Navier-Stokes Equation Derivation in LaTeX

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Continuity Equation

$$\left(\frac{dm}{dt}\right)_{syst} = 0$$

Using **Reynold's Transport Theorem [RTT]**:

Taking Intensive Property, $mass = 1$

$$\int_{cv} \frac{\partial \rho}{\partial t} d\mathcal{V} + \int_{cv} \rho(V \cdot n) dA = 0$$

$$\implies \int_{cv} \frac{\partial \rho}{\partial t} d\mathcal{V} + \sum_i (\rho_i A_i V_i)_{out} - \sum_i (\rho_i A_i V_i)_{in} = 0$$

For very small elemental control volume

$$\int_{cv} \frac{\partial \rho}{\partial t} d\mathcal{V} \approx \frac{\partial \rho}{\partial t} dxdydz$$

Face	**Inlet Mass Flow**	**Outlet Mass Flow**
x	$\rho_u dydz$	$\left[\rho_u + \frac{\partial}{\partial x}(\rho_u)dx\right] dydz$
y	$\rho_v dxdz$	$\left[\rho_v + \frac{\partial}{\partial y}(\rho_v)dx\right] dxdz$
z	$\rho_w dxdy$	$\left[\rho_w + \frac{\partial}{\partial z}(\rho_w)dx\right] dxdy$

By substituting the Inlet mass flow and Outlet mass flow in **RTT** we get

$$\frac{\partial \rho}{\partial t} dxdydz + \frac{\partial}{\partial x}(\rho_u)dxdydz + \frac{\partial}{\partial y}(\rho_v)dxdydz + \frac{\partial}{\partial z}(\rho_w)dxdydz = 0$$

Dividing the whole equation by $dxdydz$ we get

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\rho_u + \frac{\partial}{\partial y}\rho_v + \frac{\partial}{\partial z}\rho_w = 0$$

Using Laplasian operator

$$\nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

The equation can be written as:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho V) = 0$$

(or)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{V} = 0$$

This is known as the **Unsteady Term** of the equation.

For Incompressible flow, regardless of Steady or Unsteady flow:

$$\frac{\partial \rho}{\partial t} \approx 0 \implies \nabla \cdot V = 0$$

Momentum Equation

By Newton's 3^{rd} law

$$\begin{aligned} F &= ma \\ &= m\frac{dV}{dt} = \frac{d}{dt}(mV) \end{aligned}$$

Intensive property $B = mV \implies \beta = \frac{dB}{dm} = V$ (Velocity).

$$\frac{d}{dt}(mV)_{syst} = \sum F = \frac{d}{dt}\left(\int_{cv} V\rho d\mathcal{V}\right) + \int_{cs} V\rho(V \cdot n)dA \tag{1}$$

Mass flow rate $\dot{M}$ at each face of the control volume can be written as:

$$\begin{aligned} \dot{M}_{cs} &= \int_{sec} V\rho(V \cdot n)dA \\ \dot{M}_{seci} &= V_i(\rho_i V_{ni} A_i) = \dot{m}_i V_i \end{aligned}$$

Substituting $\dot{M}_{cs}$ & $\dot{M}_{seci}$ in 1 we get:

$$\implies \sum F = \frac{d}{dt}\left(\int_{cv} V\rho d\mathcal{V}\right) + \sum(\dot{m}_i V_i)_{out} - \sum(\dot{m}_i V_i)_{in} \tag{2}$$

For a small elemental control volume

$$\frac{\partial}{\partial t}(\int_{cv} V\rho d\mathcal{V} \approx \frac{\partial}{\partial t}(\rho V)dxdydz$$

Faces	**Inlet Momentum Flux**	**Outlet Momentum Flux**
x	$\rho u V dydz$	$\left[\rho u V + \frac{\partial}{\partial x}(\rho u V)dx\right] dydz$
y	$\rho v V dxdz$	$\left[\rho v V + \frac{\partial}{\partial y}(\rho v V)dx\right] dxdz$
z	$\rho w V dxdy$	$\left[\rho w V + \frac{\partial}{\partial z}(\rho w V)dx\right] dxdy$

By substituting the Inlet momentum flux and Outlet momentum flux in 2 we get:

$$\sum F = dxdydz\left[\frac{\partial}{\partial t}(\rho V) + \frac{\partial}{\partial x}(\rho u V) + \frac{\partial}{\partial y}(\rho v V) + \frac{\partial}{\partial z}(\rho w V)\right] \tag{3}$$

By splitting the terms within the brackets into 2 parts we get:

$$
\begin{aligned}
&\implies \frac{\partial}{\partial t}(\rho V) + \frac{\partial}{\partial x}(\rho u V) + \frac{\partial}{\partial y}(\rho v V) + \frac{\partial}{\partial z}(\rho w V) \\
&\implies V \left[\cancelto{0}{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho V)} \right] + \rho \left(\frac{\partial V}{\partial t} + u\frac{\partial V}{\partial x} + v\frac{\partial V}{\partial y} + w\frac{\partial V}{\partial z} \right)
\end{aligned}
\tag{4}
$$

The continuity equation becomes $Zero$ Where the Total derivative is written as:

$$
\frac{\partial V}{\partial t} + u\frac{\partial V}{\partial x} + v\frac{\partial V}{\partial y} + w\frac{\partial V}{\partial z} = \frac{dV}{dt} \tag{5}
$$

The sum of all forces can be written as:

$$
\sum F = \rho \frac{dV}{dt} dxdydz \tag{6}
$$

Which is $Total\ Force\ acting = Body\ forces + Surface\ Forces$

$$
\begin{aligned}
Body\ Force\ (Gravity) &\implies dF_{grav} = \rho g dxdydz \\
Surface\ Force &\implies Hydrostatic\ Pressure + Viscous\ Stresses
\end{aligned}
\tag{7}
$$

$$
\sigma_{ij} = \begin{bmatrix} -p + \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & -p + \tau_{yy} & \tau_{xy} \\ \tau xx & \tau xy & -p + \tau_{xz} \end{bmatrix}
$$

Gradient of these stresses causes net force

$$
dF_{x.surf} = \left[\frac{\partial}{\partial x}(\sigma_{xx}) + \frac{\partial}{\partial x}(\sigma_{xy}) + \frac{\partial}{\partial x}(\sigma_{xz}) \right] dxdydz
$$

$$
\frac{dF_x}{d\mathcal{V}} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}(\tau_{xx}) + \frac{\partial}{\partial y}(\tau_{xy}) + \frac{\partial}{\partial z}(\tau_{xz}) \tag{8}
$$

$$
\frac{dF_y}{d\mathcal{V}} = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}(\tau_{xy}) + \frac{\partial}{\partial y}(\tau_{yy}) + \frac{\partial}{\partial z}(\tau_{yz}) \tag{9}
$$

$$
\frac{dF_z}{d\mathcal{V}} = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}(\tau_{xz}) + \frac{\partial}{\partial y}(\tau_{yz}) + \frac{\partial}{\partial z}(\tau_{zz}) \tag{10}
$$

Or simply written as:

$$
\left(\frac{dF}{d\mathcal{V}} \right)_{surf} = -p\nabla + \left(\frac{dF}{d\mathcal{V}} \right)_{viscous} \tag{11}
$$

Where

$$
\left(\frac{dF}{d\mathcal{V}} \right)_{viscous} = i\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \right) + j\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} \right) + k\left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \right)
$$

Or

$$
\left(\frac{dF}{d\mathcal{V}} \right)_{viscous} = \nabla \cdot \tau_{ij}
$$

Where

$$\tau_{ij} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{bmatrix}$$

By substituting 7 and 11 in 6 we get:

$$\rho g - \nabla p + \nabla \cdot \tau_{ij} = \rho \frac{dV}{dt} \tag{12}$$

$$\begin{aligned} \rho_{gx} - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} &= \rho \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \right) \\ \rho_{gy} - \frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} &= \rho \left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \right) \\ \rho_{gz} - \frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} &= \rho \left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \right) \end{aligned} \tag{13}$$

By **Newton's law of Viscosity**

$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x} \qquad \tau_{yy} = 2\mu\frac{\partial v}{\partial y} \qquad \tau_{zz} = 2\mu\frac{\partial w}{\partial z}$$

(or)

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \qquad \tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

On substituting the above equations in 13 we get:

$$\begin{aligned} \rho_{gx} = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial w^2}\right) &= \rho\frac{du}{dt} \\ \rho_{gy} = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial w^2}\right) &= \rho\frac{dv}{dt} \\ \rho_{gz} = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial w^2}\right) &= \rho\frac{dw}{dt} \end{aligned} \tag{14}$$

We have arrived at the famous **Navier-Stokes Equation**

Navier-Stokes Equation

$$\boxed{\rho\frac{\overrightarrow{DV}}{Dt} = -\nabla p + \overrightarrow{\rho_g} + \mu\nabla^2\overrightarrow{V}}$$

Where

$$\begin{aligned} \rho\frac{\overrightarrow{DV}}{Dt} &\implies \text{Total Derivative} \\ -\nabla p &\implies \text{Pressure gradient} \\ \overrightarrow{\rho_g} &\implies \text{Body force term} \\ \mu\nabla^2\overrightarrow{V} &\implies \text{Diffusion term} \end{aligned}$$