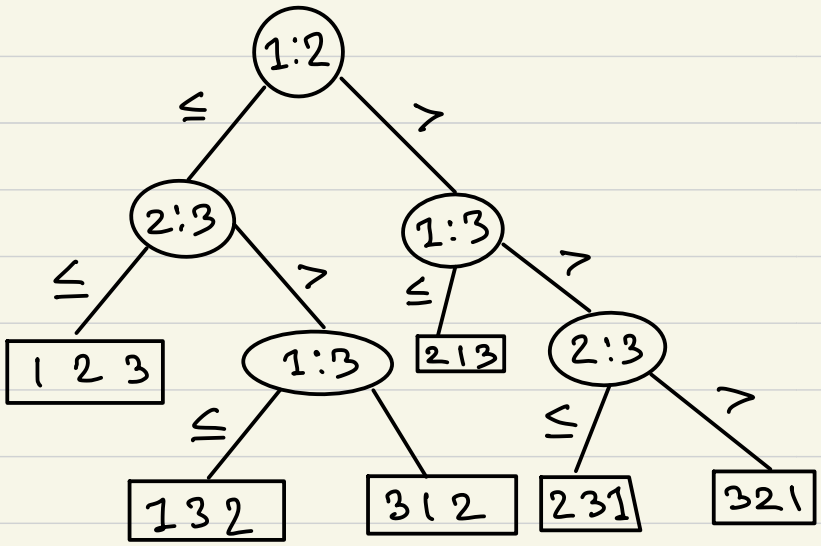
Lower Bounds

Sorting Lower Bound

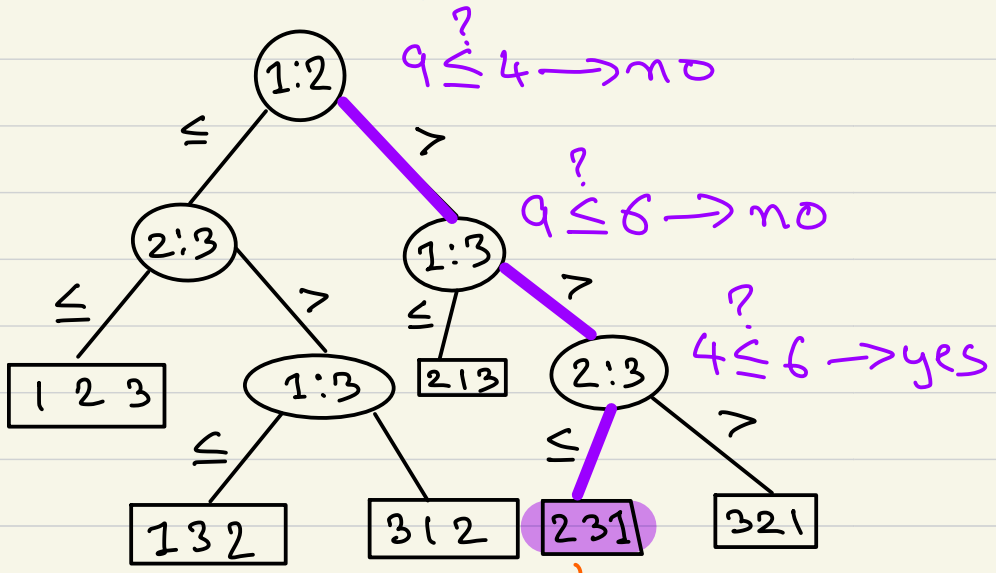
▷ Decision Tree Analysis



▷ i:j ⇒ if(A[i] ≤ A[j]); { yes → left
no → right

▷ i j k... → Leaf node

Use Case | A = <9, 4, 6>



Correct order is $A[2] \leq A[3] \leq A[1]$
or $<4, 6, 9>$

▷ Observations

- There can be multiple trees
- Runtime on A[...] = height of root → leaf path
- We're only counting comparisons, actual runtime could be worse.
- Worst case $T(n)$ = height of the entire tree.

⊙ Proof of $n \log n$ being the sorting lower bound

For $A[1 \ldots n]$

No. of leaf nodes (possible order) $= n!$

Since the tree is binary,

height of tree $(h) \geq \log(\text{no. of leaves}) = \log(n!)$

$\Rightarrow h \geq \log(n!) \longrightarrow (\Theta(n \log n)$

$\Rightarrow \boxed{h = \Omega(n \log n)}$

Since worst case $T(n) =$ height of tree

$$T(n) \geq n \log n$$

or

$$\boxed{T(n) = \Omega(n \log n)}$$

According to decision tree, no sorting algorithm can do better than $n \log n$.

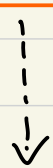
But is the decision tree the best way to find answers to complex question?

Ans: No!

▷ Element Uniqueness (EU)

Given an an array $A[1..n]$, find if there are any duplicates in the array.

Theorem: Any decision tree for elemental uniqueness has height in $\Omega(n \log n)$



Any DT, T, for EU $\Rightarrow$ DT for sorting arrays with distinct elements

▷ Every leaf node encodes all operations required to sort.

⊙ Proof | DT for EU ⇒ DT for sorting

$\forall$ T for EU

Assumptions:

▷ Input of form A[1..n], with distinct elements for all permutations

$\therefore T : [\{1..n\}, \{1..n\}..]$

$\Rightarrow A_\pi[i] = \pi(i)$

($A_\pi[i]$: Unique element; $\pi(i)$: All perm. of $A_\pi[i]$)

▷ Use notation $\pi^{-1}(k) = i$ s.t. $\pi(i) = k$

Eg: $\pi = (3, 1, 2) \Rightarrow \pi(3) = 2, \pi^{-1}(2) = 3$

($\pi^{-1}(2) = 3$: index of 2 in the perm.)

Claim:

Let $\forall \pi$ ($\Rightarrow A_\pi$), the path with all "yes" outcome be P. (ends in a "yes" leaf node)

$\forall k \in [1..n]$, let $i = \pi^{-1}(k)$, $j = \pi^{-1}(k+1)$

$\Rightarrow$ (i : j) $\in P$

(i : j): comparison b/w A[i] & A[j]

For eg: $\Pi = (4,2,1,3)$, $K=3$

$\Rightarrow i = \Pi^{-1}(3) = 4$, $j = \Pi^{-1}(4) = 1$

$\therefore$ P contains 4:1

Validation:

$\forall\ K \in [1 \ldots n]$, $i = \Pi^{-1}(K)$, $j = \Pi^{-1}(K+1)$

Let A' be $A[1 \ldots n]$, except $A'[j] = K$

For eg: If $A = (4,2,1,3, K=3 \Rightarrow A'(3,1,2,3)$

Now, $EU(A_\Pi) =$ Yes & $EU(A'_\Pi) =$ No

$\Rightarrow T(A) = \{\ldots (i:j) \rightarrow y\} \rightarrow EU(A) =$ Yes &

$T(A') = \{\ldots (i:j) \rightarrow n\} \rightarrow EU(A') =$ No

Basically, (i:j) is the only difference between T(A) & T(A').

Or, if (i:j) $\notin$ P, then T(A) = T(A'), since all other comparisons are same, $\Rightarrow$ T(A') = Yes, which is wrong.

Hence, for the unique path P, with all "yes",

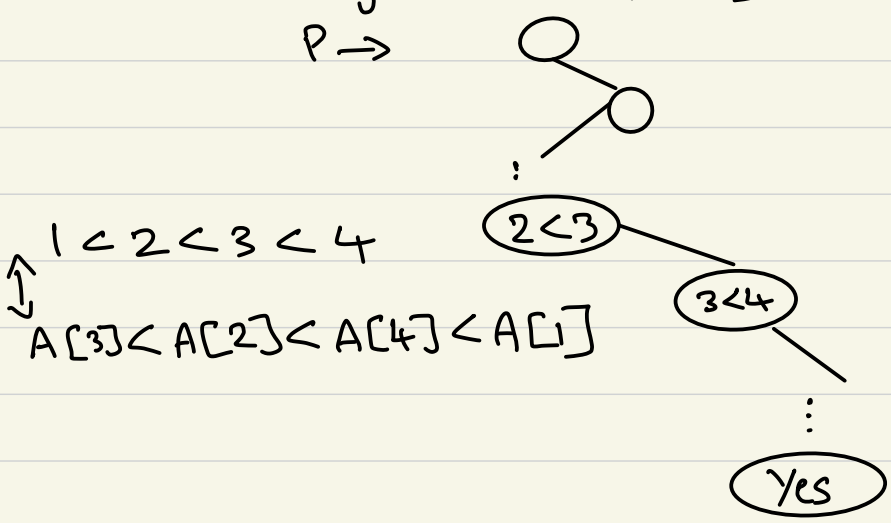
$\boxed{(i:j) \in P}$

Corollary

$\forall \pi$, if P is induced path of $T(\pi)$, then P is actually comparing numbers (not indices).

$\therefore$ All the comparisons for $A[1..n]$, i.e., $\{(1 \overset{?}{\leq} 2), (2 \overset{?}{\leq} 3), (3 \overset{?}{\leq} 4) .., (n-1 \overset{?}{\leq} n)\}$ uniquely determine π, s.t. $EU(A_\pi) = Yes$

For eg: $A[4, 2, 1, 3]$



Since all comparisons are included in P,

DT of EU $\Rightarrow$ DT of Sorting

$\Rightarrow$ Runtime of EU $\geq n \log n$

▷ Sorting In Linear Time | Counting Sort

But how do we sort in linear time?

Ans: No comparisons!

Intuition

Input: $A[1 \ldots n]$, where $A[j] \in \{1, 2, \ldots k\}$

Output: $B[1 \ldots n]$, sorted

Auxiliary storage: $C[1 \ldots k]$

Imagine an input array $A[4, 1, 3, 4, 3]$

Broadly, we do the following,

A: | 4 | 1 | 3 | 4 | 3 |

1 2 3 4

① Count occurrences

C: | 1 | 0 | 2 | 2 |

② Count values $\leq C[i]$ or $C'[j] = \sum_{i=1}^{j} C[i]$

1 2 3 4

C': | 1 | 1 | 3 | 5 |

③ Populate sorted array B.

A: | 4 | 1 | 3 | 4 | 3 |

1 2 3 4

C': | 1 | 1 | 3 | 5 |

reference index

1 2 3 4 5

B: | | | 3 | | |

insert

-1

A: | 4 | 1 | 3 | 4 | 3 |

1 2 3 4

C': | 1 | 1 | 2 | 5 |

reference index

1 2 3 4 5

B: | | | 3 | | 4 |

insert

-1

A: | 4 | 1 | 3 | 4 | 3 |

1 2 3 4

C': | 1 | 1 | 2 | 4 |

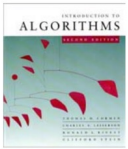
reference index

1 2 3 4 5

B: | | 3 | 3 | | 4 |

insert

Sorted → B: | 1 | 3 | 3 | 4 | 4 |



Analysis

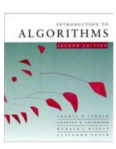
$\Theta(k)$ { **for** $i \leftarrow 1$ **to** k
do $C[i] \leftarrow 0$

$\Theta(n)$ { **for** $j \leftarrow 1$ **to** n
do $C[A[j]] \leftarrow C[A[j]] + 1$

$\Theta(k)$ { **for** $i \leftarrow 2$ **to** k
do $C[i] \leftarrow C[i] + C[i-1]$

$\Theta(n)$ { **for** $j \leftarrow n$ **downto** 1
do $B[C[A[j]]] \leftarrow A[j]$
$C[A[j]] \leftarrow C[A[j]] - 1$

$\Theta(n + k)$



Running time

If $k = O(n)$, then counting sort takes $\Theta(n)$ time.

- But, sorting takes $\Omega(n \lg n)$ time!
- Why ?

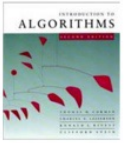
Answer:

- ***Comparison sorting*** takes $\Omega(n \lg n)$ time.
- Counting sort is not a ***comparison sort***.
- In fact, not a single comparison between elements occurs!

Stable Sorting: Counting sort is a stable sort as it preserves the input order among equal elements

▷ Radix Sort

- Sort numbers digit-by-digit.



Operation of radix sort

329	720	720	329
457	355	329	355
657	436	436	436
839	457	839	457
436	657	355	657
720	329	457	720
355	839	657	839

▷ Correctness of radix sort

For sorting on t^{th} digit, we assume that the numbers are sorted for $t-1$ digits.

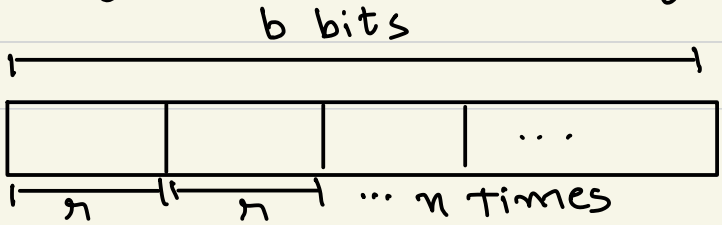
▷ We sort on t^{th} digit

① Two numbers with different t^{th} digit are correctly sorted
② Two numbers with equal t^{th} digit are put in the same order as before.

∴ The numbers are sorted till the t^{th} digit

▷ Analysis of Radix Sort

For a give "b" bit number, we can split it into "n" digits, each "r" bit long.



Hence, we have n digits in base 2^r where $n = b/r$

Recall Counting sort runtime $= \Theta(n+k)$ to sort n numbers in range $[0, k-1]$

USE CASE | $b=32$, $m=64$ numbers

① $r=32 \Rightarrow$ 1 32-bit number

$T(n) = 1 \times (2^{32} + 64) \approx 2^{32}$

② $r=16 \Rightarrow$ 2 16-bit numbers

$T(n) = 2 \times (2^{16} + 2^6) \approx 2^{17}$

③ $r = 4 \Rightarrow$ 8 4-bit numbers

$\Rightarrow T(n) = 8(2^4 + 64) = 640$

④ $r=1 \Rightarrow$ 32 1-bit numbers

$\Rightarrow T(n) = 32(2+64)$

$\approx 2^{11}$

$\therefore T(n \mid r=4) << T(n \mid r=1) << T(n \mid r=16) << T(n \mid r=32)$

Now,

$$T(n|r) = \frac{b}{r}(n + 2^r)$$

For optimal r,

① if $r << \log_2 n$, i.e., $2^r << n$

$$\Rightarrow T(n|r) = \frac{b}{r}(+2^r) \approx \frac{bn}{r}$$

$$\Rightarrow T(n|r) = O\left(\frac{bn}{r}\right)$$

push toward $\log_2 n$ to decrease runtime

② if $r >> \log_2 n$, i.e., $2^r >> n$

$$\Rightarrow T(n|r) = O\left(\frac{b \cdot 2^r}{r}\right)$$

reduce to $\log_2 n$ to decrease runtime.

By minimizing $T(n|r)$, we get

$$r \approx \log_2 n \quad \text{or} \quad \text{base } 2^r \approx$$

$$T(n) = O\left(\frac{bn}{\log n}\right)$$

Use Case | Radix sort numbers from $1 \ldots n^d$

$$r = \log_2 n \Rightarrow \text{base } n$$

$$\therefore \; b = d \log_2 n$$

$$\boxed{T(n) = O(dn)}$$

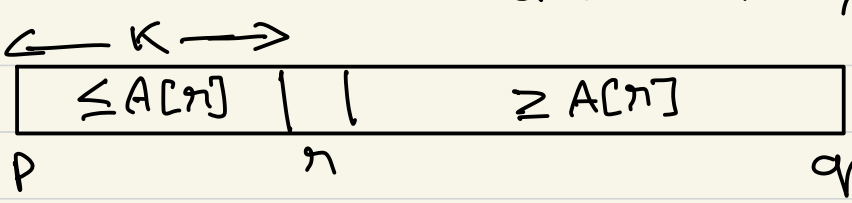
Order Statistics

Use Case | Find i^{th} smallest element in array

▷ Randomized divide-and-conquer

Essentially, we use the partition function of randomized quicksort.

```
Rand-Select(A, p, q, i) ⟶ i^th smallest in A[p..q]
    if p == q then return A[p]
    r ⟵ Rand-partition(A, p, q)
    k ⟵ r - p + 1  ⟶ k = rank(A[r])
    if i == k then return A[r]
    if i < k
        then return Rand-Select(A, p, r-1, i)
    else return Rand-select(A, r+1, q, i-k)
```



Analysis

- Worst case running time
  - Unlucky: $T(n) = T(n-1) + \Theta(n) = \Theta(n^2)$
  - Lucky: $T(n) = T(9n/10) + \Theta(n) = \Theta(n)$

- Expected running time = $O(n)$

Conclusion

- Linear expected time
- Bad worst case

Can we do better in worst case?

Ans: Yes, just generate a good pivot.

ⓟ Worst-case Linear-Time Order Statistics

	SELECT(i, n)
$\Theta(n)$	1. Divide the n elements into groups of 5. Find the median of each 5-element group by rote.
$T(n/5)$	2. Recursively SELECT the median x of the $\lfloor n/5 \rfloor$ group medians to be the pivot.
$\Theta(n)$	3. Partition around the pivot x. Let $k = \text{rank}(x)$.
$T(7n/10)$	4. **if** $i = k$ **then return** x **elseif** $i < k$ **then** recursively SELECT the ith smallest element in the lower part **else** recursively SELECT the $(i-k)$th smallest element in the upper part

$$T(n) = T\left(\frac{1}{5}n\right) + T\left(\frac{7}{10}n\right) + \Theta(n)$$

For a large enough constant "c".

$$T(n) \leq \frac{1}{5}cn + \frac{7}{10}cn + \Theta(n)$$

$$T(n) = \frac{18}{20}cn + \Theta(n)$$

$$T(n) = cn - \left(\frac{2}{20}cn - \Theta(n)\right)$$

$$\boxed{T(n) \leq cn}$$

↳ However, in practice, "c" is very large. Not very useful.

▷ Randomized select is more practical.