FUNDAMENTAL ALGORITHMS
LECTURE 1
RECURRENCE

USE CASE: $\boxed{T(n) = 8T(n/4) + n}$

Let's assume $T(1) = 1$

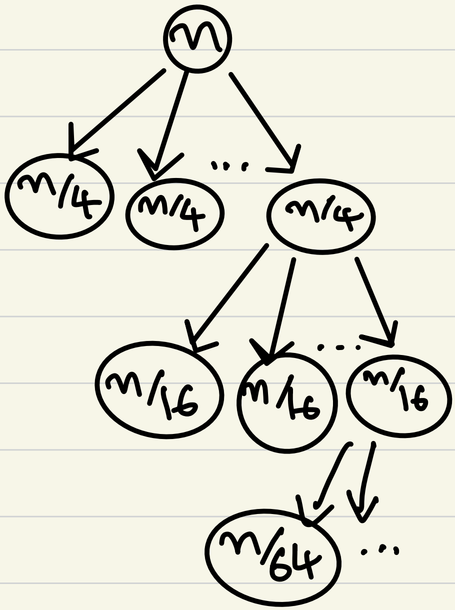
RECURSIVE TREE METHOD

1. Draw a recursion tree representing the recurrence
2. Calculate overall number of operations

Use Case | Recursion Tree

$$T(n) = \underbrace{8}_{\text{No. of subops}}\, T\big(\underbrace{n/4}_{\text{use to calc. no. of levels}}\big) + \underbrace{n}_{\text{runtime at each level}}$$

We can draw a tree as follows:



Level	No. of nodes	Operation
0	1	n
1	8	$2n$
2	64	$4n$
3	512	$8n$
⋮		
i	8^i	

Number of levels $= \log_4 n + 1$ (0^{th} level)

$$\text{Total ops} = n(1+2+4+\dots 2^{\log_4 n+1})$$
$$= n(2^{\log_4 n+2} - 1)$$
$$= n(4 \cdot 2^{\log n/\log 4} - 1)$$
$$= n(4 \cdot n^{1/2} - 1)$$
$$= 4 \cdot n^{3/2} - n$$

$$\Rightarrow \boxed{T(n) = \Theta(n^{3/2})}$$

Guess-Then-Verify

Essentially,

1. We make an educated guess
2. We use induction to verify our guess

▷ Use Case | Guess-then-verify

$$T(n) = 8(T(n/4)) + n \mid T(1) = 1$$

1) Guess: $T(n) = O(n^2) \longrightarrow 1$

Now, $T(1) = 1$

$$\Rightarrow T(1) = 1 \leq c \cdot 1^2$$

$$\Rightarrow 1 \leq c \cdot (1)^2$$

$$\Rightarrow \boxed{c \geq 1} \longrightarrow 2$$

Assume 1 is true for $i = 2 \ldots n-1$

$$\Rightarrow T(n/4) \leq (n/4)^2$$

$$\Rightarrow T(n) \leq 8(n/4)^2 + n$$

$$\Rightarrow \quad T(n) \leq \frac{n^2}{2} + n$$

Assume $n^2/2 + n \leq c' \cdot n^2$

$$\Rightarrow \quad n \leq n^2 (c' - 1/2)$$

Since $n > 2 \longrightarrow$ (3)

$$\Rightarrow \quad n(c' - 1/2) \geq 1$$

$$\Rightarrow \quad c' - 1/2 \geq n$$

From (3)

$$\Rightarrow c' - 1/2 > 2$$

$$\Rightarrow \boxed{c' > 3/2} \longrightarrow (4)$$

From (2) & (4)

$$\boxed{c > 3/2}$$

for $T(i) \leq c \cdot i^2$ s.t. $i \in [1, n]$

$$\therefore \; T(i) \leq 3/2 \, i^2$$

$$\Rightarrow \boxed{T(n) = O(n^2)}$$

But can we do better?

2) $T(n) = \Theta(n^{3/2})$ | From MT use case in the next section

$T(1) = 1$

$c_1 \cdot (1)^{3/2} \leq T(1) \leq c_2 \cdot (1)^{3/2}$

$$\boxed{c_1 \leq 1 \leq c_2} \rightarrow (5)$$

Using induction

$c_1 \cdot i^{3/2} \leq T(i) \leq c_2 \cdot i^{3/2}$ s.t. $i \in [2..n-1]$

$$c_1 \cdot (n/4)^{3/2} \leq T(n/4) \leq c_2 \cdot (n/4)^{3/2} \rightarrow (6)$$

$\therefore T(n) = 8T(n/4) + n$

From (6)

$T(n) \leq c_2 \cdot (n/4)^{3/2} + n$

Assume $c_2 \cdot (n/4)^{3/2} + n \leq c_3 \cdot n^{3/2}$

$\Rightarrow c_2 \cdot n^{3/2}/8 + n \leq c_3 \cdot n^{3/2}$

From (5) $c_2 = 1$

$$\therefore \quad \frac{n^{3/2}}{8} + n \leq c_3 \cdot n^{3/2}$$

$$\Rightarrow n^{3/2}(c_3 - 1/8) \geq n$$

$$\Rightarrow n^{1/2}(c_3 - 1/8) \geq 1$$

Since $n > 2$

$$\Rightarrow \sqrt{2}(c_3 - 1/8) \geq 1$$

$$\Rightarrow c_3 - 1/8 \geq \frac{1}{\sqrt{2}}$$

$$\Rightarrow c_3 \geq \frac{1}{2^{1/2}} + \frac{1}{2^3}$$

$$\Rightarrow c_3 \geq \frac{2^{3/2} + 1}{2^3} \simeq 0.478 \longrightarrow (7)$$

From (5) & (7)

$$T(n) \leq c \cdot n^{3/2} \quad \text{s.t. } c \geq 1$$

$$\Rightarrow \boxed{T(n) = O(n^{3/2})} \longrightarrow (8)$$

From (6)

$$c_1 \cdot n^{3/2}/8 + n \leq T(n)$$

Assume $c_1 \cdot \frac{n^{3/2}}{8} + n \geq c_4 \cdot n^{3/2}$

From (5) $c_1 \geq 1$

$$\Rightarrow \frac{n^{3/2}}{8} + n \geq c_4 \cdot n^{3/2}$$

$$\Rightarrow n \geq n^{3/2} (c_4 - 1/8)$$

$n \geq 2$

$$\Rightarrow 2 \geq 2^{3/2} (c_4 - 1/8)$$

$$\Rightarrow 2^{-1/2} \geq c_4 - 1/2^3$$

$$\Rightarrow c_4 \leq \frac{1}{2^{1/2}} - \frac{1}{2^3}$$

$$\Rightarrow c_4 \leq \frac{2^{5/2} - 1}{2^3}$$

$$\Rightarrow c_4 \leq 0.582 \longrightarrow (9)$$

From (5) & (9)

$T(n) \geq c \cdot n^{3/2}$ s.t. $c \leq 0.582$

$$\Rightarrow \boxed{T(n) = \Omega(n^{3/2})} \longrightarrow (10)$$

From 8 & 10

$$c_1 \cdot n^{3/2} \leq T(n) \leq c_2 \cdot n^{3/2} \quad \text{s.t.} \quad c_1 \leq 0.582$$

$$c_2 \geq 1$$

$$\therefore \boxed{T(n) = \Theta(n^{3/2})}$$

Master Theorem

For any eq^n of the form...

$T(n) = aT(n/b) + f(n)$, s.t. $a \geq 1, b > 1$

1. If $f(n) = O(n^{\log_b a - \epsilon}) \Rightarrow T(n) = \Theta(n^{\log_b a})$
2. If $f(n) = \Theta(n^{\log_b a}) \Rightarrow T(n) = \Theta(n^{\log_b a} * \log n)$
3. If $f(n) = \Omega(n^{\log_b a + \epsilon}) \Rightarrow T(n) = \Theta(f(n))$
   $\epsilon > 0$

NOTE: Does not work if...

- ▷ $T(n)$ is not monotone, eg: $T(n) = \sin n$
- ▷ $f(n)$ is not polynomial, eg $f(n) = 2^n$
- ▷ a is not a constant, eg $a = 2n$
- ▷ $a < 1$

▷ Use Case | Master Theorem (MT)

$$T(n) = 8(T(n/4)) + n$$

$a = 8$, $b = 4$

$$g(n) = n^{\log_4 8} = n^{\log_2 8 / \log_2 4}$$

$$= n^{3/2}$$

In terms of growth,

$$f(n) = n << g(n) = n^{3/2}$$

$$\therefore \quad f(n) = O(g)$$

$$\Rightarrow f(n) = O(n^{3/2}) \longrightarrow (1)$$

Using MT point I & (1)

$$\boxed{T(n) = \Theta(n^{3/2})}$$

Domain-Range Substitution

▷ Domain Substitution

For any $T(n) = T(n/b) + f(n)$

① Subst. n with b^k

$\Rightarrow T(b^k) = T(b^{k-1}) + f(b^k)$

② Set $S(k) = T(b^k)$

$\Rightarrow S(k) = S(k-1) + f(b^k)$

③ Solve this simpler relation $S(k)$

Use Case | Domain Substitution

$T(n) = T(n/5) + n^2, \; T(1) = 1$

Let $n = 5^k, \; k \geq 0 \rightarrow$ ①

$\Rightarrow T(5^k) = T(5^{k-1}) + 5^{2k}$

Let $S(k) = T(5^k)$

$\Rightarrow S(k) = S(k-1) + 5^{2k}$

$\Rightarrow S(k) = 25^k + 25^{k-1} + \dots 1$

$$= \frac{25^{k+1} - 1}{24}$$

$$\Rightarrow S(k) = \frac{25 \cdot 25^{k} - 1}{24}$$

from (1)

$$k = \log_5 n$$

$$\Rightarrow T(n) = S(k) = \frac{25 \cdot 5^{2\log_5 n} - 1}{24}$$

$$\Rightarrow \boxed{T(n) = \frac{25 \cdot n^2 - 1}{24}}$$

▷ Range Substitution

For any $T(n) = aT(n-1) + f(n)$

① Divide both sides with a^n

$$\therefore \frac{T(n)}{a^n} = \frac{T(n-1)}{a^{n-1}} + \frac{f(n)}{a(n)}$$

② Substitute $T(n)/a^n$ with a $P(n)$

$$\therefore P(n) = P(n-1) + f'(n)$$

③ Solve for $P(n)$

Use Case | Range Subst.

$$T(n) = 4T(n-1) + 2^n \; ; \; T(0) = 1$$

$$\Rightarrow \frac{T(n)}{4^n} = \frac{T(n-1)}{4^{n-1}} + \frac{1}{2^n}$$

Let $P(n) = \dfrac{T(n)}{4^n} \longrightarrow (1)$

$$\Rightarrow P(n) = P(n-1) + 1/2^n$$

$$\Rightarrow P(n) = (1/2)^n + (1/2)^{n-1} + \dots 1$$

$$\Rightarrow P(n) = \frac{(1/2)^{n+1} - 1}{1/2 - 1}$$

$$\Rightarrow P(n) = 2 - (1/2)^n$$

From 1

$$\Rightarrow T(n) = 4^n \cdot P(n)$$

$$= 4^n (2 - (1/2)^n)$$

$$\boxed{\Rightarrow T(n) = 2 \cdot 4^n - 2^n}$$

Use Case | Domain - Range Sub.

Domain Sub

$$T(n) = 8 \cdot T(n/4) + n \quad ; T(1) = 1$$

Let $n = 4^k$ s.t. $k \geq 0$ $\rightarrow$ 1

$$T(4^k) = 8 \cdot T(4^{k-1}) + 4^k$$

Let $S(k) = T(4^k)$ $\longrightarrow$ 2

$$S(0) = T(4^0) = T(1) = 1$$

Range Sub

$$\Rightarrow S(k) = 8 \cdot S(k-1) + 4^k$$

$$\Rightarrow \frac{S(k)}{8^k} = \frac{S(k-1)}{8^{k-1}} + \left(\frac{1}{2}\right)^k$$

Let $P(k) = \frac{S(k)}{8^k}$ $\longrightarrow$ 3

$$P(0) = \frac{S(0)}{8^0} = 1$$

$$\Rightarrow P(K) = P(K-1) + (1/2)^K$$

$$\Rightarrow P(K) = (1/2)^K + (1/2)^{K-1} + \dots 1$$

$$\Rightarrow P(K) = \frac{(1/2)^{K+1} - 1}{1/2 - 1}$$

$$\Rightarrow P(K) = 2 - (1/2)^K$$

From 3

$$S(K) = 8^K \cdot P(K)$$

$$= 2.8^K - 4^K$$

From 1 & 2

$$T(n) = 2.8^{\log_4 n} - 4^{\log_4 n}$$

$$= 2.2^{\frac{3.\log n}{\log 4}} - n$$

$$= 2.2^{\log n^{3/2}} - n$$

$$= 2.n^{3/2} - n$$

$$\Rightarrow T(n) = \Theta(n^{3/2})$$