and surface tension 'T' is given a potential of' V volt. Show that the new radius 'R' of the bubble is related to its initial radius by equation ,

$$p_0[R^3 - r^3] + 4T\,[R^2 - r^2] - \varepsilon_0 V^2 R/2 = 0$$

where, p_0 is the atmosphere pressure.

25. If the radius and surface tension of a spherical soap bubble are 'R' and 'T' respectively, then show that the charge required to double its radius would be, $8\pi R\,[\varepsilon_0 R\,[7p_0 R + 12T]]^{1/2}$, where p_0 is the atmospheric pressure.

Answers

Introductory Exercise 16.1

1. 18 cm **2.** 27 cm **3.** (a) 1.09×10^5 N/m^2 (b) 1.12×10^5N/m^2 **4.** $p = p_0 + \dfrac{mg}{A}$ **5.** 50 N **6.** 50 cm

Introductory Exercise 16.2

1. Maximum at C and minimum at B **2.** 2g **3.** 3×10^5 N / m^2

Introductory Exercise 16.3

2. 23.7 N **3.** 0.206 N **4.** (a) 14.7 m / s^2 (b) 0.63 s

Introductory Exercise 16.4

1. (a) 20 cm/s (b) 485 N / m^2 **2.** (a) 25 cm/s (b) 50 cm/s (c) 94 N / m^2 **3.** (c) **4.** 500

Introductory Exercise 16.5

1. 20 m/s **2.** $\left(\dfrac{t_0}{\sqrt{3} - 1}\right)$ **3.** (a)

Introductory Exercise 16.6

1. 0.882 mm/s **2.** $v' = (2)^{2/3} v$ **3.** $F = 0.02$ N **4.** 10^{-3} N / m^2

Introductory Exercise 16.7

1. 4.35×10^{-3}J **2.** 1.44×10^{-5} J **3.** $8\pi R^2 T$ **4.** $p = p_0 + h\rho g + \dfrac{2T}{r}$

Introductory Exercise 16.8

1. 6.0 cm **2.** $\dfrac{T_m}{T_w} = 7.23$ **3.** 1.4 mm **4.** (d)

Exercises

LEVEL 1

Assertion and Reason

1. (d) **2.** (a) **3.** (d) **4.** (a) **5.** (d) **6.** (d) **7.** (b) **8.** (a) **9.** (d) **10.** (d)
11. (c)

Single Correct Option

1. (a) **2.** (d) **3.** (d) **4.** (a) **5.** (d) **6.** (b) **7.** (a) **8.** (a) **9.** (d) **10.** (d)
11. (a) **12.** (b) **13.** (c) **14.** (a) **15.** (c) **16.** (c) **17.** (c) **18.** (b) **19.** (a) **20.** (b)
21. (b) **22.** (a) **23.** (c) **24.** (a) **25.** (a) **26.** (d) **27.** (c) **28.** (d) **29.** (a) **30.** (d)
31. (c) **32.** (b) **33.** (c) **34.** (c) **35.** (d) **36.** (d) **37.** (d) **38.** (d) **39.** (c) **40.** (c)
41. (d) **42.** (c) **43.** (a) **44.** (d) **45.** (d) **46.** (a) **47.** (d) **48.** (a) **49.** (d)

Subjective Questions

1. False **2.** Zero **3.** Vessel B **4.** $2^{2/3}W$
5. It will remain same **6.** 3840 kg/m^3 **7.** 28 cm^3 **8.** 0.33 m^3
9. 17 cm **10.** 0.6 gm/cm^3, 0.705 gm/cm^3 **11.** 6.03 g
12. 6.92×10^5 Pa **13.** 0.8
14. Mercury will rise in the arm containing sprit. The difference in level is 0.221 cm

15. (i) Absolute pressure = 96 cm of Hg, Gauge pressure = 20 cm of Hg for (a),
absolute pressure = 58 cm of Hg, gauge pressure = – 18 cm of Hg for (b)
(ii) Mercury would rise in the left limb such that the difference in the levels in the two limbs becomes 19 cm.

16. (a) 2.25 m/s (b) zero **17.** (a) 7.0×10^{-2} m^3/s (b) 2.24×10^4 Pa (c) 1.5×10^{-1} m^3/s

18. (a) $2\times 10^{-6} m^3$, 5000 kg/m^3 (b) 750 kg/m^3 **19.** 32 mm in mercury and 28 mm in water

20. $T = 20$ N, $V = 32\times 10^{-3} m^3$ **21.** (a) 24 N (b) 12 m/s^2

22. (a) 10.9 cm (b) 2.5 cm 23. (a) $\dfrac{\rho g/h^2}{2}$ (b) $\dfrac{\rho g/h^3}{6}$ (c) $\dfrac{h}{3}$ **24.** 0.58 cm **25.** $\dfrac{R^2}{4h}$

26. $v = \sqrt{\dfrac{2F}{A\rho}}$ **27.** 12.4 m/s **28.** $v_B \approx 3.28$ m/s **29.** 19.5 mm of Hg **30.** $h = 1$ cm, if seal is broken water will rise in the capillary.

31. 14.4×10^{-6} J **32.** 1860 N/m^2 **33.** 440 $dyne/cm^2$ **34.** 4.76 mm

35. 9.48 cm **37.** $\dfrac{2T}{\rho g r}$

LEVEL 2

Single Correct Option

1. (a) **2.** (d) **3.** (c) **4.** (c) **5.** (c) **6.** (a) **7.** (b) **8.** (d) **9.** (a) **10.** (d)
11. (d) **12.** (b) **13.** (a) **14.** (c) **15.** (b) **16.** (b) **17.** (c) **18.** (d) **19.** (d) **20.** (d)
21. (b) **22.** (d) **23.** (a) **24.** (b) **25.** (d) **26.** (a) **27.** (c) **28.** (c) **29.** (b) **30.** (d)
31. (c) **32.** (a) **33.** (a) **34.** (b) **35.** (a) **36.** (a) **37.** (d)

More than One Correct Options

1. (a,c) **2.** (b,c) **3.** (a,b,c) **4.** (a,c,d) **5.** (b,c,d) **6.** (a,b,c) **7.** (a,c)
8. (a,c,d) **9.** (a,d) **10.** (a,c,d)

Comprehension Based Questions

1. (b) **2.** (d) **3.** (d) **4.** (a) **5.** (b) **6.** (c) **7.** (c)

Match the Columns

1. (a) → r , (b) → p, (c) → r, (d) → p
2. (a) → s, (b) → s, (c) → p, (d) → s
3. (a) → r, (b) → r, (c) → s, (d) → s
4. (a) → r, (b) → s, (c) → p, (d) → q
5. (a) → p, (b) → q,r, (c) → s, (d) → s

Subjective Questions

1. 45° **2.** 0.354 N **3.** $\dfrac{1}{2\pi}\sqrt{\dfrac{6g}{5h_0}}$ **4.** (a) 10 m/s (b) 14.1 m/s (c) 9200 s

5. zero **6.** 0.19 **7.** (a) 9.9 m/s (b) 4.36×10^4 Pa (c) 6.6×10^4 Pa **8.** 4.51 m/s

9. $y = 0.4x^4$ **10.** 60.41% **11.** 8.33 m

12. (a) $\dfrac{t_1 d_L}{d_L - d}$ (b) No (c) The ball will continue to move with constant velocity $v = \dfrac{gt_1}{2}$ inside the liquid.

13. 1130 Pa **14.** (a) 1.5×10^{-4} N (b) 5.2×10^{-5} N (c) 32.5 m/s **15.** 3.0×10^{-4} N

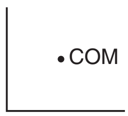
16. (a) 3 min (b) 1.5 min **17.** 9.8 m/s, 5.0 m **18.** $\dfrac{1}{2}\dfrac{\rho V^3}{s^2 t^2}$ **19.** (a) 5.5 cm (b) 5.445 J

20. 29025 J/m^3, 29400 J/m^3 **21.** 8.1796 g **22.** 5 mm **23.** $T = \dfrac{p_0(2r^3 - R^3)}{4(R^2 - 2r^2)}$

Hints & Solutions

11. Centre of Mass, Linear Momentum and Collision

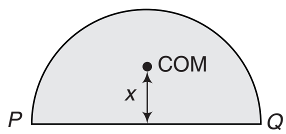
INTRODUCTORY EXERCISE 11.1

2. 

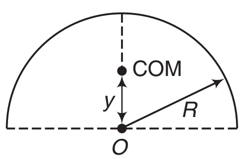
The COM of L-shaped rod shown in above figure is lying outside the body.

4. $x_{CM} = \dfrac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$

$x_1 = x_2 = 0 \Rightarrow x_{CM} = 0$

5. 

$x < r/2$, as more mass is concentrated near the line PQ.

6. 

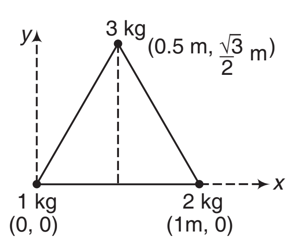
All particles of semicircular ring lie at a distance R from origin O. But its COM distance y from O is less than R.

7. $$x_{CM} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3}$$

$$= \frac{(1)(0) + (2)(1) + (3)(0.5)}{1 + 2 + 3} = \frac{7}{12} \text{ m}$$

$$y_{CM} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3}$$

$$= \frac{(1)(0) + (2)(0) + (3)\left(\frac{\sqrt{3}}{2}\right)}{1 + 2 + 3}$$



$$= \frac{\sqrt{3}}{4} \text{ m}$$

$$\therefore \quad d = \sqrt{x_{CM}^2 + x_{CM}^2}$$

$$= \frac{\sqrt{19}}{6} \text{ m} \qquad \textbf{Ans.}$$

8. $$y_{CM} = \frac{A_1 y_1 - A_2 y_2}{A_1 - A_2}$$

$$= \frac{(\pi b^2/2)(4b/3\pi) - (\pi a^2/2)(4a/3\pi)}{(\pi b^2/2) - (\pi a^2/2)}$$

$$= \frac{4}{3\pi}\left[\frac{b^3 - a^3}{b^2 - a^2}\right]$$

$$= \frac{4}{3\pi}\left\{\frac{a^2 + ab + b^2}{a + b}\right\} \qquad \textbf{Ans.}$$

9. $$x_{CM} = \frac{A_1 x_1 - A_2 x_2}{A_1 - A_2}$$

$$= \frac{(4a^2)(a) - (a^2)(3a/2)}{4a^2 - a^2} = \frac{5a}{6} \qquad \textbf{Ans.}$$

$$y_{CM} = \frac{A_1 y_1 - A_2 y_2}{A_1 - A_2}$$

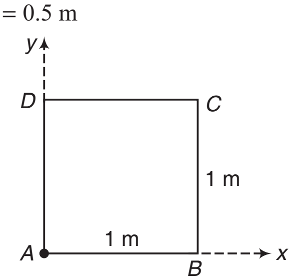
$$= \frac{(4a^2)(a) - (a^2)(3a/2)}{4a^2 - a^2}$$

$$= \frac{5a}{6} \qquad \textbf{Ans.}$$

10. $$x_{CM} = \frac{m_A x_A + m_B x_B + m_C x_C + m_D x_D}{m_A + m_B + m_C + m_D}$$

$$= \frac{(1)(0) + 2(1) + 3(1) + 4(0)}{1 + 2 + 3 + 4}$$

$$= 0.5 \text{ m}$$



$$y_{CM} = \frac{m_A y_A + m_B y_B + m_C y_C + m_D y_D}{m_A + m_B + m_C + m_D}$$

$$= \frac{(1)(0) + 2(0) + 3(1) + 4(1)}{1 + 2 + 3 + 4} = 0.7 \text{ m}$$

$$\therefore d^2 = x_{CM}^2 + y_{CM}^2$$

$$= 0.74 \text{ m}^2 \qquad \textbf{Ans.}$$

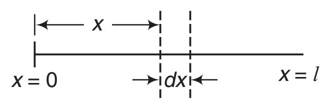
11. $d = \dfrac{A_1 d_1 + A_2 d_2}{A_1 + A_2}$

Here, d = distance of centre of mass from O.

$$\therefore \quad d = \frac{(a^2)(0) + \left(\frac{\pi}{4} a^2\right)(a)}{a^2 + (\pi/4)\, a^2}$$

$$= \left(\frac{\pi}{\pi + 4}\right) a$$

12.



Let A = area of cross section of rod.

$$x_{CM} = \frac{\int_0^l x\,dm}{\int_0^l dm} = \frac{\int_0^l (x)(A\,dx)\,\rho_0 \frac{x^2}{l^2}}{\int_0^l (A\,dx)\,\rho_0 \frac{x^2}{l^2}}$$

$$= \frac{3l}{4}$$ **Ans.**

13. $$x_{CM} = \frac{\int_0^L x\,dm}{\int_0^L dm} = \frac{\int_0^L (X)(Ax\,dx)}{\int_0^L (Ax\,dx)} = \frac{2}{3} L$$ **Ans.**

INTRODUCTORY EXERCISE 11.2

1. $$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$$= \frac{m_1 (x_i + vt)_1 + m_2 (x_i + vt)_2}{m_1 + m_2}$$

$$= \frac{(1)(10 - 6 \times 2) + (2)(12 + 4 \times 2)}{1 + 2}$$

$$= 12.67 \text{ m}$$ **Ans.**

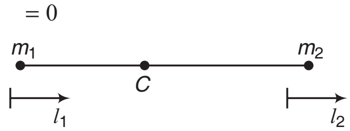
2. $$S_{CM} = \frac{m_1 S_1 + m_2 S_2}{m_1 + m_2}$$

$$= \frac{m_1 (v_1 t) + m_2 (v_2 t)}{m_1 + m_2}$$

$$= \frac{(1)(2)(1) + (2)(-1)(1)}{1 + 2}$$

$$= 0$$

3.



Displacement of centre of mass,

$$S_{CM} = \frac{m_1 l_1 + m_2 l_2}{m_1 + m_2}$$ **Ans.**

4. (a) $$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$$\Rightarrow \quad 3 = \frac{m_1 (0) + (0.10)(12)}{m_1 + 0.1}$$

Solving this equation we get,

$$m_1 = 0.3 \text{ kg}$$ **Ans.**

(b) $$\mathbf{P}_{CM} = m_{CM} \mathbf{v}_{CM}$$

$$= (0.1 + 0.3)(6\hat{\mathbf{j}})$$

$$= (2.4\hat{\mathbf{j}}) \text{ kg-m/s}$$ **Ans.**

(c) $$\mathbf{P}_{CM} = \mathbf{P}_1 + \mathbf{P}_2$$

$$\therefore \quad (2.4\hat{\mathbf{j}}) = (0.3)\mathbf{v}_1 + (0.1)(0)$$

$$\therefore \quad \mathbf{v}_1 = (8\hat{\mathbf{j}}) \text{ m/s}$$ **Ans.**

5. First stone gets 300 ms journey time and second stone gets 200 ms journey time.

(a) $$d_{CM} = \frac{m_1 d_1 + m_2 d_2}{m_1 + m_2}$$

$$= \frac{m_1\left(\frac{1}{2} g t_1^2\right) + m_2\left(\frac{1}{2} g t_2^2\right)}{m_1 + m_2}$$

$$= \frac{(m)\left(\frac{1}{2} \times 10 \times 0.3 \times 0.3\right) + 2m\left(\frac{1}{2} \times 10 \times 0.2 \times 0.2\right)}{m + 2m}$$

$$= 0.283 \text{ m}$$

$$\approx 28 \text{ cm}$$ **Ans.**

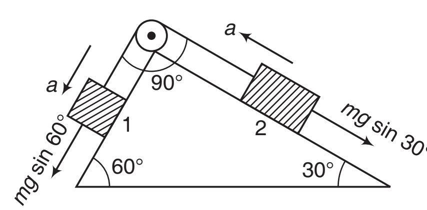
(b) $$v_{CM} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

$$= \frac{m_1 (g t_1) + m_2 (g t_2)}{m_1 + m_2}$$

$$= \frac{(m)(10 \times 0.3) + (2m)(10 \times 0.2)}{m + 2m}$$

$$= 2.3 \text{ m/s}$$ **Ans.**

6. $$a = \frac{\text{Net pulling force}}{\text{Total mass}}$$



$$= \frac{mg \sin 60° - mg \sin 30°}{2m}$$

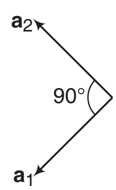
$$a = \left(\frac{\sqrt{3}-1}{4}\right) g$$

$$|\mathbf{a}_1| = |\mathbf{a}_2| = a$$

$$\therefore \quad |\mathbf{a}_1 + \mathbf{a}_2| = \sqrt{2}\, a \text{ at } 90°$$

$$\mathbf{a}_{COM} = \frac{m_1 \mathbf{a}_1 + m_2 \mathbf{a}_2}{m_1 + m_2}$$

$$= \frac{1}{2}(\mathbf{a}_1 + \mathbf{a}_2) \qquad (\text{as } m_1 = m_2)$$



$$\therefore \quad |\mathbf{a}_{COM}| = \frac{|\mathbf{a}_1 + \mathbf{a}_2|}{2}$$

$$= \frac{\sqrt{2}a}{2}$$

$$= \frac{a}{\sqrt{2}} = \left(\frac{\sqrt{3}-1}{4\sqrt{2}}\right) g \qquad \textbf{Ans.}$$

INTRODUCTORY EXERCISE 11.3

1. $\mathbf{p}_i = \mathbf{p}_f$

$$\therefore \quad (20)(20\hat{\mathbf{i}}) + 30(20\hat{\mathbf{j}}) + 40(20\hat{\mathbf{k}})$$

$$= (20)(0) + 30(10\hat{\mathbf{i}} + 20\hat{\mathbf{k}}) + 40\mathbf{v}$$

Solving this equation we get,

$$\mathbf{v} = (2.5\hat{\mathbf{i}} + 15\hat{\mathbf{j}} + 5\hat{\mathbf{k}}) \text{ cm/s} \qquad \textbf{Ans.}$$

2. $\mathbf{p}_i = \mathbf{p}_f$

$$\therefore \quad 0 = (25)(5) + (10)(v)$$

or $\quad v = -12.5$ m/s

Negative sign implies that this velocity is opposite to the direction of velocity of boy. Two velocities are in opposite directions.

Hence, $\quad |v_r| =$ Magnitude of relative velocity

$$= 12.5 + 5.0 = 17.5 \text{ m/s}$$

3. $p = \sqrt{2km}$

or $\quad p \propto \sqrt{m}$

$$\therefore \quad \frac{p_1}{p_2} = \sqrt{\frac{m_1}{m_2}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

4. $\mathbf{p}_i = \mathbf{p}_f$

$$\therefore \quad 0 = (4)(1.4 \times 10^7) + (234)v$$

$$\therefore \quad v = -2.4 \times 10^5 \text{m/s}$$

Here, negative sign implies that this velocity is opposite to the direction of velocity of alpha particle.

5. $\mathbf{p}_i = \mathbf{p}_f$

$$\therefore \quad 0 = 50(1.8) + (6 \times 10^{24})v$$

$$\therefore \quad v = -1.5 \times 10^{-23} \text{ m/s}$$

Negative sign implies the opposite direction of the velocity of man.

6. $m_1 v_1 = m_2 v_2$ (in opposite direction)

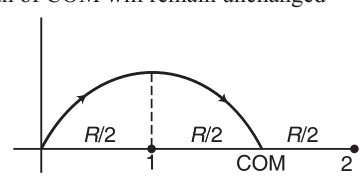
$$\therefore \quad (60)(3) = (20)\, v_2$$

$$\therefore \quad v_2 = 9 \text{ m/s} \qquad \textbf{Ans.}$$

$$\text{KE} = \frac{1}{2} \times 60 \times (3)^2 + \frac{1}{2} \times 20 \times (9)^2$$

$$= 1080 \text{J} = 1.08 \text{ kJ} \qquad \textbf{Ans.}$$

7. Path of COM will remain unchanged



$$x_2 = \frac{3R}{2} = \frac{3}{2}\left(\frac{u^2 \sin 2\theta}{g}\right)$$

$$= \frac{3}{2} \frac{(20)^2 \sin 90°}{10} = 60 \text{m} \qquad \textbf{Ans.}$$

INTRODUCTORY EXERCISE 11.4

1. Thrust force $= mg$

$$\therefore \quad v_r \left(-\frac{dm}{dt}\right) = mg$$

$$\therefore \quad \left(-\frac{dm}{dt}\right) = \frac{mg}{v_r}$$

$$= \frac{(200)(9.8)}{1.6 \times 10^3} = 1.225 \text{ kg/s}$$

(i) In this case, fuel will be finished in 90 s.

$$v = u - gt + v_r \ln\left(\frac{m_0}{m}\right)$$

$$= 0 - 9.8 \times 90 + (1.6 \times 10^3) \ln\left(\frac{200}{20}\right)$$

$$= 2.8 \text{ km/s} \qquad \textbf{Ans.}$$

(ii) In this case, fuel is finished in 9 s.

$$\therefore \quad v = 0 - 9.8 \times 10 + 1.6 \times 10^3 \ln\left(\frac{200}{20}\right)$$

$$= 3.6 \text{ km/s} \qquad \textbf{Ans.}$$

2. After 1 min total 600 kg gases will be burnt.

$$v = u - gt + V_r \ln\left(\frac{m_0}{m}\right)$$

$$= 0 - 10 \times 60 + (2000) \ln\left(\frac{1000}{400}\right)$$

$$= 1232.6 \text{ m/s}$$

3. $\left(-\dfrac{dm}{dt}\right) = \mu$

$$F_{\text{net}} = \text{thrust force} - \text{weight}$$

$$\therefore \quad ma = v_r\left(-\frac{dm}{dt}\right) - mg$$

or $\quad (m_0 - \mu t)\dfrac{d^2x}{dt^2} = u(\mu) - (m_0 - \mu t)\,g$

4. $\left(-\dfrac{dm}{dt}\right) = \dfrac{m_0}{3}$

At $t = 1$, mass will remain,

$$m = m_0 - \frac{m_0}{3} = \frac{2}{3}m_0$$

Now using the equation,

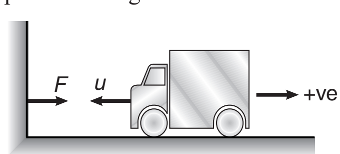
$$v = u - gt + v_r \ln\left(\frac{m_0}{m}\right)$$

$$= 0 - g(1) + u \ln\left(\frac{m_0}{2/3\, m_0}\right)$$

$$= u \ln\left(\frac{3}{2}\right) - g \qquad \textbf{Ans.}$$

INTRODUCTORY EXERCISE 11.5

1. Using impulse = change in linear momentum



We have, $\quad F \cdot t = mv_f - mv_i = m(v_f - v_i)$

or $\quad F(2) = 2 \times 10^3\,[0 - (-4)]$

or $\quad 2F = 8 \times 10^3$ or $F = 4 \times 10^3$ N

2. Using $\quad \mathbf{J} = m(\mathbf{v}_f - \mathbf{v}_i)$

$$-3m\,\hat{\mathbf{i}} = m\,[\mathbf{v}_f - (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}})]$$

or $\quad \mathbf{v}_f = -3\hat{\mathbf{i}} + (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}})$

or $\quad \mathbf{v}_f = -\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$

Note *The velocity component in the direction of $\hat{\mathbf{j}}$ is unchanged. This is because there is no impulse component in this direction.*

3. Impulse = area under F - t graph

$$= \Delta p = m(v_f - v_i)$$

$$\therefore \quad v_f = \frac{\text{Area}}{m} \qquad (\text{as } v_i = 0)$$

$$= \frac{1}{2} \times \frac{(20 \times 10^3)(8 + 16)}{12000}$$

$$= 200 \text{ m/s} \qquad \textbf{Ans.}$$

4. (a) $v^2 = u^2 - 2as$

$$\therefore \quad a = \frac{u^2}{2s} \qquad (v = 0)$$

$$= \frac{(100)^2}{2 \times 0.06} = 8.3 \times 10^4 \text{ m/s}^2$$

$$v = u - at$$

$$0 = u - at$$

or $\quad t = \dfrac{u}{a} = \dfrac{100}{8.3 \times 10^4}$

$$= 1.2 \times 10^{-3} \text{ s} \qquad \textbf{Ans.}$$

(b) Impulse $= |\Delta p| = m v_i$

$$= (5 \times 10^{-3})(100)$$

$$= 0.5 \text{ N-s} \qquad \textbf{Ans.}$$

(c) Impulse $= (F_{\text{av}})\,t$

$$\therefore \quad F_{\text{av}} = \frac{\text{Impulse}}{t}$$

$$= \frac{0.5}{1.2 \times 10^{-3}}$$

$$= 417 \text{ N} \qquad \textbf{Ans.}$$

INTRODUCTORY EXERCISE 11.6

1. At maximum extension their velocities are same. This common velocity is given by

$$v = \frac{\text{Total momentum}}{\text{Total mass}}$$

$$= \frac{2 \times 6 - 1 \times 3}{3 + 6} = 1 \text{ m/s}$$

Now, $\quad E_i = E_f$

$$\therefore \quad \frac{1}{2} \times 6 \times (2)^2 + \frac{1}{2} \times 3 \times (1)^2 = \frac{1}{2} \times 9\,(1)^2 + \frac{1}{2} \times 200 \times x_m^2$$

Solving we get, $\quad x_m = 0.3$ m

$$= 30 \text{ cm} \qquad \textbf{Ans.}$$

2. Substituting the values in above result

$$\frac{K_i - K_f}{K_i} = \frac{4\, m_1 m_2}{(m_1 + m_2)^2} = \frac{4 \times m \times 2m}{(m + 2m)^2} = \frac{8}{9} \qquad \textbf{Ans.}$$

3. $$\frac{K_i - K_f}{K_i} = 1 - \frac{K_f}{K_i} = \frac{1 - \frac{1}{2} m_1 v_1'^{\,2}}{\frac{1}{2} m_1 v_1^2} = 1 - \left(\frac{v_1'}{v_1}\right)^2$$

$$= 1 - \left[\frac{(m_1 - m_2)}{(m_1 + m_2)} v_1 / v_1\right]^2$$

$$= \frac{4\, m_1 m_2}{(m_1 + m_2)^2} \qquad \textbf{Ans.}$$

4. $$v_1' = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_1 + \left(\frac{2m_2}{m_1 + m_2}\right) v_2$$

We can see that $v_1' = v_2$ only if $m_1 = m_2$

Similarly $v_2' = v_1$ if $m_1 = m_2$.

5. In elastic collision between two equal masses, velocities are interchanged.

6. $$v_1' = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_1 + \left(\frac{2m_2}{m_1 + m_2}\right) v_2$$

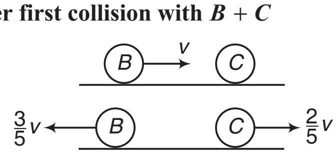
$$= \left(\frac{1-2}{1+2}\right)(+4) + \left(\frac{2 \times 2}{1+2}\right)(-6)$$

$$= -\frac{28}{3} \text{ m/s} \qquad \textbf{Ans.}$$

$$v_2' = \left(\frac{2-1}{1+2}\right)(-6) + \left(\frac{2 \times 1}{1+2}\right)(+4)$$

$$= \frac{2}{3} \text{ m/s} \qquad \textbf{Ans.}$$

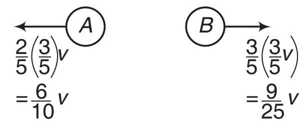
7. After first collision with $B + C$



$$v_B' = \left(\frac{m - 4m}{m + 4m}\right) v = -\frac{3}{5} v$$

$$v_C' = \left(\frac{2 \times m}{m + 4m}\right) v = \frac{2}{5} v$$

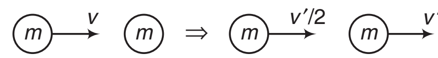
After second collision with $B + A$



Now since $\frac{9}{25} v < \frac{2}{5} v$ or v_C

So, B will not collide with C further.

Hence total collisions are only 2.

8. 

$$p_i = p_f$$

$$\therefore \quad mv = m\frac{v'}{2} + mv'$$

$$\therefore \quad v' = \frac{2}{3} v$$

Now, $$e = \frac{RVOS}{RVOA} = \frac{v' - v'/2}{v} = \frac{v'/2}{v} = \frac{(1/3\,v)}{v} = \frac{1}{3} \qquad \textbf{Ans.}$$

9. (a) $p_i = p_f$

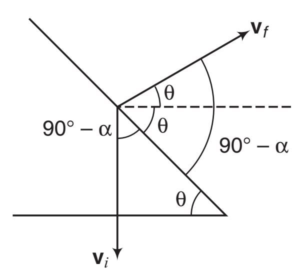
$$\therefore \quad mv = MV$$

$$\Rightarrow \quad M = \frac{mv}{V}$$

(b) $$e = \frac{RVOS}{RVOA} = \frac{V}{v}$$

10. Component parallel to wall will remain unchanged and component perpendicular to wall will become e times but in opposite direction.

11. Initial velocity vector is making an angle of $90° - \alpha$ with plane. Therefore final velocity vector will also make an angle $90° - \alpha$ with inclined plane.



$\therefore$ Angle of $\mathbf{v}_f$ with horizontal

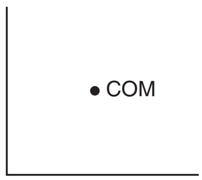
$$\theta = (90° - \alpha) - \alpha$$

$$\theta = (90° - 2\alpha) \qquad \textbf{Ans.}$$

Exercises

LEVEL 1

Assertion and Reason

1. 

In the above case, centre of mass lies outside the body.

2. $a_{CM} = \dfrac{F}{3m}$ = constant

$\therefore \qquad v_{CM} = a_{CM}t = \dfrac{F}{3m}t$

or $\qquad v_{CM} \propto t$

3. To conserve linear momentum, forces can act on a system but their vector sum should be zero.

4. Gases inside the rocket are pushed backwards.

5. Net vector sum of all internal forces = 0. So, they cannot change the linear momentum.

6. $p_1 = p_2$

$\therefore \qquad \sqrt{2K_1 m_1} = \sqrt{2K_2 m_2}$

or $\qquad \dfrac{K_1}{K_2} = \dfrac{m_2}{m_1}$

$\Rightarrow \qquad K \propto \dfrac{1}{m}$

7. Of the two masses A and B, mass A is moving downwards. Therefore net force on the system is vertically downwards and momentum is not conserved in vertical direction.

8. $\Delta p_1 + \Delta p_2 = 0$

or $\qquad \Delta p_1 = -\Delta p_2$

10. Energy can be given in the form of potential energy without giving the momentum.

11. Centre of mass remains stationary.

12. $e = \dfrac{RVOS}{RVOA}$

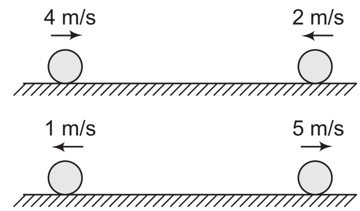
For elastic collision $e = 1$

$\therefore \qquad RVOS = RVOA$

Before collision they approach towards each other and after collision they recede from each other.

13. Centre of mass remains stationary.

15. Relative velocity of separation is equal to the relative velocity of approach. One of the case is shown below.

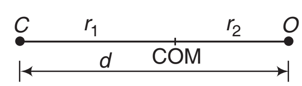


Before collision, relative speed is 2 m/s and relative velocity of approach is 6 m/s.
After collision, relative speed is 4 m/s and relative velocity of separation is 6 m/s.

Single Correct Option

10. Impulse $= F_{av}\ (\Delta t) = |\Delta P|$

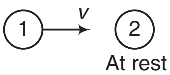
$$\therefore \quad F_{av} = \frac{|\Delta P|}{\Delta t} = \frac{m\,(v_i - v_f)}{\Delta t}$$
$$= \frac{(5)\,(65-15)}{2}$$
$$= 125 \text{ N}$$ **Ans.**

11. 

$$\frac{r_1}{r_2} = \frac{m_2}{m_1}$$
$$\therefore \quad r_1 = \left(\frac{m_2}{m_1 + m_2}\right) d$$
$$= \left(\frac{16}{12+16}\right)(1.2 \times 10^{-10})$$
$$= 0.68 \times 10^{-10}\text{m}$$ **Ans.**

12. 

$$3v_1 = 6v^2$$
$$v_2 = \frac{3v_1}{6}$$
$$= \frac{3 \times 16}{6} = 8 \text{ m/s}$$
$$(\text{KE})_{6\text{ kg}} = \frac{1}{2} \times 6 \times (8)^2$$
$$= 192 \text{ J}$$ **Ans.**

13. 

$$m_1 >> m_2$$

$$v_2' = \left(\frac{m_2 - m_1}{m_1 + m_2}\right) v_1 + \left(\frac{2m_1}{m_1 + m_2}\right) v_2$$

Substituting $m_2 \approx 0$ and $v_1 = 0$ we get,

$$v_2 \approx 2v$$

14. Velocity of dropped out mass is also 6 m/s. So its relative velocity is zero and no thrust force will act. Therefore final speed remains 6 m/s.

15. $F = \dfrac{\Delta p}{\Delta t} = n\,(mv)$

Here n = number of bullets fired per second

$$\Rightarrow \qquad n = \frac{F}{mv}$$

$$= \frac{144}{(40 \times 10^{-3})\,(1200)} = 3 \qquad \textbf{Ans.}$$

16. $|\Delta \mathbf{p}| = |\text{ Impulse }| = |\mathbf{F} \cdot \Delta t|$

Here $\qquad \Delta t = T = \dfrac{2v \sin 45^\circ}{g} = \dfrac{\sqrt{2}\, v}{g}$

and $\qquad |\mathbf{F}| = mg$

$\therefore \qquad |\Delta \mathbf{p}| = (mg)\left(\dfrac{\sqrt{2}\, v}{g}\right)$

$$= \sqrt{2}\, mv \qquad \textbf{Ans.}$$

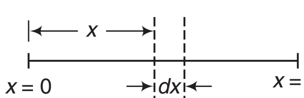
17. Velocity of ball just before collision with ground,

$$v = \sqrt{2gh} = \sqrt{2 \times 9.8 \times 4.9} = 9.8 \text{ m/s}$$

Velocity just after collision,

$$u = ev = \left(\frac{3}{4}\right)(9.8) \text{ m/s}$$

$$T = \frac{2u}{g} = \frac{2 \times (3/4) \times 9.8}{9.8} = \frac{3}{2} \text{ s}$$

18. 

dm = (Mass per unit length) (Length)

$$= \left(\frac{kx^2}{L}\right) dx \quad \Rightarrow \quad X_{\text{COM}} = \frac{\int_0^L x\,dm}{\int_0^L dm}$$

$$= \frac{\int_0^L (x)\left(\dfrac{kx^2}{L}\right) dx}{\int_0^L \left(\dfrac{kx^2}{L}\right) dx} = \frac{3}{4} L \qquad \textbf{Ans.}$$

19. 

$$450\, x = 50\,(10 - x)$$

$\therefore \qquad x = 1 \text{ m} \qquad$ **Ans.**

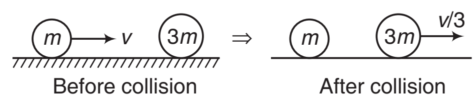
20. 

$$\frac{M}{3} x = M\,(L - x)$$

$$x = \frac{3}{4} L$$

$\therefore$ Displacement of man relative to ground

$$= L - x = \frac{L}{4} \qquad \textbf{Ans.}$$

21. 

From conservation of momentum we can see that velocity of heavier mass after collision becomes $\dfrac{v}{3}$.

Now, coefficient of restitution,

$$e = \frac{RVOS}{RVOA} = \frac{v/3}{v} = \frac{1}{3} \qquad \textbf{Ans.}$$

22. In elastic collision between equal masses velocities are interchanged. Therefore change in momentum in any one particle is mu.

Now, $\qquad |\Delta \mathbf{p}| = |\text{ Impulse }|$

$\qquad\qquad$ = area under F - t graph

$\therefore \qquad mu = \dfrac{1}{2} \times t_0 \times F_0$

$\therefore \qquad F_0 = \dfrac{2mu}{t_0} \qquad$ **Ans.**

23. Acceleration of centre of mass $= \dfrac{F}{2m}$

$$a_{\text{CM}} = \frac{m_1 a_1 + m_2 a_2}{m_1 + m_2}$$

$\therefore \qquad \dfrac{F}{2m} = \dfrac{(m)\,(a_0) + (m)\,a_2}{2m}$

$\therefore \qquad a_2 = \dfrac{F}{m} - a_0 \qquad$ **Ans.**

24. Impulse $= \Delta\, \mathbf{p} = m\,(\mathbf{v}_f - \mathbf{v}_i)$

$$= m\left[\left(-\frac{3}{4} v_0 \cos 53^\circ \hat{\mathbf{i}} + \frac{3}{4} v_0 \sin 53^\circ \hat{\mathbf{j}}\right) - (v_0 \cos 37^\circ \hat{\mathbf{i}} + v_0 \sin 37^\circ \hat{\mathbf{j}})\right]$$

$$= -\frac{5}{4} m v_0 \hat{\mathbf{i}} \qquad \textbf{Ans.}$$

25. $h_n = (e^{2n})\, hi$ ...(i)

Given that

$$(64 \text{ cm}) = (e^2)\,(100 \text{ cm})$$

$$\therefore \qquad e = 0.8$$

Substituting in Eq. (i) we have

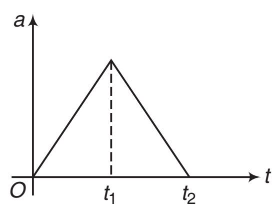
$$h_n = (0.8)^{2n}\,(1\text{m})$$
$$= (0.8)^{2n} \qquad \textbf{Ans.}$$

26. $\mathbf{p}_i = \mathbf{p}_f$

$\therefore\ (500)\,(\hat{\mathbf{i}}) = (25)\,(20\hat{\mathbf{j}}) + 475\,\mathbf{v}$

$$\therefore \qquad \mathbf{v} = \left[\left(\frac{20}{19}\,\hat{\mathbf{i}}\right) - \left(\frac{20}{19}\,\hat{\mathbf{k}}\right)\right] \text{m/s} \qquad \textbf{Ans.}$$

27. $a = \dfrac{F}{m}$ or $a \propto F$



Hence, a-t graph is similar to F-t graph.

From O to t_1

$$a \propto t$$

or $\qquad a = kt \quad (k \to \text{a positive constant})$

Integrating we get,

$$v = \frac{kt^2}{2} \quad \text{or} \quad v \propto t^2$$

i.e. v-t graph is a parabola passing through origin.

From t_1 to t_2 again acceleration is positive (in the direction of velocity). So, velocity will further increase.

28. $$x_{CM} = \frac{A_1x_1 - A_2x_2}{A_1 - A_2}$$

$$= \frac{\pi\,(3R)^2\,(0) - (\pi R^2)\,(2R)}{\pi\,(3R)^2 - \pi R^2} = -\frac{R}{4}$$

$\therefore$ Distance of centre of mass from origin $= \dfrac{R}{4}$ **Ans.**

29. $$\mathbf{r}_{CM} = \frac{A_1\mathbf{r}_1 - A_2\mathbf{r}_2 - A_3\mathbf{r}_3}{A_1 - A_2 - A_3}$$

$$= \frac{\pi\,(4R)^2\,(0) - (\pi R^2)\,(3R\hat{\mathbf{i}}) - (\pi R^2)\,(3R\hat{\mathbf{j}})}{\pi\,(4R)^2 - \pi R^2 - \pi R^2}$$

$$= \left(-\frac{3R}{14}\,\hat{\mathbf{i}} - \frac{3R}{14}\,\hat{\mathbf{j}}\right) \qquad \textbf{Ans.}$$

30. Net horizontal force on system is zero, Therefore, centre of mass does not move in horizontal. Further, all surfaces are smooth. Therefore mechanical energy of the system remains constant.

31. Ler v' is the velocity of $(M + m)$ after collision. Then from conservation of linear momentum we have,

$$mv = (M + m)\,v'$$

$$\therefore \qquad v' = \left(\frac{m}{M + m}\right) v$$

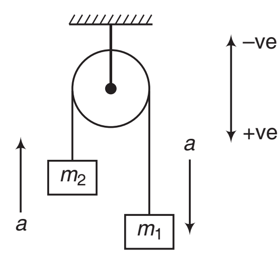
Now, $\qquad v' = \sqrt{2gh}$

$$\therefore \qquad \left(\frac{m}{M + m}\right) v = \sqrt{2gh}$$

or $\qquad v = \sqrt{2gh}\left(\dfrac{M + m}{m}\right)$

32. Net horizontal force on system is zero.

33.



$$a = \frac{\text{Net pulling force}}{\text{Total mass}}$$

$$= \frac{m_1g - m_2g}{m_1 + m_2}$$

$$= \left(\frac{m_1 - m_2}{m_1 + m_2}\right) g \qquad \text{...(i)}$$

Now, $\qquad a_{CM} = \dfrac{m_1\,a_1 + m_2\,a_2}{m_1 + m_2}$

$$a_{CM} = \frac{m_1\,(+a) + m_2\,(-a)}{m_1 + m_2}$$

$$= \left(\frac{m_1 - m_2}{m_1 + m_2}\right) a$$

Substituting the value of a from Eq. (i) we have,

$$a_{CM} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2 g \qquad \textbf{Ans.}$$

34. Using the equation,

$$v = u - gt + v_r \ln\left(\frac{m_0}{m}\right)$$

Substituting the values we get,

$$v_r = 0 - 0 + v_r \ln\left(\frac{m_0}{m}\right)$$

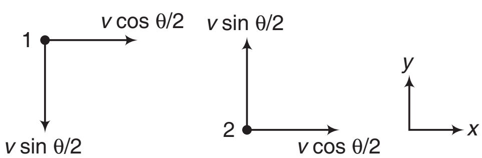
$$\therefore \qquad \frac{m_0}{m} = e = 2.178 \qquad \textbf{Ans.}$$

35. $v_i = 1\text{ m/s},\ v_f = 0$

$$|F| = \left|\frac{\Delta p}{\Delta t}\right| = \left(\frac{\Delta m}{\Delta t}\right)(v_i - v_f)$$

$$= 0.5 \times 1 = 0.5\text{ N}$$

36.



From the figure, we can see that velocity of approach is $2v \sin\frac{\theta}{2}$.

$$e = \frac{RVOS}{RVOA}$$

For completely inelastic collision, $e = 0$

$$\therefore \qquad RVOS = 0$$

$$\mathbf{p}_i = \mathbf{p}_f$$

$$\therefore \quad m\left[v\cos\frac{\theta}{2}\,\hat{\mathbf{i}} - v\sin\frac{\theta}{2}\,\hat{\mathbf{j}}\right] + m\left[v\cos\frac{\theta}{2}\,\hat{\mathbf{i}} + v\sin\frac{\theta}{2}\,\hat{\mathbf{j}}\right]$$

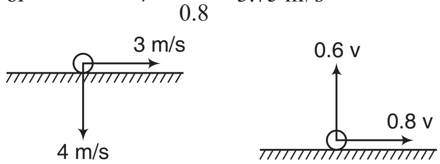
$$= 2m\mathbf{v}_c$$

$$\therefore \quad \text{Common velocity } \mathbf{v}_c = \left(v\cos\frac{\theta}{2}\right)\hat{\mathbf{i}}$$

37. Component parallel to plane remains unchanged.

$$\therefore \qquad 0.8\,v = 3$$

$$\text{or} \qquad v = \frac{3}{0.8} = 3.75\text{ m/s}$$



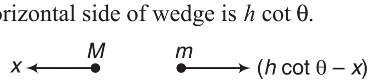
Just before collision

Just after collision

$|\Delta v| = (4 + 0.6\,v) = 6.25\text{ m/s}$ (in vertical direction)

$\therefore$ Impulse $= |\Delta p| = m\,|\Delta v| = 6.25$ N-s **Ans.**

38. Horizontal side of wedge is $h\cot\theta$.



$$Mx = m(h\cot\theta - x)$$

$$\therefore \qquad x = \frac{mh\cot\theta}{M + m} \qquad \textbf{Ans.}$$

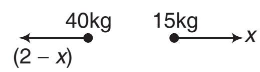
39. $$x_{CM} = \frac{A_1x_1 + A_2x_2 + A_3x_3 + A_4x_4}{A_1 + A_2 + A_3 + A_4}$$

$$= \frac{(4)(1) + (4)(1) + (4)(3) + (4)(3)}{16} = 2\text{cm}$$

$$y_{CM} = \frac{A_1y_1 + A_2y_2 + A_3y_3 + A_4y_4}{A_1 + A_2 + A_3 + A_4}$$

$$= \frac{(4)(1) + 4(3) + 4(3) + 4(3)}{16} = 2.5\text{cm}$$

40.



$$40(2 - x) = 15x$$

$$\therefore \qquad x = 1.46\text{ m} \qquad \textbf{Ans.}$$

41. $\mathbf{p}_i = \mathbf{p}_f$

$$\therefore \qquad m(v_0\hat{\mathbf{i}}) + m(-3\,v_0\hat{\mathbf{j}}) + m(5\,v_0\hat{\mathbf{k}}) = 3m\mathbf{v}$$

$$\therefore \qquad \mathbf{v} = \frac{v_0}{3}(\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 5\hat{\mathbf{k}}) \qquad \textbf{Ans.}$$

42. $$x_{CM} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

$$\therefore \qquad R = \frac{(M/5)(0) + (4M/5)D}{M}$$

$$\therefore \qquad D = \frac{5}{4}R \qquad \textbf{Ans.}$$

Subjective Questions

1. $$x_{CM} = \frac{A_1x_1 - A_2x_2}{A_1 - A_2}$$

$$= \frac{(ab)(0) - \left(\frac{ab}{4}\right)\left(\frac{a}{4}\right)}{ab - \frac{ab}{4}}$$

$$= -\frac{a}{12}$$

$$y_{CM} = \frac{A_1y_1 - A_2y_2}{A_1 - A_2}$$

$$= \frac{(ab)(0) - \left(\frac{ab}{4}\right)\left(\frac{b}{4}\right)}{ab - \frac{ab}{4}}$$

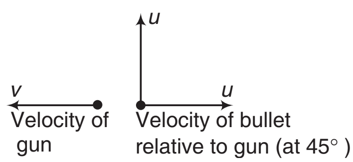
$$= -\frac{b}{12}$$

2. $x_{CM} = \dfrac{V_1x_1 - V_2x_2}{V_1 - V_2}$ $\qquad$ (V = volume)

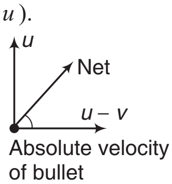
$$= \frac{\left(\frac{4}{3}\pi R^3\right)(0) - \left(\frac{4\pi a^3}{3}\right)(b)}{\left(\frac{4}{3}\pi R^3\right) - 4\left(\frac{\pi a^3}{3}\right)}$$

$$= \frac{-a^3 b}{R^3 - a^3}$$ **Ans.**

3.



At 45° means horizontal and vertical components are same (let u).



In absolute velocity, since horizontal component of velocity has been decreased ($= u - v$). Therefore, $\theta > 45°$

4. p = constant

$\therefore \qquad \sqrt{2\,Km}$ = constant

or $\qquad K \propto \dfrac{1}{m}$

5. In head on elastic collision between two equal masses, velocities are interchanged.

6. At maximum elastic potential energy, velocity of both particles is same. This common velocity will be given by

$$v = \frac{\text{Total momentum}}{\text{Total mass}}$$

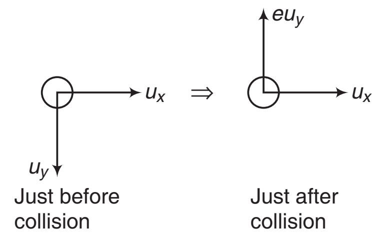
$$= \frac{\sqrt{2Km}}{2m} = \sqrt{\frac{K}{2m}}$$

$$E_i = E_f$$

$$K = \frac{1}{2} \times 2m\left(\sqrt{\frac{K}{2m}}\right)^2 + v_{max}$$

$$\therefore \qquad v_{max} = \frac{K}{2}$$ **Ans.**

7. Just after collision x-component of velocity remain unchanged but y-component of velocity becomes e times.



Now, $\quad T = \dfrac{2u_y}{g}, H = \dfrac{u_y^2}{2g}$ and $R = \dfrac{2u_x\,u_y}{g}$

or $\quad T \propto u_y\,,\; H \propto u_y^2$ and $R \propto u_y$

$$\therefore \qquad \frac{T_1}{T_2} = a = \frac{u_y}{eu_y} = \frac{1}{e}$$

$$\frac{R_1}{R_2} = b = \frac{u_y}{eu_y} = \frac{1}{e}$$

and $$\frac{H_1}{H_2} = c = \frac{(u_y)^2}{(eu_y)^2} = \frac{1}{e^2}$$

8. (a) $\mathbf{a}_{COM} = \dfrac{m_1\mathbf{a}_1 + m_2\mathbf{a}_2}{m_1 + m_2}$

$$= \frac{(1)(-10\hat{\mathbf{j}}) + (2)(-10\hat{\mathbf{j}})}{1+2}$$

$$= (-10\hat{\mathbf{j}})\ \text{m/s}^2$$ **Ans.**

(b) $\mathbf{v}_{COM} = \dfrac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2}$

$$= \frac{m_1(\mathbf{u}_1 + \mathbf{a}_1 t) + m_2(\mathbf{u}_2 + \mathbf{a}_2 t)}{m_1 + m_2}$$

$$= \frac{(1)[0 + (-10\hat{\mathbf{j}})(1)] + 2[(10\hat{\mathbf{i}} + 10\hat{\mathbf{j}}) + (-10\hat{\mathbf{j}})(1)]}{1+2}$$

$$= \frac{10}{3}(2\hat{\mathbf{i}} - \hat{\mathbf{j}})\ \text{m/s}$$ **Ans.**

(c) $\mathbf{r}_{CM} = \dfrac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$

$$= \frac{m_1\left[\mathbf{r}_i + \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2\right]_1 + m_2\left[\mathbf{r}_i + \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2\right]_2}{m_1 + m_2}$$

$$= \frac{1\left[(10\hat{\mathbf{i}} + 20\hat{\mathbf{j}}) + 0 + \frac{1}{2}(-10\hat{\mathbf{j}})(1)^2\right] + 2\left[(20\hat{\mathbf{i}} + 40\hat{\mathbf{j}}) + (10\hat{\mathbf{i}} + 10\hat{\mathbf{j}})(1) + \frac{1}{2}(-10\hat{\mathbf{j}})(1)^2\right]}{1+2}$$

$$= \left(\frac{70}{3}\hat{\mathbf{i}} + 35\hat{\mathbf{j}}\right)\ \text{m}$$ **Ans.**

9. (a) $$y_{CM} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}$$

$$\therefore \quad 24 = \frac{(0.6)(80) + (m_2)(0)}{\text{Total mass}}$$

$\therefore$ Total mass = 2 kg **Ans.**

(b) $$\mathbf{a}_{CM} = \frac{d}{dt}(\mathbf{v}_{CM}) = \frac{d}{dt}[6t^2]\,\hat{\mathbf{j}}$$

$$= (12t)\,\hat{\mathbf{j}}\ \text{m/s}^2$$ **Ans.**

(c) $$\mathbf{F}_{ext} = m_{CM}\,\mathbf{a}_{CM} = m_{CM}\frac{d}{dt}(\mathbf{v}_{CM})$$

$$= m_{CM}\,(12t)\,\hat{\mathbf{j}}$$

At $t = 3$ s, $\mathbf{F}_{ext} = (2\text{kg})(12)(3)\,\hat{\mathbf{j}}$ N

$$= (72\hat{\mathbf{j}})\ \text{N}$$ **Ans.**

10. (a) $$\mathbf{v}_c = \frac{m_1\mathbf{u}_1 + m_2\mathbf{u}_2}{m_1 + m_2} = \mathbf{0.8\,\hat{i}}\ \text{m/s}$$

(b) $\mathbf{v}_1 = -1.6\,\hat{\mathbf{i}}$ m/s

From COM, $m_1\mathbf{u}_1 + m_2\mathbf{u}_2 = m_1\mathbf{v}_1 + m_2\mathbf{v}_2$

$\Rightarrow \quad v_2 = 2.4\hat{\mathbf{i}}$ m/s

(c) $$e = \frac{v_2 - v_1}{u_1 - u_2} = \frac{4}{7}$$

11. Fuel is finished in 40 s

$$v = u - gt + v_r \ln\left(\frac{m_0}{m}\right)$$

$$= 0 - 10 \times 40 + (2 \times 10^3)\ln\left(\frac{200}{40}\right)$$

$$= 2818\ \text{m/s} = 2.82\ \text{km/s}$$ **Ans.**

12. Let v is the horizontal velocity of platform in opposite direction. Then from momentum conservation in opposite direction we have,

$$(60 + 40)\,v = (1)(10\cos 45°)$$

$$\therefore \quad v = \frac{10}{100}\cos 45°\ \text{m/s}$$

$$= \frac{10}{\sqrt{2}}\ \text{cm/s}$$

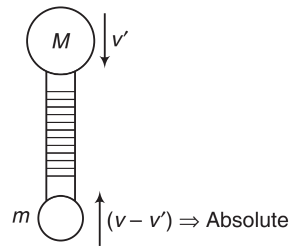
$\therefore$ Displacement of platform $= vt$

$$= \frac{(v)(2u\sin 45°)}{g}$$

$$= \left(\frac{10}{\sqrt{2}}\right) = \frac{2 \times 10 \times \frac{1}{\sqrt{2}}}{10}$$

$$= 10\ \text{cm}$$ **Ans.**

13. Since the system (or COM) was initially at rest. So it will always remain at rest. Because the net external forces are not changing.



(a) $Mv' = m(v - v')$

$$\therefore \quad v' = \frac{mv}{M + m}$$ **Ans.**

(b) If man stops climbing, then balloon will also stop to keep the system at rest.

14. $$\frac{F}{m} = \frac{v_r\left(-\frac{dm}{dt}\right)}{m} = a = \text{constant}$$

$$\therefore \quad \frac{u\left(-\frac{dm}{dt}\right)}{m} = a$$

$$\therefore \quad \int_{m_0}^{m} -\frac{dm}{m} = \int_0^t \frac{a}{u}\,dt$$

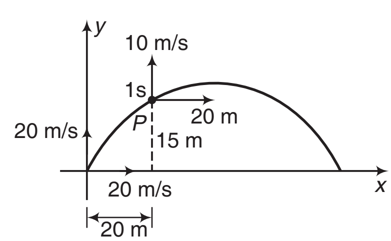
Solving this equation we get,

$$m = m_0\,e^{-at/u}$$ **Ans.**

15. At 1 seconds particle is at point P as shown in figure. Let velocity of second part is $\mathbf{v}$. Then applying momentum conservation, just before and just after explosion we have,

$$2m\,(20\hat{\mathbf{i}} + 10\hat{\mathbf{j}}) = m\,(0) + m\mathbf{v}$$

$$\therefore \quad \mathbf{v} = (40\hat{\mathbf{i}} + 20\hat{\mathbf{j}})$$



Its vertical component of velocity is still 20 m/s. So total height,

$$h = h_i + \frac{u^2}{2g}$$

$$= 15 + \frac{(20)^2}{2 \times 10}$$

$$= 35\,\text{m}$$ **Ans.**

16. $h = (1 + 1)\text{ m} = 2\text{m}$

$\therefore \quad v = \sqrt{2gh}$

$= \sqrt{2 \times 10 \times 2}$

$= 6.32\text{ m/s}$ (downwards)

Impulse $= \Delta p = m\,(v_f - v_i)$

$= (1)\,[0 - (-6.32)] = 6.32\text{ N-s}$ (upwards)

18. In elastic collision, velocities are interchanged. So, v is minimum

$\therefore \quad \frac{1}{2} \times 0.1 \times v^2 = 0.2$

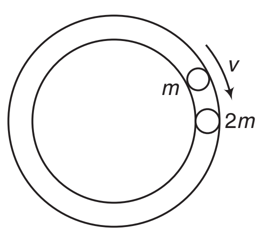
$\therefore \quad v = 2\text{ m/s} = \text{minimum value}$ **Ans.**

In perfectly inelastic collision, speed of combined mass will remain half.

$\therefore \quad \frac{1}{2} \times 0.1 \times \left(\frac{v}{2}\right)^2 = 0.2$

$\therefore \quad v = 2\sqrt{2}\text{ m/s} = \text{maximum value}$ **Ans.**

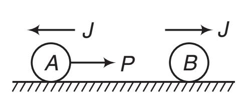
19. Since collision is elastic. It means relative velocity of separation will remain v and relative velocity means one mass is assumed at rest.



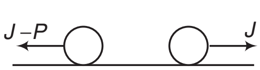
$\therefore \quad t = \frac{2\pi r}{v}$ **Ans.**

20. J = impulse = change in momentum

$e = \frac{RVOS}{RVOA}$



Initial momenta



Final momenta

$= \frac{(J/m) + (J\text{-}P/m)}{(P/m)}$

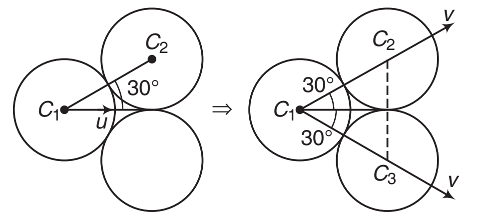
$= \frac{2J}{P} - 1$ **Ans.**

21. $p_i = p_f$

$\therefore \quad mu = 2\,mv \cos 30°$

or $\quad v = \frac{u}{\sqrt{3}}$

$e = \frac{RVOS}{RVOA}$ (along C_1C_2 or C_1C_3)



$= \frac{v}{u \cos 30°} = \frac{(u/\sqrt{3})}{\sqrt{3}\,u/2} = \frac{2}{3}$ **Ans.**

22. For the collision

$1 \times 10 = 10 \times v \quad \Rightarrow \quad v = 1\text{m/s}$

If x be the maximum compression

$\frac{1}{2} \times 10 \times 1^2 = \mu\,(m + M)\,gx + \frac{1}{2}\,kx^2$

$5 = 10x + 120x^2$

$\Rightarrow \quad x = \frac{1}{6}m$ **Ans.**

23.

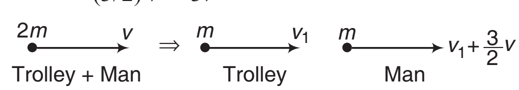
$m_A\,x_A = m_B\,x_B$

$\therefore \quad (5)\,(0.5 - x) = 30x$

$\therefore \quad x = 0.0714\text{ m}$

$= 71.4\text{ mm}$ **Ans.**

24. $t_1 = \frac{L}{(3/2)\,v} = \frac{2L}{3v}$



$t_2 = \frac{L}{(3/2)\,v} = \frac{2L}{3v}$

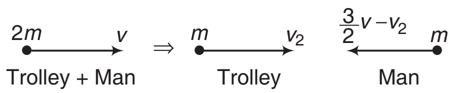
$p_i = p_f$

$\therefore \quad 2\,mv = mv_1 + m\left(v_1 + \frac{3}{2}\,v\right)$

$v_1 = \frac{v}{4}$

$\therefore \quad s_1 = v_1\,t_1 = \left(\frac{v}{4}\right)\left(\frac{2L}{3v}\right) = \frac{L}{6}$

In return journey



$$p_i = p_f$$

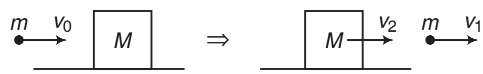
$$\therefore \quad 2\,mv = mv_2 - m\left(\frac{3}{2}v - v_2\right)$$

$$\therefore \quad v_2 = \frac{7}{4}\,v$$

or $$s_2 = v_2 t_2 = \left(\frac{7}{4}v\right)\left(\frac{2L}{3v}\right) = \frac{7L}{6v}$$

$$s_{\text{Total}} = s_1 + s_2 = \frac{4L}{3}$$ **Ans.**

25.



(a) $p_i = p_f$

$$\therefore \quad mv_0 = mv_1 + mv_2$$

$$\therefore \quad v_2 = \frac{mv_0 - mv_1}{M}$$

$$= \frac{(0.004)\,(500 - 100)}{1.0}$$

$$= 1.6 \text{ m/s}$$

Retardation due to friction,

$$a = \frac{\mu mg}{m} = \mu g = (10\,\mu)$$

$$s = \frac{v_2^2}{2a} = \frac{v_2^2}{20\,\mu}$$

$$\therefore \quad \mu = \frac{v_2^2}{20s}$$

$$= \frac{(1.6)^2}{20 \times 0.3} = 0.43$$ **Ans.**

(b) Decrease in kinetic energy of bullet

$$= \frac{1}{2}m\,(v_0^2 - v_1^2)$$

$$= \frac{1}{2} \times 0.004\,[\,(500)^2 - (100)^2\,]$$

$$= 480 \text{ J}$$ **Ans.**

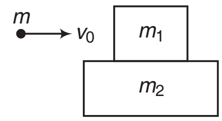
(c) Kinetic energy of block,

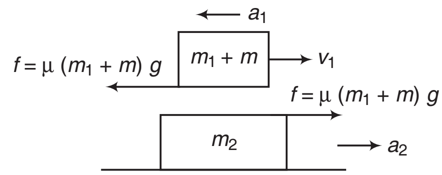
$$= \frac{1}{2}\,Mv_2^2$$

$$= \frac{1}{2} \times 1.0 \times (1.6)^2$$

$$= 1.28 \text{ J}$$ **Ans.**

26.





(i) Common velocity $= \dfrac{\text{Initial momentum}}{\text{Total mass}}$

$$v_c = \frac{0.25 \times 302}{0.25 + 37.5 + 1.25}$$

$$= 1.94 \text{ m/s}$$ **Ans.**

(ii) $v_1 = \dfrac{\text{Initial momentum}}{m_1 + m}$

$$= \frac{0.25 \times 302}{37.5 + 0.25} = 2 \text{ m/s}$$

$$a_1 = \frac{f}{m_1 + m} = \mu g = 5 \text{ m/s}^2$$

$$a_2 = \frac{f}{m_1 + m + m_2} = \left(\frac{m_1 + m}{m_1 + m + m_2}\right)\mu g$$

$$= \left(\frac{37.5 + 0.25}{37.5 + 0.25 + 1.25}\right)(0.5 \times 10)$$

$$= 4.84 \text{ m/s}^2$$

$$a_r = a_1 - a_r = 0.16 \text{ m/s}^2$$

Common velocity is achieved when, v_1 converts into v_c by a retardation a_1.

$$\therefore \quad v_c = v_1 - a_1 t$$

$$\therefore \quad t = \frac{v_1 - v_c}{a_1} = \frac{2 - 1.94}{5}$$

$$= 0.012 \text{ s}$$

Now, $$s_r = \frac{1}{2}\,a_r\,t^2$$

$$= \frac{1}{2} \times 0.16 \times (0.012)^2$$

$$= 0.011 \text{ mm}$$ **Ans.**

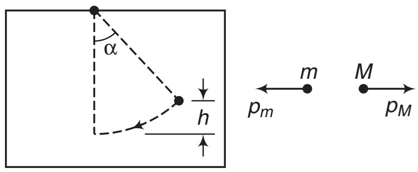
27. $h = l\,(1 - \cos\alpha)$ ...(i)

$p_m = p_M$ (in opposite directions)

$$\therefore \quad \sqrt{2K_m m} = \sqrt{2K_M M}$$

or $$\frac{K_m}{K_M} = \frac{M}{m}$$ ...(ii)

$$K_m + K_M = mgh = mgl\,(1 - \cos\alpha) \qquad ...(iii)$$



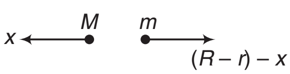
Solving these three equations we get,

$$K_M = \left(\frac{m}{m + M}\right) mgl\,(1 - \cos\alpha)$$

$$\therefore \quad \frac{1}{2} Mv_M^2 = \left(\frac{m}{M + m}\right) mgl\,(1 - \cos\alpha)$$

$$\therefore \quad v_m = 2m \sin\frac{\alpha}{2}\sqrt{\frac{gl}{M(M + m)}} \qquad \textbf{Ans.}$$

28. (i)



$$Mx = m\,(R - r - x)$$

$$\therefore \quad x = \frac{m\,(R - r)}{M + m} \qquad \textbf{Ans.}$$

(ii)

$$p_M = p_m$$

$$\therefore \quad \sqrt{2K_M M} = \sqrt{2K_m m}$$

$$\therefore \quad \frac{K_M}{K_m} = \frac{m}{M} \qquad ...(i)$$

$K_m + K_m$ = decrease in potential energy of m

$$= mg\,(R - r) \qquad ...(ii)$$

Solving these two equations we get,

$$K_M = \left(\frac{m}{M + m}\right) mg\,(R - r) = \frac{1}{2} Mv_M^2$$

$$\therefore \quad v_M = m\sqrt{\frac{2g\,(R - r)}{M\,(M + m)}} \qquad \textbf{Ans.}$$

29. (a) $|\Delta \mathbf{p}| = \sqrt{p_1^2 + p_2^2 - 2p_1p_2 \cos 90°}$

$$= \sqrt{p_1^2 + p_2^2}$$

$$= \sqrt{(0.05 \times 2)^2 + (0.05 \times 2)^2}$$

$$= 0.14 \text{ kg-m/s} \qquad \textbf{Ans.}$$

(b) Initial and final velocity of wall is zero. Therefore change in momentum is zero.

30. $v = \sqrt{2gy}$

Thrust force downwards

$$F = \lambda v^2 \qquad (\lambda = \text{mass/length})$$

or $\quad F = m\,(2gy)$ or $F = 2\,mgy$

y length is lying on table. So its weight

$$W = (ym)\,g$$

Total force on table = $F + W = (3\,mgy)$

= weight of a length $3y$ of the rope

31. Relative velocity of sand is 2 m/s in backward direction. Since mass is increasing, therefore thrust force is in the direction of relative velocity (backwards).

$$\text{Thrust force} = v_r\left(+\frac{dm}{dt}\right)$$

$$= (5)\,(2) = 10 \text{ N} \qquad \text{(backwards)}$$

$\therefore$ Force needed (F_{ext}) to move the belt with constant velocity ($F_{\text{net}} = 0$) is

$$F_{\text{ext}} = F = 10 \text{ N} \qquad \text{(in forward direction)}$$

$$P = F_{\text{ext}}\, v = 20 \text{ W} \qquad \textbf{Ans.}$$

32. Impulse = $\Delta p = p_f - p_i$

$$= m\,(v_f - v_i)$$
$$= (3)\,[40 - (-50)]$$
$$= 270 \text{ N-s}$$

i.e. impulse is 270 N-s (towards right) **Ans.**

Impulse = $F_{\text{av}}\ \Delta t$

$$\therefore \quad F_{\text{av}} = \frac{\text{Impulse}}{\Delta t} = \frac{270}{0.02}$$

$$= 13500 \text{ N}$$

$$= 13.5 \text{ kN} \qquad \text{(towards right)} \qquad \textbf{Ans.}$$

33. $v_3' = \left(\dfrac{m_3 - m_2}{m_3 + m_2}\right) v_3 + \left(\dfrac{2m_2}{m_2 + m_3}\right) v_2$

$$= \left(\frac{3 - 2}{3 + 2}\right)(2) + 0 = 0.4 \text{ m/s}$$

$$v_2' = 0 + \left(\frac{2 \times 3}{2 + 3}\right)(2) = 2.4 \text{ m/s}$$

Retardation of each block,

$$a = \frac{\mu_K mg}{m} = \mu_K g = 3 \text{ m/s}^2$$

Before coming to rest

$$s_3 = \frac{v_3'^2}{2a} = \frac{(0.4)^2}{2 \times 3} = 0.03 \text{ m}$$

$$s_2 = \frac{v_2'2}{2a} = \frac{(2.4)^2}{2 \times 3} = 0.96 \text{ m}$$

$$\therefore \quad d = s_2 - s_3 = 0.93 \text{ m} \qquad \textbf{Ans.}$$

LEVEL 2

Single Correct Option

1. Let velocity of m just after collision is v. Then, from conservation of momentum we have,

$$m_1 v_1 = mv + m_1 \frac{v_1}{3} \quad \Rightarrow \quad \therefore \quad v = \frac{2}{3}\frac{m_1 v_1}{m}$$

Now, for just completing the circle,

$$v = \sqrt{5gl}$$

$$\therefore \quad \frac{2}{3}\frac{m_1 v_1}{m} = \sqrt{5gl}$$

$$\therefore \quad v_1 = \frac{3m}{2m_1}\sqrt{5gl} \qquad \textbf{Ans.}$$

2. $v_m = \sqrt{2gl}$

From conservation of linear momentum,

$$v_{2m} = \frac{v_m}{2} = \frac{\sqrt{2gl}}{2}$$

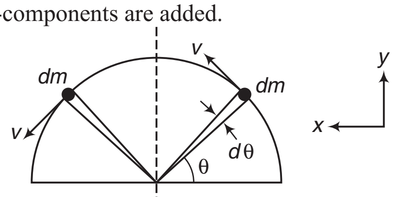
Now, $$h = \frac{v_{2m}^2}{2g} = \frac{l}{4}$$

But $$h = l(1 - \cos\theta)$$

$$\therefore \quad \frac{l}{4} = l(1-\cos\theta)$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{3}{4}\right)$$

3. y-components of momenta are cancelled and x-components are added.



$$P = \int_0^{\pi} dP_x = \int_0^{\pi} (dm)(v \sin\theta)$$

$$= \int_0^{\pi}\left(\frac{M}{\pi}\, d\theta\right)(v\sin\theta) = \frac{2Mv}{\pi} \qquad \textbf{Ans.}$$

4. Let v is the velocity of mass $2m$ in natural length of spring, then from conservation of energy we have,

$$\frac{1}{2}kx_0^2 = \frac{1}{2}(2m)v^2$$

$$\therefore \quad v = \sqrt{\frac{k}{2m}}\, x_0$$

Velocity of centre of mass at this instant,

$$v_{CM} = \frac{\text{Total momentum}}{\text{Total mass}}$$

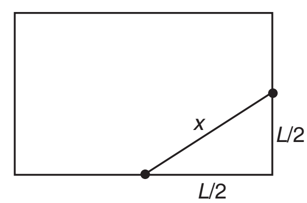
$$= \frac{2mv}{3m} = \frac{2}{3}v = \frac{1}{3}\sqrt{\frac{2k}{m}}\, x_0$$

Now, impulse on system = change in momentum of system

$$\therefore \quad F_{av}\,\Delta t = (3m)\, v_{CM} = (\sqrt{2mk})\, x_0$$

$$\therefore \quad F_{av} = \frac{(\sqrt{2mk})\, x_0}{\Delta t} \qquad \textbf{Ans.}$$

5. Retardation, $a = \dfrac{\mu\, mg}{m} = \mu g$



$$= 0.2 \times 10 = 2 \text{ m/s}^2$$

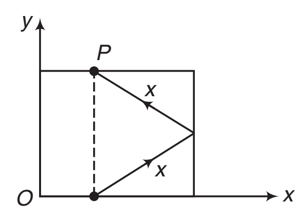
Total distance travelled before coming to rest,

$$d = \frac{u^2}{2a} = \frac{(2)^2}{2 \times 2} = 1\text{m}$$

$$x = \sqrt{2}\frac{L}{2} = \frac{L}{\sqrt{2}} = \frac{1}{2}$$

$$= 0.5 \text{ m}$$

Therefore striker will stop after travelling a distance $2x$, as shown in figure.



Co-ordinates of point P are, $\left(\dfrac{L}{2}, L\right)$

or $$\left(\frac{1}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

6. Displacement of M relative to m on reaching the other end is $2L \sin 30°$ or 1m.

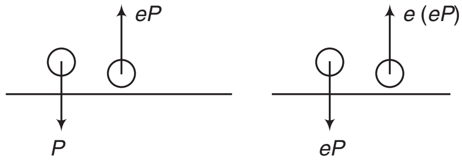
$$4x = 1\,(1-x)$$

$$\therefore \quad x = 0.2 \text{ m} \qquad \textbf{Ans.}$$

7. $\Delta P_1 = (1+e)\,P$

$\Delta P_2 = (1+e)\,eP$

and so on



$\therefore\ \Delta P_{\text{Total}} = \Delta P_1 + \Delta P_2 + \ldots$

$= (1+e)\,P\,(1 + e + e^2 + \ldots)$

$= \left(\dfrac{1+e}{1-e}\right) P$

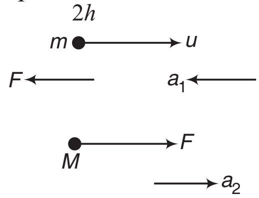
8. Assuming retardation force F to same in both cases, we have,

retardation $a = \dfrac{F}{m}$

Using, $v^2 = u^2 - 2as$, we have,

$$0 = u^2 - 2\left(\frac{F}{m}\right) h$$

$$\therefore \qquad F = \frac{mu^2}{2h} \qquad \text{...(i)}$$



In second case,

$$a_1 = \frac{F}{m},\ a_2 = \frac{F}{M}$$

Relative retardation of m,

$$a_r = a_1 + a_2 = F\left(\frac{M+m}{M\,m}\right)$$

Now, $v_r^2 = u_r^2 - 2a_r s$

$$\therefore \qquad 0 = u^2 - 2F\left(\frac{M+m}{M\,m}\right) s$$

or $s = \dfrac{u^2 M\,m}{2F\,(M+m)}$

Substituting, $\dfrac{u^2}{F} = \dfrac{2h}{m}$ from Eq. (i) we have,

$$s = \frac{M\,h}{M+m} \qquad \textbf{Ans.}$$

9. Velocity of B after collision will become,

$$\left(\frac{1+e}{2}\right) v = \left(\frac{1+e}{2}\right)(16)$$

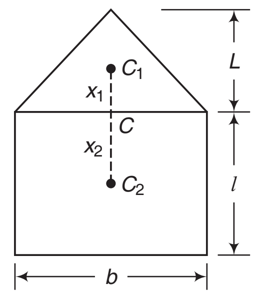
or $\qquad 8\,(1+e)$ m/s

Now, $\quad 8\,(1+e) = \sqrt{2gh}$

$= \sqrt{2 \times 10 \times 5} = 10$

$\therefore \qquad e = 0.25 \qquad$ **Ans.**

10.

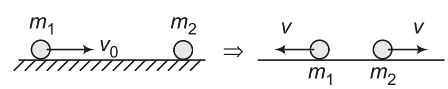


$$A_1 x_1 = A_2 x_2$$

$$\therefore \qquad \frac{1}{2}(bL)\left(\frac{L}{3}\right) = (bl)\left(\frac{l}{2}\right)$$

$$\therefore \qquad l = \frac{L}{\sqrt{3}} \qquad \textbf{Ans.}$$

11. $p_i = p_f$



$$\therefore \qquad m_1 v_0 = m_1 v + m_2 v \qquad \text{...(i)}$$

$$e = \frac{RVOS}{RVOA}$$

$e = 1$ for elastic collision

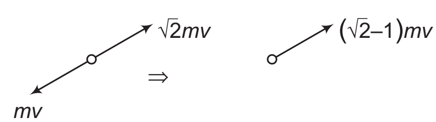
$\therefore \qquad RVOS = RVOA$

or $\qquad 2v = v_0 \qquad$ ...(ii)

From Eqs. (i) and (ii) we get,

$$\frac{m_1}{m_2} = \frac{1}{3}$$

12. Net momentum of three fragments



$$\mathbf{p}_i = \mathbf{p}_f = 0$$

$\therefore$ Momentum of fourth part is $(\sqrt{2}-1)\,mv$ in opposite direction.

So, velocity of fourth part is $(\sqrt{2}-1)\,v$

$$\therefore \qquad \text{KE} = 3\left(\frac{1}{2}mv^2\right) + \frac{1}{2}m\,[(\sqrt{2}-1)\,v]^2$$

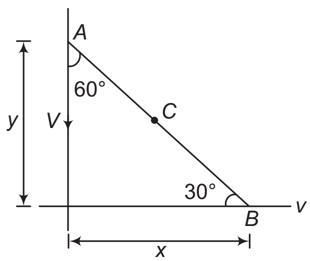
$$= (3 - \sqrt{2})\,mv^2 \qquad \textbf{Ans.}$$

13. $v_a = v_b = \dfrac{v}{2}$

$$v_c = \frac{v}{3}$$

$$\therefore \quad v_a : v_b : v_c = 3 : 3 : 2$$

14. Distance AB = constant



$\therefore$ Component of V along AB

= component of v along AB

or $$\frac{V}{2} = \frac{\sqrt{3}\,v}{2}$$

or $$V = \sqrt{3}\,v$$

Now, $$\frac{dx}{dt} = v$$

$$\left(-\frac{dy}{dt}\right) = V = \sqrt{3}\,v$$

$$\mathbf{r}_c = \frac{x}{2}\,\hat{\mathbf{i}} + \frac{y}{2}\,\hat{\mathbf{j}}$$

$$\therefore \quad \mathbf{v}_c = \frac{d\mathbf{r}_c}{dt} = \frac{1}{2}\left[\frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}}\right]$$

$$= \frac{1}{2}[v\hat{\mathbf{i}} - \sqrt{3}\,v\hat{\mathbf{j}}]$$

$$\therefore \quad |\mathbf{v}_c| = \left[\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right] v = v \qquad \textbf{Ans.}$$

15. At 5 s

$$\mathbf{r}_{COM} = \mathbf{r}_i + \mathbf{v}_{CM}\, t$$

$$= (\hat{\mathbf{i}}) + (0.2 \times 5)\,\hat{\mathbf{i}}$$

$$= 2\hat{\mathbf{i}} = \frac{m_1\mathbf{r}_i + m_2\mathbf{r}_2}{m_1 + m_2}$$

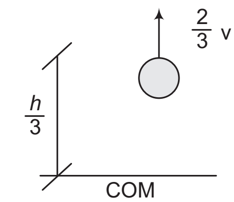
$$\therefore \quad 2\hat{\mathbf{i}} = \frac{(2/3)\,(3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - -4\hat{\mathbf{k}}) + (4/3)\mathbf{r}_2}{2}$$

$$\therefore \quad \mathbf{r}_2 = (4.5\hat{\mathbf{i}} - \hat{\mathbf{j}} - 2\hat{\mathbf{k}})$$

So, the co-ordinates are (4.5m, – 1m, – 2m) **Ans.**

16. m mass falls a distance $\dfrac{h}{2}$ in time, say t.

Then,



$$\frac{h}{2} = \frac{1}{2}gt^2 \quad \Rightarrow \quad \therefore \quad t = \sqrt{\frac{h}{g}}$$

For mass $2m$,

$$\frac{h}{2} = vt - \frac{1}{2}gt^2$$

$$= v\sqrt{\frac{h}{g}} - \frac{h}{2}$$

$$\therefore \quad v = \sqrt{gh}$$

Now, let u is the velocity with which combined mass collides with ground, then

$$u = \sqrt{\left(\frac{2}{3}v\right)^2 + 2g\left(\frac{h}{3}\right)}$$

$$= \sqrt{\frac{4}{9}gh + \frac{2}{3}gh}$$

$$= \frac{\sqrt{10}}{3}\sqrt{gh} \qquad \textbf{Ans.}$$

17. $2mv = m\,(u - v)$

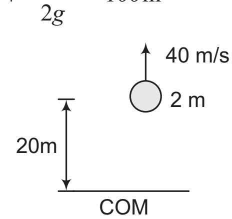
$$\therefore \quad v = \frac{u}{3}$$

$$\therefore \quad u - v = \frac{2u}{3}$$

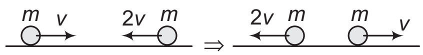
$$\text{KE} = \frac{1}{2} \times m\left(\frac{2u}{3}\right)^2 + \frac{1}{2}(2m)\left(\frac{u}{3}\right)^2$$

$$= \frac{1}{3}mu^2 \qquad \textbf{Ans.}$$

18. $h_{max} = 20 + \dfrac{(40)^2}{2g} = 100\,\text{m}$ **Ans.**



19. Let $|\mathbf{p}| = mv = p$



In elastic head on collision with equal masses, velocities are interchanged.

Gain in kinetic energy of first

$$= \frac{1}{2} m [(2v)^2 - v^2]$$

$$= \frac{3}{2} mv^2 = \left(\frac{3}{2} m\right)\left(\frac{p}{m}\right)^2 = \frac{3p^2}{2m} \quad \textbf{Ans.}$$

20. $$\frac{(4m) X_1 + (4m)\left(\frac{a}{2}\right)}{8m} = \frac{(4m) X_1 + (m)(a)}{5m}$$

Solving this equation, we get

$X_1 = X$-coordinate of COM of plate

$$= \frac{a}{6} \quad \textbf{Ans.}$$

21. $$X_{CM} = \frac{m_1 X_1 + m_2 X_2 + m_3 X_3 + m_4 X_4}{m_1 + m_2 + m_3 + m_4}$$

Similarly Y_{CM}.

22. $\mathbf{p}_i = \mathbf{p}_f$

$$\therefore \; (m_0 v_0 \hat{\mathbf{i}}) = m_0 \frac{v_0}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\hat{\mathbf{i}} + \frac{1}{\sqrt{2}}\hat{\mathbf{j}}\right) + 2m_0 \mathbf{v}$$

$$\therefore \quad \mathbf{v} = \frac{v_0}{4}\hat{\mathbf{i}} - \frac{v_0}{4}\hat{\mathbf{j}}$$

$$\therefore \quad |\mathbf{v}| = \sqrt{2}\,\frac{v_0}{4} = \frac{v_0}{2\sqrt{2}} \quad \textbf{Ans.}$$

23. $$\frac{1}{2} k x_0^2 = \frac{1}{2} m_2 v_2^2$$

$$\therefore \quad v_2 = \sqrt{\frac{k}{m_2}}\, x_0$$

$$v_{CM} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

$$= \left(\frac{\sqrt{k\, m_2}}{m_1 + m_2}\right) x_0 \quad (\text{as } v_1 = 0)$$

24. Velocity of second ball

$$= \left(\frac{m/2 - m}{m/2 + m}\right)(0) + \left(\frac{2 \times m}{m + m/2}\right) v$$

$$= \frac{4}{3} v$$

Velocity of third ball will become,

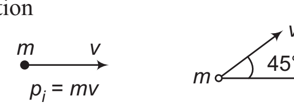
$$\frac{4}{3} \text{ times } \frac{4}{3} v \quad \text{or} \quad \left(\frac{4}{3}\right)^2 v$$

$$\therefore \quad v_n = \left(\frac{4}{3}\right)^{n-1} v = \sqrt{5gr}$$

$$\therefore \quad v = \left(\frac{3}{4}\right)^{n-1} \sqrt{5gr} \quad \textbf{Ans.}$$

More than One Correct Options

1. Momentum remains conserved in any type of equation



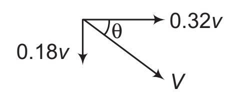
$\mathbf{p}_i = \mathbf{p}_f$

$$\therefore \quad (mv\hat{\mathbf{i}}) = \left(\frac{mv}{2\sqrt{2}}\hat{\mathbf{i}} + \frac{mv}{2\sqrt{2}}\hat{\mathbf{j}}\right) + (2m)\,\mathbf{v}$$

$\mathbf{v}$ = velocity of mass 2m

$= 0.32\hat{\mathbf{i}} - 0.35\hat{\mathbf{j}}$

$= (0.32v)\,\hat{\mathbf{i}} - (0.18v)\,\hat{\mathbf{j}}$



$V = 0.37\, v$

$$\tan\theta = \frac{0.18v}{0.32v} = 0.5625$$

$\therefore \; \theta = 29.35°$

Since $(\theta + 45°) < 90°$. Therefore, the angle of divergence between particles after collision is less than 90°.

Further, $K_i = \frac{1}{2} mv^2$ and

$$K_f = \frac{1}{2} m \left(\frac{v}{2}\right)^2 + \frac{1}{2}(2m)(0.37v)^2$$

$$K_f < K_i$$

Therefore, collision is inelastic.

2. Just after collision,

$$v_m = \left(\frac{m - 5m}{m + 5m}\right)\sqrt{2gl} = -\frac{2}{3}\sqrt{2gl}$$

$$v_{5m} = \left(\frac{2 \times m}{m + 5m}\right)\sqrt{2gl} = \frac{\sqrt{2gl}}{3}$$

$$T - mg = \frac{mv_m^2}{l} = \frac{m}{l}\left(\frac{8gl}{9}\right)$$

$$\therefore \quad T = \frac{17}{9} mg$$

$$h_m = \frac{v_m^2}{2g} = \frac{4l}{9}$$

3. $$u\cos\theta = v\cos\phi \quad \text{...(i)}$$

$$v\sin\phi = eu\sin\theta$$

or $$eu\sin\theta = v\sin\phi \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we can see that,

$$\tan\phi = e\tan\theta$$

Momentum or velocity changes only in vertical direction.

$$\therefore \quad |\text{Impulse}| = |\Delta p|$$

$$= m\,(u\sin\theta + eu\sin\theta)$$

$$= m\,(1+e)\,u\sin\theta$$

$$v = \sqrt{(v\cos\phi)^2 + (v\sin\phi)^2}$$

$$= \sqrt{(u\cos\theta)^2 + (eu\sin\theta)^2}$$

$$= \sqrt{u^2\,(\cos^2\theta + e^2\sin^2\theta)}$$

$$= u\sqrt{1-(1-e^2)\sin^2\theta}$$

$$\frac{K_f}{K_i} = \frac{\frac{1}{2}mv^2}{\frac{1}{2}mu^2} = \frac{v^2}{u^2}$$

$$= \cos^2\theta + e^2\sin^2\theta$$

4. Impulse $= \Delta\mathbf{p} = m\,(\mathbf{v}_f - \mathbf{v}_i)$

$\therefore$ Impulse received by m

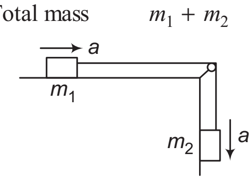
$$= m\,[(-2\hat{\mathbf{i}} + \hat{\mathbf{j}}) - (3\hat{\mathbf{i}} + 2\hat{\mathbf{j}})]$$

$$= m\,(-5\hat{\mathbf{i}} - \hat{\mathbf{j}})$$

Impulse received by $M = -$ (impulse received by m)

$$= m\,(5\hat{\mathbf{i}} + \hat{\mathbf{j}})$$

5. $a = \dfrac{\text{Net pulling force}}{\text{Total mass}} = \dfrac{m_2 g}{m_1 + m_2}$



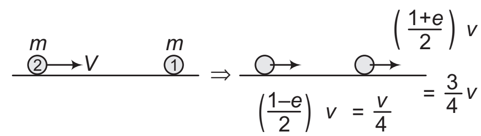
Now,

$$(a_{\text{cm}})_x = \frac{m_1 a}{m_1 + m_2} = \frac{m_1 m_2 g}{(m_1+m_2)^2}$$

$$(a_{\text{cm}})_y = \frac{m_2 a}{m_1 + m_2} = \left(\frac{m_2}{m_1+m_2}\right)^2 g$$

6. Out of two blocks, one block of mass m is moving in vertical direction also (downwards). Therefore COM is moving vertically downwards and momentum of the system is not conserved in vertical direction.

7. $|\text{Impulse}| = |\Delta p_1 \text{ or } \Delta p_2|$



$$= m\left(\frac{3}{4}v\right) = \frac{3mv}{4}$$

Loss of kinetic energy $= K_i - K_f$

$$= \frac{1}{2}mv^2 - \left[\frac{1}{2}m\left(\frac{3v}{4}\right)^2 + \frac{1}{2}m\left(\frac{v}{4}\right)^2\right]$$

$$= \frac{3}{16}mv^2$$

8. External force gravity acts on system. Therefore momentum of system is not conserved.

Mass keep on decreasing. Therefore acceleration will keep on increasing.

Comprehension Based Questions

1. Impulse $= mu$

$$\therefore \quad u = \frac{\text{Impulse}}{m} = \frac{4}{2} = 2\text{ m/s}$$

Now, $\quad E_i = E_f$

$$\therefore \quad \frac{1}{2}\times 2\times (2)^2 + \frac{1}{2}\times 4000\times (0.05)^2 = \frac{1}{2}\times 2\times v^2$$

Solving we get,

$$v = 3\text{ m/s}$$ **Ans.**

2. Again using,

$$E_i = E_f$$

$$\therefore \quad \frac{1}{2}\times 2\times (2)^2 + \frac{1}{2}\times 4000\times (0.05)^2 = \frac{1}{2}\times 4000\times x^2$$

$\therefore \quad x = 0.067$ m $= 6.7$ m $=$ compression

$\therefore \quad d = 5$ cm $+ 6.7$ cm $+ 6.7$ cm $+ 6.7$ cm

≈ 25 cm **Ans.**

3. $p_i = p_f$

$$\therefore \quad 0 = (8m)(2v) - (16m)(v) + (48m)\,v'$$

Here v' = absolute speed of rod

$= 0$

$\therefore$ Displacement of rod $= 0$ **Ans.**

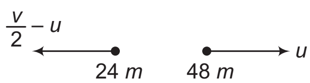
4. $x_A + x_B = 12\,L$

$\therefore \qquad 2vt + vt = 12L$

$\therefore \qquad vt = 4L$

$\Rightarrow \qquad X_B = vt = 4L$

5. $(24\,m)\left(\dfrac{v}{2} - u\right) = (48\,m)\,u$



$\therefore \qquad u = \dfrac{v}{6}$ **Ans.**

6. $t = t_1 + t_2 = \left(\dfrac{4L}{v}\right) + \left(\dfrac{12L - 4L}{v/2}\right) = \dfrac{20L}{v}$

But $\qquad \dfrac{L}{v} = T = 4\text{ s}$

$\therefore \qquad t = 80\text{ s}$ **Ans.**

7. Till t_1, rod is stationary. For time t_2 rod is moving with absolute speed $u\ (= v/6)$

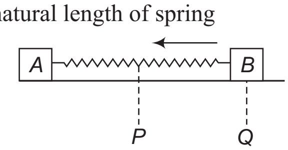
$\therefore$ Displacement of rod $= \left(\dfrac{v}{6}\right) t_2$

$= \left(\dfrac{v}{6}\right)\left(\dfrac{16L}{v}\right) = \dfrac{8L}{3}$ **Ans.**

Match the Columns

1. $P \rightarrow$ compressed state of spring

$Q \rightarrow$ natural length of spring



From P to Q

$$a_B = \frac{Kx}{m_B}$$

$$a_{CM} = \frac{m_B a_B}{m_A + m_B} = \frac{(m_B)(Kx)}{m_A + m_B}$$

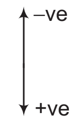
From P to Q, compression x decreases. Therefore, a_{CM} decreases. After Q, A leaves contact with wall, spring comes in its natural length. Net force on system becomes zero.

Therefore, a_{CM} becomes zero.

From P to Q velocity of B, therefore velocity of COM will increase. After that a_{CM} becomes zero. Therefore, v_{CM} becomes constant.

2. $t_1 = \dfrac{2u}{g} = \dfrac{2 \times 20}{10} = 4\text{ s}$

$t_2 = \sqrt{\dfrac{2h}{g}} = \dfrac{\sqrt{2 \times 180}}{10} = 6\text{ s}$



At $t = 0$

$$a_{CM} = \frac{m_1 a_1 + m_2 a_2}{m_1 + m_2} = \frac{(m)(10) + (m)(10)}{2m}$$

$$= 10\text{ m/s}^2$$

$$v_{CM} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} = \frac{(m)(-20) + (m)(0)}{2m}$$

$$= -10\text{ m/s}$$

At $t = 5$ s

$$a_{CM} = \frac{(m)(0) + (m)(10)}{2m} = 5\text{ m/s}^2$$

$$v_{CM} = \frac{(m)(0) + (m)(50)}{2m} = 25\text{ m/s}$$

3. (a) $p = \sqrt{2Km} = \sqrt{2 \times 0.5 \times 4} = 2\text{ kg-m/s}$

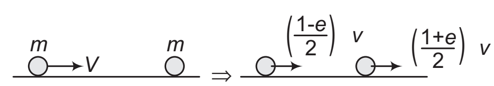
(b) $p_{CM} = p_A + p_B = 0 + 2 = 2\text{ kg - m/s}$

(c) At maximum compression, their velocities or, their momenta are same (half of p_{CM}).

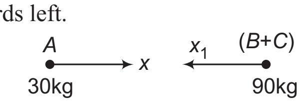
$\therefore \qquad p_A = p_B = \dfrac{p_{CM}}{2} = 1\text{ kg - m/s}$

(d) In elastic collision between two equal masses, velocities are interchanged. So, B comes to rest and A starts moving.

4.

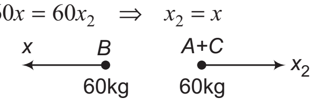


5. (a) If A moves towards right then B and C will move towards left.



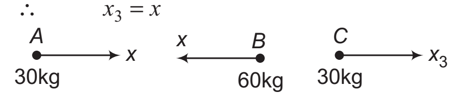
$$30x = 90x_1 \quad \Rightarrow \quad x_1 = \frac{x}{3}$$

(b) $60x = 60x_2 \quad \Rightarrow \quad x_2 = x$

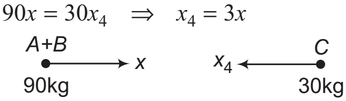


(c) $30x + 30x_3 = 60x$

$\therefore \qquad x_3 = x$



(d) $90x = 30x_4 \quad \Rightarrow \quad x_4 = 3x$



6. (a) If T = tension on string connecting m_1 and m_2, then, $2T$ = tension in other string.
Equilibrium of m_2 gives,

$$T = m_2 g = 200 \text{ N}$$

Equilibrium of $(M + m_1)$ gives,

$$3T = W_{\text{man}} + m_1 g$$

or $\quad 600 = W_{\text{man}} + 100 \quad \Rightarrow \quad W_{\text{man}} = 500 \text{ N}$

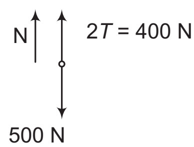
(b) Total upward force on system is $4T$ and total downward force is weight of all. To accelerate COM upwards,

$$4T > 200 + 100 + 500 \quad \text{or} \quad T > 200 \text{ N}$$

(c) To accelerate downwards,

$$T < 200 \text{ N}$$

(d) In equilibrium, forces on man



$$N + 400 = 500$$

$\therefore \quad N = 100 \text{ N}$

7. $v_{\text{CM}} = \dfrac{\text{Total momentum}}{\text{Total mass}}$

$$= \frac{2 \times 3}{3 + 6} = \frac{2}{3} \text{ m/s} = \text{constant}$$

At maximum deformation,

$$v_{3\text{kg}} = v_{6\text{kg}} = v_{\text{CM}} = \frac{2}{3} \text{ m/s}$$

Upto maximum elongation spring force on 6kg is towards left. So 6 kg block will accelerate and its velocity will be maximum.

8. $V_{\text{CM}} = \dfrac{\text{Total momentum}}{\text{Total mass}}$

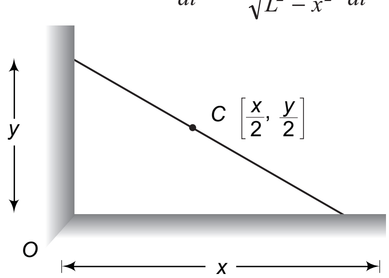
$$= \frac{(10 \times 1) - (5 \times 2)}{3} = 0 = \text{constant}$$

$P_{\text{CM}} = (m_{\text{CM}})(v_{\text{CM}}) = 0 = \text{constant}$

Finally both blocks will stop.

Subjective Questions

1. $y = \sqrt{L^2 - x^2} \quad \Rightarrow \quad \dfrac{dy}{dt} = -\dfrac{x}{\sqrt{L^2 - x^2}} \dfrac{dx}{dt}$

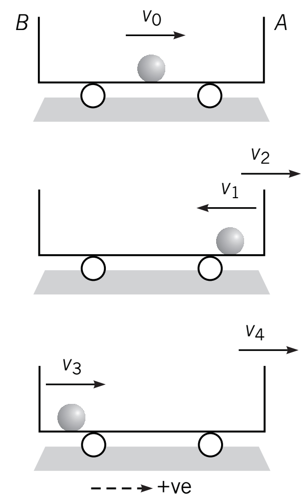


$$= -\frac{3 \times 2}{4} = -\frac{3}{2} \text{ m/s}$$

$$V_{CM} = \sqrt{\left(\frac{1}{2}\frac{dx}{dt}\right)^2 + \left(\frac{1}{2}\frac{dy}{dt}\right)^2}$$

$$= \sqrt{(1)^2 + \left(\frac{3}{4}\right)^2} = 1.25 \text{ m/s} \quad \textbf{Ans.}$$

2. Applying conservation of linear momentum and



$e = \dfrac{\text{Relative speed of separation}}{\text{Relative speed of approach}}$, we get

$$mv_0 = Mv_2 - mv_1 \qquad \ldots\text{(i)}$$

$$v_1 + v_2 = ev_0 \qquad \ldots\text{(ii)}$$

Solving these two equations, we get,

$$v_1 = \left(\frac{eM - m}{M + m}\right) v_0, \quad v_2 = m\left(\frac{e + 1}{M + m}\right) v_0 \qquad \textbf{Ans.}$$

The desired time is

$$t = \frac{d}{v_0} + \frac{2d}{ev_0} + \frac{d}{e^2 v_0}$$

or $$t = \frac{d}{v_0}\left(1 + \frac{2}{e} + \frac{1}{e^2}\right) \qquad \textbf{Ans.}$$

3. (i) $x_1 = v_0 t - A(1 - \cos \omega t)$

$$t\, x_{\text{cm}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = v_0 t$$

$$\therefore \quad x_2 = v_0 t + \frac{m_1}{m_2} A(1 - \cos \omega t)$$

(ii) $a_1 = \dfrac{d^2 x_1}{dt^2} = -\omega^2 A \cos \omega t$

The separation $x_2 - x_1$ between the two blocks will be equal to l_0 when $a_1 = 0$ or $\cos \omega t = 0$

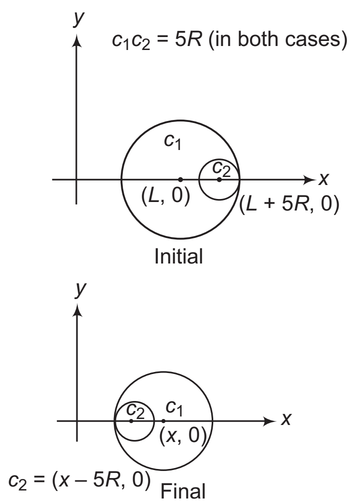
$$x_2 - x_1 = \frac{m_1}{m_2} A(1 - \cos \omega t) + A(1 - \cos \omega t)$$

or $$l_0 = \left(\frac{m_1}{m_2} + 1\right) A \qquad (\cos \omega t = 0)$$

Thus, the relation between l_0 and A is,

$$l_0 = \left(\frac{m_1}{m_2} + 1\right) A$$ **Ans.**

4. Since, all the surfaces are smooth, no external force is acting on the system in horizontal direction. Therefore, the centre of mass of the system in horizontal direction remains stationary.



x-coordinate of COM initially will given by

$$x_i = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$$x_i = \frac{(4M)(L) + M(L + 5R)}{4M + M} = (L + R) \qquad \ldots(\text{i})$$

Let $(x, 0)$ be the coordinates of the centre of large sphere in final position. Then, x-coordinate of COM finally will be

$$x_f = \frac{(4M)(x) + M(x - 5R)}{4M + M} = (x - R) \qquad \ldots(\text{ii})$$

Equating Eqs. (i) and (ii), we have

$$x = L + 2R$$

Therefore, coordinates of large sphere, when the smaller sphere reaches the other extreme position are $(L + 2R, 0)$. **Ans.**

5. (a) Chain has a constant speed. Therefore, net force on it should be zero. Thus,

P = Weight of length y of chain + thrust force

$$= \frac{m}{l} yg + \rho v_0^2 \qquad \left(\text{here } \rho = \frac{m}{l}\right)$$

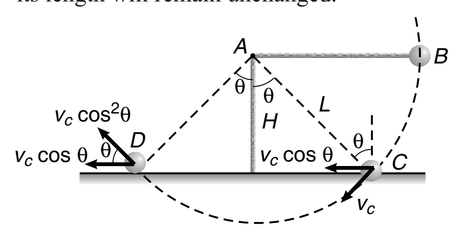
$$= \frac{m}{l}(gy + v_0^2)$$ **Ans.**

(b) Energy lost during the lifting = work done by applied force – increase in mechanical energy of chain

$$= \int_0^y P \cdot dy - \left(\frac{m}{l} y\right) g \left(\frac{y}{2}\right) - \frac{1}{2}\left(\frac{m}{l} \cdot y\right) v_0^2$$

$$= \frac{myv_0^2}{2l}$$ **Ans.**

6. In perfectly inelastic collision with the horizontal surface the component parallel to the surface will remain unchanged. Similarly when the string becomes taut again, the component perpendicular to its length will remain unchanged.



$$\cos\theta = \frac{H}{L} \quad \Rightarrow \quad v_c = \sqrt{2gH}$$

$$v_c \cos^2\theta = (\sqrt{2gH})\frac{H^2}{L^2} = v \text{ (say)}$$

$$\therefore \qquad h = \frac{v^2}{2g} = \frac{(2gH)\dfrac{H^4}{L^4}}{2g} = \frac{H^5}{L^4}$$ **Ans.**

7. Applying conservation of linear momentum at the time of collision or at $t = 1$ s,

$$m\mathbf{v} + m(0) = 2m(20\hat{\mathbf{i}} + 10\hat{\mathbf{j}})$$

$$\therefore \qquad \mathbf{v} = 40\hat{\mathbf{i}} + 20\hat{\mathbf{j}}$$

At 1 s, masses will be at height

$$h_1 = u_y t + \frac{1}{2} v_y t^2 = (20)(1) + \frac{1}{2}(-10)(1)^2 = 15 \text{ m}$$

After explosion other mass will further rise to a height :

$$h_2 = \frac{u_y^2}{2g} = \frac{(20)^2}{2 \times 10}$$

$$= 20 \text{ m}$$

$\Rightarrow$ $u_y = 20$ m/s just after collision.

$\therefore$ Total height $h = h_1 + h_2 = 35$ m **Ans.**

8. Let CT stands for common tangent direction and CN for common normal directions.

	Mass $m = eM$		Mass M	
	CT	**CN**	**CT**	**CN**
Befor collision	v_1 (let)	v_2 (let)	Zero (given)	Zero (given)
After collision	v_1	v_3 (suppose)	Zero	v_4 (suppose)

In the common tangent directions velocity components remain unchanged.

In common normal direction applying conservation of linear momentum and definition of e.

$$eMv_2 = eMv_3 + Mv_4 \qquad \ldots\text{(i)}$$

From the definition of coefficient of restitution

$$e = \frac{v_4 - v_3}{v_2} \qquad \ldots\text{(ii)}$$

Solving these two equations we get,

$$v_3 = 0 \text{ but } v_4 \neq 0$$

So, after collision velocity of m is along CT while that of M along CN or they are moving at right angles.

9. Muzzle velocity v_r is given to be constant.

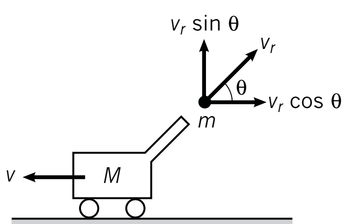
From conservation of linear momentum in horizontal direction we have,

$$Mv = m(v_r \cos\theta - v)$$

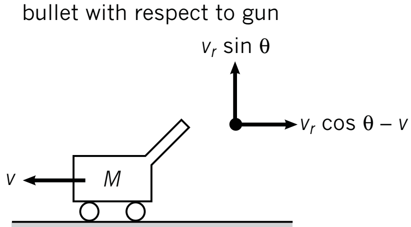
or
$$v = \frac{mv_r \cos\theta}{M + m} \qquad \ldots\text{(i)}$$

Further, range of bullet on horizontal ground

$$R = \frac{2v_r \sin\theta}{g}(v_r \cos\theta - v)$$



Components of velocity of bullet with respect to gun



Components of velocity of bullet with respect to ground

$$= \frac{2v_r \sin\theta}{g}\left(v_r \cos\theta - \frac{mv_r \cos\theta}{M + m}\right)$$

$$= \frac{2Mv_r^2 \sin\theta \cos\theta}{(M + m)g}$$

or
$$R = \left(\frac{M}{M + m}\right)\frac{v_r^2 \sin 2\theta}{g} \qquad \ldots\text{(ii)}$$

(a) From Eq. (ii) we see that maximum range is at

$$\theta = 45° \qquad \textbf{Ans.}$$

(b) At $\theta = 45°$,

$$R_{\max} = \left(\frac{M}{M + m}\right)\frac{v_r^2}{g} \qquad \textbf{Ans.}$$

10. (a) $u_r = 0, \quad a_r = g$

$\therefore \qquad v_r = \sqrt{2gh_1}$

After collision relative velocity

$$v_r' = e\sqrt{2gh_1}$$

and relative retardation is still g (downwards). Hence,

$$h_2 = \frac{(v_r')^2}{2g} = e^2 h_1 \qquad \textbf{Ans.}$$

(b) $u_r = 0, \quad a_r = g + \dfrac{g}{4} = \dfrac{5g}{4}$

$\therefore$ Just before collision $v_r = \sqrt{2\left(\dfrac{5g}{4}\right)h_1}$

Just after collision $v_r' = ev_r$.

Relative retardation is still $\dfrac{5g}{4}$.

Hence, $$h_2 = \frac{(v_r')^2}{2\left(\dfrac{5g}{4}\right)} = e^2 h_1 \qquad \textbf{Ans.}$$

11. Let the velocity of the block and the plank, when the block leaves the spring be u and v respectively.

By conservation of energy $\frac{1}{2}kx^2 = \frac{1}{2}mu^2 + \frac{1}{2}Mv^2$

[M = mass of the plank, m = mass of the block]

$$\therefore \qquad 100 = u^2 + 5v^2 \qquad \ldots\text{(i)}$$

By conservation of momentum

$$mu + Mv = 0$$

$$\Rightarrow \qquad u = -5v \qquad \ldots\text{(ii)}$$

Solving Eqs. (i) and (ii)

$$30v^2 = 100 \Rightarrow \therefore v = \sqrt{\frac{10}{3}} \text{ m/s}$$

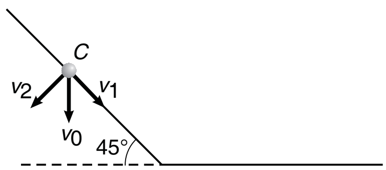
From this moment until block falls, both plank and block keep their velocity constant.

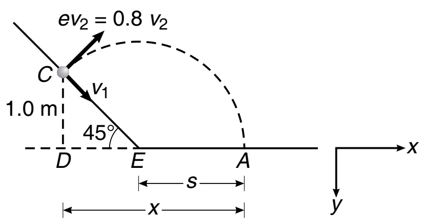
Thus, when block falls velocity of plank

$$= \sqrt{\frac{10}{3}} \text{ m/s.} \qquad \textbf{Ans.}$$

12. $v_0 = \sqrt{2gh} = \sqrt{2 \times 9.8 \times 1.5} = 5.42 \text{ m/s}$

Component of velocity parallel and perpendicular to plane at the time of collision.





$$v_1 = v_2 = \frac{v_0}{\sqrt{2}} = 3.83 \text{ m/s}$$

Component parallel to plane (v_1) remains unchanged, while component perpendicular to plane becomes ev_2, where

$$ev_2 = 0.8 \times 3.83 = 3.0 \text{ m/s}$$

∴ Component of velocity in horizontal direction after collision

$$v_x = \frac{(v_1 + ev_2)}{\sqrt{2}} = \frac{(3.83 + 3.0)}{\sqrt{2}} = 4.83 \text{ m/s}$$

While component of velocity in vertical direction after collision.

$$v_y = \frac{v_1 - ev_2}{\sqrt{2}} = \frac{3.83 - 3.0}{\sqrt{2}} = 0.59 \text{ m/s}$$

Let t be the time, the particle takes from point C to A, then

$$1.0 = 0.59t + \frac{1}{2} \times 9.8 \times t^2$$

Solving this we get,

$$t = 0.4 \text{ s} \qquad \text{(Positive value)}$$

∴ $DA = v_x t = (4.83)(0.4) = 1.93$ m

∴ $s = DA - DE$

$= 1.93 - 1.0$

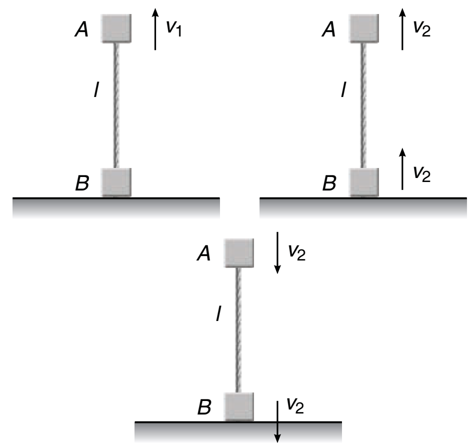
$s = 0.93$ m **Ans.**

$$v_{yA} = v_{yc} + gt$$
$$= (0.59) + (9.8)(0.4) = 4.51 \text{ m/s}$$
$$v_{xA} = v_{xC} = 4.83 \text{ m/s}$$

∴ $v_A = \sqrt{(v_{xA})^2 + (v_{yA})^2}$

$= 6.6$ m/s **Ans.**

13. String becomes tight when A moves upwards by a distance l. Let v_1 be the velocity of A at this moment, then



$$v_1^2 = (\sqrt{10gl})^2 - 2gl = 8gl \text{ or } v_1 = \sqrt{8gl}$$

Let v_2 be the common velocities of both A and B just after string becomes tight. Then from conservation of linear momentum.

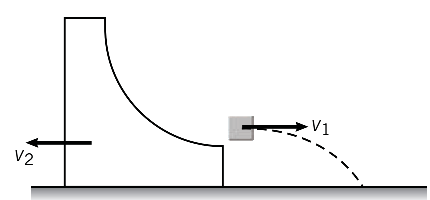
$$v_2 = \frac{v_1}{2} = \frac{\sqrt{8gl}}{2}$$

Both particles return to their original height with same speed v_2. String becomes loose after B strikes the ground and the speed v with which A strikes the ground is,

$$v^2 = v_2^2 + 2gl = \frac{8gl}{4} + 2gl$$

or $\qquad v^2 = 4gl \quad \text{or} \quad v = 2\sqrt{gl}$ **Ans.**

14.



$$mv_1 = Mv_2 \qquad \text{...(i)}$$

$$mgR = \frac{1}{2} mv_1^2 + \frac{1}{2} Mv_2^2 \qquad \text{...(ii)}$$

$$t = \sqrt{\frac{2(R/2)}{g}} = \sqrt{\frac{R}{g}} \qquad \text{...(iii)}$$

The desired distance is

$$S = (v_1 + v_2)t \qquad \text{...(iv)}$$

Solving Eqs. (i) and (ii) for v_1 and v_2 and substituting in Eq. (iv), we get

$$S = R\sqrt{\frac{2(M + m)}{M}}$$ **Ans.**

15. Let v_r be the velocity of washer relative to centre of hoop and v the velocity of centre of hoop. Applying conservation of linear momentum and mechanical energy we have,

$$m(v_r \cos\phi - v) = Mv \qquad \ldots\text{(i)}$$

$$mgr(1+\cos\phi) = \frac{1}{2}Mv^2 + \frac{1}{2}m(v_r^2 + v^2 - 2vv_r\cos\phi) \qquad \ldots\text{(ii)}$$

Solving Eqs. (i) and (ii), we have,

$$v = m\cos\phi\sqrt{\frac{2gr(1+\cos\phi)}{(M+m)(M+m\sin^2\phi)}} \qquad \textbf{Ans.}$$

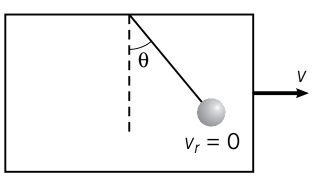
16. $H = \dfrac{u^2\sin^2\theta}{2g} = \dfrac{(20)^2 \times \frac{3}{4}}{2\times 10} = 15$ m

i.e. the shell strikes the ball at highest point of its trajectory. Velocity of (ball + shell) just after collision,

$$v = \frac{u\cos 60°}{2}$$

(from conservation of linear momentum)

$$= \frac{20}{2\times 2} = 5\,\text{m/s}$$



At highest point combined mass is at rest relative to the trolley. Let v be the velocity of trolley at this instant. From conservation of linear momentum we have,

$$2\times 5 = \left(2+\frac{4}{3}\right)v \quad \text{or} \quad v = 3 \text{ m/s}$$

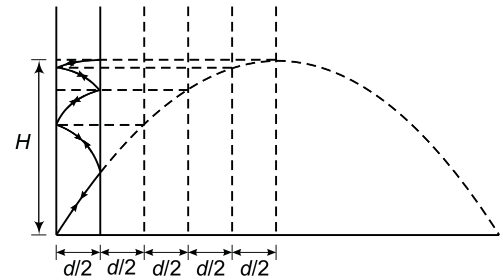
From conservation of energy, we have

$$\frac{1}{2}\times 2\times(5)^2 - \frac{1}{2}\left(2+\frac{4}{3}\right)(3)^2 = 2\times 10(1-\cos\theta)$$

Solving we get, $\cos\theta = \dfrac{1}{2}$

$\therefore$ $\theta = 60°$ **Ans.**

17. While colliding with the wall its vertical component (v_y) of velocity will remain unchanged (component along common tangent direction remains unchanged) while horizontal component (v_x) is reversed remaining same in magnitude. Thus, path of the particle will be as shown in figure.



(a) $H = \dfrac{u^2\sin^2\alpha}{2g}$

(b) Total number of collisions with the walls before the ball comes back to the ground are nine.

(c) Ball will return to point O (the starting point)

18. As the collisions are perfectly elastic, collision of the ball will not affect the vertical component of its velocity while the horizontal component will be simply reversed.

Hence, $H_{\max} = \dfrac{v_y^2}{2g} = \dfrac{[20\times\sin 45°]^2}{2\times 10} = 10$ m

Total time of flight

$$T = \frac{2v_y}{g} = \frac{2\times 20\times\left(\frac{1}{\sqrt{2}}\right)}{10} = 2\sqrt{2}\text{ s}$$

Total horizontal distance travelled before striking the ground $x = v_xT = 40$ m

$$PB + BA + AB + BA + AB = 45 \text{ m}$$

Hence, total number of collision suffered by the particle with the walls before it hits ground = 4.

Ans.

19. Let v_1 = velocity of block 2 kg just before collision

v_2 = velocity of block 2 kg just after collision

and v_3 = velocity of block M just after collision.

Applying work energy theorem

(change in kinetic energy = work done by all the forces) at different stages as shown in figure.

Figure 1.

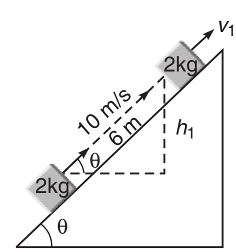


Fig. 1

$$\Delta\text{KE} = W_{\text{friction}} + W_{\text{gravity}}$$

$$\left[\frac{1}{2}m\{v_1^2-(10)^2\}\right]=-6\mu\,mg\cos\theta-mgh_1$$

$(m = 2\text{ kg})$

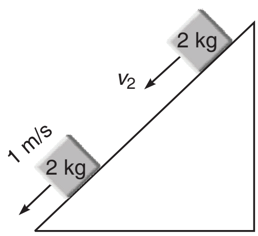
or $\quad v_1^2-100=2[6\mu g\cos\theta+gh_1]$

$$\cos\theta=\sqrt{1-\sin^2\theta}=\sqrt{1-(0.05)^2}\approx 0.99$$

$\therefore \quad v_1^2=100-2[(6)(0.25)(10)(0.99)+(10)(0.3)]$

$\Rightarrow \quad v_1\approx 8\text{ m/s}$

Figure 2.



$$\Delta KE=W_{\text{friction}}+W_{\text{gravity}}$$

$$\frac{1}{2}m[(1)^2-(v_2^2)]=-6\mu\,mg\cos\theta+mgh_1$$

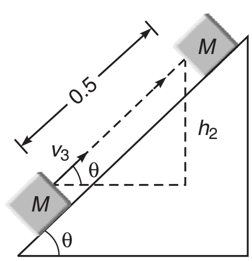
or $\quad 1-v_1^2=2[-6\mu\,g\cos\theta+gh_1]$

$$=2[(-6)(0.25)(10)(0.99)+(10)(0.3)]$$

$$=-23.7$$

$\therefore \quad v_2^2=24.7 \quad$ or $\quad v_2\approx 5\text{ m/s}$

Figure 3.



$$\Delta KE=W_{\text{friction}}+W_{\text{gravity}}$$

$$\frac{1}{2}M[0-v_3^2]=-(0.5)(\mu)(M)g\cos\theta-Mgh_2$$

or $\quad -v_3^2=-\mu\,g\cos\theta-2gh_2$

or $\quad v_3^2=(0.25)(10)(0.99)+2(10)(0.025)$

or $\quad v_3^2=2.975$

$\therefore \quad v_3\approx 1.72\text{ m/s}$

Now

(i) Coefficient of restitution

$$=\frac{\text{Relative velocity of separation}}{\text{Relative velocity of approach}}$$

$$=\frac{v_2+v_3}{v_1}=\frac{5+1.72}{8}=\frac{6.72}{8}$$

or $\quad e\approx 0.84$

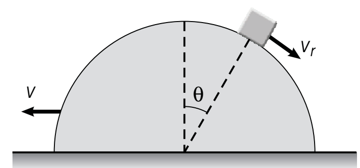
(ii) Applying conservation of linear momentum before and after collision

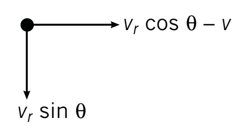
$$2v_1=Mv_3-2v_2$$

$$\therefore \quad M=\frac{2(v_1+v_2)}{v_3}=\frac{2(8+5)}{1.72}=\frac{26}{1.72}$$

$M\approx 15.12\text{ kg}$ **Ans.**

20. Let v_r be the relative velocity of block as it leaves contact with the sphere ($N=0$) and v the horizontal velocity of sphere at this instant.





Absolute components of velocity of block

Applying conservation of linear momentum in horizontal direction, we get

$$mv=m(v_r\cos\theta-v)$$

or $\quad 2v=v_r\cos\theta \quad$ ...(i)

Conservation of mechanical energy gives,

$$mgr(1-\cos\theta)=\frac{1}{2}mv^2+\frac{1}{2}m(v_r^2+v^2-2vv_r\cos\theta)$$

or $\quad gr(1-\cos\theta)=v^2+\dfrac{v_r^2}{2}-vv_r\cos\theta \quad$ ...(ii)

Equation of laws of motion gives,

$$mg\cos\theta=\frac{mv_r^2}{r} \quad \text{or} \quad gr=\frac{v_r^2}{\cos\theta} \quad \text{...(iii)}$$

Solving Eqs. (i), (ii) and (iii), we get

$$\cos^3\theta-6\cos\theta+4=0$$ **Ans.**

Note *We have not considered pseudo force while writing the equation of motion. Think why?*

21. $\dfrac{x}{u\cos\alpha}+\dfrac{x}{eu\cos\alpha}=T=\dfrac{2u\sin\alpha}{g}$

or $\quad x=\dfrac{eu^2\sin 2\alpha}{(1+e)g} \quad \Rightarrow \quad x_{\max}=\dfrac{eu^2}{(1+e)g}$

at $\quad 2\alpha=90°$ **Ans.**

12. Rotational Mechanics

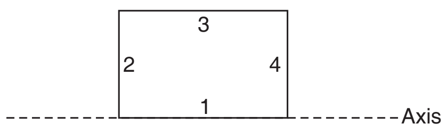
INTRODUCTORY EXERCISE 12.1

1. $K = \sqrt{\dfrac{I}{m}} = \sqrt{\dfrac{m\,(2l)^2/3}{m}} = \dfrac{2}{\sqrt{3}}\,l$ **Ans.**

2. $I = (1)\left[\sqrt{(1)^2 + (2)^2}\,\right]^2 + 2[\sqrt{(3^2) + (4)^2}\,]^2$

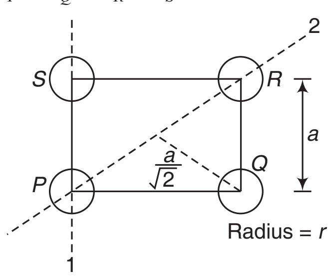
$= 55 \text{ kg-m}^2$ **Ans.**

3. $I = I_1 + I_2 + I_3 + I_4$



$= 0 + \dfrac{ml^2}{3} + ml^2 + \dfrac{ml^2}{3} = \dfrac{5}{3}\,ml^2$ **Ans.**

5. $I_1 = I_P + I_Q + I_R + I_S$



$= \left(\dfrac{2}{5}\,mr^2\right) + \left(\dfrac{2}{5}\,mr^2 + ma^2\right) + \left(\dfrac{2}{5}\,mr^2 + ma^2\right) + \dfrac{2}{5}\,mr^2$

$= \dfrac{8}{5}\,mr^2 + 2ma^2$

$I_2 = I_P + I_Q + I_R + I_S$

$= \dfrac{2}{5}\,mr^2 + \left(\dfrac{2}{5}\,mr^2 + \dfrac{ma^2}{2}\right) + \dfrac{2}{5}\,mr^2 + \left(\dfrac{2}{5}\,mr^2 + \dfrac{ma^2}{2}\right)$

$= \dfrac{8}{5}\,mr^2 + ma^2$

6. $M = \int_0^{2l} dM = \int_0^{2l} (mx) \cdot dx$

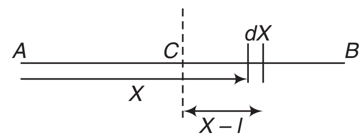
$= 2ml^2$

$\therefore \qquad m = \dfrac{M}{2l^2}$

(a) $I_A = \int_0^{2l} (dM)\, X^2 = \int_0^{2l} (mXdX)\, X^2$

$= \int_0^{2l} \left(\dfrac{M}{2l^2}\, X^3\, dX\right) = 2\, Ml^2$ **Ans.**

(b) $I_C = \int_0^{2l} (dM)\,(X - l)^2$



$= \int_0^{2l} (mXdX)\,(X - l)^2$

$= \int_0^{2l} \left(\dfrac{M}{2l^2}\right) (X)\,(X - l)^2\, dX$

$= \dfrac{1}{3}\, Ml^2$ **Ans.**

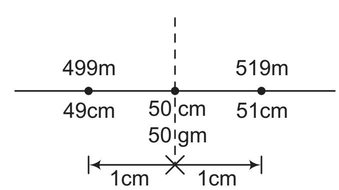
7. $I = \dfrac{5}{6}\, I_{\text{Total}} = \dfrac{5}{6}\,(0.6) \text{ kg-m}^2 = 0.5 \text{ kg-m}^2$

8. $I = \dfrac{1}{2}\, mR^2$ or $I \propto R^2$

m and thickness are same.

Therefore, radius R of the material having smaller density should be more, so the moment of inertia.

9. $I = 50\,(0)^2 + (49 + 51)\,(1)^2 + (48 + 52)\,(2)^2 + \text{.........} + (1 + 99)\,(49)^2$



$= 100\,[(1)^2 + (2)^2 + \text{.......} + (49)^2\,]$

$= 4.3 \times 10^6 \text{gm-cm}^2 = 0.43 \text{ kg-m}^2$ **Ans.**

10. $l = 2\pi R \quad \Rightarrow \quad R = \dfrac{l}{2\pi}$

$\dfrac{I_1}{I_2} = \dfrac{ml^2/12}{mR^2} = \dfrac{l^2}{12R^2}$

$= \dfrac{l^2}{12\,(l/2\pi)^2}$

$= \dfrac{\pi^2}{3}$ **Ans.**

INTRODUCTORY EXERCISE 12.2

1. $\omega = \dfrac{2\pi}{T}$

$= \dfrac{2\pi}{60} = \dfrac{\pi}{30}$ rad/s **Ans.**

2. $\omega = \dfrac{v_r}{PQ} = \dfrac{2v - v}{2R} = \dfrac{v}{2R}$

Rotation is clockwise. So, ω is perpendicular to paper inwards.

3. $\omega = \dfrac{\text{component of velocity perpendicular to } \mathbf{r}}{r}$

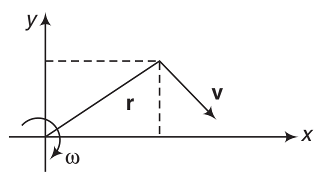
$\mathbf{r} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}$

$\mathbf{v} = 4\hat{\mathbf{i}} - 3\hat{\mathbf{j}}$

Since, $\mathbf{r} \cdot \mathbf{v} = 0$

$\therefore$ $\mathbf{v} \perp \mathbf{r}$

$\therefore$ $\omega = \dfrac{v}{r} = 1$ rad/s

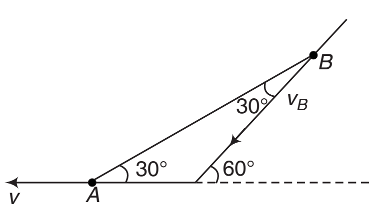


$v = |\mathbf{v}| = 5$ m/s

$r = |\mathbf{r}| = 5$ m

From the figure, we can see that ω is along negative z-axis. **Ans.**

4. Component of velocities of A and B along AB should be same



$\therefore$ $v_A \cos 30° = v_B \cos 30°$

$\therefore$ $v_B = v_A = v$

$\omega = \dfrac{\text{Relative velocity} \perp \text{to } AB}{AB}$

$= \dfrac{v_A \sin 30° + v_B \sin 30°}{l}$

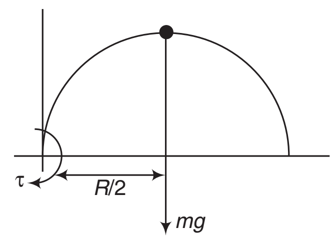
$= \dfrac{v}{l}$ (as $v_A = v_B = v$)

INTRODUCTORY EXERCISE 12.3

1. $\mathbf{r} = (2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 2\hat{\mathbf{k}})$

$\tau = \mathbf{r} \times \mathbf{F}$ **Ans.**

2. $\tau = \dfrac{mgR}{2} = \dfrac{(mg)\, u^2 \sin 2\theta}{2g}$



$= \dfrac{mu^2}{2}$ (as $\theta = 45°$)

$= \dfrac{(1)\,(20\sqrt{2})^2}{2}$

$= 400$ N-m **Ans.**

3. $\tau_{10N} = 0$

$\tau_{20N} = (20 \cos 45° \times 0.1)$ (clockwise)

$= 1.414$ N-m

$\tau_{30N} = (30 \sin 60° \times 0.05)$ (clockwise)

$= 1.299$ N-m

$\therefore$ $\tau_{\text{Total}} = 2.71$ N-m

4. $\tau_{10N} = 10 \times 0.25$

$= 2.5$ N-m (clockwise)

$\tau_{9N} = 9 \times 0.25$

$= 2.25$ N-m (clockwise)

$\tau_{12N} = 12 \cos 60° \times 0.1$

$= 0.6$ N (anticlockwise)

$\therefore$ $\tau_{\text{net}} = 4.15$ N-m

INTRODUCTORY EXERCISE 12.4

1. $\theta = \dfrac{1}{2}\alpha t^2$

$\therefore$ $\alpha = \dfrac{2\theta}{t^2} = \dfrac{2\,(50)\,(2\pi)}{(5)^2}$

$= (8\pi)$ rad/s^2

$= 25.14$ rad/s^2 **Ans.**

$\omega = (\alpha t)$

$= (8\pi)\,(5)$

$= (40\pi)$ rad/s **Ans.**

2. $\omega^2 = \omega_0^2 + 2\alpha\theta$

$$\therefore \quad \theta = \frac{\omega^2 - \omega_0^2}{2\alpha} = \frac{225 - 25}{2 \times 1}$$

$$= 100 \text{ rad} \qquad \textbf{Ans.}$$

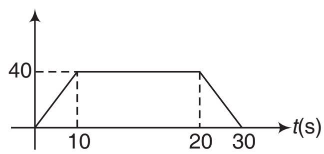
3. $0 = \omega_0 - \alpha t$

$$\Rightarrow \quad \alpha = \frac{\omega_0}{t} = \frac{10}{10}$$

$$= 1 \text{ rad/s}^2$$

$$\tau = I\alpha = 5 \text{ N-m} \qquad \textbf{Ans.}$$

4. θ = Area under ω-t graph



$$= \frac{1}{2}(30 + 10)(40) = 800 \text{ rad} \qquad \textbf{Ans.}$$

5. $\omega^2 = \omega_0^2 - 2\alpha\theta$

$$0 = \left(\frac{100}{60} \times 2\pi\right)^2 - 2\alpha\,(10 \times 2\pi)$$

$$\alpha = 0.87 N = \frac{\tau}{I}$$

$$= \frac{F \cdot R}{\frac{1}{2} mR^2}$$

$$\therefore \quad F = (0.87)\,(0.5\, mR)$$

$$= (0.87)\,(0.5)\,(10)\,(0.2)$$

$$= 0.87 \text{ N} \qquad \textbf{Ans.}$$

6. $\omega = \dfrac{d\theta}{dt} = 6 - 6t^2$

$$\alpha = \frac{d\omega}{dt} = -12t$$

$$\omega = 0$$

at $\quad t = 1$ s

(a) $$\langle\omega\rangle_{0-1} = \frac{\int_0^1 \omega dt}{1} = \int_0^1 (6 - 6t^2)\, dt$$

$$= 4 \text{ rad/s} \qquad \textbf{Ans.}$$

$$\langle\alpha\rangle_{0-1} = \frac{\int_0^1 (-12t)\, dt}{1}$$

$$= -6 \text{ rad/s}^2 \qquad \textbf{Ans.}$$

(b) At $t = 1$ s

$$\alpha = -12 \text{ rad/s}^2 \qquad \textbf{Ans.}$$

7. $\omega = \omega_0 - \alpha t$

$$-20 = 20 - 2t$$

$$\therefore \quad t = 20 \text{ s} \qquad \textbf{Ans.}$$

8. (a) Angular impulse

= change in angular momentum

$$\therefore \quad \tau_f t = I\omega$$

or $$\tau_f = \frac{I\omega}{t} = \frac{0.03 \times 20}{60} = 0.01 \text{ N-m} \qquad \textbf{Ans.}$$

(b) During acceleration

$$(\tau_e - \tau_f)\, t = I\omega$$

$$\therefore \quad \tau_e = \frac{I\omega}{t} + \tau_f = \frac{0.03 \times 20}{5} + 0.01$$

$$= 0.13 \text{ N-m} \qquad \textbf{Ans.}$$

9. (a) Angular impulse

= change in angular momentum

$$\therefore \quad \tau\, t_0 = I\omega$$

$$\therefore \quad t_0 = \frac{I\omega}{\tau} = \frac{16 \times 9}{4} = 36 \text{ s}$$

(b) $$\int_0^{t_0} \tau\, dt = I\omega$$

$$\therefore \quad \int_0^{t_0} ktdt = I\omega$$

$$\therefore \quad \frac{kt_0^2}{2} = I\omega$$

$$\therefore \quad t_0 = \sqrt{I\omega}\,\sqrt{\frac{2}{k}}$$

$$= \sqrt{16 \times 9}\,\sqrt{\frac{2}{k}} = 12\sqrt{\frac{2}{k}} \qquad \textbf{Ans.}$$

10. (a) $\int d\omega = \int \alpha\, dt$

$$\therefore \quad \int_{65}^{\omega} d\omega = \int_0^3 (-10 - 5t)\, dt$$

$$\therefore \quad \omega = 65 - [10\,t + 2.5\,t^2]_0^3$$

$$= 12.5 \text{ rad/s} \qquad \textbf{Ans.}$$

(b) $$\int_{65}^{\omega} d\omega = \int_0^t (-10 - 5t)\, dt$$

$$\therefore \quad \omega = 64 - 10t - 2.5t^2$$

$$\int_0^{\theta} d\theta = \int_0^3 \omega dt = \int_0^3 (65 - 10t - 2.5t^2)\, dt$$

$$\therefore \quad \theta = 195 - 45 - 22.5$$

$$= 127.5 \text{ rad} \qquad \textbf{Ans.}$$

11. $\theta = \displaystyle\int_0^3 \omega\, dt = \int_0^3 (12 - 3t^2)\, dt$

$$= 36 - 27 = 9 \text{ rad} \qquad \textbf{Ans.}$$

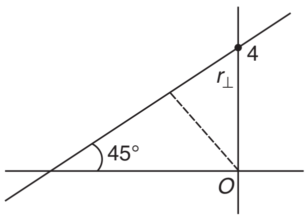
$$N = \frac{\theta}{2\pi} = \frac{9}{2 \times 3.14} = 1.43 \qquad \textbf{Ans.}$$

INTRODUCTORY EXERCISE 12.5

1. $I = \frac{m(3l)^2}{12} + m\left(\frac{l}{2}\right)^2 = ml^2$

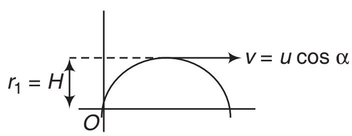
$$L = I\omega = ml^2\omega$$

2. $r_\perp = 4\cos 45^\circ = 2\sqrt{2}$ m



$$L_0 = mvr_\perp$$
$$= (1)(2)(2\sqrt{2})$$
$$= 4\sqrt{2}\ \text{kg-m}^2/\text{s}$$

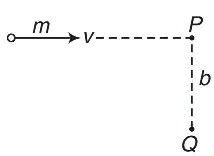
3. $L_0 = mvr_\perp$



$$= m(u\cos\alpha)\left(\frac{u^2\sin^2\alpha}{2g}\right)$$

$$= \frac{mu^3\cos\alpha\sin^2\alpha}{2g}$$ **Ans.**

4.



$$L_P = 0 \qquad (\text{as } r_\perp = 0)$$
$$L_Q = mvb \qquad (\text{as } r_\perp = b)$$

5. $L = mvr_\perp + I_c\omega$

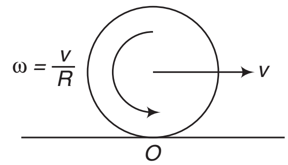
$$= mvR + \left(\frac{2}{5}mR^2\right)\left(\frac{v}{R}\right)$$

$$= \frac{7}{5}mvR$$

This is clockwise, or along negative z-axis.

$$\therefore \qquad \mathbf{L} = \left(-\frac{7}{5}mvR\right)\hat{\mathbf{k}}$$ **Ans.**

6. $L_0 = mvR - I_c\omega$



$$= mvR - \left(\frac{1}{2}mR^2\right)(vR)$$

$$= \frac{1}{2}mvR \qquad \text{(clockwise)}$$ **Ans.**

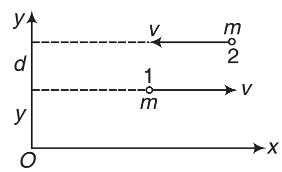
7. $L_0 = L_1 + L_2$

= (mvy in clockwise or along negative z-axis)

+ [$mv\,(y + d)$ in anticlockwise or along positive z-axis]

= mvd in anticlockwise direction or along positive z-axis.

= constant



INTRODUCTORY EXERCISE 12.6

1. $I_1\omega_1 = I_2\omega_2$

$$\therefore \qquad \omega_2 = \frac{I_1}{I_2}\omega_1$$

$$= \left(\frac{MR^2}{MR^2 + 2mR^2}\right)\omega_0$$

$$= \left(\frac{M}{M + 2m}\right)\omega_0$$ **Ans.**

2. Mass will move towards equator (away from axis). So, I will increase. Therefore ω will decrease (as $I\omega$ = constant). Hence, time period will increase $\left(\text{as } T = \frac{2\pi}{\omega}\right)$.

3. Moment of inertia I increase. Therefore, ω decreases (as $I\omega$ = constant). Hence time period increases $\left(\text{as } T = \frac{2\pi}{\omega}\right)$

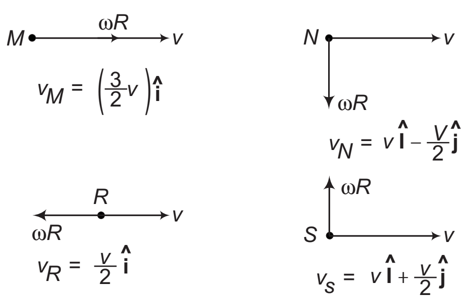
4. I will increase $\left(\text{as } I = \frac{2}{5}mR^2\right)$

$\therefore$ ω will decrease $(\text{as } I\omega = \text{constant})$

$\therefore T$ will increase $(\text{as } T = 2\pi/\omega)$

INTRODUCTORY EXERCISE 12.7

1. $\omega R = \frac{v}{2}$



2. There is only one acceleration,

$$a_r = R\omega^2 \qquad \text{(towards centre)}$$

INTRODUCTORY EXERCISE 12.8

1. $\frac{K_R}{K_T} = \frac{2}{5} \Rightarrow K_T = \frac{5}{2}K_R = \frac{5}{2} \times 10 = 25 \text{ J}$

$\therefore$ Total kinetic energy $= K_R + K_T = 35$ J

2. Under forward slip condition $v > R\omega$

$$K_T = \frac{1}{2}mv^2$$

$$K_R = \frac{1}{2}I\omega^2 = \frac{1}{2}(mR^2)\omega^2$$

$$\therefore \qquad K_T > K_R$$

3. $K_T = \frac{1}{2}mv^2, K_R = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{1}{2}mR^2\right)\omega^2$

Since, in backward slip condition, $v < R\omega$

$\therefore$ K_R may be equal to K_T.

INTRODUCTORY EXERCISE 12.9

1. $v = r\omega$, where r is the distance from A (position of instantaneous axis of rotation)

For A, $r = 0 \Rightarrow v_A = 0$

For B and D, $r = \sqrt{2}R$

$\Rightarrow \qquad v_B = v_D = \sqrt{2}R\omega = \sqrt{2}v$

For C, $r = 2R \Rightarrow v_C = 2R\omega = 2v$

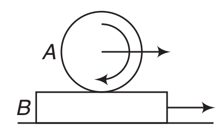
Direction of velocity of a general point P is perpendicular to AP in the sense of rotation.

2. $I_C = \frac{2}{5}mR^2, I_0 = \frac{7}{5}mR^2$ and $\omega = \frac{v}{R}$

Substituting these values in the given equation, we get the result.

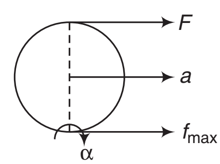
INTRODUCTORY EXERCISE 12.10

1. Net work done by friction in pure rolling is always zero.



For example, if A pure rolls over B, then work done by friction on both A and B are non-zero (one is positive and other is negative). But net work done by friction = 0.

2. Till pure rolling continues we can find α about bottommost point. About bottommost point torque of friction is zero.



$$\alpha = \frac{F\ (2R)}{(3/2)\ mR^2} = \frac{4F}{3mR}$$

$$a = R\alpha = \frac{4F}{3m}$$

$$F + f_{\max} = ma = \frac{4F}{3}$$

$$F = 3\ f_{\max} = 3\,\mu mg$$

$$= 3 \times 0.6 \times 4 \times 10 = 72 \text{ N} \qquad \textbf{Ans.}$$

This is the maximum value of applied force, as we have taken the maximum friction.

3. (a) $\mu_{\min} = \frac{\tan\theta}{1 + (mR^2/I)}$

$$= \frac{\tan 30°}{1 + 5/2} = \frac{2}{7\sqrt{3}} \qquad \textbf{Ans.}$$

(b) $a = \frac{g \sin\theta}{1 + I/mR^2}$

$$= \frac{(10)\sin 30°}{1 + 2/5}$$

$$= \frac{25}{7} \text{ m/s}^2 \qquad \textbf{Ans.}$$

(c) $a = g \sin\theta - \mu g \cos\theta$

$$= (10)\sin 30^\circ - \left(\frac{1}{7\sqrt{3}}\right)(10)(\cos 30^\circ)$$

$$= 5 - \frac{5}{7} = \frac{30}{7}\ \text{m/s}^2 \quad \textbf{Ans.}$$

4. (a) Maximum friction will act

$$\therefore \quad a = \frac{mg\sin\theta - \mu\, mg\cos\theta}{m}$$

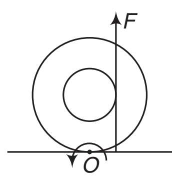
$$= g\sin\theta - \mu\, g\cos\theta$$

(b) Only friction will provide the torque.

$$\therefore \quad \alpha = \frac{\tau}{I} = \frac{(\mu\, mg\cos\theta)\,R}{\frac{2}{5}mR^2}$$

$$= \frac{5\mu\, g\cos\theta}{2R} \quad \textbf{Ans.}$$

5. τ_F about O is anticlockwise.



So, it will roll leftwards F is vertical. So, horizontal force is vertical. So, horizontal force is provided by friction (to give it linear acceleration a). So, friction acts leftwards.

INTRODUCTORY EXERCISE 12.11

1. Torque of $mg\sin\theta$ is non-zero. So, angular momentum is not constant.

2. (a) $v = \frac{J}{m}$, J and m are same for both.

(b) $\omega = \frac{Jh}{I}$

or $\qquad \omega \propto \frac{1}{I}$

J and h are same but, $I_{\text{solid}} < I_{\text{hollow}}$

$\therefore \qquad \omega_{\text{solid}} < \omega_{\text{hollow}}$

(c) $K_R = \frac{1}{2}I\omega^2$

$$\omega \propto \frac{1}{I}$$

$$\therefore \quad K_R \propto \frac{1}{I}$$

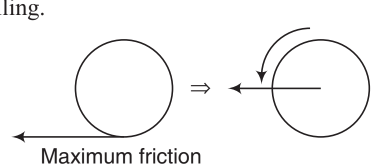
Exercise

LEVEL 1

Assertion and Reason

1. If we compare between many parallel axes, then moment of inertial is least about that axis which passes through COM.

2. Slips is towards right. So maximum friction will act towards left which will convert this slip into pure rolling.



3. Solid sphere, $\frac{K_R}{K_T} = \frac{2}{5}$

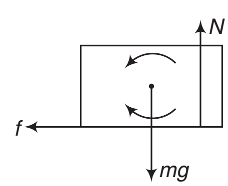
$\therefore \quad K_R = \frac{2}{7}K_{\text{Total}}$ and $K_T = \frac{5}{7}K_{\text{Total}}$

Hollow sphere, $\frac{K_R}{K_T} = \frac{2}{3}$

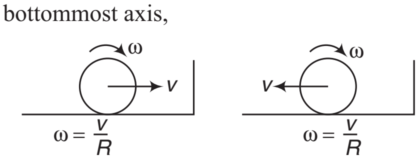
$\therefore \quad K_R = \frac{2}{5}K_{\text{Total}}$ and $K_T = \frac{3}{5}K_{\text{Total}}$

4. In case of smooth surface whole of its potential energy (at A) will convert into its translational kinetic energy (at B).

5. f = friction force



6. In second figure, net angular momentum about bottommost axis,

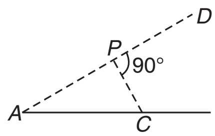


$$L = mvR - I_c\omega$$

$$= mvR - (mR^2)\left(\frac{v}{R}\right)$$

$$= 0$$

7. At P distance from C is minimum.



$$I = I_C + mr^2$$

From A to P, moment of inertia will first decrease. At P it is minimum. Then it will increase.

8. Constant linear momentum means moving with constant speed in a straight line. Therefore, in $L = mv\, r_\perp$ all three m, v and $r_\perp$ are constant.

9. At C, two terms v and $R\omega$ are at 0°. At A they are at 180° and at B they are at 90°

$$\therefore \quad v_C = v + R\omega$$

$$v_A = v \sim R\omega$$

and $$v_B = \sqrt{v^2 + (R\omega)^2}$$

11. In first case,

$$a = R\alpha$$

$$\therefore \quad \frac{F - \mu_1 mg}{m} = R\left(\frac{\mu_1 mgR}{2/5\, mR^2}\right)$$

$$= \frac{5}{2}\mu_1 g \quad \text{or} \quad \mu_1 = \frac{2}{7}\frac{F}{mg}$$

In second case,

$$a - R\alpha = a_{\text{Plank}}$$

$$\therefore \quad \left(\frac{F - \mu_2 mg}{m}\right) - R\left(\frac{\mu_2 mg}{2/5\, mR^2}\right) = \frac{\mu_2 mg}{M}$$

$$\therefore \quad \frac{F}{m} = \mu_2 g + \frac{5}{2}\mu_2 g + \frac{m}{M}\mu_2 g$$

$$\therefore \quad \mu_2 = \frac{F}{\left(7/2 + \frac{m}{M}\right) mg}$$

or $\mu_2 < \mu_1$

Net work done by friction in pure rolling is zero.

Single Correct Option

2. $K = \sqrt{\dfrac{I}{M}}$

$$= \sqrt{\frac{\frac{1}{2}mR^2}{m}} = \frac{R}{\sqrt{2}}$$

$$= \frac{25}{\sqrt{2}} \approx 18 \text{ cm}$$

3. $0 = \omega_0 - \alpha t$

$$\therefore \quad \alpha = \frac{\omega_0}{t} = \frac{(2\pi)(1725/60)}{20}$$

$$= 9 \text{ rad/s}^2$$

$$\theta = \frac{1}{2}\alpha t^2 = \frac{1}{2} \times 9 \times (20)^2 = 1800 \text{ rad}$$

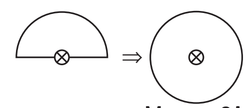
$$N = \frac{\theta}{2\pi} = 287$$ **Ans.**

4. I will decrease

$\therefore$ ω will increase (as $L = I\omega$ = constant)

KE $= \dfrac{L^2}{2I}$ will increase as I is decreasing.

5. $I_{\text{Whole disc}} = \dfrac{1}{2}(2M)\, r^2 = Mr^2$



Mass = $2M$

$$\therefore \quad I_{\text{Half disc}} = \frac{1}{2}Mr^2$$

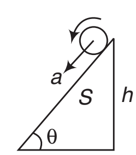
7. In case of pure rolling on ground, net work done by friction = 0

8. After melting, ice will distribute in whole pan. So, moment of inertia I will increase. Hence, angular speed ω will decrease (as $I\omega$ = constant).

9. $\dfrac{K_R}{K_T} = \dfrac{\frac{1}{2}I\omega^2}{\frac{1}{2}mv^2} = \dfrac{I\omega^2}{mv^2}$

$$= \frac{(mR^2)(v/r)^2}{mv^2} = \frac{R^2}{r^2}$$ **Ans.**

10. $a = \dfrac{g \sin\theta}{1 + I/mR^2} = \dfrac{g\sin\theta}{1 + 2/5} = \dfrac{5}{7}g\sin\theta$



$$t = \sqrt{\frac{2S}{a}} = \sqrt{\frac{2\left(\frac{h}{\sin\theta}\right)}{(5/7)\, g\sin\theta}}$$

or $$t \propto \frac{1}{\sin\theta}$$

So, time will be different.

$$v = \sqrt{2aS} = \sqrt{2\left(\frac{5}{7}g\sin\theta\right)\left(\frac{h}{\sin\theta}\right)}$$

or v is independent of θ. So, v will be same.

11. Mass is distributed at maximum distance (from the axis) in case of four rods.

12. $f = \dfrac{mg\sin\theta}{1 + mR^2/I}$

$$f_{\text{disc}} = \frac{mg\sin\theta}{1+2} = \frac{1}{3}mg\sin\theta$$

$$f_{\text{sphere}} = \frac{mg\sin\theta}{1+5/2} = \frac{2}{7}mg\sin\theta$$

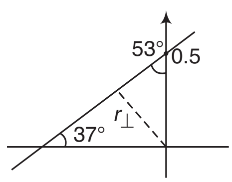
$\therefore$ The ratio is $\dfrac{7}{6}$.

13. $I_1\omega_1 = I_2\omega_2$

$$\therefore\ \omega_2 = \left(\frac{I_1}{I_2}\right)\omega_1 = \frac{\left(\frac{1}{2}mR^2\right)}{\left(\frac{1}{2}mR^2 + mR^2\right)}\omega = \frac{\omega}{3}$$

14. Friction will act (due to sliding). But torque of friction about the mentioned point in the question will be zero. Hence, angular momentum will remain constant.

15. $y = \dfrac{3}{4}x + 0.5$



$$\text{Slope} = \tan\theta = \frac{3}{4}$$

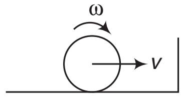
$$\therefore\quad \theta = 37°$$

$$r_\perp = 0.5\sin 53° = 0.4\ \text{m}$$

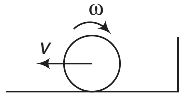
$$L = mvr_\perp = (3)(5)(0.4)$$

$$= 6\ \text{kg-m}^2/\text{s}$$

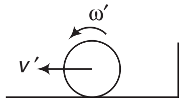
16. $v = R\omega$ and $v' = R\omega'$



(a)



(b)



(c)

From (b) to (c), forward slip will take place and backward maximum friction will convert it into pure rolling.

17. $I_\theta = \dfrac{ma^2}{12}$ (in case of square plate)

or it is independent of θ.

18. Net torque about bottommost point is clockwise. Hence, the spool will move towards right.

19. $K = \dfrac{L^2}{2I}$ or $K \propto \dfrac{1}{I}$ (as L = constant)

As I has doubled, so K will become half.

20. $$\rho = \frac{M}{\left\{\frac{4}{3}\pi(2R)^3\right\} - \left\{\frac{4}{3}\pi R^3\right\}}$$

$$= \frac{3M}{28\pi R^3}$$

$$M_{2R} = \rho\left(\frac{4}{3}\right)(\pi)(2R)^3$$

$$= \left(\frac{3}{28}\right)\left(\frac{4}{3}\right)(8)M = \frac{8}{7}M$$

$$M_R = \rho\left(\frac{4}{3}\pi R^3\right)$$

$$= \left(\frac{3}{28}\right)\left(\frac{4}{3}\right)M = \frac{1}{7}M$$

Now, $I = I_{2R} - I_R$

$$= \frac{1}{2}M_{2R}(2R)^2 - \frac{1}{2}M_R R^2$$

$$= \frac{1}{2}\left(\frac{8}{7}M\right)(4R^2) - \frac{1}{2}\left(\frac{M}{7}\right)R^2$$

$$= \frac{31}{14}MR^2$$ **Ans.**

21. S = area of cross-section

$$\rho_0 \qquad\qquad 2\rho_0$$

$$\rho_x = \rho_0 + \left(\frac{2\rho_0 - \rho_0}{L}\right)x = \rho_0 + \frac{\rho_0}{L}x$$

Now, $$M = \int_0^L dM = \int_0^L (Sdx)\left(\rho_0 + \frac{\rho_0}{L}x\right)$$

$$= \rho_0 S\left(\frac{3L}{2}\right)$$

$$\therefore\quad \rho_0 S = \frac{2M}{3L} \qquad \text{...(i)}$$

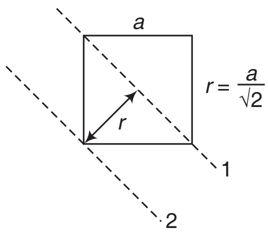
Now, $$I = \int_0^L dI = \int_0^L (dM)\,x^2$$

$$= \int_0^L (Sdx)\left(\rho_0 + \frac{\rho_0}{L}x\right)(x^2)$$

After substituting value of $\rho_0 S$ from Eq. (i), then we find

$$I = \frac{7}{18} ML^2 \qquad \textbf{Ans.}$$

22. $I_1 = \dfrac{ma^2}{12}$



$$I_2 = I_1 + mr^2$$
$$= \frac{ma^2}{12} + \frac{ma^2}{2}$$
$$= \frac{7}{12} ma^2$$

$\therefore \qquad I_1 : I_2 = 1 : 7 \qquad$ **Ans.**

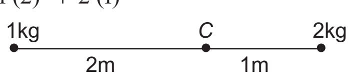
23. $I = \dfrac{ML^2}{12} + M\left(\dfrac{L}{4}\right)^2 = \dfrac{7}{48} ML^2$

24. Decrease in gravitational kinetic energy
= increase in rotational kinetic energy

$$\therefore mg\left(\frac{l}{2} - \frac{l}{2}\cos\theta\right) = \frac{1}{2} I\omega^2$$
$$= \frac{1}{2}\left(\frac{ml^2}{3}\right)\omega^2$$

$\therefore \qquad \omega = \sqrt{\dfrac{6g}{L}} \sin\dfrac{\theta}{2} \qquad$ **Ans.**

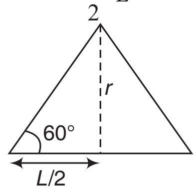
25. $I = 1\,(2)^2 + 2\,(1)^2$



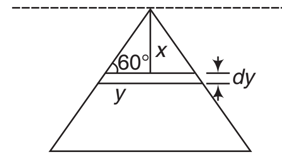
$$= 6 \text{ kg-m}^2 \qquad \textbf{Ans.}$$

26. $r = \dfrac{L}{2} \tan 60°$

$$= \frac{\sqrt{3}}{2} L$$



$$\text{Area } = \frac{1}{2} \times \text{base} \times \text{height}$$
$$= \frac{1}{2} \times L \times \frac{\sqrt{3}\,L}{2} = \frac{\sqrt{3}}{4} L^2$$



$\therefore$ Mass per unit area $= \dfrac{M}{\sqrt{3}L^2/4} = \dfrac{4M}{\sqrt{3}L^2}$

or $\qquad y = \dfrac{x}{\sqrt{3}}$

$\Rightarrow \qquad dy = \dfrac{dX}{\sqrt{3}}$

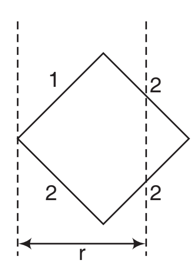
$$\text{Area of strip} = (2y)\,dy = \left(\frac{2x}{\sqrt{3}}\right)\left(\frac{dx}{\sqrt{3}}\right)$$
$$= \left(\frac{2x}{3}\right) dX$$

$\therefore \qquad$ Mass of strip $= \left(\dfrac{4M}{\sqrt{3}L^2}\right)\left(\dfrac{2x}{3}\,dX\right) = dM$

$$I = \int dI$$
$$= \int_0^{\frac{\sqrt{3}}{2}L} (dM)\, X^2$$
$$= \frac{3}{8\sqrt{3}} L^2 \qquad \textbf{Ans.}$$

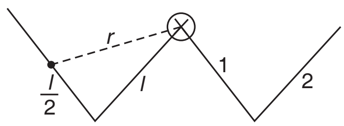
27. $r = \dfrac{l}{\sqrt{2}} + \dfrac{l}{2\sqrt{2}} = \dfrac{3l}{2\sqrt{2}}$

$$I = 2I_1 + 2I_2$$
$$= 2\,(I_1 + I_2)$$
$$= 2\left[\frac{ML^2}{3}\sin^2 45° + \left(\frac{ML^2}{12}\sin^2 45° + Mr^2\right)\right]$$
$$= \frac{8}{3} ML^2$$



28. $r = \sqrt{l^2 + \frac{l^2}{4}} = \frac{\sqrt{5}}{2}\, l$

$$I = 2\,[I_1 + I_2] = 2\left[\frac{ml^2}{3} + \left(\frac{ml^2}{12} + mr^2\right)\right]$$



$$= \frac{10}{3}\, ml^2 \qquad \textbf{Ans.}$$

29. $\frac{x}{r} = \frac{h}{R} \quad \Rightarrow \quad r = \frac{x}{h}\, R$

Volume of disc = $(\pi r^2)\, dX$

Mass of disc = $(\pi r^2)\,(dX)\,\rho = dm$

$$dI = \frac{1}{2}\,(dm)\, r^2 = \frac{1}{2}\,(\pi r^2 dx)\,\rho r^2$$

$$= \frac{\pi\rho}{2}\, r^4\, dx = \left(\frac{\pi\rho}{2}\right)\left(\frac{R^4}{h^4}\right) x^4\, dx$$

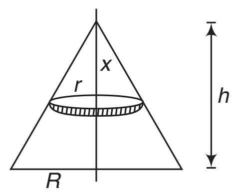
$$I = \int_0^h dI = \left(\frac{\pi\rho}{2}\right)\left(\frac{R^4}{h^4}\right)\left(\frac{h^5}{5}\right)$$

$$= \left(\frac{\pi\rho}{10}\right)(hR^4)$$

$$\rho = \frac{m}{\left(\frac{1}{3}\pi R^2 h\right)}$$

$$\therefore \qquad I = \left(\frac{\pi}{10}\right)\left(\frac{m}{\frac{1}{3}\,\pi R^2 h}\right)(hR^4)$$

So, $\qquad I \propto mR^2$



Since m and R are same. Therefore I is same.

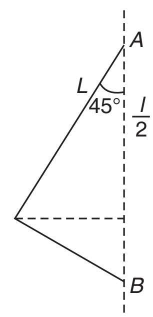
30. $L = \frac{l}{2}\sec 45° = \frac{l}{\sqrt{2}}$

$$m = Lx = \frac{lx}{\sqrt{2}}$$

$$I = 2\left[\frac{mL^2}{3}\sin^2 45°\right]$$

$$= 2\left[\left(\frac{lx}{3\sqrt{2}}\right)\left(\frac{l}{\sqrt{2}}\right)^2\left(\frac{1}{2}\right)\right]$$

$$= \frac{xl^3}{6\sqrt{2}} \qquad \textbf{Ans.}$$



Subjective Questions

1. Present angular momentum of earth

$$L_1 = I\omega = \frac{2}{5}\, MR^2\omega$$

New angular momentum because of change in radius

$$L_2 = \frac{2}{5} M\left(\frac{R}{2}\right)^2 \omega'$$

If external torque is zero then angular momentum must be conserved

$$L_1 = L_2$$

$$\frac{2}{5}\, MR^2\omega = \frac{1}{4} \times \frac{2}{5}\, MR^2\omega'$$

i.e. $\qquad \omega' = 4\omega$

$$T' = \frac{1}{4} T = \frac{1}{4} \times 24 = 6 \text{ h} \qquad \textbf{Ans.}$$

2. $\qquad R = \sqrt{\frac{I}{m}} = \sqrt{\frac{\frac{1}{2}\, mR^2 + md^2}{m}}$

$$\therefore \qquad d = \frac{R}{\sqrt{2}} \qquad \textbf{Ans.}$$

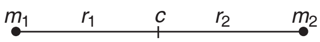
3. $I_1 = I_2 \quad \Rightarrow \quad I_1 + I_2 = I_3 = \frac{Ma^2}{6}$

$$\therefore \qquad 2I_1 \quad \text{or} \quad 2I_2 = \frac{Ma^2}{6}$$

$$\therefore \qquad I_1 = I_2 = \frac{Ma^2}{12} \qquad \textbf{Ans.}$$

4. $\frac{7}{12} ml^2 = \frac{ml^2}{12} + mr^2$

$\therefore \qquad r = \frac{l}{\sqrt{2}}$ **Ans.**

5. 

$$r_1 = \left(\frac{m_2}{m_1 + m_2}\right) r$$

$$r_2 = \left(\frac{m_1}{m_1 + m_2}\right) r$$

$$I_c = m_1 r_1^2 + m_2 r_2^2$$ **Ans.**

6. $I_r = I_c + mr^2$

$$mK_r^2 = mK_c^2 + mr^2$$

$$\therefore \quad K_c = \sqrt{K_r^2 - r^2}$$

$$= \sqrt{(10)^2 - (6)^2}$$

$$= 8 \text{ cm}$$ **Ans.**

7. Instantaneous angular velocity at time t is

$$\omega = \frac{d\theta}{dt} = \frac{d}{dt}(at^2)$$

or $\quad \omega = 2at = 0.4\, t \qquad (\text{as } a = 0.2 \text{ rad/s}^2)$

Further, instantaneous angular acceleration is,

$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}(0.4\, t)$$

or $\quad \alpha = 0.4 \text{ rad/s}^2$

Angular velocity at $t = 2.5$ s is

$$\omega = 0.4 \times 2.5 = 1.0 \text{ rad/s}$$

Further, radius of the wheel

$$R = \frac{v}{\omega}$$

or $\quad R = \frac{0.65}{1.0} = 0.65 \text{ m}$

Now, magnitude of total acceleration is,

$$a = \sqrt{a_n^2 + a_t^2}$$

Here, $\quad a_n = R\omega^2 = (0.65)(1.0)^2 = 0.65 \text{ m/s}^2$

and $\quad a_t = R\alpha = (0.65)(0.4) = 0.26 \text{ m/s}^2$

$\therefore \quad a = \sqrt{(0.65)^2 + (0.26)^2}$

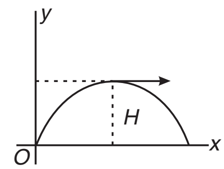
or $\quad a = 0.7 \text{ m/s}^2$ **Ans.**

8. $\omega_c = \frac{v}{R} = \frac{2}{0.1} = 20 \text{ rad/s}$

ω about any point on circumference.

$$= \frac{\omega_c}{2} = 10 \text{ rad/s}$$ **Ans.**

9. At the highest point it has only horizontal velocity $v_x = v \cos\theta$



Length of the perpendicular to the horizontal velocity from 'O' is the maximum height, where

$$H_{\max} = \frac{v^2 \sin^2\theta}{2g}$$

$\Rightarrow$ Angular momentum

$$L = \frac{mv^3 \sin^2\theta \cos\theta}{2g}$$

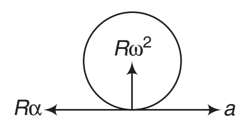
10. $I = \int_0^l dI = \int_0^l (dm)\, x^2$

$$= \int_0^l (\lambda_X\, dx)\, x^2$$

$$= \int_0^l (\alpha x + \beta)\, x^2\, dx$$

$$= \frac{\alpha l^4}{4} + \frac{\beta l^3}{3}$$ **Ans.**

11. $a = R\alpha$



So, net acceleration is $R\omega^2$ towards centre.

12. In case of pure rolling

$$\frac{K_R}{K_T} = \frac{2}{5} \qquad \text{(for a solid sphere)}$$

$$\therefore \quad K_R = \frac{2}{7} \qquad \text{(total kinetic energy)}$$

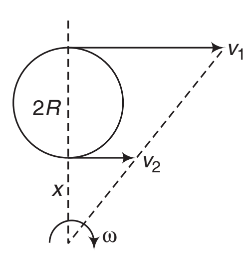
$$= \frac{2}{7}(mgh)$$ **Ans.**

13. $x\omega = v_2$ ...(i)

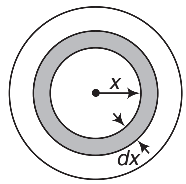
$(x + 2R)\,\omega = v_1$ ...(ii)

Solving Eqs. (i) and (ii) we get,

$$\omega = \left(\frac{v_1 - v_2}{2R}\right)$$ **Ans.**



14. $M = \int_0^a (dM)$



$$= \int_0^a (2\pi x)\, dX\, (kx^2) = \frac{\pi k a^4}{2}$$

$$\therefore \quad \pi k = \frac{2M}{a^4}$$

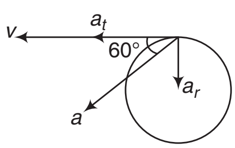
Now, $I = \int_0^a dI = \int_0^a (dM)\, X^2$

$$= \int_0^a (2\pi k x^3 dx)\, x^2 = \frac{(\pi k)\, a^6}{3}$$

Substituting the value of πk, we get

$$I = \frac{2}{3} Ma^2$$ **Ans.**

15. $\tan 60° = \frac{a_r}{a_t} = \frac{r\omega^2}{r\alpha}$



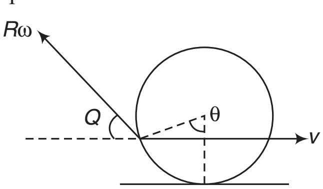
$$= \frac{\left[\int_0^t \alpha dt\right]^2}{\alpha} = \frac{(0.01\, t^2)^2}{0.02\, t}$$

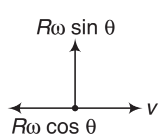
Solving we get,

$$t^3 = 346$$

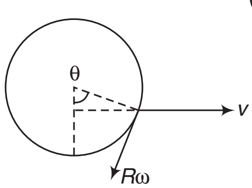
$$\therefore \quad t \approx 7 \text{ s}$$ **Ans.**

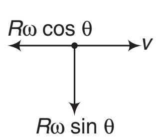
16. If $R\omega \cos\theta = v$ then velocity of point P is vertically upwards.





$$\therefore \quad \theta = \pm \cos^{-1}\left(\frac{v}{R\omega}\right)$$ **Ans.**





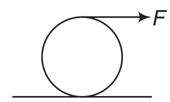
17. $a = R\alpha$ by applied forces

$$\therefore \quad \frac{F_1 + F_2}{M} = 2r\left[\frac{F_1\,(2r) - F_2\,(r)}{I}\right]$$

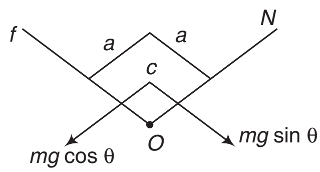
Solving these two equations we get,

$$\frac{F_1}{F_2} = \frac{I + 2\,Mr^2}{4\,Mr^2 - I}$$

18. Even a small force will produce torque about bottommost point. So, the disc starts toppling about bottommost point and here toppling means motion.



19. **For sliding** $\quad mg \sin\theta > \mu\, mg \cos\theta$



or $\quad \tan\theta > \mu \quad$ ...(i)

For toppling

$$\tau_{mg \sin\theta} > \tau_{mg \cos\theta} \quad \text{(about } O\text{)}$$

$$\therefore \quad (mg \sin\theta)\left(\frac{a}{2}\right) > (mg \cos\theta)\left(\frac{a}{2}\right)$$

or $\quad \tan\theta > 1 \quad$ ...(ii)

If $\mu > 1$, then condition (ii) is satisfied earlier. So, the cube topples before sliding.

If $\mu < 1$, then condition (i) is satisfied earlier. So, the cube slides before toppling.

20. Angular impulse = change in angular momentum

$$\therefore \quad \tau t = I\omega = \frac{1}{2} mR^2\, \omega \quad \text{...(i)}$$

$$\therefore \quad \tau = \frac{mR^2\omega}{2t}$$

$$= \frac{(20)\,(0.5)^2\left(2\pi \times \frac{240}{\pi \times 60}\right)}{2 \times 3}$$

$$= \frac{20}{3} \text{ N-m}$$ **Ans.**

During retardation using Eq. (i)

$$t = \frac{mR^2\omega}{2\tau} = \frac{mR^2\omega}{2\,(FR)}$$

$$= \frac{mR\omega}{2F}$$

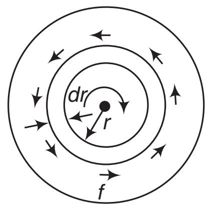
$$= \frac{(20)\,(0.5)\left(2\pi \times \frac{240}{\pi \times 60}\right)}{2 \times 10}$$

$$= 4 \text{ s}$$ **Ans.**

21. $d\tau = \mu\,(dm)\,g \cdot r$

$$= \mu\left(\frac{M}{\pi R^2}\right)(2\pi r dr)\,gr$$

$$\therefore \quad \tau = \int_0^R d\tau = \frac{2\mu MgR}{3}$$



Angular retardation, $\alpha = \dfrac{\tau}{I}$

$$= \frac{(2\mu MgR/3)}{(1/2MR^2)} = \frac{4\mu g}{3R}$$

(a) $0 = \omega_0 - \alpha t$

$$\therefore \quad t = \frac{\omega_0}{\alpha} = \frac{3\omega_0 R}{4\mu g} \qquad \textbf{Ans.}$$

(b) $0 = \omega_0^2 - 2\alpha\theta$

$$\therefore \quad \theta = \frac{\omega_0^2}{2\alpha} = \frac{3\,\omega_0^2 R}{8\,\mu g} \qquad \textbf{Ans.}$$

22. $\theta = 6t - 2t^3$

$$\omega = \frac{d\theta}{dt} = 6 - 6t^2$$

$$\alpha = \frac{d\omega}{dt} = -12t$$

$$\omega = 0$$

at $\quad t = 1$ sec

$$\langle\omega\rangle_{0-1} = \frac{\int_0^1 \omega dt}{1} = \int_0^1 (6 - 6t^2)\,dt$$

$$= 4 \text{ rad/s} \qquad \textbf{Ans.}$$

$$\langle\alpha\rangle_{0-1} = \frac{\int_0^1 \alpha dt}{1} = \int_0^1 (-12t)$$

$$= -6 \text{ rad/s}^2 \qquad \textbf{Ans.}$$

23. Centre of mass of both lies at the centre of ring.

$$I_c = mR^2 + \frac{m\,(2R)^2}{12} = \frac{4}{3}mR^2$$

$$L = (2mvr_\perp) + I_c\omega$$

$$= (2mvR) + \frac{4}{3}mR^2\left(\frac{v}{R}\right) = \frac{10}{3}mvR$$

This is clockwise, or along negative z-axis

$$\therefore \quad \mathbf{L} = \left(-\frac{10}{3}mvR\right)\hat{\mathbf{k}} \qquad \textbf{Ans.}$$

24. $I_1\,\omega_1 = I_2\,\omega_2$

$$\therefore \quad \omega_2 = \left(\frac{I_1}{I_2}\right)\omega_1$$

$$= \frac{\left(\frac{1}{2}MR^2\right)}{\left[\frac{1}{2}MR^2 + m\left(\frac{R}{\sqrt{2}}\right)^2\right]}\,\omega_1$$

$$= \left(\frac{M}{M+m}\right)\omega_1 = \left(\frac{1}{1+0.2}\right)5$$

$$= \frac{25}{6} \text{ rad/s}$$

25. $I_1\omega_1 = I_2\,\omega_2$

$\therefore \quad I_1\,(100) = (I_1 + (10)\,(9)^2\,](90)$

or $\quad I_1 + 810 = 1.11\,I_1$

$\therefore \quad I_1 = 7290$ g-cm^2

$= 7.29 \times 10^{-4}$ kg-m^2 **Ans.**

26. (a) $I_1\,\omega_1 = I_2\,\omega_2$

$$\therefore \quad \omega_2 = \frac{I_1}{I_2}\cdot\omega_1$$

$$= \frac{[1.6 + (2)\,(4)\,(0.9)^2\,]}{[1.6 + (2)\,(4)\,(0.15)^2\,]} \times 0.5$$

$$= \left(\frac{8.08}{1.78}\right)(0.5)$$

$= 2.27$ rev/s

$= 14.3$ rad/s

(b) $\quad K_i = \dfrac{1}{2}I_1\,\omega_1^2$

$$= \frac{1}{2}[1.6 + (2)\,(4)\,(0.9)^2\,]\,(0.5 \times 2\pi)^2$$

$= 39.9$ J **Ans.**

$$K_f = \frac{1}{2}[1.6 + (2)\,(4)\,(0.15)^2\,]\,(14.3)^2$$

$= 181$ J

(c) $W = K_f - K_i = 141.1$ J

27. (a) $I_1 \omega_1 = I_2 \omega_2$

$$\therefore \quad \omega_2 = \left(\frac{I_1}{I_2}\right)\omega_1 = \frac{\left(\frac{1}{2}MR^2 + mR^2\right)}{\left(\frac{1}{2}mR^2\right)} \cdot \omega_0$$

$$= \left(1 + \frac{2m}{M}\right)\omega_0$$

(b) $W = K_f - K_i$

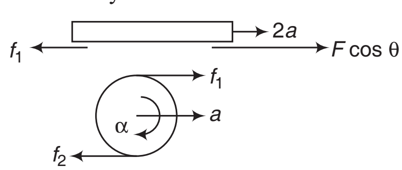
$$= \frac{1}{2} I_2 \omega_2^2 - \frac{1}{2} I_1 \omega_1^2$$

$$= \frac{1}{2}\left[\frac{1}{2}MR^2\right]\left[1 + \frac{2m}{M}\right]^2 \omega_0^2 - \frac{1}{2}\left[\frac{1}{2}MR^2 + mR^2\right]\omega_0^2$$

$$= \frac{1}{4}MR^2\left(1 + \frac{2m}{M}\right)\omega_0^2\left[1 + \frac{2m}{M} - 1\right]$$

$$= \frac{1}{2}m\omega_0^2 R^2\left(1 + \frac{2m}{M}\right) \qquad \textbf{Ans.}$$

28. Since, there is no slip at any contact. Therefore, net work done by friction = 0.



In time *t*

Work done by the applied force

= kinetic energy of plank and cylinder ($F \cos\theta$) (displacement of plank)

$$= \frac{1}{2}m\,(\text{velocity of plank})^2 + \frac{1}{2}\left(1 + \frac{1}{2}\right)M\,(\text{velocity of cylinder})^2$$

$$\therefore \quad (F\cos\theta)\left(\frac{1}{2} \times 2a \times t^2\right) = \frac{1}{2} \times m\,(2at)^2 + \frac{3}{4} \times M \times (at)^2$$

Solving, we get

$$a = \frac{4F\cos\theta}{3M + 8m} \qquad \textbf{Ans.}$$

Equation of plank gives,

$$F\cos\theta - f_1 = m\,(2a)$$

$$\therefore \quad f_1 = F\cos\theta - 2ma$$

$$= F - \frac{8mF\cos\theta}{3M + 8m}$$

$$= \frac{3MF\cos\theta}{3M + 8m} \qquad \textbf{Ans.}$$

Equation of cylinder gives

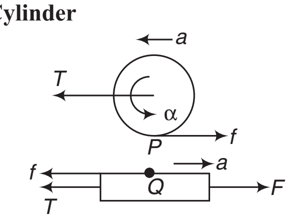
$$f_1 - f_2 = M \cdot a$$

$$\therefore \quad f_2 = f_1 - Ma$$

$$= \frac{3MF\cos\theta}{3M + 8m} - \frac{4MF\cos\theta}{3M + 8m}$$

or $$|f_2| = \frac{MF\cos\theta}{3M + 8m} \qquad \textbf{Ans.}$$

29. **For Cylinder**



$$T - f = ma \qquad \text{...(i)}$$

$$a_P = a_Q$$

$$\therefore \quad R\alpha - a = a \qquad \text{...(ii)}$$

$$\alpha = \frac{\tau}{I} = \frac{f \cdot R}{\frac{1}{1}mR^2} = \frac{2f}{mR} \qquad \text{...(iii)}$$

For Plank

$$F - T - f = Ma \qquad \text{...(iv)}$$

We have four knows f, T, a and α.

Solving the equations we get,

$$a = \frac{F}{M + 3m} \qquad \textbf{Ans.}$$

30. Angular impulse = change in angular momentum

$$\therefore \quad (J \times r_\perp) = I\omega$$

where, J = linear impulse $= \dfrac{I\omega}{r_t}$

$$= \frac{(m)\left(\frac{2l}{3}\right)^2 \cdot \omega}{l}$$

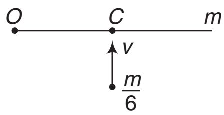
$$= \frac{4}{3}ml\omega \qquad \textbf{Ans.}$$

31. (a) Angular momentum about O remains constant just before and just after collision.

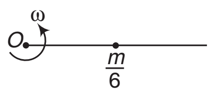
$$\therefore \quad L_i = L_f$$

or $$\left(\frac{m}{6}v\frac{L}{2}\right) = I\omega$$

$$= \left[\frac{mL^2}{3} + \frac{m}{6}\frac{L^2}{4}\right]\omega$$



⇒



Solving, we get

$$\omega = \frac{2v}{9L} \qquad \textbf{Ans.}$$

(b) $$\frac{K_f}{K_i} = \frac{\frac{1}{2}I\omega^2}{\frac{1}{2}\left(\frac{m}{6}\right)v^2}$$

$$= \frac{6\,I\omega^2}{v^2}$$

$$= \frac{6\left[\frac{mL^2}{3} + \frac{mL^2}{24}\right]\left(\frac{2v}{9L}\right)^2}{v^2}$$

$$= 6 \times \frac{9}{24} \times \frac{4}{81}$$

$$= \frac{1}{9} \qquad \textbf{Ans.}$$

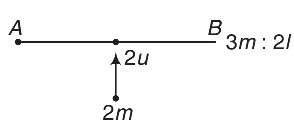
32. (a) Angular momentum of system is conserved just before and just after the impact.

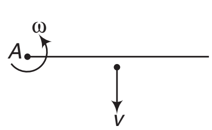
$\therefore \qquad L_i = L_f$

$\therefore \quad (2m)\,(2u)\,(l) = \dfrac{(3m)\,(2l)^2}{3} \cdot \omega - (2m)\,vl \qquad \text{...(i)}$

At the point of impact,

$$e = 1$$





$\therefore \qquad \dfrac{RVOS}{RVOA} = 1$

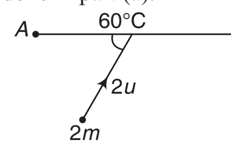
$\therefore \qquad RVOS = RVOA$

or $\qquad \omega l + v = 2u \qquad \text{...(ii)}$

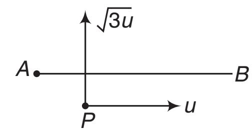
Solving Eqs. (i) and (ii) we get,

$$v = \frac{2u}{3} \qquad \textbf{Ans.}$$

(b) Component of velocity of particle $P\,(= u)$ parallel to AB remains the same. Component perpendicular to AB produces the same impact as was done in part (a).



⇒

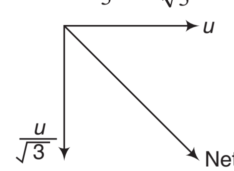


In part *(a)*, $2u$ velocity becomes

$$\frac{2u}{3}\left(\text{i.e., } \frac{1}{3}\text{ rd}\right).$$

Therefore, in this part $\sqrt{3}\,u$ will become

$$\frac{\sqrt{3}\,u}{3} \text{ or } \frac{u}{\sqrt{3}}.$$



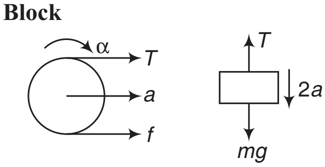
$\therefore \quad$ Net velocity $= \sqrt{u^2 + \dfrac{u^2}{3}}$

$$= \frac{2u}{\sqrt{3}} \qquad \textbf{Ans.}$$

LEVEL 2

Single Correct Option

1. **For Block**



$$mg - T = m\,(2a) \qquad \text{...(i)}$$

For Ring

$$T + f = ma \qquad \text{...(ii)}$$

$$\frac{(T - f)\,R}{mR^2} = \alpha \qquad \text{...(iii)}$$

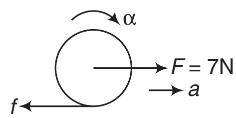
$$a = R\alpha \qquad \text{...(iv)}$$

Solving these equations we get

$$a = \frac{g}{3},\ 2a = \frac{2g}{3},\ T = \frac{mg}{3} \text{ and } f = 0$$

2. $f_{max} = \mu mg = 0.1 \times 10 \times 10 = 10$ N

Requirement of friction



$$a = R\alpha$$

$$\therefore \quad \left(\frac{7-f}{10}\right) = \frac{(R)\,(f \cdot R)}{2/5\ mR^2}$$

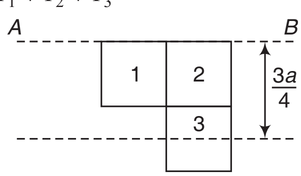
$$= \frac{2.5f}{m} = \frac{2.5f}{10}$$

Solving this equation, we get

$$f = 2$$

Since, $f < f_{max}$, therefore $2N$ frictional force will act.

3. $I_{AB} = I_1 + I_2 + I_3$



$$= \frac{(m/4)\,(a/2)^2}{3} + \frac{(m/4)\,(a/2)^2}{3} + \left[\frac{(m/4)\,(a/2)^2}{12} + (m/4)\left(\frac{3a}{4}\right)^2\right]$$

$$= \frac{3}{16} ma^2$$

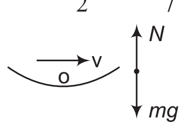
4. At bottom total kinetic energy

=decrease in potential energy

$= mgR$

The ratio, $\dfrac{K_R}{K_T} = \dfrac{2}{5}$

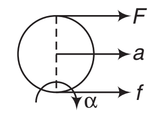
$$\therefore \quad K_T = \frac{1}{2}mv^2 = \frac{5}{7}mgR$$



$$\therefore \quad \frac{mv^2}{R} = \frac{10}{7}mg = N - mg$$

$$\therefore \quad N = \frac{17}{7}mg \qquad \textbf{Ans.}$$

5. In case of pure rolling α can be obtained about bottommost point, about which torque of friction is zero.



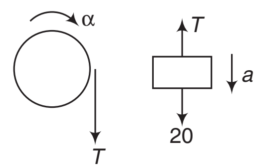
$$\alpha = \frac{F\,(2R)}{3/2\ mR^2} = \frac{4}{3}\frac{F}{mR}$$

$$\therefore \quad a = R\alpha = \frac{4F}{3m} \qquad \textbf{Ans.}$$

6. $F + f = ma = \dfrac{4F}{3}$

$$\therefore \quad f = \frac{F}{3} \quad \text{(towards right)} \qquad \textbf{Ans.}$$

7. For Block



$$20 - T = 2a \qquad \text{...(i)}$$

For Pulley

$$TR = I\alpha$$

$$\therefore \quad T\,(0.2) = (0.32)\,\alpha$$

$$T = 1.6\,\alpha \qquad \text{...(ii)}$$

$$a = R\alpha = 0.2\,\alpha \qquad \text{...(iii)}$$

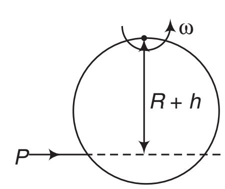
Solving these equations, we get

$$a = 2 \text{ m/s}^2 \qquad \textbf{Ans.}$$

8. $\omega = \dfrac{\text{Angular impulse}}{I}$ (about O)

$$= \frac{P\,(R+h)}{\frac{3}{2}mR^2}$$

Now, $\quad v_{COM} = R\omega = \dfrac{PR\,(R+h)}{\frac{3}{2}mR^2}$



$$P + P_{Hinge} = mv_{COM} = \frac{mpR\,(R+h)}{\frac{3}{2}mR^2}$$

Solving this equation, we get $h = \dfrac{R}{2}$

9. $T_i = \dfrac{mg}{2}$ and $T_f = \dfrac{mg}{4}$

10. In first case $T < Mg$

In second case $T = Mg$

Torques are different, so angular accelerations are different.

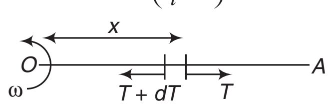
11. From energy conservation principle, maximum spring potential energy

= maximum decrease in gravitational potential energy

$$\therefore \qquad \frac{1}{2} kx_m^2 = Mgh = Mg\,(x_m \sin\theta)$$

$$\therefore \qquad x_m = \frac{2\,Mg\sin\theta}{k}$$

12. $-\,dT = (dm)\times\omega^2 = \left(\dfrac{m}{l}\,dx\right) x\omega^2$



$$\therefore \qquad -\int_0^T dT = \int_l^x \left(\frac{m}{l}\,\omega^2\right) x\,dx$$

$$\therefore \qquad T = \frac{1}{2}\, m\omega^2\, l \left(1 - \frac{x^2}{l^2}\right)$$ **Ans.**

13. $\omega = 10 + 5t$

$$\alpha = \frac{d\omega}{dt} = 5$$

At $\qquad t = 0, \omega = 10$ rad/s

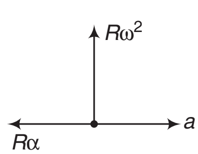
and $\qquad \alpha = 5\text{ rad/s}^2 \Rightarrow r = OA = 1$ m

$$\mathbf{v} = (r\omega)\,\hat{\mathbf{j}} = (10\hat{\mathbf{j}})\text{ m/s}$$

$$\mathbf{a} = (r\alpha)\,\hat{\mathbf{j}} - (r\omega^2)\,\hat{\mathbf{i}}$$

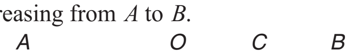
$$= (-100\,\hat{\mathbf{i}} + 5\hat{\mathbf{j}})\text{ m/s}^2$$ **Ans.**

14. $a = R\alpha$



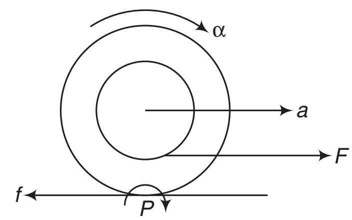
$\therefore \qquad a_{\text{net}} = R\omega^2 \qquad$ (towards centre)

15. I_C is least and C lies between O and B as density is increasing from A to B.

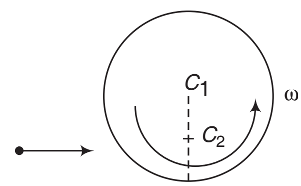


$I_A > I_B$ as more mass is concentrated towards B.

16. Torque of F about P is clockwise. So, spool will move towards right. Torque of friction will provide the clockwise torque about centre. So, f is leftwards. Further, a is towards right, so, $F > f$.



17. $C_1 C_2 = \dfrac{R}{2}$



where, C_2 = combined centre of mass of ring and particle.

From conservation of angular momentum about C_2.

$$L_i = L_f$$

$$\therefore \qquad \frac{mv\cdot R}{2} = I\omega$$

$$= \left[mR^2 + \frac{mR^2}{4} + \frac{mR^2}{4}\right]\omega$$

Solving this equation we get,

$$\omega = \frac{v}{3R}$$ **Ans.**

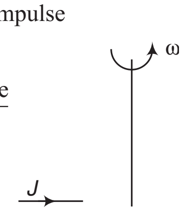
18. Mg can't provide rotational motion. This is only N which actually provides the rotational motion.

19. Angular momentum = Angular impulse

$$\therefore \qquad I\omega = Jl$$

or $$\omega = \frac{\text{Angular impulse}}{I}$$

$$= \frac{Jl}{ml^2/3} = \frac{3J}{ml}$$



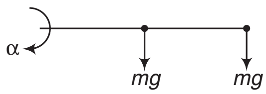
$$\text{KE} = \frac{1}{2}\,I\omega^2 = \frac{1}{2}\left(\frac{ml}{3}\right)^2\left(\frac{3J}{ml}\right)^2$$

$$= \frac{3J^2}{2m}$$

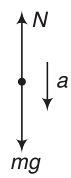
20. No rotational motion.

21. $\alpha = \dfrac{\tau_{\text{net}}}{I}$

$$= \frac{mg\left(\dfrac{l}{2}\right) + mgl}{(ml^2/3)}$$



$$= \frac{4.5\,g}{l}$$



$$a_{\text{coin}} = l\alpha = 4.5\,g$$

$$mg - N = ma_{\text{coin}}$$

$\therefore \quad N = mg - ma_{\text{coin}} = -3.5\,g$

Since, N can't be negative.

So, $N = 0$ and coin can not remain in contact with rod.

22. Let F = the applied force and f = force of friction (leftwards)

Net horizontal force = 0

$\therefore \quad F \cos\theta = f \quad$ ...(i)

Net torque about centre = 0

$\therefore \quad F \cdot r = f R$

Substituting $\dfrac{f}{R} = \dfrac{r}{R}$ in Eq. (i) we have,

$$\cos\theta = \frac{r}{R}$$

or $\quad \sin\theta = \sqrt{1 - \dfrac{r^2}{R^2}}$

$\therefore \quad \theta = \sin^{-1}\left(\sqrt{1 - \dfrac{r^2}{R^2}}\right)$

23. $a = \dfrac{mg - T}{m} \quad$ ...(i)

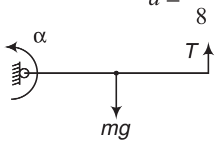
$$\alpha = \frac{\tau}{I} = \frac{T \cdot l - mg \cdot \dfrac{l}{2}}{(ml^2/3)}$$

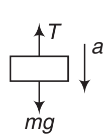
or $\quad \alpha = \dfrac{1}{ml}\left[3T - \dfrac{3mg}{2}\right] \quad$ ...(ii)

$a = l\alpha \quad$ ...(iii)

Solving these three equations we get,

$a = \dfrac{3g}{8}$ **Ans.**



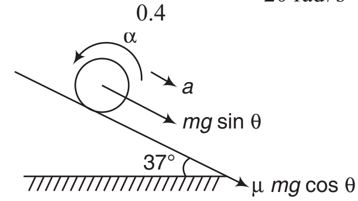


24. $a = (\mu g \cos\theta) + (g \sin\theta)$

$$= 0.5 \times 10 \times 0.8 + 10 \times 0.6 = 10\ \text{m/s}^2$$

$$\alpha = \frac{(\mu\, mg \cos\theta)\, R}{\dfrac{1}{2} mR^2} = \frac{2\,\mu g \cos\theta}{R}$$

$$= \frac{2 \times 0.5 \times 10 \times 0.8}{0.4} = 20\ \text{rad/s}^2$$



Pure rolling will start when,

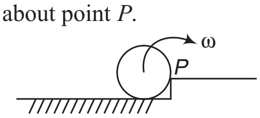
$v = R\omega \quad$ or $\quad at = R\,(\omega_0 - \alpha t)$

$\therefore \quad 10\,t = 0.4\,(54 - 20t)$

Solving this equation, we get

$t = 1.2\,\text{s}$ **Ans.**

25. Angular momentum will remain conserved at the point of impact P. and just after impact it starts rotating about point P.



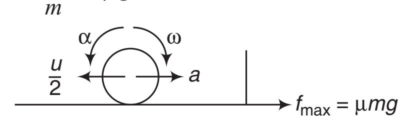
$$L_i = L_f$$

$\therefore \; Mv_0\left(R - \dfrac{R}{4}\right) + \dfrac{1}{2} MR^2 \left(\dfrac{v_0}{R}\right) = \dfrac{3}{2} MR^2 \cdot \omega$

$\therefore \quad \omega = \dfrac{5v_0}{6R}$

$\therefore \quad v_{\text{COM}} = \omega R = \dfrac{5v_0}{6}$ **Ans.**

26. $a = \dfrac{\mu mg}{m} = \mu g$



$$\omega = \frac{u}{R} \quad \Rightarrow \quad \alpha = \frac{(\mu mg)\,R}{2/5\; mR^2} = \frac{5\,\mu g}{2R}$$

Pure rolling will start when

$$v = R\omega$$

$$\left(\frac{u}{2} - at\right) = R\,(\alpha t - \omega)$$

or $\quad \left(\dfrac{u}{2} - \mu\, gt\right) = R\left(\dfrac{5\,\mu g}{2R}\, t - \dfrac{u}{R}\right)$

Solving we get, $\quad t = \dfrac{3u}{7\,\mu g}$ **Ans.**

27. Viscous liquid has only translational kinetic energy

$\therefore$ KE = translational + rotational kinetic energy of hollow sphere + translational kinetic energy of liquid.

$$= \frac{1}{2}mv^2 + \frac{1}{2}\times\frac{2}{3}mR^2\left(\frac{v}{R}\right)^2 + \frac{1}{2}mv^2$$

$$= \frac{4}{3}mv^2 \qquad \textbf{Ans.}$$

28. $a = \dfrac{g\sin\theta}{1 + I/mR^2} = \dfrac{g\sin\theta}{1+(1/2)}$

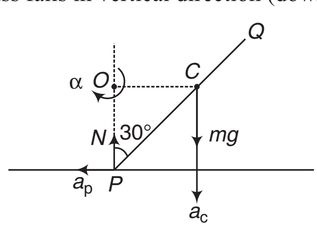
$$= \frac{2g\sin\theta}{3}$$

$\therefore \qquad g\sin\theta = \dfrac{3}{2}a$

Now, $\qquad f = \dfrac{mg\sin\theta}{1 + mR^2/I} = \dfrac{m\,(3/2\,a)}{1+2}$

$$= \frac{1}{2}ma \qquad \textbf{Ans.}$$

29. Only two forces are acting on rod, normal reaction (vertically upwards) and weight (vertically downwards). Since, both forces are vertical centre of mass falls in vertical direction (downwards).



$$OC = \frac{l}{2}\sin 30° = \frac{l}{4}$$

$$\alpha = \frac{\tau}{I} \qquad \text{(about } O\text{)}$$

$$= \frac{(mg)\left(\frac{l}{2}\sin 30°\right)}{\left(\frac{ml^2}{12} + \frac{ml^2}{16}\right)}$$

$$= \frac{12}{7}\frac{g}{l}$$

$$a_C = (OC)\,\alpha$$

$$= \left(\frac{l}{4}\right)\left(\frac{12}{7}\frac{g}{l}\right) = \frac{3}{7}g$$

Now, $mg - N = ma_C = \dfrac{3\,mg}{7}$

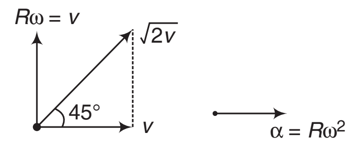
$\therefore \qquad N = \dfrac{4\,mg}{7} \qquad$ **Ans.**

30. $a_P = (OP)\,\alpha$

$$= \left(\frac{l}{2}\cos 30°\right)\left(\frac{12}{7}\frac{g}{l}\right)$$

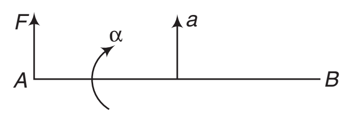
$$= \frac{3\sqrt{3}g}{7} \qquad \textbf{Ans.}$$

31. Angle between acceleration and velocity is 45°.



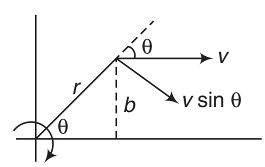
32. $a = \dfrac{F}{M}$

$$\alpha = \frac{FL/2}{ML^2/12} = \frac{6F}{ML}$$



$$a_B = \frac{L}{2}\alpha - a = \frac{2F}{M} \qquad \text{(downwards)}$$

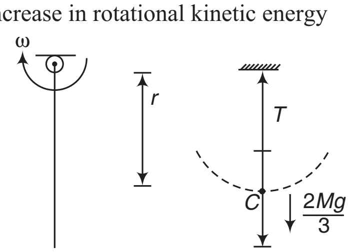
33. $\omega = \dfrac{v\sin\theta}{r} = \dfrac{v\sin\theta}{b/\sin\theta}$



$$= \frac{v\sin^2\theta}{b} \propto \sin^2\theta \qquad \text{(as } v \text{ and } b \text{ are constants)}$$

θ is decreasing. Therefore, ω will decrease.

34. Decrease in gravitational potential energy = increase in rotational kinetic energy



$\therefore \qquad Mg\dfrac{L}{2} = \dfrac{1}{2}\left(\dfrac{ML^2}{3}\right)\cdot\omega^2 \;\therefore\; \omega^2 = \dfrac{3g}{L}$

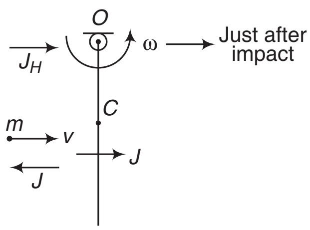
$$r = \frac{L}{3} + \frac{2L/3}{2} = \frac{2L}{3}$$

$$\frac{T-2Mg}{3}=\left(\frac{2M}{3}\right)a_C=\left(\frac{2M}{3}\right)(r\omega^2)$$

$$\therefore \quad T=\frac{2Mg}{3}+\left(\frac{2Mg}{3}\right)\left(\frac{2L}{3}\right)\left(\frac{3g}{L}\right)$$

$$=2\,Mg \qquad \textbf{Ans.}$$

35. From conservation of angular momentum, just before and just after impact about point O we have,



$$L_i=L_f$$

$$\therefore \quad mv\frac{L}{2}=\left(\frac{mL^2}{3}\right)\cdot\omega$$

$$\therefore \quad \omega=\frac{3v}{2L}$$

$$v_C=\frac{1}{2}\cdot\omega=\frac{3}{4}v$$

Impulse J has changed the momentum of particle from mv to O. Hence,

$$J=mv$$

For rod

$$J_H+J=mv_C=\frac{3}{4}mv$$

$$\therefore \quad J_H=\frac{3}{4}mv-J=\frac{3}{4}mv-mv$$

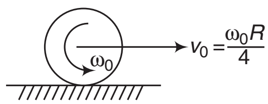
$$=-\frac{mv}{4} \text{ or } |J_H|=\frac{mv}{4} \qquad \textbf{Ans.}$$

36. $L_i=L_f$

$$(I+mR^2)\,\omega=(mvR)+I\omega'$$

$$\therefore \quad \omega'=\frac{(I+mR^2)\,\omega-mvR}{I} \qquad \textbf{Ans.}$$

37. About bottommost point,



Angular momentum,

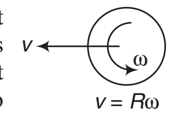
$$L=I_C\,\omega_0-mv_0R \qquad \text{(anticlockwise)}$$

$$=\frac{1}{2}mR^2\omega_0-m\left(\omega_0\frac{R}{4}\right)R$$

$$=\frac{1}{4}mR^2\omega_0$$

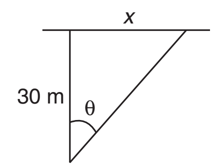
$$=+\text{ ve or anticlockwise}$$

During slip, friction acts about bottommost point. So, its torque is zero or angular momentum about bottommost point should also remain anticlockwise when pure rolling starts. So, figure should be as shown below.



So, the disc will return to its initial position for all values of μ.

38. $x=30\tan\theta$



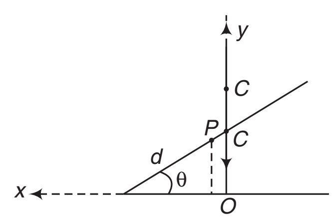
$$\therefore \quad \left(\frac{dX}{dt}\right)=(30\sec^2\theta)\cdot\frac{d\theta}{dt}$$

or $\quad v_{\text{car}}=(30\sec^2\theta)\,\omega$

$$\therefore \quad \omega=\frac{v_{\text{car}}}{30\sec^2\theta}$$

$$=\frac{40}{(30)\sec^2 30^\circ}=1\text{ rad/s}$$

39. Two forces normal reaction and weight are the only forces acting on the rod during motion. Both forces are vertical. So centre of mass will fall downwards in a vertical line.



$$X_P=x\text{ (say)}=\frac{l}{2}\cos\theta-d\cos\theta$$

or $\quad \cos\theta=\dfrac{x}{\left(\dfrac{l}{2}-d\right)} \qquad$ ...(i)

Similarly,

$$y_P=y=d\sin\theta$$

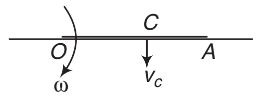
$$\therefore \quad \sin\theta=\frac{y}{d} \qquad \text{...(ii)}$$

Squaring and adding Eqs. (i) and (ii), we get

$$\frac{X^2}{\left(\dfrac{l}{2}-d\right)^2}+\frac{y^2}{d^2}=1$$

This is an equation of a circle for $d=\dfrac{l}{4}$. For any other value of 'd' it is equation of ellipse.

40. Instantaneous axis of rotation will pass through O.



$\therefore \qquad v_0 = 0$

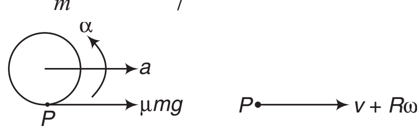
41. Decrease in gravitational potential energy
= increase in rotational kinetic energy about O

$$\therefore \qquad mg\frac{l}{2} = \frac{1}{2}\left(\frac{ml^2}{3}\right)\omega^2$$

$$\text{or} \qquad \omega = \sqrt{\frac{3g}{l}}$$

$$v_A = r\omega = l\omega = \sqrt{3gl} \qquad \textbf{Ans.}$$

42. $a = \dfrac{\mu\, mg}{m} = \mu g = \dfrac{2}{7}g$



$$\alpha = \frac{\mu\, mgR}{\frac{2}{5}mR^2} = \frac{5\mu g}{2R} = \frac{5}{7}\frac{g}{R}$$

Pure rolling will start when,

$$v_P = v + R\omega = v_0$$

$$\text{or} \qquad at + R(\alpha t) = v_0$$

$$\therefore \qquad \left(\frac{2}{7}gt\right) + \frac{5}{7}gt = v_0$$

$$\therefore \qquad t = \frac{v_0}{g}$$

$$S = \frac{1}{2}at^2$$

$$= \frac{1}{2}\left(\frac{2}{7}g\right)\left(\frac{v_0}{g}\right)^2 = \frac{v_0^2}{7g} \qquad \textbf{Ans.}$$

More than One Correct Options

1. $\dfrac{K_R}{K_T} = \dfrac{I}{mr^2}$

$$\therefore \qquad K_T = \left(\frac{mr^2}{I + mr^2}\right)K_{\text{Total}}$$

$$\text{or} \qquad K_{\text{Total}} = \left(\frac{I + mr^2}{mr^2}\right)\left(\frac{1}{2}mv^2\right)$$

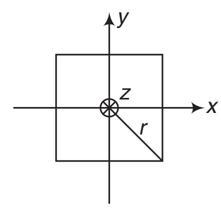
$$= mgh = mg\left(\frac{3}{4}\frac{v^2}{g}\right)$$

Solving we get $I = \dfrac{1}{2}mr^2$

So it is either solid cylinder or disc.

2. (a) $I_x = I_y = 2\left[\dfrac{ml^2}{12} + m\left(\dfrac{l}{2}\right)^2\right]$

$$= \frac{2}{3}ml^2$$



(b) $I_z = I_x + I_y = \dfrac{4}{3}ml^2$

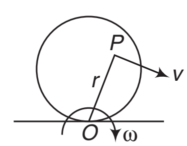
(c) $I = I_z + (4m)r^2$

$$= \frac{4}{3}ml^2 + (4m)\left(\frac{l}{\sqrt{2}}\right)^2$$

$$= \frac{10}{3}ml^2$$

(d) $I = 2\dfrac{ml^2}{3} + ml^2 = \dfrac{5}{3}ml^2$

3. $O \rightarrow$ instantaneous axis of rotation

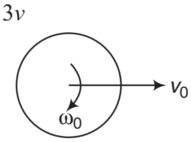


$$v = r\omega \text{ or } v \propto r$$

More the value of r from O more is the speed of point P.

4. $v_0 + R\omega_0 = v \qquad$ ...(i)

$v_0 - R\omega_0 = 3v \qquad$ ...(ii)



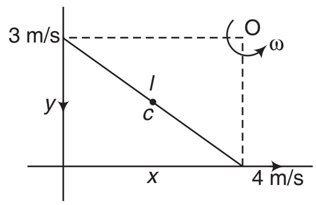
Solving these two equations,

we get $\qquad v_0 = 2v$

and $\qquad \omega_0 = -\dfrac{v}{R}$

5. $x^2 + y^2 = l^2 = (0.25)\ m^2$

$$y\omega = 4 \quad \text{...(i)}$$
$$x\omega = 3 \quad \text{...(ii)}$$



Squaring and adding, we get

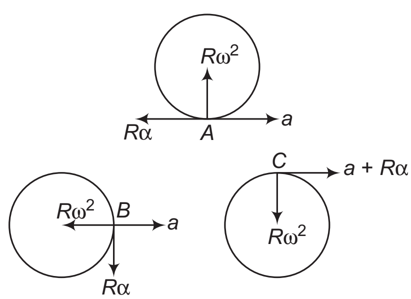
$$(x^2 + y^2)\,\omega^2 = 25$$

or $\quad (0.25)\,\omega^2 = 25$

$\therefore \quad \omega = 10$ rad/s

$$OC = \frac{l}{2} = 0.25\text{m}$$
$$v_{\text{CM}} = (OC)\,\omega = 2.5 \text{ m/s}$$
$$K_{\text{Total}} = \frac{1}{2} mv_{\text{cm}}^2 + \frac{1}{2} I_{\text{cm}}\,\omega^2$$
$$= \frac{1}{2} \times 2 \times (2.5)^2 + \frac{1}{2}\left(\frac{2 \times 0.25}{12}\right)(10)^2$$
$$= \frac{25}{3} \text{ J}$$

6. Net acceleration of any point on the rim is vector sum of a, $R\omega^2$ and $R\alpha$ with $a = R\alpha$

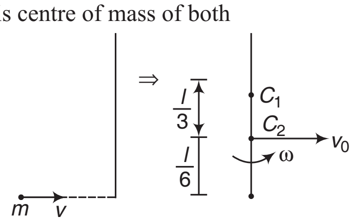


$a_A = R\omega^2 \rightarrow$ vertically upwards

If $a = R\omega^2$, a_B is vertically downwards and so on.

7. C_1 is centre of mass of rod

C_2 is centre of mass of both



$$P_i = P_f$$

$\therefore \quad mv = 3mv_0 \quad \Rightarrow \quad v_0 = \dfrac{v}{3} \quad \text{...(i)}$

$L_i = L_f$ about C_2 we have,

$$mv\,\frac{l}{6} = I_{C_2}\,\omega$$
$$= \left[\left(\frac{2ml^2}{12}\right) + 2m\left(\frac{l}{3}\right)^2 + m\left(\frac{l}{6}\right)^2\right]$$

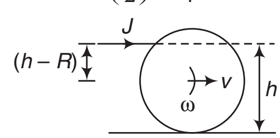
$\therefore \quad \omega = \dfrac{2}{5}\dfrac{v}{l} \quad \Rightarrow \quad K_i = \dfrac{1}{2} mv^2$

$$K_f = \frac{1}{2}(3m)\left(\frac{v}{3}\right)^2 + \frac{1}{2} I_{C_2}\,\omega^2$$
$$= \frac{1}{6} mv^2 + \frac{1}{2}\left(\frac{5}{12} ml^2\right)\left(\frac{2}{5}\frac{v}{l}\right)^2$$
$$= \frac{1}{5} mv^2$$

$\therefore \quad$ Loss of kinetic energy $= K_i - K_f = \dfrac{3}{10} mv^2$

8. $I = mK^2 \quad \left(\text{where, } K = \dfrac{R}{2} = \text{radius of gyration}\right)$

$$= m\left(\frac{R}{2}\right)^2 = \frac{1}{4} mR^2$$



Ball will roll purely if

$$v = R\omega$$

$\therefore \quad \left(\dfrac{J}{m}\right) = R\left[\dfrac{J\,(h - R)}{(1/4)\,mR^2}\right]$

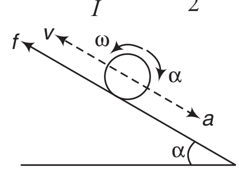
Solving this equation, we get

$$h = \frac{5R}{4}$$

Further, if ball is struck at centre of mass, there will be no rotation only translation.

9. Friction always acts upwards. If the ball moves upwards, it becomes a case of retardation with pure rolling.

$$\mu_{\min} = \frac{\tan\alpha}{1 + \dfrac{mR^2}{I}} = \frac{\tan\alpha}{1 + \dfrac{3}{2}} = \frac{2}{5}\tan\alpha$$



10. $I_1\omega_1 = I_2\omega_2$

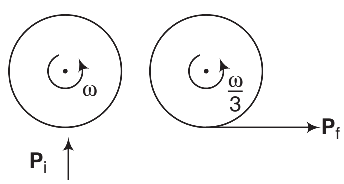
$$\therefore \quad \omega_2 = \frac{I_1}{I_2}\cdot\omega_1$$

$$= \frac{(1/2\, mR^2)}{(1/2\, mR^2 + mR^2)}\cdot\omega$$

$$= \frac{\omega}{3}$$

$$P_i = mv_i = m\,(2\omega R) = 2m\omega R$$

$$P_f = mv_f = m\left(\frac{\omega}{3}R\right) = \frac{1}{3}m\omega R$$

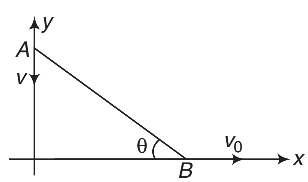


Impulse $J = |\mathbf{P}_f - \mathbf{P}_i|$

$$= \sqrt{P_i^2 + P_f^2} \qquad (\text{as}\,\theta = 90°)$$

$$= \frac{\sqrt{37}}{3}m\omega R$$

11. Velocity component along $AB = 0$



$$\therefore \quad v_0\cos\theta = v\sin\theta$$

or $\quad v = v_0\cot\theta = f_1(\theta)$

at $\quad \theta = 37°,\ v = \dfrac{4}{3}v_0$

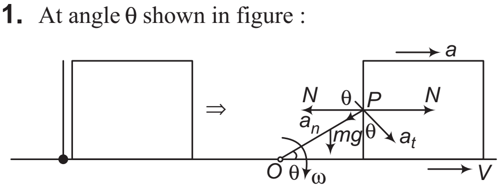
$$\omega = \frac{\text{Relative velocity} \perp \text{to } AB}{l}$$

$$= \frac{v_0\sin\theta + v\cos\theta}{l} = f_2(\theta)$$

$$= \frac{(v_0)(3/5) + (4/3\, v_0)(4/5)}{l} = \frac{5v_0}{3l}$$

Comprehension Based Questions

1. At angle θ shown in figure :



decrease in potential energy of rod
= increase in rotational kinetic energy of rod
+ translational kinetic energy of block.

But $\quad v = v_P\sin\theta = (l\omega)\sin\theta$

$$\therefore \quad mg\left(\frac{l}{2} - \frac{l}{2}\sin\theta\right) = \frac{1}{2}\left(\frac{ml^2}{3}\omega^2\right) + \frac{1}{2}M\,(\omega l\sin\theta)^2$$

From here we get,

$$l\omega^2 = a_n = \frac{mg\,(1-\sin\theta)}{m/3 + M\sin^2\theta} \qquad \text{...(i)}$$

$$a_t = l\alpha = l\left(\frac{\tau}{I}\right)_0 = l\left[\frac{mg\dfrac{l}{2}\cos\theta - Nl\sin\theta}{ml^2/3}\right]$$

$$= \frac{3}{2}g\cos\theta - \frac{3N\sin\theta}{m} \qquad \text{...(ii)}$$

For block

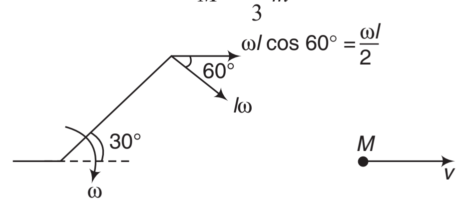
$$N = Ma = M\,[a_t\sin\theta - a_n\cos\theta] \qquad \text{...(iii)}$$

Now putting values of a_t and a_n from Eqs. (i) and (ii) in Eqs. (iii) and then putting $N = 0$ and $\theta = 30°$ in the equation we get,

$$\frac{M}{m} = \frac{4}{3} \qquad \textbf{Ans.}$$

2. $\dfrac{M}{m} = \dfrac{4}{3}$

$$M = \frac{4}{3}m$$



$$\frac{\omega l}{2} = v \quad \therefore \quad \omega = \frac{2v}{l}$$

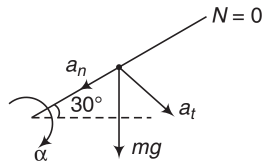
Decrease in potential energy of rod
= increase in rotational kinetic energy of rod
+ translational kinetic energy of block

$$\therefore \quad mg\left(\frac{l}{2} - \frac{l}{2}\sin 30°\right) = \frac{1}{2}\left(\frac{ml}{3}\right)^2\left(\frac{2v}{l}\right)^2 + \frac{1}{2}\left(\frac{4m}{3}\right)(v^2)$$

Solving this equation, we get

$$v = \frac{\sqrt{3gl}}{4} \qquad \textbf{Ans.}$$

3. $a_n = \dfrac{l}{2}\omega^2 = \dfrac{l}{2}\left(\dfrac{4v^2}{l^2}\right)$



$$= \left(\frac{2}{l}\right)\left(\frac{3gl}{16}\right) = \frac{3}{8}g$$

$$a_t = \frac{l}{2}\alpha = \frac{l}{2}\left(\frac{\tau}{I}\right) = \frac{l}{2}\left[\frac{mg\left(\dfrac{l}{2}\cos 30°\right)}{ml^2/3}\right]$$

$$= \frac{3\sqrt{3}\,g}{8}$$

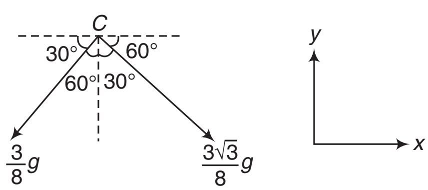
$$\therefore \qquad a = \sqrt{a_n^2 + a_t^2}$$

$$= \frac{3g}{4}$$

4. $\mathbf{a}_C = \left(\dfrac{3\sqrt{3}\,g}{8}\cos 60° - \dfrac{3}{8}g\cos 30°\right)\hat{\mathbf{i}}$

$$+ \left(\frac{3\sqrt{3}\,g}{8}g\cos 30° + \frac{3}{8}g\cos 60°\right)(-\hat{\mathbf{j}})$$

$$= -\left(\frac{3g}{4}\right)\hat{\mathbf{j}}$$



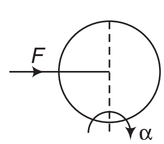
Now, $\quad mg(-\hat{\mathbf{j}}) + \mathbf{F}_{\text{Hinge}} = m\mathbf{a}_C$

Substituting the values we get

$$\mathbf{F}_{\text{Hinge}} = \left(\frac{mg}{4}\right)\hat{\mathbf{j}} \qquad \textbf{Ans.}$$

5. Bottommost point where friction actually acts is at rest.

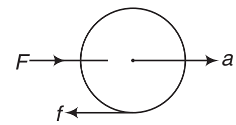
6. $\alpha = \dfrac{F\cdot R}{\frac{3}{2}mR^2} = \dfrac{2F}{3mR}$



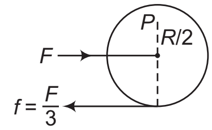
$$a = R\alpha = \frac{2F}{3m}$$

$$F - f = ma = \frac{2F}{3}$$

$$\therefore \qquad f = \frac{F}{3} = \frac{1}{2}ma$$



7. About point P net torque of F and f is zero. So, angular momentum is conserved.



8. K_{Total} after falling a height h is mgh

$$\frac{K_R}{K_T} = \frac{2}{3}$$

$$\therefore \qquad K_T = \frac{1}{2}mv^2 = \left(\frac{3}{5}\right)(mgh)$$

$$\therefore \qquad v = \sqrt{\frac{6gh}{5}} \qquad \textbf{Ans.}$$

9. $H = \dfrac{v^2\sin^2\sin 37°}{2g}$

$$= \frac{(6gh/5)(3/5)^2}{2g} = \frac{27h}{125} \qquad \textbf{Ans.}$$

10. $R = x = \dfrac{2u^2\sin 37°\cos 37°}{g}$

$$= \frac{(2)(6gh/5)(3/5)(4/5)}{g}$$

$$= \frac{144}{125}h \qquad \textbf{Ans.}$$

Match the Columns

1. (a) $t = \sqrt{\dfrac{2S}{a}}$ and $a = \dfrac{g\sin\theta}{1 + I/mR^2}$

I of hollow sphere is maximum.

So, a is minimum and t is maximum.

(b) $K_{\text{Total}} = mgh$ for all

(c) $\dfrac{K_R}{K_T} = \dfrac{2}{5}$ for solid sphere

$= \dfrac{2}{3}$ for hollow sphere

$= \dfrac{1}{2}$ for disc

It is maximum for hollow sphere.

But $\frac{K_T}{K_R}$ is maximum for solid sphere.

2. In all cases,

$$v = \frac{J}{m} \qquad (J = \text{linear impulse})$$

$$\omega = \frac{J \times r_{\perp}}{I_E}$$

Here, $r_{\perp}$ is from centre E. v is rightwards. So, for pure rolling ω should be clockwise. Hence, J should be applied at A.

If it is below A, angular velocity is anticlockwise and it will cause forward slip.

3. In case of pure roll (upwards or downwards) required value of friction acts in upward direction.

In case of slip (forward or backward) maximum friction will act in backward or forward direction.

4. (a) $K = \sqrt{I/m}$

$$= \sqrt{\frac{m\,(2a)^2}{3}} = \frac{2}{\sqrt{3}}\,a$$

(b) $K = \sqrt{\frac{m\,(2a)^2}{12}} = \frac{a}{\sqrt{3}}$

(c) $K = \sqrt{\frac{ma^2}{3}} = \frac{a}{\sqrt{3}}$

(d) $K = \sqrt{\frac{ma^2}{12}} = \frac{a}{\sqrt{12}}$

5. (a) $K_{\text{Total}} = mgh\,K \text{ (say)}$

$$\frac{K_R}{K_T} = \frac{2}{5}$$

$$\therefore \quad K_R = \frac{2}{7}K = \frac{2}{7}mgh$$

(b) $K_{\text{Total}} = mg\frac{h}{2}$

$$\frac{K_R}{K_T} = \frac{2}{5}$$

$$\therefore \quad K_T = \frac{5}{7}\left(mg\,\frac{h}{2}\right) = \frac{5}{14}mgh$$

(c) At 3

$$K_R = \frac{2}{7}\left(mg\,\frac{h}{2}\right) = \frac{1}{7}mgh$$

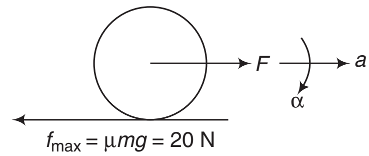
From 3 to 4, K_R will become constant.

(d) $K_T = K_{\text{Total}} - K_R$

$$= mgh - \frac{1}{7}mgh = \frac{6}{7}mgh$$

6. $a = R\alpha = R\left(\frac{\tau}{I}\right)$

$$\therefore \quad \frac{F-20}{10} = (1)\left(\frac{20 \times 1}{\frac{1}{2} \times 10 \times 1^2}\right)$$



Solving this equation, we get

$$F = 60 \text{ N}$$

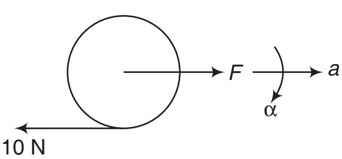
So, when $F = 60$ N, friction reaches its maximum value or slipping will start. F becomes 60 N at 6 s.

(d) $a = R\,\alpha = R\left(\frac{\tau}{I}\right)$

$$\text{or} \quad \frac{F-10}{10} = (1)\left[\frac{10 \times 1}{\frac{1}{2} \times 10 \times 1^2}\right]$$

$$F = 30 \text{ N}$$

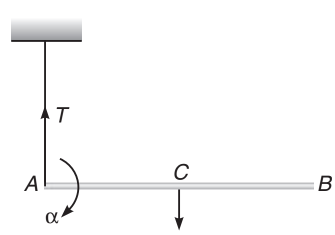
and F becomes 30 N at 3 s.



Subjective Questions

1. Torque about bottommost point in each case is clockwise.

2. $\alpha = \frac{\tau}{I} = \frac{mg\left(\frac{l}{2}\right)}{\frac{ml^2}{3}} = \frac{3}{2}\frac{g}{l}$



(a) $a_B = l\alpha = \dfrac{3g}{2}$ **Ans.**

(b) $a_C = \dfrac{l}{2}\alpha = \dfrac{3g}{4}$ **Ans.**

(c) $a_C = \dfrac{mg - T}{m}$ or $\dfrac{3g}{4} = \dfrac{mg - T}{m}$

$\therefore \quad T = \dfrac{mg}{4}$ **Ans.**

3. $mgh = K_R + K_T = \dfrac{3}{4}mv^2$

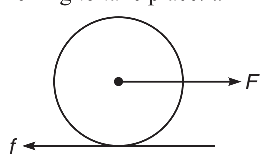
Here $h = s\sin\theta$

$\therefore gs\sin\theta = \dfrac{3}{4}v^2$ or $s = \dfrac{3v^2}{4g\sin\theta}$

$$= \frac{3\times(2.0)^2}{4\times 9.8\times\frac{1}{2}} = 0.612 \text{ m}$$ **Ans.**

4. $I = MR^2$

For pure rolling to take place. $a = R\alpha$



or $\dfrac{F - f}{M} = R\left(\dfrac{f.R}{MR^2}\right) = \dfrac{f}{M}$

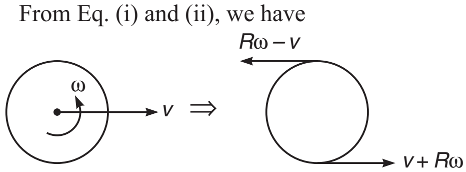
$\therefore \quad f = \dfrac{F}{2}$ **Ans.**

and $a = \dfrac{F - f}{M} = \dfrac{F}{2M}$ **Ans.**

5. $v + R\omega = 1.5$ ...(i)

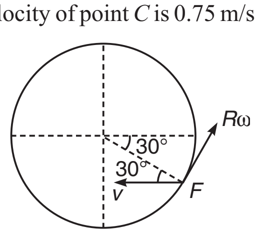
and $R\omega - v = 3.0$ ...(ii)

From Eq. (i) and (ii), we have



$R\omega = 2.25$ m/s and $v = -0.75$ m/s

Thus, velocity of point C is 0.75 m/s (towards left).



$$v_F = \sqrt{v^2 + (R\omega)^2 + 2v(R\omega)\cos(90+30)}$$

$$= \sqrt{0.5625 + 5.0625 + 2\times 0.75\times 2.25\times\left(-\frac{1}{2}\right)}$$

$= 1.98$ m/s **Ans.**

6. $a = \dfrac{Mg - T}{M}$ ...(i)

$$\alpha = \frac{TR}{\frac{1}{2}MR^2} = \frac{2T}{MR} \quad \text{...(ii)}$$

$a = R\alpha$ ...(iii)

Solving these three equations, we get

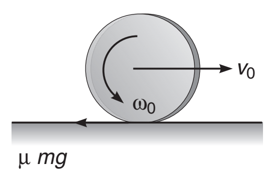
$a = \dfrac{2g}{3}$ and $T = \dfrac{Mg}{3}$ **Ans.**

7. From conservation of mechanical energy, decrease in gravitational PE = increase in rotational KE

or $mg(R) = \left[\dfrac{1}{2}MR^2 + mR^2\right]\left(\dfrac{1}{2}\omega^2\right)$

or $\omega = \sqrt{\dfrac{4mg}{(2m+M)R}}$ **Ans.**

8. Initially there is forward slipping. Therefore, friction is backwards and maximum.



Let velocity becomes zero in time t_1 and angular velocity becomes zero in time t_2.

Then, $0 = v_0 - at_1$

or $t_1 = \dfrac{v_0}{a} = \dfrac{v_0}{\mu g}$ ...(i)

and $0 = \omega_0 - \alpha t_2$

or $t_2 = \dfrac{\omega_0}{\alpha}$

Here, $\alpha = \dfrac{\mu mgR}{\frac{1}{2}mR^2} = \dfrac{2\mu g}{R}$

$\therefore \quad t_2 = \dfrac{\omega_0 R}{2\mu g}$ ...(ii)

Disk will return back when

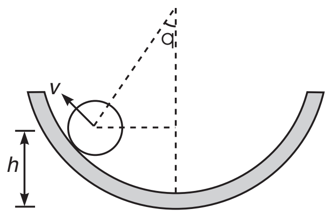
$t_2 > t_1$ or $\dfrac{\omega_0 R}{2\mu g} > \dfrac{v_0}{\mu g}$

or $\omega_0 > \dfrac{2v_0}{R}$ **Ans.**

9. $h = (R - r)(1 - \cos\theta)$ …(i)

Kinetic energy at angle θ is,

$$K = \frac{7}{5}\left(\frac{1}{2}mv_0^2\right) - mgh$$



∵ In case of pure rolling

$$K_T = \frac{5}{7}K$$

$$\therefore \quad \frac{1}{2}mv^2 = \frac{1}{2}mv_0^2 - \frac{5}{7}mgh$$

$$\therefore \quad v^2 = v_0^2 - \frac{10}{7}gh \quad \text{…(ii)}$$

Equation of motion at angle θ is,

$$N - mg\cos\theta = \frac{mv^2}{(R - r)}$$

$$\therefore \quad N = mg\cos\theta + \frac{m}{(R - r)}\left(v_0^2 - \frac{10}{7}gh\right)$$

Substituting value of h from Eq. (i)

$$N = mg\cos\theta + \left(\frac{m}{R - r}\right)\left\{v_0^2 - \frac{10}{7}g(R - r)(1 - \cos\theta)\right\}$$

$$= \frac{mg}{7}(17\cos\theta - 10) + \frac{mv_0^2}{(R - r)} \quad \textbf{Ans.}$$

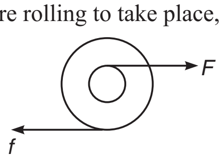
Force of friction,

$$f = \frac{mg\sin\theta}{1 + \frac{mr^2}{I}} \quad \text{(for pure rolling to take place)}$$

$$= \frac{mg\sin\theta}{1 + \frac{5}{2}} \quad \left(I = \frac{2}{5}mr^2\right)$$

$$= \frac{2}{7}mg\sin\theta \quad \textbf{Ans.}$$

10. (a) For pure rolling to take place,



$$a = R\alpha \quad \text{or} \quad \frac{F - f}{m} = R\left[\frac{Fr + fR}{\frac{1}{2}mR^2}\right]$$

Solving this equation, we get

$$f = \frac{2}{3}\left(\frac{1}{2} - \frac{r}{R}\right)F \quad \textbf{Ans.}$$

(b) Acceleration $a = \dfrac{F - f}{m}$

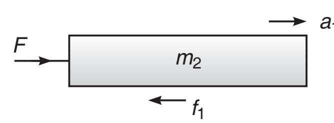
Substituting value of f from part (a), we get

$$a = \frac{2F}{3mR}(R + r) \quad \textbf{Ans.}$$

(c) $a > \dfrac{F}{m}$ if $\dfrac{2}{3R}(R + r) > 1$ or $r > \dfrac{R}{2}$ **Ans.**

(d) In this case force of friction is in forward direction. **Ans.**

11. We can choose any arbitrary directions of frictional forces at different contacts.



In the final answer, the negative value will show the opposite directions.

Let f_1 = friction between plank and cylinder

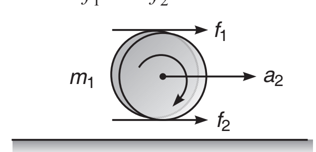
f_2 = friction between cylinder and ground

a_1 = acceleration of plank

a_2 = acceleration of centre of mass of cylinder

and α = angular acceleration of cylinder about is COM.

Directions of f_1 and f_2 are as shown here



Since, there is no slipping anywhere

$$\therefore \quad a_1 = 2a_2 \quad \text{…(i)}$$

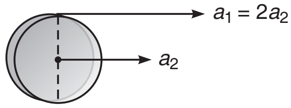
(Acceleration of plank = acceleration of top point of cylinder)

$$a_1 = \frac{F - f_1}{m_2} \quad \text{…(ii)}$$

$$a_2 = \frac{f_1 + f_2}{m_1} \quad \text{…(iii)}$$

$$\alpha = \frac{(f_1 - f_2)R}{I}$$

(I = moment of inertia of cylinder about COM)

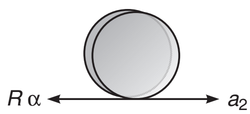


$$\therefore \qquad \alpha = \frac{(f_1 - f_2)R}{\frac{1}{2}m_1 R^2}$$

$$\alpha = \frac{2(f_1 - f_2)}{m_1 R} \qquad \ldots\text{(iv)}$$

$$a_2 = R\alpha = \frac{2(f_1 - f_2)}{m_1} \qquad \ldots\text{(v)}$$

(Acceleration of bottommost point of cylinder = 0)



(a) Solving Eqs. (i), (ii), (iii) and (v), we get

$$a_1 = \frac{8F}{3m_1 + 8m_2}$$

and
$$a_2 = \frac{4F}{3m_1 + 8m_2}$$

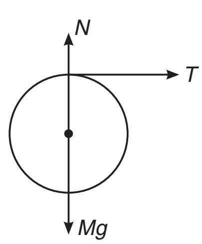
(b) $f_1 = \dfrac{3m_1 F}{3m_1 + 8m_2}$

$$f_2 = \frac{m_1 F}{3m_1 + 8m_2}$$

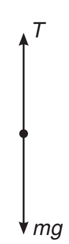
Since, all quantities are positive, they are correctly shown in figures.

Note *Above calculations have been done at $t = 0$ when $\omega = 0$.*

12. If α be the angular acceleration of the hoop and a be the acceleration of its centre, acceleration of m would be $\alpha r + a$.



FBD of hoop



FBD of block of mass m

Here, $Tr = I\alpha$ [where I = moment of inertia of the hoop about the horizontal axis passing through its centre]

Also, $\quad T = Ma \quad$ and $\quad mg - T = m[a + \alpha r]$

Solving, we get

$$a = \frac{mg}{[M + 2m]} = \frac{2}{1.4} = 1.43 \text{ m/s}^2$$

Hence, $\quad T = 1.43$ N

and $\quad \alpha = \dfrac{Tr}{I} = \dfrac{T}{Mr} = 7.15 \text{ rad/s}^2$

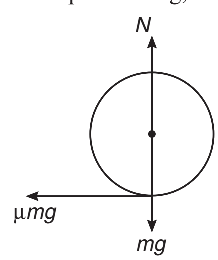
13. Initially the cylinder will slip on the plank, therefore kinetic friction will act between the cylinder and the plank.

$$a_c = -\frac{\mu mg}{m} = -\mu g$$

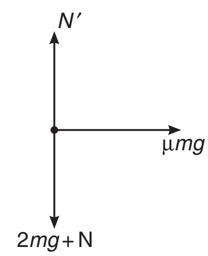
$$a_p = +\frac{\mu mg}{2m} = +\frac{\mu g}{2}$$

$$\alpha_c = +\frac{(\mu mg)(R)}{(mR^2/2)} = +\frac{2\mu g}{R}$$

For pure rolling,



FBD of cylinder



FBD of plank

$$v_p = v_c - R\omega_c$$

$$\therefore \qquad \frac{\mu g}{2} t = v_0 - \mu g t - (R)\left(\frac{2\mu g}{R}\right)(t)$$

$$\therefore \qquad t = \frac{v_0}{3.5\mu g} = \frac{7}{3.5 \times 0.1 \times 10} = 2 \text{ s}$$

$$\therefore \qquad s_c - s_p = v_0 t - \frac{1}{2} \times (\mu g)(t^2) - \frac{1}{2}\left(\frac{\mu g}{2}\right)(t^2)$$

$$= (7 \times 2) - \frac{1}{2}(0.1)(10)(4) - \frac{1}{2}\left(\frac{0.1 \times 10}{2}\right)(4) = 11 \text{ m}$$

Also, $\quad v_c - v_P = (v_0 - \mu g t) - \left(\dfrac{\mu g}{2}\right)(t)$

$$= 7 - 0.1 \times 10 \times 2 - \frac{0.1 \times 10 \times 2}{2} = 4 \text{ m/s}$$

Hence, the remaining distance $(12 - 11 = 1 \text{ m})$ is travelled in a time,

$$t' = \frac{1.0}{4} = 0.25 \text{ s}$$

$\therefore \qquad$ Total time $= 2 + 0.25 = 2.25$ s

14. In rolling without sliding on a stationary ground, work done by friction is zero. Hence

work done by the applied force

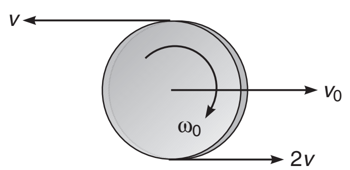
= change in kinetic energy

$$\therefore \quad (30)(0.25) = \frac{1}{2}\times 9\times v^2$$
$$+ 2\left[\frac{1}{2}\times 6\times v^2 + \frac{1}{2}\times\frac{1}{2}\times 6\times r^2\times\frac{v^2}{r^2}\right]$$

or $\quad 7.5 = 13.5v^2$

$\therefore \quad v = 0.745$ m/s **Ans.**

15. Let v_0 be the linear velocity and ω_0 the angular velocity of the disc as shown in figure then,



$$v_0 - r\omega_0 = 2v \quad \text{...(i)}$$

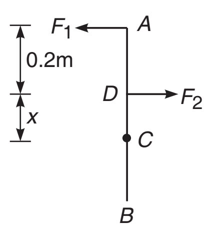
and $\quad v_0 + r\omega_0 = -v \quad$ ...(ii)

Solving Eqs. (i) and (ii), we have

$$\omega_0 = -\frac{3}{2}\frac{v}{r}$$

Hence, the angular velocity of disc is $\frac{3}{2}\frac{v}{r}$ anticlockwise. **Ans.**

16. Let x be the distance of centre point C of rod from D. Then,



$$F_2 - F_1 = ma$$

or $\quad F_1 = 3$ N

Further, $\quad \tau_c = 0$

$\therefore \quad F_2 x = F_1\ (0.2 + x)$

$\quad 5x = F_1\ (0.2 + x)$

$\therefore \quad 5x = 3\ (0.2 + x)$

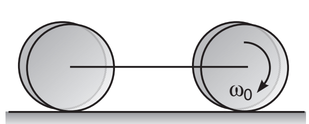
or $\quad x = 0.3$ m

$\therefore \quad$ Length of rod $= 2\ (x + 0.2) = 1.0$ m **Ans.**

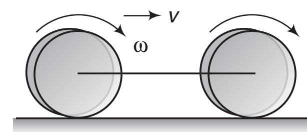
17. $L_i = L_f \quad$ (about bottommost point)

$\therefore \quad I\omega_0 = 2[I\omega + mRv]$

or $\quad \left(\frac{1}{2}mR^2\right)\omega_0 = 2\left[\frac{1}{2}mR^2\omega + mR\ (\omega R)\right]$



$\Rightarrow$



$\therefore \quad \omega = \frac{\omega_0}{6}$ **Ans.**

and $\quad v = \omega R = \frac{\omega_0 R}{6}$ **Ans.**

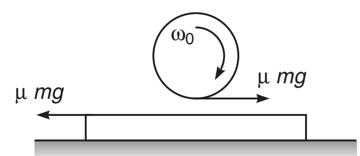
18. Let,

a_1 = linear acceleration of sphere (towards right),

a_2 = linear acceleration of plank (towards left)

and $\quad \alpha$ = angular retardation of sphere

$$a_1 = a_2 = \frac{\mu mg}{m} = \mu g \quad \Rightarrow \quad \alpha = \frac{\mu mgr}{\frac{2}{5}mr^2} = \frac{5}{2}\frac{\mu g}{r}$$



Let pure rolling starts after time 't'. Then

$$\omega r - v = v$$

$\therefore \quad \omega r = 2v$

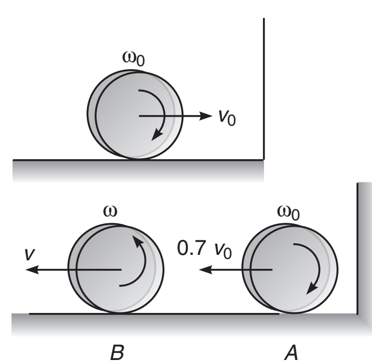
$$(\omega_0 - \alpha t)r = 2(a_1 t)$$

Substituting the values,

$$t = \frac{2}{9}\frac{\omega_0 r}{\mu g}$$

$\therefore \quad s = \frac{1}{2}(a_2)t^2 = \frac{2\omega_0^2 r^2}{81\mu g}$ **Ans.**

19. Between A and B, there is forward slipping. Therefore, friction will be maximum and backwards (rightwards). At point B where $v = R\omega$, ball starts rolling without slipping and force of friction becomes zero.



From conservation of angular momentum between points A and B about bottommost point (because torque of friction about this point is zero)

$$L_A = L_B$$

$\therefore \quad m(0.7v_0)R - I\omega_0 = mvR + I\omega$

Substituting $\omega_0 = \dfrac{v_0}{R}, \omega = \dfrac{v}{R}$

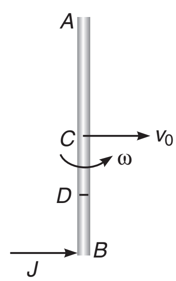
and $\quad I = \dfrac{2}{5}mR^2$, we get

$$v = \frac{3}{14}v_0 = \left(\frac{3}{14}\right)(7)\text{m/s} = 1.5 \text{ m/s} \qquad \textbf{Ans.}$$

20. Let J be the linear impulse applied at B and ω the angular speed of rod.

$$J = mv_0 \qquad \text{...(i)}$$

$$J\left(\frac{l}{2}\right) = \frac{ml^2}{12}.\omega \qquad \text{...(ii)}$$



Solving these two equations,

$$\omega = \frac{6v_0}{l}$$

Linear speed of D (mid-point of CB) relative to C,

$$v = \omega\left(\frac{l}{4}\right) = \frac{3}{2}v_0$$

$\therefore$ Force exerted by upper half on the lower half,

$$F = \frac{\left(\frac{m}{2}\right)v^2}{\left(\frac{l}{4}\right)}$$

Substituting $\quad v = \dfrac{3}{2}v_0$, we get

$$F = \frac{9}{2}\frac{mv_0^2}{l} \qquad \textbf{Ans.}$$

21. $\dfrac{I}{mR^2} = \dfrac{2}{5} = 0.4$ for sphere

$= \dfrac{1}{2} = 0.5$ for disc $\quad$ and $\quad = 1$ for hoop

$$s = \frac{2}{\sin 30°} = 4 \text{ m}$$

For sphere

$$a = \frac{g\sin\theta}{1 + \dfrac{I}{mR^2}} = \frac{9.8 \times \frac{1}{2}}{1 + 0.4} = 3.5 \text{ m/s}$$

$\therefore \quad v = \sqrt{2as} = \sqrt{2 \times 3.5 \times 4} = 5.29 \text{ m/s}$

$$f = \frac{mg\sin\theta}{1 + mR^2/I} = \frac{3 \times 9.8 \times \frac{1}{2}}{1 + \left(\frac{1}{0.4}\right)} = 4.2 \text{ N}$$

$$t = \sqrt{\frac{2s}{a}} = \sqrt{\frac{2 \times 4}{3.5}} = 1.51\text{s}$$

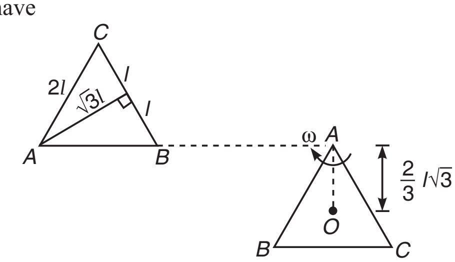
Similarly, the values for disk and hoop can be obtained.

22. $I_A = I_{AB} + I_{AC} + I_{BC}$

$= \dfrac{4}{3}ml^2 + \dfrac{4}{3}ml^2 + \left\{\dfrac{1}{3}ml^2 + m(l\sqrt{3})^2\right\}$

$= 6\,ml^2$

If ω is the angular velocity in the second position, then using conservation of mechanical energy, we have



For COM $\qquad h_i = +\dfrac{\sqrt{3}l}{3}$

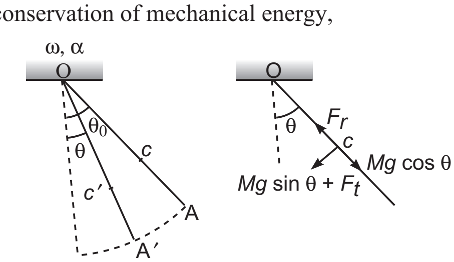
and $\qquad h_f = -\dfrac{2\sqrt{3}}{3}l$

$$3mg\left(\frac{l\sqrt{3}}{3}\right) = \frac{1}{2}(6ml^2)\omega^2 + 3mg\left(\frac{-2l\sqrt{3}}{3}\right)$$

or $\qquad \omega = \sqrt{\dfrac{g\sqrt{3}}{l}}$

Now, velocity of C at this instant is $2l\omega$ or $2\sqrt{gl\sqrt{3}}$ and maximum. **Ans.**

23. (*i*) C is the centre of mass of the rod. Let ω be the angular speed of rod about point O at angle θ. From conservation of mechanical energy,



$$Mg\,\frac{L}{2}\,(\cos\theta - \cos\theta_0) = \frac{1}{2}\left(\frac{ML^2}{3}\right)\omega^2$$

$\therefore$ $$\omega^2 = \frac{3g}{L}\,(\cos\theta - \cos\theta_0) \qquad \ldots(\text{i})$$

Now, $$F_r - Mg\cos\theta = M\left(\frac{L}{2}\right)\omega^2 \qquad \ldots(\text{ii})$$

From Eqs. (i) and (ii), we get

$$F_r = \frac{1}{2}\,Mg\,(5\cos\theta - 3\cos\theta_0)$$ **Hence proved.**

(*ii*) Angular acceleration of rod at this instant,

$$\alpha = \frac{\tau}{I} = \frac{Mg\,\frac{L}{2}\sin\theta}{\frac{ML^2}{3}} = \frac{3}{2}\,\frac{g\sin\theta}{L}$$

Tangential acceleration of COM,

$$a_t = (\alpha)\left(\frac{L}{2}\right) = \frac{3}{4}\,g\sin\theta \quad \ldots(\text{iii})$$

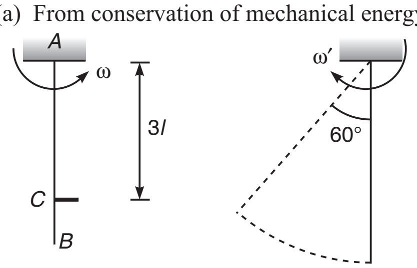
Now, $F_t + Mg\sin\theta = Ma_t$ ...(iv)

From Eqs. (iii) and (iv), we get

$$F_t = -\frac{1}{4}\,Mg\sin\theta$$ **Hence proved.**

Here negative sign implies that direction of F_t is opposite to the component $Mg\sin\theta$.

24. (a) From conservation of mechanical energy.



$$(3m)(g)(2l) = \frac{1}{2}\,I\omega^2$$
$$= \frac{1}{2}\left[\frac{(3m)(4l)^2}{3}\right]\omega^2 = 8ml^2\omega^2$$

$\therefore$ $$\omega = \frac{1}{2}\sqrt{\frac{3g}{l}}$$

Applying,

angular impulse = change in angular momentum

$$J(3l) = I\omega$$

or $$3Jl = (16ml^2)\left(\frac{1}{2}\sqrt{\frac{3g}{l}}\right)$$

$\therefore$ $$J = \frac{8}{3}\,ml\sqrt{\frac{3g}{l}}$$

or $$J = \frac{8}{3}\,m\sqrt{3gl}$$ **Ans.**

(b) Let ω' be the angular speed in opposite direction. Again applying conservation of mechanical energy,

$$(3m)(g)(l) = \frac{1}{2}\,I(\omega')^2 = 8ml^2(\omega')^2$$

$\therefore$ $$\omega' = \frac{1}{2\sqrt{2}}\sqrt{\frac{3g}{l}}$$

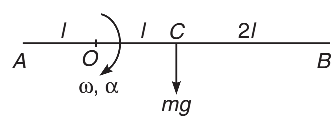
Now, applying, angular impulse = change in angular momentum

$\therefore$ $$J(3l) = I(\omega + \omega') = (16ml^2)\,\frac{1}{2}\sqrt{\frac{3g}{l}}\left(1 + \frac{1}{\sqrt{2}}\right)$$

$\therefore$ $$J = \frac{4}{3}\,m\sqrt{6gl}\,(\sqrt{2} + 1)$$ **Ans.**

25. $$\alpha = \frac{mgl}{\frac{m(4l)^2}{12} + ml^2} = \frac{3}{7}\cdot\frac{g}{l}$$

$\therefore$ $$(a_C)_V = l\alpha = \frac{3}{7}\,g \qquad \text{(downwards)}$$



Let V be the vertical reaction (upwards) at axis, then

$$mg - V = ma_C = \frac{3mg}{7}$$

$\therefore$ $$V = \frac{4}{7}\,mg \qquad \ldots(\text{i})$$

If H be the horizontal reaction (towards CO) at axis, then

$$H = ml\omega^2 \qquad \ldots(\text{ii})$$

$\therefore$ Total reaction at axis,

$$N = \sqrt{H^2 + V^2} = \frac{4}{7}\,mg\sqrt{1 + \left(\frac{7l\omega^2}{4g}\right)^2}$$ **Ans.**

(b) $$a_C = \sqrt{(a_C)_V^2 + (l\omega^2)^2}$$
$$= \sqrt{\left(\frac{3g}{7}\right)^2 + (l\omega^2)^2}$$ **Ans.**

(c) Let ω' be the angular speed of the rod when it becomes vertical for the first time. Then from conservation of mechanical energy,

$$\frac{1}{2}\,I\,(\omega'^2 - \omega^2) = mgl$$

$$\therefore \qquad \omega'^2 = \omega^2 + \frac{2mgl}{I}$$

$$= \omega^2 + \frac{2mgl}{\frac{7}{3}ml^2}$$

$$= \omega^2 + \frac{6}{7}\frac{g}{l}$$

Acceleration of centre of mass at this instance will be,

$$a_C = l\omega'^2 = l\omega^2 + \frac{6g}{7} \qquad \textbf{Ans.}$$

Let V be the reaction (upwards) at axis at this instant, then,

$$V - mg = ma_C = ml\omega^2 + \frac{6mg}{7}$$

$$\therefore \qquad V = \frac{13}{7}mg + ml\omega^2 \qquad \textbf{Ans.}$$

(d) From conservation of mechanical energy,

$$mgl = \frac{1}{2}I\omega_{\min}^2$$

$$\therefore \qquad \omega_{\min} = \sqrt{\frac{2mgl}{I}} = \sqrt{\frac{2mgl}{\frac{7}{3}ml^2}}$$

$$= \sqrt{\frac{6}{7}\frac{g}{l}} \qquad \textbf{Ans.}$$

26. Linear momentum, angular momentum and kinetic energy are conserved in the process.

From conservation of linear momentum,

$$Mv' = mv$$

or $$v' = \frac{m}{M}v \qquad \ldots\text{(i)}$$

Conservation of angular momentum gives,

$$mvd = \left(\frac{Ml^2}{12}\right)\omega$$

or $$\omega = \left(\frac{12mvd}{Ml^2}\right) \qquad \ldots\text{(ii)}$$

Collision is elastic. Hence,

$$e = 1$$

or relative speed of approach

= relative speed of separation

$$\therefore \qquad v = v' + d\omega$$

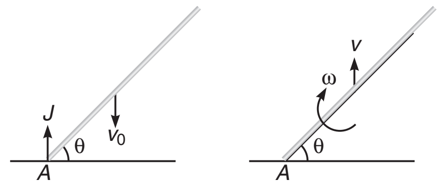
Substituting the values, we have

$$v = \frac{m}{M}v + \frac{12mvd^2}{Ml^2}$$

Solving it, we get

$$m = \frac{Ml^2}{12d^2 + l^2} \qquad \textbf{Ans.}$$

27. Let v = linear velocity of rod after impact (upwards),



At the time of impact After impact

ω = angular velocity of rod

and J = linear impulse at A during impact

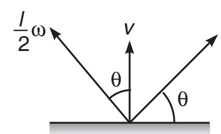
Then, $$J = \Delta P = P_f - P_i$$

$$J = mv - (-mv_0)$$

$$\therefore \qquad J = m(v + v_0) \qquad \ldots\text{(i)}$$

Angular impulse = ΔL

$$\therefore \qquad J\left(\frac{l}{2}\cos\theta\right) = I\omega = \frac{ml^2}{12}\omega \qquad \ldots\text{(ii)}$$



Collision is elastic ($e = 1$)

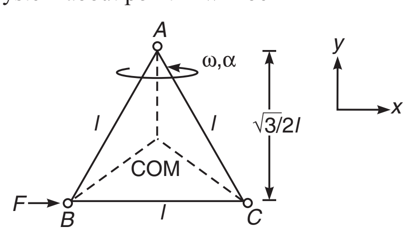
$\therefore$ Relative speed of approach = Relative speed of separation at point of impact

$$v_0 = v + \frac{l}{2}\omega\cos\theta \qquad \ldots\text{(iii)}$$

Solving above equations, we get

$$\omega = \frac{6v_0\cos\theta}{l(1 + 3\cos^2\theta)} \qquad \textbf{Ans.}$$

28. (a) The distance of centre of mass (COM) of the system about point A will be



$$r = \frac{l}{\sqrt{3}}$$

Therefore, the magnitude of horizontal force exerted by the hinge on the body is

$$F = \text{centripetal force}$$

or $$F = (3m)r\omega^2$$

or $\qquad F = (3m)\left(\dfrac{l}{\sqrt{3}}\right)\omega^2$

or $\qquad F = \sqrt{3}\, ml\omega^2$

(b) Angular acceleration of system about point A is

$$\alpha = \frac{\tau_A}{I_A} = \frac{(F)\left(\dfrac{\sqrt{3}}{2}l\right)}{2ml^2} = \frac{\sqrt{3}\,F}{4ml}$$

Now, acceleration of COM along x-axis is

$$a_x = r\alpha = \left(\frac{l}{\sqrt{3}}\right)\left(\frac{\sqrt{3}\,F}{4ml}\right) \text{ or } a_x = \frac{F}{4m}$$

Now, let F_x be the force applied by the hinge along x-axis.

Then, $\qquad F_x + F = (3m)a_x$

or $\qquad F_x + F = (3m)\left(\dfrac{F}{4m}\right)$

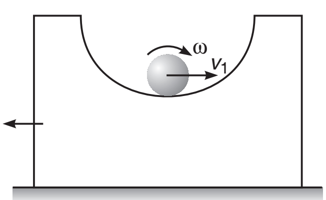
or $\qquad F_x + F = \dfrac{3}{4}F$ or $F_x = -\dfrac{F}{4}$

Further if F_y be the force applied by the hinge along y-axis. Then,

$\qquad F_y$ = centripetal force

or $\qquad F_y = \sqrt{3}\, ml\omega^2$

29. From conservation of linear momentum



$$mv_1 = Mv_2 \qquad \ldots\text{(i)}$$

$\therefore$ Velocity of cylinder axis relative to block

$$v_r = v_1 + v_2 \qquad \ldots\text{(ii)}$$

Applying conservation of mechanical energy,

$$mg(R - r) = \frac{1}{2}mv_1^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}Mv_2^2 \qquad \ldots\text{(iii)}$$

Here, $\quad I = \dfrac{1}{2}mr^2 \quad$ and $\quad \omega = \dfrac{v_r}{r}$

Solving the above equations with given data, we get

$$v_1 = 2.0 \text{ m/s}$$

and $\qquad v_2 = 1.5$ m/s **Ans.**

Further, $\qquad N - mg = \dfrac{mv_r^2}{R - r}$

$$\therefore N = mg + \frac{mv_r^2}{R-r} = (0.5)(10) + \frac{(0.5)(3.5)^2}{0.525} = 16.67 \text{ N}$$ **Ans.**

30. Given $\mu > \tan\alpha \Rightarrow \mu mg\cos\alpha > mg\sin\alpha$

$$a = (\mu g\cos\alpha - g\sin\alpha)$$

$$\alpha = \frac{(\mu mg\cos\alpha)\,r}{\frac{1}{2}mr^2} = \frac{2\mu g\cos\alpha}{r}$$

Slipping will stop when,

$$v = r\omega$$

or $\qquad at = r(\omega_0 - \alpha t)$

$$\therefore \quad t = \frac{r\omega_0}{a + r\alpha} = \left(\frac{r\omega_0}{3\mu g\cos\alpha - g\sin\alpha}\right)$$

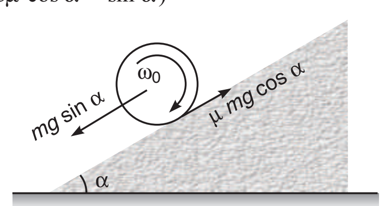
$$d_1 = \frac{1}{2}at^2$$

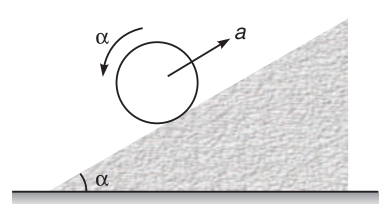
$$= \frac{1}{2}(\mu g\cos\alpha - g\sin\alpha)\left(\frac{r\omega_0}{3\mu g\cos\alpha - g\sin\alpha}\right)^2$$

$$= \frac{r^2\omega_0^2(\mu\cos\alpha - \sin\alpha)}{2g(3\mu\cos\alpha - \sin\alpha)^2}$$ **Ans.**

$$v = at = (\mu g\cos\alpha - g\sin\alpha)\left(\frac{r\omega_0}{3\mu g\cos\alpha - g\sin\alpha}\right)$$

$$= \frac{r\omega_0(\mu\cos\alpha - \sin\alpha)}{(3\mu\cos\alpha - \sin\alpha)}$$





Once slipping is stopped, retardation in cylinder,

$$a' = \frac{g\sin\alpha}{1 + \dfrac{I}{mr^2}} = \frac{g\sin\alpha}{1 + \dfrac{1}{2}} = \frac{2}{3}g\sin\alpha$$

$$d_2 = \frac{v^2}{2a'} = \frac{3r^2\omega_0^2 (\mu \cos \alpha - \sin \alpha)^2}{(3\mu \cos \alpha - \sin \alpha)^2 (4g \sin \alpha)}$$

$$\therefore \quad d_{max} = d_1 + d_2$$

$$= \frac{r^2\omega_0^2 (\mu \cos \alpha - \sin \alpha)}{2g\ (3\mu \cos \alpha - \sin \alpha)^2} \left[1 + \frac{3\ (\mu \cos \alpha - \sin \alpha)}{2 \sin \alpha}\right]$$

$$= \frac{r^2\omega_0^2 (\mu \cos \alpha - \sin \alpha)}{4g \sin \alpha\ (3\mu \cos \alpha - \sin \alpha)}$$ **Ans.**

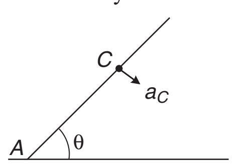
Note *Once slipping was stopped, pure rolling continues if*

$$\mu > \frac{\tan \alpha}{1 + \frac{mr^2}{I}}$$

or $\quad \mu > \frac{\tan \alpha}{1 + 2} \quad$ or $\quad \mu > \frac{\tan \alpha}{3}$

and already in the question it is given that $\mu > \tan \alpha$. That's why we have taken $a' = \frac{2}{3} g \sin \alpha$.

31. Point A is momentarily at rest.



$$\alpha = \frac{mg \frac{l}{2} \cos \theta}{\frac{ml^2}{3}} = \frac{3}{2} \frac{g \cos \theta}{l}$$

$$\therefore \quad a_C = \frac{l}{2} \alpha = \frac{3}{4} g \cos \theta$$

Now $\quad \mu N = ma_x$

or $\quad \mu N = ma_C \sin \theta$

or $\quad \mu N = \frac{3}{4} mg \sin \theta \cos \theta \quad$ ...(i)

Further, $\quad mg - N = ma_y$

or $\quad N = mg - ma_C \cos \theta$

or $\quad N = mg - \frac{3}{4} mg \cos^2 \theta \quad$ ...(ii)

Dividing Eq. (i) by Eqs. (ii), we have

$$\mu = \frac{\frac{3}{4} \sin \theta \cos \theta}{1 - \frac{3}{4} \cos^2 \theta} = \frac{3 \sin \theta \cos \theta}{4 - 3 \cos^2 \theta}$$

$$= \frac{3 \sin \theta \cos \theta}{1 + 3 \sin^2 \theta}$$ **Hence proved.**

32. **Figure (a) and (b)**

ω : Decrease in gravitational potential energy = increase in rotational kinetic energy

$$\therefore \quad mg \frac{l}{4} \sin \theta = \frac{1}{2} I_0 \omega^2$$

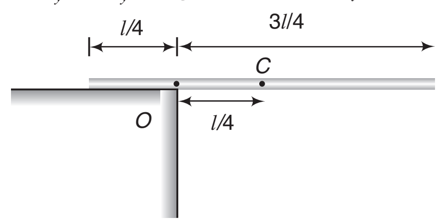
$$= \frac{1}{2} \left[\frac{ml^2}{12} + m\left(\frac{l}{4}\right)^2\right] \omega^2$$

$$\therefore \quad \omega = \sqrt{\left[\frac{24g \sin \theta}{7l}\right]} \quad \text{...(i)}$$

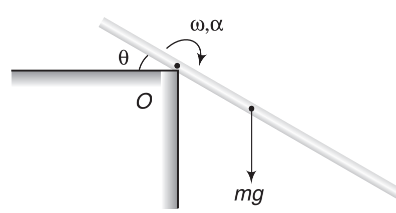
$$\alpha : \quad \alpha = \frac{\tau}{I} = \frac{mg \frac{l}{4} \cos \theta}{\left[\frac{ml^2}{12} + m\left(\frac{l}{4}\right)^2\right]}$$

$$= \frac{12\ g \cos \theta}{7l} \quad \text{...(ii)}$$

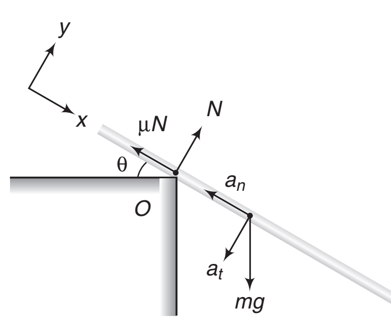
$\Sigma F_y = ma_y$ or $mg \cos \theta - N = ma_t$



(a)



(b)



(c)

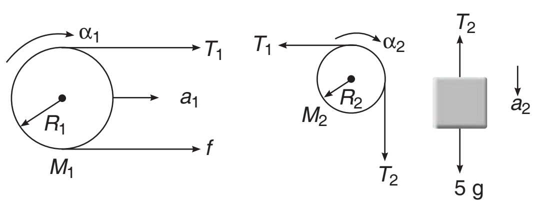
or $$N = mg\cos\theta - ma_t$$
$$= mg\cos\theta - m\frac{l}{4}\alpha$$

Substituting value of α from Eq. (ii), we get
$$N = \frac{4}{7}mg\cos\theta \qquad \ldots\text{(iii)}$$

Rod begins to slip when
$$\mu N - mg\sin\theta = ma_n$$
or $$\frac{4}{7}\mu mg\cos\theta - mg\sin\theta = m\frac{l}{4}\omega^2$$

Substitution value of ω from Eq. (i), we get
$$\tan\theta = \frac{4\mu}{13}$$
$\therefore$ $$\theta = \tan^{-1}\left(\frac{4\mu}{13}\right) \qquad \textbf{Ans.}$$

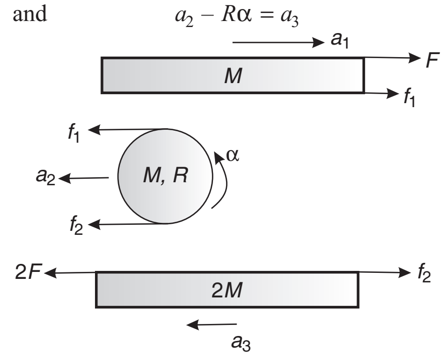
33. Writing equations of motion, we get
$$5g - T_2 = 5a_2 \qquad \ldots\text{(i)}$$
$$\frac{(T_2 - T_1)R_2}{\frac{1}{2}M_2R_2^2} = \alpha_2 \qquad \ldots\text{(ii)}$$
$$T_1 + f = M_1a_1 \qquad \ldots\text{(iii)}$$
$$\frac{(T_1 - f)R_1}{\frac{1}{2}M_1R_1^2} = \alpha_1 \qquad \ldots\text{(iv)}$$
$$a_1 = R_1\alpha_1 \qquad \ldots\text{(v)}$$
$$a_1 + R_1\alpha_1 = R_2\alpha_2 \qquad \ldots\text{(vi)}$$
$$R_2\alpha_2 = a_2 \qquad \ldots\text{(vii)}$$



We have seven unknowns, $T_1, T_2, a_1, a_2, \alpha_1, \alpha_2$ and f solving above equations, we get
$$a_2 = \frac{4}{11}g = 3.6 \text{ m/s}^2 \qquad \textbf{Ans.}$$
$$v = a_2t = \frac{4gt}{11} \qquad \textbf{Ans.}$$

34. Equations of motion are,
$$F + f_1 = Ma_1 \qquad \ldots\text{(i)}$$
$$f_1 + f_2 = Ma_2 \qquad \ldots\text{(ii)}$$
$$2F - f_2 = 2Ma_3 \qquad \ldots\text{(iii)}$$
$$\alpha = \frac{(f_1 - f_2)R}{\frac{1}{2}MR^2}$$
or $$\alpha = \frac{2(f_1 - f_2)}{MR} \qquad \ldots\text{(iv)}$$

For no slipping condition,
$$a_2 + R\alpha = -a_1 \qquad \ldots\text{(v)}$$
and $$a_2 - R\alpha = a_3 \qquad \ldots\text{(vi)}$$



We have six unknowns, f_1, f_2, a_1, a_2, a_3 and α. Solving the above six equations, we get
$$a_1 = \frac{21F}{26M} \quad \text{and} \quad a_2 = \frac{F}{26M} \qquad \textbf{Ans.}$$

35. **Angular velocity** From conservation of mechanical energy,

decrease in gravitational PE

= increase in rotational KE

or $mgr\sin 60° + mg\,(2r\sin 60°)$
$$= \frac{1}{2}\left[\frac{3mr^2}{2} + m\,(2r)^2\right]\omega^2$$
$\therefore$ $$\frac{3\sqrt{3}mgr}{2} = \frac{11}{4}mr^2\omega^2$$
$\therefore$ $$\omega = \sqrt{\frac{6g\sqrt{3}}{11r}} \qquad \textbf{Ans.}$$

Angular acceleration
$$\alpha = \frac{\tau}{I} = \frac{mgr\cos 60° + mg\,(2r\cos 60°)}{\left[\frac{3mr^2}{2} + m\,(2r)^2\right]}$$
$$= \frac{\frac{3mgr}{2}}{\frac{11}{2}mr^2} = \frac{3g}{11r} \qquad \textbf{Ans.}$$

36. $\mathbf{a}_B = \mathbf{a}_0 + \mathbf{a}_{B/0}$

Here, $\mathbf{a}_{B/0}$ has two components a_t (tangential acceleration) and a_n (normal acceleration)

$$a_t = r\alpha = (0.3)(5) = 1.5 \text{ m/s}^2$$

$$a_n = r\omega^2 = (0.3)(4)^2 = 4.8 \text{ m/s}^2$$

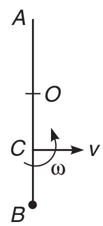
and $\quad a_0 = 2 \text{ m/s}^2$

$\therefore \quad a_B = \sqrt{(\Sigma a_x)^2 + (\Sigma a_y)^2}$

$$= \sqrt{(2 + 4.8 \cos 45° - 1.5 \cos 45°)^2 + (4.8 \sin 45° + 1.5 \sin 45°)^2}$$

$$= 6.21 \text{ m/s}^2$$

37. C is the COM of $(M + m)$



$$BC = \left(\frac{M}{M+m}\right)\left(\frac{l}{2}\right)$$

and $\quad OC = \left(\frac{m}{M+m}\right)\left(\frac{l}{2}\right)$

From conservation of linear momentum,

$$(M + m)\, v = mv_0$$

or $\quad v = \left(\frac{m}{M+m}\right) v_0 \qquad \ldots\text{(i)}$

From conservation of angular momentum about point C we have,

$$mv_0(BC) = I\omega$$

or $\quad \dfrac{mMv_0 l}{2\,(M+m)} = \left[m\left(\dfrac{M}{M+m}\right)^2\left(\dfrac{l}{4}\right)^2 + \dfrac{Ml^2}{12} + M\left(\dfrac{m}{M+m}\right)^2\left(\dfrac{l^2}{4}\right)\right]\omega$

Putting $\quad \dfrac{mv_0}{M+m} = v$

from Eq. (i), we have

$$\frac{v}{\omega} = \frac{l}{6}\left[\frac{4m + M}{M + m}\right]$$

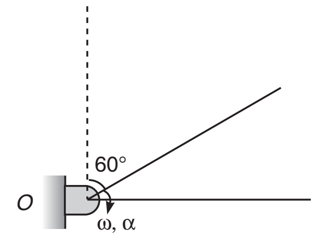
Now, a point (say P) at a distance $x = \dfrac{v}{\omega}$, from C (towards O) will be at rest. Hence, distance of point P from boy at B will be

$BP = BC + x$

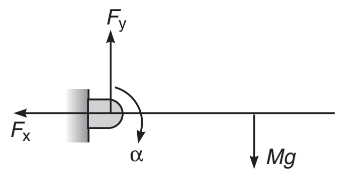
$$= \left(\frac{M}{M+m}\right)\left(\frac{l}{2}\right) + \frac{l}{6}\left[\frac{4m + M}{M + m}\right]$$

$$= \frac{2l}{3} \qquad \textbf{Ans.}$$

38. Let ω be the angular velocity and α the angular acceleration of rod in horizontal position. Then



$$\alpha = \frac{(Mg)\dfrac{l}{2}}{\dfrac{Ml^2}{3}} = \frac{3}{2}\frac{g}{l} \qquad \ldots\text{(i)}$$



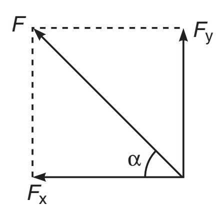
$$\frac{1}{2}\left(\frac{Ml^2}{3}\right)\omega^2 = Mg\,\frac{l}{4}$$

$\therefore \quad \omega = \dfrac{3}{2}\cdot\dfrac{g}{l} \qquad \ldots\text{(ii)}$

$$F_x = M\left(\frac{l}{2}\right)\omega^2 = M\left(\frac{l}{2}\right)\left(\frac{3g}{2l}\right) = \frac{3}{4}Mg$$

$$Mg - F_y = M(\alpha)\left(\frac{l}{2}\right)$$

$$F_y = Mg - \frac{3}{4}Mg = \frac{Mg}{4}$$



$\therefore \quad F = \sqrt{F_x^2 + F_y^2}$

$$= \frac{\sqrt{10}}{4}Mg \qquad \textbf{Ans.}$$

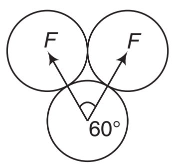
$$\tan\alpha = \frac{F_y}{F_x} = \frac{(Mg/4)}{(3Mg/4)} = \frac{1}{3}$$

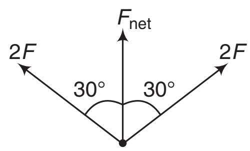
$\therefore \quad \alpha = \tan^{-1}\left(\dfrac{1}{3}\right) \qquad$ **Ans.**

13. Gravitation

INTRODUCTORY EXERCISE 13.1

1. $F_{net} = \sqrt{F^2 + F^2 + 2FF \cos 60°}$

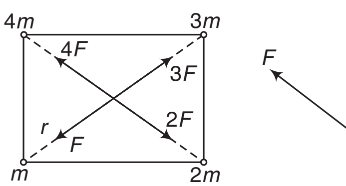


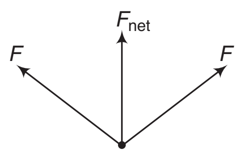


$$= \sqrt{3}\, F$$

$$= \sqrt{3}\, \frac{GMM}{(2a)^2} = \frac{\sqrt{3}\, GM^2}{4a^2}$$

2. $F_{net} = \sqrt{2}\,(2F)$





$$= 2\sqrt{2}\, \frac{G.m.m}{r^2}$$

$$= \frac{2\sqrt{2}\, Gm^2}{(a/\sqrt{2})^2} = \frac{4\sqrt{2}\, Gm^2}{a^2}$$

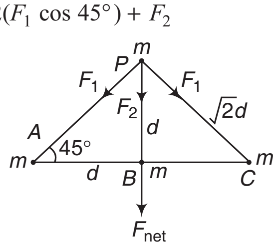
3. $a_1 = \frac{F}{m_1} = \frac{Gm_1m_2}{r^2} = \frac{Gm_2}{r^2}$

$$= \frac{(6.67 \times 10^{-11})\,(2)}{(0.5)^2}$$

$$= 5.3 \times 10^{-10}\ \text{m/s}^2$$

Similarly, $a_2 = \frac{Gm_1}{r^2}$

4. $F_{net} = 2(F_1 \cos 45°) + F_2$



$$= \sqrt{2}\, F_1 + F_2 = \frac{(\sqrt{2})\, Gmm}{(\sqrt{2}d)^2} + \frac{Gmm}{d^2}$$

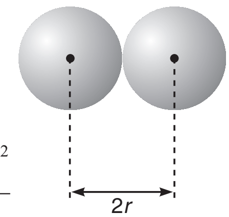
$$= \left(\frac{\sqrt{2} + 1}{\sqrt{2}}\right) \frac{Gm^2}{d^2} \quad \text{(Along } PB\text{)}$$

5. $m_1 = m_2 = \text{(volume)(density)}$

$$= \left(\frac{4}{3}\,\pi r^3\right) \rho$$

$$\therefore \quad F = \frac{Gm_1m_2}{r^2}$$

$$= \frac{G\left(\frac{4}{3}\,\pi r^3\right)\left(\frac{4}{3}\,\pi r^3\right)\rho^2}{r^2}$$



or $F \propto r^4$ **Hence proved.**

INTRODUCTORY EXERCISE 13.2

1. $g = \frac{GM}{R^2}$ or $g \propto \frac{M}{R^2}$

Mass and radius both are two times. Therefore, value of g is half.

2. (a) $g' = \frac{g}{\left(1 + \frac{h}{R}\right)^2}$

At $h = R,\ g' = \frac{g}{4}$

(d) $g' = g\left(1 - \frac{d}{R}\right)$

At $d = \frac{R}{2},\ g' = \frac{g}{2}$

3. $\frac{g}{\left(1 + \frac{h}{R}\right)^2} = g\left(1 - \frac{h}{R}\right)$

Solving this equation, we get

$$h = \left(\frac{\sqrt{5} - 1}{2}\right) R$$

4. $\Delta g = g' - g$

$$= -R\omega^2 \cos^2\theta$$

$$= -(6.37 \times 10^6)\left(\frac{2\pi}{24 \times 3600}\right)^2 \cos^2 45°$$

$$= -0.0168\ \text{m/s}^2$$

5. $g' = 0.64\, g = \frac{g}{\left(1 + \frac{h}{R}\right)^2}$

$$\therefore \quad 1 + \frac{h}{R} = \frac{5}{4}$$

$$h = \frac{R}{4} = 1600\ \text{km}$$

6. $g' = \frac{3}{5} g = g - R\omega^2 \cos^2 0°$

$\therefore \quad R\omega^2 = \frac{2}{5} g$

or $\quad \omega = \sqrt{\frac{2g}{5R}} = \sqrt{\frac{2 \times 9.8}{5 \times 6400 \times 10^3}}$

$= 7.8 \times 10^{-4}$ rad/s

7. At equator, $g' = g - R\omega^2$

$$= (9.8)^2 - (6400 \times 10^3)\left(\frac{2\pi}{24 \times 3600}\right)^2$$

$$= 9.766 \text{ m/s}^2$$

Now, $mg = 1000$

$\therefore \quad m = \frac{1000}{g}$

$$w' = mg' = (1000)\left(\frac{g'}{g}\right)$$

$$= \frac{(1000)\,(9.766)}{(9.8)} = 997 \text{ N}$$

8. $g' = g - R\omega^2$

$\therefore \quad 0 = g - R\omega^2$

$\therefore \quad \omega = \sqrt{\frac{g}{R}} = \sqrt{\frac{9.8}{6400 \times 1000}}$

$= 1.237 \times 10^{-3}$ rad/s and $T = \frac{2\pi}{\omega}$

By putting the value of ω, we get

$T \approx 84.6$ min

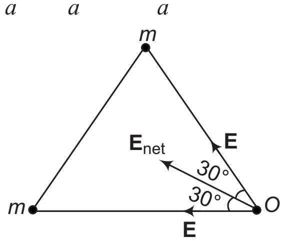
9. $\frac{g}{\left(1 + \frac{h}{R}\right)} = g - R\omega^2$

By putting the values of g, h, R and ω, we get

$h \approx 10$ km

INTRODUCTORY EXERCISE 13.3

1. $V = -\frac{Gm}{a} - \frac{Gm}{a} = -\frac{2Gm}{a}$



$$E = \frac{Gm}{a^2}$$

$$E_{net} = \sqrt{E^2 + E^2 + 2(E)(E)\cos 60°} = \sqrt{3}E$$

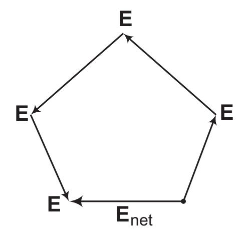
$$= \frac{\sqrt{3}Gm}{a^2}$$

2. $V = 5\left(\frac{-Gm}{a}\right) = -\frac{5Gm}{a}$

For E Five vectors of equal magnitudes, when added as per polygon law of vector addition make a closed regular pentagon. Hence, net **E** is zero.

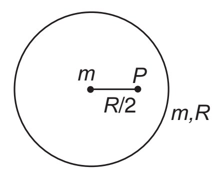
3. $V = 4\left(-\frac{Gm}{a}\right) = -4\frac{Gm}{a}$

For **E**



$$\mathbf{E}_{net} = \mathbf{E} = \frac{Gm}{a^2}$$

4. V_p = due to particle + due to shell



$$= -\frac{Gm}{R/2} - \frac{Gm}{R}$$

$$= -\frac{3Gm}{R}$$

5. $\mathbf{E} = \frac{\mathbf{F}}{m} = \frac{4\hat{\mathbf{i}}}{20 \times 10^{-3}} = (200\,\hat{\mathbf{i}})$ N/kg

INTRODUCTORY EXERCISE 13.4

1. $\mathbf{E} = -\left[\frac{\partial V}{\partial X}\hat{\mathbf{i}} + \frac{\partial V}{\partial y}\hat{\mathbf{j}} + \frac{\partial V}{\partial z}\hat{\mathbf{k}}\right]$

2. $|\mathbf{E}| = \sqrt{\left(-\frac{\partial V}{\partial x}\right)^2 + \left(-\frac{\partial V}{\partial y}\right)^2 + \left(\frac{-\partial V}{\partial z}\right)^2}$

$-\frac{\partial V}{\partial x} = 10$ J/kg-m is given

No information is given about

$-\frac{\partial V}{\partial y}$ and $-\frac{\partial V}{\partial z}$

So, $\quad |\mathbf{E}| \geq 10$ N/kg

3. $\mathbf{F} = m\mathbf{E} = m\left[\left(-\frac{\partial V}{\partial x}\right)\hat{\mathbf{i}} + \left(-\frac{\partial V}{\partial y}\right)\hat{\mathbf{j}}\right]$

$= -(10\,\hat{\mathbf{i}} + 10\,\hat{\mathbf{j}})$ N or $|\mathbf{F}| = 10\sqrt{2}$ N

4. $\mathbf{F} = m\mathbf{E} = (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}})$ N/kg

We can check that path given (therefore displacement) is perpendicular to force.

$\therefore \quad W = 0$

INTRODUCTORY EXERCISE 13.5

1. $K_i + U_i = K_f + U_f$

$$\therefore \quad 0 - \frac{Gm_1m_2}{r_i} = \frac{1}{2} \times 10 \times (2v)^2 + \frac{1}{2} \times 20\, v^2 - \frac{Gm_1m_2}{r_f}$$

$$\therefore \quad v = \sqrt{\frac{Gm_1m_2}{30}\left(\frac{1}{r_f} - \frac{1}{r_i}\right)}$$

$$= \sqrt{\frac{6.67 \times 10^{-11} \times 10 \times 20}{30}\left(\frac{1}{0.5} - \frac{1}{1}\right)}$$

$$= 2.1 \times 10^{-5} \text{ m/s}$$

and $\quad 2v = 4.2 \times 10^{-5}$ m/s

2. Total pairs $= \frac{4(4-1)}{2} = 6$. Four pairs are at a distance of a and two pairs at distance of $\sqrt{2}a$.

3. $\Delta U = \dfrac{mgh}{1 + \dfrac{h}{R}}$

Put $\quad h = R$

4. *(i)* $\Delta U = \dfrac{mgh}{1 + \dfrac{h}{R}}$

Putting $h = nR$, we get

$$\Delta U = \left(\frac{n}{n+1}\right) mgR$$

(ii) $\Delta U = \left(\dfrac{n}{n+1}\right) mgR$

$$= \frac{1}{2} mv^2$$

$$\therefore \quad v = \sqrt{\frac{2ngR}{n+1}}$$

5. $h = \dfrac{v^2}{2g - (v^2/R)}$

$$= \frac{(10^4)^2}{2 \times 9.8 - \dfrac{(10^4)^2}{6.4 \times 10^6}}$$

$$= 2.51 \times 10^7 \text{ m}$$

$$= 2.51 \times 10^4 \text{ km}$$

6. Apply $\dfrac{1}{2}mv^2 = \dfrac{mgh}{1 + \dfrac{h}{R}}$

INTRODUCTORY EXERCISE 13.6

1. $\text{KE} = \dfrac{1}{2} mv_e^2$

$$= \frac{1}{2} m \left(\frac{2GM}{R}\right)$$

$$= \frac{m}{R}(gR^2) \qquad (\text{as } GM = gR^2)$$

$$= mgR$$

2. $v_e = \sqrt{\dfrac{2GM}{R}}$

or $\quad v_e \propto \sqrt{\dfrac{M}{R}}$

3. (a) Total mechanical energy $= E_0 - 2E_0 = -E_0$.

Since, it is negative, it will not escape to infinity.

(b) $E_i = E_f \Rightarrow E_0 - 2E_0 = 0 + U \Rightarrow U = -E_0$

INTRODUCTORY EXERCISE 13.7

1. Delhi does not lie on equator.

2. $T \propto r^{3/2}$

$$\therefore \quad \left(\frac{T_2}{\pi}\right) = \left(\frac{r_2}{r_1}\right)^{3/2}$$

or $\quad T_2 = \left(\dfrac{r_2}{r_1}\right)^{3/2} T_1$

$$= \left(\frac{2 \times 10^4}{10^4}\right)^{3/2} (28) = 56\sqrt{2} \text{ h}$$

3. $r_1 = R + h_1 = R + R = 2R$

$r_2 = R + h_2 = R + 3R = 4R$

KE and PE $\propto \dfrac{1}{r}$

$$\therefore \quad \frac{K_1}{K_2} = \frac{U_1}{U_2} = \frac{r_2}{r_1} = \frac{2}{1}$$

4. (a) $v = \sqrt{\dfrac{GM}{r}} = \sqrt{\dfrac{GM}{R+h}}$

(b) $\text{KE} = \dfrac{GMm}{2r} = \dfrac{GMm}{2(R+h)}$

(c) $\text{PE} = -\dfrac{GMm}{r} = -\dfrac{GMm}{(R+h)}$

(d) $T = \dfrac{2\pi}{\sqrt{GM}} r^{3/2} = \dfrac{2\pi}{\sqrt{GM}} (R+h)^{3/2}$

5. (a) W = Energy of satellite in first orbit – energy of satellite on the surface of earth

$$= -\frac{GMm}{2\,(2R)} - \left(-\frac{GMm}{R}\right)$$

$$= \frac{3}{4}\frac{GMm}{R}$$

$$= \frac{3}{4}\,mgR = \frac{3}{4} \times 2 \times 10^3 \times 10 \times 6.4 \times 10^6$$

$$= 9.6 \times 10^{10}\ \text{J}$$

(b) W = energy of satellite in second orbit – energy in first orbit

$$= -\frac{GMm}{2\,(3R)} - \left[\frac{-GMm}{2\,(2R)}\right]$$

$$= \frac{1}{12}\frac{GMm}{R} = \frac{1}{12}\,mgR$$

$$= \frac{1}{12} \times 2 \times 10^3 \times 10 \times 6.4 \times 10^6$$

$$= 1.07 \times 10^{10}\ \text{J}$$

Exercises

LEVEL 1

Assertion and Reason

1. $U = -\dfrac{G\,m_1 m_2}{r}$

If r decreases, U also decreases.

2.
$$V_C = -1.5\,\frac{GM}{R}$$
$$V_S = -\frac{GM}{R}$$
$$E_C = 0 \quad \text{and} \quad E_S = \frac{GM}{R^2}$$

$C \to$ centre, $S \to$ surface.

3. If a mass m is displaced from centre along the line AB, force on it is away from the centre. So, it is in unstable equilibrium position at centre. So, potential energy and hence the potential at centre is maximum.

4. $\mathbf{E} = -\left[\dfrac{\partial V}{\partial x}\hat{\mathbf{i}} + \dfrac{\partial V}{\partial y}\hat{\mathbf{j}} + \dfrac{\partial V}{\partial z}\hat{\mathbf{k}}\right]$

$\dfrac{\partial V}{\partial X} = 0$ does not mean field strength is zero. Because it also depends on $\dfrac{\partial V}{\partial y}$ and $\dfrac{\partial V}{\partial z}$.

5. $h = \dfrac{u^2}{2g}$ or $h \propto u^2$ cannot be applied in this case. Because, for higher values of v, acceleration due to gravity g does not remain constant.

6. Force on planet is towards centre of sum. Hence, torque is zero only about centre of sun.

7. Polar satellites don't have equatorial plane.

8. Earth's gravity is utilised in providing the necessary centripetal force. But weight is felt due to moon's weight is felt due to moon's gravity.

9. Geostationary satellites lies about equator and Moscow does not lies over equator.

10. $V = -\dfrac{GMm}{r}$, $K = \dfrac{GMm}{2r}$, $E = -\dfrac{GMm}{2r}$

and $\qquad V = \sqrt{\dfrac{GM}{r}}$

11. By changing the radius, moment of inertia will change. Hence, angular speed ω will change. But ω has no effect on the value of g on pole.

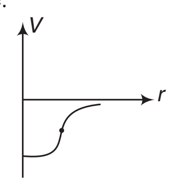
Single Correct Option

2. Angular momentum is conserved only about centre of sun.

3. $g' = g - R\omega^2 \cos^2\phi$

At $\phi = 90°$, $g' = g$, independent of ω.

5. V is negative.



6. Comparing with

$$\oint \mathbf{E}\cdot d\mathbf{S} = \left(\frac{q}{\varepsilon_0}\right)$$

We have, $\qquad \dfrac{1}{4\pi\varepsilon_0} \equiv G$

$\Rightarrow \qquad \dfrac{1}{\varepsilon_0} \equiv 4\pi G$

$\Rightarrow \qquad q \equiv m$

8. $E = 0$ inside a sheet. Therefore, gravitational force on m is zero.

9. $T \propto r^{3/2}$

10. Area velocity remains constant. If area is half time taken is half.

11. $T \propto \dfrac{1}{\sqrt{g}}$

$$\frac{T_2}{T_1} = \sqrt{\frac{g_1}{g_2}} \quad \text{or} \quad \frac{1.05\,T}{T} = \sqrt{\frac{g}{g\left(1 - \dfrac{h}{R}\right)}}$$

Solving this equation we get, $h = 64$ km

12. $g = \dfrac{GM}{R^2} \Rightarrow \therefore \dfrac{g}{G} = \dfrac{M}{R^2}$

13. $E \propto \dfrac{1}{r^2}$

$$\frac{E_2}{E_1} = \left(\frac{r_1}{r_2}\right)^2$$

or $$\frac{1}{100} = \left(\frac{R_e}{R_e + h}\right)^2$$

or $$h = 9\,R_e$$

14. $T \propto r^{3/2}$

$$\frac{T_2}{T_1} = \left(\frac{r_2}{r_1}\right)^{3/2} = \left(\frac{3R + R}{R}\right)^{3/2} = 8$$

$\therefore$ $$T_2 = 8T_1$$

15. $g_{\text{equator}} = g - R\omega^2$ and $g_{\text{pole}} - R\omega^2$

$\therefore$ $$\frac{g_{\text{pole}}}{2} \quad \text{or} \quad \frac{g}{2} = g - R\omega^2$$

or $$\omega \sqrt{\frac{g}{2R}}$$

$$= \sqrt{\frac{9.8}{2 \times 6.4 \times 10^6}}$$

$$= 8.75 \times 10^{-4} \text{ rad/s}$$

16. $U = mV$

or $$U \propto V$$

U is half, it means gravitational potential is half.

$$-\frac{GM}{(R + h)} = \frac{1}{2}\left[-\frac{3}{2}\frac{GM}{R}\right]$$

Solving we get, $h = \dfrac{R}{3}$ **Ans.**

17. $\Delta U = \dfrac{mgh}{1 + \dfrac{h}{R}}$

18. $T = \sqrt{\dfrac{GM}{r}}$

T is independent of R, the radius of earth.

19. $g = \dfrac{GM}{R^2} = \dfrac{G\left(\dfrac{4}{3}\pi R^3 \rho\right)}{R^2}$

or $$g \propto \rho R$$

or $$g_1 = g_2$$

$\therefore$ $$\rho_1 R_1 = \rho_2 R_2$$

or $$R_2 = \frac{\rho_1}{\rho_2} \cdot R_1 = \left(\frac{1}{2}\right) R = \frac{R}{2}$$ **Ans**

20. At 84.6 min, g at equator becomes zero.

$$\frac{\omega_2}{\omega_1} = \frac{2\pi/T_2}{2\pi/T_1} = \frac{T_1}{T_2}$$

$$= \frac{24 \times 60}{84.6} = 17$$ **Ans.**

21. $\omega_1\, t + \omega_2 t = 2\pi$

$$(\omega_1)\,6 + \left(\frac{2\pi}{23}\right)^6 = 2\pi$$

or $$6\omega_1 = \frac{3\pi}{2} \quad \text{or} \quad \omega_1 = \left(\frac{\pi}{4}\right) \text{rad/h}$$

23. Only due to M_1.

24. Time taken in one complete oscillations is 84.6 min.

25. $W = \Delta U = U_f - V_i$

$$= 3\left(-\frac{GMM}{2l}\right) - 3\left(-\frac{GMM}{l}\right)$$

$$= \frac{3}{2}\frac{GM^2}{l}$$

26. Change in kinetic energy = change in potential energy

$\therefore$ $$\frac{1}{2}mv^2 = -\frac{GMm}{R + R} - \left(-\frac{GMm}{R}\right)$$

$$v = \sqrt{\frac{GM}{R}} = \frac{\sqrt{2GM/R}}{\sqrt{2}}$$

$$= \frac{v_e}{\sqrt{2}}$$

where, v_e = escape velocity

27. $U = \dfrac{-GMm}{r}$

$\therefore$ $$(-GM) = \frac{Ur}{m}$$

$$E = -\frac{GM}{(2r)^2} = \frac{(Ur/m)}{4r^2} = \frac{U}{4mr}$$

$$F = mE = \frac{U}{4r}$$

Subjective Questions

1. Potential at the surface of sphere,

$$V = -\frac{GM}{R}$$
$$= -\frac{(6.67 \times 10^{-11})(20)}{1} \text{ J/kg}$$
$$= -1.334 \times 10^{-9} \text{ J/kg}$$

i.e. 1.334×10^{-9} J work is obtained to bring a mass of 1 kg from infinity to the surface of sphere. Hence, the same amount of work will have to be done to take the particle away from the surface of sphere. Thus,

$$W = 1.334 \times 10^{-9} \text{ J}$$

2. $g = \dfrac{GM}{R^2}$

$$\Rightarrow \quad \frac{dg}{dR} = \frac{-2GM}{R^3}$$
$$\Rightarrow \quad \frac{dg}{h} = \frac{-2GM}{R^2} \cdot \frac{1}{R}$$
$$\Rightarrow \quad \frac{dg}{g} = -2\left(\frac{h}{R}\right)$$

3. Let v_r be their velocity of approach. From conservation of energy,

Increase in kinetic energy

= decrease in gravitational potential energy

$$\text{or} \quad \frac{1}{2}\mu v_r^2 = \frac{Gm_1m_2}{r} \quad \ldots\text{(i)}$$

Here, μ = reduced mass

$$= \frac{m_1m_2}{m_1 + m_2}$$

Substituting in Eq. (i), we get

$$v_r = \sqrt{\frac{2G(m_1 + m_2)}{r}}$$

4. Close to earth,

$$T = 2\pi\sqrt{\frac{R}{g}} = 2\pi\sqrt{\frac{R}{GM/R^2}}$$

$$\text{or} \quad T^2 = \frac{4\pi^2R^3}{GM}$$
$$= \frac{4\pi^2R^3}{G\left(\frac{4}{3}\pi R^3\rho\right)}$$

$$\text{or} \quad \rho T^2 = \frac{3\pi}{G} = a \text{ universal constant}$$

5. (a) $v_e = \sqrt{2}\, v_o$. Given $v = \sqrt{1.5}\, v_o$

If $v_o < v < v_e$, satellite will move in elliptical orbit with initial position as the perigee position.

(b) $v = 2v_o > \sqrt{2}\, v_o$ or v_e

Hence, the satellite will escape to infinity.

6. Present angular momentum of earth

$$L_1 = I\omega = \frac{2}{5}MR^2\omega$$

New angular momentum because of change in radius

$$L_2 = \frac{2}{5}M\left(\frac{R}{2}\right)^2\omega'$$

If external torque is zero then angular momentum must be conserved

$$L_1 = L_2$$
$$\frac{2}{5}MR^2\omega = \frac{1}{4} \times \frac{2}{5}MR^2\omega'$$

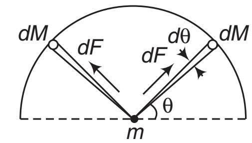
$$\text{i.e.} \quad \omega' = 4\omega$$
$$T' = \frac{1}{4}T = \frac{1}{4} \times 24 = 6 \text{ h}$$

7. Particle lies between two shells. Therefore, net force in exerted only by the inner shell.

$$\therefore \quad F = \frac{Gm_1m}{\left(\frac{R_1 + R_2}{2}\right)^2} = \frac{4\,Gm_1m}{(R_1 + R_2)^2}$$

8. $L = \pi R$

$$\therefore \quad R = \frac{L}{\pi} \quad \Rightarrow F = \int_0^{\pi} dF \sin\theta$$
$$= \int_0^{\pi} \frac{GdM.m}{R^2} \sin\theta$$



$$= \int_0^{\pi} \frac{G\left(\frac{M}{\pi}d\theta\right)m}{L^2}(\pi^2)\sin\theta$$
$$= \frac{2\pi G\,mM}{L^2}$$

9. $K_i + U_i = K_f + U_f$

$$\therefore \quad \frac{1}{2}m\,(2\sqrt{gR})^2 - \frac{GMm}{R} = \frac{1}{2}mv^2 + 0$$

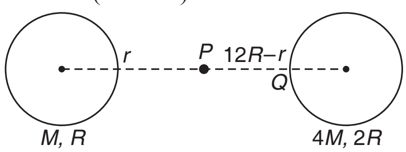
$\therefore \qquad v^2 = 4gR - \dfrac{2GM}{R}$

Putting $GM = gR^2$, we get

$$v^2 = 4gR - 2gR$$

$\therefore \qquad v = \sqrt{2gR}$

10. (a) $\dfrac{GM}{r^2} = \dfrac{G\,(4M)}{(12\,R - r)^2}$



Solving this equations, we get $r = 4R$

$$v_P = \frac{v_Q}{2}$$

$\therefore \qquad \left(-\dfrac{GM}{r}\right) = \left[\dfrac{-G\,(4m)}{12\,R - r}\right]$

$$= \frac{-G\,(4m)}{2R} - \frac{GM}{10\,R}$$

Solving this equation, we get

$r = 7.65\,R$ and $1.49\,R$

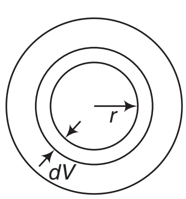
11. (a) At distance $\dfrac{3a}{2}$ which lies between solid sphere and shell field is only due to solid sphere.

$$E = \frac{GM}{(3a/2)^2} = \frac{4GM}{9a^2} \quad \text{(towards centre)}$$

(b) At distance $\dfrac{5a}{2}$ which lies outside the shell. So, field strength is due to both

$\therefore \qquad E = \dfrac{G\,(M+M)}{(5a/2)^2} = \dfrac{8\,GM}{25a^2}$ (towards centre)

12. Volume of small strip $dV = (4\pi r^2)\,dr$ mass of this strip $dm = (\rho)\,(dV)$



$$= \left(\frac{\rho_0 a}{r}\right)(4\pi r^2)dr = (4\pi\, a\rho_0)\,(rdr)$$

$\therefore$ Mass of whole sphere $= \int_0^a (4\pi a\rho_0)\,rdr$

$$m = (2\pi\, a^3\rho_0)$$

$$E = \frac{Gm}{(2a)^2} = \frac{G\,(2\pi a^3\rho_0)}{4a^2}$$

$$= \frac{\pi G\rho_0 a}{2}$$

13. By conservation of momentum, their speeds are same. Using the energy equations we have

$$K_i + U_i = K_f + U_f$$

$$0 - \frac{Gmm}{r_i} = 2\left(\frac{1}{2}mv^2\right) - \frac{Gmm}{r_f}$$

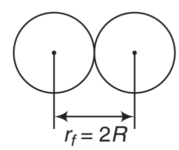
$\therefore \qquad v = \sqrt{Gm\left(\dfrac{1}{r_f} - \dfrac{1}{r_i}\right)} \qquad \ldots\text{(i)}$

$$= \sqrt{\frac{Gm}{r_i}}$$

$$= \sqrt{\frac{6.67\times 10^{-11}\times 10^{30}}{10^{10}}}$$

$$= 8.16\times 10^4 \text{ m/s}$$

$= 81.6$ km/s **Ans.**



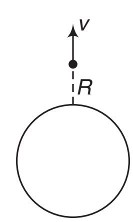
Using Eq. (i), we have

$$v = \sqrt{Gm\left(\frac{1}{r_f} - \frac{1}{r_i}\right)}$$

$$= \sqrt{6.67\times 10^{-11}\times 10^{30}\left(\frac{1}{2\times 10^5} - \frac{1}{10^{10}}\right)}$$

$= 1.8\times 10^7$ m/s $= 1.8\times 10^4$ km/s **Ans.**

14. Using the equation



$$K_i + U_i = K_f + U_f$$

$\therefore \qquad \dfrac{1}{2}mv^2 - \dfrac{GmM}{2R} = 0 + 0$

$\therefore \qquad v = \sqrt{\dfrac{GM}{R}}$

15. (a) $W = U_f - U_i = U_B - U_A$

$= mV_B - mV_A$ $\qquad$ (V = gravitational potential)

$= m\,(V_B - V_A)$

$$= (1)\left[-\frac{100}{2} - \frac{400}{8} + \frac{100}{8} + \frac{400}{2}\right](6.67\times 10^{-11})$$

$$= 7.5\times 10^{-9} \text{ J}$$

(b) First let us find the point (say) c between A and B where field strength due to 400 kg and 100 kg masses is zero. Let its distance from 400 kg is r.

Then, $$\frac{G(400)}{r^2} = \frac{G\,(100)}{(10-r)^2}$$

$$r = \frac{20}{3}\text{ m} \quad \text{and} \quad (10-r) = \frac{10}{3}\text{ m}$$

So, we have to move the body only from A to C. After that 100 kg mass will pull 1 kg mass by its own.

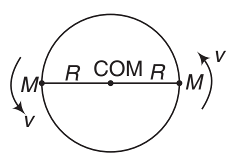
$\therefore$ Minimum kinetic energy $= \Delta U = U_C - U_A$

$= m\,(V_C - V_A)$

$= (1)\left[-\frac{400}{20/3} - \frac{100}{10/3} + \frac{100}{8} + \frac{400}{2}\right](6.67\times 10^{-11})$

$= 8.17\times 10^{-9}$ J

16. (a) $F = \frac{GM.M}{(2R)^2} = \frac{GM^2}{4R^2}$



(b) $$\frac{GM^2}{4R^2} = \frac{MV^2}{R}$$

$$\therefore \quad v = \sqrt{\frac{GM}{4R}}$$

$$T = \frac{2\pi(R)}{v} = \frac{2\pi R}{\sqrt{GM/4R}} = \frac{4\pi R^{3/2}}{\sqrt{GM}}$$

(c) $W = E_f - E_i = [U_f - U_i] + [K_f - K_i]$

$$= \left[0 - \left(\frac{-GMM}{2R}\right)\right] + \left[0 - 2\times\frac{1}{2}\times M\left(\sqrt{\frac{GM}{4R}}\right)^2\right]$$

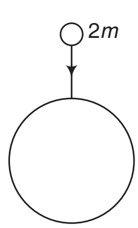
$$= \frac{GM^2}{4R}$$

17. (a) $E = 2$ (energy of one satellite)

$$= 2\left(-\frac{GMm}{2r}\right) = \frac{-GMm}{r}$$

(b) Immediately after collision velocity of combined mass = 0

Path is straight line as shown only potential energy is there.



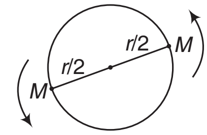
$$v = -\frac{G\,(2m)\,M}{r}$$

$$= -\frac{2GMm}{r}$$

18. $$\frac{2\pi}{\sqrt{GM}}\,r^{3/2} = 1\text{yr} \qquad \text{...(i)}$$

$$\frac{G\,(M)\,(M)}{r^2} = \frac{Mv^2}{(r/2)}$$

$$\therefore \quad v = \sqrt{\frac{GM}{2r}}$$



$$T = \frac{2\pi\,(r/2)}{v} = \frac{\pi r}{\sqrt{\frac{GM}{2r}}}$$

$$= \sqrt{2}\left(\frac{\pi}{\sqrt{GM}}\right)r^{3/2}$$

$$= \sqrt{2}\left(\frac{1}{2}\right)\text{yr} = 0.71\text{ yr} \qquad \left[\text{as } \frac{\pi r^{3/2}}{\sqrt{GM}} = \frac{1}{2}\text{ yr}\right]$$

19. $$W_1 = \Delta U = -\frac{GMm}{(R+h)} + \frac{GMm}{R} = \frac{GMmh}{R\,(R+h)}$$

$$W_2 = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\sqrt{\frac{GM}{r}}\right)^2$$

$$= \frac{1}{2}m\left(\frac{GM}{R+h}\right) = \frac{GMm}{2\,(R+h)}$$

$$W_1 > W_2$$

If $\quad \frac{h}{R} > \frac{1}{2} \quad$ or $\quad h > \frac{R}{2}$

or $\quad h > \frac{6370}{2}$ km

or $\quad h > 3185$ km

LEVEL 2

Single Correction Option

1. Just before collision,

$$v_0 = \sqrt{\frac{GM}{r}} = \sqrt{\frac{GM}{R+R}} = \sqrt{\frac{GM}{2R}}$$

From conservation of linear momentum,

$$mv_0 = \frac{m}{2}\times 0 + \frac{m}{2}\,v$$

$$\therefore \quad v = 2v_0 = \sqrt{\frac{2GM}{R}}$$

Increase in mechanical energy

$$= K_f - K_i$$

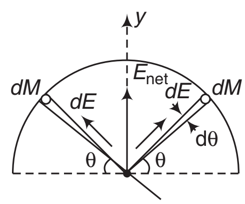
$$= \frac{1}{2}\frac{m}{2}v^2 - \frac{1}{2}mv_0^2$$

$$= \frac{1}{4}m\,(2v_0)^2 - \frac{1}{2}mv_0^2$$

$$= \frac{1}{2} m v_0^2 = \frac{1}{2} m \left(\frac{GM}{2R}\right) = \frac{1}{4} \frac{GMm}{R}$$

$$= \frac{1}{4} mgR \qquad \left(\text{as } g = \frac{GM}{R^2}\right)$$

2. $E_{\text{net}} = \int_0^{\pi} dE \sin\theta$



$$= \int_0^{\pi} \frac{G \cdot dM}{R^2} \sin\theta$$

$$= \int_0^{\pi} \frac{G\left(\frac{M}{\pi} d\theta\right)}{R^2} \sin\theta$$

$$= \frac{2GM}{\pi R^2} = \frac{2\, G\pi M}{l^2}$$

$$l = \pi R$$

$$\therefore \qquad R = \frac{l}{\pi}$$

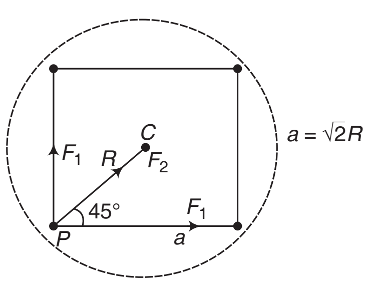
3. Net force on P towards centre

$$F = \sqrt{2}\, F_1 + F_2$$

$$\therefore \qquad \frac{Mv^2}{R} = \sqrt{2}\left[\frac{G.MM}{a^2}\right] + \frac{GMM}{(2R)^2}$$

$$\text{or} \qquad \frac{v^2}{R} = \frac{\sqrt{2}\, GM}{(\sqrt{2}R)^2} + \frac{GM}{4R^2}$$

$$\therefore \qquad v = \sqrt{\frac{GM}{R} \frac{2\sqrt{2}+1}{4}}$$



4. $K_i + U_i = K_f + U_f$

$$\frac{1}{2} m (kv_e)^2 - \frac{GMm}{R} = 0 - \frac{GMm}{r}$$

$$\therefore \quad \frac{1}{2} k^2 (2gR) - \frac{GM}{R} = -\frac{GM}{r}$$

$$\text{or} \qquad \frac{GM}{r}(1 - k^2) = \frac{GM}{r} \left(\text{as } gR = \frac{GM}{R}\right)$$

$$\text{So,} \qquad r = \frac{R}{1 - k^2}$$

5. $a = E = \left(\frac{GM}{R^3}\right) y \qquad \left(\text{as } a = E = \frac{F}{m}\right)$

$$= \frac{G\left(\frac{4}{3}\pi R^3 \rho\right)}{R^3} y = \frac{4}{3} \pi G \rho y$$

6. Earth rotates from west to east. Net velocity of train

$$= (\omega R - \theta)^2$$

$$\text{Now, } mg - N = \frac{m(\omega R - \theta)^2}{R}$$

$$\therefore \qquad N = mg - \frac{m(\omega R - v)^2}{R}$$

$$= mg\left[1 - \frac{\omega^2 R^2 + v^2 - 2\, vR\omega}{gR}\right]$$

$$= mg\left[1 - \frac{\omega(\omega R - 2v)}{g} - \frac{v^2}{Rg}\right] \quad \textbf{Ans.}$$

7. Gravitational field at any point inside the cavity is uniform (both in magnitude as well as direction). So, let us find its value at centre of cavity.

$E_R = E_T - E_C$ (at centre of cavity)

$R \to$ Remaining, $T \to$ total, $C \to$ cavity

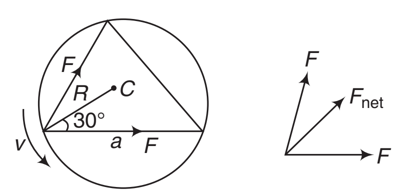
So, $\quad E_R = E_T \quad$ (as $E_C = 0$)

$$= \left(\frac{GM}{R^3}\right) a \quad \text{or} \quad E_R \propto a$$

8. Potential (and hence potential energy) at centre is $\frac{3}{2}$ times the value on surface. So, required kinetic energy is $\frac{3}{2}$ times or required speed is $\sqrt{\frac{3}{2}}$ times.

9. $\int dV = -\int E dr$

10. $a = 2R \cos 30° = \sqrt{3}\, R$



$F_{\text{net}} = \sqrt{3}\, F$ is towards centre C

$$\therefore \qquad \sqrt{3}\, F = \frac{MV^2}{R}$$

or $\sqrt{3}\,\dfrac{GMM}{a^2} = \dfrac{MV^2}{R}$

or $\dfrac{\sqrt{3}GM^2}{(\sqrt{3}R)^2} = \dfrac{MV^2}{R}$

or $v = \sqrt{\dfrac{GM}{\sqrt{3}R}}$

11. $T^2 \propto R^3 \quad \Rightarrow \quad \therefore \quad T^{2/3} \propto R \quad \text{or} \quad T^2 \propto \dfrac{1}{(1/R^3)}$

12. $W = \Delta E = E_A - E_\alpha$

$$= (K_A + U_A) - (K_\alpha + U_\alpha)$$
$$= (K_A + mV_A) - (0 + 0)$$

$\therefore \quad -5.5 = \dfrac{1}{2} \times 1 \times (3)^2 + (1)\,V_A$

or $\quad V_A = -10$ J/kg

13. $K_i + U_i = K_f + U_f$

$\therefore \quad \dfrac{1}{2}mv^2 - 2\,\dfrac{GMm}{\sqrt{2}R} = 0 + 0$

$\therefore \quad v = \sqrt{\dfrac{2\sqrt{2}GM}{R}}$

14. $W = -\Delta U = U_i - U_f$

$$= -\frac{Gmm}{a}\left[\left\{\frac{(1)\,(2)}{1} + \frac{(1)\,(3)}{1} + \frac{(2)\,(3)}{\sqrt{2}}\right\}\right.$$
$$\left. - \left\{\frac{(1\times 2)}{1} + \frac{(1\times 3)}{1} + \frac{(2\times 3)}{1}\right\}\right]$$
$$= \frac{6\,Gm^2}{a}\left(1 - \frac{1}{\sqrt{2}}\right)$$

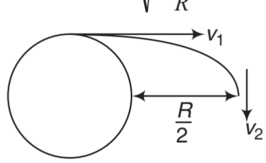
15. $K_i + U_i = K_f + U_f$

$\therefore \quad \dfrac{1}{2}mv^2 - \dfrac{3}{2}\,\dfrac{GMm}{R} = 0 - \dfrac{GMm}{R}$

Solving, we get

$$v = \sqrt{\frac{GM}{R}}$$

16. $v_1 = nv_e = n\sqrt{2gR} = n\sqrt{\dfrac{2GM}{R}}$



From mechanical energy conservative, we have

$$\frac{1}{2}m\left[n\sqrt{\frac{2GM}{R}}\right]^2 - \frac{GMm}{R} = \frac{1}{2}mv_2^2 - \frac{GMm}{R + \dfrac{R}{2}} \quad \text{...(i)}$$

From conservation of angular momentum about centre of earth

$$m\left[n\sqrt{\frac{2GM}{R}}\right]R = mv_2\left(R + \frac{R}{2}\right) \quad \text{...(ii)}$$

Solving these two equations we get,

$$n = \sqrt{0.6}$$

17. Let v is the velocity at centre then.

$$K_i + U_i = K_f + U_f$$

or $\quad 0 - \dfrac{GMm}{R} = \dfrac{1}{2}mv^2 - \dfrac{3}{2}\,\dfrac{GMm}{R}$

$\therefore \quad v = \sqrt{\dfrac{GM}{R}}$

From conservation of linear momentum.

$$2mv - mv = (2m + m)\,v$$

$\therefore \quad v' = \dfrac{v}{3} = \dfrac{1}{3}\sqrt{\dfrac{GM}{R}}$

Again applying energy conservation,

$$K_i + U_i = K_f + U_f$$

$\therefore \quad \dfrac{1}{2}(3m)\left[\dfrac{1}{3}\sqrt{\dfrac{GM}{R}}\right]^2 - \dfrac{3}{2}\,\dfrac{GM\,(3m)}{R}$

$$= 0 - 3m\left[\frac{GM}{R^3}\,(1.5\,R^2 - 0.5\,r^2\right]$$

Solving this equations, we get

$$r = \frac{R}{3}$$

So, $\dfrac{R}{3}$ is the amplitude of oscillations.

18. $v_e = \sqrt{2gR}$

or $v_e \propto \sqrt{gR}$

19. $F = -\dfrac{dU}{dr} = -6r^2$

Now $\quad \dfrac{mv^2}{r} = 6r^2$

$\therefore \quad \dfrac{1}{2}mv^2 = 3r^3$

$$E = K + U = 3r^3 + 2r^3 = 5r^3$$

At $\quad r = 5\,\text{m},\ E = 625$ J **Ans.**

20. E on the surface of earth,

$$E_1 = \frac{F}{m} = \frac{10}{1}$$
$$= 10 \text{ N/kg}$$
$$E \propto \frac{1}{r^2}$$

$$\therefore \qquad \frac{E_2}{E_1} = \left(\frac{r_1}{r_2}\right)^2 = \left(\frac{R}{3R/2}\right)^2 = \frac{4}{9}$$

$$\therefore \qquad E_2 = \frac{4}{9} E_f = \frac{40}{9} \text{ N/kg}$$

$$F_2 = m\, E_2 = (200)\left(\frac{40}{9}\right) = 889 \text{ N}$$

21. $g = \dfrac{GM}{R^2} \quad \Rightarrow \quad \therefore \quad GM = gR^2$

$$v = \sqrt{\frac{GM}{r}} = \sqrt{\frac{GM}{R + x}} = \sqrt{\frac{gR^2}{R + x}}$$

22. $C \to$ cavity, $T \to$ Total, $R \to$ remaining

$$F_1 = \frac{GMm}{(2R)^2} \qquad \ldots\text{(i)}$$

$$F_R = F_2 = F_T - F_C = F_1 - F_C$$

or
$$F_2 = \frac{GMm}{(2R)^2} - \frac{G\left(\dfrac{M}{8}\right)m}{(3R/2)^2}$$

or
$$F_2 = \frac{14\, GMm}{72R^2} \qquad \ldots\text{(ii)}$$

From Eqs. (i) and (ii) we get,

$$\frac{F_2}{F_1} = \frac{7}{9}$$

More than One Correct Options

1. $g = \dfrac{GM}{R^2} = \dfrac{G\left(\dfrac{4}{3}\pi R^3\rho\right)}{R^2}$

or $\qquad g \propto R \qquad$ (as ρ is same)

$$v_e = \sqrt{2gR} = \sqrt{\frac{2GM}{R}}$$

$$= \sqrt{\frac{2G\left(\dfrac{4}{3}\pi R^3\rho\right)}{R}}$$

So, v_e or $v \propto R$ (as ρ is same).

3. $K_i + U_i = K_f + U_f$

$$\therefore \qquad 0 - \frac{GMm}{2R} = \frac{1}{2} mv^2 - \frac{3}{2}\frac{GMm}{R}$$

or
$$v = \sqrt{\frac{2GM}{R}}$$

4. E due to point mass is $E = \dfrac{Gm}{r^2}$

As $r \to 0, E \to \infty$

So, just over the point masses, $E = \infty$. Hence, in moving from one point mass to other point mass, E first decreases and then increases.

V due to a point mass is

$$V = -\frac{Gm}{r}$$

As $\qquad r \to 0, V \to -\infty$

So, just over the point mass, V is $-\infty$. Hence, in moving from one point mass to other point mass, V first increases and then decreases.

5. Inside a shell, $V =$ constant and $E = 0$.

Between A and B, $E_{net} = 0$, $V_{net} =$ constant because these points µ inside both shells.

Between B and C

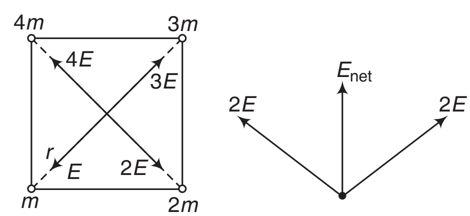
E of $m \neq 0$, E of $2m = 0$

V of $m \neq$ constant, V of $2m =$ constant.

Beyond C

E and V due to both shells are neither zero nor constant.

6. $E = \dfrac{Gm}{r^2}$

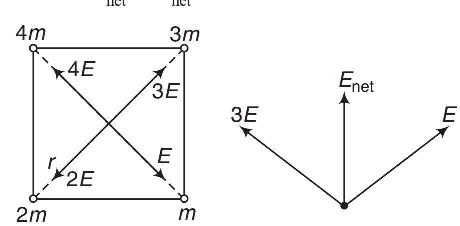


$$E'_{net} = 2\sqrt{2}\, E = \frac{2\sqrt{2}\, Gm}{r^2}$$

$$E'_{net} = \sqrt{(9E)^2 + E^2} = \sqrt{10}\, E$$

$$= \frac{\sqrt{10}\, Gm}{r^2}$$

$$E'_{net} > E_{net}$$



Potential in both cases is

$$V_{net} = -\frac{G}{r}(m + 2m + 3m + 4m)$$

$$= -\frac{10\, Gm}{r}$$

7. At highest point velocity of particle-1will becomes zero. But velocity of particle-2 is non zero.

8. At maximum distance (at A) kinetic energy is minimum. But angular momentum about centre of sun always remains constant.

9. $E_1 = -\frac{GMm}{R}$

$$U_2 = -\frac{GMm}{2R} \qquad (r = R + R = 2R)$$

$$K_2 = \frac{GMm}{4R}$$

and $$E_2 = -\frac{GMm}{4R}$$

10. $v = \sqrt{\frac{GM}{r}}$

or $$v \propto \sqrt{GM}$$

$$T = \frac{2\pi}{\sqrt{GM}} r^{3/2}$$

or $$T \propto \frac{1}{\sqrt{GM}}$$

Match the Columns

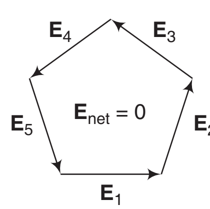
1. Inside the shell, V, U, K, and v remains constant. F, E and a are zero.

From A to B, kinetic energy increase and potential energy decreases.

2. Resultant of five field strength vectors of equal magnitude acting at same angles is zero.

(b) $V = -\frac{5\,GMm}{r}$

(c) If $\mathbf{E}_5$ is removed then resultant of rest four vectors is equal and opposite of $\mathbf{E}_5$.



$\therefore$ $$|\mathbf{E}_{net}| = |\mathbf{E}_5| = \frac{Gm}{r^2}$$

(d) $V = -\frac{4Gm}{r}$

3. $\frac{GM}{y} = x$ or $GM = xy$

(a) $E = \frac{GM}{(2y)^2} = \frac{xy}{4y^2} = \frac{x}{4y}$

(b) $|V| = \frac{GM}{y^3}\left(1.5\,y^2 - 0.5\,\frac{y^3}{4}\right)$

$$= \left(\frac{xy}{y^3}\right)\left(\frac{11}{8}y^2\right) = \frac{11}{8}x$$

(c) $E = \frac{GM}{y^3}\left(\frac{y}{2}\right) = \frac{xy}{y^3}\frac{y}{2} = \frac{x}{2y}$

(d) $|V| = \frac{GM}{2y} = \frac{xy}{2y} = \frac{x}{2}$

4. (a) $W = \Delta U = \frac{mgh}{1 + \frac{h}{R}}$

Put $h = R$, $W = \frac{1}{2}mgR$

(b) $\text{KE} = \frac{GMm}{2r} = \frac{GMm}{4R} = \frac{1}{4}mgR \quad (\text{as}\, GM = gR^2)$

(c) $E = -\frac{GMm}{2r}$

$$E_1 = -\frac{GMm}{2(2R)} = -\frac{GMm}{4R} = -\frac{1}{4}mgR$$

$$E_2 = -\frac{GMm}{2(3R)} = -\frac{1}{6}mgR$$

$$E_1 \sim E = \frac{1}{12}mgR$$

(d) $\text{KE} = \Delta U = \frac{mgh}{1 + \frac{h}{R}}$

Putting $h = R$, KE $= \frac{1}{2}mgR$

Subjective Questions

1. $W = \Delta U = U_f - U_i$

$$= 3\left[\frac{-Gmm}{2a}\right] - 3\left[-\frac{Gmm}{a}\right]$$

$$= \frac{3\,Gmm}{2a} = \frac{3Gm^2}{2a}$$ **Ans.**

2. $h = \frac{u^2}{2g_e}$

$\therefore$ $$u = \sqrt{2g_e h} \qquad \text{...(i)}$$

For the asked planet this u should be equal to the escape velocity from its surface.

$\therefore$ $$\sqrt{2g_e h} = \sqrt{2g_p R_p}$$

or $$g_e h = g_p R_p$$

$$\frac{GM_e}{R_e^2}.h = \frac{GM_p}{R_p^2}.\,R_p$$

or $\dfrac{\left(\dfrac{4}{3}\pi R_e^3\right)\rho h}{R_e^2} = \dfrac{\left(\dfrac{4}{3}\pi R_p^3\right)\rho R_p}{R_p^2}$

or $R_P = \sqrt{R_e h}$

$= \sqrt{(6.41 \times 10^6)(1.5)}$

$= 3.1 \times 10^3$ m **Ans.**

3. (a) $v_o = \dfrac{v_e}{2}$

$\therefore \quad \sqrt{\dfrac{GM}{r}} = \dfrac{\sqrt{\dfrac{2GM}{R}}}{2}$

$\therefore \quad r = 2R$

or $h = r - R = R$ or height = radius of earth.

(b) Increase in kinetic energy = decrease in potential energy

$\therefore \quad \dfrac{1}{2}mv^2 = \dfrac{mgh}{1 + \dfrac{h}{R}}$

$\therefore \quad v = \sqrt{\dfrac{2hg}{1 + \dfrac{h}{R}}}$

Substituting the values we have,

$v = \sqrt{\dfrac{2 \times 9.81 \times 6400 \times 10^3}{1 + \dfrac{R}{R}}}$

$= 7924$ m/s ≈ 7.92 km/s **Ans.**

4. (i) At point A, field strength due to shell will be zero.

Net field is only due to metal sphere. Distance between centre of metal sphere and point A is $4R$.

$\therefore \quad E_A = \dfrac{G(m)}{(4R)^2} = \dfrac{Gm}{16R^2}$

(ii) At point B, net field is due to both, due to shell and due to metal sphere.

$\therefore \quad E_B = \dfrac{Gm}{(5R)^2} + \dfrac{Gm}{(6R)^2}$

$= \dfrac{61\,Gm}{900\,R^2}$ **Ans.**

5. Radius of hollow sphere is $\dfrac{R}{2}$, so mass in this hollow portion would had been, $\dfrac{M}{8}$.

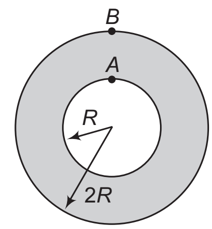
Now, net force on m due to whole sphere = force due to remaining mass + force due to cavity mass.

$\therefore$ Force due to remaining mass = force due to whole sphere − force due to cavity mass

$= \dfrac{G\,Mm}{d^2} - \dfrac{G\,Mm}{8(d - R/2)^2}$

$= \dfrac{G\,Mm}{d^2}\left[1 - \dfrac{1}{8\left(1 - \dfrac{R}{2d}\right)^2}\right]$

6. Let m_1 be the mass of the core and m_2 the mass of outer shell.



$g_A = g_B$ (given)

Then $\dfrac{Gm_1}{R^2} = \dfrac{G(m_1 + m_2)}{(2R)^2}$

$\therefore \quad 4m_1 = (m_1 + m_2)$

or $4\left\{\dfrac{4}{3}\pi R^3 \rho_1\right\} = \dfrac{4}{3}\pi R^3 \cdot \rho_1 + \left\{\dfrac{4}{3}\pi(2R)^3 - \dfrac{4}{3}\pi R^3\right\}\rho_2$

$\therefore \quad 4\rho_1 = \rho_1 + 7\rho_2$

$\therefore \quad \dfrac{\rho_1}{\rho_2} = \dfrac{7}{3}$ **Ans.**

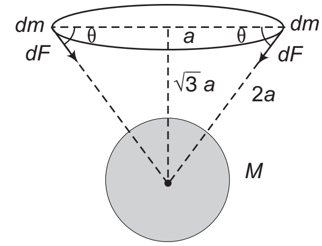
7. Total mechanical energy of a satellite in an elliptical orbit of semi major axis 'a' is $-\dfrac{GMm}{2a}$.

$E = K + U$

$\therefore \quad -\dfrac{GMm}{2a} = \dfrac{1}{2}mv^2 - \dfrac{GMm}{r}$

or $v^2 = GM\left[\dfrac{2}{r} - \dfrac{1}{a}\right]$ **Hence Proved.**

8. dF = force on a small mass 'dm' of the ring by the sphere.



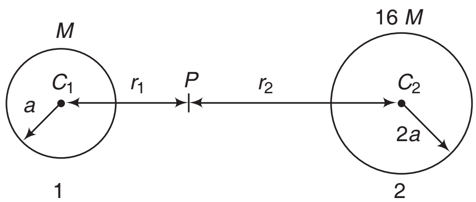
Net force on ring = $\Sigma(dF \sin\theta)$ or $\int dF \sin\theta$

$$= \Sigma \frac{GM(dm)}{(2a)^2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}GM}{8a^2} \Sigma(dm)$$

But $\Sigma(dm) = m$, the mass of whole ring.

$$\therefore \qquad \text{Net force} = \frac{\sqrt{3}GMm}{8a^2} \qquad \textbf{Ans.}$$

9. Let there are two stars 1 and 2 as shown below.



Let P is a point between C_1 and C_2, where gravitational field strength is zero. Or at P field strength due to star 1 is equal and opposite to the field strength due to star 2. Hence,

$$\frac{GM}{r_1^2} = \frac{G(16M)}{r_2^2} \quad \text{or} \quad \frac{r_2}{r_1} = 4$$

also $\qquad r_1 + r_2 = 10a$

$$\therefore \qquad r_2 = \left(\frac{4}{4+1}\right)(10a) = 8a$$

and $\qquad r_1 = 2a$

Now, the body of mass m is projected from the surface of larger star towards the smaller one. Between C_2 and P it is attracted towards 2 and between C_1 and P it will be attracted towards 1. Therefore, the body should be projected to just cross point P because beyond that the particle is attracted towards the smaller star itself.

From conservation of mechanical energy $\frac{1}{2} mv_{\min}^2$

= Potential energy of the body at P
– Potential energy at the surface of the larger star.

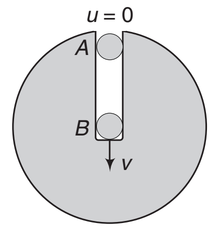
$$\therefore \quad \frac{1}{2} mv_{\min}^2 = \left[-\frac{GMm}{r_1} - \frac{16GMm}{r_2}\right] - \left[-\frac{GMm}{10a - 2a} - \frac{16GMm}{2a}\right]$$

$$= \left[-\frac{GMm}{2a} - \frac{16GMm}{8a}\right] - \left[-\frac{GMm}{8a} - \frac{8GMm}{a}\right]$$

or $$\frac{1}{2} mv_{\min}^2 = \left(\frac{45}{8}\right)\frac{GMm}{a}$$

$$\therefore \qquad v_{\min} = \frac{3\sqrt{5}}{2}\left(\sqrt{\frac{GM}{a}}\right) \qquad \textbf{Ans.}$$

10. Let mass of the ball be m.



$$\frac{1}{2} mv^2 = m(V_A - V_B)$$

$$= m\left[-\frac{GM}{R} - \left(-1.5\frac{GM}{R}\right)\right]$$

$$= \frac{GMm}{2R}$$

$$\therefore \qquad v = \sqrt{\frac{GM}{R}}$$

Velocity of ball just after collision,

$$v' = ev = \frac{1}{2}\sqrt{\frac{GM}{R}}$$

Let r be the distance from the centre upto where the ball reaches after collision. Then,

$$\frac{1}{2} mv'^2 = m\,[V(r) - V(\text{centre})]$$

or $$\frac{1}{8}\frac{GMm}{R} = m\left[\frac{3GM}{2R} - \frac{GM}{R^3}\left(\frac{3R^2}{2} - \frac{r^2}{2}\right)\right]$$

or $$\frac{1}{8} = \frac{3}{2} - \frac{3}{2} + \frac{r^2}{2R^2}$$

$$\therefore \qquad \frac{r^2}{R^2} = \frac{1}{4} \quad \text{or} \quad r = \frac{R}{2}$$

$\therefore$ The desired distance,

$$s = R + \frac{R}{2} + \frac{R}{2} = 2R \qquad \textbf{Ans.}$$

11. Applying conservation of mechanical energy,

Increase in kinetic energy

= decrease in gravitational potential energy

or $$\frac{1}{2} m_0 v^2 = U_B - U_A = m_0(V_B - V_A)$$

$$\therefore \qquad v = \sqrt{2(V_B - V_A)} \qquad \text{...(i)}$$

Potential at A

V_A = potential due to complete sphere
– potential due to cavity

$$= -\frac{1.5\,GM}{R} - \left[-\frac{Gm}{R/2}\right]$$

$$= \frac{2Gm}{R} - \frac{1.5\,GM}{R}$$

Here, $m = \dfrac{4}{3}\pi\left(\dfrac{R}{2}\right)^3 \rho = \dfrac{\pi\rho R^3}{6}$

and $M = \dfrac{4}{3}\pi R^3 \rho$

Substituting the values, we get

$$V_A = \frac{G}{R}\left[\frac{\pi\rho R^3}{3} - 2\pi\rho R^3\right] = -\frac{5}{3}\pi G\rho R^2$$

Potential at B

$$V_B = -\frac{GM}{R^3}\left[1.5R^2 - 0.5\left(\frac{R}{2}\right)^2\right] + \frac{1.5Gm}{R/2}$$

$$= -\frac{11}{8}\frac{GM}{R} + \frac{3Gm}{R}$$

$$= \frac{G}{R}\left[\frac{\pi\rho R^3}{2} - \frac{11}{6}\cdot\pi\rho R^3\right] = -\frac{4}{3}\pi G\rho R^2$$

$\therefore \quad V_B - V_A = \dfrac{1}{3}\pi G\rho R^2$

So, from Eq. (i)

$$v = \sqrt{\frac{2}{3}\pi G\rho R^2} \qquad \textbf{Ans.}$$

12. (a) Let x be the displacement of ring. Then displacement of the particle is $x_0 - x$, or $(3.0 - x)$ m. Centre of mass will not move. Hence,

$$(5.4 \times 10^9)x = (6 \times 10^8)(3 - x)$$

Solving, we get

$$x = 0.3 \text{ m} \qquad \textbf{Ans.}$$

(b) Apply conservation of linear momentum and conservation of mechanical energy.

13. (a) Mean radius of planet,

$$m_2 = \frac{r_1 + r_2}{2} = 1.4 \times 10^8 \text{ km}$$

Now, $T \propto r^{3/2}$

$\therefore$ Time period of m_2 :

$$T_2 = T_1\left(\frac{1.4 \times 10^8}{10^8}\right)^{3/2}$$

or $T_2 = 2\,(1.4)^{3/2}$

$= 3.31$ yr **Ans.**

(b) For m_2, point P is perigee position. So, speed at this point is greater than orbital speed for circular orbit.

$\therefore \quad U_{m_2} = U_{m_1}$

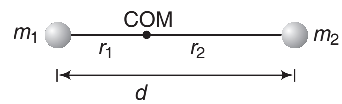
$\therefore \quad E_{m_2} > E_{m_1}$

$K_{m_2} > K_{m_1}$

(c) vr = constant

14. (a) $r_1 + r_2 = d \qquad \ldots\text{(i)}$

$m_1 r_1 = m_2 r_2 \qquad \ldots\text{(ii)}$



Solving these two equations we get,

$$r_1 = \left(\frac{m_2}{m_1 + m_2}\right)d \quad \text{or} \quad r_2 = \left(\frac{m_1}{m_1 + m_2}\right)d$$

The centripetal force is provided by gravitational force,

$$m_1 r_1 \omega^2 = m_2 r_2 \omega^2 = \frac{Gm_1 m_2}{d^2}$$

Solving these equations, we get

$$\omega = \sqrt{\frac{G(m_1 + m_2)}{d^3}}$$

$$\therefore \quad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{d^3}{G(m_1 + m_2)}} \qquad \textbf{Ans.}$$

(b) $\dfrac{K_1}{K_2} = \dfrac{\frac{1}{2}I_1\omega^2}{\frac{1}{2}I_2\omega^2} = \dfrac{I_1}{I_2} = \dfrac{m_1 r_1^2}{m_2 r_2^2}$

$$= \left(\frac{m_1}{m_2}\right)\left(\frac{r_1}{r_2}\right)^2 = \left(\frac{m_1}{m_2}\right)\left(\frac{m_2}{m_1}\right)^2 = \frac{m_2}{m_1} \qquad \textbf{Ans.}$$

(c) $\dfrac{L_1}{L_2} = \dfrac{I_1\omega}{I_2\omega} = \dfrac{I_1}{I_2} = \dfrac{m_2}{m_1}$ **Ans.**

(d) $L = L_1 + L_2 = (I_1 + I_2)\,\omega$

$= (m_1 r_1^2 + m_2 r_2^2)\,\omega$

$$= \left[\frac{m_1 m_2^2 d^2}{(m_1 + m_2)^2} + \frac{m_2 m_1^2 d^2}{(m_1 + m_2)^2}\right]\omega$$

$= \mu\omega d^2$ **Ans.**

where, $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$ = reduced mass

(e) $K = \dfrac{1}{2}(I_1 + I_2)\,\omega^2 = \dfrac{1}{2}\mu\omega^2 d^2$ **Ans.**

14. Simple Harmonic Motion

INTRODUCTORY EXERCISE 14.1

1. $a = -4x$

Comparing with $a = -\omega^2 x$, we get

$$\omega = 2 \text{ rad/s}$$

Now, $$T = \frac{2\pi}{\omega} = (\pi)\text{sec}$$

2. $E = \frac{1}{2}kA^2$

$$U = \frac{1}{2}kx^2 = \frac{1}{2}k\left(\frac{A}{4}\right)^2$$
$$= \frac{1}{16}\left(\frac{1}{2}kA^2\right)$$
$$= \frac{1}{16}E$$

Hence, $\frac{1}{16}$ fraction is potential energy and $\frac{15}{16}$ fraction is kinetic energy.

3. (a) $A =$ initial displacement from mean position

$= 15$ cm

(b) $T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{2.0}{150}} = 0.726$ s

(c) $F = \frac{1}{T} = 1.38$ Hz **Ans.**

(d) $E = \frac{1}{2}kA^2$

$= \frac{1}{2} \times 150 \times (0.15)^2 = 1.69$ J **Ans.**

(e) $v_{max} = \omega A$

$= \left(\frac{2\pi}{T}\right)A = \left(\frac{2\pi}{0.726}\right)(0.15)$

$= 1.30$ m/s **Ans.**

4. $v_{max} = \omega A = 2\pi fA$

$= (2\pi)(2)(8 \times 10^{-3})$

$= 0.101$ m/s **Ans.**

$a_{max} = \omega^2 A = (2\pi f)^2 A$

$= 4 \times \pi^2 \times 4 \times 8 \times 10^{-3}$

$= 1.264$ m/s^2 **Ans.**

$F_{max} = ma_{max}$

$= (0.5)(1.264)$

$= 0.632$ N **Ans.**

5. No, as acceleration in SHM $a = -\omega^2 x$ is variable.

INTRODUCTORY EXERCISE 14.2

1. (a) $E = \frac{1}{2}mv_{max}^2 = \frac{1}{2}(m)\omega^2 A^2$

$= \frac{1}{2} \times 2.0 \times \left(\frac{\pi}{4}\right)^2 \times (1.5)^2$

$= 1.39$ J **Ans.**

(b) $0.5 = 1.5 \sin\left(\frac{\pi t_1}{4} + \frac{\pi}{6}\right)$

From here find t.

Then, $-0.75 = 1.5 \sin\left(\frac{\pi t_2}{4} + \frac{\pi}{6}\right)$

From here find t_2.

Now, $t_1 \sim t_2$ is the required time.

2. $\omega = 3$ rad/s

$$A = 0.2 \text{ m}$$

At $x = 5$ cm

$v = \pm\omega\sqrt{A^2 - x^2}$

$= \pm 3.0\sqrt{(0.2)^2 - (0.05)^2}$

$= \pm 0.58$ m/s **Ans.**

$a = -\omega^2 x = -(3)^2(0.05)$

$= -0.45$ m/s^2 **Ans.**

At $x = 0$

$v = \pm\omega A$

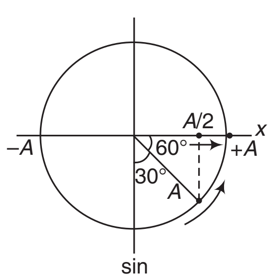
$= \pm(3.0)(0.2)$

$= \pm 0.6$ m/s **Ans.**

$a = -\omega^2 x = -(3)^2(0)$

$= 0$ **Ans.**

3.



$x = A \sin(\omega t + 30°)$

$\therefore$ $\delta = 30°$ **Ans.**

4. (b) $\omega = \frac{2\pi}{T} = \frac{2\pi}{4} = 1.57$ rad/s **Ans.**

(c) $\omega = \sqrt{\dfrac{k}{m}}$

$\therefore\ k = m\omega^2 = (0.8)(1.57)^2 = 1.97$ N/m **Ans.**

(d) At $t = 1$ s,

$v =$ slope of x -t graph $= 0$

(e) At $t = 1$ s, x is maximum ($= +A$). Therefore. magnitude of acceleration is also maximum.

$a =$ maximum $= \omega^2 A = (1.57)^2\ (0.08)$

$= 0.197\ \text{m/s}^2$ **Ans.**

5. $X = 5\sin\left(20t + \dfrac{\pi}{3}\right)$

$$v = \frac{dx}{dt} = 100\cos\left(20\,t + \frac{\pi}{3}\right)$$

$$a = \frac{dv}{dt} = -\,2000\sin\left(20t + \frac{\pi}{3}\right)$$

(a) $v = 0$ for the first time when,

$$20t + \frac{\pi}{3} = \frac{\pi}{2}$$

$\therefore$ $t = \dfrac{\pi}{120}$ s **Ans.**

(b) $a = 0$ for the first time, when

$$20t + \frac{\pi}{3} = \pi$$

$\therefore$ $t = \dfrac{\pi}{30}$ s **Ans.**

(c) When $a = 0$ for the first time, its speed will be maximum.

$t = \dfrac{\pi}{30}$ s **Ans.**

6. Simple harmonic with mean at $x = 10$, amplitude 4 and extreme positions at $x = 6$ and $x = 14$. At $t = 0$, it starts from $x = 6$. Here, x is coordinate, not displacement from mean position.

INTRODUCTORY EXERCISE 14.3

1. $a = -16x$

Comparing with $a = -\omega^2 x \Rightarrow \omega = 4$ rad/s

Now, $T = \dfrac{2\pi}{\omega} = \left(\dfrac{\pi}{2}\right)$ sec

2. $2 = 2\pi\sqrt{\dfrac{M}{k}}$ …(i)

$3 = 2\pi\sqrt{\dfrac{M+4}{k}}$ …(ii)

Solving two equations, we get

$M = 3.2$ kg

3. $2 = 2\pi\sqrt{\dfrac{(0.3+0.1)}{k}}$ …(i)

$T = 2\pi\sqrt{\dfrac{0.1}{k}}$ …(ii)

Solving these two equations we get,

$T = 1$ s

4. Comparing with $a = -\,\omega^2 x$

$$\omega^2 = p$$

$\therefore$ $\omega = \sqrt{p}$

5. $T = 2\pi\sqrt{\dfrac{M}{k}}$

$\dfrac{5T}{3} = 2\pi\sqrt{\dfrac{M+m}{k}}$

Solving these two equations, we get

$$\frac{m}{M} = \frac{16}{9}$$

6. $T \propto \sqrt{l}$

$$\frac{T'}{T} = \sqrt{\frac{l'}{l}} = \sqrt{\frac{1.21\,l}{l}} = 1.1$$

$\therefore$ $T' = 1.1\,T$

Hence, percentage increase in T is 11%.

7. $\omega^2 x = \omega\sqrt{A^2 - x^2} = \dfrac{\sqrt{A^2 - x^2}}{x} = 2\pi f$

$\therefore$ $f = \dfrac{\sqrt{A^2 - x^2}}{2\pi x}$

Substitute, $A = 2$ cm and $x = 1$ cm

INTRODUCTORY EXERCISE 14.4

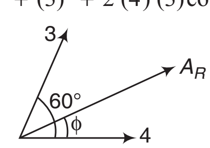
1. In such situation, amplitudes are added by vector method.

$$A_R = \sqrt{(4)^2 + (3)^2 + 2\,(4)\,(3)\cos\phi}$$

$= \sqrt{25 + 24\cos\phi}$ …(i)

Now, we can substitute different values of ϕ given in different parts in the question and can find the value of A_R.

2. $A_R = \sqrt{(4)^2 + (3)^2 + 2\,(4)\,(3)\cos 60°}$



$= 6.1$ units

$$\tan\phi = \frac{3\sin 60°}{4 + 3\cos 60°} = 0.472$$

$\therefore \qquad \phi = 25.3°$

$\therefore \qquad x = 6.1 \sin(100\pi t + 25.3°)$

(a) At $t = 0$, $x = 6.1 \sin 25.3°$

$= 2.6$ unit

(b) $v_{max} = \omega A$

$= (100\pi)(6.1)$

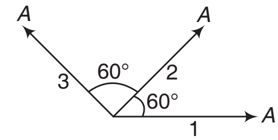
$= 1917$ unit

(c) $a_{max} = \omega^2 A$

$= (100\pi)^2 (6.1)$

$= 6.0 \times 10^5$ units

3. Resultant of 1 and 3 is also A in the direction of 2.



$\therefore \qquad A_R = 2A$

4. $A = \sqrt{A^2 + A^2 + 2A.A\cos\phi}$

Solving we get,

$$\cos\phi = -\frac{1}{2} \quad \text{or} \quad \phi = 120° \quad \text{or} \quad \frac{2\pi}{3}$$

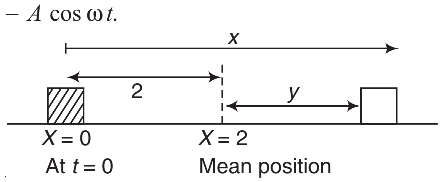
Exercises

LEVEL 1

Assertion and Reason

1. x is measured from the mean position.

2. $y = (-A\cos\omega t)$ at the particle starts from $-A$. So, displacement from the mean position will be $-A\cos\omega t$.

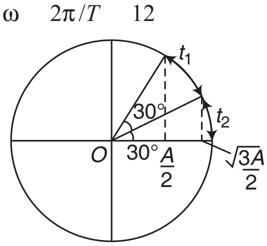


$\therefore \qquad x = y + 2$

$= -2\cos\omega t + 2$

3. By applying a constant force on spring-block system mean position is changed but time period remains unchanged.

4. $t_1 = t_2 = \dfrac{\theta}{\omega} = \dfrac{\pi/6}{2\pi/T} = \dfrac{T}{12}$



5. In angular SHM, path is not straight line.

6. $k \propto \dfrac{1}{l} \qquad \Rightarrow \qquad \omega = \sqrt{\dfrac{k}{m}}$

$\therefore \qquad \omega \propto \dfrac{1}{\sqrt{ml}}$

It m and l both are halved then ω will become 2 times.

7. $F = -kx$. For all displacements. From the mean position, whether they are small or large.

8. $\dfrac{\pi}{2\omega} = \dfrac{\pi}{2(2\pi/T)} = \dfrac{T}{4}$. In the given time particle moves from $x = A$ to $x = 0$.

9. $\left|\dfrac{x}{a}\right| = \left(\dfrac{T}{2\pi}\right)^2 = \dfrac{1}{\omega^2} =$ constant for a given SHM.

10. If a particle P rotates in a circle with constant angle speed ω and we draw a perpendicular on and diameter. This perpendicular cuts the diameter at point Q then motion of Q is simple harmonic. But motion of P is circular.

Single Correct Option

1. In equation $x = A\cos\omega t$, putting $x = \dfrac{A}{2}$ we get

$$\omega t = \frac{\pi}{3}$$

$$\therefore \qquad \left(\frac{2\pi}{T}\right)t = \frac{\pi}{3}$$

$$\text{or} \qquad t = \frac{T}{6}$$

2. Kinetic energy and potential energy in SHM oscillate with double frequency.

3. $v = \omega\sqrt{A^2 - x^2}$

$$\therefore \qquad \frac{v^2}{\omega^2} + \frac{x^2}{(1)^2} = A^2$$

Hence, $v - x$ graph is an ellipse.

4. l in the equation $T = 2\pi\sqrt{\dfrac{l}{g}}$ is measured from centre of mass and centre of mass in both cases remains at same location.

5. **At $t = 1$ s**

$$\phi_1 = \frac{\pi}{2} + \phi$$

and $$\phi_2 = \frac{2\pi}{3} + \phi$$

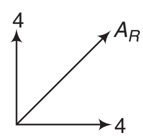
$\therefore$ $$\Delta\phi = \phi_2 - \phi_1 = \frac{\pi}{6}$$

6. $v = \omega\sqrt{A^2 - x^2}$

$\therefore$ $\frac{\omega A}{2} = \omega A\sqrt{A^2 - x^2}$ or $x = \frac{\sqrt{3}}{2}A$

7. All other equation can be converted into the from $y = a \sin(\omega t \pm \phi)$ or $a \cos(\omega t \pm \theta)$.

8. Phase difference between $\cos \pi t$ and $\sin \pi t$ is 90°.



$$A_R = 4\sqrt{2} \text{ units}$$

9. $\omega_1 t = \omega_2 t + 2\pi$ $\qquad (\theta_1 = \theta_2 + 2\pi)$

$\therefore$ $$t = \frac{2\pi}{\omega_1 - \omega_2} = \frac{2\pi}{(2\pi/T_1) - (2\pi/T_2)} = \frac{T_1T_2}{T_2 - T_1} = \frac{4 \times 4.2}{4.2 - 4} = 84$$

Number of vibration of X in this time are,

$$N_1 = \frac{t}{T_1} = \frac{84}{4} = 21$$

10. $mg = kx \quad \Rightarrow \quad \therefore \quad k = \frac{mg}{x}$

$$T = 2\pi\sqrt{\frac{M + m}{R}} = 2\pi\sqrt{\frac{(M + m)x}{mg}}$$

11. $\omega_1 A_1 = \omega_2 A_2$

$\therefore$ $$\frac{A_1}{A_2} = \frac{\omega_2}{\omega_1} = \frac{\sqrt{k_2/m}}{\sqrt{k_1/m}} = \sqrt{\frac{k_2}{k_1}}$$

12. $I = \frac{1}{2}mR^2 + mR^2 = \frac{3}{2}mR^2$

$$T = 2\pi\sqrt{\frac{I}{mgl}}$$

Here $\qquad l = R$

13. $U = \frac{1}{2}kx^2$

i.e. U *versus* x^2 graph is a straight line passing through origin.

14. $T = 2\pi\sqrt{\dfrac{1}{g\left(\dfrac{1}{l} + \dfrac{1}{R}\right)}}$

Put $\qquad l = R$

15. $x = A \cos \omega t$

$$F \propto -x$$

Hence, F is $-\cos \omega t$ graph.

16. In equilibrium, let x_0 is extension in spring then

$$kx_0 = mg \sin\theta$$

$\therefore$ $$x_0 = \frac{mg \sin\theta}{k}$$

= amplitude of oscillations

17. $\frac{d^2x}{dt^2} = -\pi^2 x$

Comparing with $\frac{d^2x}{dt^2} = -\omega^2 x$

We have, $\qquad \omega = \pi$

$\therefore$ $\qquad 2\pi f = \pi$

$$f = \frac{1}{2} \text{ Hz}$$

18. $f = \frac{1}{2\pi}\sqrt{\frac{k}{\mu}}$

where, $\qquad \mu$ = reduced mass $= \frac{M\,m}{M + m}$

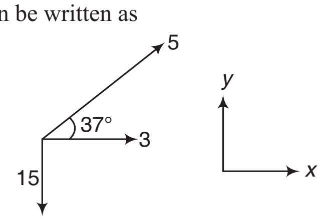
19. $T \propto \frac{1}{\sqrt{g}} \propto \frac{1}{\sqrt{M/R^2}}$

or $\qquad T \propto \frac{R}{\sqrt{M}}$

R and M both are doubled. So, T will become $\sqrt{2}$ times.

$\therefore$ $\qquad T^{-1} = \sqrt{2}\,T = 2\sqrt{2}$ s $\qquad$ (as $T = 2$ s)

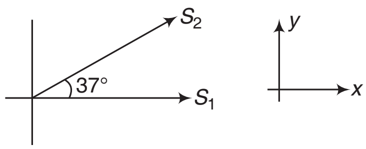
20. x_3 can be written as



$$x_3 = 15 \sin(\omega t - \pi/2)$$

$$\mathbf{A} = 3\,\hat{\mathbf{i}} + 5\cos 37°\,\hat{\mathbf{j}} + 5\sin 37°\,\hat{\mathbf{i}} - 15\,\hat{\mathbf{j}}$$
$$= 7\hat{\mathbf{i}} + 12\hat{\mathbf{j}}$$
$$\therefore \quad |\mathbf{A}| = \sqrt{(7)^2 + (12)^2} = 13.89$$

21. $X = S_1 + S_2 \cos 37°$



$$= (a \sin \omega t + 0.8\, b \sin \omega t) \qquad \ldots(i)$$
$$y = S_2 \sin 37°$$
$$= 0.6\, b \sin \omega t$$
$$\sin \omega t = \frac{y}{0.6\, b}$$

Substituting this value in Eq. (i), we have

$$x = \left(\frac{a + 0.8\, b}{0.6\, b}\right) y$$

This is equation of a straight line passing through origin.

22. (a) Kinetic energy is 0.64 times, it means speed is 0.8 times

$$0.8\,\omega A = \omega\sqrt{A^2 - x^2}$$
$$\therefore \quad x = 0.6\, A = 6\,\text{cm}$$

(b) $$\frac{\omega A}{2} = \omega\sqrt{A^2 - x^2}$$
$$\therefore \quad x = \frac{\sqrt{3}}{2} A$$

23. $$T = 2\pi\sqrt{\left|\frac{x}{a}\right|}$$
$$\therefore \quad \omega = \frac{2\pi}{T} = \sqrt{\left|\frac{x}{a}\right|} = \sqrt{\frac{0.5}{4 \times 10^{-2}}}$$
$$= 3.53 \text{ rad/s}$$

24. Let x_0 is the compression in equilibrium. Then,

$$kx_0 = ma$$
$$x_0 = \frac{ma}{k} = \frac{1 \times 2}{100}$$
$$= 0.02 \text{ m}$$

Amplitude $= x_0 = 0.02$ m **Ans.**

Subjective Question

1. Resultant of k and k is $2k$. Then, resultant of $2k$ and $2k$ is k.

Now $$T = 2\pi\sqrt{\frac{m}{k}}$$

2. $$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{0.2}{80}} = \frac{\pi}{10} \text{ s}$$
$$= 0.314 \text{ s}$$ **Ans.**

3. $F = kx$

$$\therefore \quad k = \frac{F}{x} = \frac{9}{0.05} = 180 \text{ N/m}$$
$$m = \frac{W}{g} = \frac{27}{10} = 2.7 \text{ kg}$$
$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{2.7}{180}}$$
$$= 0.78 \text{ s}$$ **Ans.**

4. $\Delta T = \frac{1}{2} T\alpha\, \Delta\theta$

$$\Rightarrow \quad \Delta t = \frac{\Delta T}{T^1} t$$

But $T' \approx T$

$$\therefore \quad \Delta t = \frac{1}{2}\alpha\, \Delta\theta\,(t)$$
$$= \frac{1}{2} \times 0.000012 \times 20 \times 24 \times 3600$$
$$= 10.37 \text{ s}$$ **Ans.**

At higher temperature, length of pendulum clock will be more. So, time period will be more and it will lose the time.

5. (a) $kx = mg$

$$\therefore \quad k = \frac{mg}{x} = \frac{(20 \times 10^{-3})\,(10)}{7 \times 10^{-2}}$$
$$= \frac{20}{7} \text{ N/m}$$ **Ans.**

(b) $$T = 2\pi\sqrt{\frac{m}{k}}$$
$$= 2\pi\sqrt{\frac{50 \times 10^{-3}}{(20/7)}}$$
$$= 0.84 \text{ s}$$ **Ans.**

6. $k \propto \frac{1}{l}$

Length is halved, so value of force constant of each part will become $2k$. Now net force constant of $2k$ and $2k$ as shown in figure will becomes $4k$.

$$T = 2\pi\sqrt{\frac{m}{k}}$$

or $$T \propto \frac{1}{\sqrt{k}}$$

k has becomes 4 times. Therefore, T will remain half.

8. $g_e = g - \dfrac{\text{Upthrust}}{\text{mass}}$

$$= g - \frac{V\,(\delta/10)\,g}{V\delta} = \frac{9}{10}g$$

$$T' = 2\pi\sqrt{\frac{l}{g_e}} = 2\pi\sqrt{\frac{10}{9}\left(\frac{l}{g}\right)}$$

$$= \left(\sqrt{\frac{10}{9}}\right)T$$

9. For small values of x, the term x^2 can be neglected.

$\therefore$ $F = -6x$

Comparing with $F = -kx$, we get

$k = 6$ N/m **Ans.**

10. $\omega = \dfrac{2\pi}{T} = \dfrac{2\pi}{0.25} = (8\pi)$ rad/s

$$x = A\sin(\omega t + \phi)$$

$$U = \frac{dx}{dt} = \omega A\cos(\omega t + \phi)$$

At $t = 0$

$$x = A\sin\phi$$

$\therefore$ $5 = A\sin\phi$ ...(i)

$$U = \omega A\cos\phi = (8\pi)\,A\cos\phi$$

$\therefore$ $218 = (8\pi\ A)\cos\phi$ ...(ii)

Solving Eqs. (i) and (ii) we get,

$A = 10$ cm and $\phi = \dfrac{\pi}{6}$ or $30°$

11. KE at mean position

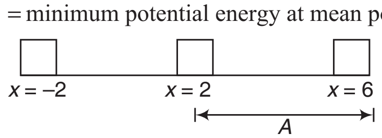
$$K = \frac{1}{2}m\,v_{max}^2 = \frac{1}{2}m\,\omega^2A^2$$

$\therefore$ $\omega = \left(\sqrt{\dfrac{2K}{m}}\right)\dfrac{1}{A}$

$$= \left(\sqrt{\frac{2\times10^{-3}\times8}{0.1}}\right)\frac{1}{0.1} = 4 \text{ rad/s}$$

Now, $y = A\sin(\omega t + \pi/4)$ is the required equation.

12. U_0 = minimum potential energy at mean position



$= 10$ J

At extreme position

U = Total mechanical energy

$= 26$ J $= 10 + (x-2)^2$

$\therefore$ $(x-2) = \pm 4$

Hence, $x = 6$ m and $x = -2$ m are the extreme positions.

(a) $K_{max} = E - U_0 = 16$ J

$\therefore$ $\dfrac{1}{2}m\omega^2A^2 = 16$

or $\dfrac{1}{2}\times2\times\omega^2\times(4)^2 = 16$

or $\omega = 1$ rad/s

(b) At mean position

$E = 26$ J

$U = U_0 = 10$ J

$\therefore$ $K = 16$ J

At extreme position

$K = 0$

$\therefore$ $U = E = 26$ J

13. $g' = \dfrac{g}{\left(1+\dfrac{h}{R}\right)^2}$

Putting $h = R$

$$g' = \frac{g}{4}$$

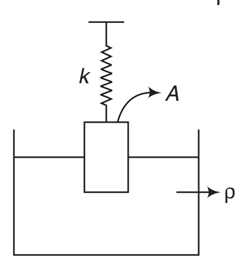
$$T = 2\pi\sqrt{\frac{l}{(g/4)}}$$

$$= 4\pi\sqrt{\frac{l}{g}}$$

$= 4\sqrt{l} = 4\sqrt{1.0}$ (as $g = \pi^2$)

$= 4$ s **Ans.**

14. In such situation, upthrust also behaves like a spring force of force constant $= \rho Ag$



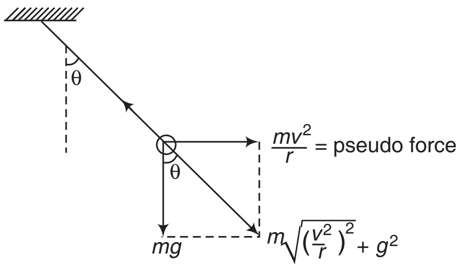
k_{net} in the given situation is $(k + \rho Ag)$

$\therefore$ $T = 2\pi\sqrt{\dfrac{m}{k+\rho Ag}}$

$$= 2\pi\sqrt{\frac{10}{100 + 1000\times20\times10^{-4}\times10}}$$

$= 1.8$ s **Ans.**

15. $g_e = \sqrt{\left(\frac{v^2}{r}\right)^2 + g^2}$



$$T = 2\pi\sqrt{\frac{g}{g_e}} \quad \Rightarrow \quad \theta = \tan^{-1}\left(\frac{v^2}{rg}\right)$$

16. $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{4 \times 10^3}{0.1}} = 200$ rad/s

$$X = 10 \sin (200\, t)$$
$$6 = 10 \sin 200\, t_1$$
$$\therefore \quad 200\, t_1 = \sin^{-1}\left(\frac{3}{5}\right) = 0.646 \text{ rad}$$
$$\therefore \quad t_1 = 3.23 \text{ ms}$$
$$8 = 10 \sin 200\, t_2$$
$$\text{or} \quad 200\, t_2 = \sin^{-1}\left(\frac{4}{5}\right) = 0.925 \text{ rad}$$
$$\therefore \quad t_2 = 4.62 \text{ ms}$$
$$\therefore \quad \Delta t = 1.4 \text{ ms} = 1.4 \times 10^{-3} \text{s}$$ **Ans.**

17. (a) For small displacement x, the term x^2 can be neglected.

$$\therefore \quad F = -100x$$

Comparing with $F = -kx$ we have,

$$k = 100 \text{ N/m}$$
$$T = 2\pi\sqrt{\frac{m}{k}}$$
$$= 2\pi\sqrt{\frac{0.2}{100}} = 0.28 \text{ s}$$ **Ans.**

(b) $|\Delta F| = 10x^2 = 10\,(0.04)^2 = 0.016$ N

$|F| = 100\,x = 100\,(0.04) = 4$N

$\therefore$ % error $= \frac{|\Delta F|}{|F|} \times 100 = 0.4\%$

18. It is a physical pendulum, the time period of which is,

$$T = 2\pi\sqrt{\frac{I}{mgl}}$$

Here, I = moment of inertia of the ring about point of suspension

$$= mr^2 + mr^2 = 2mr^2$$

and l = distance of point of suspension from centre of gravity $= r$

$$\therefore \quad T = 2\pi\sqrt{\frac{2mr^2}{mgr}}$$
$$= 2\pi\sqrt{\frac{2r}{g}}$$

$\therefore$ Angular frequency

$$\omega = \frac{2\pi}{T}$$
$$\text{or} \quad \omega = \sqrt{\frac{g}{2r}}$$ **Ans.**

19. (a) Frequency $= \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

(Frequency is independent of g in spring)

(b) Extension in spring in equilibrium

$$\text{initial} = \frac{mg}{k}$$

Extension in spring in equilibrium in accelerating lift $= \frac{m\,(g + a)}{k}$

$\therefore$ Amplitude $= \frac{m\,(g+a)}{k} - \frac{mg}{k} = \frac{ma}{k}$ **Ans.**

20. $\omega = \frac{2\pi}{T} = \frac{2\pi}{0.05} = (40\pi)$ rad/s

Since, the body starts from the extreme position, we can write

$$\theta = \theta_0 \cos\omega t$$
$$\text{or} \quad \theta = \left(\frac{\pi}{10}\right)\cos 40\pi t$$ **Ans.**

21. $\omega = \frac{2\pi}{T} = \frac{2\pi}{16} = \frac{\pi}{8}$ rad/s

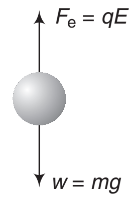
If at $t = 0$, particle passes through its mean position ($x = A \sin \omega t$) with maximum speed its v - t equation can be written as

$$v = \omega A \cos \omega t$$

Substituting the given values, we have

$$2 = \left(\frac{\pi}{8}\right) A \cos\left(\frac{\pi}{8}\right)(2)$$
$$\therefore \quad A = \frac{16\sqrt{2}}{\pi} \text{ m}$$
$$= 7.2 \text{ m}$$ **Ans.**

22. The two forces acting on the bob are shown in figure



g_{eff} in this case will be $\dfrac{w - F_e}{m}$

or $$g_{\text{eff}} = \frac{mg - qE}{m}$$

$$= g - \frac{qE}{m}$$

$\therefore$ $$T = 2\pi\sqrt{\frac{l}{g_{\text{eff}}}}$$

$$= 2\pi\sqrt{\frac{l}{g - \dfrac{qE}{m}}} \qquad \textbf{Ans.}$$

23. (a) $g_e = g + a$

(b) $g_e = g - a$

(c) $g_e = 0$

(d) $g_e = \sqrt{g^2 + a^2}$

24. (a) Before the lump of putty is dropped the total mechanical energy of the block and spring is

$$E_1 = \frac{1}{2}kA_1^2.$$

Since, the block is at the equilibrium position, $U = 0$, and the energy is purely kinetic. Let v_1 be the speed of the block at the equilibrium position, we have

$$E_1 = \frac{1}{2}Mv_1^2 = \frac{1}{2}kA_1^2$$

$\therefore$ $$v_1 = \sqrt{\frac{k}{M}}\,A_1$$

During the process momentum of the system in horizontal direction is conserved. Let v_2 be the speed of the combined mass, then

$$(M + m)v_2 = Mv_1$$

$\therefore$ $$v_2 = \frac{M}{M + m}v_1$$

Now, let A_2 be the amplitude afterwards. Then,

$$E_2 = \frac{1}{2}kA_2^2 = \frac{1}{2}(M + m)v_2^2$$

Substituting the proper values, we have

$$A_2 = A_1\sqrt{\frac{M}{M + m}} \qquad \textbf{Ans.}$$

Note *$E_2 < E_1$, as some energy is lost into heating up the block and putty.*

Further, $$T_2 = 2\pi\sqrt{\frac{M + m}{k}} \qquad \textbf{Ans.}$$

(b) When the putty drops on the block, the block is instantaneously at rest. All the mechanical energy is stored in the spring as potential energy. Again the momentum in horizontal direction is conserved during the process. But now it is zero just before and after putty is dropped. So, in this case, adding the extra mass of the putty has no effect on the mechanical energy, i.e.

$$E_2 = E_1 = \frac{1}{2}kA_1^2$$

and the amplitude is still A_1. Thus,

$$A_2 = A_1$$

and $$T_2 = 2\pi\sqrt{\frac{M + m}{k}} \qquad \textbf{Ans.}$$

25. Let v is speed of combined mass just after collision. Then, from conservation of linear momentum, we have

$$(M + m)v = mv_0$$

$\therefore$ $$v = \frac{mv_0}{M + m}$$

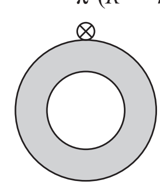
This is maximum speed of combined mass at mean position

$\therefore$ $$v = \omega A$$

or $$\left(\frac{m v_0}{M + m}\right) = \sqrt{\left(\frac{k}{M + m}\right)}\,A$$

$\therefore$ $$A = \frac{mv_0}{\sqrt{k(M + m)}}$$

26. Mass per unit area $= \dfrac{m}{\pi(R^2 - r^2)} = \sigma$ (say)



Whole mass $m_1 = (\pi R^2)\sigma$

$$= \left(\frac{R^2}{R^2 - r^2}\right)m$$

Mass of cavity $m_2 = (\pi r^2)\,\sigma$

$$= \left(\frac{r^2}{R^2 - r^2}\right) m$$

$$I = \frac{3}{2} m_1 R^2 - \left[\frac{1}{2} m_2 r^2 + m_2 R^2\right]$$

$$= \frac{3}{2}\left[\frac{R^2}{R^2 - r^2}\right] mR^2 - \left[\frac{1}{2}\left(\frac{r^2}{R^2 - r^2}\right) mr^2 + \left(\frac{r^2}{R^2 - r^2}\right) mR^2\right]$$

$$= \frac{m}{2\,(R^2 - r^2)}\,[3R^4 - r^4 - 2r^2R^2]$$

$$= \frac{m\,(3R^2 + r^2)}{2}$$

Now, $T = 2\pi \sqrt{\dfrac{I}{mgl}}$ ...(i)

Here, $l = R$

$\therefore$ $\dfrac{I}{mR} = \dfrac{3R}{2} + \dfrac{r^2}{2R}$

Substituting in Eq. (i) we have,

$$T = 2\pi \sqrt{\frac{\dfrac{3R}{2} + \dfrac{r^2}{2R}}{g}}$$

Comparing with

$$T = 2\pi \sqrt{\frac{l}{g}}$$

l of pendulum $= \dfrac{3R}{2} + \dfrac{r^2}{2R}$

$l = \dfrac{3R}{2}$ as $r \to 0$

and $l = 2R$ as $r \to R$

27. $2\pi \sqrt{\dfrac{l}{g}} = 2\pi \sqrt{\dfrac{I}{mgl'}}$

$\therefore$ $l = \dfrac{I}{ml'}$

or $I = (m)\,(l)\,(l')$

$= (200)\,(20)\,(35)$

$= 1.4 \times 10^5$ g - cm^2

28. At earth's surface, the value of time period is given by

$$T = 2\pi \sqrt{\frac{L}{g}} \quad \text{or} \quad T \propto \frac{1}{\sqrt{g}}$$

At a depth h below the surface,

$$g' = g\left(1 - \frac{h}{R}\right)$$

$\therefore$ $\dfrac{T'}{T} = \sqrt{\dfrac{g}{g'}}$

$$= \sqrt{\frac{1}{\left(1 - \dfrac{h}{R}\right)}} = \sqrt{\frac{R}{R - h}}$$

$\therefore$ $T' = T\sqrt{\dfrac{R}{R - h}}$

or $T' \propto \dfrac{1}{\sqrt{R - h}}$ **Hence proved.**

Further, $T_{R/2} = 2\sqrt{\dfrac{R}{R - R/2}}$

$= 2\sqrt{2}$ s **Ans.**

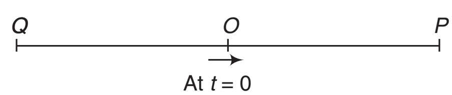
29. (a) Distance travelled in first 4 s

$= OP + PO = A + A = 2A$

Distance travelled in next 4 s

$= OQ + QO = A + A = 2A$

Two distances are equal.



(b) Distance travelled in first 2 s $= OP = A$ and distance travelled in next 2 s $= PO = A$

Again the two distances are same.

30. (a) $u = \omega\sqrt{A^2 - x_1^2}$ ... (i)

$v = \omega\sqrt{A^2 - x_2^2}$ ...(ii)

Solving Eqs. (i) and (ii), we get the result,

$$A \text{ or } a = \sqrt{\frac{v^2x_1^2 - u^2x_2^2}{v^2 - u^2}}$$

(b) $u_1 = \omega\sqrt{A^2 - x_1^2}$ ...(iii)

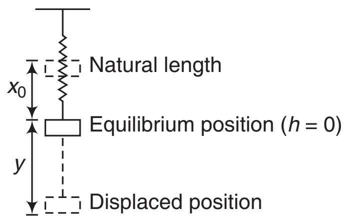
$u_2 = \omega\sqrt{A^2 - x_2^2}$ ...(iv)

Solving Eqs. (iii) and (iv), we find

$$\omega = \sqrt{\frac{u_1^2 - u_2^3}{x_2^2 - x_1^2}} = \frac{2\pi}{T}$$

$\therefore$ $T = 2\pi\sqrt{\dfrac{x_2^2 - x_1^2}{u_1^2 - u_2^2}}$

31. At equilibrium position, $kx_0 = mg$



In displaced position,

$$U = \frac{1}{2}k(x_0 + y)^2 - mg\ y$$

$$= \frac{1}{2}kx_0{}^2 + \frac{1}{2}ky^2 + kx_0y - mgy$$

Substituting $kx_0 = mg$ we get,

$$U = \frac{1}{2}kx_0^2 + \frac{1}{2}ky^2$$

$$= \frac{1}{2}kx_0^2 + \frac{1}{2}ky^2 + kx_0y - mgy$$

But $\quad \frac{1}{2}kx_0^2 =$ constant say U_0

$$\therefore \quad U = U_0 + \frac{1}{2}ky^2$$

32. While returning to equilibrium position,

$$\frac{1}{2}kd^2 = \frac{1}{2}(m_1 + m_2)v^2$$

$$\therefore \quad v = \left(\sqrt{\frac{k_1}{m_1 + m_2}}\right)d$$

Now, after mean position m_2 is detached from m_1 and keeps on moving with this constant velocity v towards right. Block m_1 starts SHM with spring and this v becomes its maximum velocity at mean position.

$$\therefore \quad v = \omega A$$

$$\therefore \quad \left(\sqrt{\frac{k}{m_1 + m_2}}\right)d = \left(\sqrt{\frac{k}{m_1}}\right)A$$

$$\therefore \quad A = \left(\sqrt{\frac{m_1}{m_1 + m_2}}\right)d \qquad \textbf{Ans.}$$

33. (a) In equilibrium, let x_0 is the elongation then,

$$F = kx_0$$

$$\therefore \quad x_0 = \frac{F}{k}$$

This x_0 is the amplitude

$$\therefore \quad A = x_0 = \frac{F}{k}$$

$$T = 2\pi\sqrt{\frac{M}{k}}$$

(b) $E = \frac{1}{2}kx_0^2 = \frac{1}{2}k\left(\frac{F}{k}\right)^2 = \frac{F^2}{2k}$

(c) Kinetic energy at mean position

$$= E = \frac{F^2}{2k}$$

34. (a) $F = kx_0$

$$\therefore \quad x_0 = \frac{F}{k} = \frac{10}{100} = 0.1\text{ m} \qquad \textbf{Ans.}$$

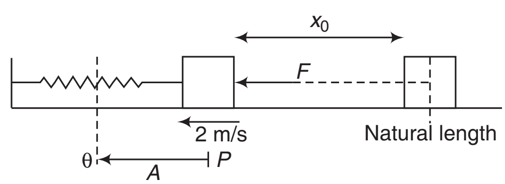
(b) $E = \frac{1}{2}mv^2 + \frac{1}{2}kx_0^2$

$$= \frac{1}{2}\times 1\times(2)^2 + \frac{1}{2}\times 100\times(0.1)^2 = 2.5\text{ J} \qquad \textbf{Ans.}$$

(c) $T = 2\pi\sqrt{\frac{M}{k}}$

$$= 2\pi\sqrt{\frac{1}{100}} = \frac{\pi}{5}\text{ sec} \qquad \textbf{Ans.}$$

(d) From P to Q



Work done by applied force = change in mechanical energy.

$$\therefore \quad FA = E_Q - E_P$$

$$\therefore \quad (10)A = \frac{1}{2}k(A + x_0)^2 - 2.5$$

$$= \frac{1}{2}\times 100(A + 0.1)^2 - 2.5$$

Solving this equation, we get

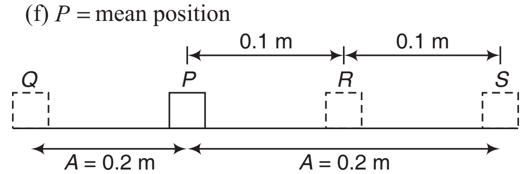
$$A = 0.2\text{ m} \qquad \textbf{Ans.}$$

(e) $U_Q = \frac{1}{2}k(A + x_0)^2$

$$= \frac{1}{2}\times 100(0.2 + 0.1)^2$$

$$= 4.5\text{ J} \qquad \textbf{Ans.}$$

(f) P = mean position



$$Q, S = \text{extreme position}$$
$$R = \text{natural length}$$
$$U_S = \frac{1}{2}k\ (0.1)^2$$
$$= \frac{1}{2} \times 100 \times (0.1)^2 = 0.5\text{ J}$$

Due to work done by the applied fore, F, answer are different.

35. $T_A = 2\pi\sqrt{\dfrac{I}{mgl}}$

$$= 2\pi\sqrt{\frac{(md^2/3)}{mg\ (d/2)}} = \left(\frac{\sqrt{2}}{3}\right) 2\pi\sqrt{\frac{d}{g}}$$

and $\quad T_B = 2\pi\sqrt{\dfrac{d}{g}}$

$\therefore \quad \dfrac{T_A}{T_B} = \sqrt{\dfrac{2}{3}} = 0.816$ **Ans.**

36. In displaced position,

$$E = \frac{1}{2}\ kx^2 + \frac{1}{2}mv^2 + \frac{1}{2}I\ \omega^2$$

Putting $\quad I = \dfrac{1}{2}mR^2$

and $\quad \omega = \dfrac{v}{R}$

we get $\quad E = \dfrac{1}{2}\ kx^2 + \dfrac{3}{4}\ mv^2$

Since, E = constant

$\therefore \quad \dfrac{dE}{dt} = 0$

or $\quad 0 = \dfrac{1}{2}k\left(\dfrac{dx}{dt}\right)(2x) + \dfrac{3}{4}\ m\ (2v)\ \dfrac{dv}{dt}$

Putting, $\quad \dfrac{dx}{dt} = v$ and $\dfrac{dv}{dt} = a$

we get, $\quad F = (ma) = -\left(\dfrac{2k}{3}\right)x$

Since, $F \propto -x$ motion is simple harmonic

$$k_e = -\frac{2k}{3}$$

$$T = 2\pi\sqrt{\frac{m}{k_e}} = 2\pi\sqrt{\frac{3m}{2k}}$$ **Hence Proved.**

37. In equilibrium

$$k\,x_0 = mg$$

$\therefore \quad x_0 = \dfrac{mg}{k} = \dfrac{0.5 \times 10}{20}$

$$= 0.25\text{ m}$$

When displaced downwards by x from the mean position, total mechanical energy,

$$E = -\ mgx + \frac{1}{2}\ mv^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}\ (x + x_0)^2$$

Substituting $I = 0.6\ MR^2$ and $\omega = \dfrac{v}{R}$

We have,

$$E = -\ mgx\ + \frac{1}{2}\ mv^2 + \frac{1}{2}\ (0.6\ MR^2)\left(\frac{v}{R}\right)^2 + \frac{1}{2}k\ (x + x_0)^2$$

Since $\quad E$ = constant

$\therefore \quad \dfrac{dE}{dt} = 0$

or $\quad 0 = -\ mg\left(\dfrac{dx}{dt}\right) + \dfrac{1}{2}\ m\left(2v \cdot \dfrac{dv}{dt}\right)$

$$+ (0.6\ M) + (v)\ \frac{dv}{dt} + \frac{1}{2}\ k\ [2\ (x + x_0)]\ \frac{dx}{dt}$$

Putting $\quad \dfrac{dx}{dt} = v,\ kx_0 = mg \quad$ and $\quad \dfrac{dv}{dt} = a$

We have

$$0 = (ma + 0.6\ Ma) + kx$$

or $\quad a = -\left(\dfrac{k}{m + 0.6\ M}\right)x$

Since, $a \propto -x$ motion is SHM.

$$f = \frac{1}{2\pi}\sqrt{\left|\frac{a}{x}\right|}$$
$$= \frac{1}{2\pi}\sqrt{\frac{k}{m + 0.6\ M}}$$
$$= \frac{1}{2\pi}\sqrt{\frac{20}{0.5 \times 0.6 \times 5}}$$
$$= 0.38\text{ Hz}$$ **Ans.**

38. $x = A\ \sin\ \omega t$ $\quad$ …(i)

$$y = A\ \sin\ (2\omega t + \pi/2)$$
$$= A\ \cos\ 2\omega t$$
$$= A\ (1 - 2\ \sin^2\omega t)$$

From Eq. (i),

$$\sin\ \omega t = \frac{x}{A}$$

$\therefore \quad y = A\left(1 - \dfrac{2x^2}{A}\right)$

39. (a) $x = x_1 + x_2$

At $\quad t = 0.0125$ s

$$x_1 = -\sqrt{2}\text{ cm}$$

and $$x_2 = -1\text{ cm}$$

$\therefore$ $$x = x_1 + x_2 = -2.41\text{ cm}$$

(b) At $t = 0.025$ s.

$$x_1 = 2\text{cm}$$

and $$x_2 = -1.73\text{cm}$$

$\therefore$ $$x = x_1 + x_2$$
$$= 0.27\text{cm}$$

LEVEL 2

Single Correct Option

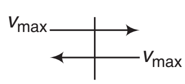
1. $\frac{1}{2}kA^2 = 1.0$

$$\frac{1}{2}k\,(0.4)^2 = 1.0$$

or $$k = 12.5 \quad \text{or} \quad \frac{25}{2}\text{ N/m}$$

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{2}{\frac{25}{2}}} = \frac{4\pi}{5}\text{ s}$$

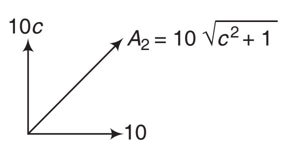
2. $v_{\max} = \omega A = \left(\sqrt{\frac{K}{m}}\right)a$



$$\Delta P = 2m\,v_{\max} = 2m\left(\sqrt{\frac{K}{m}}\right)a$$

$$F = \frac{\Delta P}{\Delta t} = \frac{\Delta P}{T/2} = \frac{2ma\sqrt{k/m}}{\pi\sqrt{m/k}} = \frac{2aK}{\pi}$$

3. $A_1 = 40$ units



Putting $A_1 = A_2$ we get,

$$c = \sqrt{15}$$

4. $T = \frac{1}{F} = \frac{1}{2.5} = 0.4$ s

$$t = 0.3\text{ s} = \frac{3T}{4}$$

At the given time, particle will be in its extreme position if at $t = 0$ it crosses the mean position.

5. $\omega = \frac{2\pi}{T} = \frac{2\pi}{16} = \frac{\pi}{8}$ rad/s

At $t = 0$, particle crosses the means position. Hence its velocity is maximum. So, velocity as a function of time can be written as

$$v = v_{\max}\cos\omega t$$

or $$v = \omega A\cos\omega t$$

$\therefore$ $$1 = \left(\frac{\pi}{8}\right)A\cos\left(\frac{\pi}{8}\right)(2)$$
$$= \left(\frac{\pi}{8\sqrt{2}}\right)A$$

$\therefore$ $$A = \frac{8\sqrt{2}}{\pi}\text{m}$$

6. $g_e = \sqrt{g^2 + a^2} = \sqrt{(10)^2 + (4)^2}$

$$= 10.77\text{ m/s}^2$$

$$T = 2\pi\sqrt{\frac{l}{g_e}} = 2\pi\sqrt{\frac{1}{10.77}}$$
$$= 1.90\text{ s}$$

7. If $v = v_0\sin\omega t$, then

$$a = \frac{dv}{dt} = v_0\omega\cos\omega t$$

$$\sin\omega t = \frac{v}{v_0} \quad \text{...(i)}$$

$$\cos\omega t = \frac{a}{v_0\omega} \quad \text{...(ii)}$$

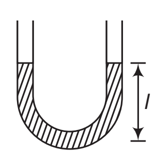
Squaring and adding these equations, we get

$$1 = \frac{v_2}{v_0^2} + \frac{a^2}{v_0\omega}$$

$\therefore$ $$v^2 = \left(-\frac{v_0}{\omega}\right)a^2 + v_0^2$$

Hence, v^2 *versus* a^2 equation is a straight line with positive intercept and negative slope.

8. $T = 2\pi\sqrt{\frac{l}{g}}$



$$2l = \frac{1000}{20}\text{ g}$$
$$= 50\text{ cm} = 0.5\text{ cm}$$

$\Rightarrow$ $$l = 0.25\text{ m}$$

Now $$T = 2\pi\sqrt{\frac{0.25}{9.8}} = 1\text{ s}$$

9. $T = 2\pi\sqrt{\dfrac{m}{k}}$

Here $\qquad m = (\rho_0)(a^3)$

and $\qquad K = \rho_l\ A\ g = \rho a^2 g$

$\therefore \qquad T = 2\pi\sqrt{\dfrac{\rho_0 a}{\rho g}}$

10. $T = 2\pi\sqrt{\dfrac{I}{mgd}}$

$$= 2\pi\sqrt{\frac{\dfrac{ml^2}{12} + md^2}{mgd}}$$

$$= 2\pi\sqrt{\frac{l^2 + 12d^2}{12\ d}}$$

$$= 2\pi\sqrt{\left(\frac{l^2}{12d} + d\right)}$$

T is minimum when first derivation of the quantity inside the root with respect to d is zero.

$$-\frac{l^2}{12d^2} + 1 = 0$$

or $\qquad \dfrac{d}{l} = \dfrac{1}{\sqrt{12}}$

11. In solved example, we have shown that

$$T_1 = 2\pi\sqrt{\frac{m}{k}}$$

$$T_2 = 2\pi\sqrt{\frac{4m}{k}}$$

and $\qquad T_3 = 2\pi\sqrt{\dfrac{m}{4k}}$

12. $T = 2\pi\sqrt{\dfrac{m + m_s/3}{k}}$ (in general)

Here $\qquad m_s =$ mass of spring

and $\qquad m =$ mass of block

13. Half the oscillation is completed with one spring and the other half with other spring. Hence,

$$T = \frac{T_1}{2} + \frac{T_2}{2} = \frac{2\pi\sqrt{\dfrac{M}{k}}}{2} + \frac{2\pi\sqrt{\dfrac{M}{4k}}}{2}$$

$$= \pi\sqrt{\frac{M}{k}}\left[1 + \frac{1}{2}\right] = \frac{3\pi}{2}\sqrt{\frac{M}{k}}$$

14. $F = -\dfrac{dU}{dx} = 20 - 10x$

$$F = 0$$

at $\qquad x = 0$

So, from $x = -3$ to $x = 2$, amplitude in 5.

Hence, other extreme position will be

$$x = 2 + 5 = 7$$

15. $k \propto \dfrac{1}{\text{Length of spring}}$

Length	Force constant
L	k
nL	k/n
$(1-n)L$	$k/(1-n)$

$$m_1 = nm$$

and $\qquad m_2 = (1-n)\ m$

$$\frac{f_1}{f_2} = \frac{\dfrac{1}{2\pi}\sqrt{\dfrac{k/n}{(1-n)\ m}}}{\dfrac{1}{2\pi}\sqrt{\dfrac{k/(1-n)}{nm}}} = 1$$

16. $\dfrac{1}{2}k\ (A^2 - x^2) = \dfrac{1}{4}\left(\dfrac{1}{2}kA^2\right)$

$\therefore \qquad x = \dfrac{\sqrt{3}}{2}A$

$\therefore \qquad CD = CB = \dfrac{\sqrt{3}}{2}R$

or $\qquad BD = 2\ (CD)$

or $\qquad 2\ (CB) = \sqrt{3}R$

17. $K_Q = \dfrac{YA}{L}$

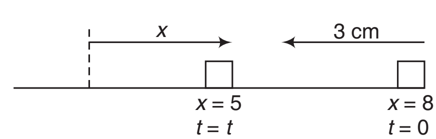
$$K_p = \frac{(2Y)\ (A/2)}{L} = \frac{YA}{L}$$

$\therefore \qquad K_P = K_Q = K = \dfrac{YA}{L}$

Half oscillation is completed with P and half with Q. But their value of K is same. Hence, we can say that in one oscillation one time period is completed with spring.

$$T = \left(\frac{2L}{v}\right) + 2\pi\sqrt{\frac{m}{K}} = \frac{2L}{v} + 2\pi\sqrt{\frac{mL}{AY}}$$

18. $X = A\cos\omega t$



$$5 = 8\cos\omega t$$

$$\omega t = \cos^{-1}(5/8) = 0.9$$

$$\therefore \quad t = \frac{0.9}{\omega} = \frac{0.9}{(2\pi/T)}$$

$$= \frac{0.9T}{2\pi} = \frac{0.9 \times 1.2}{2\pi}$$

$$= 0.17 \text{ s}$$ **Ans.**

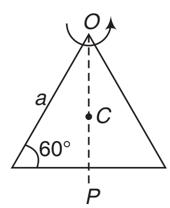
19. $OP = a \sin 60° = \dfrac{\sqrt{3}\,a}{2}$

$$OC = l = \frac{2}{3}(OP) = \frac{a}{\sqrt{3}}$$

$$I = I_0 = 2\left(\frac{ma^2}{3}\right) + m\,(OP)^2 + \frac{ma^2}{12}$$

$$= \frac{2}{3}ma^2 + \frac{3}{4}ma^2 + \frac{1}{12}ma^2 = \frac{3}{2}ma^2$$

$$T = 2\pi\sqrt{\frac{I}{(3m)\,gl}}$$



$$= 2\pi\sqrt{\frac{(3/2m\,a^2)}{(3m)\,g\left(\dfrac{a}{\sqrt{3}}\right)}}$$

$$= 2\pi\sqrt{\frac{\sqrt{3}}{2}\,\frac{a}{g}}$$

Putting $a = \dfrac{1}{\sqrt{3}}$ m and $g = 10 \text{ m/s}^2$

We get, $T = \dfrac{\pi}{\sqrt{5}}$ s **Ans.**

20. $t = \dfrac{2.5\,\pi}{\omega} = \dfrac{2.5\,\pi}{(2\pi/T)}$

$$= (2.5)\left(\frac{T}{2}\right)$$

$$= 5\left(\frac{T}{4}\right)$$

Particle starts from extreme position and in every $\dfrac{T}{4}$ time it travels a distance A. So, in time $t = 5\,(T/4)$ it will travel a distance $5\,A$.

21. Half of the oscillation is completed with length l and rest half with $l/4$.

$\therefore$ Time period $= \dfrac{T_1}{2} + \dfrac{T_2}{2}$

$$= \frac{1}{2}\left[2\pi\sqrt{\frac{l}{g}} + 2\pi\sqrt{\frac{l/4}{g}}\right]$$

$$= \frac{3}{4}\left[2\pi\sqrt{\frac{l}{g}}\right] = \frac{3}{4}T$$

22. $A = \text{radius} = \dfrac{\text{Diameter}}{2} = 0.4$ m

$$f = \frac{30}{60} = \frac{1}{2} \text{ rps or } \frac{1}{2}\text{Hz}$$

$$\therefore \quad T = \frac{1}{f} = 2 \text{ s.}$$

23. $T = 0.6\pi$

$$\therefore \quad \omega = \frac{2\pi}{T} = \frac{2\pi}{0.6\,\pi}$$

$$= \frac{10}{3} \text{ rad/s}$$

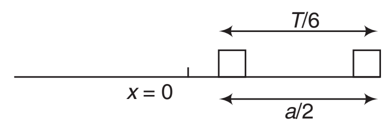
Particle starts from $y = -A = -2$ cm

$$\therefore \quad y = -A\cos\omega t$$

$$= A\sin(\omega t - \pi/2)$$

$$= 2\sin\left(\frac{10}{3}t - \pi/2\right)$$

24. Mean velocity $= \dfrac{\text{displacement}}{\text{time}}$



$$= \frac{(a/2)}{(T/6)} = \frac{3a}{T}$$

25. Maximum acceleration $= g$

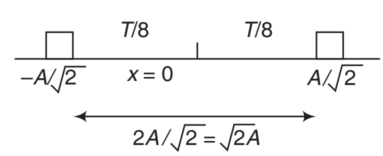
$$\therefore \quad \omega^2 A = g$$

$$\therefore \quad (2\pi f)^2\,(0.5) = g$$

$$f^2 = \frac{2g}{(2\pi)^2}$$

$$\therefore \quad f = \frac{\sqrt{2g}}{2\pi}$$

26. $\dfrac{1}{2}kX^2 = \dfrac{1}{2}k\,(A^2 - X^2)$

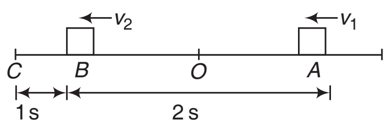


$$\therefore \quad X = \pm\frac{A}{\sqrt{2}}$$

Average speed $= \dfrac{d}{t} = \dfrac{\sqrt{2}A}{T/8 + T/8}$

$$= \frac{4\sqrt{2}\,A}{T}$$

27. $V_1 = V_2$



$$t_{AB} = 2 \text{ s}$$

$\therefore$ $$t_{OB} = 1 \text{ s}$$

$$t_{BCB} = 2 \text{ s}$$

$\therefore$ $$_{BC} = 1 \text{ s}$$

$\therefore$ $$t_{OB} = t_{BC}$$

$\therefore$ $$OB = \frac{A}{\sqrt{2}}$$

or $$\frac{OB}{A} = \frac{1}{\sqrt{2}}$$

28. $t = \dfrac{\pi}{6\omega}$

$\therefore$ $$\omega t = (\pi/6)$$

According to the equation,

$$X = A \cos \omega t$$

$$X = A \text{ at } t = 0$$

and $$X = \frac{\sqrt{3}A}{2} \text{ at } \omega t = \pi/6$$

$\therefore$ Distance travelled

$$= A - \frac{\sqrt{3}}{2}A$$

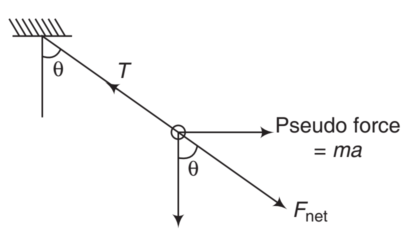
$$= (2 - \sqrt{3})\frac{A}{2}$$

Average speed $= \dfrac{\text{distance}}{\text{time}}$

$$= \frac{(2-\sqrt{3})\,A/2}{(\pi/6\omega)} = \frac{3\omega A}{\pi}(2 - \sqrt{3})$$

More than One Correct Options

1. $\tan\theta = \dfrac{ma}{mg} = \dfrac{a}{g}$

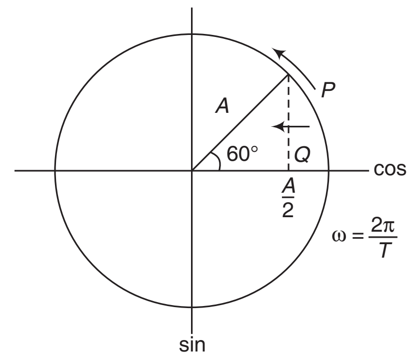


$\therefore$ $$\theta = \tan^{-1}\left(\frac{a}{g}\right)$$

$$T = F_{net}$$

$$= m\sqrt{a^2 + g^2}$$

2. $x = A \sin(\omega t + 150°)$



$$= A \sin\left(\frac{2\pi}{T}t + \frac{5\pi}{6}\right)$$

or $$x = A \cos(\omega t + 60°)$$

$$= A \cos\left(\frac{2\pi}{T}t + \frac{\pi}{3}\right)$$

3. $k x_0 = mg$

$\therefore$ $$x_0 = \frac{mg}{k} = \frac{1 \times 10}{500}$$

$$= 0.02 \text{ m} = 2 \text{ cm}$$

So, equilibrium is obtained after an extension of 2 cm of at a length of 42 cm. But it is released from a length of 45 cm.

$\therefore$ $$A = 3\text{cm} = 0.03 \text{ m}$$

(b) $v_{max} = \omega A = \sqrt{\dfrac{k}{m}}A$

$$= \left(\sqrt{\frac{500}{1}}\right)(0.03) = 0.3\sqrt{5} \text{ m/s}$$

$$= 30\sqrt{5} \text{ cm/s}$$

(c) $a_{max} = \omega^2 A$

$$= \left(\frac{k}{m}\right)(A) = \left(\frac{500}{1}\right)(0.03) = 15 \text{ m/s}^2$$

(d) Mean position is at 42 cm length and amplitude is 3 cm. Hence block oscillates between 45 cm length and 39 cm. Natural length 40 cm lies in between these two, where elastic potential energy = 0.

4. v or KE = 0 at $y = \pm A$

v or KE = maximum at $y = 0$

F or a is maximum at $y = \pm A$

F or a is zero at $y = 0$

5. $v = \omega\sqrt{A^2 - x^2}$

$$\therefore \quad \frac{v^2}{\omega^2} + \frac{x^2}{(1)^2} = A^2$$

i.e. v - x graph is an ellipse.

$$a = -\omega^2 x$$

i.e. a - x graph is a straight line passing through origin with negative slope.

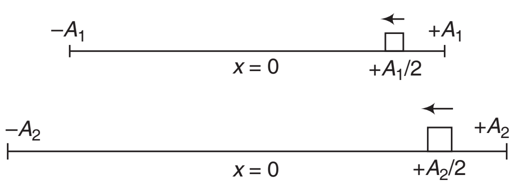
6. $a = 0$ at $x = 0.5$ m

and particle is released from $x = 2$m

Hence,

$$A = 2 - 0.5$$
$$= 1.5 \text{ m}$$
$$\omega^2 = 100$$
$$\therefore \quad \omega = 10 \text{ rad/s}$$
$$T = \frac{2\pi}{\omega} = \frac{2\pi}{10}$$
$$= 0.63 \text{ s}$$
$$v_{max} = \omega A = (10)(1.5)$$
$$= 15 \text{ m/s}$$

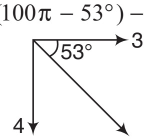
7. Two particles shown in figure are in same phase, although they have distances from the mean position at $t = 0$



8. The given equation can be written as

$$y = 3\sin 100\pi t + (4 - 4\cos 100\pi t) - 6$$
$$= 3\sin 100\pi t + 4\sin(100\pi t + \pi/2) - 2$$

or $\quad y = 5\sin(100\pi - 53°) - 2$



$$y_{max} = 5 - 2 = 3$$
$$y_{min} = -5 - 2 = -7$$

$$\text{Mean position} = \frac{y_{max} + y_{min}}{2} = -2 \text{ cm}$$

Comprehension Based Questions

1. $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{400}{2}} = 10\sqrt{2}$ rad/s

2. $Kx_0 = ma$

$$x_0 = \frac{ma}{K} = \frac{2 \times 5}{400} \text{ m}$$
$$= 0.025 \text{ m}$$
$$= 2.5 \text{cm}$$
$$\therefore \quad A = x_0 = 2.5 \text{ cm}$$

Match the Columns

1. $x_{min} = 2 - 2 = 0 \rightarrow$ extreme position

$x_{min} = 2 + 2 = 4 \rightarrow$ extreme position

Mean position $= \frac{x_{min} + x_{min}}{2} = 2$

Maximum potential energy is at extreme positions.

2. $U_0 + \frac{1}{2}KA^2 = U_{max}$

$$4 + \frac{1}{2}kA^2 = -20 \quad \text{or} \quad \frac{1}{2}kA^2 = 16\text{J}$$

(a) $\quad U = U_0 + \frac{1}{2}kX^2$

$$= U_0 + \frac{1}{2}k\left(\frac{A}{2}\right)^2$$
$$= 4 + \frac{16}{4} = 8 \text{ J}$$

(b) $\text{KE} = \frac{1}{2}k\left(A^2 - \frac{A^2}{116}\right)$

$$= \frac{15}{16}\left(\frac{1}{2}kA^2\right) = \frac{15}{16} \times 16 = 15\text{J}$$

(c) $\text{KE} = \frac{1}{2}k(A^2 - 0)$

$$= \frac{1}{2}kA^2 = 16\text{J}$$

(d) $\text{KE} = \frac{1}{2}k\left(A^2 - \frac{A^2}{4}\right)$

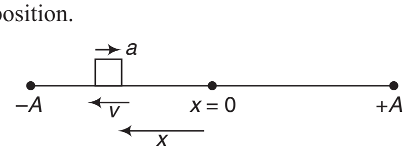
$$= \frac{3}{4}\left(\frac{1}{2}kA^2\right) = \frac{3}{4}(16\text{J}) = 12\text{J}$$

3. (a) $x \propto -\alpha$

At time t_1 acceleration is positive. Hence, X will be negative.

(b) At time $t_2 \quad a = 0 \quad \therefore \quad x = 0$

(c) At time t_1, x is negative and a is increasing. So, the particle is moving toward extreme position.



$\therefore$ Velocity is negative.

(d) At mean position, velocity is maximum. Just after few seconds acceleration becomes positive. So displacement will becomes negative. Hence, the velocity should be negative.

4. $v_{max} = \omega A = (4\pi)(2)$

$$= 25.12 \text{ m/s}$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{4\pi}$$

$$= 0.5 \text{ s}$$

Hence the given time

$$t = 1 \text{ s} = 2T$$

$$\text{Maximum KE} = \frac{1}{2} m v_{max}^2$$

$$= \frac{1}{2} \times 2 \times (25.12)^2$$

$$= 631 \text{ J}$$

Given kinetic energy of 400 J is less than this maximum kinetic energy. So in one time period kinetic energy becomes 400 J four times. In time $2T$ it becomes 400 J eight times.

(d) $|a_{max}| = \omega^2 A$

$$= (4\pi)^2(2)$$

$$= 315.5 \text{ m/s}^2$$

Given acceleration is less than this. So, it becomes two times in one time period and four times in time $2T$.

5. $x_{max} = 4 + 6 = 10 \text{m}$

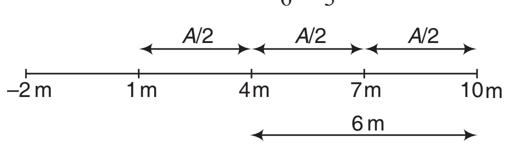
$x_{min} = 4 - 6 = -2 \text{m}$

$$\text{Mean position} = \frac{x_{max} + x_{min}}{2} = 4 \text{ m}$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\pi} = 2 \text{ s}$$

(a) $x = 10\text{m}$ to $x = 4$ m $t = \frac{T}{4} = \frac{1}{2}$ s

(b) $x = 10\text{m}$ to $x = 7\text{m}$, $t = \frac{T}{6} = \frac{1}{3}$ s



(c) $x = 7\text{m}$ to x , $t = 2\left(\frac{T}{12}\right) = \frac{1}{3}$s

(d) $x = 10\text{m}$ to $x = -2\text{m}$, $t = \frac{T}{2} = 1$ s

Subjective Questions

1. $f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{100}{4}} = \frac{2.5}{\pi} = 0.8$ Hz **Ans.**

with 1 kg mass, $f_0 = \frac{1}{2\pi}\sqrt{\frac{100}{1}} = \frac{5}{\pi}$ Hz

Further, from conservation of linear momentum (at mean position)

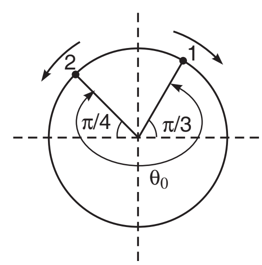
$$\omega A = \frac{1}{4}\omega_0 A_0$$

or $$fA = \frac{1}{4} f_0 A_0$$

or $$A = \frac{f_0 A_0}{4f} - \frac{(5/\pi)(0.1)}{(4)(2.5/\pi)}$$

$$= 0.05 \text{ m}$$ **Ans.**

2. (a) $\theta_0 = \omega_1 t + \omega_2 t$



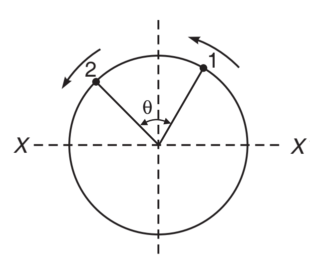
but $$\omega_1 = \omega_2 = \frac{2\pi}{T}$$

and $$\theta_0 = \pi + \frac{\pi}{3} + \frac{\pi}{4} = \frac{19}{12}\pi$$

$\therefore$ $$\frac{19\pi}{12} = \frac{2\pi}{T}t + \frac{2\pi}{T}t$$

or $$t = \frac{19}{48}T$$

(b) $\theta = 2\pi - \theta_0 = 2\pi - \frac{19}{12}\pi = \frac{5\pi}{12}$



Two particles will collide when line XX' becomes the line of bisector of angle θ.

$\therefore$ Any one of the particles (say-2) has rotated an angle

$$\omega t = \pi/4 + \theta/2$$

or $$\frac{2\pi}{T}t = \frac{\pi}{4} + \frac{5\pi}{24} = \frac{11\pi}{24}$$

$\therefore$ $$t = \frac{11T}{48}$$

3. $1.5 = \omega A = 2A \Rightarrow \therefore A = 0.75$ m

Further, the particle will start its journey from its mean position in downward direction (–ve direction).

4. (a) $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{600}{2+1}} = 10\sqrt{2}$ rad/s

From conservation of linear momentum (at mean position) velocity of combined mass will be 2 m/s.

This is the maximum velocity of combined mass.

$\therefore$ $v = \omega A$

or $$A = \frac{v}{\omega} = \frac{2}{10\sqrt{2}} \text{ m}$$

$$= 0.141 \text{ m} = 14.1 \text{ cm}$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{10\sqrt{2}} = 0.44 \text{ s}$$

(b) If the collision is elastic,

$$\omega' = \sqrt{\frac{600}{2}} = 10\sqrt{3} \text{ rad/s}$$

In elastic collision,

$$v_2' = \left(\frac{m_2 - m_1}{m_2 + m_1}\right)v_2 + \left(\frac{2m_1}{m_1 + m_2}\right)v_1$$

$$= \left(\frac{2 \times 1}{1+2}\right)(6) = 4 \text{ m/s} \qquad (\text{As } v_2 = 0)$$

This is maximum velocity.

$\therefore$ $v_2' = \omega' A'$

or $$A' = \frac{v_2'}{\omega'} = \frac{4}{10\sqrt{3}}$$

$$= 0.23 \text{ m} = 23 \text{ cm}$$

$$T' = \frac{2\pi}{\omega'} = \frac{2\pi}{10\sqrt{3}} = 0.36 \text{ s}$$

(c) In both cases journey is started from mean position

$\therefore$ $x = \pm A \sin \omega t$

will be the displacement-time equation. For impulse we can apply the equation. Impulse = change in linear momemtum.

5. $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{400}{4}} = 10$ rad/s

Let t_1 be the time from $x = 0$ to $x = 12$ cm and t_2 the time from $x = 0$ to $x = 9$ cm. Then,

$$12 = 15 \sin (10t_1)$$

or $$t_1 = 0.093 \text{ s}$$

$$9 = 15 \sin (10t_2)$$

or $$t_2 = 0.064 \text{ s}$$

$\therefore$ $$\text{Total time} = t_1 + t_2 = 0.157 \text{ s}$$

6. (a) $y = a(1 - \cos \omega t)$

$$\frac{d^2y}{dt^2} = a\omega^2 \cos \omega t$$

$$N - mg = m \cdot \frac{d^2y}{dt^2}$$

or $$N = mg + ma\omega^2 \cos \omega t$$

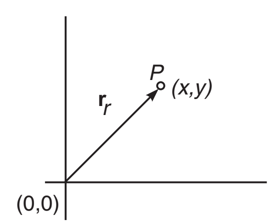
or $$N = m(g + a\omega^2 \cos \omega t)$$ **Ans.**

(b) $\left(\frac{d^2y}{dt^2}\right)_{max} = a\omega^2$ or $a\omega^2 = g$

$\therefore$ $$a = \frac{g}{\omega^2} = \frac{980}{(11)^2}$$

$$= 8.1 \text{ cm}$$ **Ans.**

7. $\mathbf{F} = -kx\hat{\mathbf{i}} - ky\hat{\mathbf{j}}$



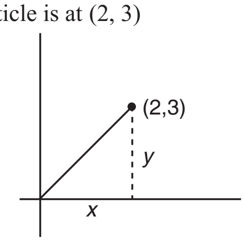
$$\mathbf{F} = 0 \quad \text{at} \quad (0, 0)$$

When it is displaced to a point P whose position vector is

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$$

Force on it is $\mathbf{F} = -k(x\hat{\mathbf{i}} + y\hat{\mathbf{j}}) = -k\mathbf{r}$

Since, $\mathbf{F} \propto -\mathbf{r}$, motion is simple harmonic. At $t = 0$ particle is at (2, 3)



$$\frac{y}{x} = \frac{3}{2} \quad \text{or} \quad 2y - 3x = 0$$

i.e. the particle will oscillate simple harmonically along this line.

8. In equilibrium, $mg \sin\theta = kx_0$ ...(i)

When displaced by x,

$$E = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}k(x + x_0)^2 - mgx \sin\theta$$

Since, $E = \text{constant}$

$$\frac{dE}{dt} = 0$$

$$0 = mv\left(\frac{dv}{dt}\right) + I\omega\left(\frac{d\omega}{dt}\right) + k(x + x_0)\frac{dx}{dt} - mg \sin\theta \frac{dx}{dt}$$

Substituting,

$$\frac{dv}{dt} = a, \quad \omega = \frac{v}{R}, \quad I = \frac{1}{2}mR^2$$

$$\frac{d\omega}{dt} = \alpha = \frac{a}{R}, \quad \frac{dx}{dt} = v$$

and $kx_0 = mg \sin\theta$

We get, $3ma = -2kx$

$$\therefore \quad f = \frac{1}{2\pi}\sqrt{\left|\frac{a}{x}\right|} = \frac{1}{2\pi}\sqrt{\frac{2k}{3m}}$$

Substituting the values,

$$f = \frac{1}{2\pi}\sqrt{\frac{2 \times 200}{\frac{3 \times 100}{9.8}}}$$

$$= 0.56 \text{ Hz}$$ **Ans.**

9. In the displaced position,

$$E = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}k(2x)^2$$

$$I = \frac{1}{2}mR^2 \quad \text{and} \quad \omega = \frac{v}{R}$$

$$\therefore \quad E = \frac{3}{4}mv^2 + 2kx^2$$

$$E = \text{constant}$$

$$\therefore \quad \frac{dE}{dt} = 0$$

or $$\frac{3}{2}mv\frac{dv}{dt} + 4kx\frac{dx}{dt} = 0$$

Substituting, $\frac{dx}{dt} = v$ and $\frac{dv}{dt} = a$

$$a = -\frac{8k}{3m}\,.\,x$$

Comparing with, $a = -\omega^2 x$

We have

$$\omega = \sqrt{\frac{8k}{3m}} = \sqrt{\frac{8 \times 1000}{\frac{3 \times 100}{9.8}}} = 16.16 \text{ rad/s}$$

$$\therefore \quad \theta = \theta_0 \cos \omega t$$

or $$\theta = 0.4 \cos(16.16t)$$ **Ans.**

10. Similar to Q.9

11. Let F be the restoring force (extra tension) on block m when displaced by x from its equilibrium position.

$$x = 2x_1 + 2x_2$$

$$= 2\left[\frac{2F}{k_1} + \frac{2F}{k_2}\right]$$

$$= 4F\left(\frac{k_1 + k_2}{k_1 k_2}\right)$$

or $$F = -\frac{k_1 k_2}{4(k_1 + k_2)}x$$

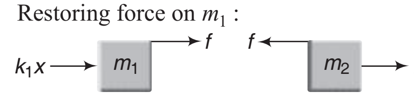
$$\therefore \quad a = -\frac{k_1 k_2}{4m(k_1 + k_2)}x$$

$$\omega = \sqrt{\frac{k_1 k_2}{4m(k_1 + k_2)}}$$ **Ans.**

12. (a) $$f = \frac{1}{2\pi}\sqrt{\frac{k_{\text{eff}}}{\text{total mass}}} = \frac{1}{2\pi}\sqrt{\frac{k_1 + k_2}{m_1 + m_2}}$$

(b) Suppose the system is displaced towards left by a distance x.

Restoring force on m_1 :



$$F = m_1\omega^2 x \quad \text{(towards right)}$$

$$= m_1\left(\frac{k_1 + k_2}{m_1 + m_2}\right)x$$

Friction f on it will be towards right if,

$$k_1 x < F$$

or $$k_1 x < m_1\left(\frac{k_1 + k_2}{m_1 + m_2}\right)x$$

or $$\frac{k_1}{k_2} < \frac{m_1}{m_2}$$ **Ans.**

(c) $$k_1 A_m + \mu m_2 g = m_1\left(\frac{k_1 + k_2}{m_1 + m_2}\right)A_m$$

$$A_m\left(\frac{m_1 k_1 + m_1 k_2}{m_1 + m_2} - k_1\right) = \mu m_2 g$$

or $$A_m = \frac{\mu(m_1 + m_2)m_2 g}{m_1 k_2 - m_2 k_1}$$ **Ans.**

13. (a) $\frac{1}{2}kx_0^2 = \frac{1}{2}\mu v_r^2$

Here, μ = reduced mass $= \frac{6 \times 3}{6+3}$

$= 2$ kg

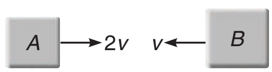
$\therefore\ v_r = \sqrt{\frac{k}{\mu}}\, x_0 = \left(\sqrt{\frac{200}{2}}\right)(3 \times 10^{-2})$

$= 0.3$ m/s $= 2v + v$

$\therefore\quad v = 0.1$ m/s

$\therefore\quad v_1 = 0.2$ m/s

and $\quad v_2 = 0.1$ m/s **Ans.**



Angular frequency

$$\omega = \sqrt{\frac{k}{\mu}} = \sqrt{\frac{200}{2}}$$

$= 10$ rad/s

(b) $v_{cm} = \dfrac{m_1v_1 + m_2v_2 + m_3v_3}{m_1 + m_2 + m_3}$

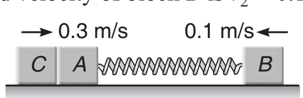
$= \dfrac{(3)(0.2) - (6)(0.1) + 3(0.4)}{3 + 6 + 3}$

$= 0.1$ m/s (towards right) **Ans.**

(c) After collision velocity of combined blocks $(A + C)$

$$v_0 = \frac{(3 \times 0.2) + (3)(0.4)}{3+3} = 0.3 \text{ m/s}$$

and velocity of block B is $v_2 = 0.1$ m/s



The spring will compress till velocity of all the blocks become equal to the centre of mass. Applying conservation of mechanical energy,

$$\frac{1}{2}(3+3)(0.3)^2 + \frac{1}{2}(6)(0.1)^2 = \frac{1}{2}(3+3+6)(0.1)^2 + \frac{1}{2}kA^2$$

Solving this we get, $A = 0.048$ m

or $\quad A = 4.8$ cm **Ans.**

(d) $\Delta E = \frac{1}{2}(3)(0.4)^2 + \frac{1}{2}(3)(0.2)^2 - \frac{1}{2}(3+3)(0.3)^2$

$= 0.24 + 0.06 - 0.27 = 0.03$ J **Ans.**

14. Restoring torque

$$\tau = -(kl\theta)l - \left(k\frac{l}{2}\theta\right)\frac{l}{2} = -\frac{5}{4}kl^2\theta$$

Now, $\left(\frac{ml^2}{3}\right)\alpha = -\left(\frac{5}{4}kl^2\right)\theta$

$$f = \frac{1}{2\pi}\sqrt{\left|\frac{\alpha}{\theta}\right|} = \frac{1}{2\pi}\sqrt{\frac{15k}{4m}}$$

15. When the mass m is displaced from its mean position by a distance x, let F be the restoring (extra tension) force produced in the string. By this extra tension further elongation in the springs are $\frac{2F}{k_1}, \frac{2F}{k_2}, \frac{2F}{k_3}$ and $\frac{2F}{k_4}$ respectively.

Then,

$$x = 2\left(\frac{2F}{k_2}\right) + 2\left(\frac{2F}{k_2}\right) + 2\left(\frac{2F}{k_3}\right) + 2\left(\frac{2F}{k_4}\right)$$

or $\quad F\left(\frac{4}{k_1} + \frac{4}{k_2} + \frac{4}{k_3} + \frac{4}{k_4}\right) = -x$

Here negative sign shows the restoring nature of force.

$$a = -\frac{x}{m\left(\frac{4}{k_1} + \frac{4}{k_2} + \frac{4}{k_3} + \frac{4}{k_4}\right)}$$

$$T = 2\pi\sqrt{\left|\frac{x}{a}\right|}$$

$$= 4\pi\sqrt{m\left(\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \frac{1}{k_4}\right)}$$ **Ans.**

16. Let F be the extra tension in the string, when the block is displaced x from its mean position.

Extension in spring-2 is

$$x_2 = \frac{F}{k_2}$$

Extension is spring-1 is

$$x_1 = \frac{2F}{k_1}$$

$$x = 2x_1 + x_2 = \frac{4F}{k_1} + \frac{F}{k_2}$$

Extra tension F will become restoring force for the block. Therefore, above equation can be written as,

$$F = -\left(\frac{1}{\frac{4}{k_1} + \frac{1}{k_2}}\right)x = \left(\frac{k_1k_2}{4k_2 + k_1}\right)x$$

or $$k_e = \frac{k_1 k_2}{4k_2 + k_1}$$

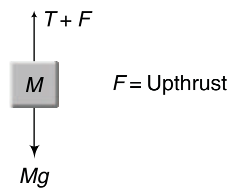
$$T = 2\pi\sqrt{\frac{m}{k_e}}$$

$$= 2\pi\sqrt{\frac{m(4k_2 + k_1)}{k_1 k_2}}$$ **Ans.**

17. In equilibrium,

$$T + F = Mg \qquad \ldots\text{(i)}$$

When the block is further depressed by x, weight Mg remains unchanged, upthrust F increases by ρAxg and let ΔT be the increase in tension.



If a is the acceleration of block then,

$$\Delta T + \rho Axg = Ma \qquad \ldots\text{(ii)}$$

Restoring torque on the cylinder,

$$\tau = \left[\frac{kx}{2}\frac{R}{2} - \Delta T\, R\right] = \left[\frac{kxR}{4} - (Ma - \rho Axg)\, R\right]$$

$$\frac{1}{2}MR^2\alpha = \left[\frac{kR^2\theta}{4} - (MR\alpha - \rho AgR\theta)\, R\right]$$

or $$\frac{3}{2}MR^2\alpha = \left[\frac{kR^2}{4} + \rho AgR^2\right]\theta$$

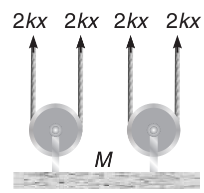
or $$\alpha = \frac{-\left[\frac{k}{4} + \rho Ag\right]}{\frac{3}{2}M}\theta$$

Here negative sign has been used for restoring nature of torque.

$\therefore$ $$f = \frac{1}{2\pi}\sqrt{\left|\frac{\alpha}{\theta}\right|}$$

$$= \frac{1}{2\pi}\sqrt{\frac{k + 4\rho Ag}{6M}}$$ **Ans.**

18. If the mass M is displaced by x from its mean position each spring further stretches by $2x$.



Net restoring force

$$F = -8kx$$

$\therefore$ $$M \cdot a = -8kx$$

$$f = \frac{1}{2\pi}\sqrt{\left|\frac{a}{x}\right|}$$

$$= \frac{1}{2\pi}\sqrt{\frac{8k}{M}}$$

$$= \frac{1}{\pi}\sqrt{\frac{2k}{M}}$$ **Ans.**

15 Elasticity

INTRODUCTORY EXERCISE 15.1

1. $\Delta l = \frac{Fl}{AY}$ or $\Delta l \propto \frac{1}{Y}$

2. $\Delta l = \frac{Fl}{AY} = \frac{1000 \times 100}{4 \times 2 \times 10^6}$

$= 0.0125\,\text{cm}$

3. $\sigma = \frac{F}{A} = \frac{F}{\pi(R^2 - r^2)}$

$$R = \sqrt{\frac{F}{\sigma\pi} + r^2}$$

$$= \sqrt{\frac{1.6 \times 10^6}{90 \times 10^6 \times 3.14} + (0.1)^2}$$

$= 0.1251\text{ m} = 125.1\text{ mm}$

Diameter $= 2R = 250.2$ mm

4. Stress $= \frac{\text{Force}}{\text{Area}}$

Strain $= \frac{\Delta l}{l}$

Modulus of elasticity $= \frac{\text{Stress}}{\text{Strain}}$

INTRODUCTORY EXERCISE 15.2

1. Energy density = energy per unit volume

2. (a) Energy stored $U = \frac{1}{2}$ (stress) (strain) (volume)

or $U = \frac{1}{2}\left(\frac{F}{A}\right)\left(\frac{\Delta l}{l}\right)(Al)$

$= \frac{1}{2} F \cdot \Delta l$

$= \frac{1}{2}(100)(0.3 \times 10^{-3})$

$= 0.015$ J **Ans.**

(b) Work done = Potential energy stored

$= \frac{1}{2} k (\Delta l)^2$

$= \frac{1}{2}\left(\frac{YA}{l}\right)(\Delta l)^2 \qquad \left(\text{as } k = \frac{YA}{l}\right)$

Substituting the values, we have

$$W = \frac{1}{2}\frac{(2.0 \times 10^{11})(10^{-6})}{(2)}(0.1 \times 10^{-3})^2$$

$= 5.0 \times 10^{-4}$ J **Ans.**

Exercises

LEVEL 1

Assertion and Reason

2. At higher pressure, it is difficult to press the gas more. So, bulk modulus is high.

3. If length is doubled, Δl will also becomes two times and Y will remain same.

4. Bulk modulus is related to volume change and volume change is possible in all three states. Young's modulus is related to length change, which is possible only in solids.

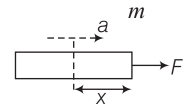
6. Stress $\propto$ strain only in proportionality limit.

7. $B_T = P$

$\Rightarrow \qquad B_S = \gamma p$

$\therefore \qquad \frac{B_T}{B_S} = \frac{1}{\gamma}$

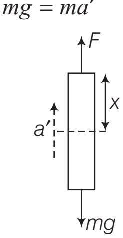
8. ***Case* I** $F = ma \Rightarrow a = \frac{F}{m}$



$F - T = m'a = \frac{m}{L} xa$

$T = F - \frac{m}{L} x \cdot \frac{F}{m} = F\left(1 - \frac{x}{L}\right) \qquad \ldots\text{(i)}$

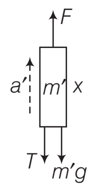
***Case* II** $F - mg = ma'$



$$a' = \frac{F}{m} - g$$

$$F - T - m'g = m'a'$$

$$F - T - m'g = m'\left(\frac{F}{m} - g\right)$$



$$F - T = m' \cdot \frac{F}{m}$$

$$F - T = \frac{m}{L} x \cdot \frac{F}{m} = \frac{x}{L} F$$

$$\Rightarrow \qquad T = F\left(1 - \frac{x}{L}\right) \qquad \text{...(ii)}$$

Tension at a point on the rod (of length L) at a distance x from point of application of force is same in both cases. [from Eqs. (i) and (ii)]

Hence, weight has no effect on tension in case (II).

Note *You can appreciate the extension of rod in first case, by comparing it with a case of many identical blocks connected by ideal springs.*

Extension in rod occurs due to force acting at any point on the rod. In certain cases, when net force acts at the centre of rod like weight, extension may not occur like the given case (II).

9. $Y_{steel} > Y_{copper}$. So, for the same strain, stress to be produced in steel is more.

Also, $\dfrac{\text{Work done}}{\text{Volume}} = \dfrac{1}{2}\text{Stress} \times \text{Strain} = \dfrac{1}{2} Y \times \varepsilon^2$

Single Correct Option

1. $B = -\dfrac{\Delta p}{\Delta V / V}$

$\Delta V = 0$ for an incompressible liquid.

$\therefore \quad B = \infty$

2. Young's modulus of elasticity is a materials property.

3. $\sigma_{max} = \dfrac{F_{max}}{A}$

$\therefore \quad F_{max} = (\sigma_{max}) A$,

which is independent of length of wire.

4. $T = 0$ is free fall.

5. $B = -\dfrac{\Delta p}{\Delta V / V}$

$$= \frac{(1.2 \times 10^5)(1 \times 10^{-3})}{0.3 \times 10^{-3}}$$

$$= 4 \times 10^5 \text{ N/m}^2$$ **Ans.**

6. Length of the wire is $l = 20$ m

Radius of the wire is $r = 2 \times 10^{-3}$ m

Increase in length is

$$\Delta l = 0.031 \times 10^{-3} \text{ m}$$

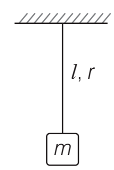
$$g = 3.1 \times \pi \text{ ms}^{-2}$$

Young's modulus is

$$Y = \frac{Fl}{A\Delta l} = \frac{(mg)l}{\pi r^2 (\Delta l)}$$

$$\therefore \quad Y = \frac{4 \times 3.1 \times \pi \times 20}{\pi (2 \times 10^{-3})^2 \times 0.031 \times 10^{-3}}$$

$$= 2 \times 10^{12} \text{ Nm}^{-2}$$



7. Volume of wire is $V = L \times \pi r^2$

$$= 1 \times \pi (10^{-3})^2$$

$$= \pi \times 10^{-6} \text{ m}^3$$

Area of square cross-section

$$= (2 \times 10^{-3})^2$$

$$= 4 \times 10^{-6} \text{ m}^2$$

Length of new wire $= \dfrac{\text{Volume}}{\text{Area}} = \dfrac{\pi \times 10^{-6}}{4 \times 10^{-6}} = \dfrac{\pi}{4}$ m

Initially extension is $x = \dfrac{FL}{AY} = \dfrac{F}{Y}\left(\dfrac{1}{\pi \times 10^{-6}}\right)$

$$\therefore \quad \frac{F}{Y} = (\pi \times 10^{-6}) x$$

Finally extension is $x' = \dfrac{FL'}{A'Y} = \dfrac{F}{Y} \cdot \dfrac{L'}{A'}$

$$x' = (\pi \times 10^{-6})\, x \cdot \frac{\pi/4}{4 \times 10^{-6}} = \frac{\pi^2}{16} x$$

8. In the figure, the reciprocal of slope of stress-strain (x and y-axes) curve, upto proportionality limit, gives Young's modulus. The measure of ductility is obtained as the length of the stress-strain curve between yield point and ultimate load.

9. $\rho' = \dfrac{\rho}{1-\dfrac{dp}{B}} \approx \rho\left(1+\dfrac{dp}{B}\right)$

$$\Delta\rho = \rho' - \rho = \frac{\rho\,(dp)}{B} \qquad \dots\text{(i)}$$

$$\frac{\Delta\rho}{\rho} \times 100 = 0.1$$

$$\therefore \qquad \frac{\Delta\rho}{\rho} = 0.001$$

From Eq. (i),

$$dp \quad \text{or} \quad \Delta p = B\frac{\Delta\rho}{\rho}$$

$$= (2\times 10^9)\,(0.001) = 2\times 10^6 \text{ N/m}^2$$

10. $W = U = \dfrac{1}{2}F(\Delta l) = \dfrac{1}{2}\cdot\dfrac{YA(\Delta l)^2}{l}$

$\therefore \qquad U \propto \dfrac{A}{l}$ (∵ In both the cases Δl is same)

$$\frac{U_2}{U_1} = \frac{A_2}{A_1}\cdot\frac{l_1}{l_2} = \left(\frac{r_2}{r_1}\right)^2\cdot\frac{l_1}{l_2} = (4)(2)$$

$\therefore \qquad U_2 = 8U_1 = 8(2) = 16$ J

11. $U = \dfrac{1}{2}\sigma\varepsilon = \dfrac{1}{2}(Y\varepsilon)\varepsilon$

$\Rightarrow \quad U = \dfrac{1}{2}Y\varepsilon^2$ is similar to $x = ky^2$

Which is a parabola passing through origin and symmetric about x-axis.

12. Change in volume, $\Delta V = \dfrac{-\Delta p\cdot V_i}{B}$

Hence, density at depth of about 11 km is

$$= \frac{\text{Mass}}{\text{Volume}} = \frac{\rho_0\times V_i}{V_i - \dfrac{\Delta p v_i}{B}} = \frac{\rho_0 B}{(B-\Delta p)}$$

$$= \frac{\rho_0}{1-\dfrac{\Delta p}{B}} = \frac{\rho_0}{1-\dfrac{1\times 10^8}{2\times 10^9}}$$

$$= \frac{\rho_0}{1-\dfrac{1}{20}} = \frac{\rho_0}{0.95} \quad\Rightarrow\quad \rho = \frac{\rho_0}{0.95}$$

$\Rightarrow \qquad \rho_0 = 0.95\rho$

$\therefore$ % change in density $= \dfrac{\rho-\rho_0}{\rho_0}\times 100$

$$= \left[\frac{\dfrac{1}{0.95}-1}{1}\right]\times 100$$

$$\approx 5\ \%$$

13. $\Delta l = \dfrac{FL}{AY} = \dfrac{FL}{(\pi r^2)Y}$

$\therefore \qquad \Delta l \propto \dfrac{L}{r^2}$

$\therefore \qquad \dfrac{\Delta l_1}{\Delta l_2} = \dfrac{L/R^2}{2L/(2R)^2} = 2$

14. The net pressure at a depth of 1 km in the ocean is

$$p = \rho g h + p_{\text{atm}}$$
$$= (10^3)\,(9.8)\,(10^3) + 10^5$$
$$= 99\times 10^5 \text{ Pa} = 9.9\times 10^6 \text{ Pa}$$

This pressure acts uniformly on all sides of the balloon (which is in equilibrium) and the restoring forces with in the balloon are equal to external forces. So, normal stress will be same as external pressure.

15. Let T be the tension in the rope. Then,

$$2T = 10 \text{ kN}$$
$$\Rightarrow \qquad T = 5000 \text{ N}$$

Longitudinal stress in the rope is

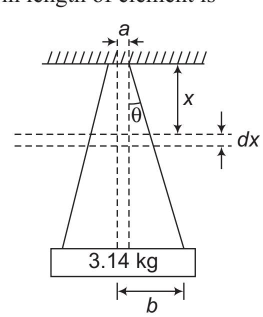
$$\sigma = \frac{T}{A} = \frac{5000 \text{ N}}{10^3 \text{ mm}^2} = 5 \text{N mm}^{-2}$$

Extension in the rope $= \dfrac{\text{Stress}}{Y}\times L$

$$= \frac{5}{10^3}\times(600+900) = 7.5 \text{ mm}$$

$\therefore$ Downward deflection of the load $= \dfrac{7.5}{2} = 3.75$ mm

16. Change in length of element is



$$d(\Delta L) = \frac{F\,dx}{AY}$$

Cross-sectional area of the element is

$$A = \pi\left[a + \frac{b-a}{L}x\right]^2$$

where, $a = \dfrac{b-a}{L}x$ is radius of the wire at the location of element.

$$d(\Delta L) = \frac{F}{\pi\left[a + \frac{b-a}{L}x\right]^2 Y} dx$$

Total change in length of wire is

$$\Delta L = \frac{F}{\pi Y}\int_0^L \frac{1}{\left(a + \frac{b-a}{L}x\right)^2} dx$$

Let $\quad a + \frac{b-a}{L}x = t$

$$\Rightarrow \quad \frac{b-a}{L}dx = dt \quad \Rightarrow \quad dx = \frac{L}{b-a}dt$$

When $x = 0$, $t = a$ and when $x = L$, $t = b$

$$\therefore \quad \Delta L = \frac{F}{\pi Y}\int_a^b t^{-2} \cdot \frac{L}{b-a} dt$$

$$= \frac{FL}{\pi Y(b-a)}(-1)\left(\frac{1}{b} - \frac{1}{a}\right)$$

$$= \frac{FL}{\pi Y(b-a)} \cdot \frac{(b-a)}{ab} = \frac{FL}{\pi Y ab}$$

$$= \frac{(3.14)(9.8)(10)}{(3.14)(2\times 10^{11})(5\times 10^{-4})(9.8\times 10^{-4})}$$

$$= 10^{-3}\,\text{m} \quad \Rightarrow \quad \therefore \quad \Delta L = 10^{-3}\,\text{m}$$

Subjective Questions

1. The changed density, $\rho' = \dfrac{\rho}{1 - \dfrac{dp}{B}}$

Substituting the value, we have

$$\rho' = \frac{11.4}{1 - \dfrac{2.0\times 10^8}{8.0\times 10^9}} \quad \text{or} \quad \rho' = 11.69\ \text{g/cm}^3$$ **Ans.**

2. $\Delta l = \dfrac{Fl}{AY} = \dfrac{Fl}{(\pi d^2/4)Y}$

$$\therefore \quad d = \sqrt{\frac{4\,Fl}{\pi\,(\Delta l)\,Y}}$$

$$= \sqrt{\frac{4\times 400\times 3}{3.14\times 0.2\times 10^{-2}\times 2.1\times 10^{11}}}$$

$$= 1.91\times 10^{-3}\,\text{m} = 1.91\ \text{mm}$$ **Ans.**

3. $\dfrac{m(g+a)}{A} = \dfrac{1}{3}\sigma_{\text{max}}$

$$\therefore \quad a = \frac{\sigma_{\text{max}} A}{3m} - g$$

$$= \frac{(3\times 10^8)(4\times 10^{-4})}{3\times 900} - 9.8$$

$$= 34.64\ \text{m/s}^2$$ **Ans.**

4. Refer solved example 1

$$\Delta l = \frac{mgl}{2AY} = \frac{\rho(\pi r^2 l)gl}{2(\pi r^2)Y}$$

$$= \frac{gl^2\rho}{2Y} = \frac{(9.8)(5)^2(8000)}{2\times 2\times 10^{11}}$$

$$= 4.9\times 10^{-6}\,\text{m}$$

$$U = \frac{1}{2}k(\Delta l)^2 = \frac{1}{2}\frac{YA}{l}(\Delta l)^2$$

$$= \frac{1}{2}\times\frac{Y\times\pi r^2}{l}(\Delta l)^2$$

$$= \frac{(2\times 10^{11})(3.14)(6\times 10^{-3})^2(4.9\times 10^{-6})^2}{2\times 5}$$

$$= 5.43\times 10^{-5}\,\text{J}$$ **Ans.**

5. σ_{max} on upper string $= \dfrac{(m_1 + m_2 + m)g}{0.006\times 10^{-4}}$

$$\therefore \quad 8\times 10^8 = \frac{(10 + 20 + m)\times 10}{0.006\times 10^{-4}}$$

or $\quad m = 18$ kg

σ_{max} on lower string $= \dfrac{(m_1 + m')g}{0.003\times 10^{-4}}$

or $\quad 8\times 10^8 = \dfrac{(10 + m')\times 10}{0.003\times 10^{-4}}$

or $\quad m' = 14$ kg

So, answer is 14 kg and lower string will break earlier.

6. (a) $\sigma = \dfrac{F}{A}$

F and A both are same. Hence, the ratio is 1.

(b) Strain $= \dfrac{\text{Stress}}{Y} \propto \dfrac{1}{Y}$

$$\therefore \quad \frac{(\text{Strain})_{\text{steel}}}{(\text{Strain})_{\text{copper}}} = \frac{Y_{\text{copper}}}{Y_{\text{steel}}} = \frac{13}{20}$$ **Ans.**

7. $\rho' = \dfrac{\rho}{1 - \dfrac{dp}{B}} \approx \rho\left(1 + \dfrac{dp}{B}\right)$

$$\therefore \quad \Delta\rho = \rho' - \rho = \frac{\rho(dp)}{B}$$

$$= \frac{\rho(\rho gh)}{B}$$

$$= \frac{(1030)^2\times 9.8\times 400}{2\times 10^9}$$

$$= 2.0\ \text{kg/m}^3$$ **Ans.**

8. $B = \dfrac{\Delta p}{\Delta V/V} = \dfrac{\rho g h}{(\Delta V/V)}$

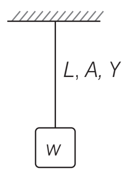
$= \dfrac{(10^3)(9.8)(180)}{(0.1/100)}$

$= 1.76 \times 10^9 \text{ N/m}^2$ **Ans.**

LEVEL 2

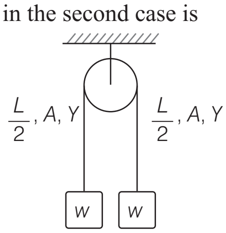
Single Correct Option

1. Extension in the first case is



$$l = \frac{FL}{AY} = \frac{wL}{AY}$$

Extension in the second case is



$$l' = \frac{w(L/2)}{AY} + \frac{w(L/2)}{AY} = \frac{wL}{AY}$$

It is clear that $l' = l$

Note *In the above problem, even if lengths of wire are unequal on two sides of the pulley, the elongation will still be l.*

2. Force on wire = Load carried

$$= m(g + a) = m\left(g + \frac{g}{2}\right) = \frac{3mg}{2}$$

Stress $\sigma = \dfrac{F}{A} \Rightarrow A = \dfrac{F}{\sigma}$

Hence, $A_{\text{min}} = \dfrac{F}{\sigma_{\text{max}}}$

$$\frac{\pi d_{\text{min}}^2}{4} = \frac{\frac{3}{2} m \cdot g}{\sigma}$$

$\Rightarrow$ $d_{\text{min}}^2 = \dfrac{\frac{3}{2} mg \times 4}{\sigma \pi}$

$$d_{\text{min}} = \sqrt{\frac{6}{\pi} \cdot \frac{mg}{\sigma}}$$

3. $T = ml\,\omega^2$

$$\frac{T}{A} = \sigma_{\text{max}} = \frac{ml\omega^2}{A}$$

$\therefore$ $\omega = \sqrt{\dfrac{\sigma_{\text{max}} A}{ml}}$

$$= \sqrt{\frac{4.8 \times 10^7 \times 10^{-6}}{10 \times 0.3}}$$

$= 4$ rad/s **Ans.**

4. $T = \dfrac{M}{2} g = \left(\dfrac{L}{2} s\rho\right) g$

$\therefore$ $\sigma = \dfrac{T}{S} = \dfrac{1}{2} \rho g L$ **Ans.**

5. $F = -\dfrac{dU}{dr} = -$ slope of U - r graph

At B, slope $= 0$

$\therefore$ $F = 0$

Hence, atoms are in equilibrium at B. From A to B or even before A slope is negative. Hence, force is positive and positive force mean repulsion.

6. $l_1 - l = \dfrac{T_1 l}{AY}$ ...(i)

$l_2 - l = \dfrac{T_2 l}{AY}$ ...(ii)

Dividing Eq. (i) by Eq. (ii) and then simplifying, we get

$l = \dfrac{l_2 T_1 - l_1 T_2}{T_1 - T_2}$ **Ans.**

7. $\Delta l = \dfrac{Fl}{AY} = \dfrac{Fl}{(V/l)\,Y}$ (V = Volume)

$$= \frac{Fl^2}{VY}$$

$\therefore$ $\Delta l \propto l^2$ (as F, V and Y = constants)

8. $\Delta l = l\alpha\,\Delta\theta$

$\therefore$ Strain $= \dfrac{\Delta l}{l} = (\alpha\Delta\theta)$

Stress $= Y \times$ Strain $= Y\alpha\Delta\theta$

Energy stored per unit volume

$$= \frac{1}{2} \times \text{stress} \times \text{strain}$$

$$= \frac{1}{2} \times Y \times (\alpha\,\Delta\theta)^2$$

$$= \frac{1}{2} \times 10^{11} \times (12 \times 10^{-6} \times 20)^2$$

$= 2880 \text{ J/m}^3$ **Ans.**

9. Δl due to temperature rise

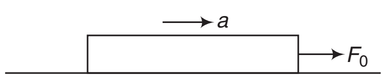
$$= l\alpha\,\Delta\theta$$
$$= (1000)(10^{-4})(20)$$
$$= 2\,\text{mm}$$

But 1 mm is allowed (1000 mm to 1001 mm)

$$\therefore \quad \Delta l_e = 1\text{ mm}$$

$$\text{Stress} = Y \times \text{Strain}$$
$$= (10^{11})\left(\frac{1\text{ mm}}{1000\text{ mm}}\right)$$
$$= 10^8\,\text{N/m}^2 \qquad \textbf{Ans.}$$

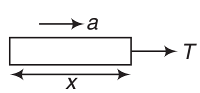
10.



$$a = \frac{F_0}{m}$$

$$T = m_x\, a = \left(\frac{m}{l}\right) x \left(\frac{F_0}{m}\right) = \left(\frac{F_0}{l}\right)x$$

$$\Delta l = \int_0^L \frac{T dx}{SY}$$



$$= \left(\frac{F_0}{lsY}\right) \times \frac{l^2}{2}$$

$$\text{strain} = \frac{\Delta l}{l} = \frac{F_0}{2SY} \qquad \textbf{Ans.}$$

More than One Correct Options

1. $U = \frac{1}{2}kx^2$. It is a parabola symmetric about U-axis.

At $x = 0.2$ mm, $U = 0.2$ J (from the figure)

$$\therefore \quad 0.2 = \frac{1}{2}k(2\times 10^{-4})^2$$

$$\Rightarrow \quad k = 10^7\ \text{Nm}^{-1}$$

$$k = \frac{YA}{L}$$

$$\Rightarrow \quad \frac{A}{L} = \frac{k}{Y} = \frac{10^7}{2\times 10^{11}} = 5\times 10^{-5} \qquad \ldots\text{(i)}$$

$$AL = \text{Volume} = 200\times 10^{-6}\ \text{m}^3 \qquad \ldots\text{(ii)}$$

On solving Eqs. (i) and (ii), we get

$$A = 10^{-4}\,\text{m}^2 \text{ and } L = 2\,\text{m}$$

2. W_F = elastic potential energy stored in the wire

$$= \frac{1}{2}k\,(\Delta l)^2$$
$$= \frac{1}{2}\left(\frac{YA}{L}\right)(l)^2 = \frac{YAl^2}{2L}$$

3.
$$\Delta x = \frac{Fl}{AY}$$

$$\therefore \quad F = \left(\frac{AY}{L}\right)\Delta x$$

i.e. F *versus* Δx graph is a straight line of slope $\frac{YA}{L}$.

$$(\text{Slope})_B > (\text{Slope})_A$$

$$\therefore \quad \left(\frac{YA}{L}\right)_B > \left(\frac{YA}{L}\right)_A$$

or $\qquad (A)_B > (A)_A$

They are of same material. Hence,

$$Y_B = Y_A$$

4. Half of energy is lost in heat and rest half is stored as elastic potential energy.

$$l = \frac{Mgl}{AY} \qquad \ldots\text{(i)}$$

$$U = \frac{1}{2}Kl^2 = \frac{1}{2}\left(\frac{YA}{L}\right)l^2 \qquad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), we can prove that

$$U = \frac{1}{2}Mgl$$

5. When, $T_Q = \frac{mg}{3}$

and $\qquad T_P = mg + \frac{mg}{3} = \frac{4mg}{3}$

Stress, $\qquad \sigma = \frac{T}{\pi r^2}$

$$\frac{\sigma_P}{\sigma_Q} = \frac{T_P}{T_Q}\cdot\left(\frac{r_Q}{r_P}\right)^2$$

If $r_P = r_Q$, $\sigma_P = 4\sigma_Q$, then P breaks.

If $r_P < 2r_Q$, $\sigma_P > \sigma_Q$, then P breaks.

If $r_P = 2r_Q$, $\sigma_A = \sigma_B$, then either P or Q may break.

6. Area of cross-section is $A = \frac{\pi d^2}{4}$

$$\Delta L = \frac{F}{\pi d^2/4}\cdot\frac{L}{Y}$$

$$\Rightarrow \quad \Delta L \propto \frac{1}{d^2}$$

$$\therefore \quad \frac{\Delta L_B}{\Delta L_A} = \frac{(2d)^2}{(d)^2}$$

$$\Rightarrow \quad \Delta L_B = 4\,\Delta L_A$$

$$\text{Strain} = \frac{\Delta L}{L} = \frac{4F}{\pi d^2 Y}$$

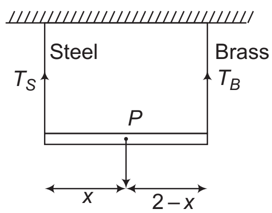
$\Rightarrow$ $$\text{Strain} \propto \frac{1}{d^2}$$

$\therefore$ $$(\text{Strain})_B = 4\,(\text{Strain})_A$$

7. If stress in steel = stress in brass,

then $$\frac{T_S}{A_S} = \frac{T_B}{A_B}$$

$$\frac{T_S}{T_B} = \frac{A_S}{A_B} = \frac{10^{-3}}{2\times10^{-3}} = \frac{1}{2} \qquad \ldots\text{(i)}$$



System is in equilibrium. So, taking moments about P

$$T_S \cdot x = T_B\,(2-x)$$

$\Rightarrow$ $$\frac{T_S}{T_B} = \frac{2-x}{x} \qquad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), we get $x = 1.33$ m

$$\text{Strain} = \frac{\text{Stress}}{Y}$$

If strain in steel = strain in brass,

then $$\frac{T_S/A_S}{Y_S} = \frac{T_B/A_B}{Y_B}$$

$\therefore$ $$\frac{T_S}{T_B} = \frac{A_S\,Y_S}{A_B\,Y_B} = \frac{(10^{-3})\,(2\times10^{11})}{(2\times10^{-3})\,(10^{11})} = 1 \qquad \ldots\text{(iii)}$$

From Eqs. (ii) and (iii), we get $x = 1\,\text{m}$

8. Area of steel rod, $A_S = 16\ \text{cm}^2$

Area of two brass rods,

$$A_B = 2\times10 = 20\ \text{cm}^2$$

$$F = 5000\ \text{kg}$$

σ_S = Stress in steel

and σ_B = Stress in brass

Decrease in length of steel rod = Decrease in length of brass rod

$$\frac{\sigma_S}{Y_S}\cdot L_S = \frac{\sigma_B}{Y_B}\cdot L_B$$

$\Rightarrow$ $$\sigma_S = \frac{Y_S}{Y_B}\cdot\frac{L_B}{L_S}\cdot\sigma_B$$

or $$\sigma_S = \left(\frac{2\times10^6}{10^6}\right)\left(\frac{20}{30}\right)\sigma_B$$

$\Rightarrow$ $$\sigma_S = \frac{4}{3}\,\sigma_B \qquad \ldots\text{(i)}$$

Now, $$F = \sigma_S\,A_S + \sigma_B\,A_B$$

or $$5000 = \sigma_S\times16 + \sigma_B\times20 \qquad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), we get

$$\sigma_B = 120.9\ \text{kg cm}^{-2}$$

and $$\sigma_S = 161.2\ \text{kg cm}^{-2}$$

Comprehension Based Questions

1. Initially, $$\Delta L = \frac{FL}{AY} = \frac{10\times1}{10^{-3}\times2\times10^5} = 0.05\ \text{m} = 5\ \text{cm}$$

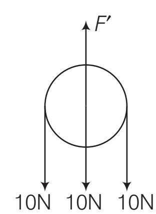
2. Force constant of string is

$$k = \frac{\text{Force}}{\text{Elongation}} = \frac{F}{FL/AY}$$

or $$k = \frac{YA}{L} = \frac{1\times10^5\times10^{-3}}{1} = 200\ \text{Nm}^{-1}$$

Initial elastic potential energy of string

$$= \frac{1}{2}k\,(0.05)^2 = \frac{1}{2}\,(200)\,(25\times10^{-4}) = 25\times10^{-2}\ \text{J}$$



Let after force $F = 10$ N is applied extra elongation is x, then

$$0.25 + 30x = \frac{1}{2}(200)\,(x^2 + 25\times10^{-4} + 0.1x)$$

$$0.25 + 30x = 100x^2 + 0.25 + 10x$$

$\therefore$ $$100x^2 = 20x$$

$\Rightarrow$ $$x = \frac{20}{100} = \frac{1}{5}\,\text{m} = 0.2\ \text{m} = 20\ \text{cm}$$

$\therefore$ $$x_{\max} = 20 + 5 = 25\text{cm}$$

3. When the displacement of the pulley is 25 cm, the string gets loosened on both the sides. So, point A moves down by 50 cm.

4. Young's modulus is a material constant.

5. On crossing the yield region, the material will experience the breaking stress and further elongation causes reduction in stress and breaking of the wire.

6. Stress-strain curve flattens on crossing elastic region.

Match the Columns

1. Stress, modulus of elasticity, energy density and pressure have SI units J/m^3 or N/m^2.

Strain, coefficient of friction and relative density are dimensionless. Force constant have SI unit N/m.

2. $\Delta l = \dfrac{Fl}{AY}$ and stress $= \dfrac{F}{A}$

3. For A and B, the rod is in equilibrium and hence internal restoring force developed per unit area across any cross-section is same and thus stress is uniform. But in C and D, the case is opposite.

Subjective Questions

1. Increase in pressure is $\Delta p = \dfrac{Mg}{A}$

Bulk modulus is $B = \dfrac{\Delta p}{(\Delta V/V)}$

$$\therefore \quad \frac{\Delta V}{V} = \frac{\Delta p}{B} = \frac{Mg}{AB} \qquad \text{...(i)}$$

Also, the volume of the sphere is

$$V = \frac{4}{3}\pi R^3 \quad \Rightarrow \quad \frac{\Delta V}{V} = \frac{3\Delta R}{R}$$

or
$$\frac{\Delta R}{R} = \frac{1}{3}\cdot\frac{\Delta V}{V},$$

Using Eq. (i), we get

$$\frac{\Delta R}{R} = \frac{Mg}{3AB} \quad \Rightarrow \quad \therefore \ \alpha = 3$$

2. Force constant of the wire is

$$k = \frac{F}{\Delta L} = \frac{F}{FL/AY} = \frac{YA}{L}$$

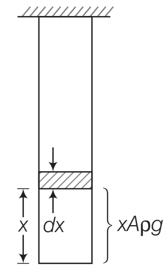
$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{YA}{Lm}} \quad \Rightarrow \quad \therefore \ \sqrt{\frac{YA}{Lm}} = 140$$

$$\Rightarrow \quad \frac{Y \times 4.9 \times 10^{-7}}{1 \times 0.1} = 140 \times 140$$

$$Y = \frac{140 \times 140}{49 \times 10^{-7}} = 4 \times 10^9$$

$$\Rightarrow \quad p = 4$$

3. Consider an element as shown in the figure.



$$\text{Stress in the element} = \frac{\text{Force}}{\text{Area}} = \frac{xA\rho g}{A} = x\rho g$$

Now, elastic potential energy stored in the wire is

$$dU = \frac{1}{2}\text{(Stress) (Strain) (Volume)}$$

$$= \frac{1}{2}\cdot\frac{(\text{Stress})^2}{Y}(\text{Volume})$$

$$dU = \frac{1}{2}\cdot\frac{(x\rho g)^2}{Y}A\,dx = \frac{1}{2}\cdot\frac{\rho^2 g^2 A}{Y}x^2 dx$$

$$\text{Total elastic potential energy} = \frac{1}{2}\cdot\frac{\rho^2 g^2 A}{Y}\int_0^L x^2 dx$$

$$= \frac{\rho^2 g^2 A L^3}{6Y}$$

4. $\Delta l = \dfrac{Fl}{\Delta Y}$

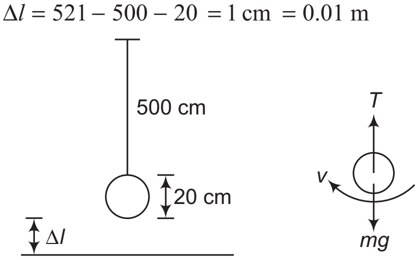
Here, $\quad F = \text{Upthrust} = V_i \rho_l g \ \Rightarrow \ A = \dfrac{\pi d^2}{4}$

$$\therefore \quad \Delta l = \frac{4V_i \rho_l g l}{\pi d^2 Y}$$

$$= \frac{(4)(10^{-3})(800)(9.8)(3)}{(3.14)(0.4 \times 10^{-3})^2 (8 \times 10^{10})}$$

$$= 2.34 \times 10^{-3}\text{m}$$ **Ans.**

5. $\Delta l = 521 - 500 - 20 = 1\text{ cm} = 0.01\text{ m}$



$$T - mg = \frac{mv^2}{R}$$

$$\therefore \quad T = m\left(g + \frac{v^2}{R}\right) = m\left(g + \frac{v^2}{l}\right)$$

$$\Delta l = \frac{Tl}{AY} = \frac{m\left(g + \frac{v^2}{l}\right)l}{(\pi d^2/4)Y}$$

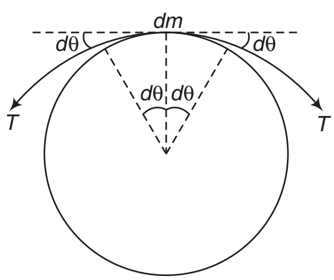
$$= \frac{mgl + mv^2}{(\pi d^2/4)\,Y}$$

$$\therefore \quad v = \sqrt{\frac{\pi d^2\,\Delta l Y}{4m} - gl}$$

$$= \sqrt{\frac{(3.14)(4\times10^{-3})^2(0.01)(2\times10^{11})}{4\times25} - 9.8\times5}$$

≈ 31 m/s **Ans.**

6. $2T\sin d\theta = (dm)\,R\omega^2$



For small angles, $\sin d\theta \approx d\theta$

$$\therefore \quad 2T(d\theta) = (2R\,d\theta)(\pi r^2)(\rho)(R)(2\pi f)$$

$$T = (4\pi^3\ f^2R^2r^2\rho)$$

Now $\quad \Delta l = \dfrac{Tl}{AY}$

$$\frac{\Delta l}{l} = \frac{T}{AY} = \frac{T}{\pi r^2 Y}$$

$$l = 2\pi R$$

$$\Delta l = 2\pi(\Delta R)$$

$$\therefore \quad \frac{\Delta l}{l} = \frac{\Delta R}{R} = \frac{T}{\pi r^2 Y}$$

$$= \frac{4\pi^3 f^2R^2r^2\rho}{\pi r^2 Y}$$

7. $T - mg = ml\omega^2 = ml\,(2\pi f)^2$

$$\therefore \quad T = mg + 4\pi^2 mlf^2$$

$$= 6\times9.8 + 4\pi^2(6)(0.6)(2)^2$$

$$= 628\text{ N}$$

Now, $\quad \Delta l = \dfrac{Tl}{AY}$

$$= \frac{(628)(0.6)}{(0.05\times10^{-4})(2\times10^{11})}$$

$= 3.8\times10^{-4}$m **Ans.**

8. $2T_C + T_S = mg$ …(i)

$$\Delta l_C = \Delta l_s$$

$$\therefore \quad \frac{T_C\,l_C}{A_C Y_C} = \frac{T_S\,l_S}{A_S Y_S}$$

$$\therefore \quad T_S = 2T_C$$

(as $Y_S = 2Y_C$, $l_S = l_C$, $A_S = A_C$)

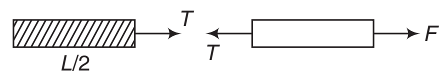
or $\quad T_S = 2T_C$ …(ii)

Solving these two equations we get,

$$T_C = \frac{mg}{4}$$ and

$$T_S = 2T_C$$

9.



(a) $T = \left(\dfrac{M}{2}\right)a_0 = \left(\dfrac{L}{2}S\rho\right)a_0$

$\therefore \quad$ Stress $= \dfrac{T}{S} = \dfrac{L\rho a_0}{2}$ **Ans.**

(b) $\quad T_x = M_x\,a_0 = x\rho Sa_0$

$$\Delta l = \int_0^L \frac{T_x dX}{SY} = \int_0^L \frac{x\rho Sa_0}{SY}\,dX$$

$= \dfrac{\rho a_0 L^2}{2Y}$ **Ans.**

10. $\Delta l = \dfrac{Fl}{AY}$

$$U = \frac{1}{2}K\,(\Delta l^2)$$

$$= \frac{1}{2}\left(\frac{YA}{l}\right)(\Delta l^2)$$

$$= \frac{1}{2}\left(\frac{YA}{l}\right)\left(\frac{Fl}{AY}\right)^2$$

$$= \frac{F^2 l}{2AY} = mS\Delta\theta$$

$$\therefore \quad \Delta\theta = \frac{F^2 l}{2AYms}$$

$$= \frac{(Mg)^2 l}{2(\pi r^2)\,Y\,(l\pi r^2)\rho s}$$

$$= \frac{M^2g^2}{2\pi^2 r^4 Y\rho s}$$

$$= \frac{(100\times9.8)^2}{2\pi^2(2\times10^{-3})^4\,(2.1\times10^{11})\,(7860)\,(420)}$$

$\approx 4.568\times10^{-3}\,{}^\circ$C **Ans.**

16. Fluid Mechanics

INTRODUCTORY EXERCISE 16.1

1. $p_0 + \rho_1 g h_1 = p_0 + \rho_2 g h_2$

$$\therefore \quad h_1 = \frac{\rho_2 h_2}{\rho_1} = \frac{20 \times 0.9}{1} = 18 \text{ cm}$$ **Ans.**

2. Suppose the atmospheric pressure $= p_0$.

Pressure at $A = p_0 + h(1000 \text{ kgm}^{-3})g$

Pressure at $B = p_0 + (0.02\text{m})(13600 \text{ kgm}^{-3})g$

These pressures are equal as A and B which are at the same horizontal level. Thus,

$$h = (0.02\text{m}) \times 13.6 = 0.27\text{m} = 27\text{cm}$$

3. (a) $p = p_0 + \rho_{Hg} g \Delta h$

(b) $p = p_0 + \rho_{Hg} \; g h_{RHS}$

Here, $\Delta h = (8 - 2) = 6 \text{ cm} = 6 \times 10^{-2}$ m

and $h_{RHS} = 8 \text{ cm} = 8 \times 10^{-2}$ m

4. Let the pressure of the liquid just below the piston be p. The forces acting on the piston are:

(a) its weight, mg (downward)

(b) force due to the air above it, $p_0 A$ (downward)

(c) force due to the liquid below it, pA (upward)

It the piston is in equilibrium,

$$pA = p_0 A + mg$$

or

$$p = p_0 + \frac{mg}{A}$$

5. In equilibrium, the pressure at the two surfaces should be equal as they lie in the same horizontal level. If the atmospheric pressure is p and a force F is applied to maintain the equilibrium, the pressure are

$$p_0 + \frac{5\text{N}}{1 \text{ cm}^2} \text{ and } p_0 + \frac{F}{10 \text{ cm}^2} \text{ respectively.}$$

This gives $F = 50$ N

6. $\Delta p_1 = \Delta p_2$

$$\frac{F}{A} = (\Delta h) \rho g$$

$$\therefore \quad \Delta h = \frac{F}{A\rho g} = \frac{45g}{(900 \times 10^{-4})(10^3)g} = 0.5 \text{ m} = 50 \text{ cm}$$

INTRODUCTORY EXERCISE 16.2

1. In vertical direction pressure increases with depth. In horizontal direction pressure decreases in the direction of acceleration.

2. $p_N + \rho_g h - \rho a \left(\frac{h}{2}\right) = p_M$

But $\quad p_N = p_M$

$\therefore \quad a = 2g$

3. $p_B = p_A + \dfrac{\rho \omega^2 x^2}{2} = p_0 + \dfrac{\rho \omega^2 x^2}{2}$

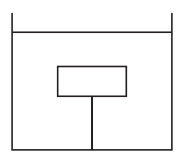
INTRODUCTORY EXERCISE 16.3

1. Weight = upthrust

$$V \rho_1 g = \left(\frac{3}{4} V\right) \rho_2 g$$

$$\therefore \quad \rho_2 = \frac{4}{3} \rho_1$$

2.



T + weight = upthrust

$$\therefore \quad T = \text{upthrust} - \text{weight} = (71.2)\left(\frac{1}{0.75}\right) - 71.2 = 23.7 \text{ N}$$ **Ans.**

3. Reading = weight of water + magnitude of upthrust on piece of metal (acting downwards)

$$= (0.02 \times 9.8) + (10^{-6})(10^3)(9.8) = 0.206 \text{ N}$$ **Ans.**

4. (a) $a = \dfrac{\text{upthrust} - \text{weight}}{\text{mass}}$

$$= \frac{(V)(1000)(9.8) - (V)(0.4 \times 10^{-3})(9.8)}{(V)(0.4 \times 10^3)} = 14.7 \text{ m/s}^2$$ **Ans.**

(b) $t = \sqrt{\dfrac{2s}{a}}$

$$= \sqrt{\frac{2 \times 2.9}{14.7}} = 0.63 \text{ s}$$ **Ans.**

INTRODUCTORY EXERCISE 16.4

1. (a) Apply continuity equation

(b) Apply Bernoulli's equation

2. (a) $\frac{dV}{dt} = Av$

(b) $\frac{dV}{dt} = Av$

(c) Apply Bernoulli's equation

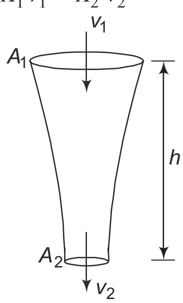
3. From conservation of energy

$$v_2^2 = v_1^2 + 2gh \quad \ldots\text{(i)}$$

[can also be found by applying Bernoulli's theorem]

From continuity equation

$$A_1 v_1 = A_2 v_2$$



$$v_2 = \left(\frac{A_1}{A_2}\right) v_1 \quad \ldots\text{(ii)}$$

Substituting value of v_2 from Eq. (ii) in Eq. (i),

$$\frac{A_1^2}{A_2^2} \cdot v_1^2 = v_1^2 + 2gh$$

or $$A_2^2 = \frac{A_1^2 v_1^2}{v_1^2 + 2gh}$$

$\therefore$ $$A_2 = \frac{A_1 v_1}{\sqrt{v_1^2 + 2gh}}$$

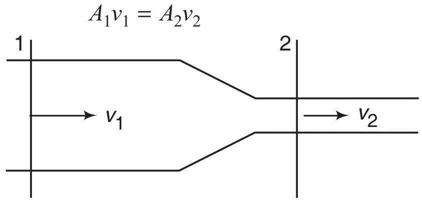
Substituting the given values, we get

$$A_2 = \frac{(10^{-4})(1.0)}{\sqrt{(1.0)^2 + 2\,(10)\,(0.15)}}$$

$$A_2 = 5.0 \times 10^{-5}\ \text{m}^2$$

4. From continuity equation,

$$A_1 v_1 = A_2 v_2$$



$\therefore$ $$v_2 = \left(\frac{A_1}{A_2}\right) v_1 = \left(\frac{10}{5}\right)(1) = 2\ \text{m/s}$$

Applying Bernoulli's theorem at 1 and 2

$$p_2 + \frac{1}{2}\rho v_2^2 = p_1 + \frac{1}{2}\rho v_1^2$$

$\therefore$ $$p_2 = p_1 + \frac{1}{2}\rho\,(v_1^2 - v_2^2)$$

$$= \left(2000 + \frac{1}{2} \times 10^3\,(1 - 4)\right)$$

$\therefore$ $$p_2 = 500\,\text{Pa}$$

INTRODUCTORY EXERCISE 16.5

1. $\Delta p = \frac{1}{2}\rho v^2$

$\therefore$ $$v = \sqrt{\frac{2\Delta p}{\rho}}$$

$$= \sqrt{\frac{2\,(3\ \text{atm} - 1\ \text{atm})}{\rho}}$$

$$= \sqrt{\frac{2 \times 2 \times 10^5}{10^3}} = 20\ \text{m/s}$$

2. $t = \frac{A}{a}\sqrt{\frac{2H}{g}}$ (to empty the complete tank)

Now, $t_{H \to 0} = t_{H \to \frac{H}{3}} + t_{\frac{H}{3} \to 0}$

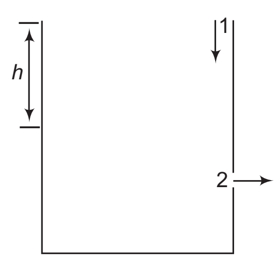
Given, $\frac{A}{a}\sqrt{\frac{2H}{g}} = t_0 + \frac{A}{a}\sqrt{\frac{2\,(H/3)}{g}}$

From here find,

$$\frac{A}{a}\sqrt{\frac{2\,(H/3)}{g}}$$

3. Applying continuity equation at 1 and 2, we have

$$A_1 v_1 = A_2 v_2 \quad \ldots\text{(i)}$$



Further applying Bernoulli's equation at these two points, we have

$$p_0 + \rho g h + \frac{1}{2}\rho v_1^2 = p_0 + 0 + \frac{1}{2}\rho v_2^2 \quad \ldots\text{(ii)}$$

Solving Eqs. (i) and (ii), we have $v_2^2 = \dfrac{2gh}{1 - \dfrac{A_2^2}{A_1^2}}$

Substituting the values, we have

$$v_2^2 = \frac{2 \times 10 \times 2.475}{1 - (0.1)^2} = 50\ \text{m}^2/\text{s}^2$$

INTRODUCTORY EXERCISE 16.6

1. Using, $V = \dfrac{2}{9} \dfrac{r^2 (\rho - \sigma) g}{\eta}$

$$\eta = 1.0\ \text{mPi} = 1.0 \times 10^{-3}\ \text{Pi}$$

$$= \frac{2}{9} \frac{(20 \times 10^{-6})^2 \times (2 \times 10^3 - 10^3) \times 9.8}{1.0 \times 10^{-3}}$$

$$= 0.871\ \text{mm/s}$$

2. $\because$ Volume of bigger drop

$$= n \times \text{volume of smaller drop}$$

$$\frac{4}{3} \pi R^3 = 2 \times \frac{4}{3} \pi r^3$$

$$R = 2^{1/3} r$$

Terminal velocity $\propto r^2$

$\therefore$ $v' = 2^{2/3} v$

3. $F = \eta A \dfrac{\Delta V}{\Delta x}$

$$= 10^{-3} \times 10 \times 2 \times 10^{-1}$$

$$= 2 \times 10^{-2}$$

$$= 0.02\ \text{N}$$

4. $\because$ $F = \eta A \dfrac{\Delta V}{\Delta x}$

$$\frac{F}{A} = \text{shearing stress} = \eta \frac{\Delta V}{\Delta x}$$

$$= 10^{-3} \times \frac{5}{5} = 10^{-3}\ \text{N/m}^2$$

INTRODUCTORY EXERCISE 16.7

1. $\dfrac{4}{3} \pi R^3 = 10^6 \left(\dfrac{4}{3} \pi r^3 \right)$

$\Rightarrow$ $r = (10^{-2}) R = 10^{-2}$ cm

$$A_i = 4\pi R^2$$

$$A_f = (10^6)(4\pi r^2)$$

$$\Delta A = A_f - A_i$$

$\therefore$ $\Delta U = (T)(\Delta A)$

2. $W = T\,(\Delta A)$

$$= T(2)(l)(\Delta d)$$

$$= 7.2 \times 10^{-2} \times 2 \times 0.10 \times 10^{-3}$$

$$= 1.44 \times 10^{-5}\ \text{J}$$

3. $W = T\,(\Delta A)$

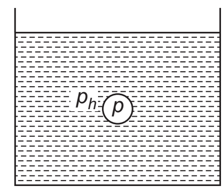
Soap bubble has two free surfaces.

$\therefore$ $W = T\,(8\pi R^2)$

4. $p - p_h = \dfrac{2T}{r}$

$\therefore$ $p = p_h + \dfrac{2T}{r}$

$$= p_0 + h\rho g + \frac{2T}{r}$$



INTRODUCTORY EXERCISE 16.8

1. $h = \dfrac{2T \cos\theta}{r\rho g}$

$\therefore$ $hr = \dfrac{2T \cos\theta}{\rho g} = \text{constant}$

$\therefore$ $h_1 r_1 = h_2 r_2$ or $h_2 = \dfrac{h_1 r_1}{r_2}$

Substituting the values, we get

$$h_2 = (2.0)(3) \qquad \left(\frac{r_2}{r_1} = \frac{1}{3} \right)$$

$$= 6.0\ \text{cm}$$ **Ans.**

2. $\because$ $h = \dfrac{2T \cos\theta}{r\rho g}$

$$T_1 = \frac{h_1 \rho_1 rg}{2 \cos\theta_1}$$

$$T_2 = \frac{h_2 \rho_2 rg}{2 \cos\theta_2}$$

$$\frac{T_2}{T_1} = \frac{h_2}{h_1} \cdot \frac{\rho_2}{\rho_1} \cdot \frac{\cos\theta_1}{\cos\theta_2}$$

$$= 7.23$$

3. $h = \dfrac{2T}{R\rho g}$ (R = radius of meniscus at the top)

$\therefore$ $R = \dfrac{2T}{h\rho g} = \dfrac{2 \times 0.07}{(10^{-2})(10^3)(10)}$

$$= 1.4 \times 10^{-3}\ \text{m}$$

$$= 1.4\ \text{mm}$$

4. $p_A = p_C = (p_B + \rho g h)$

Exercise

LEVEL 1

Assertion and Reason

1. Direction of force in $p = \frac{F}{A}$ is always perpendicular to the surface or it is always specific. Hence, it is not a vector.

3.
$$p_1 = p_0 + \rho g h$$
$$p_2 = p_0 + \rho g\,(2h) \neq 2p_1$$

5. Area of cross-section is different. So, heights are different. In pressure height is more important.

6. Speed will also depend on h.

9. On moon, atmospheric pressure is zero. Hence barometer height is zero.

10. Pressure at P is less. As liquid is flowing at P while liquid is at rest at Q.

11. Force of buoyancy,
$$U = V\rho_{air}\, g$$
Since, ρ_{air} is negligible. Hence, U is negligible.

Single Correct Option

1. $v =$ constant

$\therefore \quad a = 0$

or $\quad F_{net} = 0$

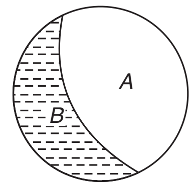
2. $$\left[\frac{\eta}{\rho}\right] = \left[\frac{ML^{-1}T^{-1}}{ML^{-3}}\right] = [M^0L^2T^{-1}]$$

9. $n = n_1 + n_2$
$$\frac{pV}{RT} = \frac{p_1V_1}{RT} + \frac{p_2V_2}{RT}$$
or $$pV = p_1V_1 + p_2V_2$$
$\therefore$ $$\left(\frac{4T}{r}\right)\left(\frac{4}{3}\pi r^3\right) = \left(\frac{4T}{r_1}\right)\left(\frac{4}{3}\pi r_1^3\right) + \left(\frac{4T}{r_2}\right)\left(\frac{4}{3}\pi r_2^3\right)$$
$\therefore$ $$r = \sqrt{r_1^2 + r_2^2} = 5 \text{ cm}$$

10. Drag force + upthrust = weight

$\therefore$ $$ku + F = mg$$
or $$u = \frac{mg - F}{k}$$

11. B will have a tendency to keep its area as low as possible.



12. $m \propto r^3$ and terminal velocity $\propto r^2$ mass is eight times. So, radius is two times and terminal velocity will be four times.

14. $$N\left(\frac{4}{3}\pi r^3\right) = \frac{4}{3}\pi R^3$$
$\therefore$ $$r = R\left(\frac{1}{N}\right)^{1/3}$$
$$\begin{aligned} W &= \sigma\,(\Delta A) \\ &= \sigma\,(A_f - A_i) \\ &= \sigma\,[N\,(4\pi r^2) - 4\pi R^2] \\ &= 4\pi\sigma\left[NR^3\left(\frac{1}{N}\right)^{2/3} - R^2\right] \\ &= 4\sigma\pi R^2\,(N^{1/3} - 1) \end{aligned}$$
Ans.

15. $$(p_2 - p_1) = \frac{4T}{R}$$
where $R =$ radius of common surface

$\therefore$ $$\frac{4T}{R_2} - \frac{4T}{R_1} = \frac{4T}{R}$$
or $$\frac{1}{R} = \frac{1}{R_2} - \frac{1}{R_1}$$

16. Pressure (and hence the level of liquid) will keep on decreasing in the direction of motion of liquid.

17. $$v_t = \frac{2}{9}\frac{r^2\,(\rho - \sigma)\,g}{\eta}$$
Ignoring the density of air σ we have,
$$v_T = \frac{2}{9}\frac{r^2\rho g}{\eta}$$
$\therefore$ $$\begin{aligned} r &= \sqrt{\frac{9\eta\, v_T}{2\rho g}} \\ &= \sqrt{\frac{9 \times 1.8 \times 10^{-5} \times 0.3}{2 \times 10^3 \times 9.8}} \\ &= 0.49 \times 10^{-4} \text{ m} \\ &\approx 0.05 \text{ mm} \end{aligned}$$
Ans.

18. If angle of contact in 90°, then liquid will neither rise nor fall.

19. $v_T \propto r^2$

Radius or diameter is half. So, uniform or terminal speed is $\frac{1}{4}$ th.

20. $$h\rho g = 10^5 \text{ Pa (given)}$$

$$\therefore \quad p_x = \left(h - \frac{h}{5}\right)\rho g = 0.8\, h\rho g$$

$$= 0.8 \times 10^5 \text{ Pa}$$

21. Fraction of volume immersed

$$f = \frac{\rho_s}{\rho_l} \quad \text{or} \quad r \propto \frac{1}{\rho_l}$$

$$\frac{f_1}{f_2} = \frac{\rho_{l_2}}{\rho_{l_1}} \quad \text{or} \quad \rho_{l_2} = \left(\frac{f_1}{f_2}\right)\rho_{l1}$$

$$= \frac{(V - V/3)}{(V/3)}\rho_{l_1}$$

$$= 2\rho_{l1}$$

22. Let ρ = density of sugar solution

$$> \rho_w$$

m = mass of ice.

In floating condition weight = upthrust

$$\therefore \quad mg = V_D \rho g$$

$$\text{or} \quad V_D = \frac{m}{\rho} \qquad \text{...(i)}$$

When ice melts, m mass of ice becomes m mass of water. Volume of this water formed.

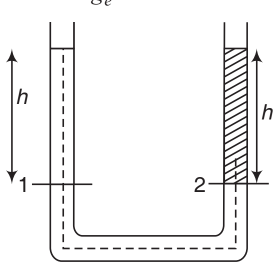
$$V_F = \frac{m}{\rho_w} \qquad \text{...(ii)}$$

Since, $\rho_w < \rho$, $V_F > V_D$. Hence, level will increase.

23. Upthrust $= V_i \rho_l g_e$

Value of g_e will decrease. So upthrust will decrease.

24.



$$p_1 = p_2$$

$$\therefore \quad p_0 + \rho_w gh = p_0 + \rho gh$$

$$\Rightarrow \quad \rho = \rho_w$$

25. F_1 will decrease and F_2 will increase. So f_1 may or may not be greater than f_2.

Total weight to system in both conditions will remain same. Hence,

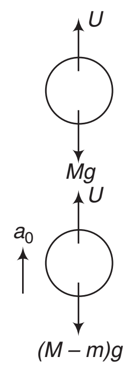
$$f_1 + f_2 = F_1 + F_2$$

26. $\Delta p = \frac{1}{2}\rho v^2$

$$\therefore \quad v = \sqrt{\frac{2\Delta p}{\rho}} = \sqrt{\frac{2 \times 0.5 \times 10^5}{10^3}}$$

$$= 10 \text{ m/s}$$

27.



$$Mg - U = Ma_0 \qquad \text{...(i)}$$

$$U = (M - m)\,g = (M - m)\,a_0 \qquad \text{...(ii)}$$

Solving these two equations we get,

$$m = \frac{2Ma_0}{g + a_0}$$

28. Diameter of left hand side is $\frac{1}{5}$ times. So area will be $\frac{1}{25}$ times.

$$\Delta p_{\text{LHS}} = \Delta p_{\text{RHS}} \quad \Rightarrow \quad \frac{F}{(A/25)} = \frac{(Mg)}{A}$$

$$\therefore \quad F = \frac{Mg}{25}$$

Equating the volumes

Displacement on LHS = 5 times

Displacement on RHS

$$\therefore \quad S = (5)\,(0.5) = 2.5 \text{ cm}$$

$$W = FS$$

$$\therefore \quad (500) = \left(\frac{Mg}{25}\right)(2.5 \times 10^{-2})$$

$$\therefore \quad M = 5 \times 10^4 \text{ kg} \qquad (\text{with } g = 10 \text{ m/s}^2)$$

29. $4 = \frac{Vd_1 + Vd_2}{2V}$

$$\text{or} \quad d_1 + d_2 = 8 \qquad \text{...(i)}$$

$$3 = \frac{m + m}{(m/d_1) + (m/d_2)} \quad \text{or} \quad \frac{2d_1d_2}{d_1 + d_2} = 3 \qquad \ldots\text{(ii)}$$

Solving Eqs. (i) and (ii), we get

$$d_1 = 6 \quad \text{and} \quad d_2 = 2$$

30. Total weight = total upthrust

$(A \times 0.5 \times 900)\, g + (100)\, g = A \times 0.5 \times 1000 \times g$

$\therefore \qquad A = 2\ \text{m}^2$ **Ans.**

31. Change in weight = upthrust.

$\therefore (38.2 - 36.2)\, g = (V_{\text{gold}} + V_{\text{cavity}})\, \rho_w g$

$$\therefore \qquad 2 = \left(\frac{38.2}{19.3}\right) + V_{\text{cavity}} \left(\text{as } \rho_w = \frac{1\ \text{g}}{\text{cm}^3}\right)$$

$\therefore \qquad V_{\text{cavity}} = 0.02\ \text{cm}^3$ **Ans.**

32. See the hint of Q-No. 23 (a) of subjective questions of Level 1.

33. $v = \sqrt{2gH} \quad \Rightarrow \quad \dfrac{dV}{dt} = av = a\sqrt{2gH}$

This is independent of ρ_{liquid}.

34. $(\Sigma Q)_{\text{inflow}} = (\Sigma Q)_{\text{outflow}}$

$\therefore (4 \times 10^{-6}) + Q + (4 \times 10^{-6})$

$= (8 \times 10^{-6}) + (2 \times 10^{-6}) + (5 \times 10^{-6}) + (6 \times 10^{-4})$

$\therefore \qquad Q = 13 \times 10^{-6}\ \text{m}^3/\text{s}$

35. Weight = Upthrust

$$\therefore \qquad Mg = \left(\frac{3}{4} a^3\right) \rho g$$

$$\therefore \qquad a = \left(\frac{4M}{3\rho}\right)^{1/3}$$

36. $h = \dfrac{2T \cos\theta}{r\rho g}$

$$10 = \frac{2T_1 \cos 0^\circ}{r\rho_1 g} \qquad \ldots\text{(i)}$$

$$-3.42 = \frac{2T_2 \cos 135^\circ}{r\rho_2 g} \qquad \ldots\text{(ii)}$$

From these two equations, we get

$$\frac{T_1}{T_2} = \frac{10.\ (\cos 135^\circ)}{(-3.42)\ (\cos 0^\circ)\ \rho_2}$$

$$= \frac{(10)\ (-1/\sqrt{2})\ (1)}{(-3.42)\ (1)\ (13.6)}$$

$$= 1 : 6.5$$

37. Area is halved, means radius of tube is made $\dfrac{1}{\sqrt{2}}$ times.

$$h = \frac{2T \cos\theta}{r\rho g} \quad \text{or} \quad h \propto \frac{1}{r}$$

Hence, h will become $\sqrt{2}$ times.

38. In steady state,

Volume flow rate entering the vessel

= volume flow rate leaving the vessel

$$\therefore \quad Q = av = a\sqrt{2gh} \quad \text{or} \quad h = \frac{Q^2}{2ga^2}$$

$$= \frac{(10^2)^2}{(2 \times 1000)\ (1)^2} = 5\ \text{cm}$$ **Ans.**

39. Area at other point is half. So, speed will be double. Now,

$$p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2$$

$$\therefore \qquad p_2 = p_1 + \frac{1}{2}\rho\ (v_1^2 - v_2^2)$$

$$= (8000) + \frac{1}{2} \times 1000\ (4 - 16)$$

$$= 2000\ \text{Pa}$$ **Ans.**

40. Equating the volume, we have

$$8\left(\frac{4}{3}\pi r^3\right) = \frac{4}{3}\pi R^3 \quad \text{or} \quad R = 2r$$

$$v_T \propto (\text{radius})^2$$

Radius has become two times.Therefore, terminal velocity will become 4 times.

41. Under normal conditions, upward surface tension force supports the weight of liquid of height h in the capillary. In artificial satellite effective weight will be zero, but upward surface tension force will be there. Hence, liquid rises upto the top.

42. Pressure inside a soap bubble is given by,

$$P = P_0 + \frac{4T}{r}$$

So, pressure inside a smaller soap bubble will be more and air will flow from smaller drop to bigger drop.

43. $h = \dfrac{2T \cos\theta}{R\rho g} \quad \text{or} \quad h \propto \dfrac{1}{R}$

Now, $\qquad M = (\pi R^2 h)\, \rho$

or $\qquad M \propto R^2 h$

or $\qquad M \propto R$

Radius is doubled, so mass in the capillary tube will also become two times.

44. $\rho g h = \dfrac{4T}{r} \quad \Rightarrow \quad T = \dfrac{rh\rho g}{4}$

45. $h\rho g = \dfrac{2T}{r} \Rightarrow h = \dfrac{2T}{r\rho g}$

$$= \frac{2 \times 70}{0.005 \times 1 \times 1000} = 28 \text{ cm}$$

46. $N\left(\dfrac{4}{3}\pi a^3\right) = \dfrac{4}{3}\pi b^3 \Rightarrow N = \dfrac{b^3}{a^3}$

$$A_i = N\,(4\pi a^2) = \frac{4\pi b^3}{a}$$

$$A_f = 4\pi b^2$$

$$\Delta A = A_i - A_f = 4\pi b^2\left(\frac{b}{a} - 1\right)$$

$$W = T\,(\Delta A) = 4\pi b^2\left(\frac{b}{a} - 1\right) T$$

$$= \frac{1}{2} mv^2 = \frac{1}{2}\left(\frac{4}{3}\pi b^3 \rho\right) v^2$$

Solving we get

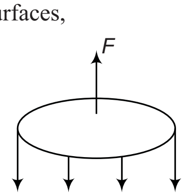
$$v = \sqrt{\frac{6T}{\rho}\left(\frac{1}{a} - \frac{1}{b}\right)}$$

47. $p = \dfrac{2T}{r} + p_0$

$$= \frac{2 \times 0.7}{0.14 \times 10^{-3}} + 10^5$$

$$= 1.01 \times 10^5 \text{ N/m}^2$$

48. From two surfaces,



S.T force

$$F = 2\,(Tl) = 2\,(T)\,(2\pi r)$$

$$= 2\,(T)\,(2\pi)\,(1.5)$$

$$= (6\pi T) \text{ dyne}$$

49. $\Delta p = h\rho g + \dfrac{2T}{r}$

$$= 2 \times 1 \times 980 + \frac{2 \times 70}{0.05}$$

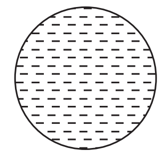
$$= 4760 \text{ dyne/cm}^2$$

Subjective Questions

1. In floating condition,
weight of liquid displaced = weight of solid

2. Weight $= V\rho_w g$

Upthrust $= V\rho_e g = V\rho_w g$



Since, weight = upthrust

Apparent weight = 0

3. $F = pA\,(\rho gh)\,B$

$\therefore \quad F \propto h$

4. If volume is doubled, then radius becomes $(2)^{\frac{1}{3}}$ times.

$$W = T\,(8\pi R^2)$$

or $\quad W \propto R^2$

Therefore $\quad W' = (2)^{\frac{2}{3}}\, W.$

5. Concept is same as in Q.No-2 of same exercise.

6. Relative density of metal in this case is given by

$$\text{RD} = \frac{\text{weight in air}}{\text{change in weight in water}}$$

$$= \frac{0.096}{0.096 - 0.071} = 3.84$$

$\therefore$ Density of metal $= 3.84\,\rho_w$

$= 3840 \text{ kg/m}^3$ **Ans.**

7. Weight = upthrust

$\therefore \quad (25 + 5)\,g = (V + 2)\,\rho_w g$

Putting $\quad \rho_w = 1 \text{ g/cm}^3$

We get, $\quad V = 28 \text{ cm}^3$ **Ans.**

8. Weight of both (block + woman)

= upthrust on 100% volume of block

$\therefore \quad 50\,g + V\,(850)\,g = V\,(1000)\,g$

$\therefore \quad V = 0.33 \text{ m}^3$ **Ans.**

9. Weight of ice block + weight of metal piece
= upthrust on 100% volume of ice cube.

Let a = side of ice cube. Then,

$$[(a^3) \times 900 \times g + 0.5\text{ g}] = (a^3)\,(1000)\,(g)$$

$\therefore \quad a^3 = (5 \times 10^{-3})\text{m}^3$

or $\quad a \approx 0.17$ m or 17 cm **Ans.**

10. Fraction of volume immersed is given by

$$f = \frac{\rho_s}{\rho_l}$$

In first case $\quad 0.6 = \dfrac{\rho_s}{\rho_w} = \dfrac{\rho_s}{1}$

or $\rho_s = 0.6 \text{ g/cm}^3 =$ density of wood

In second case

$$0.8 = \frac{\rho_s}{\rho_l} = \frac{0.6}{\rho_e}$$

or $$\rho_l = \frac{0.6}{0.85}$$

$$= 0.705 \text{ g/cm}^3$$ **Ans.**

11. Extra weight = extra upthrust on extra immersed volume.

$\therefore$ $(\Delta m)\, g = (\pi r^2 \Delta h)\, \rho_w g$

$\therefore$ $\Delta m = \pi r^2 \Delta h\, \rho_w$

$= (\pi)\,(0.8)^2\,(3)\,(1)$

$= 6.03$ g **Ans.**

12. $$p = \frac{F}{A} = \frac{3000 \times 9.8}{425 \times 10^{-4}}$$

$= 6.92 \times 10^5 \text{ N/m}^2$ **Ans.**

13. $p_0 + \rho_w h_w g = p_0 + \rho_s h_s g$

$\therefore$ $$\frac{\rho_s}{\rho_w} = \frac{h_w}{h_s}$$

$$= \frac{10}{12.5} = 0.8$$

= relative density of mercury

14. $(p_0 + \rho_w h_w g) - (p_0 + \rho_s h_s g) = \rho_{Hg} h_{Hg} g$

$\therefore$ $$h_{Hg} = \frac{\rho_w \rho_w - \rho_s h_s}{\rho_{Hg}}$$

$$= \frac{(1)\,(25) - (0.8)\,(27.5)}{13.6}$$

$= 0.221$ cm **Ans.**

15. (i) Absolute pressure in part (*a*)

= 76 cm of Hg + 20 cm of Hg

= 96 cm of Hg

Gauge pressure in part (*a*) = 20 cm of Hg

Absolute pressure in part (*b*)

= 76 cm of Hg – 18 cm of Hg = 58 cm of Hg

Gauge pressure in part (*b*) = – 18 cm of Hg.

(ii) 13.6 cm of water is equivalent to 1 cm of Hg. So the new level difference will become 19 cm.

16. $$v = \frac{p_1 - p_2}{4\eta L}\,(R^2 - r^2)$$

or $v \propto R^2 - r^2$

(a) $$\frac{v_1}{v_2} = \frac{R^2 - r_1^2}{R^2 - r_2^2}$$

or $$\frac{3}{v_2} = \frac{400 - 0}{400 - 100}$$

$\therefore$ $v_2 = 2.25$ m/s **Ans.**

(b) $$v_2 = \left(\frac{R^2 - r_2^2}{R^2 - r_1^2}\right) v_1$$

$v_2 = 0$ at $r_2 = R$

17. (a) $$Q = \frac{\pi}{8}\left(\frac{R^4}{\eta}\right)\left(\frac{p_1 - p_2}{L}\right)$$

$$= \left(\frac{\pi}{8}\right)\frac{(0.04)^4}{1.005 \times 10^{-1}} \times \frac{1400}{0.2}$$

$= 7 \times 10^{-2} \text{ m}^3\text{/s}$

(b) $Q_1 = Q_2$

$\therefore$ $[(p_1 - p_2)\, R^4]_i = [(p_1 - p_2)\, R^4]_f$

Since, diameter or radius has decreased to half. Therefore gauge pressure should become 16 times.

or $(p_1 - p_2)_f = 16 \times 1400$

$= 2.24 \times 10^4$ Pa **Ans.**

(c) $$Q \propto \frac{1}{\eta}$$

$\therefore$ $$\frac{Q_1}{Q_2} = \frac{\eta_2}{\eta_1}$$

or $$Q_2 = \left(\frac{\eta_1}{\eta_2}\right) Q_1$$

$$= \left(\frac{1.005}{0.469}\right)(7 \times 10^{-2})$$

$= 0.15 \text{ m}^3\text{/s}$ **Ans.**

18. (a) Relative density of metal

$$= \frac{\text{weight in air}}{\text{change in weight in water}} = \frac{10}{2} = 5$$

$\therefore$ Density of metal $= 5\rho_w = 5000 \text{ kg/m}^3$

Now, $$\text{volume} = \frac{\text{mass}}{\text{density}}$$

$$= \frac{10 \times 10^{-3}}{5000}$$

$= 2 \times 10^{-6} \text{ m}^3$

(b) Change in weight

= upthrust on 100% volume of solid

or $\Delta w = V_s \rho_l g$

$\therefore$ $\Delta w \propto \rho_l$

$\therefore \quad \dfrac{\Delta w_l}{\Delta w_w} = \dfrac{\rho_l}{\rho_w}$

or $\quad \rho_l = \left(\dfrac{\Delta w_l}{\Delta w_w}\right)\rho_w = \left(\dfrac{1.5}{2}\right)(1000)$

$= 750 \text{ kg/m}^3$

19. Let (x) mm is in water and $(60 - x)$ mm is in mercury.

Total upthrust = weight

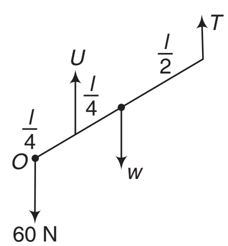
$= (0.06)^2 (x \times 10^{-3})(10^3) g + (0.06)^2 (60 - x)(10^{-3})(13.6 \times 10^3) g$

$= (0.06)^3 (7.7 \times 10^{-3})(g)$

Solving this equation, we get

$x = 28 \text{ mm} \quad \text{and} \quad 60 - x = 32 \text{ mm}$

20. $T + U = 60 + w \qquad \ldots(\text{i})$



Σ (Moments) about = O

$\therefore \quad = U\left(\dfrac{l}{4}\right) + T(l) = W\left(\dfrac{l}{2}\right)$

or $\quad U + 4T = 2W \qquad \ldots(\text{ii})$

$w = 120 \text{ N} \qquad \ldots(\text{iii})$

Solving these three equations we get $T = 20$ N and $U = 160$ N

Now, $160\left(\dfrac{V}{2}\right)\rho_w g = \left(\dfrac{V}{2}\right)(1000)(10)$

$\therefore \quad V = 32 \times 10^{-3} \text{ m}^3$ **Ans.**

21. (a) Upthrust – Weight – $T = ma$

$\therefore \quad T =$ Upthrust – Weight – ma

$= \left(\dfrac{2}{500}\right)(1000)(10 + 2) - 20 - 4$

$= 48 - 20 - 4 = 24 \text{N}$ **Ans.**

(b) Downward force T suddenly becomes zero. Therefore

$a = \dfrac{\text{upthrust} - \text{weight}}{m} = \dfrac{48 - 20}{2}$

$= 14 \text{ m/s}^2$

$\therefore$ Acceleration w.r.t. tank.

$= 14 - 2$

$= 12 \text{ m/s}^2$ **Ans.**

22. (a) Force from left hand side

= force from right hand side

$\therefore \quad kx = \Delta p A$

or $\quad x = \dfrac{(\Delta p) A}{k}$

$= \dfrac{(30 + 101) \times 10^3 \times 0.5 \times 10^{-4}}{60}$

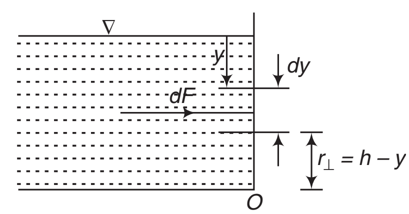
$= 0.109$ m or 10.9 cm **Ans.**

(b) $\quad x = \dfrac{(\Delta p) A}{k}$

$= \dfrac{30 \times 10^3 \times 0.5 \times 10^{-4}}{60}$

$= 0.025 \text{ m} = 2.5 \text{ cm}$ **Ans.**

23. (a) $dF = (p_y)(dA) = (\rho g y)(l dy)$



$\therefore \quad F = \int_0^h dF = \dfrac{\rho g l h^2}{2}$

(b) Perpendicular distance of about force from point O is

$r_\perp = h - y$

$\therefore \quad d\tau = (dF) r_\perp = (\rho g l y)(h - y) dy$

$=$ total torque $= \int_0^h d\tau$

After substituting the values we get,

$\tau = \dfrac{\rho g l h^3}{6}$ **Ans.**

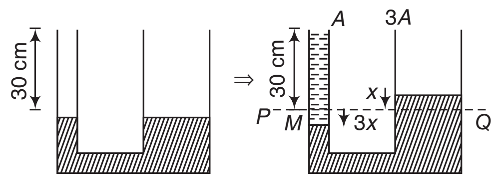
(c) $F \times r_\perp = \tau$

$\therefore \quad r_\perp = \dfrac{\tau}{F}$

Substituting the values we get,

$r_\perp = \dfrac{h}{3}$ **Ans.**

24. PQ is original level of mercury. Equating the pressures at the level of M from left hand side and right hand side.



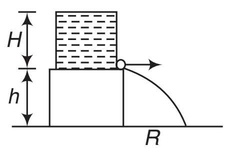
$$p_0 + \rho_w g\, h_w = p_0 + \rho_{Hg}\, g h_{Hg}$$

or $$\rho_w\, h_w = \rho_{Hg}\, h_{Hg}$$

$\therefore$ $(1)\,(30 + 3x) = (13.6)\,(4x)$

Solving we get, $x = 0.58$ cm **Ans.**

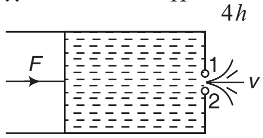
25.



$$R = 2\sqrt{Hh}$$

$\therefore$ $$H = \frac{R^2}{4h}$$ **Ans.**

26.



$$p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2$$

$$\left(p_0 + \frac{F}{A}\right) + 0 = p_0 + \frac{1}{2}\rho v^2$$

$\therefore$ $$v = \sqrt{\frac{2F}{\rho A}}$$ **Ans.**

27. $\frac{1}{2}\rho_{air}\, v^2 = \rho_w g h_w$

$$v = \sqrt{\frac{2\rho_w g\, h_w}{\rho_{air}}}$$

$$= \sqrt{\frac{2\times 10^3 \times 10 \times 10^{-2}}{1.3}}$$

$= 12.4$ m/s **Ans.**

28. $\left(\frac{Mg}{A}\right) + \rho g h = \frac{1}{2}\rho v^2$

$\therefore$ $$v = \sqrt{\frac{2Mg}{\rho A} + 2gh}$$

$$= \sqrt{\frac{2\times 20\times 10}{0.5\times 10^3} + 2\times 10\times 0.5}$$

$= 3.28$ m/s **Ans.**

29. $$v_{max} = \frac{(p_1 - p_2)\, R^2}{4\eta L}$$

$\therefore$ $$(p_1 - p_2) = \frac{4\eta L\, v_{max}}{R^2}$$

$$= \frac{(4\times 4\times 10^{-3})\,(10^{-3})\,(0.66\times 10^{-3})}{(2\times 10^{-6})^2}$$

$= 2.64 \times 10^3$ N/m^2 $= h\rho g$

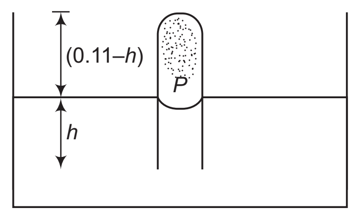
$$h = \frac{2.64\times 10^3}{\rho g}$$

$$= \frac{2.64\times 10^3}{13.6\times 10^3\times 9.81} \text{ m of Hg}$$

$= 0.0195$ m of Hg

≈ 19.5 mm of Hg

30.



$$p - \frac{2T}{R} = p_0$$

$\therefore$ $$P = \frac{2T}{R} + P_0$$

$$R = \frac{r}{\cos\theta}$$

For water $\theta = 0°$

$\therefore$ $R = r$ or $P = \frac{2T}{r} + P_0$

As temperature is constant.

$$p_1 V_1 = p_2 V_2$$

$\therefore$ $p_0\,(0.11) = p\,(0.11 - h)$

or $p_0\,(0.11) = \left(p_0 + \frac{2T}{r}\right)(0.11 - h)$

Substituting the values, we get

$$(1.01\times 10^5\;(0.11) = \left[1.01\times 10^5 + \frac{2\times 5.06\times 10^{-2}}{10^{-5}}\right](0.11 = h)$$

Solving this equation we get,

$h = 0.01$ m or 1 cm

If seal is broken, then water will rise in the capillary.

31. $W = T\,(\Delta A)$

$= T\,(2ld)$

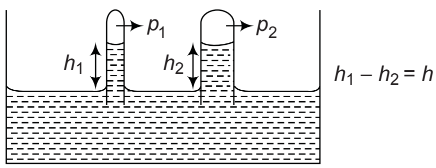
$= (72\times 10^{-3})\,(2)\,(0.1)\,(10^{-3})$

$= 14.4 \times 10^{-6}$ J **Ans.**

32. Let the pressures in the wide and narrow capillaries of radii r_1 and r_2 respectively be p_1 and p_2. Then pressure just below the meniscus in the wide and narrow tubes respectively are

$$\left(p_1 - \frac{2T}{r_1}\right) \text{ and } \left(p_2 - \frac{2T}{r_2}\right)$$

$$\left[\text{excess pressure} = \frac{2T}{r}\right]$$



Difference in these pressures

$$\left(p_1 - \frac{2T}{r_1}\right) - \left(p_2 - \frac{2T}{r_2}\right) = h\rho g$$

$\therefore$ True pressure difference $= p_1 - p_2$

$$= h\rho g + 2T\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

$$= 0.2 \times 10^3 \times 9.8 + 2 \times 72 \times 10^{-3} \left[\frac{1}{1.44 \times 10^{-3}} - \frac{1}{7.2 \times 10^{-4}}\right]$$

$$= 1.86 \times 10^3 = 1860 \text{ N/m}^2$$

33. The surface tension of the liquid is

$$T = \frac{rh\rho g}{2}$$

$$= \frac{(0.025\text{cm})(3.0\text{cm})(1.5\text{gm}/\text{cm}^3)(980\text{cm}/\text{s}^2)}{2}$$

$= 55$ dyne/cm

Hence, excess pressure inside a spherical bubble

$$p = \frac{4T}{R} = \frac{4 \times 55\text{dyne/cm}}{(0.5\text{cm})}$$

$$= 440 \text{ dyne}/\text{cm}^2$$

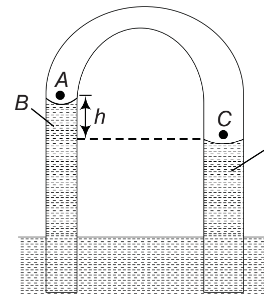
34. Suppose pressure at the points, A, B, C and D be p_A, p_B, p_C and p_D respectively.

The pressure on the concave side of the liquid surface is greater than that on the other side by $\frac{2T}{R}$.

An angle of contact θ is given to be 0°, hence

$$R\cos 0° = r \quad \text{or} \quad R = r$$

$$\therefore \quad p_A = p_B + \frac{2T}{r_1} \quad \text{and} \quad p_C = p_D + \frac{2T}{r_2}$$



where, r_1 and r_2 are the radii of the two limbs

But $\quad p_A = p_C$

$$\therefore \quad p_B + \frac{2T}{r_1} = p_D + \frac{2T}{r_2}$$

or $\quad p_D - p_B - 2T\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$

where, h is the difference in water levels in the two limbs

Now, $\quad h = \frac{2T}{\rho g}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$

Given that $\quad T = 0.07 \text{ Nm}^{-1}, \rho = 1000 \text{ kgm}^{-3}$

$$r_1 = \frac{3}{2}\text{mm} = \frac{3}{20}\text{cm} = \frac{3}{20 \times 100}\text{m}$$

$$= 1.5 \times 10^{-3}\text{m}, \; r_2 = 3 \times 10^{-3} \text{ m}$$

$$\therefore \quad h = \frac{2 \times 0.07}{1000 \times 9.8}\left(\frac{1}{1.5 \times 10^{-3}} - \frac{1}{3 \times 10^{-3}}\right)\text{m}$$

$$= 4.76 \times 10^{-3}\text{m} = 4.76\text{mm}$$

35. The total pressure inside the bubble at depth h_1 is (p is atmospheric pressure)

$$= (p + h_1\rho g) + \frac{2T}{r_1} = p_1$$

and the total pressure inside the bubble at depth h_2 is

$$= (p + h_2\rho g) + \frac{2T}{r_2} = p_2$$

Now, according to Boyle's law

$$p_1 V_1 = p_2 V_2$$

where, $\quad V_1 = \frac{4}{3}\pi r_1^3 \quad$ and $\quad V_2 = \frac{4}{3}\pi r_2^3$

Hence, we get

$$\left[(p + h_1\rho g) + \frac{2T}{r_1}\right]\frac{4}{3}\pi r_1^3$$

$$= \left[(p + h_2\rho g) + \frac{2T}{r_2}\right]\frac{4}{3}\pi r_2^3$$

or $\left[(p + h_1\rho g) + \frac{2T}{r_1}\right]r_1^3 = \left[(p + h_2\rho g) + \frac{2T}{r_2}\right]r_2^3$

Given that $h_1 = 100$ cm, $r_1 = 0.1$ mm $= 0.01$ cm, $r_2 = 0.126$ mm $= 0.0126$ cm, $T = 567$ dyne/cm, $p = 76$ cm of mercury. Substituting all the values, we get

$$h_2 = 9.48\,\text{cm}.$$

36. Let n be the number of little droplets.

Since, volume will remain constant, hence volume of n little droplets = volume of single drop

$$\therefore \quad n \times \frac{4}{3}\pi r^3 = \frac{4}{3}\pi R^3$$

or $$nr^3 = R^3$$

Decrease in surface area $= n \times 4\pi r^2 - 4\pi R^2$

or $\Delta A = 4\pi[nr^2 - R^2]$

$$= 4\pi\left[\frac{nr^3}{r} - R^2\right] = 4\pi\left[\frac{R^3}{r} - R^2\right]$$

$$= 4\pi R^3\left[\frac{1}{r} - \frac{1}{R}\right]$$

Energy evolved $W = T \times$ decrease in surface area

$$= T \times 4\pi R^3\left[\frac{1}{r} - \frac{1}{R}\right]$$

Heat produced, $Q = \frac{W}{J} = \frac{4\pi TR^3}{J}\left[\frac{1}{r} - \frac{1}{R}\right]$

But $$Q = msd\theta$$

where, m is the mass of big drop, s is the specific heat of water and $d\theta$ is the rise in temperature.

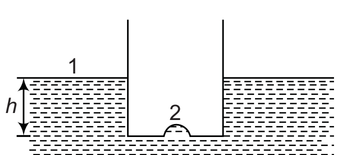
$\therefore \quad \frac{4\pi TR^3}{J}\left[\frac{1}{r} - \frac{1}{R}\right] =$ Volume of big drop

$\times$ density of water $\times$ sp. heat of water $\times\, d\theta$

or $$\frac{4}{3}\pi R^3 \times 1 \times 1 \times d\theta = \frac{4\pi TR^3}{J}\left(\frac{1}{r} - \frac{1}{R}\right)$$

or $$d\theta = \frac{3T}{J}\left[\frac{1}{r} - \frac{1}{R}\right]$$

37.



Applying pressure equations between 1 and 2,

$$p_0 + \rho gh - \frac{2T}{r} = p_0$$

$$\Rightarrow \quad h = \frac{2T}{r\rho g}$$

LEVEL 2

Single Correct Option

1. When ice melts into water its volume decreases. Hence, over all level should decrease.

Now suppose m is the mass of ice, V_1 is volume immersed in water and V_2 the volume immersed in oil. Then in floating condition,

weight = upthrust

$$\therefore \quad mg = V_1\rho_w g + V_2\rho_0 g$$

$$\therefore \quad V_1 = \frac{m - V_2\rho_0}{\rho_w}$$

$$= \frac{m}{\rho_w} - \frac{V_2\rho_0}{\rho_w} \quad \text{...(i)}$$

When ice melts, m mass of ice converts into m mass of water. Volume of water so formed is

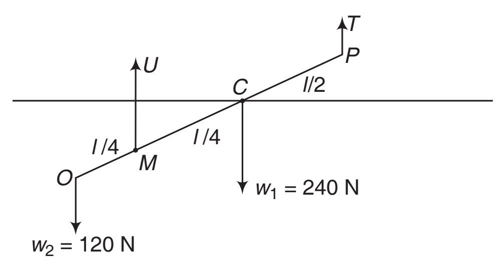
$$V_3 = \frac{m}{\rho_w} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), $V_3 > V_1$

$\therefore$ Interface level will rise.

2. $g_e = 0$ so, pressure inside and pressure outside the hole will be same.

3.



$$T + U = w_1 + w_2 = 360 \text{ N} \quad \text{...(i)}$$

$$U = \frac{V}{2}\rho_w g = (V/2)\,(10^3)\,(10)$$

or $$U = 0.5 \times 10^4\, V \quad \text{...(ii)}$$

(Σ Moments) about $M = 0$

$$\therefore \quad 120\left(\frac{l}{4}\right) + T\left(\frac{3l}{4}\right) = 240\left(\frac{l}{4}\right)$$

or $$T = \frac{240 - 120}{3}$$

$$= 40 \text{ N} = 4\text{g}$$

Now from Eqs. (i) and (ii), we get

$$V = 6.4 \times 10^{-2} \text{ m}^3$$ **Ans.**

4. Volume flow rate $= av = \frac{V}{t}$

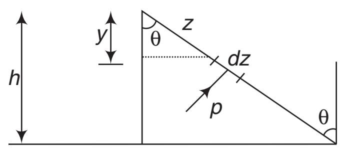
$$\therefore \quad v_1 = \frac{V}{A_1 t} = \frac{120 \times 10^{-3}}{5 \times 10^{-4} \times 2 \times 60}$$

Now, $A_1v_1 = A_2v_2$

A_2 is $\frac{1}{5}$ th of A_1.

Hence, v_2 is five times of v_1 or 10 m/s.

$$t_0 = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2\times 1}{10}}$$
$$= 0.447 \text{ s}$$
$$R = v_2t_0 = 4.47 \text{ m}$$

5. 

$$z = y \sec\theta$$
$$dz = dy \sec\theta$$
$$p = (\rho g y)$$
$$dA = (b)\,dz = (b)\,dy \sec\theta$$
$$dF = p\,dA = (\rho g b \sec\theta)\;y\,dy$$
$$\therefore \quad F = \int_0^h dF = \frac{1}{2}\rho b h^2 g \sec\theta$$

6. Velocity of body just before touching the lake surface is,

$$v = \sqrt{2gh}$$

Retardation in the lake,

$$a = \frac{\text{upthrust} - \text{weight}}{\text{mass}}$$
$$= \frac{V\rho g - V\rho g}{V\rho} = \left(\frac{\sigma - \rho}{\rho}\right) g$$

Maximum depth $d_{\max} = \frac{v^2}{2a} = \frac{h\rho}{\sigma - \rho}$

7. $\rho g h = \rho a L$

$$\therefore \quad h = \frac{aL}{g}$$

8. Radius of meniscus $= \frac{R}{\cos\theta} = r$

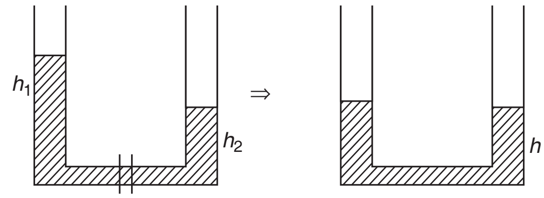
Δp due to spherical surface $= \frac{2\sigma}{r}$

$$= \frac{2\sigma\cos\theta}{R}$$
$$p_B = p_{\text{atm}}$$

9. $a_1v_1 = a_2v_2$

$$\therefore \quad (L^2)\sqrt{2gh} = (\pi R^2)\sqrt{2g\,(4h)}$$
$$R = \frac{L}{\sqrt{2\pi}}$$

10. Let h = final height of liquid.



Equating the volumes we have,

$$Ah_1 + Ah_2 = A\,(2h)$$
$$\therefore \quad h = \frac{h_1 + h_2}{2}$$
$$W_{mg} = -\Delta U = U_i - U_f$$
$$= \left[\left(Ah_1\rho g\frac{h_1}{2}\right) + \left(Ah_2\rho g\frac{h_2}{2}\right)\right] - \left[2A\left(\frac{h_1+h_2}{2}\right)\rho\left(\frac{h_1+h_2}{4}\right)\right]$$

Simplifying this expression we get the result.

11. $\Delta p = \rho_{0\text{il}}\; g\; h_{0\text{il}} + \rho_w\, g h_w$

$$= (600)(10)(10\times10^{-2}) + (1000\times10\times2\times10^{-2})$$
$$= 800 \text{ N/m}^2$$

12. $\Delta p = \frac{1}{2}\rho v^2$

$$\therefore \quad v = \sqrt{\frac{2\Delta p}{\rho}} = \sqrt{\frac{2\times5\times10^5}{1000}}$$
$$= 31.5 \text{ m/s}$$

13. From continuity equation,

$$v_A = v_B = v_0$$
$$\therefore \quad p_A + \rho g h = p_B + 0$$
$$\therefore \quad p_B - p_A = \rho g h \qquad \text{...(i)}$$

Now, let us make pressure equation from manometer.

$$p_A + \rho g\,(h + H) - \rho_{\text{Hg}}\; gh = p_3$$

Putting $p_B - p_A = \rho g h$ we get $h = 0$

14. $F = \frac{\Delta p}{\Delta t} = \left(\frac{\Delta m}{\Delta t}\right)(\Delta v)$

$$= \rho\left(\frac{\Delta V}{\Delta t}\right)(2v)$$
$$= \rho\,(Av)\,(2v) = 2\rho A v^2$$

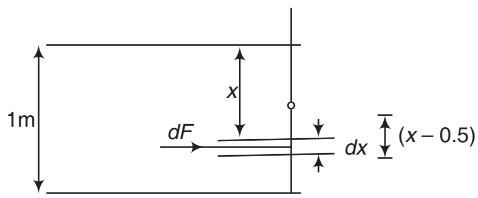
15. $\Delta p_A = \rho g h$ or $\rho a l$

16. At every instant half of the length remains up the surface and half below the surface of liquid.

Since, $D >> d$, over all level will remain unchanged.

Let at $t = 0$, length of candle is 10 cm.

Time (s)	Length of candle (cm)	Length about the surface (cm)	Length below the surface (cm)
0	10	5	5
1	8	4	4

17. 

Net torque about hinge = 0

$$dF = (\rho g x)(1 \cdot dx)$$

$$d\tau = (\rho g x)(x - 0.5)\,dx$$

$\therefore$ Net anticlockwise torque

$$= \int_0^1 d\tau = \left(\frac{\rho g}{12}\right)$$

Net clockwise torque of applied force $= F\,(0.5) = \dfrac{F}{2}$

Equating the two torques we get,

$$F = \frac{\rho g}{6}$$

18. See the typed example 12.

$$\tan\theta = \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} = \tan 30° = \frac{1}{\sqrt{3}}$$

Solving this equation we get,

$$\frac{\rho_1}{\rho_2} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

19. $F = \dfrac{\Delta p}{\Delta t} = \left(\dfrac{\Delta m}{\Delta t}\right)(\Delta v)$

$$= \rho\left(\frac{\Delta V}{\Delta t}\right)(v_1 + v_2) = \rho V\,(v_1 + v_2)$$

20. $v_T = \dfrac{2}{9}\dfrac{r^2(\rho - \sigma)g}{\eta}$

$$= \frac{2 \times (0.003)^2\,(1260 \times 2 - 1260) \times 10}{9 \times 1.26}$$

$$= 0.02 \text{ m/s}$$

$$t = \frac{d}{v_T} = \frac{10 \times 10^{-2}}{0.02} = 5 \text{ s.}$$

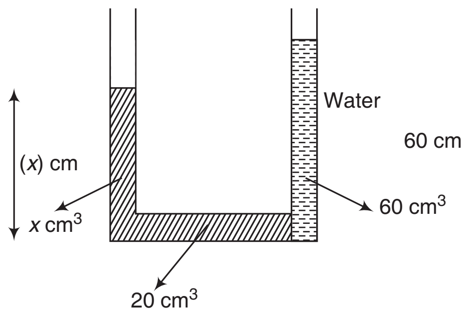
21. Torque of hydrostatic force about centre of sphere is already zero as hydrostatic force passes through the centre.

$$\therefore \quad F_{\text{LHS}} = F_{\text{RHS}}$$

For finding force refer Q-No. 23 (a) subjective questions of section Level 1.

$$\therefore \quad \frac{1}{2}(2\rho)(g)h^2 = \frac{1}{2}(3\rho)(g)R^2$$

$$\therefore \quad h = \sqrt{\frac{3}{2}}\,R$$

22. 

$$\rho_1 g h_1 = \rho_2 g h_2$$

$$\therefore \quad (4)(g)(x) = (1)(g)(60)$$

or $\quad x = 15$ cm

Total volume of liquid $= (20 + 15)$ cm^3

$= 35$ cm^3

23. Viscous force $= mg \sin\theta$

$$\therefore \quad \eta\,(a^2)\frac{v}{t} = mg \sin 37° = \frac{3}{5}mg$$

$$= \frac{3}{5}(a^3\rho)\,g \Rightarrow \quad \therefore \quad \eta = \frac{3\rho a g t}{5v}$$

24. Reading = weight of bucket of water + magnitude of upthrust on block

$$= (10g) + \frac{1}{2}\left(\frac{7.2}{7.2\,\rho_w}\right)\rho_w g$$

$$= 10.5 \text{ g} = 10.5 \text{ kg}$$

25. Sum of all three terms are different at three points A, B and C.

26. $a = \dfrac{\text{upthrust} - \text{weight}}{\text{mass}}$ (upwards)

$$= \frac{V\sigma g - V\rho g}{V\rho} = \left(\frac{\sigma}{\rho} - 1\right)g \qquad \text{(upwards)}$$

27. $h_{\text{Top}} = \dfrac{2H + H}{2} = 1.5H$

Since, this point lies in the tank. So hole should be made at this point.

28. Viscous force $F \propto$ (area)

So let $\quad F = kA$

$$F_0 = k\,(A_1 + A_2) \qquad \ldots\text{(i)}$$

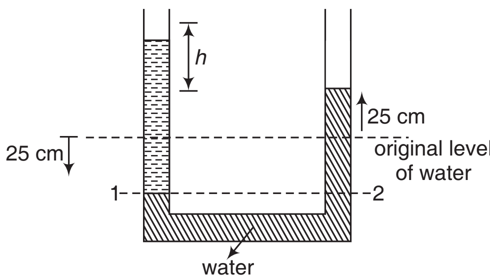
and $\quad T = k\,A_1 \qquad \ldots$(ii)

Dividing Eq. (ii) by Eq.(i) we get,

$$\frac{T}{F_0} = \frac{A_1}{A_1 + A_2}$$

$$\therefore \quad T = \frac{F_0 A_1}{A_1 + A_2}$$

29.



$$p_1 = p_2$$

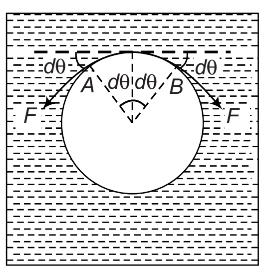
$$\therefore \quad p_0 + \rho_0 g h_0 = p_0 + \rho_w g\, h_w$$

or $\quad \rho_0 h_0 = \rho_w h_w$

or $\quad (0.8)(h + 50) = (1)(50)$

or $\quad h = 12.5$ cm

30.



F = Tension

Surface tension force is radially outwards.

On AB :

$2F \sin d\theta$ = Surface tension force from two films from two sides

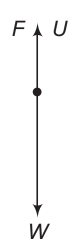
For small angle,

$$\sin d\theta \approx d\theta$$

$$\therefore \quad 2F\, d\theta = 2\,(Tl) = 2\,(T)\,(2Rd\theta)$$

$$\therefore \quad F = 2\,TR$$

31. Surface tension force on liquid is downwards. But on the disc it is upwards.



In the figure, F = Surface tension force

$$= Tl \cos\theta = T\,(2\,\pi r)\cos\theta$$

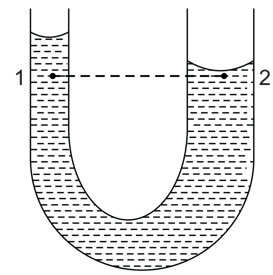
$$U = \text{upthrust} = \omega$$

$$W = \text{weight of disc}$$

For equilibrium of disc,

$$W = U + F = \omega + 2\,\pi T r \cos\theta$$

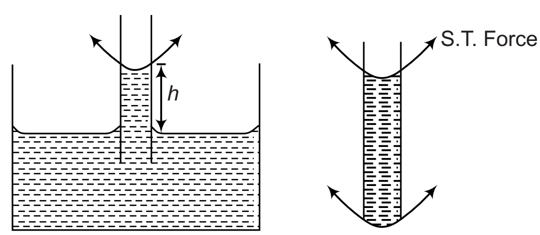
32. $p_1 = p_2$



$$p_0 - \frac{2T}{r_1} + \rho g h = p_0 - \frac{2T}{r_2}$$

$$\therefore \quad T = \frac{\rho g h r_1 r_2}{2\,(r_2 - r_1)}$$

33. When the capillary is inside the liquid, the surface tension force supports the weight of liquid of height 'h'.



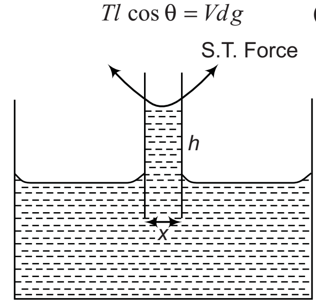
When the capillary is taken out from the liquid similar type of surface tension force acts at the bottom also, as shown in second figure. Hence, now it can support weight of a liquid of height $2h$.

34. Let L be the width of plates (perpendicular to paper inwards).

Surface tension force in upward direction

= weight of liquid of height h

$$\therefore \quad Tl\cos\theta = Vdg \qquad (V = \text{volume})$$



or $\quad T\,(2L)\cos\theta = (Lxh)\,dg$

$$\therefore \quad h = \frac{2T\cos\theta}{xdg}$$

35. Let R_1 and R_2 are the radii of soap bubbles before and after collapsing. Then given that,

$$V = 2\left(\frac{4}{3}\pi R_1^3\right) - \frac{4}{3}\pi R_2^3 \qquad \text{...(i)}$$

$$S = 2(8\pi R_1^2) - 8\pi R_2^2 \qquad \text{...(ii)}$$

$$P_1V_1 = P_2V_2$$

$$\therefore \left(P_0 + \frac{4T}{R_1}\right)\left(2 \times \frac{4}{3}\pi R_1^3\right) = \left(P_0 + \frac{4T}{R_2}\right)\left(\frac{4}{3}\pi R_2^3\right) \qquad \text{...(iii)}$$

Solving these three equation we get the desired result.

36. $p_iV_i = p_fV_f$

$$\therefore \left(p_1 + \frac{4\sigma}{r}\right)\left(\frac{4}{3}\pi r^3\right) = \left(p_2 + \frac{4\sigma}{r/2}\right)\left(\frac{4}{3}\pi\right)\left(\frac{r}{2}\right)^3$$

Solving this equation we get,

$$p_2 = 8p_1 + \frac{24\sigma}{r}$$

37. $$W = 2\pi(r_1 + r_2)T$$

$$\therefore \quad T = \frac{W}{2\pi(r_1 + r_2)} = \frac{7.48 \times 10 \times 10^{-3}}{2\pi \times 17 \times 10^{-2}}$$

$$= 70 \times 10^{-3} \text{ N/m}$$

More than One Correct Options

1. Velocity gradient $= \frac{\Delta v}{\Delta h} = \frac{2\text{m/s}}{1\text{ m}} = 2\text{s}^{-1}$

$$F = \eta A \frac{\Delta v}{\Delta h} = (10^{-3})(10)(2)$$

$$= 0.02 \text{ N}$$

2. For contact angle $\theta = 90°$, liquid neither rises nor falls.

3. Restoring force $= -(\rho A g)x$ or $F \propto -x$

This is just like a spring-block system of force constant

$$K = \rho A g$$

4. From continuity equation $Av =$ constant

$$v_2 > v_2 \quad \text{as} \quad A_2 < A_1$$

From Bernoulli's equation,

$$p + \frac{1}{2}\rho v^2 = \text{constant} \quad (\text{as } h = \text{constant})$$

$$p_2 < p_1 \quad \text{as} \quad v_2 > v_2$$

5. Fraction of volume immersed,

$$f = \frac{\rho_s}{\rho_l}$$

This fraction is independent of atmospheric pressure. With increase in temperature ρ_s and ρ_l both will decrease.

6. $U = \sqrt{2gh_T}$: $t = \sqrt{\frac{2h_B}{g}}$, $R = 2\sqrt{h_T h_B}$

Here, h_T = distance of hole from top surface of liquid

and h_B = distance of hole from bottom surface

7. Pressure increase with depth in vertical direction and in horizontal direction it increases in opposite direction of acceleration based on this concept pressure is maximum at point D and minimum at B.

8. In air, $a_1 = g$ (downwards)

In liquid, $a_2 = \frac{\text{upthrust} - \text{weight}}{\text{mass}}$

$$= \frac{(V)(2\rho)(g) - (V)(\rho)(g)}{(V\rho)}$$

$$= g \text{ (upwards)}$$

$\therefore \quad a_1 \neq a_2$

9. Initially

$$\frac{dV_1}{dt} = v_1 a_1 = (\sqrt{2gh})(\pi)(2R)^2 \qquad \text{...(i)}$$

$$\frac{dV_2}{dt} = v_2 a_2 = (\sqrt{2g(16h)})(\pi)(R)^2 \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we can see that

$$\frac{dV_1}{dt} = \frac{dV_2}{dt}$$

After some time v_1 and v_2 both will decrease, but decrease in the value of v_1 is more dominating. So,

$$a_1 v_1 \quad \text{or} \quad \frac{dV_1}{dt} < a_2 v_2 \quad \text{or} \quad \frac{dV_2}{dt}$$

10. Fraction of volume immersed is given by

$$f = \frac{\rho_s}{\rho_l}$$

ρ_s and ρ_l are same. Hence :

$$f_1 = f_2 = f_3$$

Base area in third case is uniform. Hence h_3 is minimum.

Comprehension Based Questions

2. $x_3 = 2\sqrt{h_{\text{Top}} \times h_{\text{bottom}}}$

$$= 2\sqrt{3a \times a} = 2\sqrt{3}\,a$$

3. $x_1 = 2\sqrt{a \times 3a} = 2\sqrt{3}\,a$

$$x_2 = 2\sqrt{2a \times 2a} = 2\sqrt{4}\,a$$

$$x_3 = 2\sqrt{3}a$$

$$\therefore \quad x_1 = x_3 < x_2$$

4. Weight = upthrust

$$\therefore \left(L \frac{A}{5} Dg\right) = \frac{L}{4} \times \frac{A}{5} \times 2d \times g + \frac{3L}{4} \times \frac{A}{5} \times d \times g$$

$$\therefore \qquad D = \frac{5d}{4}$$

5. $pA = p_0A$ + weight of two liquids + weight of cylinder

$$= p_0A + \left(\frac{H}{2}\right)(A)(d)(g) + \left(\frac{H}{2}\right)(A)(2d)(g) + L\left(\frac{A}{5}\right)\left(\frac{5d}{4}\right)g$$

$$\therefore \qquad p = p_0 + \frac{(L + 6H)}{4} dg$$

6. Applying Bernoulli's equation just inside and just outside the hole,

$$p_0 + \left(\frac{H}{2}\right)(d)g + \left(\frac{H}{2} - h\right)(2d)(g) = \frac{1}{2}(2d)v^2 + p_0$$

$$\therefore \qquad v = \sqrt{\frac{g}{2}(3H - 4h)}$$

7. $t = \sqrt{\dfrac{2h}{g}} \Rightarrow \therefore\ x = vt = \sqrt{(3H - 4h)\, h}$

Match the Columns

1. $v = \sqrt{2gh_T}$ and $R = 2\sqrt{h_T h_B}$

Here, h_T = height of hole from top surface

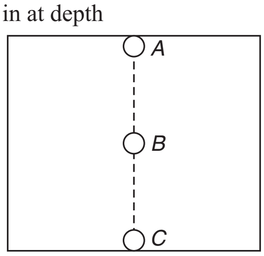
and h_B = height of hole from bottom

2. (a) and (b) In floating condition upthrust = weight

By increasing temperature of density liquid, weight remains unchanged. Hence, upthrust is unchanged.

(c) When density of solid is increased (with constant volume) mass and hence weight of solid will increase. So, upthrust will increase.

3. Ball will oscillate simple harmonically. The mean position in at depth



$$\rho = \alpha h \quad \text{or} \quad h = \frac{\rho}{\alpha} = AB$$

$$h_{max} = \frac{2\rho}{\alpha} = AC$$

$$\text{Amplitude} = \frac{\rho}{\alpha} = AB \quad \text{or} \quad BC$$

From A to B : $\rho > \rho_l$, weight > upthrust

At B, $\rho = \rho_l$, weight = upthrust

From B to C, $\rho_e > \rho$, upthrust > weight.

From A to $C \rightarrow$ upthrust will increase and gravitational potential energy will decrease.

From C to A, upthrust will decrease and gravitational potential energy will increase.

From A to B, speed will increase.

From B to C, speed will decrease.

4. Surface tension $= \dfrac{F}{e}$

Viscous force $f = \eta A \dfrac{\Delta v}{\Delta y}$

Energy density $= \dfrac{\text{Energy}}{\text{Volume}}$

Volume flow rate $= \dfrac{\text{Volume}}{\text{Time}}$

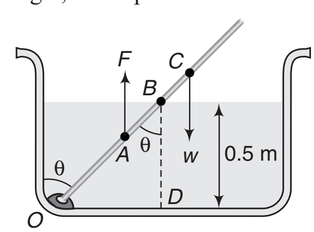
5. $F_2 - F_1$ = upthrust

= weight of cylinder in equilibrium or floating condition.

From liquid-1, horizontal pair of forces cancel out. So, net force = 0

Subjective Questions

1. w = Weight, F = Upthrust



$$OA = \frac{OB}{2} = \frac{0.5 \sec\theta}{2} = 0.25 \sec\theta$$

About point O, clockwise moment of w = anticlockwise moment of F.

$$\therefore\ w\left(\frac{L}{2}\sin\theta\right) = F(OA \sin\theta) = F(0.25 \sin\theta \cdot \sec\theta)$$

Given, $\qquad L = 1$ m

$$\therefore \qquad \cos\theta = \frac{F}{2w} = \frac{(0.5\sec\theta)(A)(1.0)g}{2(1)(A)((0.5)g}$$

$$\text{or} \qquad \cos^2\theta = \frac{1}{2}, \cos\theta = \frac{1}{\sqrt{2}}$$

$$\therefore \qquad \theta = 45°$$ **Ans.**

2. Fraction of volume immersed before putting the new weight $= \dfrac{\rho_{block}}{\rho_{water}} = \dfrac{800}{1000} = 0.8$

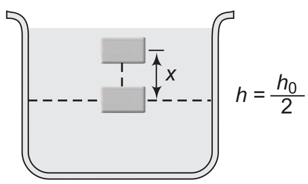
i.e. 20% of 3 cm or 0.6 cm is above water. Let w is the new weight, then spring will be compressed by 0.6 cm.

$\therefore$ w + weight of block = Upthrust on whole volume of block + spring force

or $$w = (3 \times 10^{-2})^3 \times 1000 \times 10 + 50 \times (0.6 \times 10^{-2}) - (3 \times 10^{-2})^3 \times 800 \times 10$$

$\therefore$ $w = 0.354$ N **Ans.**

3. Net force on the block at a height h from the bottom is



$$F_{net} = \text{upthrust} - \text{weight} \quad \text{(upwards)}$$

$$= \left(\frac{m}{\frac{5}{2}\rho_0}\right)\rho_0\left(4 - \frac{3h}{h_0}\right)g - mg$$

$$F_{net} = 0 \text{ at } h = \frac{h_0}{2}$$

So, $h = \dfrac{h_0}{2}$ is the equilibrium position of the block.

For $h > \dfrac{h_0}{2}$, weight > upthrust

i.e. net force is downwards and for $h < \dfrac{h_0}{2}$

weight < upthrust

i.e. net force is upwards.

For upward displacement x from mean position, net downward force is

$$F = -\left[\left(\frac{m}{\frac{5}{2}\rho_0}\right)\rho_0\left\{4 - \frac{3(h+x)}{h_0}\right\}g - mg\right]\left(h = \frac{h_0}{2}\right)$$

$\therefore$ $$F = -\frac{6mg}{5h_0}x \qquad \text{...(i)}$$

(because at $h = \dfrac{h_0}{2}$ upthrust and weight are equal)

Since $F \propto -x$

Oscillations are simple harmonic in nature. Rewriting Eq. (i)

$$ma = -\frac{6mgx}{5h_0}$$

or $$a = -\frac{6g}{5h_0}x$$

$\therefore$ $$f = \frac{1}{2\pi}\sqrt{\left|\frac{a}{x}\right|}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{6g}{5h_0}}$$ **Ans.**

4. (a) $v = \sqrt{2gh} = \sqrt{2 \times 10 \times 5}$

$= 10$ m/s

(b) From conservation of energy,

$$v'^2 = v^2 + 2gH$$

$$= 100 + 2 \times 10 \times 5 = 200$$

$$v' = 14.1 \text{ m/s}$$

(c) $$t = \frac{2A}{a\sqrt{2g}}\left[\sqrt{H} - \sqrt{\frac{H}{2}}\right]$$

$$= \frac{2 \times \pi \times (1)^2}{10^{-4}\sqrt{2 \times 10}}[\sqrt{5} - \sqrt{2.5}]$$

$= 9200$ s

5. Writing equation of motion for the block

$$T - mg \sin 30° = ma \qquad \text{...(i)}$$

For the sphere

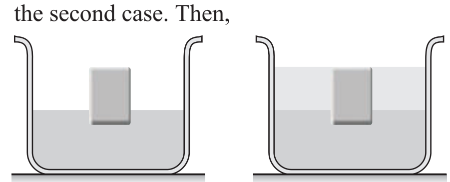
Weight – Buoyant force – $T = ma$ ...(ii)

or $$mg - \frac{mg}{2} - T = ma$$

Solving, we get $a = 0$

6. $w = (0.25)V\rho_{Hg}\, g$ ...(i)

Let x fraction of volume is immersed in mercury in the second case. Then,



$$w = xV\rho_{Hg}g + (1-x)V\rho_w g \qquad \text{...(ii)}$$

Equating Eqs. (i) and (ii), we have

$$\frac{\rho_{Hg}}{4} = x\rho_{Hg} + (1-x)\rho_w$$

$\therefore$ $12.6x = 2.4$

or $x = 0.19$ **Ans.**

7. (a) $p_A - \frac{1}{2}\rho v^2 + \rho g h = p_D$

But $\quad p_A = p_D = p_0$

$\therefore \quad \frac{1}{2}\rho v^2 = \rho g h$

$\therefore \quad v = \sqrt{2gh}$

Here, $\quad h = (4 + 1) = 5$ m

$\therefore \quad v = \sqrt{2 \times 9.8 \times 5} = 9.9$ m/s

(b) Applying Bernoulli's equation at A and B,

$$p_A + 0 + 0 = p_B + \frac{1}{2}\rho v^2 + \rho g(1.5)$$

or $\quad p_0 = p_B + \frac{1}{2}\rho v^2 + 1.5\,\rho g$

$$\therefore \quad p_B = 1.01 \times 10^5 - \frac{1}{2} \times 900 \times (9.9)^2 - 1.5 \times 900 \times 9.8$$

$$= 4.36 \times 10^4 \text{ Pa}$$ **Ans.**

(c) Applying Bernoulli's equation at A and C,

$$p_0 = p_C + \frac{1}{2}\rho v^2 - \rho g(1.0)$$

$$\therefore \quad p_C = p_0 + \rho g - \frac{1}{2}\rho v^2$$

$$= 1.01 \times 10^5 + 900 \times 9.8 - \frac{1}{2} \times 900 \times (9.9)^2$$

$$= 6.6 \times 10^4 \text{ Pa}$$ **Ans.**

8. $\frac{1}{2}\rho v^2 = \rho g h + \frac{mg}{A}$

Here, $\quad h = 1.0 - 0.5 = 0.5$ m,

A = Area of piston = 0.5 m^2

$$\therefore \quad v = \sqrt{2gh + \frac{2mg}{\rho A}}$$

$$= \sqrt{2 \times 9.8 \times 0.5 + \frac{2 \times 20 \times 9.8}{10^3 \times 0.5}}$$

$$= 3.25 \text{ m/s}$$

$\therefore$ Speed with which it hits the surface is

$$v' = \sqrt{v^2 + 2gh'}$$

$$= \sqrt{(3.25)^2 + (2 \times 9.8 \times 0.5)}$$

$$= 4.51 \text{ m/s}$$ **Ans.**

9. $a\sqrt{2gy} = \pi x^2 \left(-\frac{dy}{dt}\right) \quad \ldots\text{(i)}$

Here, $\quad -\frac{dy}{dt} = \frac{4 \times 10^{-2}}{3600} = 1.11 \times 10^{-5}$ m/s

$$a = \pi r^2 = \pi\,(2 \times 10^{-3})^2$$

$$= 1.26 \times 10^{-5} \text{ m}^2$$

Substituting these values in Eq. (i), we have

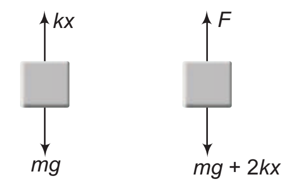
$$(1.26 \times 10^{-5})\sqrt{2 \times 9.8 \times y} = \pi(1.11 \times 10^{-5})x^2$$

or $\quad y = 0.4x^4$

This is the desired x-y relation.

10. Initially, $kx = mg \quad \ldots\text{(i)}$

In the second case,



$$F = mg + 2kx$$

or $\quad kx = \frac{F - mg}{2} \quad \ldots\text{(ii)}$

From Eqs. (i) and (ii), we have

$$mg = \frac{F - mg}{2}$$

or $\quad F = 3\,mg$

Let V be the total volume then,

$$V\rho_w g = 3mg$$

$$\therefore \quad V = \frac{3mg}{\rho_w g} = \frac{(3)(8)}{10^3} \text{ m}^3$$

$$= 0.024 \text{ m}^3$$

Volume of wood $= \frac{\text{mass}}{\text{density}}$

$$= \frac{8}{840} = 0.0095$$

$\therefore$ Volume of cavity $= 0.024 - 0.0095$

$= 0.0145$

$\therefore$ Percentage volume of cavity

$$= \frac{0.0145}{0.024} \times 100$$

$$= 60.41\%$$

11. $v = \sqrt{2g(10 - h)} \quad \ldots\text{(i)}$

Component of its velocity parallel to the plane is $v \cos 30°$.

Let the stream strikes the plane after time t. Then

$$0 = v \cos 30° - g \sin 30°\, t$$

$$\therefore \quad t = \frac{v \cot 30°}{g}$$

Further $\quad x = vt = \dfrac{v^2 \cot 30°}{g}$

$= \sqrt{3}\, y$

or $\quad \dfrac{v^2 \cot 30°}{g} = \sqrt{3}\left(h - \dfrac{1}{2} g t^2\right)$

$\therefore \quad \dfrac{\sqrt{3} v^2}{g} = \sqrt{3}\left(h - \dfrac{g}{2}\dfrac{v^2 \cot^2 30°}{g^2}\right)$

or $\quad \dfrac{v^2}{g} = h - \dfrac{3}{2}\dfrac{v^2}{g}$

$\therefore \quad \dfrac{5}{2}\dfrac{v^2}{g} = h \quad$ or $\quad 5(10 - h) = h$

$\therefore \quad h = 8.33$ m **Ans.**

12. In elastic collision with the surface, direction of velocity is reversed but its magnitude remains the same.

Therefore, time of fall = time of rise.

or $\quad$ time of fall $= \dfrac{t_1}{2}$

Hence, velocity of the ball just before it collides with liquid is

$$v = g\frac{t_1}{2} \qquad \text{...(i)}$$

Retardation inside the liquid

$$a = \frac{\text{upthrust} - \text{weight}}{\text{mass}} = \frac{Vd_L g - Vdg}{Vd}$$

$$= \left(\frac{d_L - d}{d}\right) g \qquad \text{...(ii)}$$

Time taken to come to rest under this retardation will be

$$t = \frac{v}{a} = \frac{gt_1}{2a} = \frac{gt_1}{2\left(\dfrac{d_L - d}{d}\right)g}$$

$$= \frac{dt_1}{2(d_L - d)}$$

Same will be the time to come back on the liquid surface.

Therefore,

(a) t_2 = time the ball takes to came back to the position from where it was released

$$= t_1 + 2t = t_1 + \frac{dt_1}{d_L - d}$$

$$= t_1\left[1 + \frac{d}{d_L - d}\right] \quad \text{or} \quad t_2 = \frac{t_1 d_L}{d_L - d}$$

(b) The motion of the ball is periodic but not simple harmonic because the acceleration of the ball is g in air and $\left(\dfrac{d_L - d}{d}\right) g$ inside the liquid which is not proportional to the displacement, which is necessary and sufficient condition for SHM.

(c) When $d_L = d$, retardation or acceleration inside the liquid becomes zero (upthrust = weight). Therefore, the ball will continue to move with constant velocity $v = \dfrac{gt_1}{2}$ inside the liquid.

13. Pressure inside the bubble $= p_0 + \rho g h + \dfrac{2T}{r}$

$\therefore$ Amount of pressure inside the bubble greater than the atmospheric pressure $= \rho g h + \dfrac{2T}{r}$

Substituting the values we get,

$$\Delta p = 10^3 \times 9.8 \times 0.1 + \frac{2 \times 0.075}{10^{-3}}$$

$$= 980 + 150 = 1130 \text{ N/m}^2$$

14. (a) $F_r = 6\pi\eta r v$

$= 6\pi(0.8)(10^{-3})(10^{-2})$

$= 1.5 \times 10^{-4}$ N **Ans.**

(b) Hydrostatic force = Upthrust

$= \left(\dfrac{4}{3}\pi r^3\right)\rho g = \dfrac{4}{3}\pi\,(10^{-3})^3 \times 1260 \times 9.8$

$= 5.2 \times 10^{-5}$ N **Ans.**

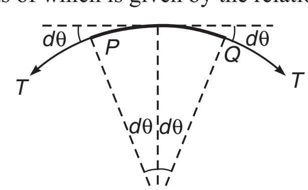
(c) At terminal velocity

w = Upthrust + viscous force

or $\quad (50 \times 10^{-3} \times 9.8) = (5.2 \times 10^{-5}) + 6\pi(0.8)(10^{-3})\, v_T$

Solving we get $\quad v_T = 32.5$ m/s **Ans.**

15. The loop will take circular shape after pricking. Radius of which is given by the relation.



$$l = 2\pi R \quad \text{or} \quad R = \frac{l}{2\pi} = \frac{6.28}{2 \times 3.14} = 1 \text{ cm}$$

$$= 10^{-2} \text{ m}$$

$2T\sin(d\theta)$ force in inward direction is balanced by surface tension force in outward direction.

$\therefore$ $2T\sin(d\theta)$ = (Surface tension) × (length of arc)

For small angles, $\sin d\theta \approx d\theta$

$\therefore \quad 2T\,d\theta = S(2R\,d\theta) \quad (S = \text{Surface tension})$

$$\therefore \quad T = SR = (0.030)(10^{-2}) = 3.0\times10^{-4}\ \text{N}$$ **Ans.**

16. (a) Time taken to empty the tank (has been derived in theory) is

$$t = \frac{2A}{a\sqrt{2g}}\sqrt{H}$$

Given, $\dfrac{A}{a} = 400$

Substituting the values we have,

$$t = \frac{2\times400}{\sqrt{2\times9.8}}\sqrt{1}$$

$$= 180\ \text{s} = 3\ \text{min}$$ **Ans.**

(b) Rate of flow of water $Q = a\sqrt{2gH}$ = constant

Total volume of water $V = AH$

$\therefore$ Time take to empty the tank with constant rate

$$t = \frac{V}{Q} = \frac{AH}{a\sqrt{2gH}} = \frac{400\times1}{\sqrt{2\times9.8\times1}}$$

$$= 90\ \text{s} = 1.5\ \text{min}$$ **Ans.**

17. (a) $\Delta p = h(\rho_w - \rho_0)g = (10)(1000-500)\,9.8$

$$= 49000\ \text{N/m}^2$$

Now, $\Delta p = \dfrac{1}{2}\rho_w v^2$

$$\therefore \quad v = \sqrt{\frac{2\Delta p}{\rho_w}} = \sqrt{\frac{2\times49000}{1000}}$$

$$= 9.8\ \text{m/s}$$ **Ans.**

The flow will stop when,

(b) $(10+5)\rho_0 g = 5\rho_0 g + h\rho_w g$

$$10\rho_0 = h\rho_w$$

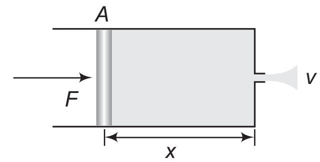
$$\therefore \quad h = \frac{10\times500}{1000} = 5\ \text{m}$$

i.e. flow will stop when the water-oil interface is at a height of 5.0 m. **Ans.**

18. $$\frac{F}{A} = \frac{1}{2}\rho v^2$$

or $$F = \frac{1}{2}\rho A v^2 \qquad \text{...(i)}$$

Here, v is the velocity of liquid, with which it comes out of the hole.



Further $$V = Ax \qquad \text{...(ii)}$$

$$t = \frac{V}{sv} \qquad \text{...(iii)}$$

and $$w = F\cdot x \qquad \text{...(iv)}$$

From the above four equations,

$$w = \left(\frac{1}{2}A\rho v^2\right)\left(\frac{V}{A}\right)$$

$$= \frac{1}{2}\rho\frac{V^2}{s^2t^2}\cdot V = \frac{1}{2}\frac{\rho V^3}{s^2t^2}$$ **Ans.**

19. (a) $p_0 A = kx$

$$\therefore \quad x = \frac{p_0 A}{k} = \frac{(p_0)(\pi r^2)}{k}$$

$$= \frac{(1.01\times10^5)(\pi)(0.025)^2}{3600}$$

$$= 0.055\ \text{m} = 5.5\ \text{cm}$$ **Ans.**

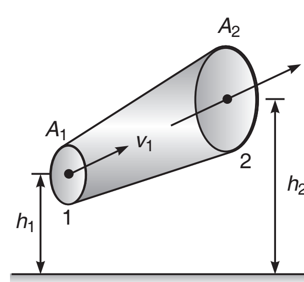
(b) Work done by atmospheric pressure

$$W = \frac{1}{2}kx^2 = \frac{1}{2}(3600)(0.055)^2$$

$$= 5.445\ \text{J}$$ **Ans.**

20. Given, $A_1 = 4\times10^{-3}\ \text{m}^2$, $A_2 = 8\times10^{-3}\ \text{m}^2$,

$h_1 = 2$ m, $h_2 = 5$ m, $v_1 = 1$ m/s and $\rho = 10^3\ \text{kg/m}^3$



From continuity equation, we have

$$A_1 v_1 = A_2 v_2$$

or $v_2 = \left(\frac{A_1}{A_2}\right) v_1$ or $v_2 = \left(\frac{4 \times 10^{-3}}{8 \times 10^{-3}}\right)(1 \text{ m/s})$

$$v_2 = \frac{1}{2} \text{ m/s}$$

Applying Bernoulli's equation at sections 1 and 2

$$p_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = p_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2$$

or $$p_1 - p_2 = \rho g(h_2 - h_1) + \frac{1}{2}\rho(v_2^2 - v_1^2) \qquad \text{...(i)}$$

(i) Work done per unit volume by the pressure as the fluid flows from P to Q.

$$w_1 = p_1 - p_2$$

$$= \rho g(h_2 - h_1) + \frac{1}{2}\rho(v_2^2 - v_1^2) \qquad \text{[From Eq. (i)]}$$

$$= \left\{(10^3)(9.8)(5-2) + \frac{1}{2}(10)^3\left(\frac{1}{4} - 1\right)\right\} \text{ J/m}^3$$

$$= [29400 - 375] \text{ J/m}^3 = 29025 \text{ J/m}^3$$

(ii) Work done per unit volume by the gravity as the fluid flows from P to Q.

$$W_2 = \rho g(h_2 - h_1) = \{(10^3)(9.8)(5-2)\} \text{ J/m}^3$$

or $W_2 = 29400 \text{ J/m}^3$

21. Volume of the portion of the plate immersed in water is

$$10 \times \frac{1}{2}(1.54) \times 0.2 = 1.54 \text{ cm}^3$$

Therefore, if the density of water is taken as 1, then upthrust

= weight of the water displaced

$= 1.54 \times 1 \times 980 = 1509.2$ dynes

Now, the total length of the plate in contact with the water surface is $2(10 + 0.2) = 20.4$ cm

$\therefore$ Downward pull upon the plate due to surface tension

$= 20.4 \times 73 = 1489.2$ dynes

$\therefore$ Resultant upthrust

$= 1509.2 - 1489.2$

$= 20.0 \text{ dynes } = \frac{20}{980}$

$= 0.0204$ gm-wt

$\therefore$ Apparent weight of the plate in water

= weight of the plate in air – resultant upthrust

$= 8.2 - 0.0204 = 8.1796$ gm **Ans.**

22. Given that $r_1 = \frac{3.0}{2} = 1.5 \text{ mm} = 1.5 \times 10^{-3} \text{ m}$,

$r_2 = \frac{6.0}{2} = 3.0 \text{ mm} = 3.0 \times 10^{-3} \text{m}$,

$T = 7.3 \times 10^{-2} \text{ Nm}^{-1}$, $\theta = 0°$ $\rho = 1.0 \times 10^3 \text{ kg/m}^3$, $g = 9.8 \text{ m/s}^2$

When angle of contact is zero degree, the radius of the meniscus equals radius of bore.

Excess pressure in the first bore, $p_1 = \frac{2T}{r_1}$

$$= \frac{2 \times 7.3 \times 10^{-2}}{1.5 \times 10^{-3}} = 97.3 \text{ Pa}$$

Excess pressure in the second bore, $p_2 = \frac{2T}{r_2}$

$$= \frac{2 \times 7.3 \times 10^{-2}}{3 \times 10^{-3}} = 48.7 \text{ Pa}$$

Hence, pressure difference in the two limbs of the tube

$$\Delta p = p_1 - p_2 = h\rho g$$

or $$h = \frac{p_1 - p_2}{\rho g}$$

$$= \frac{97.3 - 48.7}{1.0 \times 10^3 \times 9.8}$$

$$\approx 5 \times 10^{-3} \text{ m}$$

$$= 5.0 \text{ mm}$$

23. $p_1 V_1 = p_2 V_2$

$$\therefore \left(p_0 + \frac{4T}{r}\right)\left(2 \times \frac{4}{3}\pi r^3\right) = \left(p_0 + \frac{4T}{R}\right)\left(\frac{4}{3}\pi R^3\right)$$

From here we can find expression of T.

24. $p_1 V_1 = p_2 V_2$

$$\therefore \left(p_0 + \frac{4T}{r}\right)\left(\frac{4}{3}\pi r^3\right) = \left(p_0 + \frac{4T}{R} - \frac{\sigma^2}{2\varepsilon_0}\right)\left(\frac{4}{3}\pi R^3\right)$$

Here, $$V = \frac{\sigma R}{\varepsilon_0}$$

or $$\sigma = \frac{V\varepsilon_0}{R}$$

25. $p_1 V_1 = p_2 V_2$

$$\left(p_0 + \frac{4T}{R}\right)\left(\frac{4}{3}\pi R^3\right) = \left(p_0 + \frac{4T}{2R} - \frac{\sigma^2}{2\varepsilon_0}\right)\left(\frac{4}{3}\pi\right)(2R)^3$$

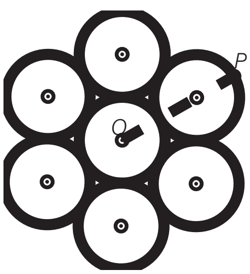
Here, $$\sigma = \frac{q}{A} = \frac{q}{4\pi\,(2R)^2}$$

JEE Main and Advanced

Previous Years' Questions (2018-13)

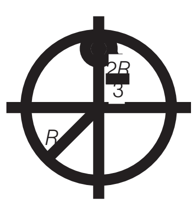
JEE Main

1. Seven identical circular planar discs, each of mass M and radius R are welded symmetrically as shown in the figure. The moment of inertia of the arrangement about the axis normal to the plane and passing through the point P is **(2018)**



(a) $\frac{19}{2}MR^2$ (b) $\frac{55}{2}MR^2$
(c) $\frac{73}{2}MR^2$ (d) $\frac{181}{2}MR^2$

2. From a uniform circular disc of radius R and mass $9M$, a small disc of radius $\frac{R}{3}$ is removed as shown in the figure. The moment of inertia of the remaining disc about an axis perpendicular to the plane of the disc and passing through centre of disc is **(2018)**



(a) $4MR^2$ (b) $\frac{40}{9}MR^2$ (c) $10MR^2$ (d) $\frac{37}{9}MR^2$

3. A solid sphere of radius r made of a soft material of bulk modulus K is surrounded by a liquid in a cylindrical container. A massless piston of area a floats on the surface of the liquid, covering entire cross-section of cylindrical container. When a mass m is placed on the surface of the piston to compress the liquid, the fractional decrement in the radius of the sphere, $\left(\frac{dr}{r}\right)$ is **(2018)**

(a) $\frac{Ka}{mg}$ (b) $\frac{Ka}{3mg}$ (c) $\frac{mg}{3Ka}$ (d) $\frac{mg}{Ka}$

4. The mass of a hydrogen molecule is 3.32×10^{-27} kg. If 10^{23} hydrogen molecules strike per second, a fixed wall of area $2\ cm^2$ at an angle of 45° to the normal and rebound elastically with a speed of 10^3 m/s, then the pressure on the wall is nearly **(2018)**

(a) $2.35 \times 10^3\ N/m^2$ (b) $4.70 \times 10^3\ N/m^2$
(c) $2.35 \times 10^2\ N/m^2$ (d) $4.70 \times 10^2\ N/m^2$

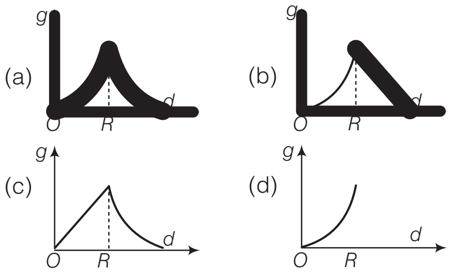
5. The moment of inertia of a uniform cylinder of length l and radius R about its perpendicular bisector is I. What is the ratio l/R such that the moment of inertia is minimum? **(2017)**

(a) $\frac{\sqrt{3}}{2}$ (b) 1 (c) $\frac{3}{\sqrt{2}}$ (d) $\sqrt{\frac{3}{2}}$

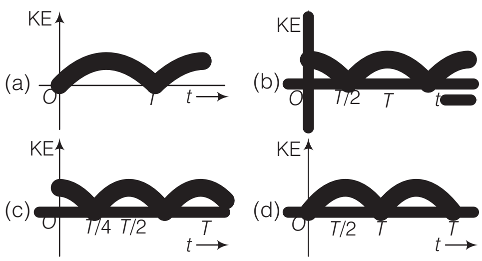
6. A slender uniform rod of mass M and length l is pivoted at one end so that it can rotate in a vertical plane (see the figure). There is negligible friction at the pivot. The free end is held vertically above the pivot and then released. The angular acceleration of the rod when it makes an angle θ with the vertical, is **(2017)**

(a) $\frac{2g}{3l}\sin\theta$ (b) $\frac{3g}{2l}\cos\theta$
(c) $\frac{2g}{3l}\cos\theta$ (d) $\frac{3g}{2l}\sin\theta$

7. The variation of acceleration due to gravity g with distance d from centre of the Earth is best represented by (R = Earth's radius) **(2017)**



8. A particle is executing simple harmonic motion with a time period T. At time $t = 0$, it is at its position of equilibrium. The kinetic energy-time graph of the particle will look, like **(2017)**



9. A magnetic needle of magnetic moment 6.7×10^{-2} Am^2 and moment of inertia 7.5×10^{-6} kg m^2 is performing simple harmonic oscillations in a magnetic field of 0.01 T. Time taken for 10 complete oscillations is **(2017)**

(a) 8.89 s (b) 6.98 s (c) 8.76 s (d) 6.65s

10. The following observations were taken for determining surface tension T of water by capillary method. Diameter of capillary, $d = 1.25 \times 10^{-2}$ m rise of water, $h = 1.45 \times 10^{-2}$m. Using $g = 9.80$ m/s^2 and the simplified relation $T = \frac{rhg}{2} \times 10^3$ N/m, the possible error in surface tension is closest to **(2017)**

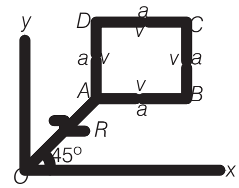
(a) 1.5% (b) 2.4% (c) 10% (d) 0.15%

11. A man grows into a giant such that his linear dimensions increase by a factor of 9. Assuming that his density remains same, the stress in the leg will change by a factor of **(2017)**

(a) $\frac{1}{9}$ (b) 81

(c) $\frac{1}{81}$ (d) 9

12. A particle of mass m is moving along the side of a square of side a, with a uniform speed v in the xy-plane as shown in the figure.



Which of the following statements is **false** for the angular momentum **L** about the origin? **(2016)**

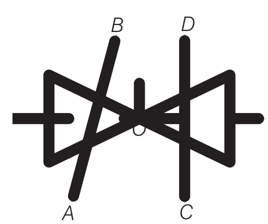
(a) $\mathbf{L} = \frac{-mv}{\sqrt{2}} R\hat{\mathbf{k}}$ when the particle is moving from A to B

(b) $\mathbf{L} = mv\left(\frac{R}{\sqrt{2}} - a\right)\hat{\mathbf{k}}$ when the particle is moving from C to D

(c) $\mathbf{L} = mv\left(\frac{R}{\sqrt{2}} + a\right)\hat{\mathbf{k}}$ when the particle is moving from B to C

(d) $\mathbf{L} = \frac{mv}{\sqrt{2}} R\hat{\mathbf{k}}$ when the particle is moving from D to A

13. A roller is made by joining together two corners at their vertices O. It is kept on two rails AB and CD which are placed asymmetrically (see the figure), with its axis perpendicular to CD and its centre O at the centre of line joining AB and CD (see the figure). It is given a light push, so that it starts rolling with its centre O moving parallel to CD in the direction shown. As it moves, the roller will tend to **(2016)**



(a) turn left
(b) turn right
(c) go straight
(d) turn left and right alternately

14. A satellite is revolving in a circular orbit at a height h from the Earth's surface (radius of earth $R, h << R$). The minimum increase in its orbital velocity required, so that the satellite could escape from the Earth's gravitational field, is close to (Neglect the effect of atmosphere) **(2016)**

(a) $\sqrt{2gR}$ (b) $\sqrt{gR}$
(c) $\sqrt{gR/2}$ (d) $\sqrt{gR}\,(\sqrt{2}-1)$

15. A student measures the time period of 100 oscillations of a simple pendulum four times. The data set is 90s, 91s, 92s and 95s. If the minimum division in the measuring clock is 1s, then the reported mean time should be **(2016)**

(a) (92 ± 2 s) (b) (92 ± 5 s)
(c) (92 ± 1.8s) (d) (92 ± 3 s)

16. A particle performs simple harmonic motion with amplitude A. Its speed is trebled at the instant that it is at a distance $\frac{2}{3}A$ from equilibrium position. The new amplitude of the motion is **(2016)**

(a) $\frac{A}{3}\sqrt{41}$ (b) $3A$ (c) $A\sqrt{3}$ (d) $\frac{7}{3}A$

17. Distance of the centre of mass of a solid uniform cone from its vertex is z_0. If the radius of its base is R and its height is h, then z_0 is equal to **(2015)**

(a) $\frac{3h}{4}$ (b) $\frac{h^2}{4R}$ (c) $\frac{5h}{8}$ (d) $\frac{3h^2}{8R}$

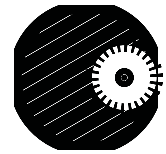
18. A particle of mass m moving in the x-direction with speed $2v$ is hit by another particle of mass $2m$ moving in the y-direction with speed v. If the collision is perfectly inelastic, the percentage loss in the energy during the collision is close to **(2015)**

(a) 50 % (b) 56 % (c) 62 % (d) 44 %

19. From a solid sphere of mass M and radius R, a cube of maximum possible volume is cut. Moment of inertia of cube about an axis passing through its centre and perpendicular to one of its faces is **(2015)**

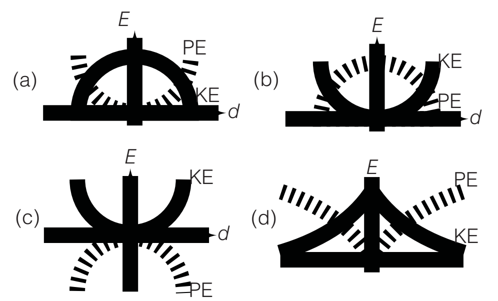
(a) $\frac{MR^2}{32\sqrt{2}\pi}$ (b) $\frac{4MR^2}{9\sqrt{3}\pi}$ (c) $\frac{MR^2}{16\sqrt{2}\pi}$ (d) $\frac{4MR^2}{3\sqrt{3}\pi}$

20. From a solid sphere of mass M and radius R, a spherical portion of radius $\left(\frac{R}{2}\right)$ is removed as shown in the figure. Taking gravitational potential $V = 0$ at $r = \infty$, the potential at the centre of the cavity thus formed is (G = gravitational constant) **(2015)**



(a) $\frac{-GM}{R}$ (b) $\frac{-GM}{2R}$
(c) $\frac{-2GM}{3R}$ (d) $\frac{-2GM}{R}$

21. For a simple pendulum, a graph is plotted between its Kinetic Energy (KE) and Potential Energy (PE) against its displacement d. Which one of the following represents these correctly? (graphs are schematic and not drawn to scale) **(2015)**



22. A pendulum made of a uniform wire of cross-sectional area A has time period T. When an additional mass M is added to its bob, the time period changes to T_M. If the Young's modulus of the material of the wire is Y, then $1/Y$ is equal to (g = gravitational acceleration) **(2015)**

(a) $\left[1-\left(\frac{T}{T_M}\right)^2\right]\frac{A}{Mg}$

(b) $\left[\left(\frac{T_M}{T}\right)^2-1\right]\frac{Mg}{A}$

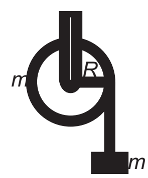
(c) $\left[1-\left(\frac{T_M}{T}\right)^2\right]\frac{A}{Mg}$

(d) $\left[\left(\frac{T_M}{T}\right)^2-1\right]\frac{A}{Mg}$

23. A bob of mass m attached to an inextensible string of length l is suspended from a vertical support. The bob rotates in a horizontal circle with an angular speed ω rad/s about the vertical support. About the point of suspension **(2014)**

(a) angular momentum is conserved
(b) angular momentum changes in magnitude but not in direction
(c) angular momentum changes in direction but not in magnitude
(d) angular momentum changes both in direction and magnitude

24. A mass m supported by a massless string wound around a uniform hollow cylinder of mass m and radius R. If the string does not slip on the cylinder, with what acceleration will the mass fall on release? **(2014)**



(a) $2g/3$ (b) $g/2$
(c) $5g/6$ (d) g

25. Four particles, each of mass M and equidistant from each other, move along a circle of radius R under the action of their mutual gravitational attraction, the speed of each particle is **(2014)**

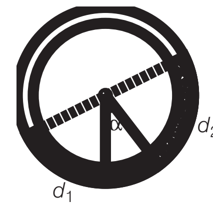
(a) $\sqrt{\dfrac{GM}{R}}$ (b) $\sqrt{2\sqrt{2}\dfrac{GM}{R}}$

(c) $\sqrt{\dfrac{GM}{R}(1+2\sqrt{2})}$ (d) $\dfrac{1}{2}\sqrt{\dfrac{GM}{R}(1+2\sqrt{2})}$

26. A particle moves with simple harmonic motion in a straight line. In first τ sec, after starting from rest it travels a distance a and in next τ sec, it travels $2a$, in same direction, then **(2014)**

(a) amplitude of motion is $3a$
(b) time period of oscillations is 8π
(c) amplitude of motion is $4a$
(d) time period of oscillations is 6π

27. There is a circular tube in a vertical plane. Two liquids which do not mix and of densities d_1 and d_2 are filled in the tube. Each liquid subtends 90° angle at centre. Radius joining their interface makes an angle α with vertical. Ratio d_1 / d_2 is **(2014)**



(a) $\dfrac{1+\sin\alpha}{1-\sin\alpha}$ (b) $\dfrac{1+\cos\alpha}{1-\cos\alpha}$

(c) $\dfrac{1+\tan\alpha}{1-\tan\alpha}$ (d) $\dfrac{1+\sin\alpha}{1-\cos\alpha}$

28. On heating water, bubbles beings formed at the bottom of the vessel detach and rise. Take the bubbles to be spheres of radius R and making a circular contact of radius r with the bottom of the vessel. If $r << R$ and the surface tension of water is T, value of r just before bubbles detach is (density of water is ρ) **(2014)**

(a) $R^2\sqrt{\dfrac{2\rho_w g}{3T}}$ (b) $R^2\sqrt{\dfrac{\rho_w g}{6T}}$

(c) $R^2\sqrt{\dfrac{\rho_w g}{T}}$ (d) $R^2\sqrt{\dfrac{3\rho_w g}{T}}$

29. The pressure that has to be applied to the ends of a steel wire of length 10 cm to keep its length constant when its temperature is raised by 100°C is (For steel, Young's modulus is $2\times10^{11}\,\mathrm{Nm^{-2}}$ and coefficient of thermal expansion is $1.1\times10^{-5}\,\mathrm{K^{-1}}$) **(2014)**

(a) 2.2×10^{8} Pa
(b) 2.2×10^{9} Pa
(c) 2.2×10^{7} Pa
(d) 2.2×10^{6} Pa

30. An open glass tube is immersed in mercury in such a way that a length of 8 cm extends above the mercury level. The open end of the tube is then closed and sealed and the tube is raised vertically up by additional 46 cm. What will be length of the air column above mercury in the tube now? (Atmospheric pressure = 76 cm of Hg) **(2014)**

(a) 16 cm (b) 22 cm
(c) 38 cm (d) 6 cm

31. This question has statement I and statement II. Of the four choices given after the statements, choose the one that best describes the two statements. **(2013)**

Statement I A point particle of mass m moving with speed v collides with stationary point particle of mass M. If the maximum energy loss possible is given as $f\left(\frac{1}{2}mv^2\right)$, then $f=\left(\frac{m}{M+m}\right)$.

Statement II Maximum energy loss occurs when the particles get stuck together as a result of the collision.

(a) Statement I is true, Statement II is true, and Statement II is the correct explanation of Statement I
(b) Statement I is true, Statement II is true, but Statement II is not the correct explanation of Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

32. A hoop of radius r and mass m rotating with an angular velocity ω_0 is placed on a rough horizontal surface. The initial velocity of the centre of the hoop is zero. What will be the velocity of the centre of the hoop when it ceases to slip? **(2013)**

(a) $r\omega_0/4$ (b) $r\omega_0/3$
(c) $r\omega_0/2$ (d) $r\omega_0$

33. What is the minimum energy required to launch a satellite of mass m from the surface of a planet of mass M and radius R in a circular orbit at an altitude of $2R$? **(2013)**

(a) $\frac{5GmM}{6R}$ (b) $\frac{2GmM}{3R}$ (c) $\frac{GmM}{2R}$ (d) $\frac{GmM}{3R}$

34. Assume that a drop of liquid evaporates by decrease in its surface energy, so that its temperature remains unchanged. What should be the minimum radius of the drop for this to be possible? The surface tension is T, denstiy of liquid is ρ and L is its latent heat of vaporisation **(2013)**

(a) $\frac{\rho L}{T}$ (b) $\sqrt{\frac{T}{\rho L}}$ (c) $\frac{T}{\rho L}$ (d) $\frac{2T}{\rho L}$

35. A uniform cylinder of length L and mass M having cross-sectional area A is suspended, with its length vertical from a fixed point by a massless spring such that it is half submerged in a liquid of density σ at equilibrium position. The extension x_0 of the spring when it is in equilibrium is **(2013)**

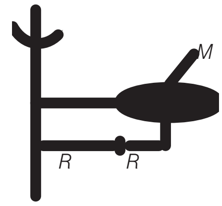
(a) $\frac{Mg}{k}$ (b) $\frac{Mg}{k}\left(1-\frac{LA\sigma}{M}\right)$
(c) $\frac{Mg}{k}\left(1-\frac{LA\sigma}{2M}\right)$ (d) $\frac{Mg}{k}\left(1+\frac{LA\sigma}{M}\right)$

Answer *with* Explanations

1. (d) **Key Idea** *First we found moment of inertia (MI) of system using parallel axis theorem about centre of mass, then we use it to find moment of inertia about given axis.*

Moment of inertia of an outer disc about the axis through centre is

$$=\frac{MR^2}{2}+M(2R)^2=MR^2\left(4+\frac{1}{2}\right)=\frac{9}{2}MR^2$$



For 6 such discs,

moment of inertia $=6\times\frac{9}{2}MR^2=27MR^2$

So, moment of inertia of system

$$=\frac{MR^2}{2}+27MR^2=\frac{55}{2}MR^2$$

Hence, $I_P=\frac{55}{2}MR^2+(7M\times 9R^2)$

$$\Rightarrow \quad I_P=\frac{181}{2}MR^2 \text{ and } I_{\text{system}}=\frac{181}{2}MR^2$$

2. (a) Moment of inertia of remaining solid
= Moment of intertia of complete solid
– Moment of inertia of removed portion

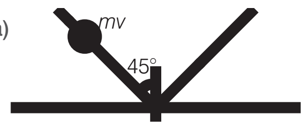
$$\therefore \quad I=\frac{9MR^2}{2}-\left[\frac{M(R/3)^2}{2}+M\left(\frac{2R}{3}\right)^2\right]\Rightarrow I=4MR^2$$

3. **(c)** $\therefore$ Bulk modulus, $K = \dfrac{\text{Volumetric stress}}{\text{Volumetric strain}} = \dfrac{\Delta p}{\dfrac{\Delta V}{V}}$

$$\Rightarrow \quad K = \frac{mg}{a\left(\dfrac{3\Delta r}{r}\right)} \qquad \left[\because V = \frac{4}{3}\pi r^3, \text{ so } \frac{\Delta V}{V} = \frac{3\Delta r}{r}\right]$$

$$\Rightarrow \quad \frac{\Delta r}{r} = \frac{mg}{3Ka}$$

4. **(a)**



Momentum imparted due to first collision

$$= 2mv \sin 45°$$

$$= \sqrt{2}mv \qquad \left[\because \sin 45° = \frac{1}{\sqrt{2}}\right]$$

$$\therefore \text{Pressure on surface} = \frac{n\sqrt{2}mv}{\text{Area}}$$

$$= \frac{10^{23} \times \sqrt{2} \times 3.32 \times 10^{-27} \times 10^3}{(2 \times 10^{-2})^2}$$

$$p = 2.35 \times 10^3 \text{ N/m}^2$$

5. **(d)** MI of a solid cylinder about its perpendicular bisector of length is

$$I = M\left(\frac{l^2}{12} + \frac{R^2}{4}\right)$$

$$\Rightarrow \quad I = \frac{mR^2}{4} + \frac{ml^2}{12} = \frac{m^2}{4\pi\rho l} + \frac{ml^2}{12} \qquad [\because \rho\pi r^2 l = m]$$

For I to be maximum,

$$\frac{dI}{dl} = -\frac{m^2}{4\pi\rho}\left(\frac{1}{l^2}\right) + \frac{ml}{6} = 0 \Rightarrow \frac{m^2}{4\mu\pi\rho} = \frac{ml^3}{6}$$

$$\Rightarrow \quad l^3 = \frac{3m}{2\pi\rho} \Rightarrow l = \left(\frac{3}{2}\right)^{1/3}\left(\frac{m}{\pi\rho}\right)^{1/3}$$

$$\rho = \frac{m}{\pi R^2 l} \Rightarrow R^2 = \frac{m}{\pi\rho l}$$

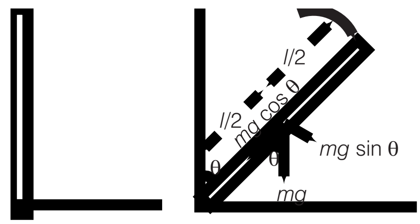
$$\Rightarrow \quad R^2 = \frac{m}{\pi\rho}\left(\frac{2}{3}\right)^{1/3}\left(\frac{\pi\rho}{m}\right)^{1/3} = \left(\frac{m}{\pi\rho}\right)^{2/3}\left(\frac{2}{3}\right)^{1/3}$$

$$\Rightarrow \quad R = \left(\frac{m}{\pi\rho}\right)^{1/3}\left(\frac{2}{3}\right)^{1/6}$$

$$\frac{l}{R} = \frac{\left(\dfrac{3}{2}\right)^{1/3}\left(\dfrac{m}{\pi\rho}\right)^{1/3}}{\left(\dfrac{m}{\pi\rho}\right)^{1/3}\left(\dfrac{2}{3}\right)^{1/6}}$$

$$= \left(\frac{3}{2}\right)^{1/3} + \left(\frac{3}{2}\right)^{1/6} \Rightarrow \frac{l}{R} = \sqrt{\frac{3}{2}}$$

6. **(d)** As the rod rotates in vertical plane so a torque is acting on it, which is due to the vertical component of weight of rod.



Initial condition   At any time t

Now, Torque τ = force × perpendicular distance of line of action of force from axis of rotation

$$= mg \sin\theta \times \frac{l}{2}$$

Again, Torque, $\tau = I\alpha$

where, I = moment of inertia $= \dfrac{ml^2}{3}$

[Force and Torque frequency along axis of rotation passing through in end]

α = angular acceleration

$$\therefore \quad mg \sin\theta \times \frac{l}{2} = \frac{ml^2}{3}\alpha \Rightarrow \alpha = \frac{3g \sin\theta}{2l}$$

7. **(c)** Inside the earth surface

$$g = \frac{GM}{R^3} r \quad \text{i.e.} \quad g \propto r$$

Out the earth surface $g = \dfrac{Gm}{r^2}$ i.e. $g \propto \dfrac{1}{r^2}$

So, till earth surface g increases linearly with distance r, shown only in graph (c).

8. **(c)** KE is maximum at mean position and minimum at extreme position $\left(\text{at } t = \dfrac{T}{4}\right)$.

9. **(d)** Time period of oscillation is

$$T = 2\pi\sqrt{\frac{I}{MB}}$$

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{7.5 \times 10^{-6}}{6.7 \times 10^{-2} \times 0.01}} = 0.665 \text{ s}$$

Hence, time for 10 oscillations is $t = 6.65$ s.

10. **(a)** By ascent formula, we have surface tension,

$$T = \frac{rhg}{2} \times 10^3 \frac{\text{N}}{\text{m}} = \frac{dhg}{4} \times 10^3 \frac{\text{N}}{\text{m}}$$

$$\Rightarrow \quad \frac{\Delta T}{T} = \frac{\Delta d}{d} + \frac{\Delta h}{h} \qquad [\text{given, } g \text{ is constant}]$$

$$\text{So, percentage} = \frac{\Delta T}{T} \times 100 = \left(\frac{\Delta d}{d} + \frac{\Delta h}{h}\right) \times 100$$

$$= \left(\frac{0.01 \times 10^{-2}}{1.25 \times 10^{-2}} + \frac{0.01 \times 10^{-2}}{1.45 \times 10^{-2}} \right) \times 100 = 1.5\%$$

$\therefore \quad \frac{\Delta T}{T} \times 100 = 1.5\%$

11. (d) Stress $= \frac{\text{Weight}}{\text{Area}} = \frac{9^3 \times W_0}{9^2 \times A_0} = 9\left(\frac{W_0}{A_0}\right)$

Hence, the stress increases by a factor of 9.

12. (b, d) We can apply $\mathbf{L} = m(\mathbf{r} \times \mathbf{v})$ for different parts.

For example :

In part (a), coordinates of A are $\left(\frac{R}{\sqrt{2}}, \frac{R}{\sqrt{2}}\right)$

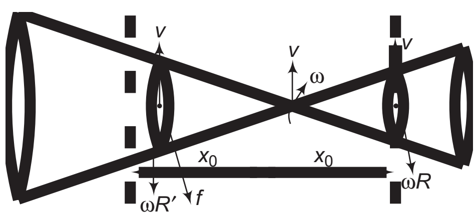
Therefore, $\quad \mathbf{r} = \frac{R}{\sqrt{2}}\hat{\mathbf{i}} + \frac{R}{\sqrt{2}}\hat{\mathbf{j}}$ and $\mathbf{v} = v\hat{\mathbf{i}}$

So, substituting in $\quad \mathbf{L} = m(\mathbf{r} \times \mathbf{v})$ we get,

$$\mathbf{L} = -\frac{mvR}{\sqrt{2}}\hat{\mathbf{k}}$$

Hence, option (a) is correct. Similarly, we can check other options also.

13. (a)



At distance x_0 from O, $v = \omega R$

Distance less than x_0, $v > \omega R$

Initially, there is pure rolling at both the contacts. As the cone moves forward, slipping at AB will start in forward direction, as radius at left contact decreases. Thus, the cone will start turning towards left. As it moves, further slipping at CD will start in backward direction which will also turn the cone towards left.

14. (d) $v_{\text{orbital}} = \sqrt{\frac{GM}{R}} = \sqrt{gR} \quad \Rightarrow \quad v_{\text{escape}} = \sqrt{2gR}$

$\therefore$ Extra velocity required

$$= v_{\text{escape}} - v_{\text{orbital}} = \sqrt{gR}\ (\sqrt{2} - 1)$$

15. (a) True value $= \frac{90 + 91 + 95 + 92}{4} = 92$

Mean absolute error

$$= \frac{|92 - 90| + |92 - 91| + |92 - 95| + |92 - 92|}{4}$$

$$= \frac{2 + 1 + 3 + 0}{4} = 1.5$$

Value = (92 ± 1.5)

Since, least count is 1 sec

$\therefore \qquad$ Value = $(92 \pm 2\ \text{s})$

16. (d) $v = \omega\sqrt{A^2 - x^2}$ At, $x = \frac{2A}{3}$

$$v = \omega\sqrt{A^2 - \left(\frac{2A}{3}\right)^2} = \frac{\sqrt{5}}{3}\omega A$$

As, velocity is trebled, hence $v' = \sqrt{5}A\omega$

This leads to new amplitude A'

$$\therefore \qquad \omega\sqrt{A'^2 - \left(\frac{2A}{3}\right)^2} = \sqrt{5}A\omega$$

$$\Rightarrow \qquad \omega^2\left[A'^2 - \frac{4A^2}{9}\right] = 5A^2\omega^2$$

$$\Rightarrow A'^2 = 5A^2 + \frac{4}{9}A^2 = \frac{49}{9}A^2 \Rightarrow A' = \frac{7}{3}A$$

17. (a) Centre of mass of uniform solid cone of height h is at a height of $\frac{h}{4}$ from base. Therefore from vertex it's $\frac{3h}{4}$.

18. (b) In all type of collisions, momentum of the system always remains constant. In perfectly inelastic collision, particles stick together and move with a common velocity.

Let this velocity is $\mathbf{v}_c$. Then,

initial momentum of system = final momentum of system

or $\qquad m(2v)\hat{\mathbf{i}} + 2m(v)\hat{\mathbf{j}} = (m + 2m)\mathbf{v}_c$

$$\therefore \qquad \mathbf{v}_c = \frac{2}{3}(v\hat{\mathbf{i}} + v\hat{\mathbf{j}})$$

$$|\mathbf{v}_c| \text{ or } v_c \text{ or speed} = \sqrt{\left(\frac{2}{3}v\right)^2 + \left(\frac{2}{3}v\right)^2} = \frac{2\sqrt{2}}{3}v$$

Initial kinetic energy

$$K_i = \frac{1}{2}(m)(2v)^2 + \frac{1}{2}(2m)(v)^2 = 3mv^2$$

Final kinetic energy

$$K_f = \frac{1}{2}(3m)\left(\frac{2\sqrt{2}}{3}v\right)^2 = \frac{4}{3}mv^2$$

$$\text{Fractional loss} = \left(\frac{K_i - K_f}{K_i}\right) \times 100$$

$$= \left[\frac{(3mv^2) - [(4/3)mv^2]}{(3mv^2)}\right] \times 100 = 56\%$$

19. (b) Maximum possible volume of cube will occur when

$$\sqrt{3}a = 2R \qquad (a = \text{side of cube})$$

$$\therefore \qquad a = \frac{2}{\sqrt{3}}R$$

Now, density of sphere, $\rho = \frac{M}{\frac{4}{3}\pi R^3}$

Mass of cube, $m =$ (volume of cube)$(\rho) = (a^3)(\rho)$

$$= \left[\frac{2}{\sqrt{3}}R\right]^3 \left[\frac{m}{(4/3)\pi R^3}\right] = \left(\frac{2}{\sqrt{3}\pi}\right)M$$

Now, moment of inertia of the cube about the said axis is

$$I = \frac{ma^2}{6} = \frac{\left(\frac{2}{\sqrt{3}\pi}\right)M\left(\frac{2}{\sqrt{3}}R\right)^2}{\sigma} = \frac{4MR^2}{9\sqrt{3}\pi}$$

20. **(a)** $V_R = V_T - V_C$

$V_R =$ Potential due to remaining portion

$V_T =$ Potential due to total sphere

$V_C =$ Potential due to cavity

Radius of cavity is $\frac{R}{2}$. Hence, volume and mass is $\frac{M}{8}$.

$$\therefore V_R = -\frac{GM}{R^3}\left[1.5R^2 - 0.5\left(\frac{R}{2}\right)^2\right] + \frac{G(M/8)}{(R/2)}\left(\frac{3}{2}\right)$$

$$= -\frac{GM}{R}$$

21. **(a)** Taking minimum potential energy at mean position to be zero, the expression of KE and PE are

$$\text{KE} = \frac{1}{2}m\omega^2(A^2 - d^2) \text{ and PE} = \frac{1}{2}m\omega^2 d^2$$

Both graphs are parabola. At $d = 0$, the mean position,

$\text{PE} = 0$ and $\text{KE} = \frac{1}{2}m\omega^2A^2 =$ maximum

At $d = \pm A$, the extreme positions,

$\text{KE} = 0$ and $\text{PE} = \frac{1}{2}m\omega^2A^2 =$ maximum

Therefore, the correct graph is (a).

22. **(d)**

$$T = 2\pi\sqrt{\frac{L}{g}} \quad \ldots\text{(i)}$$

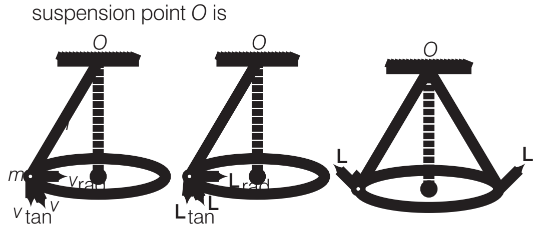
$$T_M = 2\pi\sqrt{\frac{L + \Delta L}{g}}$$

Here, $\Delta L = \frac{FL}{AY} = \frac{MgL}{AY} \Rightarrow T_M = 2\pi\sqrt{\frac{L + \frac{MgL}{AY}}{g}}$ $\quad\ldots$(ii)

Solving Eqs. (i) and (ii), we get

$$\frac{1}{Y} = \frac{A}{Mg}\left[\left(\frac{T_M}{T}\right)^2 - 1\right]$$

23. **(c)** Angular momentum of the pendulum about the suspension point O is

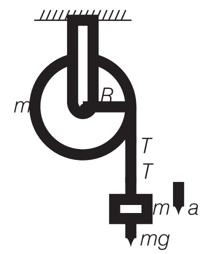


Then, v can be resolved into two components, radial component r_{rad} and tangential component r_{tan}. Due to v_{rad}, L will be tangential and due to v_{tan}, L will be radially outwards as shown. So, net angular momentum will be as shown in figure whose magnitude will be constant ($|L| = mvl$). But its direction will change as shown in the figure.

$$\mathbf{L} = m(\mathbf{r} \times \mathbf{v})$$

where, r = radius of circle.

24. **(b)** For the mass m, $mg - T = ma$



As we know, $\quad a = R\alpha$

So, $\quad mg - T = mR\alpha \quad \ldots$(i)

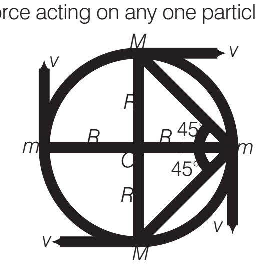
Torque about centre of pully

$$T \times R = mR^2\alpha \quad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), we get, $a = g/2$

Hence, the acceleration with the mass of a body fall is $g/2$.

25. **(d)** Net force acting on any one particle M,



$$= \frac{GM^2}{(2R)^2} + \frac{GM^2}{(R\sqrt{2})^2}\cos 45° + \frac{GM^2}{(R\sqrt{2})^2}\cos 45°$$

$$= \frac{GM^2}{R^2}\left(\frac{1}{4} + \frac{1}{\sqrt{2}}\right)$$

This force will equal to centripetal force.

So, $\quad \frac{Mv^2}{R} = \frac{GM^2}{R^2}\left(\frac{1}{4} + \frac{1}{\sqrt{2}}\right)$

$$v = \sqrt{\frac{GM}{4R}(1 + 2\sqrt{2})}$$

$$= \frac{1}{2}\sqrt{\frac{GM}{R}(2\sqrt{2} + 1)}$$

Hence, speed of each particle in a circular motion is

$$\frac{1}{2}\sqrt{\frac{GM}{R}(2\sqrt{2} + 1)}.$$

26. **(d)** In SHM, a particle starts from rest, we have

i.e. $x = A\cos\omega t$, at $t = 0$, $x = A$

When $t = \tau$, then $x = A - a$ ...(i)

When $t = 2\tau$, then $x = A - 3a$ ...(ii)

On comparing Eqs. (i) and (ii), we get

$$A - a = A\cos\omega\tau \Rightarrow A - 3a = A\cos 2\omega\tau$$

As $\cos 2\omega\tau = 2\cos^2\omega\tau - 1$

$$\Rightarrow \frac{A-3a}{A} = 2\left(\frac{A-a}{A}\right)^2 - 1$$

$$\Rightarrow \frac{A-3a}{A} = \frac{2A^2 + 2a^2 - 4Aa - A^2}{A^2}$$

$$A^2 - 3aA = A^2 + 2a^2 - 4Aa$$

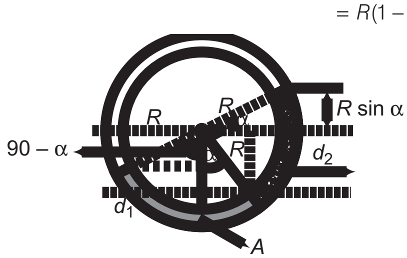
$$a^2 = 2aA \Rightarrow A = 2a$$

Now, $A - a = A\cos\omega\tau$

$$\Rightarrow \cos\omega\tau = 1/2 \Rightarrow \frac{2\pi}{T}\tau = \frac{\pi}{3} \Rightarrow T = 6\pi$$

27. **(c)** Equating pressure at A, we get

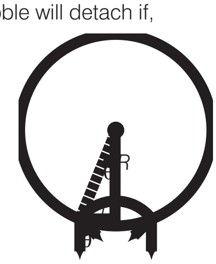
$$R\sin\alpha\, d_2 + R\cos\alpha\, d_2 + R(1-\cos\alpha)\, d_1 = R(1-\sin\alpha)\, d_1$$



$$(\sin\alpha + \cos\alpha)\, d_2 = d_1(\cos\alpha - \sin\alpha)$$

$$\Rightarrow \frac{d_1}{d_2} = \frac{1+\tan\alpha}{1-\tan\alpha}$$

28. **(a)** The bubble will detach if,



$$\int \sin\theta T \times dl = T(2\pi r)\sin\theta$$

Buoyant force $\geq$ Surface tension force

$$\frac{4}{3}\pi R^3 \rho_w g \geq \int T \times dl \sin\theta$$

$$(\rho_w)\left(\frac{4}{3}\pi R^3\right) g \geq (T)(2\pi r)\sin\theta \Rightarrow \sin\theta = \frac{r}{R}$$

Solving, $r = \sqrt{\frac{2\rho_w R^4 g}{3T}} = R^2\sqrt{\frac{2\rho_w g}{3T}}$

29. **(a)** If the deformation is small, then the stress in a body is directly proportional to the corresponding strain.

According to Hooke's law i.e.

$$\text{Young's modulus } (Y) = \frac{\text{Tensile stress}}{\text{Tensile strain}}$$

So, $Y = \dfrac{F/A}{\Delta L/L} = \dfrac{FL}{A\Delta L}$

If the rod is compressed, then compressive stress and strain appear. Their ratio Y is same as that for tensile case.

Given, length of a steel wire $(L) = 10$ cm

Temperature $(\theta) = 100°$C

As length is constant.

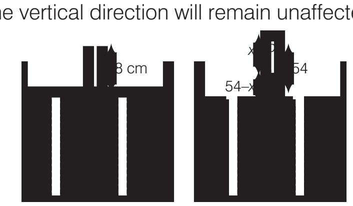
$\therefore$ Strain $= \dfrac{\Delta L}{L} = \alpha\Delta\theta$

Now, pressure = stress = $Y \times$ strain

[Given, $Y = 2\times 10^{11}$ N/m^2 and $\alpha = 1.1\times 10^{-5}$ K^{-1}]

$= 2\times 10^{11} \times 1.1\times 10^{-5} \times 100 = 2.2\times 10^{8}$ Pa

30. **(a)** **Thinking Process** In this question, the system is accelerating horizontally i.e. no component of acceleration in vertical direction. Hence, the pressure in the vertical direction will remain unaffected.



i.e. $p_1 = p_0 + \rho g h$

Again, we have to use the concept that the pressure in the same level will be same.

For air trapped in tube, $p_1 V_1 = p_2 V_2$

$p_1 = p_{atm} = \rho g 76 \Rightarrow V_1 = A \cdot 8$

[A = area of cross-section]

$p_2 = p_{atm} - \rho g(54 - x) = \rho g(22 + x)$

$V_2 = A \cdot x \qquad \rho g 76 \times 8A = \rho g(22 + x)Ax$

$x^2 + 22x - 78 \times 8 = 0 \Rightarrow x = 16$ cm

31. **(d)** Maximum energy loss

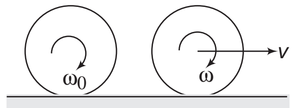
$$= \frac{p^2}{2m} - \frac{p^2}{2(m+M)} \qquad \left(\because \text{KE} = \frac{p^2}{2m}\right)$$

Before collision the mass m and after collision the mass is $m + M$

$$= \frac{p^2}{2m}\left[\frac{M}{(m+M)}\right]$$

$$= \frac{1}{2}mv^2\left\{\frac{M}{m+M}\right\} \qquad \left(f = \frac{M}{m+M}\right)$$

32. (c)



$$\omega = v/r$$

From conservation of angular momentum about bottommost point

$$mr^2\omega_0 = mvr + mr^2 \times v/r \Rightarrow v = \frac{\omega_0 r}{2}$$

33. (a) E = Energy of satellite – energy of mass on the surface of planet

$$= -\frac{GMm}{2r} - \left(-\frac{GMm}{R}\right)$$

Here, $r = R + 2R = 3R$

Substituting in about equation we get, $E = \dfrac{5GMm}{6R}$

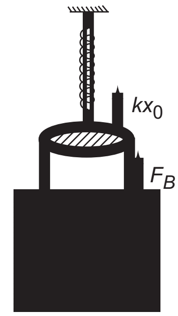
34. (d) Decrease in surface energy = heat required in vaporisation.

$\therefore \quad T(dS) = L(dm) \Rightarrow T(2)(4\pi r)dr = L(4\pi r^2 dr)\rho$

$\therefore \quad r = \dfrac{2T}{\rho L}$

35. (c) In equilibrium, Upward force = Downward force

$$kx_0 + F_B = mg$$



Here, kx_0 is restoring force of spring and F_B is buoyancy force.

$$kx_0 + \sigma\frac{L}{2}Ag = Mg$$

$$x_0 = \frac{Mg - \dfrac{\sigma LAg}{2}}{k}$$

JEE Advanced

1. The potential energy of mass m at a distance r from a fixed point O is given by $V(r) = kr^2/2$, where k is a positive constant of appropriate dimensions. This particle is moving in a circular orbit of radius R about the point O. If v is the speed of the particle and L is the magnitude of its angular momentum about O, which of the following statements is (are) true ?

(More than One Correct Option, 2018)

(a) $v = \sqrt{\dfrac{k}{2m}}R$ (b) $v = \sqrt{\dfrac{k}{m}}R$

(c) $L = \sqrt{mk}R^2$ (d) $L = \sqrt{\dfrac{mk}{2}}R^2$

2. Consider a body of mass 1.0 kg at rest at the origin at time $t = 0$. A force $\mathbf{F} = (\alpha t\,\hat{\mathbf{i}} + \beta\,\hat{\mathbf{j}})$ is applied on the body, where $\alpha = 1.0\,\text{Ns}^{-1}$ and $\beta = 1.0\,\text{N}$. The torque acting on the body about the origin at time $t = 1.0\,\text{s}$ is τ. Which of the following statements is (are) true ?

(More than One Correct Option, 2018)

(a) $|\tau| = \dfrac{1}{3}\,\text{N-m}$

(b) The torque τ is in the direction of the unit vector $+\hat{\mathbf{k}}$

(c) The velocity of the body at $t = 1\,\text{s}$ is $\mathbf{v} = \dfrac{1}{2}(\hat{\mathbf{i}} + 2\hat{\mathbf{j}})\,\text{ms}^{-1}$

(d) The magnitude of displacement of the body at $t = 1\,\text{s}$ is $\dfrac{1}{6}\,\text{m}$

3. A uniform capillary tube of inner radius r is dipped vertically into a beaker filled with water. The water rises to a height h in the capillary tube above the water surface in the beaker. The surface tension of water is σ. The angle of contact between water and the wall of the capillary tube is θ. Ignore the mass of water in the meniscus. Which of the following statements is (are) true ?

(More than one Correct Option, 2018)

(a) For a given material of the capillary tube, h decreases with increase in r

(b) For a given material of the capillary tube, h is independent of σ

(c) If this experiment is performed in a lift going up with a constant acceleration, then h decreases

(d) h is proportional to contact angle θ

4. A ring and a disc are initially at rest, side by side, at the top of an inclined plane which makes an angle 60° with the horizontal. They start to roll without slipping at the same instant of time along the shortest path. If the time difference between their reaching the ground is $(2-\sqrt{3})/\sqrt{10}$ s, then the height of the top of the inclined plane, in metres, is .... (Take, $g = 10\text{ms}^{-2}$) **(Numerical Value, 2018)**

5. Consider a thin square plate floating on a viscous liquid in a large tank. The height h of the liquid in the tank is much less than the width of the tank. The floating plate is pulled horizontally with a constant velocity u_0. Which of the following statements is (are) true? **(More than One Correct Option, 2018)**

(a) The resistive force of liquid on the plate is inversely proportional to h

(b) The resistive force of liquid on the plate is independent of the area of the plate

(c) The tangential (shear) stress on the floor of the tank increases with u_0

(d) The tangential (shear) stress on the plate varies linearly with the viscosity η of the liquid

6. A steel wire of diameter 0.5 mm and Young's modulus 2×10^{11} N m^{-2} carries a load of mass m. The length of the wire with the load is 1.0 m. A vernier scale with 10 divisions is attached to the end of this wire. Next to the steel wire is a reference wire to which a main scale, of least count 1.0 mm, is attached. The 10 divisions of the vernier scale correspond to 9 divisions of the main scale. Initially, the zero of vernier scale coincides with the zero of main scale. If the load on the steel wire is increased by 1.2 kg, the vernier scale division which coincides with a main scale division is .... (Take, $g = 10$ ms^{-2} and $\pi = 3.2$). **(Numerical Value, 2018)**

7. A planet of mass M, has two natural satellites with masses m_1 and m_2. The radii of their circular orbits are R_1 and R_2, respectively. Ignore the gravitational force between the satellites. Define v_1, L_1, K_1 and T_1 to be respectively, the orbital speed, angular momentum, kinetic energy and time period of revolution of satellite 1; and v_2, L_2, K_2 and T_2 to be the corresponding quantities of satellite 2. Given, $m_1/m_2 = 2$ and $R_1/R_2 = 1/4$, match the ratios in List-I to the numbers in List-II. **(Matching Type, 2018)**

List-I	List-II
P. v_1/v_2	1. 1/8
Q. L_1/L_2	2. 1
R. K_1/K_2	3. 2
S. T_1/T_2	4. 8

(a) P → 4; Q → 2; R → 1; S → 3

(b) P → 3; Q → 2; R → 4; S → 1

(c) P → 2; Q → 3; R → 1; S → 4

(d) P → 2; Q → 3; R → 4; S → 1

8. In the List-I below, four different paths of a particle are given as functions of time. In these functions, α and β are positive constants of appropriate dimensions and $\alpha \neq \beta$. In each case, the force acting on the particle is either zero or conservative. In List-II, five physical quantities of the particle are mentioned: **p** is the linear momentum, **L** is the angular momentum about the origin, K is the kinetic energy, U is the potential energy and E is the total energy. Match each path in List-I with those quantities in List-II, which are conserved for that path **(Matching Type, 2018)**

	List-I	List-II
P.	$\mathbf{r}(t) = \alpha t\ \mathbf{i} + \beta t\ \mathbf{j}$	1. **p**
Q.	$\mathbf{r}(t) = \alpha \cos \omega t\ \mathbf{i} + \beta \sin \omega t\ \mathbf{j}$	2. **L**
R.	$\mathbf{r}(t) = \alpha (\cos \omega t\ \mathbf{i} + \sin \omega t\ \mathbf{j})$	3. K
S.	$\mathbf{r}(t) = \alpha t\ \mathbf{i} + \frac{\beta}{2} t^2\ \mathbf{j}$	4. U
		5. E

(a) P → 1, 2, 3, 4, 5; Q → 2, 5; R → 2, 3, 4, 5; S → 5
(b) P → 1, 2, 3, 4, 5; Q → 3, 5; R → 2, 3, 4, 5; S → 2, 5
(c) P → 2, 3, 4; Q → 5; R → 1, 2, 4; S → 2, 5
(d) P → 1, 2, 3, 5; Q → 2, 5; R → 2, 3, 4, 5; S → 2, 5

Passage (Q. Nos. 9-10)

One twirl is a circular ring (of mass M and radius R) near the tip of one's finger as shown in Fig. 1. In the process the finger never loses contact with the inner rim of the ring. The finger traces out the surface of a cone, shown by the dotted line. The radius of the path traced out by the point where the ring and the finger is in contact is r. The finger rotates with an angular velocity ω_0.

The rotating ring rolls without slipping on the outside of a smaller circle described by the point where the ring and the finger is in contact (Fig. 2). The coefficient of friction between the ring and the finger is μ and the acceleration due to gravity is g.

(Passage Type, 2017)

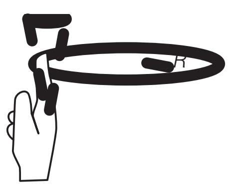


Fig. 1

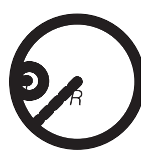


Fig. 2

9. The total kinetic energy of the ring is

(a) $M\omega_0^2(R-r)^2$ (b) $\frac{1}{2}M\omega_0^2(R-r)^2$
(c) $M\omega_0^2R^2$ (d) $\frac{3}{2}M\omega_0^2(R-r)^2$

10. The minimum value of ω_0 below which the ring will drop down is

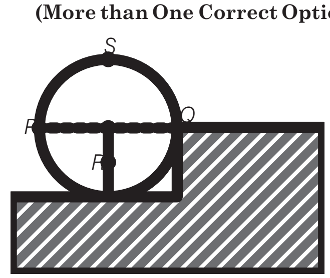
(a) $\sqrt{\frac{g}{2\mu(R-r)}}$ (b) $\sqrt{\frac{3g}{2\mu(R-r)}}$
(c) $\sqrt{\frac{g}{\mu(R-r)}}$ (d) $\sqrt{\frac{2g}{\mu(R-r)}}$

11. Consider regular polygons with number of sides $n = 3, 4, 5$ ...... as shown in the figure. The centre of mass of all the polygons is at height h from the ground. They roll on a horizontal surface about the leading vertex without slipping and sliding as depicted. The maximum increase in height of the locus of the centre of mass for each each polygon is Δ. Then, Δ depends on n and h as **(Single Correct Option, 2017)**

(a) $\Delta = h\sin^2\left(\frac{\pi}{n}\right)$ (b) $\Delta = h\sin\left(\frac{2\pi}{n}\right)$
(c) $\Delta = h\tan^2\left(\frac{\pi}{2n}\right)$ (d) $\Delta = h\left(\frac{1}{\cos\left(\frac{\pi}{n}\right)} - 1\right)$

12. A wheel of radius R and mass M is placed at the bottom of a fixed step of height R as shown in the figure. A constant force is continuously applied on the surface of the wheel so that it just climbs the step without slipping. Consider the torque τ about an axis normal to the plane of the paper passing through the point Q. Which of the following option(s) is/are correct?

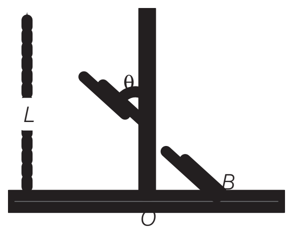
(More than One Correct Option, 2017)



(a) If the force is applied normal to the circumference at point P, then τ is zero
(b) If the force is applied tangentially at point S, then $\tau \neq 0$ but the wheel never climbs the step
(c) If the force is applied at point P tangentially, then τ decreases continuously as the wheel climbs
(d) If the force is applied normal to the circumference at point X, then τ is constant

13. A rigid uniform bar AB of length L is slipping from its vertical position on a frictionless floor (as shown in the figure). At some instant of time, the angle made by the bar with the vertical is θ. Which of the following statements about its motion is/are correct?

(More than One Correct Option, 2017)



(a) Instantaneous torque about the point in contact with the floor is proportional to $\sin\theta$

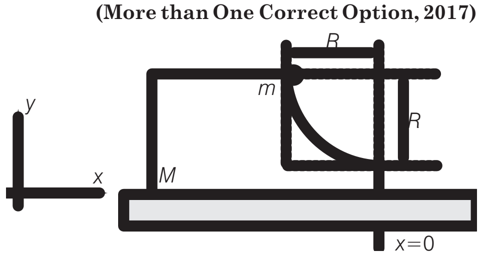
(b) The trajectory of the point A is parabola

(c) The mid-point of the bar will fall vertically downward

(d) When the bar makes an angle θ with the vertical, the displacement of its mid-point from the initial position is proportional to $(1-\cos\theta)$

14. A block of mass M has a circular cut with a frictionless surface as shown. The block rests on the horizontal frictionless surface of a fixed table. Initially, the right edge of the block is at $x = 0$, in a coordinate system fixed to the table. A point mass m is released from rest at the topmost point of the path as shown and it slides down. When the mass loses contact with the block, its position is x and the velocity is v. At that instant, which of the following option(s) is/are correct?

(More than One Correct Option, 2017)



(a) The velocity of the point mass m is $v = \sqrt{\dfrac{2gR}{1+\dfrac{m}{M}}}$

(b) The x component of displacement of the centre of mass of the block M is $-\dfrac{mR}{M+m}$

(c) The position of the point mass is $x = -\sqrt{2}\dfrac{mR}{M+m}$

(d) The velocity of the block M is $v = -\dfrac{m}{M}\sqrt{2gR}$

15. A rocket is launched normal to the surface of the Earth, away from the Sun, along the line joining the Sun and the Earth. The Sun is 3×10^5 times heavier than the Earth and is at a distance 2.5×10^4 times larger than the radius of Earth. The escape velocity from Earth's gravitational field is $v_e = 11.2\ \text{km s}^{-1}$. The minimum initial velocity (v_s) required for the rocket to be able to leave the Sun-Earth system is closest to (Ignore the rotation and revolution of the Earth and the presence of any other planet)

(Single Correct Option, 2017)

(a) $v_s = 72\ \text{km s}^{-1}$

(b) $v_s = 22\ \text{km s}^{-1}$

(c) $v_s = 42\ \text{km s}^{-1}$

(d) $v_s = 62\ \text{km s}^{-1}$

16. Consider an expanding sphere of instantaneous radius R whose total mass remains constant. The expansion is such that the instantaneous density ρ remains uniform throughout the volume. The rate of fractional change in density $\left(\dfrac{1}{p}\dfrac{d\rho}{dt}\right)$ is constant. The velocity v of any point of the surface of the expanding sphere is proportional to **(Single Correct Option, 2017)**

(a) R (b) $\dfrac{1}{R}$

(c) R^3 (d) $R^{\frac{2}{3}}$

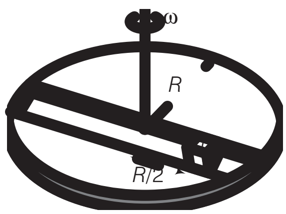
17. A drop of liquid of radius $R = 10^{-2}$ m having surface tension $S = \dfrac{0.1}{4\pi}\ \text{Nm}^{-1}$ divides itself into K identical drops. In this process the total change in the surface energy $\Delta U = 10^{-3}$ J. If $K = 10^{\alpha}$, then the value of α is

(Single Integer Type, 2017)

Passage (Q. Nos. 18-19)

A frame of the reference that is accelerated with respect to an inertial frame of reference is called a non-inertial frame of reference. A coordinate system fixed on a circular disc rotating about a fixed axis with a constant angular velocity ω is an example of a non-inertial frame of reference.

The relationship between the force $\mathbf{F}_{\text{rot}}$ experienced by a particle of mass m moving on the rotating disc and the force $\mathbf{F}_{\text{in}}$ experienced by the particle in an inertial frame of reference is, $\mathbf{F}_{\text{rot}} = \mathbf{F}_{\text{in}} + 2m\,(\mathbf{v}_{\text{rot}} \times \vec{\omega}) + m\,(\vec{\omega} \times \mathbf{r}) \times \vec{\omega}$, where, $\mathbf{v}_{\text{rot}}$ is the velocity of the particle in the rotating frame of reference and $\mathbf{r}$ is the position vector of the particle with respect to the centre of the disc



Now, consider a smooth slot along a diameter of a disc of radius R rotating counter-clockwise with a constant angular speed ω about its vertical axis through its centre.

We assign a coordinate system with the origin at the centre of the disc, the X-axis along the slot, the Y-axis perpendicular to the slot and the Z-axis along th rotation axis $(\omega = \omega\mathbf{k})$. A small block of mass m is gently placed in the slot at $\mathbf{r} = (R/2)\,\hat{\mathbf{i}}$ at $t = 0$ and is constrained to move only along the slot. **(Passage Type, 2016)**

18. The distance r of the block at time t is

(a) $\frac{R}{2}\cos 2\omega t$ (b) $\frac{R}{2}\cos\omega t$

(c) $\frac{R}{4}(e^{\omega t} + e^{-\omega t})$ (d) $\frac{R}{4}(e^{2\omega t} + e^{-2\omega t})$

19. The net reaction of the disc on the block, is

(a) $m\omega^2 R \sin\omega t\,\hat{\mathbf{j}} - mg\hat{\mathbf{k}}$

(b) $\frac{1}{2}m\omega^2 R(e^{\omega t} - e^{-\omega t})\hat{\mathbf{j}} + mg\hat{\mathbf{k}}$

(c) $\frac{1}{2}m\omega^2 R(e^{2\omega t} - e^{-\omega t})\hat{\mathbf{j}} + mg\hat{\mathbf{k}}$

(d) $-m\omega^2 R\cos\omega t\,\hat{\mathbf{j}} - mg\hat{\mathbf{k}}$

20. The position vector $\mathbf{r}$ of particle of mass m is given by the following equation
$\mathbf{r}(t) = \alpha t^3\hat{\mathbf{i}} + \beta t^2\hat{\mathbf{j}}$ where, $\alpha = \frac{10}{3}\,\text{ms}^{-3}$, $\beta = 5\,\text{ms}^{-2}$ and $m = 0.1$ kg.
At $t = 1$ s, which of the following statement(s) is (are) true about the particle? **(More than One Correct Option, 2016)**

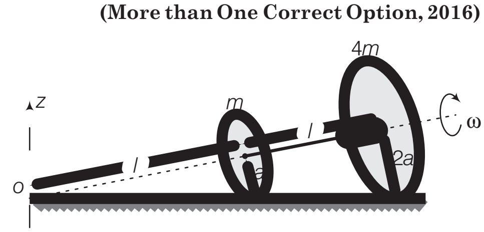
(a) The velocity $\mathbf{v}$ is given by $\mathbf{v} = (10\hat{\mathbf{i}} + 10\hat{\mathbf{j}})\text{ms}^{-1}$

(b) The angular momentum $\mathbf{L}$ with respect to the origin is given by $\mathbf{L} = (5/3)\hat{\mathbf{k}}$Nm-s

(c) The force $\mathbf{F}$ is given by $\mathbf{F} = (\hat{\mathbf{i}} + 2\hat{\mathbf{j}})$ N

(d) The torque τ with respect to the origin is given by $\tau = -\frac{20}{3}\hat{\mathbf{k}}$N-s

21. Two thin circular discs of mass m and $4m$, having radii of a and $2a$, respectively, are rigidly fixed by a massless, rigid rod of length $l = \sqrt{24}\,a$ through their centres. This assembly is laid on a firm and flat surface and set rolling without slipping on the surface so that the angular speed about the axis of the rod is ω. The angular momentum of the entire assembly about the point 'O' is $\mathbf{L}$ (see the figure). Which of the following statement(s) is (are) true?
(More than One Correct Option, 2016)



(a) The magnitude of the z-component of $\mathbf{L}$ is 55 $ma^2\omega$

(b) The magnitude of angular momentum of centre of mass of the assembly about the point O is $81ma^2\omega$

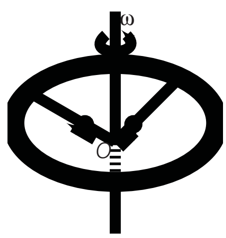
(c) The centre of mass of the assembly rotates about the Z-axis with an angular speed of $\omega/5$

(d) The magnitude of angular momentum of the assembly about its centre of mass is $17\,ma^2\omega/2$

22. A block with mass M is connected by a massless spring with stiffness constant k to a rigid wall and moves without friction on a horizontal surface. The block oscillates with small amplitude A about an equilibrium position x_0. Consider two cases : (i) when the block is at x_0 and (ii) when the block is at $x = x_0 + A$. In both the cases, a particle with mass m ($< M$) is softly placed on the block after which they stick to each other. Which of the following statement(s) is (are) true about the motion after the mass m is placed on the mass M?
(More than One Correct Option, 2016)

(a) The amplitude of oscillation in the first case changes by a factor of $\sqrt{\frac{M}{m+M}}$, whereas in the second case it remains unchanged
(b) The final time period of oscillation in both the cases is same
(c) The total energy decreases in both the cases
(d) The instantaneous speed at x_0 of the combined masses decreases in both the cases

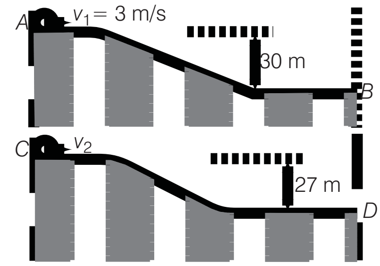
23. A ring of mass M and radius R is rotating with angular speed ω about a fixed vertical axis passing through its centre O with two point masses each of mass $M/8$ at rest at O. These masses can move radially outwards along two massless rods fixed on the ring as shown in the figure. At some instant, the angular speed of the system is $(8/9)\omega$ and one of the masses is at a distance of $\frac{3}{5}R$ from O. At this instant, the distance of the other mass from O is



(More than One Correct Option, 2015)

(a) $\frac{2}{3}R$ (b) $\frac{1}{3}R$ (c) $\frac{3}{5}R$ (d) $\frac{4}{5}R$

24. Two identical uniform discs roll without slipping on two different surfaces AB and CD (see figure) starting at A and C with linear speeds v_1 and v_2, respectively, and always remain in contact with the surfaces.

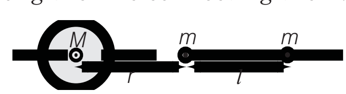


If they reach B and D with the same linear speed and $v_1 = 3$ m/s, then v_2 in m/s is ($g = 10$ m/s^2) **(Single Integer Type, 2015)**

25. The densities of two solids spheres A and B of the same radii R vary with radial distance r as $\rho_A(r) = k\left(\frac{r}{R}\right)$ and $\rho_B(r) = k\left(\frac{r}{R}\right)^5$, respectively, where k is a constant. The moments of inertia of the individual spheres about axes passing through their centres are I_A and I_B, respectively. If $\frac{I_B}{I_A} = \frac{n}{10}$, the value of n is
(Single Integer Type, 2015)

26. A bullet is fired vertically upwards with velocity v from the surface of a spherical planet. When it reaches its maximum height, its acceleration due to the planet's gravity is $1/4$ th of its value at the surface of the planet. If the escape velocity from the planet is $v_{\text{sec}} = v\sqrt{N}$, then the value of N is (ignore energy loss due to atmosphere) **(Single Integer Type, 2015)**

27. A large spherical mass M is fixed at one position and two identical masses m are kept on a line passing through the centre of M (see figure). The point masses are connected by a rigid massless rod of length l and this assembly is free to move along the line connecting them.



All three masses interact only through their mutual gravitational interaction. When the point mass nearer to M is at a distance $r = 3l$ from M the tension in the rod is zero for $m = k\left(\frac{M}{288}\right)$. The value of k is **(Single Integer Type, 2015)**

28. A spherical body of radius R consists of a fluid of constant density and is in equilibrium under its own gravity. If $P(r)$ is the pressure at $r(r < R)$, then the correct options is/are

(More than One Correct Option, 2015)

(a) $p(r = 0) = 0$

(b) $\dfrac{p\left(r = \frac{3R}{4}\right)}{p\left(r = \frac{2R}{3}\right)} = \dfrac{63}{80}$

(c) $\dfrac{p\left(r = \frac{3R}{5}\right)}{p\left(r = \frac{2R}{5}\right)} = \dfrac{16}{21}$

(d) $\dfrac{p\left(r = \frac{R}{2}\right)}{p\left(r = \frac{R}{3}\right)} = \dfrac{20}{27}$

29. A person in a lift is holding a water jar, which has a small hole at the lower end of its side. When the lift is at rest, the water jet coming out of the hole hits the floor of the lift at a distance d of 1.2 m from the person.

In the following, state of the lift's motion is given in Column I and the distance where the water jet hits the floor of the lift is given in Column II. Match the statements from Column I with those in Column II and select the correct answer using the code given below the columns.

Column I	Column II
P. Lift is accelerating vertlcally up.	1. $d = 1.2m$
Q. Lift is accelerating with an acceleration loss than the gravitational acceleration.	2. $d > 1.2m$
R. Lift is moving vertically up with constant speed.	3. $d < 1.2m$
S. Lift is falling freely.	4. No water leaks out of the jar

(More than One Correct Option, 2015)

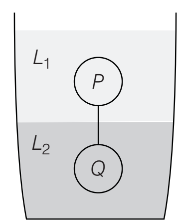
Codes

	P	Q	R	S
(a)	2,	3,	2,	4
(b)	2,	3,	1,	4
(c)	1,	1,	1,	4
(d)	2,	3,	1,	1

30. Two spheres P and Q for equal radii have densities ρ_1 and ρ_2, respectively. The spheres are connected by a massless string and placed in liquids L_1 and L_2 of densities σ_1 and σ_2 and viscosities η_1 and η_2, respectively. They float in equilibrium with the sphere P in L_1 and sphere Q in L_2 and the string being taut (see figure).

If sphere P alone in L_2 has terminal velocity $\mathbf{v}_P$ and Q alone in L_1 has terminal velocity $\mathbf{v}_Q$, then

(Single Correct Option, 2015)



(a) $\dfrac{|\mathbf{v}_P|}{|\mathbf{v}_Q|} = \dfrac{\eta_1}{\eta_2}$

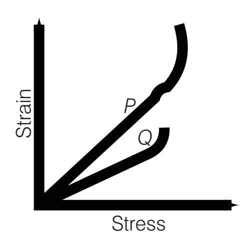
(b) $\dfrac{|\mathbf{v}_P|}{|\mathbf{v}_Q|} = \dfrac{\eta_2}{\eta_1}$

(c) $\mathbf{v}_P \cdot \mathbf{v}_Q > 0$

(d) $\mathbf{v}_P \cdot \mathbf{v}_Q < 0$

31. In plotting stress *versus* strain curves for two materials P and Q, a student by mistake puts strain on the y-axis and stress on the x-axis as shown in the figure. Then, the correct statements is/are

(More than One Correct Option, 2015)



(a) P has more tensile strength than Q

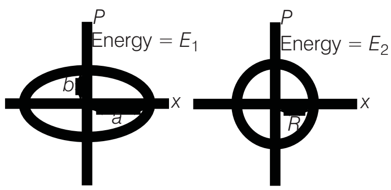
(b) P is more ductile than Q

(c) P is more brittle than Q

(d) The Young's modulus of P is more than that of Q

32. Two independent harmonic oscillators of equal masses are oscillating about the origin with angular frequencies ω_1 and ω_2 and have total energies E_1 and E_2, respectively. The variations of their momenta p with positions x are shown in the figures. If $\frac{a}{b} = n^2$ and $\frac{a}{R} = n$, then the correct equations is/are

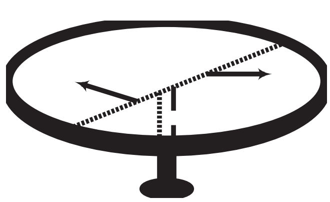
(More than One Correct Option, 2015)



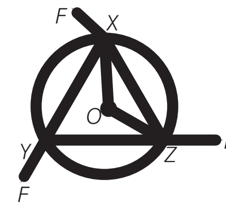
(a) $E_1\omega_1 = E_2\omega_2$ (b) $\frac{\omega_2}{\omega_1} = n^2$

(c) $\omega_1\omega_2 = n^2$ (d) $\frac{E_1}{\omega_1} = \frac{E_2}{\omega_2}$

33. A horizontal circular platform of radius 0.5 m and mass 0.45 kg is free to rotate about its axis. Two massless spring toy-guns, each carrying a steel ball of mass 0.05 kg are attached to the platform at a distance 0.25 m from the centre on its either sides along its diameter (see figure). Each gun simultaneously fires the balls horizontally and perpendicular to the diameter in opposite directions. After leaving the platform, the balls have horizontal speed of $9\,\text{ms}^{-1}$ with respect to the ground. The rotational speed of the platform in rads^{-1} after the balls leave the platform is **(Single Integer Type, 2014)**



34. A uniform circular disc of mass 1.5 kg and radius 0.5 m is initially at rest on a horizontal frictionless surface. Three forces of equal magnitude $F = 0.5\,\text{N}$ are applied simultaneously along the three sides of an equilateral triangle XYZ with its vertices on the perimeter of the disc (see figure). One second after applying the forces, the angular speed of the disc in rad s^{-1} is



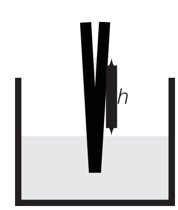
(Single Integer Type, 2014)

35. A planet of radius $R = 1/10 \times$ (radius of earth) has the same mass density as earth. Scientists dig a well of depth $\frac{R}{5}$ on it and lower a wire of the same length and of linear mass density 10^{-3} kgm^{-1} into it. If the wire is not touching anywhere, the force applied at the top of the wire by a person holding it in place is (take the radius of earth $= 6 \times 10^6$ m and the acceleration due to gravity of earth is 10 ms^{-2}) **(Single Correct Option, 2014)**

(a) 96 N (b) 108 N
(c) 120 N (d) 150 N

36. A glass capillary tube is the shape of truncated cone with an apex angle α so that its two ends have cross-sections of different radii. When dipped in water vertically, water rises in it to a height h, where the radius of its cross-section is b. If the surface tension of water is S, its density is ρ, and its contact angle with glass is θ, the value of h will be (g is the acceleration due to gravity)

(Single Correct Option, 2014)



(a) $\frac{2S}{b\rho g}\cos(\theta - \alpha)$ b) $\frac{2S}{b\rho g}\cos(\theta + \alpha)$

(c) $\frac{2S}{b\rho g}\cos(\theta - \alpha/2)$ (d) $\frac{2S}{b\rho g}\cos(\theta + \alpha/2)$

37. During Searle's experiment, zero of the vernier scale lies between 3.20×10^{-2} m and 3.25×10^{-2} m of the main scale. The 20th division of the vernier scale exactly coincides with one of the main scale divisions.

When an additional load of 2 kg is applied to the wire, the zero of the vernier scale still lies between 3.20×10^{-2} m and 3.25×10^{-2} m of the main scale but now the 45th division of vernier scale coincides with one of the main scale divisions. The length of the thin metallic wire is 2 m and its cross-sectional area is 8×10^{-7} m^2. The least count of the vernier scale is 1.0×10^{-5} m. The maximum percentage error in the Young's modulus of the wire is **(Single Integer Type, 2014)**

Passage (Q. Nos. 38-39)

A spray gun is shown in the figure where a piston pushes air out of nozzle. A thin tube of uniform cross-section is connected to the nozzle. The other end of the tube is in a small liquid container. As the piston pushes air through the nozzle, the liquid from the container rises into the nozzle and is sprayed out. For the spray gun shown, the radii of the piston and the nozzle are 20 mm and 1 mm respectively. The upper end of the container is open to the atmosphere. **(Passage Type, 2014)**

38. If the piston is pushed at a speed of $5\,\text{ms}^{-1}$, the air comes out of the nozzle with a speed of

(a) $0.1\,\text{ms}^{-1}$ (b) $1\,\text{ms}^{-1}$ (c) $2\,\text{ms}^{-1}$ (d) $8\,\text{ms}^{-1}$

39. If the density of air is ρ_a and that of the liquid ρ_l, then for a given piston speed the rate (volume per unit time) at which the liquid is sprayed will be proportional to

(a) $\sqrt{\frac{\rho_a}{\rho_l}}$ (b) $\sqrt{\rho_a\rho_l}$ (c) $\sqrt{\frac{\rho_l}{\rho_a}}$ (d) ρ_l

40. A particle of mass m is projected from the ground with an initial speed u_0 at an angle α with the horizontal. At the highest point of its trajectory, it makes a completely inelastic collision with another identical particle, which was thrown vertically upward from the ground with the same initial speed u_0. The angle that the composite system makes with the horizontal immediately after the collision is **(Single Correct Option, 2013)**

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{4} + \alpha$ (c) $\frac{\pi}{4} - \alpha$ (d) $\frac{\pi}{2}$

41. A bob of mass m, suspended by a string of length l_1, is given a minimum velocity required to complete a full circle in the vertical plane, At the highest point, it collides elastically with another bob of mass m suspended by a string of length l_2, which is initially at rest. Both the strings are massless and inextensible. If the second bob, after collision acquires the minimum speed required to complete a full circle in the vertical plane, the ratio l_1/l_2 is **(Single Integer Type, 2013)**

42. A uniform circular disc of mass 50 kg and radius 0.4 m is rotating with an angular velocity of 10 rad/s about its own axis, which is vertical. Two uniform circular rings, each of mass 6.25 kg and radius 0.2 m, are gently placed symmetrically on the disc in such a manner that they are touching each other along the axis of the disc and are horizontal. Assume that the friction is large enough such that the rings are at rest relative to the disc and the system rotates about the original axis. The new angular velocity (in rad s^{-1}) of the system is **(Single Integer Type, 2013)**

43. Two bodies, each of mass M, are kept fixed with a separation $2L$. A particle of mass m is projected from the mid-point of the line joining their centres, perpendicular to the line. The gravitational constant is G. The correct statement(s) is (are) **(Single Correct Option, 2013)**

(a) The minimum initial velocity of the mass m to escape the gravitational field of the two bodies is $4\sqrt{\frac{GM}{L}}$

(b) The minimum initial velocity of the mass m to escape the gravitational field of the two bodies is $2\sqrt{\frac{GM}{L}}$

(c) The minimum initial velocity of the mass m to escape the gravitational field of the two bodies is $\sqrt{\frac{2GM}{L}}$

(d) The energy of the mass m remains constant

44. One end of a horizontal thick copper wire of length $2L$ and radius $2R$ is welded to an end of another horizontal thin copper wire of length L and radius R. When the arrangement is stretched by applying forces at two ends, the ratio of the elongation in the thin wire to that in the thick wire is **(Single Correct Option, 2013)**

(a) 0.25 (b) 0.50 (c) 2.00 (d) 4.00

45. A solid sphere of radius R and density ρ is attached to one end of a massless spring of force constant k. The other end of the spring is connected to another solid sphere of radius R and density 3ρ. The complete arrangement is placed in a liquid of density 2ρ and is allowed to reach equilibrium. The correct statement(s) is (are) **(Single Correct Option, 2013)**

(a) the net elongation of the spring is $\frac{4\pi R^3 \rho g}{3k}$

(b) the net elongation of the spring is $\frac{8\pi R^3 \rho g}{3k}$

(c) the light sphere is partially submerged

(d) the light sphere is completely submerged displayed by the sensor when the temperature of the metal surface is raised to 2767°C?

46. A particle of mass m is attached to one end of a mass less spring of force constant k, lying on a frictionless horizontal plane. The other end of the spring is fixed. The particle starts moving horizontally from its equilibrium position at time $t = 0$ with an initial velocity u_0. When the speed of the particle is $0.5\,u_0$, it collides elastically with a rigid wall. After this collision **(More than One Correct Option, 2013)**

(a) the speed of the particle when it returns to its equilibrium position is u_0

(b) the time at which the particle passes through the equilibrium position for the first time is $t = \pi\sqrt{\frac{m}{k}}$

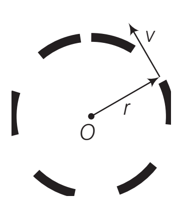
(c) the time at which the maximum compression of the spring occurs is $t = \frac{4\pi}{3}\sqrt{\frac{m}{k}}$

(d) the time at which the particle passes through the equilibrium position for the second time is $t = \frac{5\pi}{3}\sqrt{\frac{m}{k}}$

Answer *with* Explanations

1. (b, c) $V = \frac{kr^2}{2}$

$$F = -\frac{dV}{dr} = -kr \text{ (towards centre)} \quad \left[F = -\frac{dV}{dr}\right]$$



At $r = R$,

$$kR = \frac{mv^2}{R} \text{ (Centripetal force)}$$

$$v = \sqrt{\frac{kR^2}{m}} = \sqrt{\frac{k}{m}}R \Rightarrow L = mvR = \sqrt{\frac{k}{m}}R^2$$

2. (a, c) $\mathbf{F} = (\alpha t)\hat{\mathbf{i}} + \beta\hat{\mathbf{j}}$ [at $t = 0, v = 0, \mathbf{r} = 0$]

$$\alpha = 1, \beta = 1$$

$$\mathbf{F} = t\hat{\mathbf{i}} + \hat{\mathbf{j}}$$

$$m\frac{d\mathbf{v}}{dt} = t\hat{\mathbf{i}} + \hat{\mathbf{j}}$$

On integrating, $m\mathbf{v} = \frac{t^2}{2}\hat{\mathbf{i}} + t\hat{\mathbf{j}}$ [$m = 1$ kg]

$$\frac{d\mathbf{r}}{dt} = \frac{t^2}{2}\hat{\mathbf{i}} + t\hat{\mathbf{j}} \quad [\mathbf{r} = 0 \text{ at } t = 0]$$

Again, on integrating,

$$\mathbf{r} = \frac{t^3}{6}\hat{\mathbf{i}} + \frac{t^2}{2}\hat{\mathbf{j}}$$

At $t = 1$ s, $\tau = (\mathbf{r} \times \mathbf{F}) = \left(\frac{1}{6}\hat{\mathbf{i}} + \frac{1}{2}\hat{\mathbf{j}}\right) \times (\hat{\mathbf{i}} + \hat{\mathbf{j}})$

$$\tau = \frac{1}{3}\hat{\mathbf{k}}$$

$$v = \frac{t^3}{2}\hat{\mathbf{i}} + t\hat{\mathbf{j}}$$

At $t = 1$ s, $\mathbf{v} = \left(\frac{1}{2}\hat{\mathbf{i}} + \hat{\mathbf{j}}\right) = \frac{1}{2}(\hat{\mathbf{i}} + 2\hat{\mathbf{j}})\,\mathrm{m/s}$

At $t = 1$ s, $\mathbf{r}_1 - \mathbf{r}_0 = \left[\frac{1}{6}\hat{\mathbf{i}} + \frac{1}{2}\hat{\mathbf{j}}\right] - [0]$

$$\mathbf{s} = \frac{1}{6}\hat{\mathbf{i}} + \frac{1}{2}\hat{\mathbf{j}}$$

$$|\mathbf{s}| = \sqrt{\left(\frac{1}{6}\right)^2 + \left(\frac{1}{2}\right)^2} \Rightarrow \frac{\sqrt{10}}{6}\,\mathrm{m}$$

3. **(a, c)** $h = \dfrac{2\sigma\cos\theta}{r\rho g}$

(a) $\rightarrow h \propto \dfrac{1}{r}$

(b) h depends upon σ.

(c) If lift is going up with constant acceleration.

$$g_{\text{eff}} = (g + a) \quad \Rightarrow h = \frac{2\sigma\cos\theta}{r\rho(g + a)}$$

It means h decreases.

(d) h is proportional to $\cos\theta$.

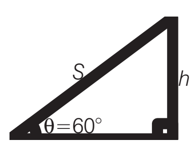
4. **(0.75 m)** $a = \dfrac{g\sin\theta}{1 + \dfrac{I}{MR^2}}$

$$a_{\text{ring}} = \frac{g\sin\theta}{2} \qquad (I = MR^2)$$

$$a_{\text{disc}} = \frac{2g\sin\theta}{3} \qquad \left(I = \frac{MR^2}{2}\right)$$

$$s = \frac{h}{\sin\theta} = \frac{1}{2}at^2$$

$$= \frac{1}{2}\left(\frac{g\sin\theta}{2}\right)t_1^2$$



$$\Rightarrow \quad t_1 = \sqrt{\frac{4h}{g\sin^2\theta}} = \sqrt{\frac{16h}{3g}}$$

$$s = \frac{h}{\sin\theta} = \frac{1}{2}at^2 = \frac{1}{2}\left(\frac{2g\sin\theta}{3}\right)t_2^2$$

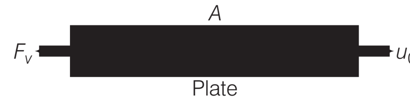
$$\Rightarrow \quad t_2 = \sqrt{\frac{3h}{g\sin^2\theta}} = \sqrt{\frac{4h}{g}}$$

$$t_2 - t_1 = \sqrt{\frac{16h}{3g}} - \sqrt{\frac{4h}{g}} = \frac{2 - \sqrt{3}}{\sqrt{10}}$$

$$\sqrt{h}\left[\frac{4}{\sqrt{3}} - 2\right] = 2 - \sqrt{3}$$

Soving this equation we get, $h = 0.75$ m.

5. **(a, c, d)**



$$F_v = -\eta A\left(\frac{dv}{dy}\right)$$

Since, height h of the liquid in tank is very small.

$$\Rightarrow \quad \frac{dv}{dy} = \frac{\Delta v}{\Delta y} = \left(\frac{u_0}{h}\right) \quad \Rightarrow \quad F_v = -(\eta)A\left(\frac{u_0}{h}\right)$$

$$F_v \propto \left(\frac{1}{h}\right), F_v \propto u_0, F \propto A, F_v \propto \eta$$

6. **(3)** Given, $d = 0.5$ mm,

$Y = 2 \times 10^{11}$ Nm^{-2} ,

$l = 1$ m

$$\Delta l = \frac{Fl}{AY} = \frac{mgl}{\frac{\pi d^2}{4}Y}$$

$$= \frac{1.2 \times 10 \times 1}{\frac{\pi}{4} \times (5 \times 10^{-4})^2 \times 2 \times 10^{11}}$$

$= 0.3$ mm

LC of vernier $= \left(1 - \dfrac{9}{10}\right)$ mm $= 0.1$ mm

So, 3rd division of vernier scale will coincide with main scale.

7. **(b)** $v = \sqrt{\dfrac{GM}{R}}$

Let $R_1 = R$, then $R_2 = 4R$

If $m_2 = m$, then $m_1 = 2m$

List-I

(P) $\dfrac{v_1}{v_2} = \sqrt{\dfrac{R_2}{R_1}} = \sqrt{\dfrac{4R}{R}} = 2:1$

(Q) $L = mvR$

$$\frac{L_1}{L_2} = \frac{R(2m)v_1}{4R(m)v_2} = \frac{1}{2}(2) = 1:1$$

(R) $\dfrac{K_1}{K_2} = \dfrac{\frac{1}{2}(2m)v_1^2}{\frac{1}{2}(m)v_2^2} = 2\,(4) = 8:1$

(S) $\dfrac{T_1}{T_2} = \left(\dfrac{R_1}{R_2}\right)^{3/2} = \left(\dfrac{1}{4}\right)^{3/2} = 1:8$

8. **(a)** When force $F = 0 \Rightarrow$ potential energy U = constant

$F \neq 0 \Rightarrow$ force is conservative $\Rightarrow$ Total energy E = constant

List-I

(P) $\mathbf{r}(t) = \alpha t\hat{\mathbf{i}} + \beta t\hat{\mathbf{j}}$

$$\frac{d\mathbf{r}}{dt} = \mathbf{v} = \alpha\hat{\mathbf{i}} + \beta\hat{\mathbf{j}} = \text{constant} \Rightarrow \mathbf{p} = \text{constant}$$

$$|\mathbf{v}| = \sqrt{\alpha^2 + \beta^2} = \text{constant}$$

$\Rightarrow K = \text{constant}$

$$\frac{d\mathbf{v}}{dt} = \mathbf{a} = 0 \Rightarrow F = 0 \Rightarrow U = \text{constant}$$

$E = U + K = \text{constant}$

$\mathbf{L} = m(\mathbf{r} \times \mathbf{v}) = 0$

$\mathbf{L} = \text{constant}$

$P \to 1, 2, 3, 4, 5$

(Q) $\mathbf{r}(t) = \alpha\cos\omega t\hat{\mathbf{i}} + \beta\sin\omega t\hat{\mathbf{j}}$

$$\frac{d\mathbf{r}}{dt} = \mathbf{v} = \alpha\omega\sin\omega t(-\hat{\mathbf{i}}) + \beta\omega\cos\omega t\hat{\mathbf{j}} \neq \text{constant}$$

$\Rightarrow \mathbf{p} \neq \text{constant}$

$$|\mathbf{v}| = \omega\sqrt{(\alpha\sin\omega t)^2 + (\beta\cos\omega t)^2} \neq \text{constant}$$

$\Rightarrow K \neq \text{constant}$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = -\omega^2\mathbf{r} \neq 0$$

$\Rightarrow E = \text{constant} = K + U$

But $K \neq \text{constant} \Rightarrow U \neq \text{constant}$

$\mathbf{L} = m(\mathbf{r} \times \mathbf{v}) = m\omega\alpha\beta\,(\hat{\mathbf{k}}) = \text{constant}$

$Q \to 2, 5$

(R) $\mathbf{r}(t) = \alpha\,(\cos\omega t\hat{\mathbf{i}} + \sin\omega t\hat{\mathbf{j}})$

$$\frac{d\mathbf{r}}{dt} = \mathbf{v} = \alpha\omega\,[\sin\omega t(-\hat{\mathbf{i}}) + \cos\omega t\hat{\mathbf{j}}] \neq \text{constant}$$

$\Rightarrow \mathbf{p} \neq \text{constant}$

$|\mathbf{v}| = \alpha\omega = \text{constant} \Rightarrow K = \text{constant}$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = -\omega^2\mathbf{r} \neq 0 \Rightarrow E = \text{constant}, U = \text{constant}$$

$\mathbf{L} = m(\mathbf{r} \times \mathbf{v}) = m\omega\alpha^2\hat{\mathbf{k}} = \text{constant}$

$R \to 2, 3, 4, 5$

(S) $\mathbf{r}(t) = \alpha t\hat{\mathbf{i}} + \frac{\beta}{2}t^2\hat{\mathbf{j}}$

$$\frac{d\mathbf{r}}{dt} = \mathbf{v} = \alpha\hat{\mathbf{i}} + \beta t\hat{\mathbf{j}} \neq \text{constant} \Rightarrow \mathbf{p} \neq \text{constant}$$

$$|\mathbf{v}| = \sqrt{\alpha^2 + (\beta t)^2} \neq \text{constant} \Rightarrow K \neq \text{constant}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \beta\hat{\mathbf{j}} \neq 0 \Rightarrow E = \text{constant} = K + U$$

But $K \neq \text{constant}$

$\therefore U \neq \text{constant}$

$$\mathbf{L} = m(\mathbf{r} \times \mathbf{v}) = \frac{1}{2}\alpha\beta t^2\hat{\mathbf{k}} \neq \text{constant}$$

$S \to 5$

9. Question is not very clear.

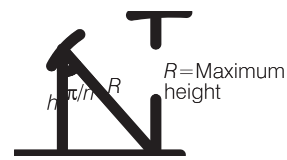
10. **(c)** If height of the cone $h >> r$

Then, $\mu N = mg$

$$\mu m(R - r)\omega_0^2 = mg$$

$$\omega_0 = \sqrt{\frac{g}{\mu(R - r)}}$$

11. **(d)**

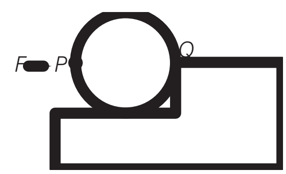


$$\cos\left(\frac{\pi}{n}\right) = \frac{h}{R}$$

$$\Delta = R - h = \frac{h}{\cos(\pi/n)} - h$$

$$= h\left[\frac{1}{\cos(\pi/n)} - 1\right]$$

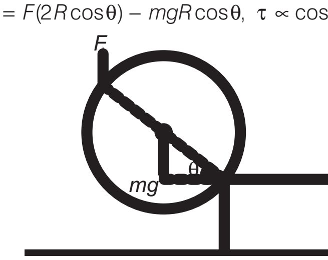
12. **(a, c)**



(a) If force is applied normal to surface at P, then line of action of force will pass from Q and thus, $\tau = 0$.

(b) Wheel can climb.

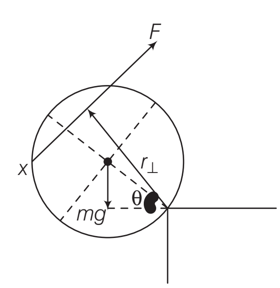
(c) $\tau = F(2R\cos\theta) - mgR\cos\theta$, $\tau \propto \cos\theta$



Hence, as θ increases

τ decreases. So its correct.

(d)



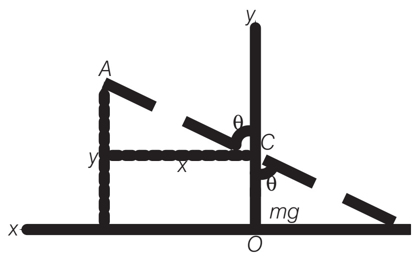
$\tau = Fr_\perp - mg\cos\theta$; τ increases with θ.

13. **(a, c, d)** When the bar makes an angle θ, the height of its COM (mid-point) is $\frac{L}{2}\cos\theta$.

$\therefore$ Displacement $= L - \frac{L}{2}\cos\theta = \frac{L}{2}(1-\cos\theta)$

Since, force on COM is only along the vertical direction, hence COM is falling vertically downward. Instantaneous torque about point of contact is

$$\tau = mg \times \frac{L}{2}\sin\theta \quad \text{or} \quad \tau \propto \sin\theta$$



Now, $x = \frac{L}{2}\sin\theta$

$y = L\cos\theta$

$$\frac{x^2}{(L/2)^2} + \frac{y^2}{L^2} = 1$$

Path of A is an ellipse.

14. **(a, b)** Δx_{cm} of the block and point mass system $= 0$

$\therefore$ $m(x+R) + Mx = 0$

where, x is displacement of the block.

Solving this equation, we get

$$x = -\frac{mR}{M+m}$$

From conservation of momentum and mechanical energy of the combined system

$$0 = mv - MV$$

$$mgR = \frac{1}{2}mv^2 + \frac{1}{2}MV^2$$

Solving these two equations, we get

$\therefore$ $$v = \sqrt{\frac{2gR}{1+\frac{m}{M}}}$$

15. **(c)** Given, $v_e = 11.2 \text{ km/s} = \sqrt{\frac{2GM_e}{R_e}}$

From energy conservation,

$$K_i + U_i = K_f + U_f$$

$$\frac{1}{2}mv_s^2 - \frac{GM_s m}{r} - \frac{GM_e m}{R_e} = 0 + 0$$

Here, r = distane of rocket from sun

$\Rightarrow$ $$v_s = \sqrt{\frac{2GM_e}{R_e} + \frac{2GM_s}{r}}$$

Given, $M_s = 3\times 10^5 M_e$

and $r = 2.5 \times 10^4 R_e$

$$\Rightarrow v_s = \sqrt{\frac{2GM_e}{R_e} + (2G)\left(\frac{3\times 10^5 M_e}{2.5\times 10^4 R_e}\right)}$$

$$= \sqrt{\frac{2GM_e}{R_e}\left(1 + \frac{3\times 10^5}{2.5\times 10^4}\right)}$$

$$= \sqrt{\frac{2GM_e}{R_e}\times 13}$$

$$\Rightarrow v_s \simeq 42 \text{ km/s}$$

16. **(a)** $m = \frac{4\pi R^3}{3}\times\rho$

On taking log both sides, we have

$$\ln(m) = \text{in}\left(\frac{4\pi}{3}\right) + \ln(\rho) + 3\ln(R)$$

On differentiating with respect to time,

$$0 = 0 + \frac{1}{\rho}\frac{d\rho}{dt} + \frac{3}{R}\frac{dR}{dt}$$

$$v = \frac{dR}{dt} = KR \quad \Rightarrow \quad v \propto R$$

17. **(6)** From mass conservation,

$$\rho\cdot\frac{4}{3}\pi R^3 = \rho\cdot K\cdot\frac{4}{3}\pi r^3 \quad \Rightarrow \quad R = K^{1/3}r$$

$\therefore$ $$\Delta U = T\Delta A = T(K\cdot 4\pi r^2 - 4\pi R^2)$$

$$= T(K\cdot 4\pi R^2 K^{-2/3} - 4\pi R^2)$$

$$\Delta U = 4\pi R^2 T[K^{1/3} - 1]$$

Putting the values, we get

$$10^{-3} = \frac{10^{-1}}{4\pi}\times 4\pi\times 10^{-4}[K^{1/3}-1]$$

$$100 = K^{1/3} - 1 \quad \Rightarrow \quad K^{1/3} \cong 100 = 10^2$$

Given that $K = 10^\alpha$

$\therefore$ $$10^{\alpha/3} = 10^2 \Rightarrow \frac{\alpha}{3} = 2 \Rightarrow \alpha = 6$$

18. **(c)** Force on block along slot

$$= m\omega^2 r = ma = m\left(\frac{vdv}{dr}\right)$$

$$\int_0^v vdv = \int_{R/2}^r \omega^2 rdr$$

$$\Rightarrow \frac{v^2}{2} = \frac{\omega^2}{2}\left(r^2 - \frac{R^2}{4}\right)$$

$$\Rightarrow v = \omega\sqrt{r^2 - \frac{R^2}{4}} = \frac{dr}{dt}$$

$$\Rightarrow \int_{R/4}^r \frac{dr}{\sqrt{r^2 - \frac{R^2}{4}}} = \int_0^t \omega\, dt$$

$$\ln\left(\frac{r+\sqrt{r^2-\frac{R^2}{4}}}{\frac{R}{2}}\right)-\ln\left(\frac{R/2+\sqrt{\frac{R^2}{4}-\frac{R^2}{4}}}{\frac{R}{2}}\right)=\omega t$$

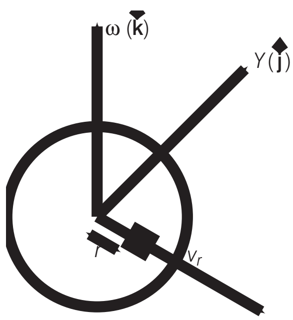
$$\Rightarrow \quad r+\sqrt{r^2-\frac{R^2}{4}}=\frac{R}{2}e^{\omega t}$$

$$\Rightarrow \quad r^2-\frac{R^2}{4}=\frac{R^2}{4}e^{2\omega t}+r^2-2r\frac{R}{2}e^{\omega t}$$

$$\Rightarrow \quad r=\frac{\frac{R^2}{4}e^{2\omega t}+\frac{R^2}{4}}{Re^{\omega t}}$$

$$=\frac{R}{4}(e^{\omega t}+e^{-\omega t})$$

19. **(b)**



$$\mathbf{F}_{\text{rot}}=\mathbf{F}_{\text{in}}+2m(v_{\text{rot}}\,\hat{\mathbf{i}})\times\omega\hat{\mathbf{k}}+m(\omega\hat{\mathbf{k}}\times r\,\hat{\mathbf{i}})\times\omega\hat{\mathbf{k}}$$

$$mr\omega^2\hat{\mathbf{i}}=\mathbf{F}_{\text{in}}+2mv_{\text{rot}}\omega(-\hat{\mathbf{j}})+m\omega^2 r\,\hat{\mathbf{i}}$$

$$\mathbf{F}_{\text{in}}=2mv_r\omega\hat{\mathbf{j}}$$

$$r=\frac{R}{4}[e^{\omega t}+e^{-\omega t}]$$

$$\frac{dr}{dt}=v_r=\frac{R}{4}[\omega e^{\omega t}-\omega e^{-\omega t}]$$

$$\mathbf{F}_{\text{in}}=2m\frac{R\omega}{\cancel{4}}[e^{\omega t}-e^{-\omega t}]\omega\hat{\mathbf{j}}$$

$$\mathbf{F}_{\text{in}}=\frac{mR\omega^2}{2}[e^{\omega t}-e^{-\omega t}]\hat{\mathbf{j}}$$

Also, reaction is due to disc surface then

$$\mathbf{F}_{\text{reaction}}=\frac{mR\omega^2}{2}[e^{\omega t}-e^{-\omega t}]\hat{\mathbf{j}}+mg\hat{\mathbf{k}}$$

20. **(a,b,d)** $\mathbf{r}=\alpha t^3\,\hat{\mathbf{i}}+\beta t^2\,\hat{\mathbf{j}}$

$$\mathbf{v}=\frac{d\mathbf{r}}{dt}=3\alpha t^2\,\hat{\mathbf{i}}+2\beta t\,\hat{\mathbf{j}}$$

$$\mathbf{a}=\frac{d^2\mathbf{r}}{dt^2}=6\alpha t\,\hat{\mathbf{i}}+2\beta\,\hat{\mathbf{j}}$$

At $t=1$ s,

(a) $\mathbf{v}=3\times\frac{10}{3}\times 1\hat{\mathbf{i}}+2\times 5\times 1\hat{\mathbf{j}}$

$=(10\hat{\mathbf{i}}+10\hat{\mathbf{j}})$m / s

(b) $\mathbf{L}=\mathbf{r}\times\mathbf{p}=\left(\frac{10}{3}\times 1\hat{\mathbf{i}}+5\times 1\hat{\mathbf{j}}\right)\times 0.1(10\hat{\mathbf{i}}+10\hat{\mathbf{j}})$

$=\left(-\frac{5}{3}\hat{\mathbf{k}}\right)$N - ms

(c) $\mathbf{F}=m\mathbf{a}=m\times\left(6\times\frac{10}{3}\times 1\hat{\mathbf{i}}+2\times 5\hat{\mathbf{j}}\right)$

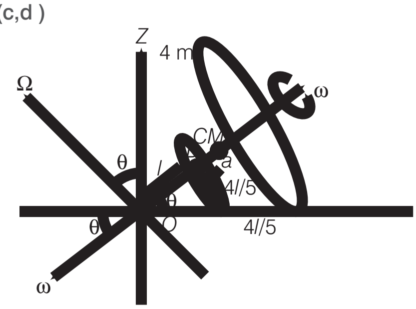
$=(2\hat{\mathbf{i}}+\hat{\mathbf{j}})$ N

(d) $\tau=r\times\mathbf{F}=\left(\frac{10}{3}\hat{\mathbf{i}}+5\hat{\mathbf{j}}\right)\times(2\hat{\mathbf{i}}+\hat{\mathbf{j}})$

$=+\frac{10}{3}\hat{\mathbf{k}}+10(-\hat{\mathbf{k}})$

$=\left(-\frac{20}{3}\hat{\mathbf{k}}\right)$ N - m

21. **(c,d)**



$$\cos\theta=\frac{l}{\sqrt{l^2+a^2}}=\frac{\sqrt{24}}{5}$$

(a) $L_z=L_{\text{CM}-O}\cos\theta-L_{D-\text{CM}}\sin\theta$

$$=\frac{81\sqrt{24}}{5}a^2m\omega\times\frac{\sqrt{24}}{5}-\frac{17\;ma^2\omega}{2}\times\frac{1}{\sqrt{24}}$$

$$=\frac{81\times 24ma^2\omega}{25}-\frac{17ma^2\omega}{2\sqrt{24}}$$

(b) $L_{\text{CM}-O}=(5m)\left[\frac{9l}{5}\,\Omega\right]\frac{9l}{5}=\frac{81ml^2\omega}{5}=\frac{81ml^2}{5}\times\frac{a\omega}{l}$

$$L_{\text{CM}-O}=\frac{81mla\omega}{5}=\frac{81\sqrt{24}\,a^2m\omega}{5}$$

(c) Velocity of point P : $a\omega=1\,\Omega$ then

$\Omega=\frac{a\omega}{1}$ = Angular velocity of C.M. w.r.t point O.

Angular velocity fo CM w.r.t Z-axis

$=\Omega\cos\theta$

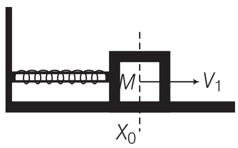
$$\omega_{\text{CM}-z}=\frac{a\omega}{1}\,\frac{\sqrt{24}}{5}$$

$$\omega_{\text{CM}-z}=\frac{a\omega}{5}=\frac{a\omega}{\sqrt{24}\,a}\,\frac{\sqrt{24}}{5}$$

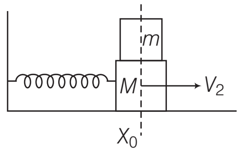
(d) $L_{D-\text{CM}}=\frac{ma^2}{2}\omega+\frac{4m(2a)^2}{2}\omega=\frac{17\,ma^2\omega}{2}$

22. **(a,b,d)** **Case 1**

Just before m is placed



Just after m is placed



In case 1, $Mv_1 = (M + m)v_2$

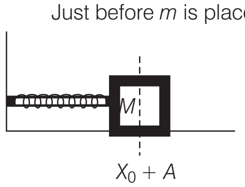
$$v_2 = \left(\frac{M}{M+m}\right)v_1$$

$$\sqrt{\frac{k}{M+m}}\,A_2 = \left(\frac{M}{M+m}\right)\sqrt{\frac{k}{M}}\,A_1$$

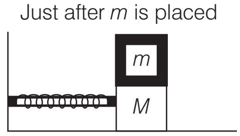
$$A_2 = \sqrt{\frac{k}{M+m}}\,A_1$$

Case 2

Just before m is placed



Just after m is placed



In case 2, $A_2 = A_1$

$$T = 2\pi\sqrt{\frac{M+m}{k}} \text{ in both cases.}$$

Total energy decreases in first case whereas remain same in 2nd case. Instantaneous speed at x_0 decreases in both cases.

23. **(d)** Let the other mass at this instant is at a distance of x from the centre O.

Applying law of conservation of angular momentum, we have $I_1\omega_1 = I_2\omega_2$

$$\therefore (MR^2)(\omega) = \left[MR^2 + \frac{M}{8}\left(\frac{3}{5}R\right)^2 + \frac{M}{8}x^2\right]\left(\frac{8}{9}\omega\right)$$

Solving this equation, we get $x = \frac{4}{5}R$.

24. **(7)** In case of pure rolling, mechanical energy remains constant (as work-done by friction is zero). Further in case of a disc,

$$\frac{\text{translational kinetic energy}}{\text{rotational kinetic energy}} = \frac{K_T}{K_R} = \frac{\frac{1}{2}mv^2}{\frac{1}{2}I\omega^2}$$

$$= \frac{mv^2}{\left(\frac{1}{2}mR^2\right)\left(\frac{v}{R}\right)^2} = \frac{2}{1}$$

or, $K_T = \frac{2}{3}$ (Total kinetic energy)

or, Total kinetic energy

$$K = \frac{3}{2}K_T = \frac{3}{2}\left(\frac{1}{2}mv^2\right) = \frac{3}{4}mv^2$$

Decrease in potential energy = increase in kinetic energy

or, $mgh = \frac{3}{4}m(v_f^2 - v_i^2)$ or $v_f = \sqrt{\frac{4}{3}gh + v_i^2}$

As final velocity in both cases is same.

So, value of $\sqrt{\frac{4}{3}gh + v_i^2}$ should be same in both cases.

$$\therefore \sqrt{\frac{4}{3} \times 10 \times 30 + (3)^2} = \sqrt{\frac{4}{3} \times 10 \times 27 + (v_2)^2}$$

Solving this equation we get, $v_2 = 7$ m/s

25. **(6)** Consider a shell of radius r and thickness dr

$$dI = (dm)r^2 \Rightarrow dI = \frac{2}{3}(\rho 4\pi r^2 dr)r^2 \Rightarrow I = \int dI$$

$$\frac{I_B}{I_A} = \frac{\int_0^R \frac{2}{3}k\frac{r^5}{R^5} \cdot 4\pi r^2 dr\, r^2}{\int_0^R \frac{2}{3}k\frac{r}{R}\, 4\pi r^2 dr\, r^2} = \frac{6}{10}$$

So, $n = 6$

26. **(2)** At height h

$$g' = \frac{g}{\left(1 + \frac{h}{R}\right)^2} \qquad \ldots\text{(i)}$$

Given, $g' = \frac{g}{4}$

Substituting in Eq. (i) we get, $h = R$

Now, from A to B,

decrease in kinetic energy = increase in potential energy

$$\Rightarrow \quad \frac{1}{2}mv^2 = \frac{mgh}{1+\frac{h}{R}}$$

$$\Rightarrow \quad \frac{v^2}{2} = \frac{gh}{1+\frac{h}{R}} = \frac{1}{2}gR \qquad (h = R)$$

$$\Rightarrow \quad v^2 = gR$$

or $\quad v = \sqrt{gR}$

Now, $\quad v_{\text{esc}} = \sqrt{2gR} = v\sqrt{2}$

$$\Rightarrow \quad N = 2$$

27. **(7)** For point mass at distance $r = 3l$

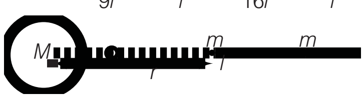
$$\frac{GMm}{(3l)^2} - \frac{Gm^2}{l^2} = ma \qquad ...(i)$$

For point mass at distance $r = 4l$

$$\frac{GMm}{(4l)^2} + \frac{Gm^2}{l^2} = ma \qquad ...(ii)$$

Equating the two equations we have,

$$\frac{GMm}{9l^2} - \frac{Gm^2}{l^2} = \frac{GMm}{16l^2} + \frac{Gm^2}{l^2}$$



$$\frac{7GMm}{144} = \frac{2Gm^2}{l^2}$$

$$m = \frac{7M}{288}$$

$$\therefore \quad k = 7$$

28. **(c)** Gravitational field at a distance r due to mass 'm'

$$E = \frac{G\rho\frac{4}{3}\pi r^3}{r^2} = \frac{4G\rho\pi r}{3}$$

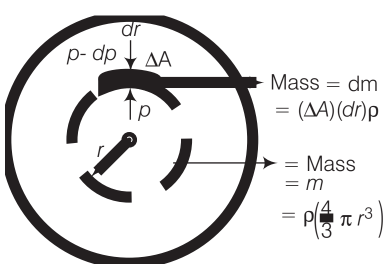
Consider a small element of width dr and area ΔA at a distance r.

Pressure force on this element outwards

= gravitational force on 'dm' from 'm' inwards

$$\Rightarrow \quad (dp)\Delta A = E(dm)$$

$$\Rightarrow \quad -dp \cdot \Delta A = \left(\frac{4}{3}G\pi\rho r\right)(\Delta A\, dr \cdot \rho)$$



$$-\int_O^P dp = \int_R^r \left(\frac{4G\rho^2\pi}{3}\right) r\,dr$$

$$-p = \frac{4G\rho^2\pi}{3\times 2}[r^2 - R^2]$$

$$\Rightarrow \quad p = c(R^2 - r^2)$$

$$r = \frac{3R}{4}, \quad p_1 = c\left(R^2 - \frac{9R^2}{16}\right)$$

$$= c\left(\frac{7R^2}{16}\right)$$

$$r = \frac{2R}{3}, \quad p_2 = c\left(R^2 - \frac{4R^2}{9}\right) = c\left(\frac{5R^2}{9}\right)$$

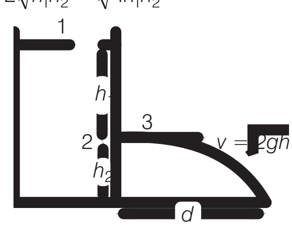
$$\frac{p_1}{p_2} = \frac{63}{80}\, r = \frac{3R}{5}, \; p_3 = c\left(R^2 - \frac{9}{25}R^2\right)$$

$$= c\left(\frac{16R^2}{25}\right)$$

$$r = \frac{2R}{5}, \; p_4 = c\left(R^2 - \frac{4R^2}{25}\right) = c\left(\frac{21R^2}{25}\right)$$

$$\Rightarrow \quad \frac{p_3}{p_4} = \frac{16}{21}$$

29. **(c)** $d = 2\sqrt{h_1h_2} = \sqrt{4h_1h_2}$



This is independent of the value of g.

(P) $g_{\text{eff}} > g$

$$d = \sqrt{4h_1h_2} = 1.2 \text{ m}$$

(Q) $g_{\text{eff}} < g$

$$d = \sqrt{4h_1h_2} = 1.2 \text{ m}$$

(R) $g_{\text{eff}} = g$

$$d = \sqrt{4h_1h_2} = 1.2 \text{ m}$$

(S) $g_{\text{eff}} = 0$

No water leaks out of jar. As there will be no pressure difference between top of the container and any other point. $P_1 = P_2 = P_3 = P_0$

30. **(a)** For floating, net weight of system = net upthrust

$$\Rightarrow \quad (\rho_1 + \rho_2)Vg = (\sigma_1 + \sigma_2)Vg$$

Since string is taut, $\rho_1 < \sigma_1$ and $\rho_2 > \sigma_2$

$$v_P = \frac{2r^2g}{2\eta_2}(\sigma_2 - \rho_1) \text{ (upward terminal velocity)}$$

$$v_Q = \frac{2r^2 g}{9\eta_1}(\rho_2 - \sigma_1)$$

(downward terminal velocity)

$$\left|\frac{v_P}{v_Q}\right| = \frac{\eta_1}{\eta_2}$$

Further, $\bar{v}_P \cdot \bar{v}_Q$ will be negative as they are opposite to each other.

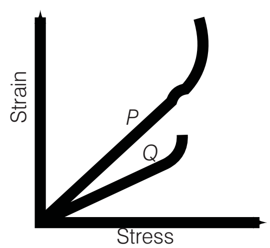
31. **(a, b)** $Y = \frac{\text{stress}}{\text{strain}}$ or $Y \propto \frac{1}{\text{strain}}$

(for same stress say σ)

$$(\text{strain})_Q < (\text{strain})_P$$

$$\Rightarrow \quad Y_Q > Y_P$$

So, P is more ductile than Q. Further, from the given figure we can also see that breaking stress of P is more than Q. So, it has more tensile strength.



32. **(b,d)** **Ist Particle**

$$P = 0 \text{ at } x = a$$

$\Rightarrow$ 'a' is the amplitude of oscillation 'A_1'.

At $x = 0$, $P = b$ (at mean position)

$$\Rightarrow mv_{\max} = b \Rightarrow v_{\max} = \frac{b}{m}$$

$$E_1 = \frac{1}{2}mv_{\max}^2 = \frac{m}{2}\left[\frac{b}{m}\right]^2 = \frac{b^2}{2m}$$

$$A_1\omega_1 = v_{\max} = \frac{b}{m}$$

$$\Rightarrow \quad \omega_1 = \frac{b}{ma} = \frac{1}{mn^2} \qquad \left(A_1 = a, \frac{a}{b} = n^2\right)$$

IInd Particle

$$P = 0 \text{ at } x = R \Rightarrow A_2 = R$$

At $\quad x = 0, P = R$

$$\Rightarrow \quad v_{\max} = \frac{R}{m}$$

$$E_2 = \frac{1}{2}mv_{\max}^2 = \frac{m}{2}\left[\frac{R}{m}\right]^2 = \frac{R^2}{2m}$$

$$A_2\omega_2 = \frac{R}{m}$$

$$\Rightarrow \quad \omega_2 = \frac{R}{mR} = \frac{1}{m}$$

(b) $\frac{\omega_2}{\omega_1} = \frac{1/m}{1/mn^2} = n^2$

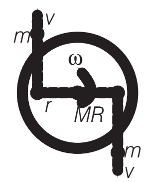
(c) $\omega_1\omega_2 = \frac{1}{mn^2} \times \frac{1}{m} = \frac{1}{m^2n^2}$

(d) $\frac{E_1}{\omega_1} = \frac{b^2/2m}{1/mn^2} = \frac{b^2n^2}{2} = \frac{a^2}{2n^2} = \frac{R^2}{2}$

$$\frac{E_2}{\omega_2} = \frac{R^2/2m}{1/m} = \frac{R^2}{2} \Rightarrow \frac{E_1}{\omega_1} = \frac{E_2}{\omega_2}$$

33. **(4)** Applying conservation of angular momentum

$$2mvr - \frac{MR^2}{2}\omega = 0, \omega = \frac{4mvr}{MR^2}$$



Substituting the values, we get

$$\omega = \frac{(4)(5 \times 10^{-2})(9)\left(\frac{1}{4}\right)}{45 \times 10^{-2} \times \frac{1}{4}}$$

$$\omega = 4 \text{ rad/s}$$

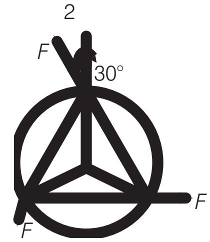
34. **(2)** Angular impulse = change in angular momentum

$$\therefore \int \tau \, dt = I\omega$$

$$\Rightarrow \quad \omega = \frac{\int \tau \, dt}{I} = \frac{\int_0^t 3F \sin 30° \, R \, dt}{I}$$

Substituting the values, we have

$$\omega = \frac{3\,(0.5)\,(0.5)\,(0.5)\,(1)}{\frac{1.5\,(0.5)^2}{2}} = 2 \text{ rad/s}$$



35. **(b)** Given, $R_{\text{planet}} = \frac{R_{\text{earth}}}{10}$ and density,

$$\rho = \frac{M_{\text{earth}}}{\frac{4}{3}\pi R_{\text{earth}}^3} = \frac{M_{\text{planet}}}{\frac{4}{3}\pi R_{\text{planet}}^3}$$

$$\Rightarrow \quad M_{\text{planet}} = \frac{M_{\text{earth}}}{10^3}$$

$$g_{\text{surface of planet}} = \frac{GM_{\text{planet}}}{R_{\text{planet}}^2} = \frac{GM_e \cdot 10^2}{10^3 \cdot R_e^2}$$

$$= \frac{GM_e}{10R_e^2} = \frac{g_{\text{surface of earth}}}{10}$$

$$g_{\text{depth of planet}} = g_{\text{surface of planet}}\left(\frac{x}{R}\right)$$

where, x = distance from centre of planet.

$\therefore$ Total force on wire

$$F = \int_{4R/5}^{R} \lambda\, dx\, g\left(\frac{x}{R}\right) = \frac{\lambda g}{R}\left[\frac{x^2}{2}\right]_{4R/5}^{R}$$

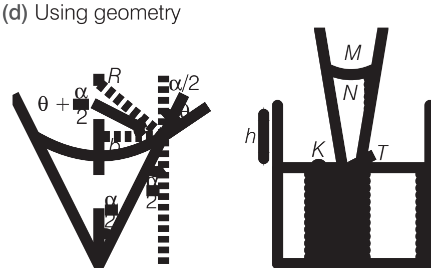
Here, $g = g_{\text{surface of planet}}$,

$R = R_{\text{planet}}$

Substituting the given values, we get

$$F = 108\text{ N}$$

36. **(d)** Using geometry



(a) (b)

$$\frac{b}{R} = \cos\left(\theta + \frac{\alpha}{2}\right)$$

$$\Rightarrow \qquad R = \frac{b}{\cos\left(\theta + \frac{\alpha}{2}\right)}$$

Using pressure equation along the path $MNTK$

$$p_0 - \frac{2S}{R} + h\rho g = p_0$$

Substituting the value of R, we get

$$h = \frac{2S}{R\rho g} = \frac{2S}{b\rho g}\cos\left(\theta + \frac{\alpha}{2}\right)$$

37. **(4)** $Y = \dfrac{F/A}{\frac{\Delta l}{l}}$, $\Delta l = 25 \times 10^{-50}$ m

$$\frac{\Delta Y}{Y} \times 100 = \frac{10^{-5}}{25 \times 10^{-5}} \times 100 = 4\%$$

38. **(c)** From continuity equation,

$$A_1 v_1 = A_2 v_2$$

Here, $A_1 = 400 A_2$

because $r_1 = 20 r_2$

and $A = \pi r^2$

$$\therefore \qquad v_2 = \frac{A_1}{A_2}(v_1) = 400 v_1$$

$= 400(5)$ mm/s

$= 2000$ mm/s

$= 2$ m/s

39. **(a)** $p_1 - p_2 = \frac{1}{2}\rho_a v_a^2$

$$p_3 - p_2 = \frac{1}{2}\rho_l v_l^2$$

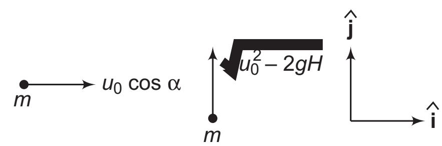
$$p_3 = p_1$$

$$\therefore \qquad \frac{1}{2}\rho_l v_l^2 = \frac{1}{2}\rho_a v_a^2$$

$$\Rightarrow \qquad v_l = \sqrt{\frac{\rho_a}{\rho_l}}\, v_a$$

$$\therefore \qquad \text{Volume flow rate} \propto \sqrt{\frac{\rho_a}{\rho_l}}$$

40. **(a)** From momentum conservation equation, we have,



$$\mathbf{p}_i = \mathbf{p}_f$$

$$\therefore\ m(u_0\cos\alpha)\hat{\mathbf{i}} + m\left(\sqrt{u_0^2 - 2gH}\right)\hat{\mathbf{j}}$$

$$= (2m)\mathbf{v} \qquad \ldots\text{(i)}$$

$$H = \frac{u_0^2 \sin^2\alpha}{2g} \qquad \ldots\text{(ii)}$$

From Eqs. (i) and (ii)

$$\mathbf{v} = \frac{u_0\cos\alpha}{2}\hat{\mathbf{i}} + \frac{u_0\cos\alpha}{2}\hat{\mathbf{j}}$$

Since both components of $\mathbf{v}$ are equal. Therefore, it is making 45° with horizontal.

41. **(5)** Velocity of first bob at highest point

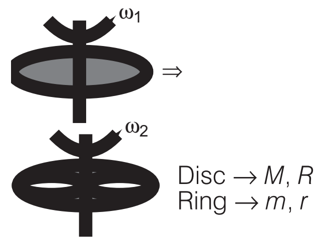
$.v_1 = \sqrt{gR} = \sqrt{gl_1}$ (to just complete the vertical circle)

= velocity of second bob just after elastic collision.

= velocity of second bob at the bottommost point

$= \sqrt{5gl_2} \quad \Rightarrow \quad \frac{l_1}{l_2} = 5$

42. **(8)** $I_1\omega_1 = I_2\omega_2$

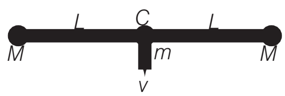


$$\therefore \omega_2 = \left(\frac{I_1}{I_2}\right)\omega_1 = \left[\frac{\frac{1}{2}MR^2}{\frac{1}{2}MR^2 + 2(mr^2)}\right]\omega_1$$

$$= \left[\frac{50(0.4)^2}{50(0.4)^2 + 8\times(6.25)\times(0.2)^2}\right](10)$$

$$= 8 \text{ rad/s}$$

43. **(b)**



Let v is the minimum velocity. From energy conservation,

$$U_c + K_c = U_\infty + K_\infty$$

$$\therefore \quad mV_c + \frac{1}{2}mv^2 = 0 + 0$$

$$\therefore \quad v = \sqrt{-2V_c} = \sqrt{(-2)\left(\frac{-2GM}{L}\right)} = 2\sqrt{\frac{GM}{L}}$$

44. **(c)** $\Delta l = \dfrac{FL}{AY} = \dfrac{FL}{(\pi r^2)Y}$

$$\therefore \quad \Delta l \propto \frac{L}{r^2}$$

$$\therefore \quad \frac{\Delta l_1}{\Delta l_2} = \frac{L/R^2}{2L/(2R)^2} = 2$$

45. **(a)**

On small sphere

$$\frac{4}{3}\pi R^3(\rho)g + kx = \frac{4}{3}\pi R^3(2\rho)g \qquad \text{...(i)}$$

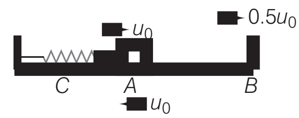
On second sphere (large)

$$\frac{4}{3}\pi R^3(3\rho)g = \frac{4}{3}\pi R^3(2\rho)g + kx \qquad \text{...(ii)}$$

By Eqs. (i) and (ii), we get

$$x = \frac{4\pi R^3 \rho g}{3k}$$

46. **(a,d)** (a) At equilibrium ($t = 0$) particle has maximum velocity u_0. Therefore velocity at time t can be written as



$$u = u_{max}\cos\omega t = u_0 \cos\omega t$$

when, $\quad u = 0.5\,u_0 = u_0\cos\omega t$

$$\therefore \quad \omega t = \frac{\pi}{3}$$

$$\therefore \quad \frac{2\pi}{T}t = \frac{\pi}{3}$$

$$\therefore \quad t = \frac{T}{6}$$

(b) $t = t_{AB} + t_{BA} = \dfrac{T}{6} + \dfrac{T}{6}$

$$= \frac{T}{3} = \frac{2\pi}{3}\sqrt{\frac{m}{k}}$$

(c) $t = t_{AB} + t_{BA} + t_{AC}$

$$= \frac{T}{6} + \frac{T}{6} + \frac{T}{4}$$

$$= \frac{7}{12}T = \frac{7\pi}{6}\sqrt{\frac{m}{k}}$$

(d) $t = t_{AB} + t_{BA} + t_{AC} + t_{CA}$

$$= \frac{T}{6} + \frac{T}{6} + \frac{T}{4} + \frac{T}{4} = \frac{5}{6}T$$

$$= \frac{5\pi}{3}\sqrt{\frac{m}{k}}$$