Notes on Axler's Linear Algebra Done Right

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Contents

1 Vector Spaces

1.1 R and C

1.2 Definition of a Vector Space

1.3 Subspaces

A subset U of a vector space V is called a **subspace** if U is also a vector space, using the same addition and scalar multiplication defined on V.

1.3.1 Conditions to a be a subspace

- Additive Identity: $0 \in U$.
- Closed under addition.
- Closed under multiplication.

1.3.2 Sums of Subspaces

Let $U_1 \ldots U_m$ be subsets of V. The sum of $U_1 \ldots U_m$ is the set of all possible sums of elements of $U_1 \ldots U_m$.

$$U_1 + \ldots + U_m = \{u_1 + u_2 + \ldots + u_m : u_1 \in U_1, u_2 \in U_2 \ldots, u_m \in U_m\}$$

The sum of subspaces is the smallest subspace containing all the summands.

1.3.3 Direct Sums

A sum $U_1 \ldots U_m$ is called a direct sum if each element of $U_1 + \ldots + U_m$ can be written in only one way. To simplify, there is only one way to write each element of the resulting space using a sum of elements of $U_1 \ldots U_m$.

1.3.4 Conditions for a Direct Sum

- Suppose that $U_1 \ldots U_m$ are subspaces of V. Then $U_{1+} \ldots + U_m$ is a direct sum if and only if the only way to create the 0-vector is to take each u_j in the sum expression $u_1 + \ldots + u_m = 0$ to be 0.
- Corollary: Suppose that U and W are subspaces of V. Then $U_1 + \ldots + U_m$ is a direct sum if and only if $U \cap W = 0$.

2 Finite Dimensional Vector Spaces

2.1 Span and Linear Independence

2.1.1 Span

The set of all linear combinations of a list of vectors $v_1 \ldots v_m$ in V is called the span of $v_1 \ldots v_m$. The span of the empty list () is defined to be $\{0\}$.

2.1.2 Finite Dimension

A vector space is finite dimensional if there is a list of vectors that spans the space. Note that lists are, by definition, finite dimensional.

2.1.3 Polynomials

- A function $p : \mathbb{F} \mapsto \mathbb{F}$ is called a polynomial with cefficients in F if there exists $a_0 \dots a_m \in \mathbb{F}$ such that
$$p(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_m z^m \quad \forall z \in \mathbb{F}$$
.
- P(F) is the set of all polynomials in **F**.

2.1.4 Linear Dependence

- A list $v_1 \dots v_m$ is linearly independent if the only choice of $a_1 \dots a_m \in \mathbb{F}$ that makes $a_1 v_1 + \dots + a_m v_m = 0$ is $a_1 = \dots = a_m = 0$. That is to say, all the coefficients in the sum must be 0.
- The empty list is declared to be linearly independent.

2.2 Bases

2.2.1 Basis

A basis of V is a list of vectors in V that is linearly independent and spans V.

2.2.2 Criterion for basis

A list $v_1 \dots v_n$ of vectors in V is a basis of V if and only if every $v \in V$ can be written in the form $v = a_1 v_1 + \dots + a_n v_n$ where $a_1 \dots a_n \in \mathbb{F}$ and the list is linearly independent.

2.2.3 Direct Sums, Subspaces, and Bases

If V is finite dimensional and U is a subspace of V, there exists a subspace W of V such that $V = W \oplus U$. To simply, every subspace of a finite dimensional vector space has a partner, which is also a subspace of V, which it forms a direct sum equal to V with.

3 Linear Maps

3.1 The Vector Space of Linear Maps

3.1.1 Definition of Linear Maps

A **linear map** from V to W is a function $T : V \mapsto W$ such that the following properties are true:

- Additivity:
$$T(u+v) = T(u) + T(v)$$
- Homogeneity:
$$T(\lambda v) = \lambda(Tv)$$

The set of all linear maps from V to W is denoted $\mathcal{L}(V, W)$.

3.1.2 Linear maps and basis of domain

Theorem 1. *Suppose $v_1 \dots v_n$ is a basis of V and and $w_1 \dots w_n \in W$. Then $\exists T : V \mapsto W$ such that $Tv_j = w_j$ for each j in $1 \dots n$.*

This theorem asserts that once we know the behavior of a linear map over the basis of vectors, the linear map is uniqely defined for all the vectors in the space.

3.1.3 Algebraic Operations on L(V,W)

- Addition: $(S + T)(u) = Su + Tu$
- Scalar Multiplication: $(\lambda T)(v) = \lambda(Tv)$

Theorem 2. *$\mathcal{L}(V, W)$ is a vector space with the addition and scalar multiplication defined above.*

3.1.4 Product of Linear Maps

The product of two linear maps is just function composition when the domains make sense.

3.1.5 Algebraic Properties of Linear Maps

- Associativity $(T_1T_2)T_3 = T_1(T_2T_3)$
- Identity
- Distributivity
- Linear maps take 0 to 0.

Multiplication of linear maps is not commutative.

3.2 Null Spaces and Ranges

3.2.1 Null Space

- For $T \in L(V, W)$, the **null space** of T, denoted **null T** is the subset of V containing those vectors that T maps to 0. This can also be called the kernel.
- Suppose $T \in L(V, W)$. Then null T is a subspace of V.

3.2.2 Injectivity and Null Spaces

- A function $T : V \mapsto W$ is called injective if Tu=TV implies u=v.
- Let $T \in L(V, W)$. Then T is injective iff null T = 0.

3.2.3 Definition of Range

For T a function for V to W, the range of T is the subset of W consisting of those vectors which are of the form Tv for some $v \in V$. This means the same as image.

3.2.4 Ranges and Subspaces

If $T \in L(V, W)$, then range T is a subspace of W.

3.2.5 Definition of Surjective

A function $T : V \mapsto W$ is called surjective if its range equals W. This means the same as onto.

3.2.6 Fundamental Theorem of Linear Maps

Suppose V is finite dimensional and $T \in L(V, W)$. Then range T is finite-dimensional and dim V = dim null T + dim range T.